Proceedings
of the
31st Conference
of the
International Group for the
Psychology of Mathematics Education

PME 31

Seoul, Korea, July 8-13, 2007

Volume 3

Research Reports

Han – Miy

Editors

Jeong-Ho Woo **Hee-Chan Lew**
Kyo-Sik Park **Dong-Yeop Seo**

The Korea Society of Educational Studies in Mathematics

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The Korea Society of Educational Studies in Mathematics
The Republic of Korea

The Proceedings are also available on CD-ROM

ISSN 0771-100X

Cover Design: Hyun-Young Kang

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Research Reports

ELEMENTARY EDUCATION STUDENTS' MEMORIES OF MATHEMATICS IN FAMILY CONTEXT

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This study looks at an interesting relationship that was found between elementary education students' advancement in mathematics and encouragement they felt to have received from home, the higher scoring students advancement correlating negatively with encouragement. In a qualitative analysis we identified different aspects of mathematics in family context and found some possible clues for explaining the negative correlation between encouragement and advancement: we found strong positive role models among those who advanced and conflicts with parent during mathematics tutoring among those who declined.

INTRODUCTION

Family background and mathematics

Studies have repeatedly confirmed the effect of family background in students' success in schools. In TIMMS study it was possible to predict a significant part of the test result variation of US and Western European students by only a few family background variables, such as parent' education, amount of books at home and presence of both parents in family. However, the effect of family background varied between countries (9% in Iceland; 26.4% in Switzerland). (Woessmann, 2004)

There is also evidence for the effect of parental attitudes and beliefs on child's attitudes and beliefs. Catsambis (2001) found a consistent conclusion across studies that parents' educational aspirations are strongly associated to students' levels of achievement in both primary and secondary education. She also found some negative relationships between parental supervision of children's homework and students' achievement. These results were confirmed in her study of 13 580 parents and their children. Multivariate analyses compared grade 12 students of similar socio-economic backgrounds, family configuration, student characteristics and prior achievement in the 8th grade. Catsambis' findings suggest that active encouragement for preparing for college may be one of the ways by which parents influence their adolescents' academic success. However her studies shed no light on the negative influence of some parental practices, which has been repeatedly attributed to parents' attempts to deal with already existing academic or behavioural problems.

In this report we will have a different perspective to this question and look at how teacher education students perceive retrospectively their childhood and family influence.

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 1-8. Seoul: PME.

Mathematical identity

How students engage with mathematics is to a large degree determined by their mathematical identity (Kaasila, Hannula, Laine & Pehkonen, 2005). Mathematical identity is constructed on the basis of student experiences in mathematics and their interpretation of these experiences. The interpretation is largely a social phenomenon, and is influenced by mathematics teacher, friends, and family. Op 't Eynde (2004) suggests that students' identity emerges in the situation: learning in the mathematics education community is characterised by an actualisation of (mathematical) identity through interactions with the teacher, the books, and the peers one engages with. We further propose that different identities suggested by Sfard & Prusak (2005) may emerge in different situations. However, identities are not only situational but each person brings one's own history to the situation and that will influence to a large degree what kind of identities are likely to actualise in the situation.

People often develop their sense of identity by seeing themselves as protagonists in different stories. What creates the identity of the character is the identity of the story and not the other way around (Ricoeur, 1992). Sfard & Prusak (2005) define identities as collections of those narratives that are reifying, enforceable and significant. In this article we shall look at the kinds of narratives student teachers have about themselves and mathematics within family context. Furthermore, these narratives will be contrasted with their success in mathematics.

Project description

In teacher education, there is the problem of low mathematical competencies and negative affective disposition of many students who enter the education. In Finland, teacher education is a popular field of study and less than 10 % of applicants are accepted for the education (NBE, 2005). Yet, roughly one fifth of the accepted students have a negative affective disposition towards mathematics (Hannula, Kaasila, Laine & Pehkonen, 2005b) and 10 % have poor grades in mathematics. The problem remains for a number of those who finish their teacher education.

This report is part of a research project "Elementary teachers' mathematics" (project #8201695), financed by the Academy of Finland (see Hannula, Kaasila, Laine & Pehkonen 2005a; 2005b). The project draws on data collected of 269 trainee teachers at three Finnish universities (Helsinki, Turku, Lapland). In this report we shall look at students' mathematics achievement and their advancement since the first test. As possible predicting variables, we shall look at the mathematics achievement and beliefs in the beginning of studies and student's gender.

Earlier results

In earlier analyses (Hannula et al., 2005b) we had identified eight principal components of students' affect, the correlations between these components, and six typical affective profiles of student teachers. In the core of student affect there were three components that were closely correlated with each other: mathematical self-confidence, liking of mathematics, and perceived difficulty of mathematics. Correlated to this core, were five additional components: positive expectation of future success, view of earlier teacher(s), perceived own diligence, insecurity as a teacher in mathematics, and encouragement from own family (in order of declining correlation with the core). Based on these components, six typical profiles were identified primarily according to their core affect (positive, neutral, negative), and, secondarily, mainly according to diligence and received encouragement. The two positive profiles were autonomous (hard working, not encouraged) and encouraged (hard working, encouraged). The two neutral profiles were pushed (encouraged, not hard working) and diligent (not encouraged, rather hard working). The remaining two negative profiles were lazy (not encouraged, not hard working, insecure as teacher) and hopeless (hard working, not encouraged, no positive expectation).

Affect was related to performance, a positive correlation was found between positive affective disposition and high scores in the test. The test result was also affected by gender (male students scoring higher), previous mathematics studies (better achievement and more advanced course selection predicting higher scores), and the enrolment procedure (students at different universities scored differently; for details see Hannula et al., 2005).

The six clusters did continue to have differences in their post-test results. That means that the clusters do reflect relevant student types with respect to their relationship with mathematics. There was even some difference in the advancement of clusters, although only one difference was statistically significant. The 'diligent' students had advanced and 'encouraged' students regressed so that their success was now on equal level. A closer analysis of the quantitative data revealed that the student encouragement from their family had a different effect on student advancement according to their achievement in the pre-test: among successful students the family encouragement was negatively correlated with advancement and among least successful students there was a positive correlation between encouragement and advancement (Hannula, Kaasila, Pehkonen & Laine, In print). We will now focus on the qualitative data In order to find some possible explanations to this finding.

METHODS

Two questionnaires were administered in autumn 2003 to measure students' situation in the beginning of their mathematics education course. The aim of the questionnaires was to measure students' experiences connected to mathematics, their views of mathematics, and their mathematical skills. Another mathematics test was administered next spring after the course. This post-test consisted of four tasks that measured understanding of infinity, division, scale and percentage

According to the preliminary analysis of the first questionnaire, 21 student teachers were selected for a qualitative study. They represented different universities and three different student types with respect to mathematics: successful with high self-confidence, unsuccessful with low self-confidence, and average performers with indifferent attitude towards mathematics. The focus students were interviewed in the beginning and at the end of their mathematics education course.

In the semi-structured interview we asked - among other topics - student teachers about their memories of mathematics in their home. If necessary, we specifically asked about their parents' and siblings' roles in their learning of mathematics. Many of them brought these issues spontaneously when they told about their mathematical school memories.

In this analysis we focus on a theme 'mathematics and home', which was identified in the quantitative analysis as an interesting issue (For details, see Hannula et al, In print). In the first phase we read all the transcribed interviews and coded all instances where they talked about home, parents, siblings or other relatives. These were read again to identify different aspects relating to home. Each focus student's relationship with home was summarised as a short memo. Finally, these analyses were contrasted with each student's success in pre- and post-test.

RESULTS

From the interviews we identified the following six themes relating to home. Within each of the topics there was also variation between students.

1) Help from home. Most students had received some help from their family members at some stage of their mathematics education. Some parents had not been able to help - at least not all the way through. Some students had not wanted help.

> Interviewer: What kind of home-related mathematics experiences do you have, and, for example, parents and siblings.
>
> Heidi: Well, from childhood I do remember and, that's elementary school time, mother and father helped as much as, as, hah, they could.
>
> Interviewer: Uhm, yes.
>
> Heidi: And there was always help available.

2) Role models at home. Students had different opinions about their family members' mathematical competencies. Some family members were 'positive' role models of successful and interested mathematics learners/users, some 'negative' role models of unsuccessful mathematics failures or uninterested mathematics avoiders.

> Ella: Mother just told that she had never been good in mathematics, but had just worked insanely hard so she has been able to get good grades.
>
> Mia: My sister, she was a real top genius in mathematics [...]

3) Value of mathematics at home. In some homes mathematics was highly valued, in others there was indifferent or even devaluing attitude.

> Sini: Father, being an engineer, is, is, appreciates mathematics, and mother is teacher, so in that sense also appreciates all subjects and mathematics among others.

4) Encouragement and/or demands from home. Some homes demanded student to put effort in schoolwork, some gave encouragement at the times of success or failure, and some did neither.

Ella: Father said that if I choose to take the more advanced mathematics in high school, then I would get like the most expensive graphic calculator available.

Aila: I've been always really diligent student [...] and had high demands from home [...] so father especially, hah, hah, would have demanded mathematics.

5) Independence from home. Some students expressed clearly that they took independently care of their studying, while others indicated clear dependence for the help available at home.

Kati: From primary grade I remember when we always did with my father the extra tasks from behind of the book. Then, I felt somehow embarrassed going to the board, because I knew that father had checked my work and I knew it to be correct. And then, eventually, I asked my father not the check my work, [...] not to say if it's wrong. I felt, somehow, dishonest going to the board.

6) Helping siblings with mathematics. Most students who had younger siblings had helped them with mathematics.

Interviewer: Have you spoken about mathematics with your siblings?

Leo: Sometimes helped my sisters with homework or with a topic; brother is so young that I have not helped him. Oldest of my sisters is two years younger. I've helped her somewhat. High school especially. [...] She is not as interested in mathematics as I am, so she chose the less advanced mathematics in high school, and she's had a lot of trouble with it. I felt it was nice that I could help her, but at times I felt it was very difficult to help her understand. When many things are clear as day for me but it was not the case with her.

When student interview data was contrasted with quantitative data, we noticed some interesting regularities. Most of the focus students who participated in both pre- and post-test achieved better in the post-test.

There were only three students whose relative achievement regressed from pre-test to post-test and one of them was only a minimal regression (from 1.44 to 1.14). Julia had scored well in the pre-test (1.75) and her success had dropped significantly by the post-test (0.12). She is especially interesting, because her family had been encouraging her strongly. Tina scored slightly below average in the pre-test (-0.20) and her performance dropped in the post-test (-0.66). Both Julia and Tina had experienced emotional conflicts with their family members when they tried to help them with mathematics.

Interviewer: You mentioned that your father helped you during high school.

Julia: Yes, somewhat, but, it's not, not very much, it's a bit like, you know, a difficult situation, because you so easily lose your nerves then

Interviewer: Uhm

Julia: both do, and if, if you start, if you really try to help, my older sister also sometimes, she was one year older, and she was really good in math, too, so she, like, also tried to help sometimes, so it, it became so easily a fight.

Tina: ...and because he [father] has never been good in school, so when he has tried, then, during elementary school to teach me, so, I lost my nerves a little bit, because he doesn't really know it himself, or he used words that were no longer used at school ...

There was another interesting regularity among those who achieved well in the pre-test and were able to improve for the post-test. Many of those had strong mathematical role models whom they respected a lot.

Interviewer: Why do you think that you can [do mathematics well].

Pekka: Don't know, hah, must be, well, genes do have their own effect.

Interviewer: Uhm.

Pekka: And then, must be father, father in his time, well, made me play and

Interviewer: Uhm

Pekka: and think and

Interviewer: Uhm

Pekka: he was a carpenter and mastered numbers and [...] from there has come the positive side...

DISCUSSION

The qualitative data revealed six major topics that appeared in students' narratives of home and mathematics: 1) getting help, 2) role models, 3) value of mathematics, 4) encouragement/demands, 5) independence, and 6) helping siblings. Each of these topics showed variation between students. However, it is difficult to tease out the causality. How much these differences in home experiences were due to the different affect and achievement of the student? How much these differences influenced the different development of affect and achievement of students? Nevertheless, the data does suggest some explanation to the different advancement of high achieving students.

One might hypothesize that the regression of encouraged high-achievers might relate to them leaving home and lacking the support they had earlier had. In this data there was no evidence for that. On the contrary, those two students who had a relative decline in their advancement from high pre-test score to average post-test had both experienced emotional confrontation with their family when they had been helped with mathematics during school years. This might be related with an inability to utilize the positive effect of working with a peer or acting as a tutor, which was identified as an important facilitator for development among teacher students (Kaasila, Hannula, Laine & Pehkonen, In Print).

On the other hand, among the students who had scored well in pre-test and advanced in the post-tests, many had strong positive role model at home and mathematical competence seemed to form an important aspect of their identity. As they had already internalised a positive disposition towards mathematics, they possibly did not experience any strong encouragement from home.

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MISTAKE-HANDLING ACTIVITIES IN THE MATHEMATICS CLASSROOM: EFFECTS OF AN IN-SERVICE TEACHER TRAINING ON STUDENTS' PERFORMANCE IN GEOMETRY

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In a quasi-experimental study with 619 students from 29 classrooms (grades 7/8) we investigated the effects of a teacher training on teachers' mistake-handling activities and students' learning of reasoning and proof in geometry. Teachers of the experimental group classrooms received a combined training in mistake-handling and teaching reasoning and proof, whereas the teachers of the control group classrooms only took part in a training on teaching reasoning and proof. Their students participated in a pre- and post-test. Moreover, they were asked to evaluate how the teachers handled their mistakes. Our findings show that the teacher training was successful: the teachers of the experimental group classrooms changed their mistake-handling behavior and, compared to the control group classrooms the students in the experimental group performed significantly better in the post-test.

INTRODUCTION

For many students and teachers mistakes are associated with negative feelings. Despite the fact that „mistakes are the best teachers", according to a well-known everyday proverb, teachers and students hardly take advantage of mistakes in class. In this paper we present a study on an in-service teacher training regarding the role of mistakes for the learning process. The findings indicate that students' mathematics achievement benefits from a change in their teachers' behavior regarding mistakes.

THEORETICAL FRAMEWORK

Our study particularly takes into account the work of Oser and colleagues on the role of mistakes for learning processes (Oser & Spychiger, 2005). Accordingly, we postulate that mistakes are necessary to elaborate the individual idea about what is false and what is correct. According to the theory of negative expertise individuals accumulate two complementary types of knowledge: positive knowledge on correct facts and processes, and negative knowledge on incorrect facts and processes (Minsky, 1983). Learning by mistakes is regarded as the acquisition of negative knowledge. Detecting one's own errors helps to revise faulty knowledge structures. Storing past errors and the cues that predict failure in memory may prevent individuals from repeating mistakes (Hesketh, 1997).

Mistake-handling in mathematics classroom – empirical results

In particular, in the 1970s and 1980s many research studies were conducted in mathematics education analyzing underlying patterns of students' mistakes in different

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 9-16. Seoul: PME.

mathematical domains (e.g., Radatz, 1979). This research followed a diagnostic perspective and aimed at the identification of reasons for typical students' mistakes.

Beyond this diagnostic research approach, there are only few research studies on mistake-handling activities and on the question what might be a promising way to deal with mistakes in the mathematics lessons. Video based research in different countries indicates that students' mistakes appear only rarely during classwork in mathematics lesson (cf. Heinze, 2005). Findings of a video study in Germany show that the average number of mistakes made publicly in a grade 8 geometry lesson dealing with reasoning and proof is less than five mistakes in 45 minutes. This is surprising since nearly all teachers in the study followed a discursive teaching style (teacher question, student answer, teacher feedback). According to questionnaire based studies in Switzerland and Germany mistake-handling activities of mathematics teachers are evaluated comparatively positive by their students (Oser & Spychiger, 2005; Heinze, 2005). In particular, students hardly fear making mistakes in "public" lesson phases. Hence, the conditions for integrating error management in the mathematics classroom are comparatively good. Nevertheless, it seems that teachers do not use errors as a chance to create learning opportunities for their students; instead they are following an implicit behavioristic style that avoids the occurrence and discussion of mistakes. Hence, it is not surprising that findings of Oser and Spychiger (2005) and Heinze (2005) show that Swiss and Germans students do not recognize the potential of their own mistakes in mathematics.

Effectiveness of error management trainings

Despite the fact that errors are regarded as important aspects for learning, research in mathematics education gives only few hints how to accomplish this task in the classroom. An exception is the teaching experiment of the Italian group Garuti, Boero and Chiappini (1999) for detecting and overcoming conceptual mistakes. They used the "voice and echoes game" as a special approach to deal with conceptual mistakes. An alternative way to use mistakes as learning opportunities is described in the research program of Borasi (1996). She conducted a series of case studies and developed the strategy of capitalizing on errors as springboards for inquiry. Her taxonomy describes three levels of abstraction in the mathematics discourse (performing a specific mathematics task, understanding technical mathematics content and understanding the nature of mathematics) and three stances of learning (remediation, discover and inquiry). For each of the possible combinations of levels and stances she gives a description how errors can be used productively in the specific situation. As Borasi (1996) summarizes the case studies and teaching experiments provide "anecdotal evidence" that learners can benefit from her approach.

There is hardly any quantitatively oriented empirical research about the effectiveness of error management trainings for mathematics classrooms. However, research in other disciplines indicates that error trainings are rather successful. Studies on the acquisition of word processing skills for example give evidence that a training in error management improves performance significantly better than a training based on error avoidance (Nordstrom, Wendland, & Williams, 1998). During error management training, the learners commit errors, either themselves (active errors) or by watching someone else commit errors (vicarious errors), and receive feedback about their mistakes. In error avoidance training, however, the learner is prevented from experiencing errors; in a behavioristic manner the aim of training is to allow learners to practice skills correctly and focus on the positive. Similar findings concerning the positive role of mistakes for the learning process are reported by Joung, Hesketh, and Neal (2006) for a training program with fire fighters.

Error management activities as part of the problem solving process

In the mathematics classroom errors and particularly the individual error management may play different roles depending on the mathematical activities. If a learner wants to acquire a principle, a formula, or an algorithm, she or he has a clear learning goal: something has to be memorized and understood, such that one can apply this knowledge in specific tasks. Reasoning and proof differs. According to the model of Boero (1999) the proving process consists of different stages in which the exploration activities play an outstanding role. Exploring a given problem situation, investigating given assumptions, retrieving suitable facts from memory etc. should be based on heuristic strategies. Here we have a situation of a systematic trial-and-error approach, which requires a permanent evaluation and drawing consequences from mistaken working steps. Hence, the ability to manage errors can be considered as a particular aspect of metacognition and is a prerequisite for solving complex reasoning and proof tasks.

RESEARCH QUESTIONS

As outlined in the previous sections we consider mistakes as a necessary part of the learning process. Moreover, error management skills are of particular importance when applying heuristic strategies in the mathematical problem solving process. As empirical findings for Germany indicate teachers and students hardly use mistakes as learning opportunities. Error management trainings in other domains show positive effects for the learning process. In the present study we are interested whether a special in-service teacher training on mistake-handling in mathematics classroom has positive effects on students' performance. Presently, we are not aware of empirical studies about the effectiveness of a teacher training in this field. Since we expect that error management abilities foster particularly problem solving competencies we focus particularly students' performance in reasoning and proof in geometry.

In our study we address the question to what extent a teacher training about the role of mistakes for the learning process in mathematics has an effect on

(1) students' perception regarding their teachers' mistake-handling activities in mathematics lessons and

(2) students' performance in geometric reasoning and proof.

DESIGN OF THE STUDY

The sample consists of 619 students (311 female and 308 male) from 29 grade 7 classrooms (about 13 years old students). At the end of grade 7 a pre-test on geometry (basic skills, reasoning and proof) and questionnaires on motivation regarding mathematics and the mistake-handling activities were administered. Based on the results in the pre-test and the motivation questionnaire the classes were assigned to an experimental (10 classes, N = 240) and to a control group (19 classes, N = 379).

At the beginning of grade 8 the teachers of the sample classes took part in an in-service teacher training. The training for the experimental and the control group were organized separately at the university and took two days for each group. The teachers of the experimental group received a training about mistakes and in teaching reasoning and proof, whereas the teachers of the control group had a training in reasoning and proof supplemented by aspects of the new German educational standards for mathematics. The training about mistakes included aspects of negative expertise, students' learning by mistakes, and the productive use of mistakes in the mathematics classroom. The training about reasoning and proof encompassed a model of the proving process, teaching material regarding reasoning and proof and typical student problems in this field. After the first training day the teachers got as an exercise to analyze their own instruction with respect to certain criteria. Their observations were included in the second part of the training two weeks later.

Two months after the teacher training, the students of both groups took part in a post-test on reasoning and proof in geometry and they filled in the questionnaire on mistake-handling activities again. During the two months among others the regular teaching unit on reasoning and proof in geometry was conducted by the teachers.

The pre- and post-test on reasoning and proof in geometry are approved instruments from our own research (e.g. Reiss, Hellmich, & Reiss, 2002). They contain curriculum related items for example on properties of triangles and quadrangles and congruence theorems. Data concerning the students' perspective on mistake-handling activities in the classroom were collected via an approved questionnaire adapted from Spychiger et al. (1998). Students had to rate statements on a four point Likert scale (see examples in table 1).

RESULTS

Students' perception of mistake-handling activities

A principal component analysis led to four factors for the 27 items of the mistake-handling questionnaire explaining 51% of the variance (see table 1).

Though there are four common items loading on the two factors concerning the affective and cognitive aspects of the teacher behaviour in mistake situations (e.g., *Our mathematics teacher is patient when a student has problems to understand.*), we decided to distinguish between these two aspects of teacher behavior. By the screeplot criterion of the principal component analysis the four factor solution was better than

Factor	Item example	Reliability (Cronbach's α)
Individual use of mistakes for the learning process	*In mathematics I explore my mistakes and try to understand them.*	0.83 (9 items)
Affective aspects of the teacher behavior in mistake situations	*Sometimes our math teacher gets an appalled face when a student makes a mistake.*	0.88 (9 items)
Cognitive aspects of the teacher behavior in mistake situations	*If I make a mistake in maths lessons my teacher handles the situation in such a way that I can benefit.*	0.87 (7 items)
Fear of making mistakes in the mathematics lessons	*I become scared when I make a mistake in mathematics classroom.*	0.74 (6 items)

Table 1: Factors of the mistake-handling questionnaire.

the three factor solution. However, one must have in mind that these two factors base partly on the same items. In Spychiger et al. (1998) a three factor solution was preferred with only one factor for the teacher behavior.

The results of the questionnaire indicate that students were not afraid of making mistakes and appreciated their teachers' affective attitude. The individual use of mistakes and teacher behavior regarding cognitive aspects were rated moderately by the students (see table 2).

Mean (SD)	individual use of mistakes	positive teacher behavior affective	positive teacher behavior cognitive	fear of making mistakes
pre-test				
total	2.47 (0.56)	3.06 (0.71)	2.79 (0.71)	1.85 (0.61)
experimental	2.48 (0.55)	2.87** (0.77)	2.65** (0.72)	1.97** (0.63)
control	2.47 (0.56)	3.17** (0.64)	2.88** (0.69)	1.78** (0.59)
post-test				
total	2.49 (0.57)	3.20 (0.63)	2.95 (0.69)	1.79 (0.59)
experimental	2.56* (0.57)	3.32** (0.53)	3.09** (0.59)	1.79 (0.56)
control	2.45* (0.57)	3.12** (0.68)	2.86** (0.73)	1.79 (0.61)
Likert scale: 1=strongly disagree, 2=disagree, 3=agree, 4=strongly agree				
* $p < 0.05$ ** $p < 0.01$				

Table 2: Mistake-handling in students' perception – pre- and post-test results.

Though the experimental group and the control group were parallelized after the pre-test with respect to their achievement in geometry and their motivation towards mathematics, they significantly differed in their perception of mistake-handling situations. Students of the control group judged their teachers as more positive than the students of the experimental group. However, in the post-test the students of the experimental group rated their teachers significantly better in the mistake questionnaire than the students of the control group. In particular, there was hardly a change for the control group regarding students' perception of mistake-handling situations (table 2). The development from pre- to post-test (as difference) differed significantly between experimental group and control group for the components "affective aspects of the teacher behavior in mistake situations" ($t(617) = 8.325$, $p < 0.001$, $d = 0.67$), "cognitive aspects of the teacher behavior in mistake situations" ($t(617) = 7.049$, $p < 0.001$, $d = 0.57$) and "fear of making mistakes" ($t(617) = -3.942$, $p < 0.001$), $d = 0.32$).

Though the control group teachers were rated better in the pre-test than the experimental group teachers, they were judged worse in the post-test. The results show that the teacher training was successful: the teacher behavior changed and became apparent to the students.

Students' performance in reasoning and proof

Both geometry tests on reasoning and proof consist of items of three competency levels: (1) basic knowledge and calculations, (2) one-step argumentation and (3) argumentation with several steps (see Reiss, Hellmich, & Reiss, 2002, for details). As described previously experimental and control group were parallelized, i.e. there is no significant difference between students' pre-test results, neither for the total test score nor for the results of the different competency levels (see table 3).

Mean (SD) Score (percentages)	Items competency level 1	Items competency level 2	Items competency level 3	Test score
pre-test				
total	79.3 (19.5)	62.8 (36.4)	26.1 (24.1)	57.4 (18.9)
experimental	79.0 (19.3)	66.1 (37.1)	26.0 (24.2)	58.1 (18.8)
control	79.4 (19.6)	60.7 (35.8)	26.1 (24.0)	57.0 (18.9)
post-test				
total	74.0 (23.0)	56.7 (23.4)	26.5 (22.9)	52.0 (17.6)
experimental	74.3 (22.9)	59.0* (21.1)	29.6** (21.9)	53.9* (17.0)
control	73.9 (23.1)	55.2* (24.7)	24.5** (23.3)	50.9* (17.8)
* $p < 0.05$ ** $p < 0.01$				

Table 3: Students performance in the geometry pre- and post-test.

For the post-test after the treatment we observe significant differences between the two groups (experimental group M = 53.9%, control group M = 50.9%, $t(617) = 2.08$, $p < 0.05$, $d = 0.17$). The better improvement of the experimental group was mainly based on geometrical proof items on competency level 3 (control group: M = 24.5%, experimental group M = 29.6%, $t(617) = 2.69$, $p < 0.01$, $d = 0.22$). This means that students of the experimental group achieved their better results particularly for the complex multi-step proof items.

DISCUSSION

In our study we trained in-service teachers regarding the role of mistakes in the teaching and learning process. Our findings indicate that this teacher training was successful from two points of view: On the one hand, the teacher of the experimental group changed their mistake-handling behavior in such a manner that it was recognized by the students. The effect sizes indicated moderate effects. On the other hand, the performance of the students in the experimental group improved significantly better in comparison to that of the control group. This improvement is mainly based on a better performance in solving geometrical proof items, i.e. items on a high competency level.

Analyzing the data in detail we can observe, that there is an improvement of the affective and cognitive teacher behavior in the perspective from the students. As described before these two factors base partly on common items, however, if we consider only the specific affective or cognitive related items we can observe the same tendency. Hence, it seems that the in-service teacher training has a positive effect for the teacher behavior which is noticed by the students.

In spite of this change in the teacher behavior we cannot observe a clear improvement in the self reported students' behavior concerning their own mistakes. There is no significant difference in the pre-post-test development between students of the experimental and the control group. It seems that the effect of the teacher training is restricted to a modification of the teacher reaction in mistake situations. There is no clear evidence for a successful further step: the instruction of students how to use mistakes as individual learning opportunities.

Investigating the development of the achievement in geometric reasoning and proof we see that students of the experimental group outperformed their mates from the control group. The advantage of the experimental group particularly goes back to the complex proof items in the pre- and post-tests; however, we can observe only a small effect size ($d = 0.22$). Though we did not control the mathematics instruction in the 29 classes, we hypothesize that this effect is particularly influenced by the improved mistake-handling activities of the teachers. Since the teachers in the experimental group were more open-minded about students' mistakes in mathematics classroom, a better improvement of students' achievement for complex mathematics tasks is in line with the theoretical assumptions. Moreover, teachers from the experimental and the control group got the same teacher training on reasoning and proof and they taught mathematics on the basis of a detailed prescribed curriculum, i.e. there was a clear frame for their teaching.

The results of our intervention study give evidence that an in-service teacher training on mistake-handling activities has positive effects on the mathematics classroom. Nevertheless, further research studies are necessary to optimize the outcome of such training sessions. In particular, one has to think about methods how to guide students to use their individual mistakes for improving their learning in mathematics. The creation and evaluation of specific learning material for this purpose may be one possible way.

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GENDER SIMILARITIES INSTEAD OF GENDER DIFFERENCES: STUDENTS' COMPETENCES IN REASONING AND PROOF

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The discussion of gender differences in mathematics competition has a long tradition, being challenged recently by Hyde's gender similarities hypothesis. In a reanalysis of four quantitative empirical studies with 2809 students of lower secondary schools we investigated performance in geometry proof from a differential perspective. Only a few significant differences could be found in proof performance as well as in growth of achievement within this field. They have only small effect sizes and do not show a uniform tendency in favour of one sex.

THEORETICAL BACKGROUND

Gender similarities versus gender differences

Many approaches in gender research presume differences between males and females not only in biological, but also in psychological respects. This assumption is supported by a large number of studies in the past decades. In contrast to this view Hyde (2005) stressed the *"gender similarities hypothesis"*, based on a meta-analysis of 46 meta-analyses:

> The gender similarities hypothesis holds that males and females are similar on most, but not all, psychological variables. That is, men and women, as well as boys and girls, are more alike than they are different. (Hyde 2005, p. 581)

In her analysis, Hyde argues that it does not suffice to identify statistical significant differences, but that also the relevance of these differences must be taken into account. She uses Cohen's distance measure d to describe the strength of the effects found. The result of her analysis is that most of the described effects are small:

> In terms of effect sizes, the gender similarities hypothesis states that most psychological gender differences are close-to-zero ($d < 0.10$) or small ($0.11 < d < 0.35$), a few are in moderate range ($0.36 < d < 0.65$) and very few are large (0.66 – 1.00) or very large ($d > 1.00$). (Hyde 2005, p. 581)

It is important to note that Hyde does not negate differences between sexes. She argues for a realistic view on differences and similarities.

With respect to mathematical achievement several investigations over the past decades suggest a trend of decreasing gender differences. First indicated by Senk and Usiskin (1983), Friedman (1989) confirmed in a meta-analysis of 98 studies from the years 1974 to 1987 that "sex difference in favor of males is decreasing over short periods of time" (Friedman, 1989, p. 205). Hyde, Fennema and Lamon (1990) found similar results in a meta-analysis of over 100 studies.

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 17-24. Seoul: PME.

For Germany this trend could not be found by Klieme (1997). He conducted a meta-analysis similar to the one of Hyde, Fennema and Lamon (1990) with more than 90 studies, but he could not replicate the results.

Mathematics achievement and gender in Germany

Several large scale studies examined mathematics achievement from a differential point of view in Germany during the past years. Some of them were coupled with international studies (TIMSS, PISA), some focused on single states (LAU, MARKUS, QuaSUM). Summarizing, these studies showed significant effects in favour of the boys, but effects are small for students of lower secondary classes. In upper secondary classes gender differences increase up to moderate effect sizes. Taking into account the particular school track gives a more detailed picture. For schools aiming at higher education (the German Gymnasium) the effects are larger for lower and upper secondary classes.

MATHEMATICAL PROOF AND ARGUMENTATION

Why investigate argumentation and proof competence in view of gender effects?

Viewing mathematical reasoning and proof as a special form of mathematical problem solving we can identify some basic competences required for these activities. For problem solving Schoenfeld (1992) specifies the following factors: Knowledge of mathematical facts, knowledge on problem solving strategies, metacognitive abilities and affective factors such as beliefs, interest and motivation. In the case of reasoning and proving an additional requirement is knowledge on the specifics of mathematical argumentation. Moreover the interplay of these competences must be controlled to solve proof-problems successfully (Weber 2001). There exist several models to grasp the complexity of the proving process (e.g. Boero 1999), showing the spectrum of competences required. Also international studies like PISA and TIMSS indicate that proving is a mathematical competence asking for skills in several areas and, moreover, a high level of general mathematical competence.

In addition to the necessary high level of competence there is a second reason for choosing proving skills as subject for gender studies. The individual ability of mathematical argumentation, especially to do geometry proofs, depends essentially on learning processes in mathematics classes. Mathematical reasoning is not part of student's every-day life, but appears almost exclusively within mathematics lessons. Senk and Usiskin (1983, p. 198) argue that this moderates the influence of gender differences via the students' interest in the measured competence.

Summarizing, we can view construction of proofs in geometry as a good indicator for learning processes regarding general mathematical performance. This means that studies of gender differences would be particularly interesting if they produce different results for geometrical proof competence and basic mathematical competence.

Studies on mathematical reasoning and proof

There are few quantitative empirical studies focussing mathematical reasoning and proof which deal exclusively with the aspect of gender differences. Most general studies, on the other hand, provide separate data for female and male students, making comparison possible.

Several studies in lower secondary schools were conducted by Celia Hoyles and colleagues in Great Britain in the project “Justifying and proving in school mathematics”. About 2500 grade 10 students were tested for their proof competence. While no gender differences were found for geometrical proofs, there occurred a significant difference in favour of the girls for algebraic proofs. A detailed analysis showed that also the way of argumentation differed for algebraic proofs between female and male students (Healy and Hoyles, 1999).

In a follow-up project (Longitudinal Proof Project) following students from grade 8 to 10 these results were confirmed (sample sizes 1500 to 2800). No difference between sexes was found in grades 8 and 10, but in grade 9 the girls scored better for algebraic proofs (Küchemann and Hoyles, 2003). This study also used a test for general mathematical competences in grades 8 and 9. Controlling these basic competences the girls achieve better results in the proof test than the boys.

Another study was conducted by Senk and Usiskin (1983) in the USA as a part of a larger project investigating the learning of geometry. Almost 1400 senior high school students at the ages of 14 to 17 years were examined using three sets of geometrical proof problems (e.g. congruence geometry). No significant differences in solution rates were found for all three problem sets. It should be noted that more female students were found in courses for weak students and more male students in courses for high achieving students. This differentiation was confirmed by a geometry test in the study. Calculating results under control of the geometry test showed a significant but small advantage for the girls in one of the three problem sets ($p < 0.05$, $d = 0.23$).

Cronje (1997) reports a study on proving from South Africa. She presented a set of proof-problems from Euclidean geometry to grade 11 students from five high schools. She could not find gender differences in test performance.

Using proof problems in the context of geometry as an indicator for general mathematical competence it cannot be neglected that there is a well established advantage for male students regarding space perception. This factor is often used to argue for gender differences in mathematics achievement and in particular regarding geometry. It is possible that this effect causes significant differences between sexes that are not due to the pure mathematical learning process.

> (…) one might expect significant sex differences in performance on doing geometry proof, which requires some spatial ability, qualifies as a high-level cognitive task, and is considered among the most difficult processes to learn in the secondary school mathematics curriculum. (Senk and Usiskin, 1983, p. 188)

Nevertheless Senk and Usiskin (1983) could not find consistent patterns of gender differences that would support this hypothesis.

All in all the empirical studies show that girls do not score worse than boys in geometrical proof problems, in spite of lower general mathematical performance. These studies were conducted during the last twenty years. As we cannot assume that gender differences in mathematics competence are stable over time a comparison is problematic.

RESEARCH QUESTIONS AND DESIGN OF THE STUDY

Research Questions

The state of research poses the question whether sex-specific differences in proof competence can be found in Germany. As described above, there are differences in general mathematics performance on lower secondary level if specific tracks of school (e.g. Gymnasium) are considered. It is an open question if these differences do also occur for proof competence. In this article we address the following questions:

- Are there differences in geometrical proof competence between female and male students at the beginning of lower secondary level?
- Does sex influence the growth of these competences during instruction on reasoning and proof?

For both questions we consider students of schools aiming at higher education (Gymnasium), on the one hand because larger effects can be expected here and on the other hand because proof and reasoning play a more important role in Gymnasium than in other school tracks.

Sample and methodology

The investigation is based on a reanalysis of data available from four studies from the past six years. These are quantitative-empirical studies within the priority program “quality of education in school” of the German Research foundation (c.f. Heinze, Reiss and Groß, 2006) with German Gymnasium students (age 13-14 years). Basic data of the sample can be found in table 1.

	study 1		study 2	study 3	study 4	
sample size	751	527	232	706	1120	870
female	407	300	118	331	552	459
male	344	227	114	375	568	411
grade	7th end	8th middle	8th beginning	7th end	7th end	8th middle
month of survey	june 2001	december 2001	september 2003	july 2004	july 2005	december 2005

Table 1: Sample data.

In all studies data was collected at two dates. Students in studies 2, 3 and 4 had a special treatment on geometrical reasoning and proof between first and second test. In study 4 the treatment was the same for all students, studies 2 and 3 evaluated different learning environments. Sample sizes were too small for a reliable gender-differentiated analysis for studies 2 and 3, so these second tests were excluded from the reanalysis. In study 1 no treatment was applied, but the lessons between the tests were videotaped and analysed.

The first test was the same for all four studies. It consisted of problems testing basic knowledge in geometry and problems on reasoning and proof. The problems on basic knowledge dealt with elementary concept knowledge and application of simple rules. Proof items asked for argumentations with one or more steps. The items were chosen in view of the curriculum and standard school books for grade 7 (e.g. sum of angles in a triangle). In studies 1 and 4 the second test took place in the middle of grade 8 and resembled the first test. The content of the items was adjusted to the curriculum of grade 8 (e.g. congruence theorems). Some anchor items were identical or similar for both tests.

The tests are based on a competence model for reasoning, argumentation and proving (Reiss, Hellmich and Reiss 2002) defining three theoretical competence levels. Problems on level I cover basic qualifications and the application of simple rules. On level II and level III there are argumentation problems requiring one step or more than one step of reasoning, respectively. All studies confirmed the theoretical model.

RESULTS

Table 2 lists the results of the first set of tests. It contains sample data, the mean M of the overall score (percent of maximum score) and the standard deviation (sd).

	study 1		study 2		study 3		study 4	
	female	male	female	male	female	male	female	male
N	407	344	118	114	331	375	552	568
m	47.5	50.1	59.4	55.1	60.3	58.3	56.2	53.8
sd	19.8	19.4	17.4	16.7	18.1	19.4	18.9	19.4
p (t-test)	0.069		0.056		0.172		0.039	

Table 2: Results of the first test.

Study 1 did not show effects of gender difference for the overall score. Differentiating the three levels of competence shows significant effects in favour of the boys on competence level I (basic problems, $p < 0.01$, $d = 0.22$) and on competence level III (argumentation with more than one step, $p < 0.05$, $d = 0.14$). Both effects are small. In study 2 there were no significant differences between the performance of girls and boys at all, concerning average score as well as single competence levels. Study 3 revealed a small significant difference in favour of the girls on competence level II (p <

0.001, d = 0.22). Results of study 4 showed a significant difference in favour of the girls for the overall score ($p < 0.05$, $d = 0.12$) and for competence level II ($p < 0.001$, $d = 0.20$).

As described above data of the second test is only considered for studies 1 and 4. Results can be found in table 3. Study 1 in grade 8 did not show any significant gender differences. On competence levels II and III the performance of girls and boys was almost identical.

	study 1		study 4	
	female	male	female	male
N	300	227	459	411
m	37.1	38.0	50.3	51.4
sd	16.0	15.0	16.9	17.8
p (t-test)	0.781		0.110	

Table 3: Results of the second test.

In Study 4 as well no effects for the overall performance could be found. Considering only items on competence level I uncovers a significant, but small effect ($p < 0.01$, $d = 0.18$) in favour of the boys.

Altogether in the overall performance of female and male students no relevant differences could be found. As regards competence levels, there are significant gender differences in only five of 18 cases, twice in advantage of the girls, three times in advantage of the boys. The values of *d* vary between 0.12 and 0.22, showing small effects of little practical relevance.

Results on the growth of achievement

The growth of achievement in studies 1 and 4 was determined based on results of the first and the second test. Using linear regression an average growth was modelled and the individual difference was calculated for each test person (z-standardized residuals). As table 4 shows no significant difference was established in both studies.

	study 1		study 4	
	female	male	female	male
N	297	226	389	355
mean residual	0.0017	-0.0017	-0.0080	0.0087
sd	0.1431	0.1343	0.1470	0.1370
p (t-test)	0.512		0.362	

Table 4: Growth of achievement in studies 1 and 4.

DISCUSSION

The results of the reanalysis show significant gender-specific effects in geometric proof performance only in one of six tests (first test in study 4). Breaking down the data into different levels of competence reveals significant effects in favour of both female and male students. If the corresponding values of the effect size *d* (0.12 – 0.22) are taken into account all effects turn out to be small. The effects found are of small practical relevance and moreover they do not appear consistently in favour of one sex.

The results confirm expectations in view of the studies described at the beginning of the article. They support the hypothesis that there are no relevant gender differences in geometric proof competence on lower secondary school level. This is remarkable as our sample consisted of Gymnasium-students, the school track in which moderate differences were found for general mathematical competences in the past.

A reason for these contrasting results is proposed by Senk and Usiskin (1983). They assume that students learn how to prove exclusively in mathematics lessons, away from out-of-school influences. Thus girls and boys have the same starting conditions in this area. In other areas of mathematics competence they see an advantage for boys gaining informal experience with mathematical contents out of school. For the choice of measures for mathematical competence they find the following pattern:

> (…) the more an instrument directly measures students' formal education experiences in mathematics, the less the likelihood of sex differences. (Senk and Usiskin, 1983, p. 198)

This explanation may be suitable for the situation in our study. Nevertheless we only considered students of a specific age at lower secondary level (grade 7/8) attending a school aiming at higher education. Thus transfer of our results to other situations is problematic.

Though we found no relevant differences between sexes in performance it should be noted that these findings cannot be transferred to all aspects of mathematics classroom. It is obvious that there are differences between girls and boys in the learning process. Jahnke-Klein (2001) notices that the form of discourse dominating classroom interaction in Germany does not meet the requirements of female students. Following her they need more support during lessons and profit from repeated explanations, sets of similar problems – with the possibility to evaluate their own results – and repetition of difficult homework problems in school. Affective factors – caused by individual or social processes – are also out of focus of our investigation (Hyde, Fennema, Ryan et. al. 1990).

The described reanalysis shows that also in the specific area of proof and reasoning female and male students show similar performance. In view of gender differences in basic mathematical achievements found by other studies it is an open question which exact processes lead to the balance in the field of reasoning and proof and if such effects can also be found for other fields.

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STUDYING LESSON STRUCTURE FROM THE PERSPECTIVE OF STUDENTS' MEANING CONSTRUCTION: THE CASE OF TWO JAPANESE MATHEMATICS CLASSROOMS

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International comparisons have shown different structural features of mathematics lesson across countries. This paper explores the lesson pattern identified in Japanese lessons from the side of students. An attention is paid to the role of Jiriki-Kaiketu activity - Solving Problem by Oneself - in students' construction of mathematical meaning. Two major roles are found from quantitative and qualitative analyses of LPS data. On the one hand, Jiriki-Kaiketsu serves as a time in which puzzlement, questions and conjectures arise within the students. On the other hand, it enables students to make sense of the development of their work that followed. The paper also discusses influences of organization of lesson on students' meaning construction by drawing on observations of teacher's careful management of students' thinking and some realities of students' difficulties.

INTRODUCTION AND BACKGROUND

The TIMSS video study identified the lesson patterns as cultural scripts for teaching in Germany, Japan and the USA (Stigler & Hiebert, 1999). They identified Japanese pattern of teaching a lesson as a sequence of five activities: Reviewing the previous lesson, Presenting the problem for the day, Students working individually or in groups, Discussing solution methods, and Highlighting and summarizing the major points (p. 79). Here, a distinct feature of Japanese lesson pattern, compared with the other two countries, was that presenting a problem set the stage for students to work on developing solution procedures. In contrast, in the USA and in Germany students work on problems after the teacher demonstrated how to solve the problem (USA) or after the teacher directed students to develop procedures to solve the task (Germany).

This feature of pattern of lesson is supported by pedagogical principles that are shared by teachers in Japan. We have the vocabulary with which to describe the practice of teaching in class. One of them, which is relevant to the pattern, is the term 'Jiriki-Kaiketsu.' Jiriki-Kaiketsu (J-K) means 'solving problem by oneself,' in which students work on the problem prepared by teacher to introduce new content. J-K can be interpreted as seatwork activity because it is the time for students to work independently on assigned tasks. Still, important conditions of J-K are that the problem is the key problem in the lesson and that student's work is presented and discussed later for the purpose of introducing new content.

The purpose of this paper is to investigate the role of J-K in students' construction of mathematical meaning in the lessons. By using data that were gathered to obtain

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 25-32. Seoul: PME.

information on students in the lessons, an attempt is made to grasp the meaning of J-K activity in the eyes of students. There are three reasons for this investigation. First, there is not enough information on the side of students under specific structural features of lesson. Articulation of different lesson patterns has produced researchers' interest in pedagogical decisions underlying the structures (e.g., Stigler & Hiebert, 1999). A missing part is the examination of students who are learning mathematics in the classroom where teacher's instructional intentions are reflected (Fujii, 2002). Interest of this paper is meaning and role of J-K from the side of students.

Second, there is a growing interest in the process of construction of mathematical meanings in the classroom. Different perspectives on understanding the meaning construction in the classroom have been presented. One important finding would be a reflexive relationship between individual students' activity and social dimensions in the classroom (e.g., Cobb & Bauersfeld, 1995). However, mechanism of individual students' meaning construction under constrains of social environment in the classroom is still an open question (Cobb et al., 1997; Waschescio, 1998). In this paper, by examining the real-time learning of individual students it is tried to reveal more about the relationship between the lesson organization and their process of learning.

Third, Japanese pattern of lesson seems to reflect our emphasis on teaching mathematical thinking via 'structured problem solving' (Shimizu, 1999). In Japan, although the style of problem solving is widely spread, the problem of 'deal letter' has been pointed out (e.g., Tsubota et al., 2006). Therefore, it would be meaningful for us to analyse strong/weak points of the Japanese pattern of lesson by carefully collecting information about process and product of students' learning.

METHOD

This study used data of Japanese mathematics lessons in two classrooms taught by two competent teachers. The data were collected in the Learner's Perspective Study (LPS) (Clarke, 2006). The LPS generated data for ten consecutive mathematics lessons in each of three participating eighth grade classrooms. The reason for selecting these two classrooms out of three is that they fell more in line with the Japanese pattern of lesson (Hino, 2006). Throughout this paper, I will use J1 and J3 to refer to these sites. Mathematical topics that were dealt with in these sites are: linear functions (J1) and simultaneous equations (J3).

As described by Clarke, data generation in LPS used a three-camera approach (Teacher camera, Student camera, Whole Class camera). Data included the onsite mixing of the Teacher and Student camera images into a picture-in-picture video record. This vide record was used in post-lesson interviews as stimulus for student reconstructions of classroom events. In post-lesson interviews, focus students were asked to identify and comment on classroom events of personal importance. In addition, photocopies of focus student written work, photocopies of textbook pages, worksheets or other written materials were also generated. These multiple sources of data were used in an integrated way to analyse students' meaning construction in the two classrooms.

In the analysis, I put in order the activities of focus students (two students in each lesson) during the lesson. I summarized their thinking activities, especially in what way student started her/his work after the teacher presented the key problem, what activities the student engaged during J-K and how the student participated in whole-class discussion that followed. In order to catch student's thinking and feeling during the lesson, it was found useful to use data from post-lesson interview. Therefore, I took classroom events of personal importance that students chose as targets and examined the role of J-K by developing categories by constantly comparing pieces of information (Glaser & Strauss, 1967).

RESULTS

J-K Viewed from Classroom Events of Student's Personal Importance

In the post-lesson video-stimulated interviews, focus students were given control of the video replay and asked to identify and comment upon classroom events of personal importance. These events were classified into four types, i.e., seatwork (including J-K) activity, whole-class discussion activity, instruction by teacher and other.

Activities mentioned by students (%)	*J1*	*J3*
Seatwork activity (including J-K activity)	*33*	*39*
Whole-class discussion	*32*	*37*
Instruction by the teacher	*35*	*18*
Other (unrelated to classroom activities)	*0*	*6*

Table 1: Percentages of four categories of classroom events identified by students as personally important

Table 1 shows the percentage of these types in J1 and J3. It shows that seatwork activity was one of the most important in the eyes of students. More detailed classification of events that belong to "seatwork activity" further shows that many students identified events of discussion with their neighbours as important. This is also the case in J-K. Informal discussion with their neighbours during J-K activity was as important as discussion with the teacher. Here, it is possible to say that the time for J-K was important as a time to exchange information and opinions with their neighbours and to think about the problematic points together.

Moreover, table 1 shows that the percentages of "whole-class discussion" were also not small. Here, it is interesting to know that not a few students commented on the important points of the work presented by their friends by comparing or connecting them to their own methods and thinking: e.g.,

> IW: Um, as I took this way of DAINYU-HOU (substitution-method), and I didn't notice there was another way, / … So, I took this way, but there were about five people who took another way of moving member of formula, … I noticed there was another way of solving that question, I mean, with KAGEN-HOU (addition-subtraction-method), … I thought everyone was using their brains well. (J3-L7)

It indicates that students remembered what they had thought about the problem during J-K activity and listened to the later development based on their thoughts. From this observation, it is also possible to say that J-K was important to the students because it serves as a basis to make sense of the development of the work that followed.

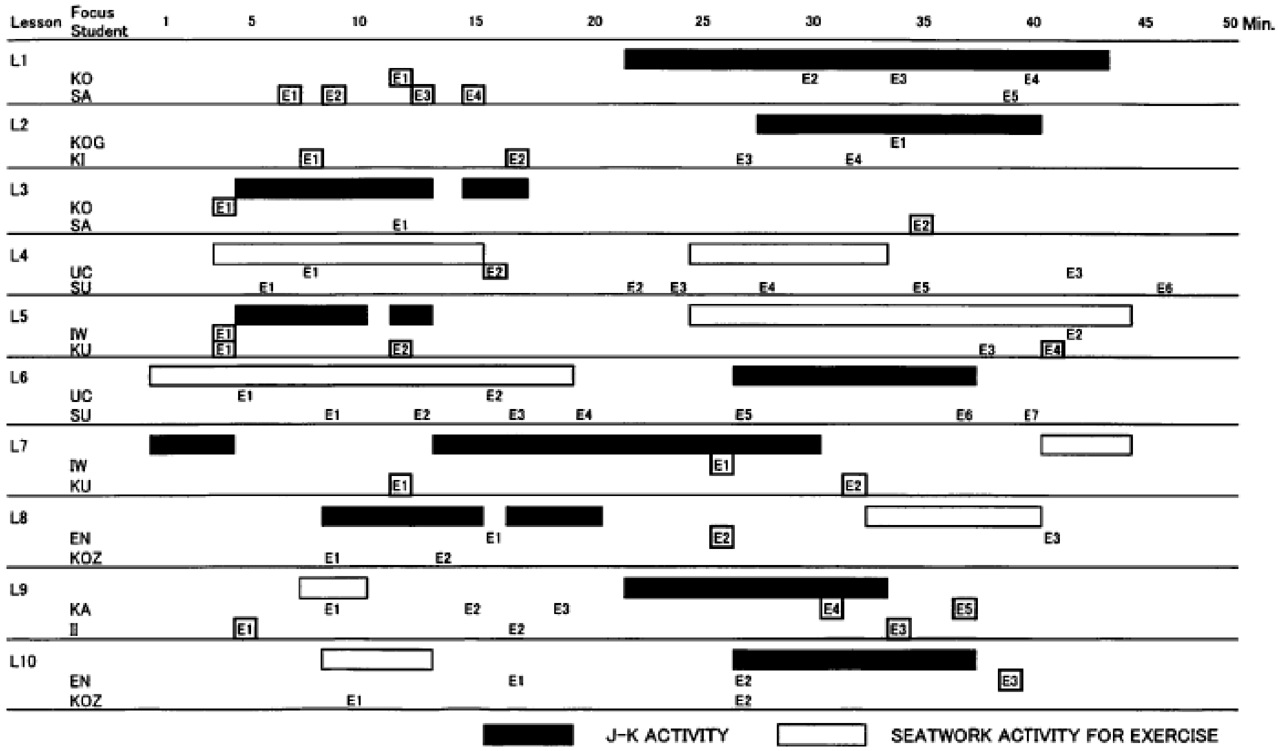


Figure 1: Chronological relationship between time for J-K and classroom events of personal importance in J3

Figure 1 shows the chronological relationship between the time for J-K and the classroom events of personal importance in J3. Locations of classroom events of personal importance students chose in the post-lesson interviews were noted by labels such as E1 and E2. Boxed labels show the events in which student either mentioned about the relationship between the work presented by their friends and her/his own work during J-K or can be interpreted as using her/his work during J-K as a knowledge base to make sense of the development in the whole-class activity that followed.

Figure 1 and another figure on J1 show that teachers took time for J-K activity at least once in almost all the lessons. In addition to spending time on seatwork activity for exercises, when introducing new content, these teachers took time to let students think about the content beforehand. Boxed labels were scattered about the lessons. Nevertheless, if we looked at their locations carefully, they tended to be located around the J-K activity. Some boxed labels, such as those in Lesson 1 (Figure 1), were seen before the J-K activity. This is because students made connections with their work that they did in the previous lesson. It also shows that teachers were flexible in organizing the lessons by reflecting their planning of a sequence of several lessons as part of the teaching unit (Shimizu, 1999). Moreover, it is noticeable that students differ in their ways of participation in the classroom. For example, most of the classroom events of personal importance chosen by TA (J1) were boxed and located after J-K activity, which suggested that she participated in the whole-class discussion via her work and thinking during J-K. On the other hand, SU and KOZ (J3)'s classroom events were not boxed and located either during J-K or what the teacher stated. Their primary concern was to solve problems before their eyes with the help of his neighbours and the teacher.

An Overview of Meaning Construction by Individual Students via J-K Activity

Some results of qualitative analysis are described here. For reasons of space, I only give an overview of two categories generated by data.

Students noticed differences in thinking between their friends and their own.

After J-K activity, both teachers spent time for students' presenting their solutions and examining/discussing them. It gave students opportunities with knowing/sharing work and thinking by their friends. The students not only checked correctness of their answers but also recognized a variety of differences between others and their own. In J3, students were learning how to solve simultaneous equations; and they compared and recognized difference in solution procedures (see the transcript by IW above). In J1, the teacher stressed presentation of students' ideas and discussion after J-K activity. Her questions were mostly open with the intention of eliciting different ideas form students. Under such circumstance, the students noticed differences not only in ideas and ways of solving problems but also in focal points of exploring functional relationship, ways of expressing ideas and solution methods, reasons behinds opinions, and features noticed in the graphs (see a transcript below). Here, TA made aware of differences between her graph and graphs by two students (whether plotting simply as 1,2,3,… or including 0.5 in between). She was developing meaning of plotted points which she had not noticed previously.

TA	What he did, was different it was different although they examined the same thing. So, there, how that happened.
Interviewer	How it happened. Yes. You mean how the shape of the graph came out differently?
TA	Yes.
Interviewer	How the points were plotted?
TA	Plotting, we plotted down one, two, three, four but NO included zero point five in between and UM and I just figured out one, two, three, four. How the answers differ. (J1-L5)

It was rather complicated how students evaluated the difference they noticed. At one time, they did not think anything but just admitting the difference. At other time, they thought about the difference further and tried to get the reason for that. Some students showed their desire to incorporate the different point proposed by their friends, such as,

SU: here I think it was good for me that I was able to hear opinions besides the ones we've discussed with our partners. I think we will keep studying about this theme next time, so if I pay attention to those points, I would be able to find new points (J1-L9).

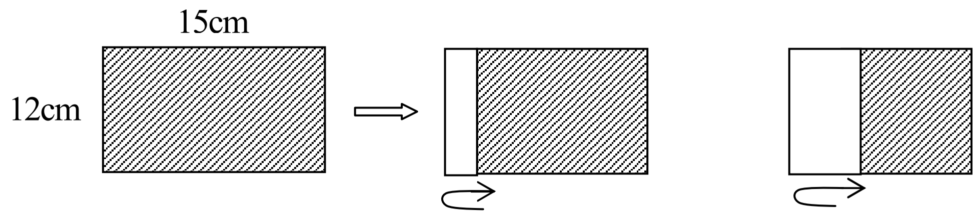
Students were concerned with the consequences of their questions and conjectures.

From video records and post-lesson interviews, it was revealed that question was a big driving force in their learning. Here, question means a specific consideration/interest such as “What does it mean?” or “Why does it happen?” than a general puzzlement “I don’t know how.” When students had questions while they were engaging in J-K activity, they tried to resolve them with their neighbours and the teacher. Still, their concern about the question was also carried over in the subsequent part of the lesson.

I show an episode from the side of one student, SH, in J-1. In L4, the teacher presented the problem of investigating the relationship between two variables. Two students presented the results of their investigation in L5. The problem and the result of investigation presented by TA (only her table is shown) are in Figure 2:

<u>The Problem of Origami (Paper folding) (L4-L7)</u>

Like the figure below, we fold into a rectangle (one side is coloured) with 12 cm in length and 15 cm in width. What changes when we change the location of the folds?



<u>Result of Investigation by TA</u>

X: Width of coloured rectangle	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
Y: Perimeter of coloured rectangle	54	52	50	48	46	44	42	40	38	36	34	32	30	28	26	0

Figure 2: The problem and result of investigation by TA

In L6, results given by the two students were discussed. Here, the students were engaged in the J-K activity to develop their opinions. During the J-K, SH had a question that the value of y may not be 0 when x=0. This question seems natural for him because he had already paid attention to the fact that the length of vertical side of the rectangle does not change (L4). SH inquired of the teacher about the question.

SH Mis, the height is not changing, is it?

Teacher No, it isn't.

SH The height. It will change when it's ( ), so (it'll be twenty-four, when it's twelve?)

Teacher Ok, if you think it means that, then you need to correct that. It's ok, isn't it?

During the time for discussion that lasted over two lessons, several students talked that y=0 when x=0. SH gave counter argument that y=24 when x=0 three times. It is worth noting that SH argued about the validity of y=24 when x=0 by interpreting the graph from different perspectives, i.e., physical embodiment and consistency in the shape of graph: e.g., “We used origami in solving this problem. And that is why such idea like calling this shape ‘a stick’ had come up in the first place,” and “Uh, if you’d like to say *y* should be zero when *x* is zero, then saying 'constantly increasing' in this graph would be wrong, huh?" SH's effort of making meaning of his argument started from his question that the value of *y* may not be 0 when x=0. He was concerned with the consequence of his question and challenged to argue against his friends. This heated discussion was finally converged to exclude x=0 and formulate y=2x+24 (x>0). Concept of domain was addressed to contribute to resolving the problematic situation.

DISCUSSION

In this paper, students' meaning construction in two classrooms was reported from the viewpoint of J-K activity. The J-K had two roles in student's meaning construction. On the one hand, J-K served as a time in which puzzlement, questions and conjectures arouse within the students. On the other hand, it enabled to create students' knowledge base so that they could participate in later part of the lesson. A qualitative analysis gave more information about individual students' meaning construction. The students were constructing their mathematical meaning by comparing and connecting between their own work and thinking during J-K and their friends' work and thinking presented later in the whole-class discussion. Such comparison was observed in various points of view. Questions and conjectures students had during J-K were also driving forces for them to make meaning in the whole-class discussion.

O'Keefe et al. (2006) compared a lesson event, Between Desks Instruction, in mathematics classrooms in six countries. They found that although this form was evident across all the classrooms, participants could attribute different characteristics to the activity. In this paper, a focus was put more on an idiosyncratic type of event in one country. Nevertheless, results also suggested particular characteristics and functions attributed by both teachers and students. Especially, by delving into the side of students, the results gave information about the possibility of idiosyncratic mechanisms of learning in classrooms in different countries. Further analysis would be interesting if similar roles are discerned in the lessons in other countries.

Based on the results of comparisons across seven countries, Hiebert et al. (2003) stated that the way in which the mathematics lesson environments were organized constrains both the mathematics content that is taught and the way that content is taught. Results in this paper put a mark on this by indicating it is not organization per sue but teacher's specific approach to students' thinking under such organization that has the influence on student's meaning construction. They were seen in teacher's dealing with a variety of students' ideas, encouraging their reflections based on actual experience and handling questions. For example, teacher utilized different kinds of questions. At one time, they anticipated questions that would arise from their students, and mentioned or discussed them when it is necessary. At other time, they went deeper and raised their questions to the students. These specific approaches resulted in strong impacts on students under their organization of lessons.

By investigating the side of students, some of the realities of students were also documented. One is the existence of individual differences. As described earlier, there were students whose concern was more for solving the problem before their eyes with their neighbours and the teacher. One student said in the interview that he often got confused when someone said something new after he solved the problem on his own way. Analysis also revealed students' difficulties in understanding the meaningfulness of J-K in learning mathematics and in reflecting on their thinking from teacher's perspective. These realities suggest deeper influence of lesson pattern on students by requiring them an interactive attitude toward learning mathematics. More information is needed because they are expected to contribute to articulating significant role of teacher in the 'structured problem solving' in Japan.

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A FRAMEWORK FOR CREATING OR ANALYZING JAPANESE LESSONS FROM THE VIEWPOINT OF MATHEMATICAL ACTIVITIES: A FRACTION LESSON

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The paper aims to (1) give a framework by which teachers can analyze Japanese mathematics lessons or create their own lessons from the standpoint of mathematical activities [CALMA Framework] and (2) discuss a fraction lesson based on the framework. First, mathematical activities, problem solving, mathematical richness, and creativity were considered for the creation of the framework, based on the current state of Japanese mathematics lessons. Second, through lesson analysis, teachers' crucial roles were derived, such as the raising of levels of mathematical richness.

INTRODUCTION

Lessons, or teaching/learning activities, depend on their respective cultures, and accordingly they can differ to varying extents. In Japan, one lesson of mathematics should be "planned as complete experiences – as stories with a beginning, a middle, and an end" (Stigler & Hiebert, 1999, p.95). In other words, the beginning and end of a lesson usually have meaningful connections via the middle.

Furthermore, Japanese "lesson study" (*jugyou kenkyuu*) has been attracting attention in recent years as the benefits gained from the improvement of lessons have become more widely known (e.g. Stigler & Hiebert, 1999; Curcio & Billay, 2003). Lesson study basically consists of the following series of three steps:

- Creating a "lesson plan" (*gakushuu-shidou-an*) before a lesson;
- Giving/observing an "open lesson" based on the lesson plan; and
- "Reflecting on and discussing" interpretations and improvements of the contents of mathematics, children's ways of thinking, teacher reactions to them, and so on, referring to the lesson plan after the lesson.

Such lesson study is characterized by the fact that a teacher giving a lesson and the other teachers and researchers observing the lesson collect and interpret data in collaboration (Takahashi, 2006). However, experienced teachers usually offer different interpretations of the data compared to novice teachers because of differences in teaching experiences. Hence this paper focuses on a replacement for such skilled teachers, by which novice teachers can develop their lessons effectively by themselves.

Accordingly, this paper aims to (1) give a framework by which teachers can analyze mathematics lessons or create their own lessons from the standpoint of mathematical activities, and (2) analyze a fraction lesson in the six grades in Japan using the

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framework and consequently show that the framework has the possibility to contribute to the better understanding of mathematics lessons.

FRAMEWORK FOR UNDERSTANDING A JAPANESE LESSON WELL

This section illustrates the four main aspects of what (should) make up a mathematics lesson in Japan; i.e., mathematical activities, problem solving, mathematical richness, and creativity, and proposes a framework for creating or analyzing Japanese lessons from the viewpoint of mathematical activities.

Mathematical Activities in a Lesson

"Mathematical activities" is the key idea characterizing current Japanese mathematics lessons. In Japan, the Ministry of Education, Culture, Sports, Science and Technology (MEXT) publishes Courses of Study (*gakushuu-shidouyouryou*), in which MEXT prescribes the national standards for education, such as the objectives for each subject.

According to the Japan Society of Mathematical Education (JSME) (2000, p.4), the main features of mathematics curricula in Japan have changed over time to have the following characteristics as follows: (1) children centered (from the second half of the 1940s); (2) unit learning; (3) mathematical ways of thinking; (4) systematic learning (1960s); (5) mathematical modernization; (6) basics, problem solving; (7) individualization, informatization; and (8) mathematical activities (2000s).

Therefore, the key term "mathematical activities" appears in the latest Courses of Study, describing the objectives of mathematics for elementary, lower secondary, and upper secondary schools (see Table 1).

	Objectives of teaching/learning mathematics
Elementary Schools **2002 revision**	*Through mathematical activities* concerning numbers, quantities and geometrical figures, to help children get basic knowledge and skills, to establish their abilities to think logically and think with good perspectives on everyday phenomena, and to help children notice the pleasure of doing activities and appreciate the value of mathematical methods, thereby to foster attitudes to make use of mathematics in daily life situations.
Upper Secondary Schools **2003 revision**	To help students deepen their understanding of the basic concepts, principles, and laws of mathematics, and to develop their abilities to think and deal mathematically with various phenomena, thereby *to cultivate their basic creativity through mathematical activities;* and to help students appreciate mathematical ways of observing and thinking, thereby to foster attitudes which seek positively to apply the qualities and abilities mentioned above.

Note. Objectives for lower secondary schools are omitted because of the limited space, the translation in English is based on JSME (2000, pp.7, 28), and the italics are our emphasis.

Table 1: Objectives of teaching/learning mathematics for elementary and upper secondary schools described in the Courses of Study in Japan

A Problem-Solving-Style Lesson

According to MEXT (1999), the mathematical activities shown in Table 1 can be interpreted as the following repetition of a sequence:

> (A) Correlating everyday phenomena with mathematics and mathematizing them, (B) Solving the problems mathematically, (C) Utilizing or appreciating the results while returning to the everyday phenomena, and (A) ...

Such a series of mathematical activities can be likened to the process of problem solving. For example, Polya (1954/1975) posed four steps of problem solving: 1) understanding the problem, 2) devising a plan, 3) carrying out the plan, and 4) looking back. Furthermore, Krulik (1977) gave the following set of steps: 1) locating a concrete problem, 2) making a mathematical model, 3) solving the mathematical model, and 4) applying it to the concrete problem.

As a result, it is assumed that in the current Courses of Study in Japan students' mathematical activities in a class are involved with such problem solving processes. Indeed, teachers in Japan, and in particular in elementary schools, often give mathematics lessons following a style of problem solving. A mathematics class in general consists of three (or four in more detail) stages: "grasping" (*tsukamu*) a problem (introduction), "solving" the problem by oneself and "developing (*neriageru*)" it in collaboration with everyone (development and turn), and "deepening (*fukameru*) and concluding" the problem (conclusion). Therefore students are sometimes encouraged to wrestle with only one or a few problems in a lesson.

In other words, a teacher has to prepare a problem which is worth solving during a 45 to 50 minute class to develop the students' concepts of mathematics. As a result, such Japanese mathematics lessons have been described as "well structured problem solving" (Stigler & Hiebert, 1999, p.40).

Three Levels of Mathematical Richness and Structures Contained in Contexts

In a problem-solving-style lesson of mathematics incorporating a variety of mathematical activities, students usually solve only a few problems. However, even though the number of problems they confront in a class may be few, they are expected to have deepened their understanding of mathematics at the end of a lesson compared to before they started. Hence, different levels of mathematics should be arranged in purposeful ways within a lesson, and it seems that Vygotsky's and Treffer's studies are useful to this end.

Referring to Köhler's chimpanzee experiments, Vygotsky and Luria (1930/1993) pointed out that grasping a structure in an unfamiliar situation and transferring it to another situation characterized intellectual behavior of higher anthropoid apes, including humans, which they referred to as "law of structure" (p.67). That is to say, once students grasp the structure of a problem in a concrete or mathematical situation, they can transfer that structure to a different mathematical or even a broader situation, although each element may be changed.

Moreover, Treffers (1987) defined mathematization as follows:

> the organizing and structuring activity in which acquired knowledge and abilities are called upon in order to discover still unknown regularities, connections, structures (p.247).

Furthermore, mathematization can be separated into horizontal mathematization and vertical mathematization. The former concerns transforming a problem situation into a mathematical context, while the latter relates to "mathematical processing and level raising in the structure of the problem field under consideration"(p.247).

In conclusion, students have the ability to transfer the mathematical structure of a problem within a particular context to another context as far as the structure is unchanged according to Vygotsky, and vertical mathematization as described by Treffers contributes to raising the level of mathematical richness.

Creativity in Mathematics Lessons

As shown in Table 1, developing creativity through mathematical activities is one of the objectives when students in Japan learn mathematics, and in particular in upper secondary schools.

Guilford (1959) viewed intellect from three aspects: operations, products, and contents. Of the five factors comprising operations, here convergent thinking and divergent thinking should be given significant attention.

For instance, the giving of various solutions or the exploring of open-ended problems helps to foster students who are rich in creativity, or, more specifically, in divergent thinking (e.g., Sinclair & Crespo, 2006). Indeed, in Japanese mathematics lessons students are encouraged to give a lot of different solutions or explanations for a problem using different modes of representation such as diagrams, number lines, and mathematical signs and expressions.

This paper places importance on the development of student abilities and attitudes, taking into account both convergent and divergent thinking. In other words, students should be required to discover a mathematical problem in everyday phenomena and solve it (convergent thinking), and, moreover, apply and develop the solution methods used to other contexts, producing broader richness of mathematical understanding (divergent thinking).

A Framework for *C*reating and *A*nalyzing *L*essons from the Viewpoint of *M*athematical *A*ctivities in Japan [CALMA Framework]

With due consideration of the discussion above, we propose a framework for creating and analyzing Japanese mathematics lessons from the viewpoint of mathematical activities (CALMA Framework) (see Figure 1) in addition to a definition of mathematical activities.

The Five Mathematical Activities Arranged in a Lesson: (1) *mathematizing*: to interpret concrete phenomena mathematically; (2) *formulating*: to formulate problems mathematically based on the concrete phenomena, i.e., to idealize or simplify them mathematically; (3) *exploring and processing*: to explore and handle the problems mathematically based on acquired knowledge and abilities; (4) *looking back and applying*: to reflect on the prior concrete situations (or problems) in broader contexts and/or apply the acquired mathematical ways of thinking and mathematical solutions to broader areas; (5) *developing, creating, and appreciating*: to develop the findings and/or create something new (for students), becoming aware of the mathematical richness of the new ways of approaching problems, and/or to appreciate the culture of mathematics.

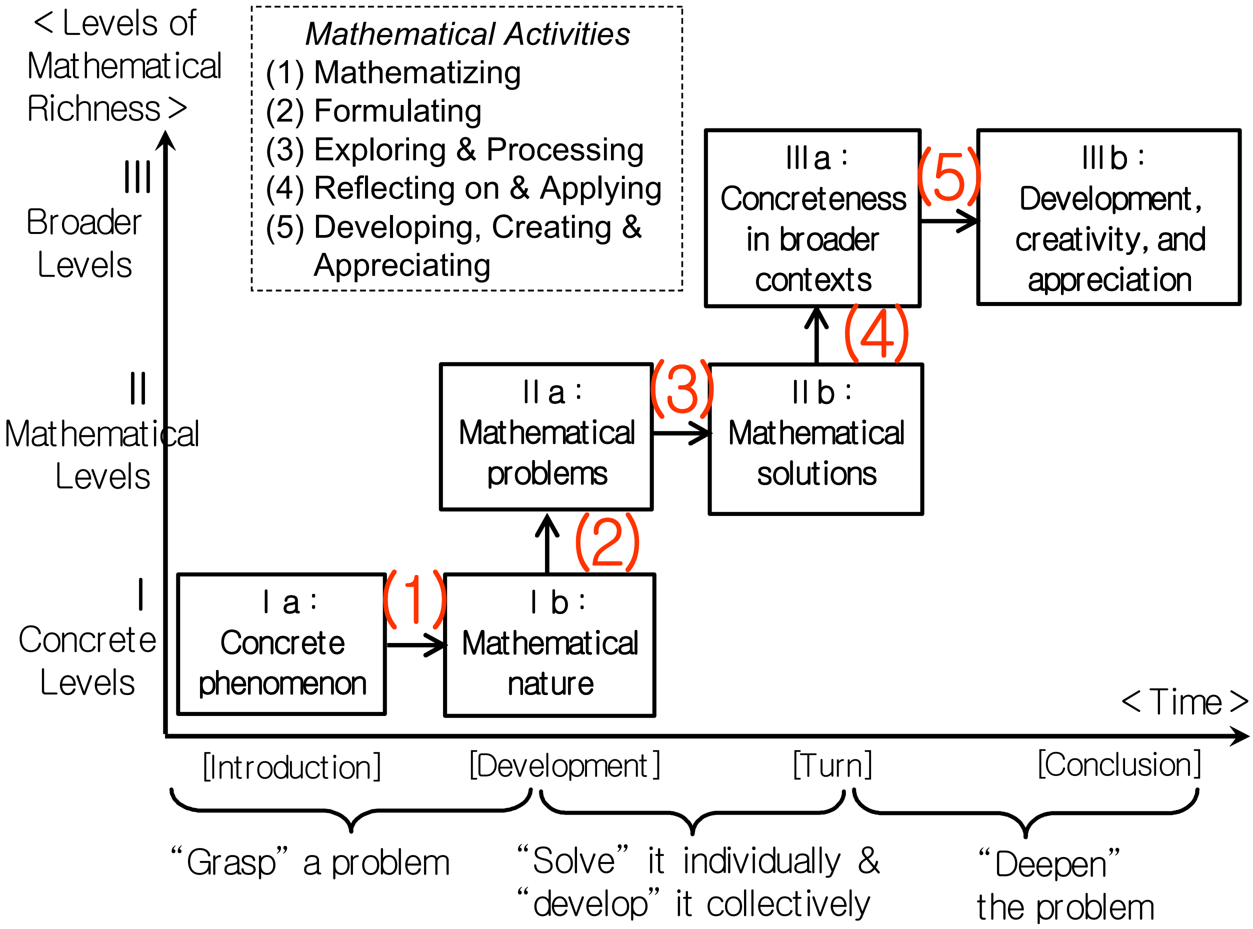

Figure 1: Japanese CALMA Framework to be used in mathematics lessons for elementary, lower secondary, and upper secondary schools

ANALYSIS OF A FRACTION LESSON USING THE *CALMA* FRAMEWORK

A fraction lesson on October 26, 2006, was observed for the purpose of this analysis. It was an open lesson conducted as part of a lesson study in a series of teacher trainings. The lesson was observed by about 15 teachers, ranging from novices to principals, in two elementary schools and by five people belonging to a university, including the authors. Because the school was in a remote area on a small island in Japan, only nine sixth graders (three girls and six boys) were in the class.

Ms. T created a lesson plan in advance according to the method of lesson study and discussed it with one of the authors several weeks before the lesson. The lesson was placed within a unit on the multiplication and division of fractions. The lesson presented a situation in which a base quantity [B] and a proportion [P] expressed by a fraction were known, while a comparative quantity [C] was unknown, in the structure of C = B * P. In the previous lesson the fractional proportion was unknown (P = C/B), while in the subsequent lesson the base quantity was unknown (B = C/P). In the fifth grade the students had learned these structures of the relationships among B, P, and C, although at the time the proportions had been decimal numbers and not fractions.

The lesson started with a concrete problem given by Ms. T as follows. This was the only problem the students attempted during the class:

> "Cookies cost 600 yen. Chocolates cost 6/5 the price of cookies. Candies cost 3/5 the price of cookies. How much are the chocolates and the candies, respectively?"

This fraction lesson was analyzed according to the Japanese CALMA framework, as shown in Appendix 1.

DISCUSSION

A central finding from Appendix 1 is the importance of the teachers' role to raise the level of mathematical richness from I to II and from II to III. When the students were formulating the problem [Ib → IIa], Ms. T asked students to predict whether the answers were higher than the basic value (of the cookies).

Ms. T:	Let's predict. The basic value is the value of the cookies. How much is it?
S1 (student):	It's 600 yen.
Ms. T:	So do you think the chocolates are more expensive or cheaper than 600 yen? (1)
Ss (many):	They should be higher.
Ms. T.:	Can you give a reason?
S2:	You can see that 600 yen is 1 in relation to 6/5(2), and that 6/5 is bigger than 5/5. So, the chocolates should be more expensive.
Ms. T.:	What about the candies?
S3:	I think they are cheaper than the cookies because when you regard 600 yen as proportion 1, 1 is equal to 5/5 (3) and 3/5 is smaller than 1. That's why the candies are cheaper than 600 yen.

The teacher's question (underline 1) triggered a raise in the level of mathematical richness from a concrete to a mathematical level. More specifically, in the beginning students talked about some elements at a concrete context (Level I) (i.e., 600 yen, 6/5, cookies, etc.), but when they had to justify why the chocolates were more expensive than the cookies, they transferred the objects of their attention from concrete contexts to mathematical ones and reinterpreted the elements from the viewpoint of mathematics (in Level II) (underlines 2 & 3).

Moreover, the teacher's question triggered the opportunity for students to "use fractions." When people think, or justify etc., they require some kind of "means" or "tools." In this case, therefore, the students are forced to use fractions when giving explanations for their predictions. As a result, students, including those who have difficulty processing such problems on a number line by themselves, can share the idea that "1 has to be divided into five equal parts in this case" (underline 3), which is crucial for handling number lines skillfully.

Next, Ms. T's question during Mathematical Activity 4 brought students to broader levels of thought. Students focused on each element (600, 3/5, and the price of candies [= x]) on a concrete level, and on a mathematical level they explored the structure of the problem, i.e., the relationships among those elements ($x = 600 * 3/5$). On the other hand, her question also lead to reflective thinking to reconsider whether using multiplication was really appropriate for expressing the relationships instead of using division on more general level. Consequently, the broader level focus was on more general structures or relationships, i.e., C = B * P and on the validity of the multiplicative structure.

Acknowledgments

This research was in part supported by the Japan Society for the Promotion of Science (Grant-in-Aid for Scientific Research 18830052, recipient: Kaori Miyauchi).

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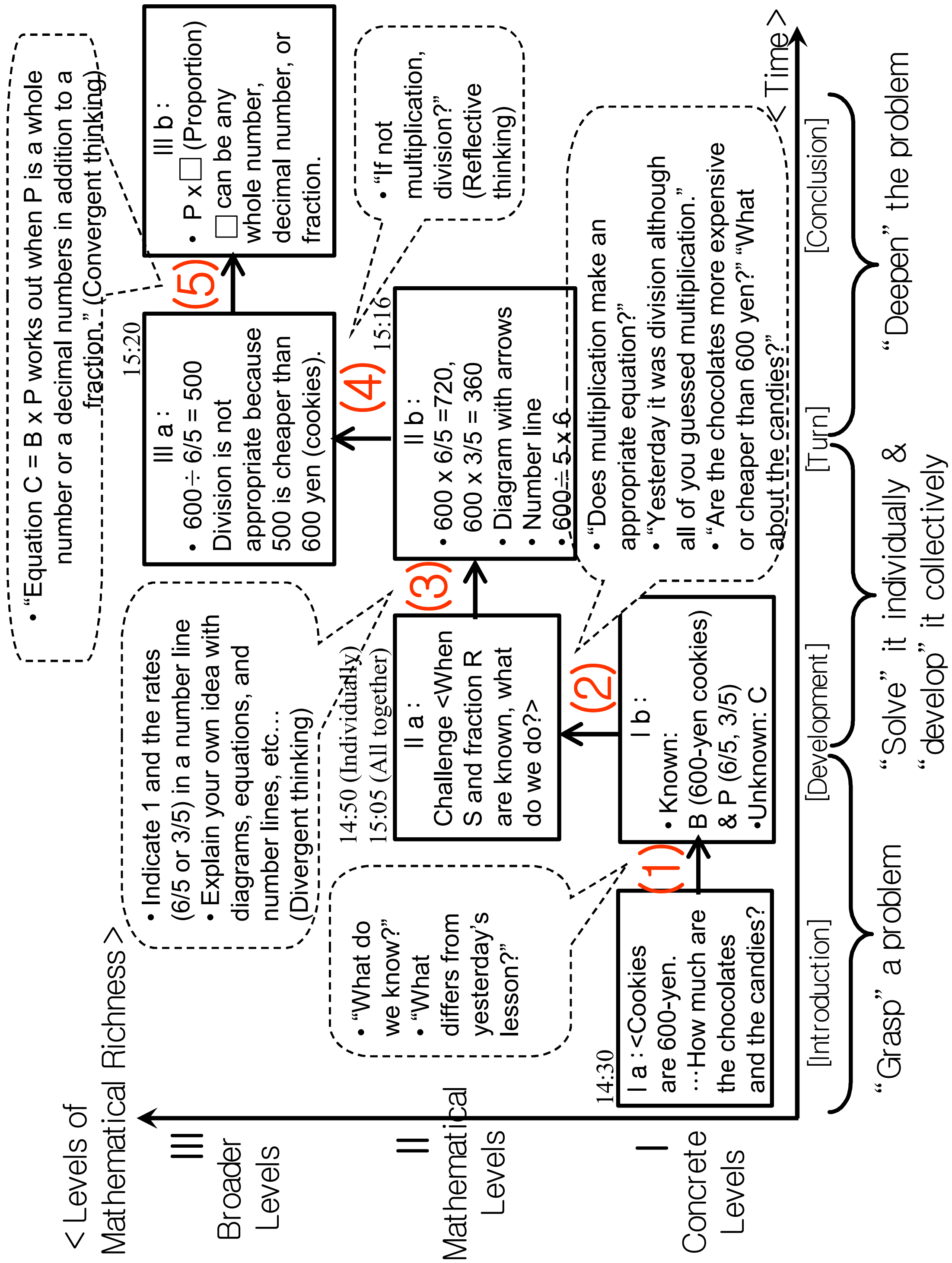
Appendix 1: Analyzing a fraction lesson in the six grades in Japan using the CALMA framework



REVISITING DISCOURSE AS AN INSTRUCTIONAL RESOURCE: PRACTICES THAT CREATE SPACES FOR LEARNING AND STUDENT CONTRIBUTIONS

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In this paper, we revisit the overall idea of classroom discourse as an instructional resource. We take the perspective of access in describing how discourse can create opportunities for students to learn important concepts and to share their ideas as part of the legitimate and sanctioned space of the classroom. We provide illustrations from a middle grades class in order to investigate how discourse can be drawn upon to advance the instructional agenda and to support students in experiencing voice in class.

INTRODUCTION

Today, mathematics teachers are faced with challenging decisions regarding the kinds of resources on which they will draw in developing lessons and teaching particular concepts. The focus of this paper is on one resource that is often under utilized in mathematics classrooms. This resource is the discourse – conversations and discussions – that develops in a class. The idea of discourse is not a new one. Reform recommendations (NCTM Standards, 1989; 1991; 2000) have emphasized the importance of discourse to all mathematics classrooms. The first Standards documents (1989) defined discourse in this way:

> Discourse refers to the ways of representing, thinking, talking, and agreeing and disagreeing that teachers and students use to engage in those tasks. The discourse embeds fundamental values about knowledge and authority. Its nature is reflected in what makes an answer right and what counts as legitimate mathematical activity, argument, and thinking. Teachers, through the ways in which they orchestrate discourse, convey messages about whose knowledge and ways of thinking and knowing are valued, who is considered about to contribute, and who has status in the group. (p. 20)

This definition implies that on one hand discourse has to do with the moment-to-moment interactions and conversations in mathematics class. On the other hand, discourse importantly reflects values and status within a class. The Standards have drawn attention to the critical role that discourse plays in mathematical learning as well as how discourse pervades every aspect of the classroom. Many scholars have described classroom discourse as an instructional resource for teachers and students, and in doing so have documented how it might play out in particular classrooms (Cobb, 2000; Gutierrez et al., 1999; Lampert, 1990; Thompson, Philipp et al. 1994). Building on previous researchers, our purpose in this paper is to illustrate how particular discourse practices can serve as resources in responding to all learners in a mathematics class. We take the perspective of *access* and provide illustrations from a middle school statistics project in order to examine

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 41-48. Seoul: PME.

the kinds of practices that have the potential to create opportunities for all students to gain access to more positive views of their mathematical learning.

ACCESS, DISCOURSE, AND LEARNING MATHEMATICS

In this paper, we take the perspective that particular classroom practices can create access to substantial learning and a sense of value in learning mathematics (Nasir & Cobb, 2002). In this view, aspects of the classroom social context are seen as supporting (or delimiting) students' opportunities to engage meaningfully with the discipline. Students' orientations toward learning mathematics can then be seen to stem from the arrangement of lessons and the participation structure that come to define life in classrooms. This perspective calls attention to how instructional practices can create spaces that give students opportunities to share their ideas, have their ideas valued, and at the same time learning important ideas.

Research in math education has emphasized how classroom discourse can afford students access to mathematical ideas and task interpretations (Cobb, 2000; Lampert, 1990) as well as a better understanding of the real world contexts that are presented in instructional tasks (Boaler, 2002). However, the issue of how discourse can provide access to more positive orientations towards learning math has not been investigated as fully. Guterriez and her colleagues show the significance of examining discourse practices in such a way.

In their analysis, Gutierrez, Baquedano-Lopez, and Tejeda (1999) draw attention to a third discursive space in an elementary class that sheds light on how discourse can support students' access to an affiliation with learning in particular classrooms. Gutierrez et al. analyze literacy practices of an immersion Spanish elementary school class and illustrate the notion of a *third space* that can emerge in discourse. In their ethnographic study, Gutierrez et al. define the official space of the classroom as consisting of the sanctioned, legitimate ways of participating in classroom discourse while the unofficial space includes students' ways of participating that do not comply with the teacher's view of appropriate participation. For example, Gutierrez et al. describe how students often engage in a counter narrative within the unofficial space of the classroom through practices such as giggling about sensitive topics, using colloquialisms, and using home and local knowledge in their comments. They describe how the classroom teacher included students' comments as part of discussions and in doing so sanctioned them as legitimate aspects of the classroom. Their analysis raises explicitly the central issue of how such practices, as guided by the teacher, can create access to experiences of voice in class while simultaneously advancing the instructional agenda in mathematics classroom.

The relationship between students' experiences of voice and their developing sense of affiliation with classroom learning hinges on Wenger's (1998) notion of *field of negotiability*. This idea refers to the realm over which students perceive themselves to have control in the classroom. This field of negotiability relates to students' perceptions of the extent to which they can contribute to the ideas that matter in the classroom. In practice, *contributing to the ideas that matter* in the mathematics classroom may involve making decisions about the legitimacy of task interpretations and the relative efficiency and sophistication of the methods used in approaching problems. Students who have an expansive field of negotiability view themselves to have significant roles in the classroom in justifying interpretations and conjectures. They do not confine decision making about the mathematical content to the role of the teacher. Students with restrictive fields of negotiability view themselves as having little voice or involvement in such decisions in the classroom. More often, they view the teacher and the textbook as having intellectual authority, not students. Of significance is that Wenger describes how changes in field of negotiability have the most profound impact on students' identification and affiliation with a community of practice in the long term (1998). Therefore, isolated classroom experiences that contribute to more expansive fields of negotiability can build and place students on a trajectory toward growing affiliation with mathematics.

BACKGROUND

In the middle-school class we studied, students participated in a statistics project that focused on supporting their understanding of statistical data analysis (Cobb, McClain, & Gravemeijer, 2003). A member of the research team served as the teacher in the project that spanned two years. Twenty-nine seventh-grade students participated in the project for the first year that took place over a twelve-week period and involved 34 classroom sessions of approximately 40 minutes in length. The following school year, a smaller contingent of students from the same class (now eighth graders) participated in a fourteen-week project continuation involving 41 classroom sessions of 40 minutes. This second year of the project addressed students' understanding of bivariate data. Both field notes and videorecordings of all classroom sessions were collected during the project.

During the second year, we conducted interviews with students to examine their perspective on their learning in the class. An analysis of the interview data indicated that, for the most part, students came to value their experiences and learning in the class (Cobb, Hodge, Visnovska, & Zhao, submitted). The findings from this analysis prompted us to return to the data of classroom videorecordings and field notes. We examined the data systematically in order to identify classroom practices that contributed to the students' affiliation with the type of mathematical learning that became constituted in the classroom.

DATA ANALYSIS

We analysed the classroom data by drawing on methods outlined by Glaser & Strauss (1967). Our analysis involved multiple overlapping phases. First, we moved through field notes and videorecordings in order to identify critical sessions in which aspects of discourse or instructional practices seemed to contribute to students' participation in whole-class discussions. We were interested in situations in which students' contributions became significant topics of conversation. Additionally, we were interested in situations in which students might have perceived themselves to be silenced. We examined moments in discussions in which students' seemingly unrelated comments emerged and how they played out in interaction. Second, we examined data across sessions at a meta-level in order to identify discursive practices that were consistent or that changed over time, noting their implications for students' participation. In both these phases, we focused on the opportunities that classroom discourse created for discussing important ideas and for students' participation in these discussions.

THE ROLE OF INTRODUCTORY DISCUSSIONS

We confine our illustrations from the middle grades class to whole-class discussions that introduced instructional activities. The rationale for our focus has to do with the important role of introductory discussions in the statistics project. The teacher drew on discussions to introduce instructional tasks to students and to develop their interest in the issue to be investigated. These discussions provided students with reasons to analyse specific datasets. We should mention that, overall, instructional tasks in the statistics project were designed to capture the authentic investigative spirit of analyzing data. As part of this effort, students were invited to analyze data that served the purpose of answering questions that were relevant from their perspective. Most of the instructional tasks involved comparing two data sets in order to make a decision or judgment (e.g., determining whether installing airbags in cars does have an impact on automobile safety or investigating the effectiveness of two Aids treatments on raising T-cell counts).

In the course of these introductory discussions, which were often times quite lengthy, the teacher and students together delineated the particular issue under investigation, clarified its significance, identified relevant aspects of the issue that should be measured, and considered how they might be measured. In this way, the introductory discussion served to clarify aspects of the problem context and their relationship to the question at hand. At the same time, the introductory discussion also created opportunities for students to understand the relevance of the issues presented in the tasks. Following the introductory discussion, the teacher then introduced the data as having been generated by a systematic process and the students conducted their analyses individually or in small groups. The final phase of an instructional activity consisted of a whole-class discussion of the students' analyses.

INCLUDING STUDENTS' CONTRIBUTIONS

One of the challenges that Gutierrez et al.'s (1999) work has indicated is how to include students in discussions while at the same time addressing important ideas (Ball, 1993). In moment-to-moment interactions, it is often difficult to make decisions quickly about how to include students' comments in such a way as to move in the direction of goals for a particular lesson, and additionally, how to respond to students' seemingly extraneous comments. We address both sides of this coin as we examine illustrations from the introductory discussions: Including and treating students' ideas as valued aspects of whole-class discussion and drawing on students' contributions strategically to advance the instructional agenda. As we discuss this tension, we emphasize one practice that supported students' access to mathematical learning in class.

From the description we have given thus far, we hope to be clear that the introductory discussions are quite different from teacher explanations of the directions students are to follow as they solve problems. These discussions involve making the activity or problems "come to life" for students by having students contribute to the construction of contexts and issues within the classroom. These introductory discussions can be described in terms of various phases that clarify:

- The overall problem or broader issue
- The significance of the problem to particular audiences
- The students' job in working on the problem and specific products that will result
- The different aspects of the problem to consider

During many introductory discussions that occurred in the initial stages of the project, many of the students' shared personal experiences that related to the overall topic of the problem (e.g. batteries; braking distances and automobile safety but not to the specifics of data to compare the two makes of cars). The following comment is an example of the personal comments that students would make during that time:

K: They have a commercial where it's like a big long line and they have the tires and says this is what happens when the tire like stops. And showing how fast it takes to go from 60 to 0.

Teacher: Yeah as opposed to 0 to 60. Usually, yeah, usually the cars that, you know, the sports cars say from 0 to 60 in you know so many seconds. They don't talk a whole lot about from 60 to 0 which is pretty important when you have to stop.

As the discussion continued, students made similar comments:

Gary: When my sister was taking her driving test, I used to watch her videos along with her. And they said on an icy road it could take a car 275 feet to stop.

Teacher: To stop. That's exactly correct. Because it just starts sliding. There's just nothing for the wheels to grab hold of.

In the early stages of the project, the teacher revoiced (O'Conner & Michaels, 1996) students' personal comments and included them as a part of the ongoing discussion. In revoicing, the teacher drew attention to different aspects of the problem situation and the issue the students would be asked to investigate. Her intent in including students' comments and supporting their participation was to cultivate their interest in the issue that the problem presented. In doing so, she supported a space in which students could share ideas as part of the teacher's goals of clarifying the problem context. We conjecture that this practice of building on students' comments may have contributed to students' interest in investigating the issue through data analysis. Additionally, we speculate that this practice placed students on a path in experiencing more expansive fields of negotiability within the context of the class. We note that this illustration offers only a glimpse of a lengthy process that occurred during the project.

As the project continued, the teacher became less accepting of such comments and she and the students together coined a term, "random comments," to label comments that did not relate to the problem at hand. These personal comments became increasingly rare, and students who made these comments were typically challenged by the teacher and in some cases, other students. This explicit attention to talking about classroom discourse emphasizes the idea of students as decision makers and resources for their own learning. From the perspective of access, the practice of building on students' comments toward the instructional agenda, provided opportunities to share their ideas and possibly view their ideas as part of the sanctioned and legitimate space of the class. This shift required that the students have a sense of what the relevant aspects of the problem task included. Additionally, this treatment of random comments created opportunities for students to reflect on their own participation and more generally on the nature of conversations.

STUDENTS' CONTRIBUTIONS AS RESOURCES

Students' contributions in introductory discussions can be viewed as reflecting student engagement and participation, and potentially more positive orientations, in the long term, about mathematical learning. At the same time, we would emphasize, students' contributions served to advance the agenda of introductory discussions. They were used as resources by the teacher to create involve the students in co-constructing the problem context, and thereby providing many students with access to the task and their purpose in analyzing related datasets. As an illustration of the role of student contributions in constructing the problem context, consider a task that occurred in the latter half of the first year of the project in which students were to assess the effectiveness of installing airbags in cars. The teacher initiated this discussion by asking the students if they had seen television commercials that show crash test dummies being used to test the safety of cars. In opening the discussion with a non-mathematical question, she offered students a way "in" to the conversation. From there, she then presented a brief introduction of the overall issue of car safety testing. Drawing on a student's question, she then focused the discussion specifically on the effectiveness of airbags, the issue that the students were to investigate. She was able to create a space for students' contributions to be legitimate and for important topics, relevant to the task, to become topics of conversation.

Dan: What are we trying to find out?

Teacher: How do you think people made this decision about it, to put airbags in cars? Rob?

Rob: A lot of people were getting in wrecks.

Teacher: How do you make a decision about whether or not to have an airbag in the car or not? How would you make that decision? Tyler?

Tyler: You'd find out the safety of the car. You put a crash dummy in there and you see if he hits his head on the dashboard or the steering wheel or whatever.

The previous excerpt offers a glimpse of how students' comments were used to support care safety to become an explicit topic of conversation and, more generally, the role of students' contributions in introductory discussions. As illustrated in this brief exchange, the teacher, through discourse, was able to provide opportunities for students to share their ideas, but at the same time drew on these comments to bring out relevant pieces of the problem. In such discussions, the students were then encouraged to anticipate information that would be necessary to investigate the question at hand. This provided students with situations in which they would have to assess the appropriateness of their own comments and that of others. In this case, this information included the nature of the data to be collected that would be necessary to convince people to install airbags in cars.

Later in the discussion, the teacher introduced data from crash tests conducted on all models of new cars available in the United States in 1993, some of which were not fitted with airbags.

> Teacher: These are the cars that were manufactured in 1993. These are the cars that had airbags (points to one of the data sets) and these are the ones that did not have airbags in them (points to the second data sets). And those are the results of the crash test, just looking at head injuries and trying to make a decision about airbags. Wes?
>
> Wes: What about the rest of the bodies?
>
> Teacher: No, they were just, the primary function of an airbag is to prevent head injuries so that's what they were focusing on that. Good question. Rob?
>
> Rob: How could they know how severe the head injury was if they were dummies?
>
> Teacher: They put sensors in the dummies that could register the impact of what's happening to them in there and slow motion cameras that watch it.

In the previous exchange, students' contributions (in this case their clarifying questions) were included to construct more fully the problem context. The students' and teacher's comments about measuring injuries suffered in car accidents together made the overall context and issue more accessible to the class. More importantly, by introducing the data that the students were to analyze as generated through a systematic process, the teacher and students co-constructed the meaning of the numbers in data sets with regard to the phenomenon to be investigated. As a consequence, the mathematical aspects of following whole-class discussions about the students' analyses of the data were made more accessible to students. The introductory discussions can then be viewed as serving the dual purposes of both making the problem context come to life and constructing meaning for the numbers presented in data sets. Therefore, students' contributions were drawn upon as resources in making the phenomenon and the mathematical ideas of problem tasks accessible to the classroom community.

DISCUSSION

In this paper, we have described classroom discourse as an instructional resource for supporting students' access to important ideas and to more positive views toward mathematics learning. We have described a central challenge to teaching, that of advancing the instructional agenda and including students' ideas as aspects of the legitimate and sanctioned space of mathematics class and how this challenge played out in one instance in a middle grades class. In navigating this tension, we have illustrated a shift in responsibility to include students in making decisions about relevancy and appropriateness in whole-class discussion. The classroom discourse we have described is significant in that it stresses the importance of creating opportunities for students to share their ideas and experience their ideas as valued. However, it is also important to create situations for students to understand what constitutes relevancy and the purpose for engaging in specific activities in mathematics class.

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AN ILLUSTRATION OF STUDENTS' ENGAGEMENT WITH MATHEMATICAL SOFTWARE USING REMOTE OBSERVATION

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Students using three types of spreadsheet calculators for understanding expected value were observed remotely. This remote observation involves the use of webcams and application sharing for observing students learning mathematics. The study illustrates how remote observation can be used for collecting mathematical education data and raises questions about the extent to which such a method can be used in future experiments.

INTRODUCTION

Various studies have investigated how students learn mathematics with software such as computer algebra systems (e.g. Bardini, Pierce, & Stacey, 2004; Berry, Graham, & Smith, 2006) and spreadsheets (e.g. Clarke, Ayres, & Sweller, 2005). However, traditional observation studies of students using software occurs when students are invited to a specially-configured computer laboratory or "user-lab" where they are video and audio recorded or the researcher visits and sets up audio and video recording facilities at the student's place of study (e.g. San Diego, Aczel, & Hodgson, 2006; Vale & Leder, 2004). Whilst user-labs provide controlled recording conditions and the possibility of more sophisticated technology such as eye-tracking (e.g. San Diego et al., 2006) these either remove or intrude on students in their natural studying environments. Less intrusive observation practices have included the logging of students' computer strokes and mouse clicks (e.g. Berry et al., 2006; Thomas & Paine, 2002) but this means rich video data is lost.

A method for observing students using software via the internet has recently been investigated called remote observation (Hosein, Aczel, Clow, & Richardson, 2007) which records both audio and video data, mouse clicks and keyboard entry. In remote observation, students use a remote application facility on their computer to connect to the researcher's computer where they are able to interact and use software on it (see Figure 1). Through the students' webcams and video conversation facilities in instant messengers (IMs), students are observed and interviewed whilst using the software. By using screen and audio capture software, students' on-screen actions, webcam video and audio can all be recorded. Hosein et al. (2007) indicated that students eventually forgot about being video recorded and observed since the window showing the webcam image was covered up. This perhaps may help in providing a more naturalistic approach to observing the students (Guba & Lincoln, 1981). This paper reports on proof-of-concept work on the use of remote observation of students using mathematical software.

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 49-56. Seoul: PME.

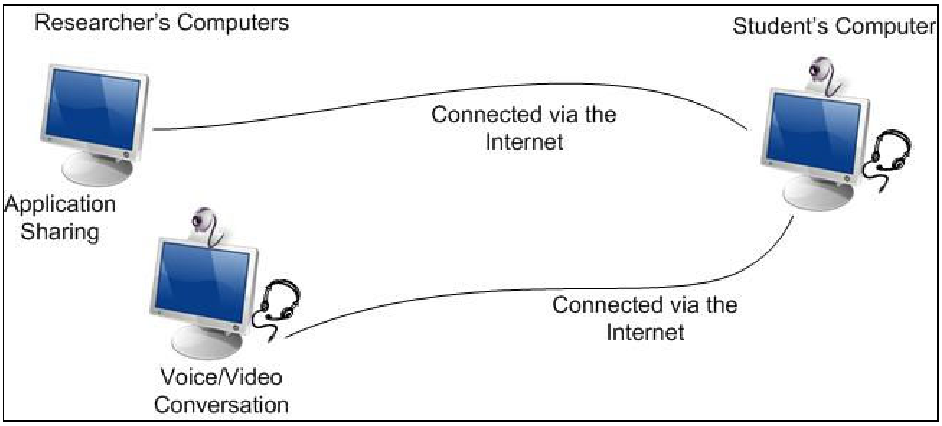

Figure 1: Remote Observation Process

METHODOLOGY

To understand how remote observation can be used for investigating students' learning of mathematics, a method was used to encompass both quantitative and qualitative data collection. The method followed that of quasi-experimental methods used in mathematical cognitive load theory (CLT) literature (e.g. Große & Renkl, 2006; Renkl, Atkinson, & Große, 2004; Schworm & Renkl, 2006). The quasi-experimental methods in CLT use a five-part procedure, usually to investigate to what extent students have learnt a topic (see Figure 2).

Steps	Instructions
1.Demographic Questionnaire	Students are asked to fill in a demographic questionnaire, including questions asking for mathematical level, age and gender
2.Instructional/ Study Materials	Students peruse materials to understand the fundamental concepts required for the learning of the topic
3. Pre-test	Students to determine what extent they have prior knowledge of the topic before the stimulus is provided for the experiment. The pre-test problems is at a lower difficulty level than the post-test problems
4. Experiment	Students are provided with the interventions/ factors that are being studied
5. Post-test	Students work on a set of questions to acquire quantitative data to compare the investigated interventions/ factors

Figure 2: Quasi-experimental method used by Atkinson, Renkl and colleagues

There is sometimes a variation in the literature, in that the second and third step of this method may be interchanged (e.g. Große & Renkl, 2006; Renkl et al., 2004). The preference for this paper is the way it is presented as this means that the learning from the instructional/ study materials do not have to be taken into account when comparing data between the pre-test and the post-test. This quasi-experimental design is used for collecting mainly quantitative data but by added on talk-aloud strategies Ericsson & Simon (1984), interviews and videoing, qualitative data is also collected.

Data collection in remote observation

In order to investigate remote observation as a method for observing students learning when using mathematical software, a simple mathematical topic was chosen: expected values. Expected values area is part of decision theory in operations research where probabilities are used to compare and determine best options. The aim of the study was to determine to what extent students may learn differently depending on the problem-solving software they employ. The software chosen for learning expected values was an Excel spreadsheet in which three types of spreadsheet calculators were used (coded using Visual Basic for Applications, VBA). Excel was chosen as it is familiar to many students and thus minimized the effect that familiarity with the software might have on the learning of the topic. The three types of spreadsheet calculator were black-box, white-box and grey-box. Black-box calculators are considered to be software in which calculations are performed without showing steps whilst grey-box calculators perform calculations showing the steps. White-box calculators allow the students to interact with the software at each step to determine the next action when calculating the answer.

The consent form for students participating in a remote observation study is problematic as signed consent is difficult to obtain when students are at a distance. In this study prior to the scheduled experimental time, students were required to fill in their names in a web-consent form and then submit the webpage. However, this meant there were no guarantees that this was indeed the student filling in the form. Perhaps, to circumvent this problem, the participants should also enter their email address, so that a confirmation email of their consent can be sent to them. However, to remedy this problem during the actual experimentation period students were asked for permission again as to whether they consented to be video and audio recorded via instant messaging and there was no objection.

The demographic and pre-test questionnaires were also produced as web pages. The links to the consent form and demographic questionnaires were emailed to the students prior to experimental period to fill in and submit. The pre-test was based solely on simple probability since Renkl et al. (2004) suggested using a level of difficulty that was lower than the post-test. Only when these two questionnaires were completed, an email was sent to the student to set up a date and time for the experiment. This was done to minimize experimental time required by the student and provided more flexibility. The pre-test questionnaire link was provided to the student via an IM and was filled in during the experimental period. The instructional/ study materials included information on how to use the spreadsheet calculators and guidance on expected values. The instructional materials, the practice questions and post-test materials were sent prior to the experiment so that students could print these and use it as a reference during the experiment. They were also told that it was not necessary to read these materials prior to the experiment. This reminder was placed to minimize students preparing or learning the topic prior to using the software. During the experiment, students were given time to read through the instructional materials on expected values and the software materials. Although this study used only 6 students for understanding the remote observation process, a rotational confounded study design (Campbell & Stanley, 1963) was tested, where each student used the three spreadsheet calculators in the 6 permutations (see Figure 3).

Student	Calculator	Calculator	Calculator
1	White	Black	Grey
2	Grey	White	Black
3	White	Grey	Black
4	Grey	Black	White
5	Black	Grey	White
6	Black	White	Grey

Figure 3: The order that the spreadsheet calculators were used by each student

The students were allowed to use a practice question for testing the three spreadsheet calculators and also practice the talk-aloud strategy which constituted step 4 of the CLT method. The practice session is similar to that done by San Diego et al. (2006) in their user-lab work. Students were required to solve the various problems using the three spreadsheets calculators and entering their answers into a spreadsheet. There were 9 problems in the post-test: the first 6 problems were multiple-choice whilst the last three questions required the entering of the answer along with an explanation. The answer sheet for the post-test used a spreadsheet for this purpose. Following the post-test a short interview was conducted with the students to elicit their opinions on the three types of calculators and on expected values.

ILLUSTRATION OF DATA COLLECTED

Quantitative Data

Firstly from the pre-test and post-test questionnaires, quantitative data was collected in which a marking scheme was used to allocate points to the student. These points can be used for further statistical analysis if a large sample is used to compare the different factors. Although 6 questions were multiple-choice, the researcher can revisit the video and audio recordings to determine how students acquired their answers for allocating points, as in some cases the students provided the correct answer, although their reasoning and method were sometimes wrong. Interestingly, students often neglected to use the spreadsheet calculators and opted instead to use pen and paper or a calculator. This data was thus lost and makes it difficult to compare spreadsheet factors, highlighting an important limitation of remote observation.

Qualitative Data

However, the qualitative observational data proved to be quite useful and can be used to triangulate with the quantitative data. From the six students, an episode is illustrated on the type of data that can be collected and what analysis can be performed. Figure 4 presents data from a student (no. 6) doing the practice question during the experimental session.

Which of the following games would I get the best expected value for?

Game 1:
1st prize: 20% probability of winning £90
2nd prize: 80% probability of winning £46

Game 2:
1st prize: 30% probability of winning £60
2nd prize: 70% probability of winning £40

Game 3:
1st prize: 10% probability of winning £100
2nd prize: 90% probability of winning £10

Student/ Researcher	Time	Action	Utterance
Student	14:02	Checks data on screen against the practice question on paper	Right, so I've got me data in …20, 80, 30, 70, 10, 90, 90, 46, 60, 40, 100, 10
Student	14:12		That looks alright. Ok, So now I can go to the black-box?
Researcher	14:15		Yep
Student	14:17	Clicks black-box button	Right.
Student	14:19	Reads question on the practice question paper	Ok. Which of the following games will I get the best expected value for?
Student	14:29	Looks for definition in instructional material	Let's look and see what expected value meant again
Student	14:32	Reads from instructional materials	Tell's me what my average prize will be
Student	14:36	Looks at the black-box calculator	OK.
Student	14:42		So….
Student	14:50		Right
Student	14:54		And I'm not doing Game 1, Game 2 or Game 3, I'm doing the practice. Wait I am doing the Game 1, Game 2 and Game 3. So, if I click on Game 1…
Student	15:05	Clicks on the buttons representing Game 1, Game 2 and Game 3	Oh it gives me an answer. Right, that's really interesting. Game 2 gives me an answer and Game 3 gives me an answer.
Student	15:14		Right so Game 1 would obviously give me the best expected value and I wasn't paying any attention to what I was doing there at all and I've got an answer.
Researcher	15:25		Right, can you just click best just to show you what you can do with the best button if you want to
Student	15:30		Oh ok
Researcher	15:33		Yeah …you don't have to but…
Student	15:35	Clicks on the Best Button	Right the best game can tell me what the biggest number is … very clever.
Student	15:40		Right, I appeared to have done that, don't I? Without any comprehension whatsoever.

Figure 4: Transcript data corresponding to audio and video data recorded from the remote observation exercise for the Excel spreadsheet

The upper left-hand corner of the figure shows the practice question whilst the upper right-hand corner shows the Excel spreadsheet that both the researcher and the student can see. Below this, a transcript of the student's utterance is shown along with the timeline in the experimental period. The actions of the student are also noted after the experiment. These actions, such as the clicking and entering of data, can be seen from watching the screen capture video, whilst the actions such as reading printed materials are noted through the webcam video. A webcam picture of a student reading printed materials is shown to the side of the transcript. In this particular episode, we note that in this practice question the student is looking at the black-box spreadsheet and there seems to be some confusion as to what to do. The data shows that from time 14:17 upon entering the black-box calculator spreadsheet, the student decides to read back the question and then tries to understand what the term ‘expected value’ means before proceeding to click the buttons to see what happens (15:05). We note that the student was able to achieve the answer (“I wasn’t paying any attention to what I was doing there at all and I’ve got an answer”, 15:14). Although the student claims later that they did this “without any comprehension whatsoever” (15:40), we note that at 15:14 they were able to tell which was the best game without clicking the ‘best button’, and this was part of the object of the task. Thus, this task shows that for the black-box spreadsheet calculator, although a student may be uncertain what the command buttons are used for with their limited understanding of the mathematical concepts and the ease of use of software which comes with a black-box type spreadsheet, the student can still work towards achieving the answer.

Looking at other students utterances using the three calculators, all students felt some amount of confusion when starting with all three calculators, but students were less likely to know what to do when they started off with the white-box calculator (students 1 and 3). However, most students who used white-box after the black-box and grey-box spreadsheets, were still uncertain on how to calculate expected values and had to check back the instructional materials (student 6) or intuitively guess what to (student 4 felt that multiplication would be the best arithmetic operation). It appears from this limited study that whilst black-box and grey-box calculators may help the students in calculating the answer, it does not help in understanding the steps. Even though the grey-box showed the steps, only two students (students 2 and 4) took time to look through to see what the steps meant, this may mean since the solution was provided for them that students did less self-explanations to seek understanding (Schworm & Renkl, 2006). Also, when using the white-box calculator students found that after understanding the steps, that the iterations became tedious and this may impede learning (Renkl et al., 2004).

DISCUSSION

Remote observation provided some challenges when trying to observe students learning new areas without them having any prior indication of the materials. Although students here were asked to print out the instructional materials, students could have easily been redirected to another webpage where they could read the materials. However, this would require them switching between windows when doing the post-test questions and perhaps creating a higher split-attention effect Mayer & Moreno, 1998; Sweller & Chandler, 1991) than that of between paper and screen. When using paper and screen, students are able to have a direct comparison without the need to hold information in their working memory between one window and the next. Students can divide their screens to accommodate both of these windows, but would only be successful if their screen is large enough to accommodate sufficient information to be seen on both windows without requiring them to hold information in their working memory whilst they scroll down the windows.

Further, although Excel is used here, in classroom/course situations more sophisticated software such as computer algebra systems may be investigated. It was noted that some students chose to use pen and paper for working out some problems or the calculator on the computer. In face-to-face observation environments, such actions can be recorded in field notes (e.g. Pirie, 1996), but in remote observation the actions might be out of the field of view of the webcam. Meanwhile, in their user-lab, San Diego et al. (2006) used a Tablet PC to record writing and sketches, but this equipment is not available in typical student settings. So unfortunately under this remote observation process this data is lost unless special requests are made that the student post or scan these and send them to the researcher. Or a directive can be made to ensure that students only use software but this may hamper their natural learning process as well as defeat the purpose of observing students in their natural learning environment (Guba & Lincoln, 1981). Also, remote observation for quasi-experimental methods does not lend itself easily to statistical analysis which requires large sample sizes. In this paper, students generally took between 1 ½ to 2 hours to complete the exercise and thus if a larger number of students is expected, a rotational design should be used to minimize the number of remote observations and also decrease the time required for tasks to be accomplished to probably between ½ to 1 hour if possible.

CONCLUSION

Remote observation for capturing students' use of software when learning mathematics seems a viable option where there is an inability to bring students to user-labs and other laboratory settings or go to them. Useful qualitative and quantitative data can be collected. Particularly for the qualitative data, talk-aloud strategies can still be employed and the actions that students undertake in the mathematical software is able to be observed and recorded, however, the recording of students activities outside of the sphere of the shared application software is lost. Therefore, in research such as this for understanding students use software for problem-solving, researchers are not limited to students in a particular setting but to any student connected to the internet that will allow them to collect rich qualitative and quantitative data.

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GEOMETRIC CALCULATIONS ARE MORE THAN JUST THE APPLICATION OF PROCEDURAL KNOWLEDGE

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Calculations are often viewed as the application of procedural knowledge or algorithms. In this paper, I argue that geometric calculations can be more than that. To prove this assumption, firstly I analyze a geometric calculation using the theoretical framework of proof schemes proposed by Harel and Sowder (1998). The theoretical analysis shows that solving a geometric calculation can provide the opportunities to learn transformational observations just as geometric proofs do. Secondly, by analyzing instruction data, I report how a teacher conveys the opportunities to learn transformational observations in terms of scaffolding students in recognizing the geometric properties the diagram possesses and in inferring new measurements inside the diagram. In doing so, the teacher expects students to see the need for using transformational observations to form a solution.

INTRODUCTION

Geometric proofs and reasoning are important topics in mathematics learning and teaching. The National Council of Teachers of Mathematics (2000) recommends helping students learn reasoning from an early age and become capable of reasoning and performing proofs before the end of 12th grade. Among the materials for learning geometric proofs, geometric calculations are pervasively used in geometry lessons and are often used as a type of questions on achievement tests (e.g., TIMSS, 1999; NEAP, 1989). Some studies in psychology also employ geometric calculations as task designs for investigating human thinking (Ayres & Sweller, 1990; Sweller, 1988, 1989). The definition of a geometric calculation can be derived from literature as the numerical calculations within a geometric environment. (Healy & Hoyles, 1998; Katz, 1993; Küchemann & Hoyles, 2002)". Because of the geometric setting, solving a geometric calculation requires solvers to manage several types of information (given verbal information, visual information, calculating or writing information) and may also require solvers to reason several steps before obtaining the final answer. The complex problem-solving process drives a geometric calculation away from the procedural calculation.

However, while research on teaching and learning proofs has become the main stream in mathematics education, research on geometric calculations has still received little attention, especially its relationship with geometric proofs. Hence, the first purpose of this paper is to analyze geometric calculations to prove that geometric calculations can provide the learning opportunities just as geometric proofs do. The second purpose is to investigate how the instruction conveys these learning opportunities to students when teaching a geometric calculation.

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 57-64. Seoul: PME.

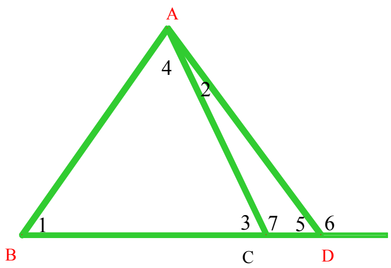
THEORETICAL FRAMEWORK AND ANALYSIS

Harel and Sowder (1998) propose a conception map in which they categorize different levels of students' schemes about proofs. Among these proof schemes, the transformation proof scheme is the one that students are expected to have when learning proofs. Transformational proof scheme is described as follows:

> Transformational observations involve operations on objects and anticipations of the operations' results. The operations are goal-oriented. They may be carried out for the purpose of leaving certain relationships unchanged, but when a change occurs, the observer intends to anticipate it and, accordingly, intends to apply operations to compensate for the change (p. 258).

From the explanation above, the major characteristic of the transformational proof scheme is the ability to dynamically manage mental images for forming a valid solution. Before finding a valid solution, solvers may have some initial thoughts. These initial thoughts are verified by the operations of transforming these images and are governed by deductive reasoning.

The following example illustrates how a geometric calculation can possess the opportunities to learn or apply transformational observations.



Given information:

$\overline{AD} = \overline{AB} = \overline{BC}$.

Given that $\angle 6 = 130^\circ$

Find the measurement of $\angle 2$.

Figure 1: A geometric calculation for finding angle measurements

Solving the problem in Figure 1 requires students to find several unknowns before obtaining the final answer. Students need to visualize the diagram in terms of decomposing the diagram and searching for their corresponding geometric properties that are used to find the final answer. For example, the position of triangle ABC is not typical as most of isosceles triangles do. Students may need to recognize ΔABC as an isosceles triangle by transforming the sub-diagram mentally for obtaining the property that $\angle 4$ and $\angle 3$ are congruent.

The transformational observation used to complete this task is to identify the relationship between $\angle 3$ and $\angle 1$ as well as the relationship between AC and AD. By looking at the diagram, one may think that $\angle 3 = \angle 1$ because AB and AC look congruent. However, students who possess the transformational observation can identify that this assumption of AB=AC and $\angle 3 = \angle 1$ cannot be true because AB equals AD, $\angle 5$ equals $\angle 1$, and triangle ΔABD is isosceles. If $\angle 3 = \angle 1$, then the segment AC should be dynamically moved and overlapped with segment AD as Figure 2 shows. Rotating segment AC to segment AD allows students to reject the possibility that AB=AC as well as $\angle 1 = \angle 3$.

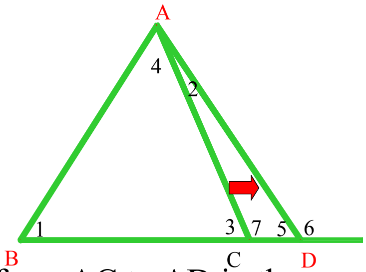


Figure 2: The movement from AC to AD in the case that AB=AC and AB=AD

In addition, using transformational observation to identify the relationship between AC and AD helps students to evaluate and monitor their own problem-solving strategies. The dynamic observations on the diagram that AB≠AC and ∠1≠∠3 may also inform solvers that ∠4 must be smaller than ∠BAC as well as that ∠BAD equals the sum of ∠4 and ∠2.

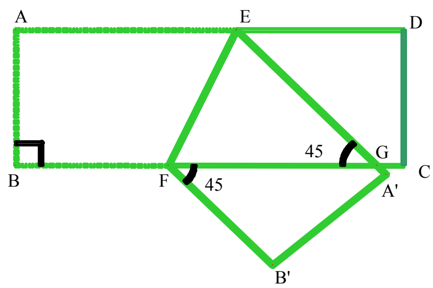
Hence, this example verifies that geometric calculations can possess the opportunities to learn transformational observations.

METHODS

Data

The data sources in this study were the videos collected from Taiwanese 9th grade geometry classes. Totally 8 lessons were taped during the summer in 2006. Among these video lessons, I picked one lesson in which several geometric calculation tasks were discussed by the teacher, Nancy[1], and her students. The instructional goal of this lesson was to discuss several geometric calculation problems students had in the test sheet. The most common *activity type* Nancy used in this lesson was *triadic dialogue* (Lemke, 1990), the *dialogues* that "teachers ask questions, call on students to answer them, and evaluate the answers (p. 217)". Sometimes, Nancy also called students to the blackboard to share their solutions with whole class.

Furthermore, I narrowed the analysis to one special episode where Nancy gave instruction on a geometric calculation that possessed the opportunities to learn transformational observations. The geometric calculation in this episode is described as follows.



As the diagram on the left shows, a rectangle ABCD can be folded along the segment EF to move point A and point B to their new positions A' and B'. If angle EGB=45°, GFB'=45°, and AB=8 centimeters, find the area of ΔEFG=_________.

Figure 3: The task requires the application of transformational observations

[1] Nancy is a pseudo name.

The problem above was not easy. Before Nancy's explanation, only 2 students could solve it by themselves. One of the difficulties in solving this problem was to apply transformational observations in terms of constructing auxiliary lines and finding the measurements by applying their corresponding geometric properties. The diagram in Figure 4 illustrates an example of the application of transformational observations.

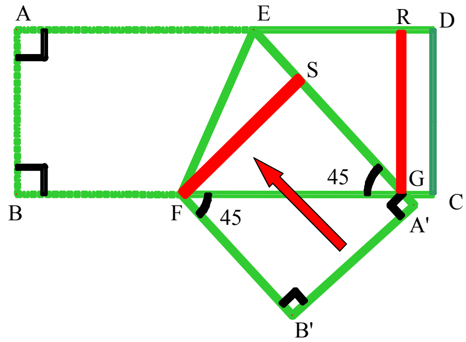


Figure 4: One of the solutions with the application of transformation observations

Knowing where to construct auxiliary lines on the diagram and anticipating the operations results are demanding. For finding the area of ΔEFG, one may need to construct the segment FS as the height of the ΔEFG by moving B'A' parallel to FS such that FS=B'A'. Another construction is to draw segment GR perpendicular AD to form ΔERG. This construction is more difficult than constructing FS because the need for constructing GR cannot directly be seen. Students have to infer $\angle GED = \angle EGF = 45^\circ$ firstly by applying alternative interior angles theorem. Then they recognize the need to construct GR because GE can be found only in the condition that one applies the isosceles right triangle property with the information of length GR. After all, the area of ΔEFG can be known.

Data Analysis

The videos of lessons were analyzed by using discourse analysis tools. I used the *manifest content approach* defined by Erickson (2007) that derived from subject pedagogical knowledge. When reviewing videos, I focused on the subject matter content manifested in talk and in written symbols as well as the physical actions, gestures, and nonverbal information (Erickson, 2007).

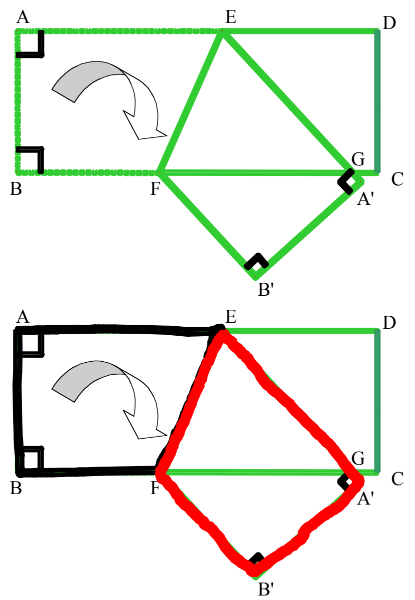
In order to investigate the meaning of instruction with respect to multiple aspects, I used the multimodal framework (Thibault, 2000) to analyze the transcription. The multimodal transcription revealed the text's meaning in which the distinct semiotic resources were co-deployed and co-contextualized in making a text specific meaning (Thibault, 2000). The multimodal transcription I used included the visual frames of the instructions, the interaction between diagrams and speakers, kinetic actions speakers took, and nonverbal information in the transcription.

FINDINGS AND DISCUSSION

How Nancy scaffolded students in learning transformational observations

This episode of how Nancy scaffolded students in learning transformational observation when solving the geometric calculation can be divided into three major stages.

The first stage was to help students to have background knowledge of geometric properties that would be used to infer new measurements on the diagram later.



Nancy: Pay attention to here, students. Like this <u>folding</u>[2] paper problem [uses gesture to show the action of folding]. There are many we need to know. For example, we do not know any numbers here [points to the diagram]. But because this part [points to ABFE] flips and turns and becomes this piece [points to A'B'FE]. So, these angles should be right angles [draws right angle marks on angle B', angle A', angle B and angle A]. (.) The quadrilateral [uses gesture to outline four sides of ABFE] and this quadrilateral [uses gesture to outline four sides of A'B'FE] should be what? (.)

SG: Congruent.

Nancy: <u>Congruent</u>.

The instructional goal in this stage was to help students to recognize the congruence of quadrilateral ABFE and A'B'FE as well as the corresponding sides and angles (e.g., ∠AEF=∠A'EF). The way Nancy conveyed the congruent properties to students was by using gestures (e.g., flipping, pointing, re-drawing) with the diagram. The actions helped students to visualize the diagram and to recognize where Nancy was talking about. In doing so, the cognitive load of re-interpreting the meaning of labels by "looking" and "reading" the diagram was also reduced.

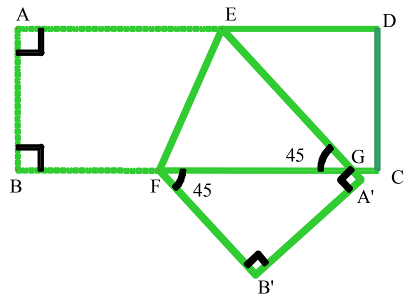
In the second stage, the instruction focused on searching for new measurements. Nancy asked students to infer any new measurements by applying these geometric properties students acquired in the first stage.

> Nancy: Now, let us see it. What are the angle measures this problem gives us? [Watches test sheet] This angle is 45˚ [writes 45˚ on angle A'FB'] and this angle is also 45˚ [writes the 45˚ on angle EGF] (1.0). I will get some information. (.) We can find a lot of new angle measurements. Now I think we should not just focus on this problem. Students, please tell me. (.) According to what we have done and the given information, what can you find in the diagram? (2.0)

[2] Notes for all transcription in this paper: (.) short pause; (number) longer pause lasts the number of seconds; (inaudible) words can not be clearly heard; _____emphasis of the words; [] teacher or students' writing or physical actions; SG: students group.

Although the purpose of searching new measurements was to find the area of ΔEFG, Nancy told students that they should not be constrained by this problem. They could infer any new measurements based on the given information and the geometric properties they found in first stage. One assumption of why Nancy did this in this episode may be that she viewed that the ability to infer new measurements by applying geometric properties was basic and important. She might think that even though the problem-solving strategies students used might not be the best ones, students might still be able to find the final answer eventually.

In addition, after one student failed in finding the measurement of angle EFG, Nancy used the gesture of pointing to inform whole class that they could find new measurements in another place.



Nancy: There are many other obvious angle measurements we can find, aren't there? [Points to angle A', angle B', angle C, and angle G] (1.0) Ok, Lily [3], can you find something?

Lily: …….

Nancy: The unknowns of angle measurements in this problem have not been found. (.) What angles can you find? (.)

Lily: These angles [points to angle AEF and FEG]

Nancy's pointing here implies that her dialogue of "what can you find" in previous transcription was not really an open-ended question. While Nancy told students not to constrain themselves in inferring new measurements because of the unknown asked by this problem, the searching for new measurements was still under the control of helping students recognize the need to construct the auxiliary lines.

The final stage was the instruction of how to construct auxiliary lines. After students recognized that the finding of angle measurements of ΔEFG was not enough for completing the task, Nancy started to instruct students how to construct the auxiliary lines.

Nancy: Is this a right triangle?

SG No.

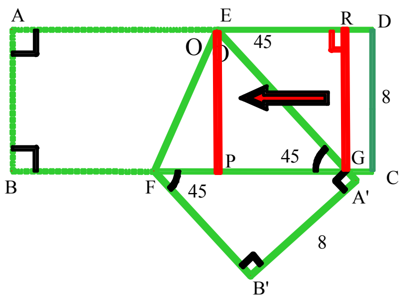
Nancy: This is an isosceles triangle [outlines three sides of ΔEFG]. But the angles of this isosceles triangle are not "perfect". Right? (.) You can see 67.5° [points to angle FEG]. 67.5° [points to angle EFG], and 45°[points to angle EGF]. (.) So, how can you find the area of this triangle? (1.0) Then, can you tell me the formula of triangle area?

SG Base times height divided by 2.

[3] Lily is a pseudo name.

Nancy Base times height divide 2. Ok, the base. (.) What is the base? What side should we use as the base? (.) The height. (.) What side should we use as the height? (.) Now the problem is that we <u>need</u> to do more. (.) The base is unknown. Height is unknown. Right? So we need to keep looking.

The transcription above shows that knowing all angle measurements of ΔEFG was not enough for finding this triangle area. Students needed to infer more information by applying transformational observations. The following transcription is an example of how Nancy taught students to construct an auxiliary line and use it as the height of the triangle EFG.



Nancy: 45° [writes 45° on angle GRE]. Is that ok? (.) 45°. Pay attention here. This side [writes 8 with CD] is 8 and this side [writes 8 with A'B'] is also 8. So, the key point in this problem is to (.) [Picks up another color pen] do what? (1.0)

SG: Draw a line.

Nancy: Draw a line. [Draws a perpendicular line through point G]

…………..

Nancy: Do you see the height? (.) [Points to RG]

SG …. (inaudible)

Nancy Height is this segment [uses gesture to move RG parallel to EP].

The gesture of moving RG parallel to EP helped students to visualize how to construct the lines and its relationship with relevant geometric properties. What's more. After Nancy guided students to find the area of ΔEFG, she also asked one student to share her solution that was different from her demonstration to whole class. In doing so, students were able to see different solutions with different application of transformational observations when solving a geometric calculation.

CONCLUSION AND REMARK

The ability to use transformational observations to solve this geometric calculation is *heuristic* (Lakatos, 1976; Pólya, 1945). The theoretical analysis shows that geometric calculations can provide the opportunities to familiarize students with transformational observations and *heuristic* reasoning. The empirical analysis also shows that the way a teacher scaffold students in learning transformational observations as well as *heuristic reasoning* (Pólya, 1945) is to make acquainted the geometric properties and to infer many new measurements in the diagram. In doing so, the teacher expects students to see the need of applying transformational observations and to learn the *heuristic reasoning* as well.

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CONSTRUCTING PEDAGOGICAL REPRESENTATIONS TO TEACH LINEAR RELATIONS IN CHINESE AND U.S. CLASSROOMS

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This study investigates Chinese and U.S. teachers' constructions of pedagogical representations by analyzing the video-taped lessons from the Learner's Perspective Study, involving 10 Chinese and 10 U.S. consecutive lessons on the topic of linear equations or linear relations. This study allows not only for the examination of what pedagogical representations Chinese and U.S. teachers construct, but also for the examination of the changes and progressions of constructed representations in these Chinese and U.S. lessons. This study is significant because it contributes to our understanding of the cultural differences involving U.S. and Chinese students' mathematical thinking and has practical implications for constructing pedagogical representations to maximize students' learning.

THEORETICAL BASIS OF THE STUDY

Cross-national studies provide unique opportunities for understanding how classroom teaching affects students' mathematical thinking, and then such studies provide diagnostic and decision-making information about how we can improve students' learning (Bradburn & Gilford, 1990; Cai, 2001; Ma, 1999; Stigler & Hiebert, 1999). Previous studies have revealed remarkable differences between U.S. and Chinese students' mathematical thinking and reasoning (Cai, 2000). Yet, we are just beginning to uncover how teaching in the two cultures may contribute to the cross-national performance differences. Because the use of mathematics representations is an important instructional feature that exerts great influence on students' mathematical thinking and reasoning, studies comparing the use of mathematics representations in U.S. and Chinese classrooms can provide us with insights into how teaching in different cultures may affect students' learning and mathematical thinking.

Pedagogical representations refer to the representations that teachers and students use in their classroom as expressions of mathematical knowledge. They help explain concepts, relationships, connections, or problem solving processes. In mathematics instruction, some representations might be more effective than others as expressions of knowledge and thinking tools to explain problem-solving processes (Cobb et al., 1992; Leinhardt, 2001). Choosing pedagogically sound representations is an important decision to make when a teacher selects instructional strategies for the mathematics classroom. To select a desirable pedagogical representation, a teacher should integrate at least two perspectives for consideration: the nature of the mathematical content being taught and the minds of students learning the content (Ball, 1993). First, the representation should highlight the features of the mathematics content the teacher

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 65-72. Seoul: PME.

wants to teach. Second, the representation should provide students with a familiar and accessible context in which they can extend and develop their capacity to reason and understand the idea. In mathematics classroom practice, Perkins and Unger (1994) also found that a powerful and effective representation often bears these two features. On the one hand, they argued, its extraneous clutter is often “stripped” in order to highlight the critical mathematical characteristics. On the other hand, it is also “concrete” to learners. Although it is not clear yet if there is a universally “good pedagogical representation” in terms of its *strippedness* and *concreteness* in teaching a mathematics idea to students in different cultural contexts, it is generally agreed that the teachers’ selection of desirable pedagogical representations of specific mathematics knowledge reflects the teachers’ conceptions, knowledge of mathematics, and their beliefs about learning and teaching (NCTM, 2001). Put another way, the pedagogical representations that teachers develop is related not only to the theory and research about student understanding, but also to teachers’ beliefs about the functions of particular representations in students’ learning and understanding (Greeno, 1987). Pedagogical representations are effective in classroom instruction only if they are either known by students or easily knowable (Leinhardt, 2001).

Recently, an attempt has been made to compare Chinese and U.S. teachers’ conceptions and constructions of pedagogical representations in mathematics instruction (Cai, 2005; Cai & Wang, 2006). For example, Cai (2005) examined U.S. and Chinese teachers’ construction, knowledge, and evaluation of representations to teach the concept of arithmetic average and found that the Chinese teachers and U.S. teachers in the study used representations differently. For example, while the Chinese teachers used concrete representations exclusively to mediate students’ understanding of the concept of average, the U.S. teachers tended to use concrete representations not only to foster students’ understanding of the concept but also to generate data. Cai and Wang (2006) further examined U.S. and Chinese teachers’ construction, knowledge, and evaluation of representations to teach the concept of ratio and found the generalities of U.S. and Chinese teachers’ construction of pedagogical representation across the content areas.

In this paper, we examined how U.S and Chinese teachers construct representations to teach linear relations over a sequence of video-taped lessons. This allows not only for the examination of what pedagogical representations Chinese and U.S. teachers construct, but also for the examination of the changes and progressions of constructed representations in these Chinese and U.S. lessons.

METHODOLOGY

Data resources

The data for this study came from the Learner’s Perspective Study (LPS for short), which examines the patterns of participation in competently taught seventh or eighth grade mathematics classrooms in thirteen countries in a more integrated and comprehensive fashion than has been attempted in previous international studies

(Clarke et al., 2006). In this study, we selected one Chinese school data set and one U.S. school data set. The main topics taught over 10 consecutive lessons in the Chinese and U.S. classroom are shown in Table 1.

Phase	Chinese lessons	U.S. Lessons
1	Concept of linear equations with two unknowns and its solution (CH1)	Concepts of linear and non-linear relations (US1, US2)
2	Concepts of coordinate plane and the coordinates; graph of linear equations with two unknowns (CH2-CH4)	General features of linear relations and multiple representations for linear and non-linear relations (US3, US4)
3	Concepts of system of linear equations with two unknowns and their solutions (CH5)	Comparison and contrast of different representations (US5-US8)
4	Methods of solving linear equation with two unknowns such as the elimination method and the graph method(CH6-CH10)	Comparison and contrast of different representations; application of linear and non-linear relations (US9, US10)

Table 1. Knowledge construction in Chinese and U.S. lessons

The Chinese school was from Shanghai, which is a key school at the district level, and the U.S. school was in top 20 percentile of schools in the state of California. The Chinese teacher, Mr. Tang, has a bachelor degree in mathematics from Teacher' Education Institute and has 24 years of teaching experience. There were 55 seventh-grade students in the classroom; the textbook used was the only unified official textbook in Shanghai. The duration of each lesson was approximately 45 minutes. The U.S. teacher, Ms. Nancy has a bachelor degree in mathematics with some teacher education training. She has more than 15 years of teaching experience. There were 37 grade eight students in the classroom. The textbook used was the Integrated Mathematics (Algebra) published by McDougal Little Inc. The duration of each lesson was approximately 50-minutes.

Data analysis

The data was analysed in two dimensions. First, we analyzed how knowledge was constructed during ten Chinese and ten U.S. consecutive video-taped lessons. Then we focused on the first four Chinese lessons (CH1- CH4) and the middle six U.S. lessons (US3 - US8), which included extensive coverage of linear relations, for further examination of the instructional tasks and the pedagogical representations involved in these Chinese and U.S. lessons. The instructional tasks, or mathematical tasks, can be defined broadly as projects, questions, problems, constructions, applications, and exercises in which students engage. The instructional tasks provide an intellectual environment for students' learning and development of mathematical thinking. Pedagogical representations of mathematics concepts were put into four categories: symbolic representation, numeric representation, tabular representation, graphic representation and verbal/literal representation. With respect to the code of pedagogical representations, one researcher developed a coding system by using special video data analysis software, Studio-code through carefully watching the video-taped lessons. Then the first author did a careful check. If the first author did not agree the code in certain episodes, then, a discussion with the research assistant was conducted until an agreement was achieved.

RESULT

Knowledge construction

Table 1 shows the sequence of how the topic was presented in Chinese and U.S, lessons. The Chinese teacher started with an introduction of the concept of linear equations and its solutions. He introduced the concept of rectangle coordinate planes to graph linear equations and then explained the concept of system of linear equations with two unknowns and its solution. After that, several methods to solve a system of linear equations with two unknowns were introduced and consolidated. It should be indicated that the Chinese teacher emphasizes the procedures for solving linear equations more than the concept involved. The U.S teacher started by introducing the concept of linear and non-linear relations in general, and then the teacher discussed extensively the features of linear relations and focused on transformation of multiple representations of linear and non-linear relations through group activities. Finally she applied the knowledge to solve word problems. The U.S. teacher intended to develop the concepts (linear and non-linear relations) and foster understanding of the features of linear and non-linear relations through multiple representations and students' group work. However, it is clear that the various activities that she used were to help students recognize the different representations, instead of using the representations to actually foster understanding of linear equations.

Comparing the Chinese and U.S. lessons, there are a number of differences in terms of lesson structures. Chinese lessons were dominated with whole class instruction, while group activity dominated the U.S. lessons. In the U.S. classroom, the students were divided into several groups, and the lessons were delivered through group activities. In the Chinese classroom, the lessons were delivered through whole classroom teaching, although there was frequent peer discussion. Each U.S. lesson included "warm-ups," which were related to the new topic to be learned in the lesson, but not related to the topics in the previous lessons. In the Chinese lessons, all lessons started with a review of knowledge learned in the previous lessons. This suggests that there were better connections between the Chinese lessons than between the U.S. lessons. In the U.S. lessons, the teacher usually did not present a summary for each lesson, while the Chinese teacher regularly summarized the key points of each lesson.

In addition, the Chinese teacher emphasized on the procedures for solving linear equations. That was not the case in the U.S. lessons. The U.S. teacher put heavy emphasis on multiple representations, and transforming among different representations is the goal in several of her lessons. That was not the case in the Chinese lessons. We will examine the representations further in the next section.

Representation construction

In this section, we examine representation constructions by comparing instructional tasks in six U.S. lessons (from US3 to US8), totalling around 300 minutes, with four Chinese lessons (from CH1 to CH4), totalling around 180 minutes. These lessons were chosen because of their extensive coverage of linear relations. The six U.S. lessons include 10 instructional tasks, and the four Chinese lessons include 23 instructional tasks. To examine the kinds of representations used, we looked at the total duration for solving each task as a whole. The proportion of different representations for each task in the U.S. classroom is depicted in Figure 1.

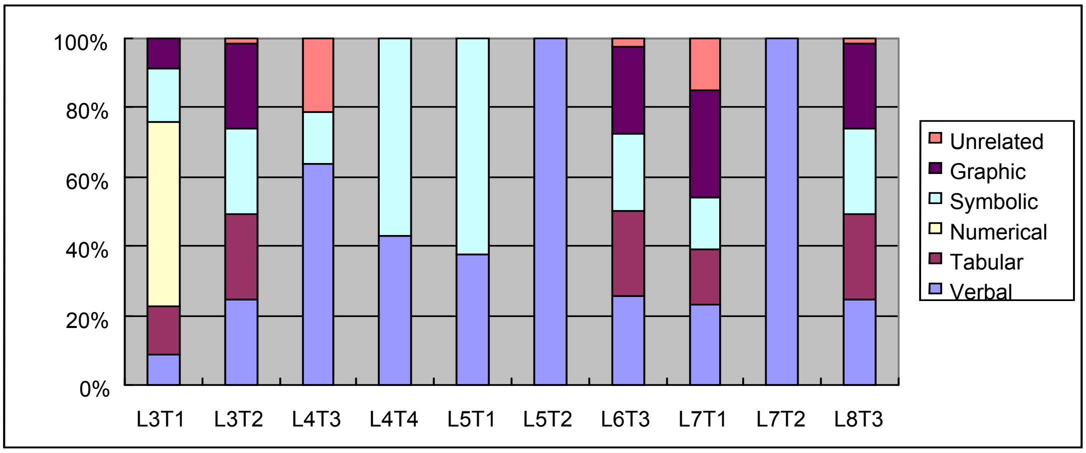

Figure 1. Distribution of representations in U.S. lessons

The above figure shows that there is one task L3T1 for which five presentations (verbal, tabular, numerical, symbolic and graphic) were used. There are four tasks (L3T2, L6T3, L7T1 and L8T3) for which four representations were used. There are three tasks for which two representations were used, while there are two tasks for which one representation was used. The order of popularity of using representations is as follows: verbal (100%), symbolic (80%), graphic (50%), tabular (50%), and numerical (10%) .

Similarly, we can show the proportion of different representations for each task in the Chinese classroom in Figure 2.

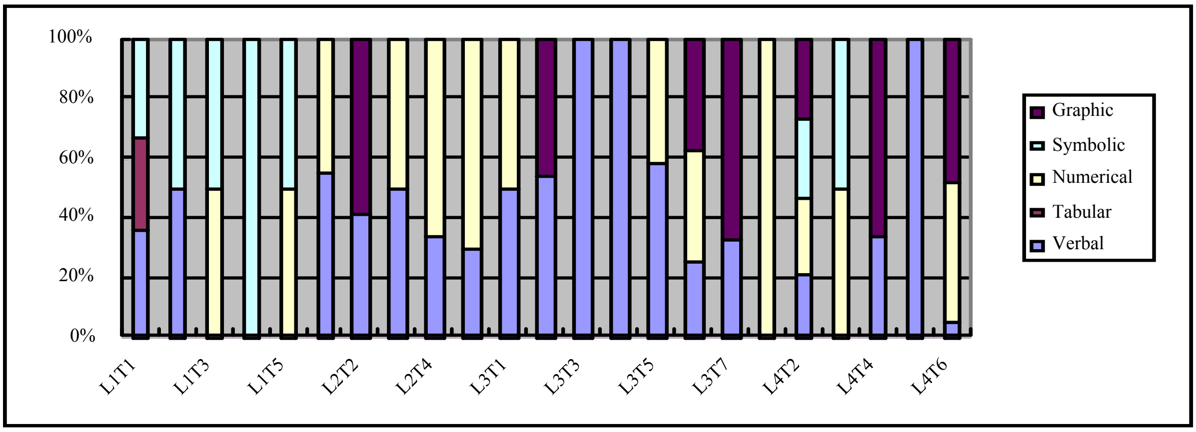

Figure 2. Distribution of representations in Chinese classroom

Obviously, the Chinese figures have a relatively "simpler" appearance. In one task (L4T2), there are four representations. There are three tasks (L1T1, L3T6 and L4T6) for which three representations were used. There are fourteen tasks for which two representations were used, and there are five tasks for which only one representation was used. The order of popularity of using representations is verbal (70%), numerical (56%), symbolic (30%), graphic (26%), and tabular (4%).

When comparing the construction of representation in the U.S and Chinese classrooms, it was found that the U.S. teacher preferred using multiple representations simultaneously (in 50% of the cases, more than three presentations were used), while the Chinese teacher preferred using one or two representations (in 83% of the cases, only one or two representations were used). In addition, verbal and numerical presentations were most commonly used and tabular representations were least commonly used in the Chinese classroom; however, the verbal, symbolic, graphic, and tabular representations were most commonly used and the numerical representations were the least used in the U.S. classroom.

Ways of constructing representations

Overall, we can present the development of representations of linear equations in the Chinese and U.S classrooms in the following diagram (Figure 3).

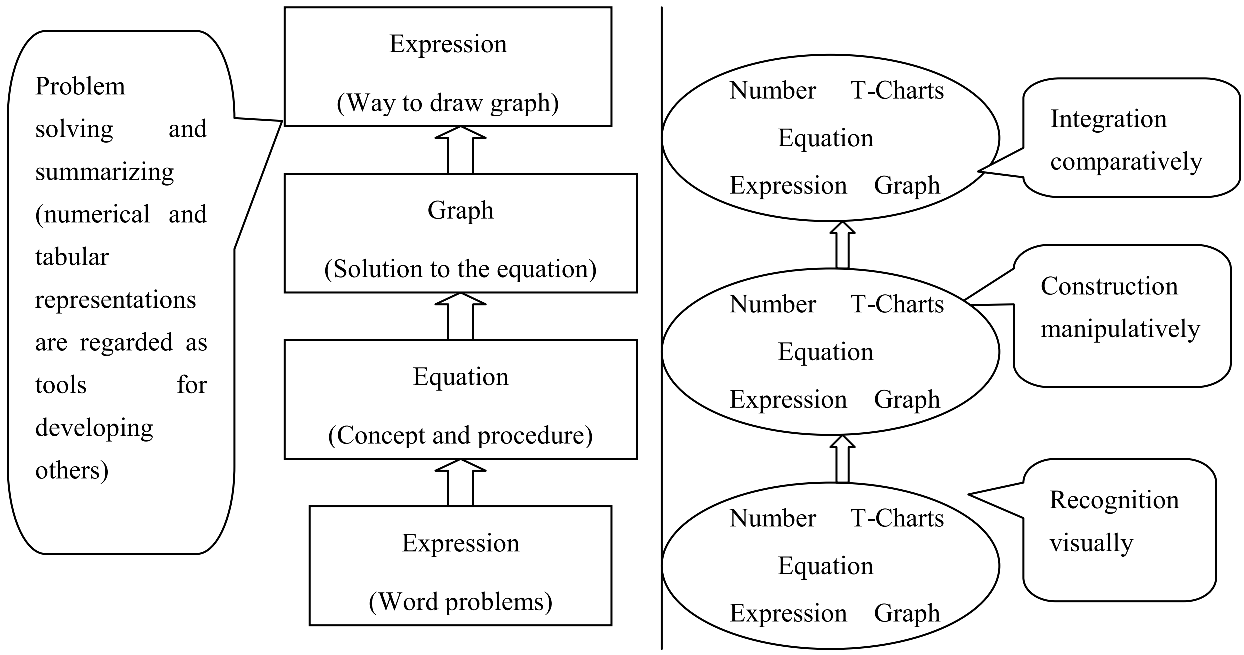

Figure 3. Representation development of linear relation in Chinese and U.S. lessons

The U.S teacher developed multiple representations simultaneously over sequential lessons through different activities, such as visual sorting, manipulative drawing and matching, and comparative and contrasting exploration. The teacher tried to deepen the students' understanding and integration of different representations through progressive activities. This diagram shows that the Chinese teacher tried to develop the concept of linear equations (symbolic representations) and graphs of linear equations (graphic representations) through solving problems (verbal daily problem and symbolic linear equations) by making use of tabular and numerical representations. Thus, students may have a deep understanding of linear equations and its figures and also understand the ways of drawing a figure of linear equation by plotting two points for the linear equation. However, they may not realize that numerical and tabular representations are of the same importance as other representations.

CONCLUSION

With regard to the construction of representations when implementing instruction tasks, it was found that the U.S. teacher preferred using multiple representations simultaneously, while the Chinese teacher preferred using only one or two representations. In addition, verbal, symbolic, graphic and tabular presentations were most popularly used by the U.S. teacher while for the Chinese teacher verbal, numeric, symbolic and graphic representations were more popular. Numerical representation was least frequently used by the U.S. teacher, while tabular representation was least frequently used by the Chinese teacher. In addition, the U.S teacher seemed to develop multiple representations simultaneously over subsequent lessons through different activities, such as visual sorting, manipulative drawing and matching, while the Chinese teacher tried to develop the concept of linear equations (symbolic representation) and graphs of linear equation (graphic representation) through solving problems (daily verbal problems and symbolic linear equations) by making use of tabular and numerical representations. Thus, the use of multiple representations appears to be an instructional goal for the U.S. teachers, while she is intended to use multiple representations as a means for students' understanding of linear relations. For the Chinese teacher, the multiple representations were used as a means to understand linear equations.

The finding that the U.S teacher tried to treat all four representations equally and develop them simultaneously through different activities may explain why the U.S. students preferred to choose concrete strategies, drawing representations both for fostering understanding of concept and also for applying knowledge. However, the Chinese teacher paid more attention to developing symbolic and graphic representations by treating numerical and tabular representations as tools for developing other representations. This finding may explain why Chinese students preferred using symbolic and abstract representation to solve problems (Cai, 2005; Cai & Lester, 2005).

Acknowledgment. We would like to express our appreciation to Professor Terry L. Wood and Dr. Ida A.C. Mok for their valuable input on this paper. This study was funded by Academic Committee, University of Macau, China (RG016/05-06S, 3691).

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TEACHERS AS RESEARCHERS: PUTTING MATHEMATICS AT THE CORE

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This paper analyses the teachers-as-researchers movement, showing that teachers research mainly focus on pedagogical issues. It reports on a study where teachers were researching aspects of mathematics for teaching limits of functions where mathematical and pedagogical issues were intertwined. It shows that it is uncomfortable for a teacher to deeply challenge his own teaching.

THE TEACHERS AS RESEARCHERS MOVEMENT

According to Elliott (1991), the teachers-as-researchers movement emerged in England during the 1960s, in the context of curriculum reform. Initially it focused on the teaching of humanities subjects, teachers working together in cross-subject teams. The research was not systematic, but occurred as a response to particular questions and issues as they arose. It aimed to improve practice rather than to produce knowledge.

This movement extended in the 1980s in what is usually known as *the teacher research movement*, which main feature is that teachers are no longer considered as mere consumers of knowledge produced by experts, but as producers and mediators of knowledge, even if it is a local knowledge. In most of their research, teachers focussed on their own classroom practice.

In Mathematics Education, research has now become an important part of many teacher education programmes all around the world. It also has been the subject of debate within the mathematics educators' community and of several papers presenting results of these programs or discussing certain aspects of teacher research. Most of these publications focus on teachers' practices.

In 1988, a working group called "Teachers as Researchers" started at PME. This group met annually during nine years and published a book based on contributions from its members (Zack, Mousley & Breen, 1997), presenting different experiences of teachers' enquiry in several countries and using several methods, which aim was basically to improve practice.

Adler (1992) reports the case study of a middle-class mathematics teacher researching his interactions with learners and their interaction with each other, during his postgraduate studies. Through this research, he realised that he dominated classroom interaction and that his mediation was gendered.

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 73-80. Seoul: PME.

Hatch & Shiu (1998) reports the case study of a primary school teacher researching her own practice through the analysis of class transcript and a reflective journal as part of an in-service course. They argue that she contributed not only to developing knowledge of her own practice but potentially to the accumulated knowledge of the research community.

Halai (1999) reports on action research conducted by mathematics teachers in Karachi and involving university researchers as facilitators. They also used participant observation, field notes, and reflective journals. She claims that this action research project promoted learning and professional growth not only of the teachers but also of the university researchers.

Edwards & Hensien (1999) describes action research collaboration between a middle school mathematics teacher and a mathematics teacher educator, involving observation and discussion of lessons and exchanging roles in the classroom. The analysis of the teacher's narrative of this collaboration as well as the teacher's regular reflections on her beliefs and practices were important to her process of change.

Jaworski (1998) describes the MTE (Mathematics Teacher Enquiry) project, which involved six secondary mathematics teachers undertaking their own research independently of an academic programme. These teachers were invited to identify a question they were interested in researching. Jaworski points out that, during this research, the teachers focused their attention on pedagogical issues, rather than on mathematical issues.

> Decisions about what mathematics should be done, what classroom tasks would be appropriate, and what outcomes would be desired, were a normal part of the teaching process, hard to extract as problematically related to the research issues. (1998: 25)

She asks the question "How might mathematics issues become more overt in the research project?" (1998: 29).

In most of the papers presented above, the focus was on teachers' classroom practices, independently of the knowledge to be taught. In all these projects, it seems that the mathematical content to be taught is taken for granted, and that teachers are not supposed to challenge it. They are only supposed to try to improve their teaching practices. A few articles mention some change, or some possible change, in teachers' knowledge of mathematics.

Mousley (1992) reports the results of a one-year course in an off-campus mode called MATHEMATICS CURRICULA. Course participants used cycle of action research in a chosen area of their change of practice. They were required to work with colleagues. A representative sample of sixty teachers was then contacted by mail, telephone or a personal interview about the impact of the course. It was found that there was

> not only some ongoing restructuring of pedagogy, in terms of content, organisation and classroom interaction, but also growth of understanding about (1) the nature of mathematics, (2) the processes of teaching and learning of maths, (3) the power of institutional contexts of teaching and learning, and (4) the processes of pedagogical change. (Mousley, 1992: 138)

Although the aim of this project was to improve practice, it also shows that through their research teachers' knowledge on mathematics evolved, and that they became aware of the weight of institutional constraints.

> The notion of mathematics as a stable body of knowledge and skills to be transmitted and practiced became problematic. Questioning traditional classroom practices provided an incentive for teachers to confront given curriculum content. (1992: 139)

Mousley concluded that

> participatory, experience-based research has the power to emancipate some teachers from taken-for-granted classroom routines which constrain and control mathematical learning. The dialectical interaction of reflection combined with social interaction allowed innovation in the nature of teaching and learning mathematics as well as in curriculum content. (1992: 143)

This experience shows that through research and interaction teachers can be led to challenge institutional relations to mathematics.

In the first edition of the *International Handbook of Mathematics Education*, Crawford & Adler suggest that:

> It seems possible if teachers and student-teachers act in generative, research-like ways, they may learn about the teaching/learning process, and about mathematics, in ways that empower them to better meet the needs of their students. (1996: 1187)

These authors seem to avoid the distinction between practical inquiry and more formal research, using the term "research-like ways". The focus is on teachers' personal learning by researching, not only their own practice, but also mathematics. They argue that, the quality of teachers' mathematical knowledge being strongly influenced by their own experience as students, they need to unlearn the old conceptions of mathematics derived from their schooling experience. The experiences of "teachers' voices" in South Africa and of a program of action-research with student teachers in Australia lead Crawford & Adler (1996) to conclude that research helps teachers to challenge their practice and their conception of mathematics. Student teachers doing action research "learn a great deal about mathematics as they work with their students to define and refine mathematical ideas and use them actively as a means to inquiry" (1996: 1200).

Another research project reporting changes in teachers' knowledge of mathematics is the PLESME project (Graven, 2005), where mathematical knowledge and mathematics pedagogical knowledge were intertwined. "PLESME focused on the development of mathematical meaning and pedagogical forms simultaneously" (2005:219). Using this two-year INSET project as an empirical field for her research, Graven investigated the nature of mathematics teachers learning within a community of practice (2005:207). She argues that most of the literature on teacher development indicates a focus on teacher change. In the South-African context, the curriculum support materials call "for radical teacher change where old practice is completely replaced by new practice". This view of teacher change is disempowering for teachers (2005: 223). On the other hand, the PLESME programme was based on a conception of learning as a life-long process, where teachers were expected to build their own knowledge.

This non exhaustive review of papers about the teachers-as-researchers movement shows how different the experiences in this domain are, in terms of research topics and methodology. However, some common trends can be found in these reports.

In the first place, they seem to share a common conception of teacher as a producer of knowledge and not as a mere consumer of knowledge produced by other individuals, particularly academics.

Secondly, in most of these research projects, teachers worked together in groups, the research team being composed of either pre-service or in-service teachers. Interaction between teachers, or between teachers and mathematics educators, allowed them to deepen the analysis of their practices and difficulties.

Finally, in all projects discussed above, teachers chose to investigate some pedagogical issue or some problem of student learning. It seems that when asked to choose a research topic, teachers question their own teaching, or their students' performance and difficulties, but take for granted the content usually taught within the institution.

LEARNING MATHEMATICS THROUGH RESEARCH

In the research project reported here, teachers were researching different aspects of limits of functions. This project is based on the one hand on the study of the institutional relation (Chevallard, 1992) of the Mozambican secondary school with this concept, and on the other hand on a study of the mathematical knowledge which would enable a teacher to extend this institutional relation (*mathematics for teaching* limits of functions).

The study of the institutional relation of Mozambican secondary school with limits shows a dichotomy between a formal part, the ε-δ definition, which students are not asked to use in practice; and an algebraic part, the determination of limits, what most of students' tasks are based on.

Mathematics for teaching limits of functions includes the following aspects: (i) Scholarly mathematical knowledge on the concept; (ii) Knowledge about the social justification to teach this concept; (iii) Knowledge on how to organise the students' first encounter with the concept; (iv) Knowledge on the practical block (tasks and techniques); (v) Knowledge about students' conceptions and difficulties when studying this concept. In each of these components mathematical and pedagogical knowledge are intertwined.

Four teachers researched a different aspect of *mathematics for teaching* limits involving both mathematical and pedagogical issues, and shared their findings in periodic seminars. One of them was an experienced Grade 12 teacher, who had taught limits at school, while the others were teachers in lower grades. All of them used their research for their Degree dissertation. I was their supervisor and the facilitator of the seminars. The teachers were also interviewed three times during the research process. All interviews and seminars were audio-taped and transcribed.

Data analysis focused on five main aspects of *mathematics for teaching* limits: how to organise students' first encounter with this concept, the social justification for teaching limits at school, the essential features of the limit concept (part of the scholarly mathematical knowledge), the graphical register (from the practical block), and the ε-δ definition (also from the scholarly mathematical knowledge).

FINDINGS

Data analysis for the five aspects mentioned above indicate that teachers' knowledge evolved substantially for the first three aspects, but that difficulties remained for the two last aspects, the graphical register and the ε-δ definition. These difficulties were explained by a lack of deep understanding of basic mathematical concepts. For the first three aspects (which only involved general mathematical knowledge), reading books and mathematics education papers, and discussing their findings within the research group seemed to allow teachers to reflect on these issues and to make links between the limit concept and other mathematical concepts. However, when a deep understanding of basic mathematical concepts was required, such as for the use of graphs to study limits or the ε-δ definition, reading books and papers and discussing these issues within the research group did not allow teachers to overcome their difficulties.

Furthermore, the Grade 12 experienced teacher faced more difficulties during the whole process. In fact, this teacher was in a less comfortable position than his colleagues. While the other three teachers were researching and challenging the institutional relation of Mozambican secondary school to limits of functions, he was also researching and challenging his own practice. For example, at some point he realised that he had taught L'Hôpital's Rule before teaching derivatives and that students could not understand it.

> I remember that, well I gave tasks about limits, er … mainly, they were polynomial functions I think, well, for me, the practical way was, you know, use what we usually call L'Hôpital's Rule, because it was practical and [*sighing*] but … after all, now I get to know that, well, how could I use that L'Hôpital's Rule if the students did not learn derivatives? And limits come before derivatives … But … I saw that after all I was doing a mistake by that time … (Abel, 3rd interview)

He then explained how he introduced limits to his students.

> I gave the definition, ok, I gave the rules, we go to the tasks. (…) Well, I was myself reduced to ... to that knowledge, thus, it's how I learnt and it's also what the textbook shows, and I'm going to pass it on [to students]. (Abel, 3rd interview)

According to his discourse, it is clear that this teacher's mathematical knowledge did not allow him to teach in a different way. It is now very hard for him to realise that he taught in a way students could not understand. This possibly explains why teachers researching their own practice seem to prefer to look at pedagogical issues or students' difficulties. In that way they do not need to challenge their own personal relation to mathematics as much. This result highlights a limitation of teachers learning through research.

CONCLUSION

This paper reviews papers on teachers as researchers, showing that they mainly focus on pedagogical issues. It then reports a study where teachers researched aspects of *mathematics for teaching* limits of functions involving both mathematical and pedagogical aspects. This study puts into evidence that the teachers' mathematical knowledge could not be taken for granted. For those aspects of *mathematics for teaching* which required a deep understanding of some basic mathematical concepts, the evolution of teachers' knowledge through the research process was limited. Furthermore, the research process was more challenging for the experienced Grade 12 teacher, whose research also challenged his own practice.

I suggest that teachers be involved in research putting mathematics at the core: research on *mathematics for teaching*, based in both mathematical and pedagogical issues. In that way they will produce knowledge that helps them evolve their personal relation to mathematics and its teaching and learning, as well as hopefully improve their practice. Obviously I do not claim that they would necessarily teach in a different way, as they would be exposed to institutional constraints, but that their personal relation to mathematics would enable them to teach differently.

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CAN YOU CONVINCE ME: LEARNING TO USE MATHEMATICAL ARGUMENTATION

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In the current mathematics education reform efforts, teachers are challenged to develop discourse communities in which the students learn to construct and evaluate mathematical arguments collectively. In this paper I examined the interactional strategies used by a teacher to constitute a classroom context in which the students participated in the discourse of collective argumentation. I report on the way the teacher used student explanations as the foundations for building justification and validation of reasoning.

INTRODUCTION

Over the past twenty-five years within both an international and New Zealand context the teaching and learning of mathematics has undergone substantial changes in the way it is conceptualised. Increased focus has been placed on student communication of mathematical reasoning including the development and presentation of mathematical arguments. Advocated in the New Zealand policy document is a requirement that teachers “provide opportunities for students to develop the skills of presentation and critical appraisal of mathematical argument or calculation” (Ministry of Education, 1992, p. 23). Similarly, the American policy document emphasises the need for teachers to create classroom environments in which their students learn to “construct mathematical arguments and respond to others’ arguments” (National Council of Teachers of Mathematics, 2000, p. 18). These documents promote ambitious and challenging goals for change in the teaching and learning practices of many mathematics classrooms. They challenge traditional beliefs toward mathematics and its discourse as a non-contentious (Weingrad, 1998). They also challenge the media prevalent view of argumentation as oppositional behaviour considered to be interference to learning (Andriessen, 2006). Moreover, many teachers have not themselves experienced learning in such environments and nor is their role in them clear (Huferd-Ackles, Fuson, & Sherin, 2004). The research reported in this paper examines how one teacher engaged in a collaborative research project, purposely transformed the discourse practices used by her and that of her students. The focus of the paper is on the interactional strategies. I examine how those strategies were used to shift the discourse toward all students participating in collaborative argumentation.

For the purposes of this study collaborative argumentation is a form of mathematical dialogue in which all parties work together to critically explore and resolve issues which they all expect to reach agreement on ultimately (Andriessen, 2006). Considerable evidence is now available of the beneficial effects of students

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 81-88. Seoul: PME.

articulating their mathematical reasoning and inquiring and challenging the reasoning of others when engaged in productive collaborative argumentation (e.g., Manouchehri, & St John, 2006; Mercer, 2000; Wood, Williams, & McNeal, 2006). These many studies have provided evidence that when opportunities are made available to students to participate in rich forms of inquiry and argumentation, the quality of their own mathematical explanations and justification are enhanced. This is because argumentation is a powerful reasoning tool which allows the participants in the dialogue to refute, criticise, elaborate and justify mathematical concepts and facts and develop an understanding of the opposing perspectives as all participants work towards constructing collective consensus. Andriessen (2006) maintains that "when students collaborate in argumentation in the classroom, they are arguing to learn (p. 443).

Creating a classroom culture in which value is placed on collaborative inquiry and argumentation requires altering the students' perceptions and beliefs about what mathematics is and how it is used (Manouchehri, & Enderson, 1999) but also their attitudes and perceptions of argumentation. Andriessen (2006) describes how many individuals link argumentation to an aggressive and oppositional form in which the goal is not to work together but rather to score points—a form of arguing which has little to contribute to mathematics education. However, engaging in collaborative interaction and using inquiry and argumentation is not something many students can accomplish easily without specific adult intervention (Rojas-Drummond, Perez, Velez, Gomez, & Mendoza, 2003). Therefore, it is vital that teachers as the more expert members of the classroom community take an active role in orchestrating a social environment in which the students "listen to one another, respect one another and themselves, accept opposing views, and participate in a genuine give-and-take of ideas and perspectives" (Manouchehri, & Enderson, p. 6).

When students engage in 'arguing to learn' they are participating in activity grounded in the social and cultural practices of the classroom community and are learning to use a social language or speech genre which denotes a particular socially situated identity (Gee, 1992). However, without direct discussion of the structure of collaborative argumentation and its rules and norms some students may not be able to access the mathematical discourse and learning of the classroom. This paper explores the defining features of a classroom climate in which the teacher developed and extended student participation in mathematical argumentation.

The theoretical framework of this study is derived from a sociocultural perspective. From this perspective mathematical teaching and learning is inherently social and embedded in active participation in communicative reasoning processes (Lerman, 2001). In this environment, students successively gain increased levels of "legitimate peripheral participation" (Lave & Wenger, 1991, p. 53) as they access and use the discourse of inquiry and argumentation.

RESEARCH DESIGN

This research reports on one teacher case study from a study which involved four teachers in a one-year collaborative teaching experiment. The study was conducted at a New Zealand urban primary school where students came from predominantly low socio-economic home environments. Students were predominantly of Pacific Nations and New Zealand Maori ethnic groupings with many of whom spoke English as their second language.

Collaborative teaching experiment design (Cobb, 2000) was used in order to direct teacher and researcher attention on the social and analytical structuring (Williams & Baxter, 1996) of the mathematical discourse. In recognition of the two central characteristics of teaching experiment design research; the iterative cycles of analysis, and an improved process or product; a tentative communication and participation trajectory was used to map the progression of the discourse toward argumentation and to provide focus for the subsequent shifts in participation and communication. For example, after Ava (pseudonym for the teacher) had completed teaching a an early algebraic unit of work and before she returned to teaching a fractional number unit, the types of questions Ava and the students could use and the interactions anticipated to scaffold a further shift toward collaborative mathematical argumentation were considered and mapped out.

Data collection over one year included three semi-formal teacher interviews, classroom artefacts, field notes, twice weekly video captured observations of lessons, diary notes of informal discussions during and after lesson observations, written and recorded teacher reflective statements and teacher recorded reflective analysis of video excerpts. The on-going data collection and analysis maintained a focus on the developing mathematical discourse and argumentation. This supported the iterative cycles and revision of the communication and participation strategies. Data analysis occurred chronologically using a grounded approach in which codes, categories, patterns and themes were created. Through use of a constant comparative method which involved interplay between the data and theory, trustworthiness was verified and refuted.

RESULTS AND DISCUSSION

In the early stages of the study the participation and communication structure that Ava made available to students operated as a scaffold to begin to develop argumentation. As the study progressed the close relationship between a shift in the roles Ava and her students took and the changes enacted in the participation and communication structure is evident.

Creating a context for collaborative argumentation

Ava immediately worked with the students to establish a set of mutual expectations for behaviour as participants in a discourse community. She directly addressed the new ‘rules’ for talk, discussing with the students how they were required to work together to build a mathematical community. She emphasised that working together involved an increase in collaborative participation in mathematical dialogue, both as listeners and as talkers. She repositioned herself from the central position of ‘mathematical authority’ to that of ‘participant in the dialogue’. She modelled the shift explicitly placing emphasis on words which placed her as a participant also.

> Ava: Can you show us with your red pen what would happen? We want to know.

In the first instance, Ava aimed to develop the students’ skills to work collectively to build mathematical explanations. She stressed that all group members needed to engage in construction of mathematical explanations and be able to explain them to a wider audience. She outlined not only how these explanations needed to make sense for a listening audience but also how listeners needed to make sense of the explanations offered by others. To develop their skill in the examination and analysis of explanations she provided opportunities for small groups to construct, explain, and in turn question and clarify explanations step-by-step through her directives:

> Ava: They might say I think it is 59. That’s cool but they have to back it up, explain how they came up with it. They have to say why. I want you before you even begin to go around in your group and actually talk about it. Someone in your group may ask you a question. For example, that’s an interesting solution, why do you think that? Could you show us how you got it?

In this early stage, although mathematical argumentation was not a strong feature of how the students interacted Ava initiated discussion about the need for agreement and disagreement in the construction of reasoned explanations. For example, when a student stated that working as a group required agreement Ava responded:

> Ava: Yes you could be agreeing with what the person says…but are you always agreeing, do you think?

In accord with the trajectory, she carefully structured ways in which the students could approach disagreement and challenge. When the students worked together she pressed them toward considering the use of arguing productively:

> Ava: Arguing is not a bad word…sometimes I know that you people think to argue is…I am talking about arguing in a good way. Please feel free to say if you do not agree with what someone else has said. You can say that as long as you say it in an okay sort of way. If you don’t agree then a suggestion could be that you might say I don’t actually agree with you. Could you show that to me? Could you perhaps write it in numbers? Could you draw something to show that idea to me? That’s fine because sometimes when you go over and you do that again you think…oh maybe that wasn’t quite right and that’s fine. That’s okay.

Questioning, clarifying and beginning to challenge

The careful attention Ava gave to socially scaffolding the discourse led to the growth in student confidence to question and clarify sections of explanations when required. For example after a group had modelled an explanation using equipment and described their actions of repeatedly adding three sticks as ‘squares times three’ they are challenged:

Jo: Isn't that just plusing three sticks not timesing it? You are not timesing you're adding.

Pania: Well what she sort of means it is like it is going up.

Alan: Is that timesing going up?

Ava: When we talk about timesing what do we actually mean?

Jo: We mean multiplying not adding. Adding is plus [indicates a + with fingers] that sign.

Sandra: You mean when you add two more squares on, that is multiplying?

Ava intercedes and uses the reasoning under discussion to extend their thinking. Through use of the interactional strategy of revoicing (O'Connor & Michaels, 1993) Ava deepens their understandings of multiplication and enriches their language.

Ava: Rachel was saying she is adding three, adding another three, so that's three plus three plus three. So if you keep adding three all the time what is another way of doing it?

Alan: You can just times instead of adding. It won't take as long and it is more efficient.

Ava: Yes you are right. Did you all hear that? Alan said that you can just times it, multiply by three because that is the same as adding on three each time. What word do we use instead of timesing?

Alan: Multiplication, multiplying.

Pressing for multiple ways to justify and validate explanations

A need for active sense-making and a press to provide conceptual explanations provided the students with the foundations with which to build collaborative argumentation. Ava pressed the students toward constructing multiple explanatory means to justify and validate their reasoning. Problems were also used which required that the students develop multiple ways to convince others. Before they began constructing their explanations in small groups, Ava placed direct emphasis on a need for them to ask specific questions. With the students she listed the questions they could use to elicit more information about mathematical explanations. Then she introduced a second set of questions which related to their need to be convinced through mathematical argumentation. She recorded an initial set of questions and then regularly recorded additional questions which arose during the classroom dialogue—questions which asked why and led to justification and validation of reasoning. She also assisted them by asking that they prepare responses in their small groups to the types of questions they might be asked in the large group situation:

Ava: Think about the questions that you might be asked. Practise using some of those questions like why does that work or how can you know that is true. Try to see what happens when you say if I do that… then that will happen.

In this climate of intellectual autonomy Ava and her students began to regularly ask the question, 'can you convince us?' This press toward need for convincing through mathematical argumentation was accepted and modelled by the students. They recognised that this supported possibilities for confirmation or reconstruction of their reasoning. They would closely examine and rehearse each step in an explanation and if required for clarity or ease of sense-making rework, reformulate and re-present sections. Ava also instituted a further shift in how the students participated in communicating their mathematical reasoning introducing the concept of 'no hands up', particularly when there were many questions and challenges for an explanation. Her direct prompting for student interjection led to an interactive flow of conversation in which collaborative forms of argumentation were used to closely examine, analyse and validate the mathematical reasoning. When needed she intervened and facilitated slower exchanges, if she considered the mathematical concepts under consideration particularly challenging. For example, Ava participated in the following collective construction of an explanation, related to a problem which required naming a point which represented 5/100 on a numberline. She prompted for interjection but also intervened to maintain a focus on inquiry and challenge and ensure that the students reflectively considered and reconsidered their reasoning.

> Tipani [Draws a numberline, marks 0 then 9 and marks 1/10: Here is 0 and 1/10.
>
> Pania [Interjects]: Why are you doing those lines?
>
> Tipani [Records 5/100 in the middle]: Because each of those lines is representing one tenth, I mean ten tenths. I am thinking that this one is meant to be 5/100.
>
> Mahaki [Interjects]: Why?
>
> Tipani: Basically because of what you said Mahaki.
>
> Ava: Which was? Explain it in your own words and see if Mahaki agrees.
>
> Tipani: That if you times… ten by ten…well I am not actually that sure. I just think that it is five one hundredths. I don't think that it is five one thousandths.

Ava intervenes and attributes ownership back to the explainer but presses for further clarification.

> Ava: Well what do you think Mahaki, and you other people who heard what Mahaki explained? Let's take a look at these fractions and think about what Tipani and Mahaki were saying. What do you see when you look at these fractions…what other ways can they be represented apart from that?

Ava referred the argument back to the explainer. But within the context of collaborative argumentation another member appropriates and revoices what has been explained.

> Chanal [Looks at Mahaki who nods at him to speak, points at the 5/100 mark]: Mahaki said that one tenth can be ten percent because if you times one by ten you get ten and you times the ten by ten you get one hundred. So that will be one tenth is like ten percent. So in the middle that will be five percent there.

Ava revoices to ensure that all participants are able to access what is being argued. She then probes further, progressing the reasoning toward collective construction of rich conceptual knowledge using multiple levels of representations.

> Ava [Points at 5/10]: You accept that Mahaki? So what you are saying is that that means five parts out of a hundred and the one tenth there means ten out of a hundred. So what is this one?

Chanal: That is fifty percent.

Pania: So how?

Recognising that further arguments are required Ava facilitates an alternative view.

Ava: Yes you jump in here Chanal, if you can explain it in a different way.

Chanal [Points at the numberline showing one tenth divided into ten segments and points at the first segment on the numberline]: I know what. If you go back to there and just pretend you shrink that down to there. There's a hundred right? So that half way mark in brackets would be right there [Points at the position it would be in if you had a whole number line not just to one tenth and it represented 1/10] and that would be ten percent and if you halved that ten percent it would be five.

Pania: Five what?

Chanal [Records 5% and 5/100]: Five percent or five hundredth.

Ava: Are you all convinced? Or do you want to ask some more questions?

Mahaki: It is five hundredth because as Chanal said that thing there would be just like a little piece of this line…But the other way is to go the percent way. You get ten percent and then half that. That's the quickest way to explain it.

Ava and the students had maintained an extended flow of productive argumentation in which rich communal understandings of the equivalent relationship of rational numbers were constructed. Ava had positioned and repositioned the students to make visible their reasoning so that claims were collectively validated.

CONCLUSIONS

The teaching experiment was designed to successively press how the students participated in communicating their mathematical reasoning. The direct focus placed on collective construction and sense-making of explanatory reasoning acted as a scaffold to shift the discourse toward justification. Through these actions the social norms of sense-making were established in the community.

The introduction of notions of 'arguing' and disagreement were important to lay the foundations for further shifts toward argumentation. The adoption by Ava and her students of a metaphorical view of the need for 'convincing' provided motivation for the students to engage in the communal activity. Direct teacher actions built on the notion of convincing and supported the constitution of an environment in which the students participated in collective argumentation. The use of specific questions to frame inquiry and challenge, and the increased student autonomy in when and how to participate were important factors.

The findings of this study support Andriessen (2006) contention that young children can participate in collective argumentation when carefully scaffolded. Moreover, Wood and her colleagues (2006) identify differences in the cognitive demand and student participation in collective reasoning in classrooms where the use of discourse extends to justification and argumentation and this was evident in this study.

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ON THE MATHEMATICAL KNOWLEDGE UNDER CONSTRUCTION IN THE CLASSROOM: A COMPARATIVE STUDY

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The present article reports on an attempt to identify the epistemological status of the mathematical knowledge interactively shaped in the classroom. To this purpose, three theoretical approaches are utilized in order to comparatively analyze a lesson provided by a well-experienced teacher on algebra, aiming at identifying the epistemological status of the knowledge under construction through the lenses offered by them. The results show that this parallel and complementary exploitation is especially valid for deepening our understanding of the mathematical knowledge under construction in the classroom.

INTRODUCTION

School mathematics and experts' mathematics are two epistemologically distinct bodies of knowledge as they differ in form, context and use from one another (e.g., Sfard, 1998). However, the former draws from the latter, thus preserving certain connections with it, which are though rather blurred. As a result, one could hardly justify why a meaning, an activity or an outcome emerging in the school context can be characterized as 'mathematical'.

The research dealing with this issue is very limited and mathematics education does not still appear to have detailed criteria of whether what is personally or socially constructed in the classroom is or is not mathematical. The study of teaching and learning phenomena and, in particular, the study of the interaction in the mathematics classroom remotely focuses on the nature of the mathematical knowledge shaped in it, which is greatly determined by this interaction (Steinbring, 1998).

In searching for criteria to analyze didactical phenomena within the perspective of the nature of the knowledge constructed in the classroom, the requirement of the underpinning fundamental and operational characteristics of mathematics, namely of its epistemological features, seems absolutely essential (e.g., Rouchier & Steinbring, 1989). To this direction, the focus of the present work is on the nature of the meaning emerging in the classroom characterized as 'mathematics' in connection with the classroom phenomena which determine this construction. In particular, in an attempt to identify the nature of the mathematical knowledge interactively constructed in the classroom contexts, we utilize the analytical tools offered by three relevant theoretical approaches, i.e., of socio-mathematical norms (Yackel & Cobb, 1996), of the epistemological triangle (Steinbring, 2005) and of the management of the epistemological features (Kaldrimidou, et al, 2000). The comparative reading of the same lessons through the lenses offered by these three approaches allows us to sharpen the analysis related to this nature.

THE THEORETICAL APPROACHES

Our purpose was to exploit the possibilities offered by each of the three approaches in order to identify the particular epistemological features of the subject matter knowledge they claim that is shaped in the classroom, as a consequence of the personal, social and epistemological constraints present. These approaches are briefly described below.

a. Sociomathematical Norms: The notion of sociomathematical norms was conceived in order to analyze the mathematical aspects of teachers' and students' activity in the mathematics classroom (Cobb & Yackel, 1996). These norms are collective criteria of values with respect to mathematical activities, which are interactively constituted (Voigt, 1995), not predetermined, but continually regenerated and modified by the interactions taking place between the teacher and the pupils. The sociomathematical norms are established in all types of classrooms and they are context dependent.

The most common sociomathematical norms reported in the literature are related to explanations, justifications and solutions. With respect to explanations and justifications, the main sociomathematical norm detected is related to 'what counts as an acceptable mathematical explanation' (Yackel & Cobb, 1996). Concerning solutions, the relevant sociomathematical norms refer to 'what is valued mathematically; what a more sophisticated solution is; what is mathematically efficient and/or different' (Yackel & Cobb, 1996).

b. The epistemological triangle: Steinbring (1998), adopting the view that knowledge is represented by a specific way of constructing relations (Rouchier & Steinbring, 1989), advocates that the epistemological status of what is interactively constructed by the students in the classroom can be identified through a relational structure called 'the epistemological triangle'. In particular, he argues that in the course of classroom interaction, students have to actively construct relationships between signs/symbols and reference contexts. This construction becomes 'official' in negotiations with the teacher and the other students. As a result, the analysis of the classroom production of mathematical meaning from an epistemological point of view needs to take into account the relationship between two interrelated dimensions: (a) the construction of meaningful relations for sign systems is regulated by the reference contexts exploited and b) the meaning construction processes are embedded and at the same time interfere with the social conditions at work in the instruction process.

In the course of the interaction between the sign system and the reference context, the role of which can be exchanged, the production of mathematical meaning can be seen as a process of meaning transition from a rather familiar situation (the reference context) to a still unfamiliar sign system.

c. Classroom management of the epistemological features of mathematics: Mathematics science functions with concepts, which are theoretical objects, with definitions as means of recognition and differentiation of objects, with theorems as means of presentation of attributes and relations and follows certain processes as means of management of objects, relationships and results. These aspects are not easily developed in students' minds, but doing mathematics or acquiring a mathematical culture is unavoidably connected with functioning with the same "means" as mathematics does (Brousseau, 2006).

Relying on the above, we claim that in order to identify the nature of the mathematical knowledge constructed in the classroom, we need to analyze classroom interaction on the basis of the management of these specific epistemological characteristics of mathematics by both teachers and students. Obviously, these elements are not always explicitly identified by the students. However, the teacher needs to control and handle them in ways that support students' understanding with respect to the nature, the meaning and the role of these features in the mathematical activity. We argue that this aspect constitutes an important dimension of the teaching/learning process if students are to learn how to work mathematically.

Hence, it is of great importance to look at how the teacher and the students deal with a concept, a definition or a theorem, how they function in solving, proving or validating procedures and, in general, if and in what degree these important characteristics of the scientific activity are valued in the classroom. To this purpose, there is a need to focus on each discursive contribution made by both teachers and pupils in the course of classroom interaction, examining the characteristics (a) assigned to it from a scientific mathematics point of view and (b) attributed to it in the context of the specific interaction. Collating these two aspects, we can identify congruencies or misrepresentations existing between the contribution made and the mathematical meaning or function underpinned, thus deepening our understanding about the nature of the knowledge shaped in the classroom.

THE STUDY

In order to study what emerges interactively in the everyday classroom as mathematical knowledge, we exploited the analytical tools suggested by the above three approaches. Our intention was to provide a comparative reading of the status of the mathematical knowledge under construction in the context of the interaction taking place in the classroom, through the lenses offered by these approaches.

For the purposes of the present study, a videotaped and transcribed 'regular' lesson in algebra taught by a teacher with a university degree in Mathematics and more than fifteen years of teaching experience is exploited. The class consisted of 21 students of 15 years old pupils (third year of a gymnasium located in the northern part of Greece).

The lesson is focused on solving quadratic equations, but the teacher begins by reminding to the students what an algebraic fraction is, a topic that they had discussed in the previous lesson (with linear expressions as denominators), with the intention of moving to algebraic fractions with quadratic expressions as denominators.

Analyzing the data in the light of the above three perspectives, we followed an interpretive approach. Specifically, we focused on the classroom interaction, trying to identify episodes which could be discussed simultaneously from the point of view of the three theoretical perspectives. We then considered the nature of the knowledge emerging, claimed to be 'mathematical', by resorting to the epistemological features of the knowledge shaped.

DATA ANALYSIS AND DISCUSSION

The analysis that follows concentrates first on the notion of the sociomathematical norms, then on that of the epistemological triangle and, finally, on the management of the epistemological features of mathematics.

Sociomathematical norms: Looking at the way the teacher poses questions, provides explanations or justifications and promotes 'better' solutions, it can be argued at first that the sociomathematical norms established in the classroom are mainly guided by her. While she proposes to the students to take initiatives and formulate their own ideas, she immediately corrects or rejects their contributions or provides the correct answer (e.g., lines 82-84 & 98).

Her main concern 'to avoid errors' ("don't lose any root", lines 94 & 98) leads her to emphasizing procedural and morphological elements in her explanations (e.g. lines 86 and 94) and to suggesting approaches, even contradictory, to ensure 'correct' solutions (e.g., lines 71-72 and lines 81- 82).

Thus, the fundamental norms about what is mathematical dominating in the classroom are either of descriptive character or concern procedures; explanations on objects are avoided. For example, in lines 71-72, the student proposes a procedure, but the teacher rejects it as 'unsafe'. No exploration of the context within which the procedure could be utilized is carried out.

Epistemological triangle: Analyzing the lesson from the perspective of the epistemological triangle, that is, in terms of the relationship between reference context and sing system, a change of the former is noticed through the lesson: from rational algebraic fractions (introductory part of the lesson) to rational numbers (lines 61- 62), then to operations with whole numbers (line 63) and finally to solving quadratic equations that can be factorized (the rest of the lesson).

Similarly, the sign system exploited changes in the course of the lesson development, without this becoming clear. Thus, in some parts of the teaching the focus is on the left hand side of the equation (algebraic expressions), while in others on substituting values for x to find out whether the equation is true. It is apparent that the above changes make the relation between reference contexts and sign systems rather problematic.

Furthermore, it should be noted that each time the teacher herself (line 84) or a student (lines 91, 95) make a contribution, which allows for the discussion to focus on relations of general or theoretical objects, the teacher avoids taking the opportunity of doing so by either rejecting the chance or by providing explanations and justifications of mainly procedural or morphological type (e.g., lines 83-86, 94, 96, 98).

Management of epistemological features: Within this perspective, a dominance of procedural and practical directions as well the lack of complete justifications based on the nature and the attributes of mathematical objects can be identified. This results in different mathematical objects appearing in a homogeneous manner. For example, in lines 61- 65, three different mathematical objects (fractions, division, algebraic fractions) are implicitly connected to equations (looking for denominators ≠0). These objects are mainly presented in a morphological manner and without any connection to definitions or properties, which could support students' identification of the new object under consideration (equation). This interplay between different mathematical objects, not clearly defined and vaguely interconnected cannot but lead to the distortion of the mathematical meaning of quadratic equations. Similarly, in lines 84–86, the incomplete reasoning utilized limits students' thinking concerning the solution of quadratic equations and two different mathematical concepts are treated as one (in Pythagoras Theorem, the equations represented relations between lengths of line segments and not only line segments; thus, only the positive solutions had meaning). Moreover, the use of 'rules' ("here it needs ±√4", line 86) instead of an argumentation based on properties results in the outcomes to appear as the result of statements. So, rules, properties and statements are treated in a homogeneous way, without any differentiation as for their nature or role.

Finally, the emphasis on descriptive elements in various parts of the lesson results in the downgrading of the meaning of concepts (e.g., lines 92-94, "algebraic are the numbers which have + and –"). This, together with the dominating undifferentiated use of rules, properties and procedures prevents the attributes of the mathematical objects to function as frameworks for dealing with mathematical objects, as well as with mathematical relations. This is apparent in lines 97-98, where the student's suggestion is rejected and not discussed on the basis of an argument which is based on the results of the strategy ("you lose a root") and not with reference to the restrictions and the ways in which an algebraic expression can be simplified.

DISCUSSION AND CONCLUSION

It is widely accepted today that students' learning of mathematics is greatly shaped by the meanings constructed through negotiations in the classroom. Thus, a systematic analysis of the interaction taking place within the mathematics classroom in relation to the mathematical meaning under construction is of particular interest. The three analyses presented above offer a different way of looking at this issue, highlighting different aspects of it.

The first approach focuses on the processes adopted in the classroom, which shape what is appointed as 'mathematical' within it. On the basis of this approach, we can argue that, in the classroom under consideration, the rules are placed by the teacher, who accepts or rejects students' contributions. The relevant criteria, often contradictory, remain ambiguous (e.g., the teacher first rejects and then accepts the student's proposal "to separate known from unknown terms" with no justification, lines 71-82). As far as what 'counts as mathematical' in this classroom is concerned, it seems to be overtly determined by the teacher and we can only implicitly talk about its nature and whether it is or not mathematical.

Steinbring's approach allows us for an epistemological analysis of the mathematical knowledge interactively constructed in the classroom with reference to the nature and the character of the different objects involved in this interplay (whether or not this knowledge is relational and context-free). Using this approach to read the transcript of the lesson at hand, we can claim that the teacher pursues to arrive at a general idea (the solution of quadratic equations) via specific reference contexts. However, the way these contexts are handled as well as the different resolutions suggested (factorizing/separating terms) do not allow for a relational view of this idea to be developed.

Finally, the third perspective explicitly focuses on the status that the knowledge shaped in the classroom acquires through the particular way it is managed, offering a lens to deciding whether what is developed in the classroom bears mathematical characteristics or involves students in genuine mathematical activity. On the basis of this analysis, we identified in the classroom under consideration the same homogeneous way in which mathematical objects, relations or procedures are treated in many other classrooms, as shown in earlier studies (Kaldrimidou, et al, 2000). This undifferentiated presentation of the various distinct objects, which are engaged in the interaction, as well as of their characteristics does not elevate properties and relations in a manner that would facilitate the management of new objects or relationships. Thus, in this particular case, the teacher places emphasis on morphological aspects or on earlier procedures, which are often used in different and even undefined ways in the new situation. This manner of dealing with mathematical objects and their properties distorts their nature and role, possibly leading students to difficulties in approaching the substance of the mathematical activity.

The points raised above suggest that the parallel exploitation of the three approaches is especially valid. The first highlights the way in which the mathematical features are shaped in the classroom, the second focuses on whether the knowledge emerging is general or context-specific, while the third allows for the identification of the nature, the status and the function of the various ingredients of the mathematics shaped in the classroom. Moreover, the preceding analysis highlights the need to look closer at the particular epistemological features of the mathematical knowledge under construction in the classroom. The complexity of the didactical phenomena framing this construction imposes the need for a multiple approach to analyzing it, which will carefully incorporate the issues raised by all three perspectives.

APPENDIX

61. **T.** Watch it, children. In order for the fraction to have meaning, what should the denominator always be?
62. **Vasos.** Different from zero.
63. **T.** That's it! Because the division by zero is what?
64. **Students.** Impossible!
65. **T.** That is, before you simplify, you should place the denominator ≠ from 0...
71. **George.** To separate known terms from unknown terms.
72. **T.** You suggest we should separate. It cannot be done because both terms are unknown. Anyone else? ……………
81. **Argyro.** The 4 will not be moved to the other side and …
82. **T.** That's right. Then, what do we have from here? $x^2=4$. Let me hear now. What am I to write? *x*…, I am listening to the rest of you. What am I to write?
83. **Argyro.** *x* equals square root of 4.
84. **T.** Bravo! Be careful children! This is what we were saying up to last year. Because, when we learnt how to solve this type of equations, *x* was a line segment. We saw this in Pythagoras Theorem, do you remember? And line segments are always…?
85. **Students.** Positive
86. **T.** Positive! What did we put then? Simply square root of 4. Here needs $\pm\sqrt{4}$. Then, what can be concluded from here? $x=\pm 2$. Because *x* takes both values, -2 and +2. If you substitute $x=+2$ in the initial equation, is the equation true?
87. **Students.** Yes
88. **T.** It is confirmed. Thus, $x=+2$ is a solution. However, if we substitute $x=-2$, is the equation also true?
89. **Students.** Yes.
90. **T.** It is again true. That is, we should not lose solutions. We should write $\pm\sqrt{4}$.
91. **Kostas.** When we say $x^2=4$, isn't it $x^2=2^2$? Thus, since the two squares are equal, should their bases be also equal?
92. **T.** Well, look. You will learn in Lyceum that if a^v is equal to b^v, then we can say that $a=b$ only if a and b are positive numbers. Our problem is different. We need to solve an equation. And what do we notice in this equation? Both +2 and -2 give us 4. Thus, the equation is true for both these values. So, we should not lose -2. From now onwards, we should always write it this way. Last year, in geometry,

we wrote √4. And what does 'algebra' mean? What numbers does algebra deal with? Algebraic numbers. And which numbers are algebraic?

93. **Theodora.** The number which have plus and minus.
94. **T.** That's it! The ones which have plus and minus. We found this example last year, when working on Pythagoras Theorem, on line segments. There, it was not necessary to put both signs. It is here. Because we ended up to a square root. When we factorize the left hand side, it becomes clear which solution is which. Whereas, when we use this way, it is not clear. So, be careful! Don't be carried away and lose a solution. That is, the negative root. Kostas?
95. **Kostas.** In x^2-2, if we write $x.x = 2x$? The x is cancelled and then x=2
96. **T.** Be careful! Which x's is going? Priority of operations... We first multiply…
97. **Kostas.** Madam, we will do x^2=2x … $x.x$ = 2x…
98. **T.** But you have a root! It is forbidden! Ok? You lose a root. Don't do this kind of cancellations, because you lose roots. All right? However, when we take out the common factor, we don't lose the root.

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STUDENTS' BELIEFS AND ATTITUDES ABOUT STUDYING AND LEARNING MATHEMATICS

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In this paper we focus on students' beliefs and attitudes which concern studying and learning mathematics. The sample of this study was 1645 students of 10th, 11th and 12th grade. From our data two factors of beliefs and three factors of attitudes were traced. We investigate whether these factors correlate, whether there are any differences of students' beliefs and attitudes according to their social status and gender and whether they influence students' performance and ability to understand mathematical proofs.

INTRODUCTION

There are many studies concerning students' beliefs and attitudes about mathematics. In Shoenfeld (1989), Mc Leod (1992) and Broun et al. (1988), it is verified that there is a link between students' attitudes and their performance in mathematics. According to Cobb (1986) there is a relation between beliefs and learning of mathematics. In Schoenfeld (1989) it is demonstrated that students' beliefs about Euclidean Geometry is a consequence of the teaching of mathematics. Some researchers agree that students' attitudes can be changed into more positive ones. Regna and Dalla (1992) assert that when teachers are enthusiastic in their teaching and plan activities which are accessible to students, then students' attitudes can be improved. In Kifer & Robitaille (1989) and in Philipou & Christou (2000) it is verified that students' beliefs are influenced by their social surrounding. According to Dematte et al. (1999) it seems that students' beliefs about mathematics are influenced by the educational system of their country. In Pehkonen (1995) students' beliefs from eight countries are investigated. In Christou C. & Philipou G. (1999) factorial structure of 13 years old students' beliefs among four countries (Cyprus, Finland, U.S.A., and Russia) are investigated. In this paper we investigate 10th, 11th, 12th grade students' beliefs and attitudes about studying and learning mathematics and we examine their correlation. We also investigate whether they influence students' performance and ability to understand mathematical proofs.

THEORETICAL BACKGROUND

As it comes from the literature, there are various opinions concerning the notion of "beliefs". According to Goldin (1999), a belief may be "the multiply encoded cognitive configuration to which the holder attributes a high value, including associated warrants". Cooney (1999), asserts that a belief is "a cluster of dispositions to do various things under various circumstances", which leads to the acceptance that "different circumstances may evoke different clusters of beliefs" (Presmeg 1988). It is widely accepted that beliefs are the individual's personal cognitions, theories and conceptions that one forms for subjective reasons. Their nature is partly logical and partly emotional. According to Mc Leod (1992) "beliefs are largely cognitive in nature and are developed over a long period of time". We will use the term "beliefs" in the meaning of personal judgments and views, which constitute one's subjective knowledge, which does not need formal justification.

As it happens with the notion "beliefs", there is also luck of consensus about the notion of "attitudes". Many researchers use attitudes as a term which includes beliefs about mathematics and about self. Mc Leod (1992) accepts that attitudes "refer to affective responses that involve positive or negative feelings of moderate intensity and reasonable stability"; they may appear as a result of the automation "of a repeated emotional reaction to mathematics" or of "the assignment of an already existing attitude to a new but related task". According to Hannula (2002) "attitude is not seen as a unitary psychological construct but as a category of behavior that is produced by different evaluative processes. Students may express liking or disliking of mathematics because of emotions, expectations or values". Hannula declared that attitudes can change under appropriate circumstances. In this study we investigate 10th, 11th, 12th grade students' beliefs and attitudes, which mainly concern studying and learning mathematics and we explore their factorial structure; we investigate whether there are any differences in student's beliefs and attitudes, concerning their social status and gender; we examine whether these factors correlate and influence students' performance at school and their ability to understand mathematical proofs.

THE STUDY

Methodology

Data reported in this paper was collected by a questionnaire administered to 1645 students of 10th, 11th and 12th grade. These students were from 25 high schools in the district of Athens in Greece, which were selected by the stratified - two stages cluster sampling method. This study is a part of a broader one, the aim of which is to investigate students' beliefs and attitudes concerning mathematics, how they are evoked and affect students' understanding, performance and ability in mathematics. We constructed the questionnaire taking into account analogous questionnaires from the literature, as in Schoenfeld, (1989). The questionnaire consists of 28 questions (statements), 10 of which concern beliefs and 14 concern attitudes about mathematics. The 25th question concerns students' performance in mathematics at school in the previous year. There are three more tasks, the 26th, 27th and 28th, called *mathtest* in this paper, which measure students' ability to understand mathematical proofs. These last three tasks were differentiated according to the students' grade. Below we present one task of this type for each grade, because of lack of space. Students were asked to choose one of the numbers 1, 2, 3,…, 9 that best describes what they feel or think about each one of the first 24 statements, using number 1 to declare "I don't agree at all" and number 9 to declare "I absolutely agree". We used a scale range from 1 to 9, because we believed that with this scaling, students would express their views precisely.

Twenty one of the questions-statements of our questionnaire are presented in table 1. These are the ones which constitute the five factors (see table 1 below).Three of the statements of the questionnaire are omitted, because of their low loadings in the factors, while statements 25, 26, 27 and 28 are presented below:

Q25. Your overall grade average in mathematics last year was : …………

Q26. For α, b>0, if α > b, then α+4>b+4 (1). So, $\frac{(a+4)a}{b} > b+4 (2)$. Thus $\frac{b+4}{a+4} < \frac{a}{b}$ (3). Explain why relations (1), (2) and (3) hold. (This task was for 10th grade students).

Q27. Let a, b, c be real numbers such that $|a-b| \le 5$ and $|b-c| \le 5$. Then the following hold: $b-5 \le a \le b+5$ (1), $-b-5 \le -c \le -b+5$ (2). So, we obtain $-10 \le a-c \le 10$ (3). Therefore $|a-c| \le 10$ (4). Explain why relations (1), (2), (3) and (4) hold. (This task was for 11th grade students).

Q28. Let f be a real function, defined by $f(x) = x^3+1, x \in R$. We observe that $f(-1)=0$. We suppose that there is $p \in R$, with $p \ne -1$, such that $f(p)=0$.Then, if $p<-1$ it holds that $f(p)<f(-1)$ (1) and if $p>-1$, it holds that $f(p)>f(-1)$ (2). In any case there is a contradiction. Explain why the relations (1) and (2) hold and what the contradiction is. (This task was for 12th grade students).

Data analysis

Exploratory factor analysis which was applied, led us to five factors, with sufficient internal consistency and reliability. Factors F_1 and F_2 concern beliefs and factors F_3, F_4 and F_5 concern attitudes. In order to investigate whether there are differences in students' beliefs and attitudes concerning their social status and gender, we applied multivariate analysis of variance (manova).We also calculated Pearson correlations for these factors and variables 25 and the mathtest, in order to investigate which of them correlate and whether they correlate positively or negatively.

RESULTS

Table 1 shows the five factors, the related items, means, standard deviations, factor loadings and Cronbachs' alpha.

Factors	Cronbach's a	Mean	St.D	Load-ings
F_1 "Utility of proofs and mathematics'"	0.604	6.584	1.58	
Q24 "You study the proof of a theorem, because you believe that the understanding of proofs can give you ideas, which will help you in problem solving"				0.665

Q3 “Mathematics which I learn at school contributes to improving my thinking”				0.634
Q23 “You study the proof of a theorem, because you believe that the understanding of the proof will help you to understand the theorem”				0.631
Q4 “Mathematics which I learn at school is useful only for those who will study mathematics, sciences and engineering in the university”(reversed)				-0.573
F_2 “Mathematical understanding through procedures”	0.639	5.812	1.35	
Q20 “If you are able to write down the proof of a theorem, then you have understood it”				0.751
Q21 “If you are able to express a definition, then you have understood it”				0.717
Q19 “Studying mathematics means you learn to apply formulas and procedures”				0.575
F_3 “Love of mathematics”	0.735	5.642	2.23	
Q6 “You loved mathematics in junior high school”				0.869
Q5 “You loved mathematics in elementary school”				0.812
Q7 “You love mathematics nowadays in senior high school”				0.665
F_4 “First level of studying mathematics-studying mathematics with understanding”	0.783	7.110	1.52	
Q10 “Whenever you study mathematics you try to understand the proofs of theorems”				0.726
Q9 “Whenever you study mathematics you try to understand what the theorems say”				0.690
Q8 “Whenever you study mathematics you try to understand definitions”				0.650
Q12 “Whenever you study the proof of a theorem you try to understand the successive steps of the proof”				0.648
Q11 “Whenever you study mathematics you try to prove the theorems by yourself”				0.599

Q13 "Whenever you study the proof of a theorem you try to understand the reason for which we follow this procedure towards the proof"				0.510
F_5 "Second level of studying mathematics-studying mathematics with reflection"	0.703	5.09	1.59	
Q15 "When you have done an exercise you examine whether it could be done in a different way"				0.794
Q18 "When you have done an exercise you examine whether you could extend it by adding some new questions"				0.736
Q16 "When you have done an exercise you think again about the steps you have taken, reflecting on them"				0.645
Q14 "When you have studied a proof of a theorem you think again about the whole proof, reflecting on it"				0.514
Q17 "When you have done an exercise you examine whether the result you have found is logical"				0.468

Table 1: The five factors

Table 2 shows the results of manova analysis with factors F_1 – F5 as dependent variables and "gender" and "social status" as independent variables. As it is shown in this table there is a significant statistical difference between female and male students concerning factors F_2 (p=.03<.05) and F_4 (p=.00<.05). More specifically, it emerges (by comparing the respective means) that female students have a stronger belief that mathematical understanding is achieved through procedures than male students do. It also emerges that females study mathematics more carefully than males do. This finding correlates with another finding of our broader study according to which girls have higher performance at school in mathematics than boys do.

Factors	Gender				Social status	
	F 4.612	P **.00**	Means female	male	F 1.433	P 0.08
F_1	3.328	.07	6.68	6.48		
F_2	8.707	**.03**	5.95	5.67		

F_3	1.130	.28	5.51	5.76		
F_4	20.323	**.00**	7.37	6.85		
F_5	2.159	.142	5.11	5.08		

Table 2: Results of manova analysis

However, according to this study, there is no significant statistical difference between boys and girls concerning mathtest. Manova analysis showed also, that there is no significant statistical difference for all the factors concerning the social status of the students.

We also traced correlations among the factors as well as variables 25 and mathtest (See table 3).

	F_1	F_2	F_3	F_4	F_5	25	Mathtest
F_1	1						
F_2	.155**	1					
F_3	.343**	.039	1				
F_4	.378**	.242**	.259**	1			
F_5	.320**	.180**	. 305**	.418**	1		
25	.155**	-.080**	.343**	.202**	.087**	1	
Mathtest	.203**	-.080**	.370**	.232**	.143**	.395**	1

Table 3: Correlations between the factors and variables 25, mathtest

As it is shown from the above table factor F_1 correlates positively with factors F_2, F_3, F_4 and F_5, as well as variables 25 and mathtest. That is, students who believe in the utility of proofs and mathematics, first study mathematics in a way that will enable them to understand (first level of studying) and then they continue on whatever they have studied by reflecting (second level of study). These students love mathematics, have high performance and ability to understand proofs.

Factor F_2 correlates positively with factors F_4, F_5 and negatively with variables 25 and the mathtest. Factor F_3 correlates positively with factors F_4, F_5 and variables 25 and the mathtest. Factor F_4 correlates positively with factor F_5 and variables 25 and the mathtest. Factor F_5 correlates positively with variables 25 and the mathtest. It seems that procedural studying and learning of mathematics is not conducive to high performance or to the ability to understand proofs. Love of mathematics correlates positively with studying of mathematics involving understanding and reflection, with high performance at school and with the ability to understand mathematical proofs. Finally it seems that high performance in mathematics correlates with high ability to understand proofs.

CONCLUSIONS

The results of this study clarify the structure of upper high school students' beliefs and attitudes about studying and learning mathematics and the way in which mathematical performance and ability are influenced by them. Two different factors concerning beliefs and three factors concerning attitudes were traced. It has been made clear that, students' beliefs and attitudes are independent from their social status. This finding would probably be different if we compared students from agricultural districts of Greece with students from an urban area as Athens. It is clarified that girls believe more than boys that mathematical understanding is achieved through procedures. They are also more careful and hardworking in studying and learning mathematics than boys are. Strong belief in the utility of proofs and mathematics as well as love of mathematics correlate positively with studying mathematics in such a way, that ensures good and deep understanding (studying with understanding and reflection). They correlate positively with high performance and mathematical ability as well. Studying mathematics with understanding (first level of studying) and with reflection (second level of studying), correlate positively with high performance and ability to understand proofs as well. On the other hand procedural view and procedural studying of mathematics correlate negatively with performance in mathematics and the ability to understand proofs. That is, performance in mathematics and ability to understand proofs depend on the way in which students study mathematics. It is remarkable that, love of mathematics, is the factor which correlates most positively with performance and mathematical ability.

ACKNOWLEDGEMENTS

The present study was funded through the programme EPEAEK II in the framework of the project "Pythagoras II – Support of University Research Groups" with 75% from European Social Funds and 25% from National Funds.

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"HOW CAN WE DESCRIBE THE RELATION BETWEEN THE FACTORED FORM AND THE EXPANDED FORM OF THESE TRINOMIALS? – WE DON'T EVEN KNOW IF OUR PAPER-AND-PENCIL FACTORIZATIONS ARE RIGHT": THE CASE FOR COMPUTER ALGEBRA SYSTEMS (CAS) WITH WEAKER ALGEBRA STUDENTS

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A small comparative study was carried out with two classes of 10th grade students in need of remedial help in algebra – one class being provided with CAS technology and the other class not. Two sets of parallel tasks were designed with the main difference between the two being the use of the CAS tool. Both classes were taught by the same teacher over the course of one month. Results indicate that the CAS class improved much more than the non-CAS class with respect to both technique and theory. The CAS technology played three roles that were instrumental in increasing students' motivation and confidence: generator of exact answers, verifier of students' written work, and instigator of classroom discussion. These findings suggest that the algebra learning of weaker students can benefit greatly from the integration of CAS technology.

PAST RESEARCH IN THIS AREA

While research evidence is beginning to accumulate regarding the positive roles that Computer Algebra Systems (CAS) can play in the learning of school algebra by academically oriented pupils (e.g., Kieran & Drijvers, 2006; Thomas, Monaghan, & Pierce, 2004; Zbiek, 2003), considerably fewer CAS studies have been specifically identified as being carried out with weaker students. Thus, little is known of the benefits of CAS technology for weak algebra students. Even though Heid and Edwards (2001) have proposed that "computer symbolic algebra utilities may encourage weak students to examine algebraic expressions from a more conceptual point of view" (p. 131), they did not refer to specific studies that could support this claim. However, Lagrange (2003) has emphasized from the research his group carried out with precalculus students that easier symbolic manipulation did not automatically enhance student reflection and understanding. In contrast, Jakucyn and Kerr (2002) have pointed out that precalculus students who lacked certain procedural skills could apply their conceptual understanding of the same procedures toward the solving of related problem situations, when provided with CAS technology. Similarly, in a study involving low-ability grade 12 students, who were using CAS in a unit on differentiation, McCrae, Asp, and Kendal (1999) noted that CAS technology led to improved strategy choice for solving calculus problems. In addition, Shaw, Jean, and Peck (1997) found that college students who were enrolled in a developmental, CAS-based, intermediate algebra course not only seemed to develop some of the skills that

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they had not mastered from previous mathematics courses, but also performed better in a follow-up mathematics course than those students who took the traditional intermediate algebra course.

Heid (2002), in a review of arguments against CAS use in the secondary algebra classroom, including the idea that they lead to a loss of by-hand skills, argued for the opposite view, that is, that CAS enhances students' understanding of the symbolic aspects of algebra rather than supplanting such skills. However, as Driver (2001) pointed out, students who are weak in algebra continue to be barred from access to CAS due to concerns that such students may be "unable to benefit from the use of an algebraic calculator or become over-reliant on it and not develop the necessary knowledge and procedures required by the course" (p. 229). Thus, while the evidence is extremely scanty with respect to weaker algebra students, the main issue appears to be whether the use of CAS permits these students to develop a stronger symbol sense than would otherwise be the case in a paper-and-pencil environment – a symbol sense that can in fact lead to improved by-hand skills. To adequately address this issue, a comparative study involving two comparable classes of weak algebra students was designed, one class having access to CAS technology and the other class not. The construction of the tasks and instructional sequences to be used in the study was underpinned by a theoretical framework based on the instrumental approach to tool use: the Task-Technique-Theory framework.

THEORETICAL FRAMEWORK OF THE STUDY

The instrumental approach to tool use encompasses elements from both cognitive ergonomics (Vérillon & Rabardel, 1995) and the anthropological theory of didactics (Chevallard, 1999). The instrumental approach has been recognized by French mathematics education researchers (e.g., Artigue, 2002; Lagrange, 2002; Guin & Trouche, 2002) as a potentially powerful framework in the context of using CAS in mathematics education. As Monaghan (2005) has pointed out, however, one can distinguish two directions within the instrumental approach. In line with the cognitive ergonomic framework, some researchers (e.g., Trouche, 2000) see the development of schemes as the heart of instrumental genesis. More in line with the anthropological approach, other researchers (e.g., Artigue, 2002; Lagrange, 2002) focus on techniques that students develop while using technological tools and in social interaction. The advantage of this focus is that instrumented techniques are visible and can be observed more easily than mental schemes. Still, it is acknowledged that techniques encompass theoretical notions. In this regard, Lagrange (2003, p. 271) has argued that: "Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration. It also serves as an object for a conceptual reflection when compared with other techniques and when discussed with regard to consistency." It is this epistemic role played by techniques that is essential to understanding our perspective on CAS use, that is, the notion that students' mathematical theorizing develops as their techniques evolve within technological environments. However, the nature of the tasks presented to students –

tasks that include a focus on the theoretical while the technical aspects are developing – is crucial. Thus, the triad Task-Technique-Theory served as our framework not only for gathering data during the teaching sequences and for analyzing the resulting data, but also for constructing the tasks and tests of this study.

METHODOLOGICAL ASPECTS OF THE STUDY

Research Questions

The central research questions of this study were the following: Do students who are weak in algebraic technique and theory benefit more from CAS-based instruction in algebra than from comparable non-CAS-based instruction? If so, what are the specific benefits, and what roles does the CAS play that can account for these benefits?

Participants

The participants were two classes of weak Grade 10 algebra students (15 to 16 years of age) who were required by the school to take one month of supplementary algebra classes in May 2005 (50 minutes per day, every 2nd day). The teacher of these two classes (the second author of this report) was enrolled in a master's program at the first author's university and so arranged that her master's research project would involve the students of these two classes. One class had access to CAS technology (TI-92 Plus calculators) during the month-long teaching sequence on algebra and the other class did not.

Task and Test Design

A set of parallel activities was developed for the two classes – focusing mainly on factoring and expanding, an area where these students were particularly weak. Every effort was made to have identical tasks for the two classes, except that where one class would use paper-and-pencil only, the other class would use CAS or a combination of CAS and paper-and-pencil. Some of the task questions were technique-oriented, while others were theory-oriented. Tasks that asked students to interpret their work, whether it was CAS-based or paper-and-pencil-based, aimed to focus students on structural aspects of algebraic expressions and to bring mathematical notions to the surface, making them objects of explicit reflection and discourse in the classroom. An example of one of these task activities is presented in the following section on the analysis of student work. Each pupil was provided with activity sheets containing the task questions, where he/she either gave answers to the technical questions or offered interpretations, explanations, and reflections for the theoretically-oriented questions.

In addition to generating two parallel sets of task activities, we also constructed one pretest and one posttest. The questions of these two tests focused primarily on factoring and expanding algebraic expressions, on describing the reasoning involved in carrying out these procedures, on describing the structural features of factored and expanded forms, and on explaining the relation between them. Test questions were divided for purposes of analysis into two types: technical and theoretical; students' tests were scored according to these two dimensions.

Unfolding of the Study

Both classes were administered the paper-and-pencil pretest at the start of the study. There was no significant statistical difference between the pretest scores of the two classes on either the technical or theoretical dimensions. However, the class that had the marginally weaker technical score was the class that was designated the CAS class. Because the students of the CAS class had not had any prior experience with symbol-manipulation technology, a few periods were then spent in initiating them to this technology, in particular to the commands that would be used during the teaching sequence. Each student was provided with a CAS calculator for the duration of the study. The same teacher taught both classes. She had not had any prior experience with using CAS technology in her algebra teaching. She taught both classes in a similar manner: introducing the topic of the day at the blackboard; describing briefly the content of the given worksheet; circulating and answering questions while students engaged with the tasks of the worksheets; and bringing all the students together during the last 15 minutes of class in order to discuss the material that they had been working on during that period. Students in the CAS class were sometimes encouraged to use the view-screen to present their work during the discussion period. At the close of the month-long instructional sequence, both classes wrote the paper-and-pencil posttest, which was an alternate version of the pretest. Neither class had access to CAS technology for the writing of the posttest.

Data Sources

The data sources, which permitted a combination of qualitative and quantitative analyses, included: (a) all the task worksheets of each student from the two classes; (b) the pretest and posttest of each student; (c) the daily summaries in the teacher's logbook, which she entered at the close of each class; here she kept track of the discussions that had occurred, and also recorded individual students' comments, concerns, difficulties, high and low points of the classroom activities, and any other items worthy of note.

ANALYSIS OF STUDENT ACTIVITY AND WRITTEN WORK

Analysis of Pretest and Posttest

An analysis of the pretest and posttest scores of the two classes of students was first carried out (see Table 1).

	Pretest Technique	Posttest Technique	Pretest Theory	Posttest Theory
CAS class	74.9%	91.2%	19%	39%
non-CAS class	75.9%	85.6%	15.2%	23.8%

Table 1: Mean percentage scores for the technical and theoretical components of the pretest and posttest by the CAS and non-CAS classes.

The wide discrepancy in the pretest scores between the technical and theoretical components is attributable to the fact that neither class had had experience with theoretically-oriented questions in their algebra classes prior to the unfolding of this study. (Furthermore, while the pretest-technique scores may appear to be quite strong, they were considered weak in a school where mastery learning was the goal.) In any case, it is clear that the posttest improvement in the CAS class on the Theory dimension was considerably greater than was the case for the non-CAS class. With respect to the Technique dimension, again both classes improved as a result of the teaching sequence that occurred between pretest and posttest, but the CAS class improved more. While this was a small study involving only two classes of students, the results of this first analysis indicate that the CAS class benefited more from the remedial instructional sequence than did the non-CAS class (see Damboise, 2006, for a detailed analysis of student responses to the two tests). To try to find explanations that could account for the greater improvement in the CAS class, we then analyzed the teacher's logbook entries and students' worksheets.

Analysis of Teacher's Logbook Entries and Students' Worksheets

The analysis of the entries in the teacher's logbook led to several conjectures regarding the mechanisms at play in the CAS class – mechanisms that could account for the superior performance of the CAS class on the posttest. These conjectures were supported by the analysis of students' technical and theoretical responses to the worksheet questions. In brief, the technology was found to play several roles in the CAS class: it provoked discussion; it generated exact answers that could be scrutinized for structure and form; it helped students to verify their conjectures, as well as their paper-and-pencil responses; it motivated the checking of answers; and it created a sense of confidence and thus led to increased interest in algebraic activity. As space constraints do not permit the presenting of data to support each of these results, we will confine ourselves to what we believe is one of our most important findings with regard to the role that CAS can play in helping weaker algebra students.

The CAS generates exact answers that can be scrutinized for structure and form. Of all the roles that the CAS played in this study, this was found to be the most crucial to the success of the weaker algebra student. It proved to be the main mechanism underlying the evolution in the CAS students' algebraic thinking. Ironically, the crucial nature of this role was first made apparent to us by the voicing of a frustration by one of the students in the non-CAS class – a frustration that we will share shortly. First, we present the CAS version (see Figure 1), then the non-CAS version (Fig. 2) of the task that led to this finding.

Activity 3 (CAS): Trinomials with positive coefficients and $a = 1$ ($ax^2 + bx + c$)

1. Use the calculator in completing the table below.

Given trinomial (in "dissected" form)	Factored form using FACTOR	Expanded form using EXPAND
(a) $x^2 + (3+4)x + 3 \bullet 4$		
(b) $x^2 + (3+5)x + 3 \bullet 5$		
(c) $x^2 + (4+6)x + 4 \bullet 6$		
(d) $x^2 + (3+5)x + 3 \bullet 3$		
(e) $x^2 + (3+4)x + 3 \bullet 6$		

2(a) Why did the calculator not factor the trinomial expressions of 1(d) and 1(e) above?
2(b) How can you tell by looking at the "dissected" form (left-hand column) if a trinomial is factorable?
2(c) If a trinomial is not in its "dissected" form but is in its expanded form, how can you tell if it is factorable? Explain and give an example.
2(d) How would you describe the relation between the factored form and the expanded form of the above trinomials in 1(a) – 1(c)?

Figure 1: A task drawn from Activity 3 (CAS version).

Activity 3 (non-CAS): Trinomials with positive coefficients and $a = 1$ ($ax^2 + bx + c$)

1. Complete the table below by following the example at the beginning of the table.

Given trinomial (in "dissected" form)	Factored form	Expanded form
Example: $x^2 + (3+4)x + 3 \bullet 4$	$x^2 + (3+4)x + 3 \bullet 4$ $= x^2 + 3x + 4x + 3 \bullet 4$ $= x(x+3) + 4(x+3)$ $= (x+3)(x+4)$	$x^2 + 7x + 12$
(a) $x^2 + (5+6)x + 5 \bullet 6$		
(b) $x^2 + (3+5)x + 3 \bullet 5$		
(c) $x^2 + (4+6)x + 4 \bullet 6$		
(d) $x^2 + (3+5)x + 3 \bullet 3$		
(e) $x^2 + (3+4)x + 3 \bullet 6$		

2(a) Why could you not factor the trinomial expressions in 1(d) and 1(e) above?
2(b) How can you tell by looking at the "dissected" form (left-hand column) if a trinomial is factorable?
2(c) If a trinomial is not in its "dissected" form but is in its expanded form, how can you tell if it is factorable? Explain and give an example.
2(d) How would you describe the relation between the factored form and the expanded form of the above trinomials in 1(a) – 1(c)?

Figure 2: The non-CAS version of the same task that was presented in Figure 1.

Note that, in the CAS version of Question 1, students are asked to enter onto their worksheet the output produced by the CAS, while in the non-CAS version they are to record their paper-and-pencil factorizations and expansions. (N.B.: The "dissected" form of the first column was one that both classes were quite familiar with by the time they met this Activity.) The problematic nature of this task, and the potential of the CAS for assisting with such tasks, showed up when the students in the non-CAS class tried to tackle Questions 2c and 2d.

Students in the non-CAS class were at a loss to answer these explanation-oriented questions. They stated emphatically that they were not sure of their answers to Question 1, and could hardly use these as a basis for answering, say, Question 2d. As one student put it so forcefully: *"How can we describe the relation between the factored form and the expanded form of these trinomials? – we don't even know if our factorizations and expansions from Question 1 are right."* In contrast, the students in the CAS class had at their disposal a set of factored and expanded expressions that had been generated by the calculator. They thus had confidence in these responses and could begin to examine them for elements related to structure and form.

CONCLUDING REMARKS

This study analyzed the improvements of two classes of weak algebra students in both *technique* (being able to do) and *theory* (i.e., being able to explain why and to note some structural aspects), in the context of tasks that invited technical and theoretical development. One of the two participating classes had access to CAS technology for the study. At the outset, both the CAS class and the non-CAS class scored at the same levels in a pretest that included technical and theoretical components. However, the CAS class improved more than the non-CAS class on both components, but especially on the theoretical component.

This is an interesting finding for several reasons. Many teachers insist that students learn to do algebraic work with paper-and-pencil first and only later use CAS – and then simply to verify the paper-and-pencil work. However, we found that the students' paper-and-pencil technical work actually benefited from the interaction with CAS. The CAS provided insights that transferred to their paper-and-pencil algebraic work and enhanced their learning. Secondly, and this is quite an exciting finding: Being able to generate exact answers with the CAS allowed students to examine their CAS work and to see patterns among answers that they were sure were correct. This kind of assurance, which led the CAS students to theorize, was found to be lacking in the uniquely paper-and-pencil environment where students made few theoretical observations. The theoretical observations made by CAS students worked hand-in-hand with improving their technical ability. Last but not least, the CAS increased students' confidence in their algebra. This confidence boosted their interest and motivation. These findings suggest that the algebra learning of weaker students can benefit greatly from the integration of CAS technology.

ACKNOWLEDGMENTS

The research presented in this report was supported by a grant from the Social Sciences and Humanities Council of Canada – INE Grant # 501-2002-0132. We also thank Texas Instruments for their generous loan of TI-92 Plus calculators.

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WHAT IS A BEAUTIFUL PROBLEM? AN UNDERGRADUATE STUDENTS' PERSPECTIVE

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In this paper, we present an approach to exploring students' aesthetical preferences in mathematics. Based on analysis of 9 undergraduate students' responses and behaviors in two problem solving workshops, we report essential elements of a preliminary student-centred model of the notion "beautiful mathematical problem." The preliminary model includes cognitive, metacognitive and social factors and seeks to appreciate the complexity of the students' aesthetical judgements.

THEORETICAL BACKGROUND

This paper is part of a series of reports on results of a large-scale project, in progress, whose purpose is to investigate high school students, undergraduate students and mathematics teachers' beliefs and actions through the lens of mathematical aesthetics and to check the possibility of incorporating an aesthetic dimension in learning and teaching mathematics. The goal of this report is two-fold. First, we describe a research approach that seems to be useful in revealing complicated mechanisms involved in undergraduate students' aesthetical judgement of a mathematical problem. Second, we suggest and illustrate elements of a preliminary model of the notion "beautiful problem" that seems to reasonably explain the (ostensibly) controversial evidence collected at the pilot stage of the project as well as some intriguing results of past research.

The study is oriented within a theoretical framework that gradually emerges during the last 20 years from research on the role of aesthetics in doing, learning and teaching mathematics (e.g., Dreyfus & Eisenberg, 1986; 1996; Silver & Metzger, 1989; Sinclair, 2004; Koichu & Berman, 2005; Sinclair & Crespo, 2006) and seeks to contribute to that framework.

Twenty years ago, Dreyfus & Eisenberg (1986) pioneered the student-centric approach to exploration of aesthetics of mathematical thoughts. They suggested that the aesthetical considerations involve a personal internalized metric upon a solution to a particular problem. The factors contributing to the aesthetic appeal of a solution are clarity, simplicity, brevity, conciseness, structure, power, cleverness and surprise. This view fits that of many recognized mathematicians, including Hadamard (1945), Poincaré (1946) and Hardy (1956). In time, it figured out that what is viewed by the experts as "beauty of mathematics" does not look the same for high school, undergraduate and even graduate students. Dreyfus and Eisenberg explored university level students' aesthetical preferences using a set of fairly difficult mathematical problems. Each problem had more than one solution, including a solution that had been

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previously evaluated by the experts as "slickly" or "elegant" one. The problems were given in different formats to pre-service teachers of mathematics in Israel and graduate students in the USA. Not surprisingly, the students rarely came up with the elegant solutions. More surprisingly, when the elegant solution was presented to them and understood, most of the students "did not find the elegant [solution] any more attractive than the ones they had come with on their own—they failed to grasp its aesthetic superiority" (Dreyfus & Eisenberg, 1986, p. 7) even after probing. Dreyfus and Eisenberg concluded that the students showed no inner sense of feeling for the elegancy of a solution. This conclusion is in line with Krutetskii's (1976) point that the tendency to appreciate the elegancy of a mathematical problem is an attribute of only exceptional mathematical giftedness. It is also in line with even more radical opinion of Von Glasersfeld who noted that we cannot expect students to show an appreciation for the beauty of mathematics (cited in Dreyfus & Eisenberg, 1986). In what follows, we suggest reconsidering a typical student's "aesthetical blindness" on the ground of the evidence collected in the new experimental format and by calling into play more comprehensive explanations of the observed behaviors. At this point, we only point the reader's attention to the fact that past research shows that the elite world of professional mathematicians and exceptionally gifted learners, on one side, and the rest of the world, on the other, seem antagonistic with respect to appreciating the beauty of mathematics.

The situation within the world of professional mathematicians and the exceptionally gifted is also far from being simple or fully understood (see, for example, the responses to the Dreyfus and Eisenberg's paper in the same 1986 issue of "For the learning of mathematics"). Two points from the research focusing on that elite population are particularly relevant to this paper. First, Silver & Metzger (1989) found that affect and aesthetics appear to serve as a basis for linking metacognitive processes, such as planning and monitoring in mathematicians' problem solving (see also Goldin, 2002, and Sinclair, 2004, for analysis of a *generative* role of beliefs). Second, Koichu & Berman (2005) found that both social and cognitive factors interplay in effectiveness-elegancy conflict encountered by exceptionally gifted students when solving olympiad-style mathematics problems.

On one hand, these findings point to complexity of the ways by which expert problem solvers call into play aesthetic considerations. On the other hand, they give a clue that metacognitive and social factors should be taken into account when exploring aesthetical judgements of different categories of problem solvers. This idea works well in research on problem solving strategies (e.g., Schoenfeld, 1987). Therefore, it can work also in research on aesthetical aspects of problem solving. The latter implication has driven our study and, specifically, our thinking of the following question: Which factors affect the undergraduate students' conception of the notion "beautiful mathematical problem"?

THE METHOD

The research setting and participants

The data presented in the paper were collected during two consecutive workshops in the context of the undergraduate course "Selected problems in mathematics." The notion "beautiful problem" had not been deliberately discussed in the course till the lessons described below. The participants were 9 third-year undergraduate students from the department of education, the department of mathematics and several engineering departments at the Technion. All the participants had solid background in formal mathematics. The authors of the paper designed the workshops, and the first author served as an instructor. The workshops were videotaped with the camera trained at the participants; all written work of the students was collected.

The first workshop

The first workshop was designed to explore which characteristics of the given problems appear in the students' aesthetical judgements. At the beginning of the workshop, the students were given 3 problems, each one consisting of three parts:

Pr. 1. In the letters shop, one can buy letters. The cost of the letters needed to write the word ONE is \$6. The cost of TWO is \$9 and the cost of ELEVEN is \$15.

a) What is the cost of the word TWELVE?
b) What is the cost of THIRTEEN?
c) What is the cost of TEN?

Pr. 2. A series of numbers is formed in the following way: The first number is 1, and then every number is obtained from the previous one according to the rule described below. Danny computed the first 2007 numbers in each of the three cases. How many (in each case) are divisible by 5?

a) A number is obtained from the preceding number by adding 2.
b) A number is obtained by multiplying the preceding one by 2 and adding 1.
c) A number is obtained by multiplying the preceding one by 2.

Pr. 3. A pedestrian and a bicyclist left Haifa and Atlit at 7:00 moving towards each other along the beach. The pedestrian walked from Haifa to Atlit while the bicyclist rode from Atlit to Haifa. Both of them moved in a constant velocity. When did the pedestrian reach Atlit in each of the following three cases?

a) The rider's velocity is 3 times the walker's one and they passed each other at 8:15.
b) At 8:00 the walker was in the middle between the rider and Haifa and the two passed each other at 8:15.
c) The rider reached Haifa at 8:40.

All the three problems are formulated in different styles to which we refer as "unconventional story", "no story" and "conventional story", respectively. Besides, they were carefully designed by the authors to meet several conditions: each problem contains items having more than one solution, one of which is shorter and more "slickly" than others; the items in each problem look similarly, but the solutions and ways of finding them are different with respect to their difficulty and heuristic arsenals involved. To illustrate these conditions, consider Pr. 1 in some detail. The "unconventional story" can be represented as a system of 3 linear equations with 7 variables. The "slickly" part follows. In item (a), there is no need for solving the system fully – it is enough to find the cost of the sum T+W+2E+L+V, which can be done very quickly; the only answer is "$18". The answer to item (b) depends on the letters I and R, which are not mentioned among the givens. Thus, the answer "the cost is not determined by the givens" can be deduced with no technical effort. In item (c), the letters T, E and N appear among the givens, but the cost of their sum T+E+N cannot be found straightforwardly. The answer to (c) is the same as to (b), but to obtain it one should use apparatus learned in the first-year linear algebra course.

The students were given about 40 minutes to individually approach/solve the problems. Afterwards, the students filled in the questionnaire, in which they were asked to individually evaluate *difficulty*, *challenge* and *beauty* of the problems using 1 to 10 scales. The three features were evaluated separately for each item, thus, each student indicated $9 \times 3 = 27$ numerical responses. The students were also encouraged to briefly explain the responses. When the questionnaires were completed, each student explained orally his or her opinion about the problems to the classmates and the instructor, and then the whole class discussion emerged. It was focused on the relationship among *difficulty*, *challenge* and *beauty* of the given problems. Finally, the students were asked to come back to their questionnaires and indicate whether or not they reconsider their previous responses.

The second workshop

The second workshop was designed to explore to which extent knowing the expert-provided "elegant" solution to a problem affects the students' aesthetical judgement. Three problems from Pólya and Kilpatrick's (1974) "*The Stanford mathematics problem book with hints and solutions*" were used for this purpose (Pr. 58-1, 58-2 and 58-3, p. 17). There is no space to present the problems here. We only note that: the problems included an "unconventional story", "no story" and "conventional story" in the meaning explained above; each problem had several solutions, including the "slickly" one; the problems were more difficult than those of the first workshop.

The students were asked to read the problems, and, based on the first impression only, evaluate each problem's *beauty* using 1 to 10 scales. They were also encouraged to explain their numerical responses. Then the students were given the written solutions to the problems from the Pólya and Kilpatrick's book. When the solutions were fully understood (this was evident from the brief discussions of the solutions), the students were asked to consider whether or not they want to reconsider their initial evaluation of the *beauty* of the problems. This was followed by a whole group discussion, in which the students' expressed their beliefs about what a beautiful problem is.

ANALYSIS

The data analysed consist of the students' written responses to the questionnaires, transcripts of the videotaped workshops and notes the students made. Because of the small number of the participants, we treated the data chiefly as a set of individual cases, in which we looked for patterns particularly interesting with respect to the research question. Following Pierce, Clement (2000) refers to such a method of analysis as *abduction* – a process of producing a model that, if it were true, would account for the observed phenomena. Thus, the concern about viability rather than validity of the findings is relevant in our research.

In addition, correlation analysis was conducted to explore the relationships among the students' numerical evaluation of *difficulty*, *challenge*, and *beauty* of the problems given in the first workshop. To avoid overestimation of small fluctuations in the students' responses, we converted the responses from 1 to 10 into 1 to 3 scales. Namely, numerical responses 1-3 to the questions "To which extent the problem is difficult/challenging/beautiful?" were interpreted as "the problem is easy/not challenging/not beautiful" and re-denoted "1". Responses 4-7 were interpreted as "the problem is fairly difficult/challenging/beautiful" and re-denoted "2". Responses 8-10 were interpreted as "the problem is very difficult/challenging/beautiful" and re-denoted "3". The quantitative results below concern the converted responses.

RESULTS

The students expressed controversial opinions about *beauty* of the problems given at the first workshop. Correlations between *difficulty* and *beauty* as well as *challenge* and *beauty* were close to 0 for all the problems. Styles of the problems' formulation did not show themself as a relevant factor either. The students alluded rather to *novelty* and *unexpectedness* as associates of *beauty*. Consider, for instance, the students' responses and statements concerning Pr. 1. One student indicated that Pr. 1(a) is "very beautiful", 3 – that it is "fairly beautiful", and 5 – that it is "not beautiful;" nobody mentioned the "unconventional story" as a factor affecting the judgement. Three students found a "slickly" solution to Problem 1a (see the previous section), and the rest solved it by more than one page long manipulations of the initial system of equations. The student that evaluated Pr. 1(a) as "very beautiful" solved it in the "slickly" way and explained: "I like it as I've not met such problems before." Two others, who found the same short solution to the problem, did not find it beautiful for two reasons. First, "It was clear what to do," and, second, "The problem is a technical one anyway". These arguments were shared by most of the students. For example, one of the students, who solved the problem in a long way, reflected on his solution as follows: "It is not difficult. You just try different combinations, sums and differences [of the equations]. You are not looking for a new idea…"

A remarkable discussion emerged from the students' reflection on Pr. 1(b) and 1(c).

Alex: You can see immediately that there is no specific solution to 1(b), so it is not too beautiful. But when you solve 1(c), you just work and work, and think that you have a way, the same one as in 1(a), and finally you haven't, and must think why… This is nicer than in a problem that can be solved in a regular way.

Instructor: You gave us an excellent explanation about the difference between a beautiful problem and a difficult one. But, perhaps, the "beauty" [of a problem] equals to [its] "challenge"?

Alex: Perhaps…

Baruch: No, not equals. If a problem is beautiful, it is also challenging, but if a problem is challenging, it is not necessarily beautiful.

Eli: I disagree, not every beautiful problem is challenging. There are some geometry problems…very beautiful and not challenging… Or number theory problems…They can be very challenging, but are not really beautiful. I think there is no connection…

Gila: Yes, beauty is not a challenge and not a difficulty; it is more than that…

Interestingly, all the students showed keen interest in the whole group discussion, but nobody changed his or her aesthetical judgement of Pr. 1(a) and 1(b) by the end of the discussion; only two students changed their opinions regarding 1(c) from "not beautiful" and "fairly beautiful" to "very beautiful".

The additional phenomena deduced from the analysis of the first workshop include:

- Six students did not change any of their opinions about the beauty of any problem during the workshop.

- Pr. 2(b) was considered the most beautiful one (mean=2.88, SD=0.35), but when a simple solution was presented, 3 students changed their opinion from "very beautiful" to "fairly beautiful" and "not beautiful" (mean=2.22, SD=0.83)

- The students' opinions about beauty of Pr. 2 and Pr. 3 varied, but 7 of them gave the same rates to items (a), (b) and (c) of these problems, even when knowing that the solutions are very different. It seemed that the mathematical affiliations of the problem (e.g., a series problem and a word problem) affected the students' aesthetical judgments more than inter-item differences.

Additional phenomena were observed in the second workshop. Namely, the students were able to evaluate to which extent the given problems are beautiful based on their first impression. All the students, but two, did not change their opinions after understanding the expert-provided solutions. Pre-post explanations of those who did not change their opinions include:

Hava: [Pre:] It looks like a tricky question. [Post:] I was right, it is a tricky question.

Tamar: [Pre:] I've solved a similar problem in the past, that's why the level of beauty is not high. [Post:] The way of the solution is as I expected.

Baruch: [Pre:] It looks like a challenging problem. [Post:] I did not appreciate the solution as I could discover it myself.

Two students, who essentially changed their opinions, explained:

Uri: [Pre:] It is not nice and unsolvable. [Post:] When I looked at the solution, I realized that there was a nice solution.

Eli: [Pre:] It is an interesting problem, it says something general about triangles. [Post:] It is even more beautiful than I thought as it is very general, but relies on simple and basic geometry facts.

In the follow-up discussion, the students elaborated their written responses. A brief summary of the discussion is this: A student's aesthetical judgment of a problem is based mostly on the first impression and cannot be easily changed. The changes, if any, are based on acknowledgement that an idea of an expert-provided solution could be discovered by the student independently, but had not come to his or her mind when reading the problem.

DISCUSSION AND CONCLUDING REMARKS

The research question under exploration was: Which factors affect the undergraduate students' conception of the notion "beautiful mathematical problem"? To address this question, we designed a research setting, in which undergraduate students' aesthetical preferences in mathematical problem solving could be evoked. We hope that the setting can be used in the growing body of research on mathematical aesthetics.

In response to the research question, we suggest: (1) a student's perception of a notion "beautiful problem" is deeply individual and involves more sophisticated considerations than difficulty, challenge or a style of a problem's formulation; (2) from a student's perspective, a problem can be *beautiful* if it is characterized by the following traits: it has a mathematical affiliation associated with a high level of aesthetic value (e.g., geometry for one student and number theory for another); it looks new; its solution is accessible, but includes elements of surprise, for instance, it is easier or based on more elementary mathematical tools than it was expected by the student when reading the problem.

These suggestions are in line with those by Koichu & Berman (2005), who utilized the *principle of parsimony* to explain the conflict between the mathematically gifted students' conceptions of elegancy and effectiveness in problem solving. They are also in a good agreement with Silver & Metzger's (1989) point about the role of aesthetics in metacognitive processes, such as planning and monitoring, of mathematicians. Thus, the presented findings may imply that the gap, with respect to mathematical aesthetics, between mathematicians and the gifted, on one side, and university level students, on the other, is smaller than it seems.

The presented findings may also lead to reconsideration of the "aesthetical blindness" of university level students indicated in past research. On one hand, in our research there were participants who refused to acknowledge "the aesthetic superiority" of the "slickly" solutions suggested by the experts (see discussion of Dreyfus & Eisenberg, 1986, in *Theoretical Background* section). On the other hand, the findings points to the importance of socially-based factors like self-esteem of the students as problem solvers, which has not been taken into account in past research. We suggest that the latter factor could in part explain why, in the Dreyfus & Eisenberg's (1986) study, the students' aesthetical judgments were not apparent.

In closing, let us note that we are fully aware of the limitations of the implemented method and the preliminary character of the findings. We hope that in the near future the viability of the presented elements of the student-centered model of the notion "beautiful mathematical problem" will be tested by additional observations.

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CAN LESSONS BE REPLICATED?

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If you have found a powerful way of teaching a particular topic – such as the infinity of decimal numbers, for example – how can you enable other teachers to use the same approach in an equally successful way. This is a general problem in the attempts to improve teaching. The main question is how to describe the successful way of teaching – the powerful lesson design. Using a certain theory of learning, the essential structure of a lesson on the infinity of decimal numbers that proved to be highly effective previously is described, in terms of a pattern of variation and invariance. In this study I am showing that when the teacher succeeds in replicating that pattern, the effects on learning are replicated also.

INTRODUCTION

If you search the Internet for the word *lesson plan* over two hundred thousand sites are found. On those web pages teachers share their good teaching ideas with other colleagues. But in what way could lesson plans be a resource for teachers? Can other teachers' lessons be replicated with similar results, even in terms of learning outcomes? Results from a research project, done in collaboration with teachers, found that two *lesson designs* about infinity of decimal numbers generated different learning outcomes. With the first lesson design, 21% of the students learned that there were infinite number of decimal numbers, compared to 94% and 88% in the classes with the second lesson design. The second lesson design appeared to create better possibilities for learning in this study (Kullberg, 2004). But is it possible to replicate a lesson and the result with other groups of students? In this new study, which could be described as a teaching experiment, the second lesson design was tested by two teachers. The result showed that in one class (class B) 100% of the students learned compared to the other class (class A) in which the improvement was much smaller. In this paper this result is explained by an analysis of the students' group activity and the analysis of how the content was handled in the interaction between the teacher and students and between students. The result of the study shows that it is not the lesson design that predicts the students' learning but rather the content specific features that need to be brought up. From the theoretical framework taken, it is therefore not the activities or the organization of teaching, for example if the teaching is student centred or not that is of decisive importance, but which features of the content that are brought out during the lessons. What happened during these lessons and what where the students able to discern? The aim with this paper is to show why the *same lesson design* generated different learning outcomes in the two classes.

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 121-128. Seoul: PME

LESSON DESIGNS

A Learning study (LS) is an intervention model in which teachers work together in teams with designing a lesson on a specific topic, in collaboration with a researcher. The aim of this collaborative effort is to improve students' learning. Marton (Marton & Tsui, 2004) developed this model with insights from Lesson study (Lewis, 2002; Stigler & Hiebert, 1999) and Design experiments (Brown, 1992; Cobb *et al.*, 2003; Collins, 1992) in the year 2000. Lessons from LS were in this new study used to define a lesson design. In this research project a *lesson design* is not seen as a technology of teaching. It is not a manuscript with specific instructions for the teacher but instead a frame of *critical features* that is needed to be brought out by the teacher together with the students, with the aim to be discerned by the student. In a lesson design it is therefore not the methods or the organization of teaching that are pointed out. So, what features of the content needs to be brought out when teaching about infinity of decimal numbers?

A considerable amount of research has been done investigating students' understanding and conceptions of decimal numbers (Sackur- Grisvard & Léonard, 1985; Hiebert & Wearne, 1986; Steinle, 2004; Roche & Clarke, 2006). Steinle (2004) refers to eleven inadequate strategies that students use when dealing with decimals, for example, students see a decimal number with many digits after the decimal point as a smaller number than a decimal number with fewer digits. Some students use the opposite 'rule', fewer digits after the decimal point indicates a smaller number. From these studies it becomes clear that students often treat decimal numbers as whole numbers. Roche and Clarke (2006) claim that the use of fractional language to describe decimals more often may contribute to a clearer conception of the decimal numeration system. Many other studies also point out that the relationship between fractions and decimals can be used to support the development of decimal knowledge (Moskal & Magone, 2001 p 317). Though, it has been found that students fail to establish the connection between the two. Moskal and Magone imply that knowledge of the whole number system and knowledge of fractions can both assist and confuse students' understanding of the decimal system. The previous research contributed in the LS to the finding of important features for teaching about infinity of decimal numbers. In the first lesson design from the LS the decimal numbers were treated and seen as numbers on a number line. Since the result after this lesson showed that only 21% of the students in that class had learned what was planned, a second design was made. In the second lesson design decimal number was presented in different forms of rational numbers such as fractions and percentage. Another difference was that in the second lesson design the part-part-whole relationship of a decimal number was emphasised. The part-part-whole relationship, meaning that you can see a decimal number, for example 0.97, as a part of a whole, since it is possible to take 0.97 of something, a ruler, a human being or a pen. The part-part- whole relationship emphasises the relationships of the parts in similar ways in to other forms of rational numbers. Fractions show in an explicit way the different parts and can express smaller and smaller parts. In conclusion the second designed lesson included the following features;

- Decimal numbers as numbers on a number line
- Different forms of rational numbers such as fractions and percentage
- The part-part-whole relationship of a decimal number

The different designs can be described as two different ways for the students to discern decimal numbers, as numbers in an interval or as number of parts in an interval (figure 1).

Lesson design 1: Numbers in the interval

Lesson design 2: Number of parts in the interval

Figure 1. Different ways of discerning the interval of two decimal numbers.

The interest of this study is to test the lesson design that generated strong learning outcomes. Therefore lesson design 2 has been chosen to be replicated in this study.

THEORETICAL FRAMEWORK

Variation theory (Marton & Tsui, 2004), developed from phenomenographic research (Marton & Booth, 1997), is the theoretical framework for this study. Variation theory is a theory about how we learn and experience the world around us. Within this framework learning is seen as differentiation. To learn is to discern specific features of an object of learning. People discern different features/aspects and therefore have different learning outcomes. To be able to discern an aspect it must be varied in order to be noticed. In other words, every concept, situation or phenomena have particular features or aspects and if an aspect is changed or varies and another remains unchanged, the changed aspect will be noticed. From this viewpoint teachers could make it possible for the students to experience necessary variation and invariance for a particular object of learning, the *critical aspects* or *features*. Marton, Runesson and Tsui (2004) argues that "The critical features have, at least in part, to be found empirically- for instance through interviews with learners and through the analysis of what is happening in the classroom- and they also have to be found for every object of learning specifically, because the critical features are critical features of specific objects of learning" (p 24). Critical features have been found in the analysis of pre-and post-tests and/or in the analysis of lessons. Research literature could also contribute to the finding of critical features. However, it is not beneficial to learning to just tell the students the critical features, these must be discerned by the learner. This means that the teacher must focus the students' attention to critical features. If the student can not differentiate between specific features, they will have difficulties learning. Therefore experiencing variation concerning critical features for learning is, according to variation theory, essential for learning.

METHOD AND DESIGN OF THE STUDY

The research design for this study consisted of two teachers that conducted one lesson each with the same lesson design. This is an experimental design and could be seen as an example of design research or design experiments (Brown, 1992; Collins, 1992; Cobb *et al*, 2000). Single lessons were the units of analysis to see how the content was handled in each lesson. The lessons were implemented in ordinary classrooms, with 6th grade students (12 year old), and were video recorded. Besides the video recorded lessons individual pre- and post-test were used to capture students' learning outcomes. The pre- and post-test were the same and several items tested the same ability. The pre-test was given a week before the research lesson and the post-test was given one day after the lesson. Material from the students' group activity is also analysed in this study. It should be noted that in this study, the two teachers jointly planned the lesson together with the researcher. The teachers had therefore the same intentions, tasks and activities for the lesson. Thus, none of the teachers gave the right answer to the question in the group activity to the students.

RESULTS

The two tasks (*"Anne claims that there is a number between 0.97 and 0.98. John says there is no such number. Who is right and why?"* and "*Are there numbers between 0.5 and 0.6?*") were used in the pre- and post-test. In the students' written explanations it was possible to analyse if they thought that there were infinite number (expressed by the students in some cases as millions of numbers, or many, many numbers or infinite numbers) of decimal numbers. Students that did not answer correct on these items answered, for example, that there were no numbers, or ten numbers in the intervals. The result from the post-test showed that 100% of the students in class B answered correct on the tasks about infinity (see table 1) compared to 69% of the students in class A. Considering that the difference between the pre- and post-test in class A was only 4%, the result from the LS was not replicated in this class, only in class B. The analysis of the lesson and the analysis of the students' group activity indicate that the connection to fractions was not brought out in the lesson in a sufficient way in class A (see tentative analysis of the lessons below).

Table 1. Percentage of students given correct answers on post-test items on the infinity of rational numbers.

	Class A (n=13)	**Class B (n=13)**
Result on post-test	69%	100%
Difference in percentage between pre- and post-test	**+4%**	**+62%**

Tentative analysis of the lessons

The two lessons were organized the same way and used the same tasks. They started with an introduction, followed by a group activity and ended with an oral account and discussion over the activity. Despite that the learning outcomes were different (see table 1). Within the variation theory framework critical features of the content must be brought out and varied for learning a specific content. In the following descriptions of the lessons the features that were brought out in the lessons are expressed. In class A and B the teachers started the lessons with the same question, namely if there were numbers between two whole numbers, for example 2 and 3. The students contributed by naming many numbers in the interval (e.g. 2.2, 2.5 and 2.35). The teachers in both class A and B then took two decimal numbers, e.g. 0.17 and 0.18 and wanted to students to come up with other names for the decimal numbers. The students contributed to naming the decimal numbers with fractions for example as seventeen and eighteen hundreds.

In class A also the written form, 17/100 was drawn on the board. The teacher in class A raised the question if 17/100 and 0.17 was the same thing, but this question was never answered by the students in this class. Another feature that was present was the part-part-whole relationship of a decimal number. The teacher in class A used a ruler to show this and asked where the two decimal numbers 0.17 and 0.18 could be seen on it. In this case 0.17 and 0.18 was seen as different parts of a one meter ruler, namely, 17 and 18 centimetres. Another 'whole', the human body was also used for showing the same thing. The decimal numbers were representing different parts (cut off) from the human body. After that the teacher in class A focused on the space between the two numbers 0.17 and 0.18 on the number line and asked if there are any numbers in the interval? This was also the question for the students' group activity (see below). In conclusion, in class A fractions was used to represent the decimal numbers but the meaning of fractions as parts was not made explicit.

In class B on the other hand this relationship was made clearer. This was done by naming the same number 0.30 with different number of parts in the fractions 3/10 and 30/100. A drawn cucumber was used to illustrate the part-part-whole relationship of a rational number. In this case 0.29 and 0.30 was seen as parts of a cucumber. In a comparison of the numbers the space between 0.29 and 0.30 were expressed as different amount of parts in the interval such as 1/100, 10/1000, 100/10000 etc. The same interval was in this case seen as smaller and smaller parts of the cucumber. This was done before the students started their group activity.

Students' group activity

The task that was discussed during the group activity in both classes was *if there are any numbers between the two decimal numbers 0.17 and 0.18.* The students were asked to draw a picture on paper of their answer and present it to the whole class in the end of the lesson. A follow up question to the task was *if there are more numbers, less numbers or the same number of decimal numbers between 1 and 2 than between the two decimal numbers.* In class A this later question was focused on more than in class B. The account of the students' group activity showed that the only group that used fractions to explain their solution of the task was students in class B, where two out of three groups did this. The fact that the students used fractions and showed the connection between fractions and decimals in the account of the group activity in whole class probably contributed to the good result for all students in class B (table 1), where 100% of the students answered correct on tasks about infinity of decimal numbers. One group in class B only presented numbers in the interval on their paper, for example 0.171, 0.172, 0.173… up to 0.1718, and showed that even more numbers came after that number. (In this group and many other groups in this study the number 0.1718 was seen as a higher number than 0.172, but this was never discussed in the lessons.) In all groups in class A (4 groups), only decimal numbers in the interval were presented in the students' group work.

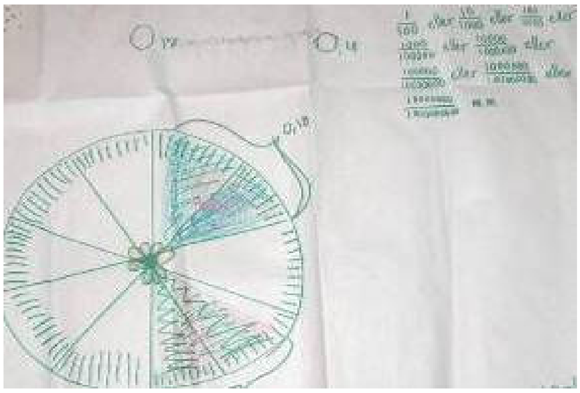

Figure 2. An example of a student answer from the group activity in class B, showing different number of parts in the interval.

As is shown in figure 2 one of the groups that (in class B) used both fractions and decimals in their answer also wrote decimal numbers on a number line to show numbers between O.17 and 0.18 (0.171, 0.172, 0.173 0.174, 0.175, 0.176, 0.177, 0.178, 0.179). Additionally they had also drawn a circle showing the numbers 0.17 and 0.18, as a part of a circle of hundreds (see figure 2). This had not been promoted or shown by the teacher. This group also wrote that between 0.17 and 0.18 there where 1/100, 10/1000, 100/10000, 1000/100000, 10000 /1000000, 100000/ 10000000, 1000000/ 100000000 or 10000000/ 1000000000 etc. In this case the group showed the interval 0.17 and 0.18 as invariant/constant but the number of parts in the interval varied. In the groups in class A, the interval was also constant (since they used the same interval) but the numbers in the interval varied. The later way of experience rational numbers promotes the view that rational numbers are countable. This view probably makes it harder to understand infinity of rational numbers while bringing up the feature of parts in an interval promotes the understanding of infinity in a more adequate way.

CONCLUSIONS

Although the relation between teaching and learning is not 'one to one', similar results in terms of learning outcomes with the same lesson designs appear. What has been showed in this study is that when the teacher succeeds in replicating the specific pattern of variation and invariance, the effects on learning are replicated also. It is therefore not the lesson design that should be replicated but rather the pattern of variation and invariance of the critical features. In a teaching experiment like this, it is profoundly important to analyse what happens in the classroom in the interaction between the teacher and students and between students. A good lesson design is therefore not a guarantee for student learning.

In this study about infinity of decimal numbers the connection between fractions and decimal numbers, seen as parts in an interval was not discerned by all students in class A. In this case the teacher used fractions to represent decimal numbers but did not bring up the feature of parts in the interval. The pattern, where the interval was invariant and the number of parts in the interval varied was replicated in class B and in that case also the result of the learning outcome from previous studies was replicated.

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PROBLEM POSING AS A MEANS FOR DEVELOPING MATHEMATICAL KNOWLEDGE OF PROSPECTIVE TEACHERS

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Abstract

In the present study we aim at exploring the development of mathematical knowledge and problem solving skills of prospective teachers as result of their engagement in problem posing activity. Data was collected through the prospective teachers' reflective portfolios and weekly class discussions. Analysis of the data shows that the prospective teachers developed their ability to examine definition and attributes of mathematical objects, connections among mathematical objects, and validity of an argument. However, they tend to focus on common posed problems, being afraid of their inability to prove their findings. This finding suggests that overemphasizing the importance of providing a formal proof prevents the development of inquiry abilities.

Introduction

Problem posing (PP) is recognized as an important component of mathematics teaching and learning (NCTM, 2000). In order that teachers will gain the knowledge and the required confidence for incorporating PP activities in their classes, they have to experience it first. While experiencing PP they will acknowledge its various benefits. Hence such an experience should start by the time these teachers are being qualified towards their profession. Therefore, while working with prospective teachers (PT) we integrate activities of PP into their method courses. In addition, accompanying the process with reflective writing might make the PT be more aware to the processes they are going through (Campbell et al, 1997), and as a result increase the plausibility that the PT will internalize the effect of the processes that are involved in PP activities. This reflective writing also enables teacher educators to evaluate the PT progress and performance (Arter & Spandel, 1991).

In the present study we aim at exploring the effects of experiencing PP on the development of PT's mathematical knowledge and problem solving skills. For that purpose we employed two evaluative tools – portfolio and class discussion.

Theoretical Background

This section includes a brief theoretical background regarding problem posing, with a special focus on the "what if not?" (WIN) strategy, and regarding the educational value of integrating PP into PT's training programs.

Problem posing. Problem posing is an important component of the mathematics curriculum, and is considered to be an essential part of mathematical doing (Brown & Walter, 1993, NCTM, 2000). PP involves generating of new problems and questions aimed at exploring a given situation as well as the reformulation of a problem during the process of solving it (Silver, 1994). Providing students with opportunities to pose

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 129-136. Seoul: PME

their own problems can foster more diverse and flexible thinking, enhance students' problem solving skills, broaden their perception of mathematics and enrich and consolidate basic concepts (Brown & Walter, 1993, English, 1996). In addition, PP might help in reducing the dependency of students on their teachers and textbooks, and give the students the feeling of becoming more engaged in their education. Cunningham (2004) showed that providing students with the opportunity to pose problems enhanced students' reasoning and reflection. When students, rather than the teacher, formulate new problems, it can foster the sense of ownership that students need to take for constructing their own knowledge. This ownership of the problems results in a highly level of engagement and curiosity, as well as enthusiasm towards the process of learning mathematics.

The 'What If Not?' strategy. Brown & Walter (1993) suggested a new approach to problem posing and problem solving in mathematics teaching, using the 'What If Not?' (WIN) strategy. The strategy is based on the idea that modifying the attributes of a given problem could yield new and intriguing problems which eventually may result in some interesting investigations. In this problem posing approach, students are encouraged to go through three levels starting with re-examining a given problem in order to derive closely related new problems. At the first level, students are asked to make a list of the problem's attributes. At the second level they should address the "What If Not?" question and than suggest alternatives to the listed attributes. The third level is posing new questions, inspired by the alternatives. The strategy enables to move away from a rigid teaching format which makes students believe that there is only one 'right way' to refer to a given problem. The usage of this problem posing strategy provides students with the opportunity to discuss a wide range of ideas, and consider the meaning of the problem rather than merely focusing on finding its solution.

The educational value of integrating problem posing into PT's training programs. Teachers have an important role in the implementation of PP into the curriculum (Gonzales, 1996). However, although PP is recognized as an important teaching method, many students are not given the opportunity to experience PP in their study of mathematics (Silver et al., 1996). In most cases teachers tend to emphasize skills, rules and procedures, which become the essence of learning instead of instruments for developing understanding and reasoning (Ernest, 1991). Consequently, mathematics teachers miss the opportunity to help their students develop problem solving skills, as well as help them to build confidence in managing unfamiliar situations. Teachers rarely use PP because they find it difficult to implement in classrooms, and because they themselves do not possess the required skills (Leung & Silver, 1997). Therefore, PT should be taught how to integrate PP in their lessons. Southwell (1998) found that posing problems based on given problems could be a valuable strategy for developing problem solving abilities of mathematics PT. Moreover, incorporating PP activities in their lessons enables them to become better acquainted with their students' mathematical knowledge and understandings.

The study

The study participants. 25 mathematics PT (8 male and 17 female) from an academic college participated in the course. They are in their third year of studying towards a B.A. degree in mathematics education. The students represented all talent levels. The students are graduated towards being teachers of mathematics and computer science or teachers of mathematics and physics in secondary and high school.

The course. The course in which the research was carried out is a two-semester course and is a part of the PT training in mathematics education. This course is the first mathematical method course the PT attend, and takes place in the third year of their studies (out of four). The research was carried out during the first semester. Problem posing and problem solving, based on the WIN strategy, is one of the main issues discussed in this semester. The lessons usually bear a constant format: By the end of the lesson the PT were given an assignment, which was aimed at supporting them progressing in their work (more details about the phases of the work are presented at the next section). During the week the PT had to accomplish the assignment and describe their work in a reflective manner using a written portfolio. A copy of the portfolio was sent to us by e-mail, reporting their progress, indecisions, doubts, thoughts and insights. When we got the impression that the PT were not progressing we suggested them with new view-points, which were consistent with their course of work. These portfolios served as a basis for class discussion at the successive lesson. Part of each class discussion was allocated to presentations of the PT's works.

The problem. The described process was the first time the PT experienced PP. We demonstrated its various phases using Morgan's theorem (Watanabe, Hanson & Nowosielski, 1996). Afterwards, the PT experienced it themselves, starting with the problem described at Figure 1. The reason for choosing this specific problem was its variety of attributes and its many possible directions of inquiry which might end with generalizations.

Triangle ABC is inscribed in circle O_1.
D is a point on circle O_1. Perpendiculars are drawn from D to AB and AC. E and F are the perpendiculars' intersection points with the sides, respectively. Where should D be located so that EF will be of a maximum length?

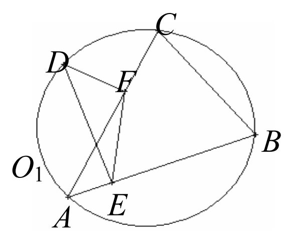

Figure 1

Employing the WIN strategy. The PT were asked to go through the following phases: (1) Solve the given geometrical problem; (2) Produce a list of attributes; (3) Negate each attribute and suggest alternatives; (4) Concentrate on one of the alternatives, formulate a new problem, and solve the new problem; (5) Raise assumptions and verify/refute them; (6) Generalize the findings and draw conclusions; (7) Repeat phases 4-6, up to the point in which the PT decide that the process had been completed.

Data Collection and analysis. In this paper we focus on data concerning the development of mathematical knowledge and inquiry abilities as a result of employing the WIN strategy, based on the above problem. Two main sources of data informed the study: The PT's portfolios, and the class discussions in which the PT presented their works and discussed various issues that were raised while reflecting on their experience. When the data collection phase was completed, we followed the process of analytic induction (Goetz & LeCompte, 1984), reviewing the entire corpus of data to identify themes and patterns and generate initial assertions regarding the effect of the PP on the PT mathematical and didactical knowledge. These research tools enabled us to study the PT's development of mathematical knowledge as well as their inquiry abilities.

RESULTS AND DISCUSSION

In this section we focus on results obtained at phases 2, 3 and 4.

Phases 2 and 3. Figure 2 summarizes the list of attributes (in bold) the PT related to the given problem; the number of students that suggested a certain attribute and in parentheses the percentage of students who suggested it, out of 25 PT. Then appear the suggested alternatives and the number of students that proposed the alternative together with percentages (out of the total students who suggested this alternative). For example, the attribute "**any triangle**" was suggested by 18 PT which is 72% of the 25 study participants. Out of the 18 PT, 9 suggested the alternative "Acute triangle". Namely, (50) designate the fact that 50% of the 18 PT suggested this alternative.

Observation of Figure 2 reveals the following: (a) In case the attribute includes a geometrical shape, most of the suggested alternatives were either a common geometrical shapes or a shape that belongs to the same family as the negated shape. For example, in case of negating the triangle (attribute no. 1), all PT suggested the common shape - quadrangle. Most of them (20) suggested the square. In case of negating attribute no. 5, most PT suggested as alternatives for the circle, shapes which belong same family, namely – square, quadrangle, rectangle and trapezoid. Only minority of them suggested polygons with more than four sides. Moreover, only two of them referred to a three dimensional shape. (b) In case the attribute includes a numerical value, most of the PT suggested as an alternative to this attribute another specific numerical value. For example, 11 PT listed attribute no. 7. All of them suggested as an alternative "Three segments are drawn from the point". These findings are consistent with Lavy & Bershadsky (2003), who found that while PT are posing problems on the basis of spatial geometrical problem, they tend to replace numerical values with other numerical values, and geometrical shapes with shapes that belong to the same family. (c) Part of the alternatives given by the PT includes generalization. Though only minority of the alternatives was formed as generalization, two types of generalization can be observed: generalization of a numerical value of an attribute and generalization of a geometrical shape. As to the former, a specific numerical value was replaced by *n*-value. For example, a generalization of attribute no. 7 is 7.2, and of attribute no. 8 is 8.2. As to the latter, the generalization of a geometrical shape can be divided into two sub-categories: generalization of the number of the shape's sides (e.g. attribute no. 1 is generalized by "*n*-sided polygon"), and generalization of shape's dimensions – a shift from a planar into a spatial shape. In attribute 5, for example, the planar shape "circle" was replaced by "sphere" up to "any spatial body".

1. **Triangle** 25(100) *alternatives*: (1.1) Quadrangle 25 (100); (1.2) Square 20 (80); (1.3) Pentagon 14 (56); (1.4) *n*-sided polygon 4 (16). 2. **Placing a point on the circle perimeter** 25(100) *alternatives*: (2.1) Placing a point inside the circle 25 (100); (2.2) Placing a point outside the circle 25 (100). 3. **Two heights are drawn from the point to the sides of the triangle** 25(100); *alternatives*: (3.1) Two angle bisectors are drawn 25 (100); (3.2) Two medians are drawn 25 (100); (3.3) Two perpendicular bisectors are drawn 10 (40); (3.4) One height and one median are drawn 3 (12). 4. **Looking for a location to point D in order for EF to be a maximum** 25(100);) *alternatives*: (4.1)Looking for a location to point D in order for EF to be a minimum 25 (100); (4.2) Looking for a location to point E in order for EF to be a maximum 17 (68); (4.3) Looking for a location to point E in order for EF to be a minimum 17 (68); (4.4) Looking for a location to point F in order for EF to be a maximum 17 (68); (4.5)Looking for a location to point F in order for EF to be a minimum 17 (68); (4.6) Looking for a location to point D in order for ratio between the area of ABC and DEF to be maximal/minimal 5 (20); (4.7) Looking for a location to point D in order for ratio between the perimeter of ABC and DEF to be maximal/minimal 5 (20); (4.8) Looking for a location to point D in order for ABC and DEF to similar triangles 4 (16); (4.9) Looking for a location to point D in order for DEF to be isosceles/equilateral/right/ acute/obtuse triangle 2 (8). **5. Triangle inscribed in a circle** 25(100) *alternatives*: (5.1)Triangle inscribed in a square25 (100); (5.2) Square inscribed in a circle25 (100); (5.3) Rectangle inscribed in a circle22 (88); (5.4) Triangle inscribed in a quadrangle21 (84); (5.5) Triangle inscribed in a rectangle 16 (64); (5.6)Triangle inscribed in a trapezoid 14 (56); (5.7) Pentagon inscribed in a circle12 (48); (5.8) Triangle inscribed in a pentagon 9 (36); (5.9) Triangle inscribed in a polygon 4 (16); (5.10) Triangle inscribed in a sphere 2(8); (5.11) Triangle inscribed in a cube 2(8); (5.12) Triangle inscribed in a pyramid 2(8); (5.13) Triangle inscribed in any spatial body 2(8**). 6. Any triangle** 18(72) *alternatives*: (6.1) Isosceles triangle 18 (100); (6.2) Equilateral triangle 18 (100); (6.3) Right triangle 18 (100); (6.4) Acute triangle 9 (50); (6.5) Obtuse triangle 9 (50).**7. Two segments are drawn from the point** 11 (44) *alternatives*: (7.1) Three segments are drawn from the point 11 (100); (7.2) *n* segments are drawn from the point 2 (18.18). **8. The segments that are drawn from the point are perpendicular to the sides of the triangle** 11 (44) *alternatives*: (8.1) The segments bisect the sides 11 (100); (8.2) The segments divide each side into *n* equal parts 2 (18.18). **9. Inscribed triangle** 7 (28) *alternatives*: (9.1) Circumscribed triangle 7 (100). **10. Circumcircle** 7 (28) *alternatives*: (10.1) Quadrangle inscribed in a triangle 7 (100); (10.2) Triangle inscribed in a triangle 3 (42.85). **11. Polygon** 6 (24) *alternatives*: (11.1) Circle 6 (100); (11.2) Parabola 1 (16.6)

Figure 2: Attributes and alternatives suggested by the PT

Although Phases 2 and 3 appear to require merely a technical work, in order to perform it well, the PT had to demonstrate mathematical knowledge concerning the formal definitions and characteristics of the negated attributes. For example, only six PT related to the attribute "polygon". This fact implies that most of the PT did not consider the formal definition of triangle, namely, "a 3-sided polygon". Referring to this issue during the class discussion, we realized that the PT related primarily to the visual aspects of triangle and not to its definition. This finding is consistent with Tall & Vinner (1981) regarding concept image and concept definition, and with the prototype phenomenon described by Hershkowitz (1989). From the PT's portfolios we realized that this phase enabled them to rethink geometrical objects, their definition and attributes. As can be seen from some of the PT's reflections: Anna (end of Phase 2): "*Analyzing the attributes helped me realize that there is much more in a problem then merely givens. Discussing each data component and its definition enables to rethink of definitions of mathematical objects and some interconnections between them".* Roy (end of Phase 2): "*The class discussion made me realize that there are so many attributes in one problem. Indeed I listed most of the attributes, but it is those which I didn't list that made me appreciate the richness that one can find in any mathematical problem*".

Phases 4. In this phase the PT had to concentrate on one of the alternatives, formulate a new problem, and solve it. Examination of the PT's portfolios reveals that 16 PT (64%) chose to focus on alternatives to attribute no. 5 (among them 14 chose "square inscribed in a circle"), 4 PT (16%) chose to focus on alternatives to attribute no. 3 (2 PT chose "two medians are drawn" and 2 chose "one height and one median are drawn"), 3 PT (12%) chose to focus on alternatives to attribute no. 7 (2 PT chose "Three segments are drawn from the point" and 1 on "*n* segments are drawn from the point"), 2 PT (8%) chose to focus on alternatives to attribute no. 4 (both replaced "maximum" by "minimum"). This finding implies that most of the PT chose as alternatives **common geometrical objects,** as square, median and height. Wondering why the PT demonstrated such a 'conservatives' behavior, we found the answer in their portfolios: Noa (beginning of phase 4): "*At the beginning Shir and I thought of choosing the alternative of triangle inscribed in a sphere [5.10], and later perhaps even a tetrahedron inscribed in a sphere. But then we realized that perhaps you expect us to prove what we are about to discover and whatever we will formulate as a conjecture. We were afraid that we will discover something that we will not be able to prove, and then all our efforts would be worthless*". From this excerpt it can be seen that teachers tend to overemphasize the importance of proof. As a result, the students feel insecure, and they do not dare to relate to mathematical inquiry as an 'adventurous' process. Rather, they tend to approach it in a hesitancy manner, believing that proof is the most important aspect of the mathematical doing. This attitude prevents the development of inquiry abilities.

Additional aspects on which the PT reflected concerned the validity of an argument and the meaning of definition. Gil (during Phase 4): "*At the beginning of the inquiry process we felt like we must first rethink of the mathematical meaning of the concepts* *involved. We asked ourselves whether the new problem situation is mathematically valid and everything is well defined. The truth is that we had never done this before. We were used to solve problems given by our teachers or the textbooks. In these cases there was no need to check the validity of any argument or to probe into the definitions of the objects involved. It was certain that everything is valid, well defined and solvable".*

To sum, reflective portfolios and class discussions turned out to be a useful tool for reflecting on processes, and tracing the PT's development of mathematical knowledge. We found that involvement in PP has the potential to develop the mathematical knowledge, and consolidate basic concepts, as suggested by Brown & Walter (1993) and English (1996). This development of knowledge came to fruition especially in the ability to examine definitions and attributes of mathematical objects, connections among mathematical objects, the validity of an argument, and appreciation of the richness that underlines mathematical problems. However, one major weak point was discovered. The PT tended to attach to familiar objects, and were not 'daring'. This tendency actually prohibits the development of problem solving skills and inquiry abilities instead of developing it. We found that this tendency can be explained by the redundant emphasize of the importance of providing formal proof. Teacher educators need to identify ways for reducing PT's fears from handling formal proves, and remove the focus to the analysis of a given situation, connections among mathematical objects and looking for generalization. This, in turn will develop their problem solving skills and their insights as regards to mathematical objects.

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ACTIVITY-BASED CLASS: DILEMMA AND COMPROMISE

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Experiencing mathematics through activity rather than rote memory and repeated practice has been emphasized. Textbooks introduce a variety of activities that helps students understand mathematical knowledge. Teachers should understand the content of, and the idea behind, activities first and then make them play out vividly in the classroom. However, activities often distract students' attention and divert the focus of the classes away from its original purposes. This paper aims to identify the dilemma and compromise of activity-based mathematics classes by examining the reflective review of a teacher on his classes about geometric figure movement and the discussion by three incumbent teachers on the classes.

INTRODUCTION

Korean mathematics curriculum puts an emphasis on activity-based mathematics classes (Ministry of Education, 1998). This is in line with research results claiming that activities, especially those based on manipulative material and technology facilitate mathematical learning (Thompson, 1994; Edwards, 1998). As Ball pointed out (1992), however, students don't always go beyond activities themselves and learn what their teachers intend them to learn. Cobb et al. (1992) argue that when teachers and students use manipulative material, their communication tend to focus on interpretation of the newly introduced material, not mathematical knowledge. In activity-based mathematics classes, it is not always possible to gain knowledge from activities and there is a constant danger of interest shift to activities themselves in a significant way. Of course, proper communication on concrete materials may give students an opportunity to discover mathematics on their own.

Dowling (1995) also argues that mathematics education as it is widely practiced is a mythologizing activity in the sense that it regards activities themselves as instances of, or representable by, mathematics. It is necessary to determine whether activities in class exactly represent mathematics or how distant they are from mathematics. The belief that activities alone will guide students towards mathematical knowledge is no longer sustainable.

Activity-based mathematics class has been emphasized and implemented in Korea since the 1990s. Now, it is time to analyse a variety of activity-based classes and find a new direction. This research aims to analyse classes on geometric figure movement by focusing on a teacher's analysis of activities in textbook, use of the activities in classes and students' perspectives. It also takes a look at the discussion by incumbent teachers about the classes on geometric figure movement and the teacher's reflective review. To achieve these objectives, this research was conducted focusing on the following two questions:

- What kind of use and understanding of textbook activities do teachers develop?
- What kind of difficulties or dilemmas do teachers find in mathematics classes while using activities to teach mathematical knowledge and what compromise do they reach?

THEORETICAL FRAMEWORK

The content of the teacher's analysis on textbook activities will be re-analysed in comparison with research results by Kang & Kilpatrick (1992), Dowling (2001) and Kulm et al. (2000). This research will identify how the teacher recognizes the changed nature of knowledge with the introduction of activities (Kang & Kilpatrick, 1992); how much he understands the goals and characteristics of activities in the textbook, and directness and indirectness of experience (Kulm et al., 2000); and whether he considers different interpretive frameworks between him and his students (Dowling, 2001).

The didactic situation concept introduced by Brousseau (1997) consists of the learners, the teacher, the mathematical content and the culture of classes, as well as the social and institutional forces acting upon that situation, including government directives in the national curricula documents, inspection and testing regimes, parental and community pressures and so on. He also introduced didactic contract which is about a kind of pressure or tension existing between teacher and learners. It is very important to decide if teachers and students abide by this contract while teaching and learning. If a teacher doesn't offer learners opportunities to explore mathematical knowledge, then he or she violates the didactic contract and that situation is not didactically appropriate Brousseau, 1997). This research examines what kinds of didactic situation and didactic contract do Korean elementary mathematics teachers seem to have. Use of activities in classes and students' perspectives will be analysed by using didactic situation concept introduced by Brousseau (1997); the didactic pole and the cognitive pole coined by Bartolini Bussi et al. (2005); the didactic transposition by Chevallard (1985).

METHOD

To find out how a teacher understand textbook activities and use in his or her classes, this research was performed following the research method that intentionally conducts the sampling of proper cases, observes and makes an in-depth analysis (Strauss & Corbin, 1990). K, the teacher who participated in this research and provided materials and issues on activity-based mathematics classes, has been teaching for 12 years but is not confident in activity-based mathematics classes. He says he is increasingly less confident as greater emphasis has been put on activities in curricula and textbooks. According to his fellow teachers, however, K always spends significant amount of time and energy in preparing for his classes and materials for activities. He wanted to know whether he has the right understanding of activity-based mathematics classes and what a typical activity-based math classes that most teachers are able to implement would look like. The three incumbent teachers, C, L and P, discussed K's classes from the viewpoint of potential and limitation of activity-based classes and tacit compromise among teachers. They have taught for 3, 7 and 15 years respectively, covered chapters taught by K and are very active in reviewing research results at a variety of teachers' study groups.

This research focuses on six hours of classes on geometric figure movement. The classes, interviews with K before and after the classes and a discussion by C, L and P were all videotaped and analysed. In addition, curriculum guidelines on geometric figure movement, activity sheets prepared for classes; activity sheets filled in and submitted by students and questionnaires given to the students were collected.

RESULTS AND DISCUSSION

K decided that he needed to teach his students about parallel transposition, symmetric transposition and rotational displacement without using coordinate in a way that is easily understood by elementary school kids by the appropriate didactic transposition (Chevallard, 1985). He analysed the textbook, the teacher's guide, and the curriculum documents to prepare lessons.

K's Analysis of Textbook and Reconstruction

K determined he would use activities because he had to teach third graders parallel transposition, symmetric transposition and rotational displacement without the use of coordinate. He recognized that a certain change was attempted in textbook to make mathematical knowledge easily understandable by elementary school students as analysed by Kang & Kilpatrick (1992).

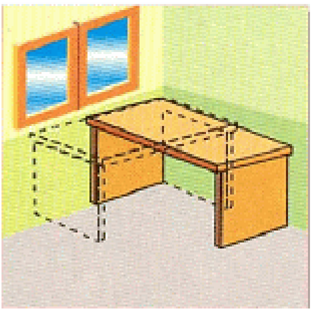

Figure 1

For example, a current textbook presents an activity of turning the desk as illustrated in Figure 1. K pointed out the activity doesn't represent rotational displacement in the mathematical sense. He also expressed his concern over the gap between textbook activities and mathematical knowledge they are supposed to represent cases of parallel transposition and symmetric transposition. K said none of the 11 textbook activities faithfully serves the original purpose and that he was worried about meta-cognitive shift as pointed by Kang & Kilpatrick (1992).

K decided it was crucial to teach the basic concept of moving geometric figures through activities although he was aware of the gap between textbook activities and mathematical knowledge. He thought moving geometric figures directly and checking the result was very important in classes. K's position could be interpreted as supporting textbooks that emphasize direct experience - one of the criteria suggested by Kulm et al (2000). He didn't regard learning from illustration or demonstration by the teacher as activity-based mathematics education. This implies that he knows the characteristics of the knowledge to be taught, in particular the dependence between the mathematical objects which must be taken into account in the creation of the coherence in the content to be taught and recognizes some constraints in didactic transposition of the knowledge in geometric figure movement (Brousseau, 1997; Laborde, 1989; Chevallard, 1985).

K often mentioned different interpretative frameworks between him and his students (Dowling, 2001). The current textbook encourages students to check the result of geometric figure movement by asking, “Has the geometric figure changed?” and this approach is perfectly reasonable from K’s perspective. However, K suspected his students would think that geometric figure movement and change in geometric figure are closely related and that the geometric figure has changed after movement, rather than remained the same.

K pointed out that while it is good for textbooks to present activities, they fail to explain why students should be engaged in the activities. This is why he decided to mention the reasons for studying geometric figure movement at the beginning of his classes and to illustrate how geometric figure movement is used in real life with a variety of examples. These examples included photos of cultural properties and buildings, and design-related materials on the Web.

Dilemma from simplified activities

At the beginning of his first class, K showed his students different patterns collected from the Web, especially those depicting the same repeated geometric figure and asked “what they feel and whether they recognize order in beauty.” He presented Korean traditional designs in tile roof, the *dancheong* patterns on the edge of the eaves, the *taegeuk* pattern and so on (*See* Figure 2), and pointed out that the “beauty of orderly patterns” has long been the focus of attention. K also asked his students, “How do you think the original creators came up with the designs? What might they have done before completing the designs? For example, they might have started by drawing something first on a piece of paper.”

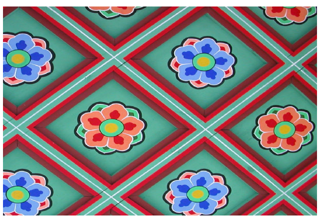
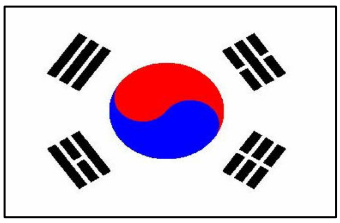

Figure 2: Pictures shown on the screen (*Dancheong* and *Taegeuk*)

Then, K asked his students to think about the learning points of the chapter on geometric figure movement and the meaning of the word movement, and told them about how the knowledge in the chapter is applied such as in making wall papers and floor materials and designing scarfs and clothes. After that, he projected examples of geometric figure movement on a screen. With the screen showing activities in textbook, K had his pupils check whether the shape of the geometric figure on the screen changes when it is moved in four different directions-upward, downward, left and right. K then handed out a piece of paper with a geometric figure drawn on it to each of his students and asked them to move the geometric figure in four different directions and determine whether the shape of the geometric figure changes. This was when an interesting debate took place between two kids at the back of the classroom.

Student 1: (She moves the geometric figure diagonally.) This is also geometric figure movement, right?

Student 2: No, you can only move it upward, downward, left and right.

Student 1: No, I don't think so. It is just that the textbook shows only four ways of moving the geometric figure. Let's ask him. (In a loud voice) Mr. K, this is also geometric figure movement, isn't it?

K: Huh? Did you move it to the right?

Student 1: No, to the upper right.

K: To the right?

Student 1: Not just to the right. I move it to the right and also upward.

K: (He approaches to Student 1.) What do you mean by that? (He looks the student's action.) Yes, that is also geometric figure movement in reality….(After a pause), but you need to choose between right and upward in this mathematics class. (After a pause) OK, has the geometric figure changed or not after being moved?

Student: It hasn't.

K: Then it's OK. It's alright as long as you understand that the shape of the geometric figure has not changed after movement.

The current textbook does not clearly explain what movement means in the mathematical sense. It only focuses on geometric figures not changing after being moved upward, downward, left and right and doesn't specify how much and in what direction the geometric figures should be moved. It is a simplified form of knowledge in geometric figure movement so that teachers hardly find appropriate further explanation to add.

K didn't seem to control the situation in the above dialogue and almost gave up teaching about the reason why the word movement is interpreted in that way, which can be interpreted as the situation closes to a-didactical one in the sense that K transmitted the responsibility of handling knowledge to the learners (Brousseau, 1997). C, one of the discussants, recalled a day when he taught in a second grade class while analysing this episode. When he asked, "What has changed after we placed a geometric figure we drew on the windowpane and open the window?" a student answered, "It is cooler because of the breeze coming in." This episode shows the students didn't understand the intention of the question and the activity. In K's classes, the definition of the geometric figure movement activity was changed by the students' knowledge and experiences regarding the linguistic meaning of movement (Bartolini Bussi et al., 2005).

Dilemma from the distance between mathematics and activities

K decided it was important to draw the result of geometric figure movement on section paper. The textbook has students transfer a geometric figure to transparent paper, place the transparent paper on section paper and transfer the geometric figure again by plotting points. K thought this process of transfer to transparent paper and re-transfer to section paper would have no meaning to his students since it doesn't give any insights into the underlying mathematics. So he changed the activity to drawing the moved geometric figure without transparent paper while stressing the concept of basis line. At first, K explained by using transparent paper but introduced the basis line concept.

K:	What do we need to draw a moved geometric figure?
Students:	A triangle.
K:	No, I mean what we need to know before movement. A basis line, right? (He shows the basis line on the screen.) Then the only thing we need to consider is the number of squares located between the basis line and the figure. How do we do it? (Projecting the picture on the screen) Take a look at this geometric figure. As I mentioned, this line serves as the basis. Look at the line. How do we draw the moved geometric figure here? Think of it as being reflected in the mirror. We should leave one column blank on each side and draw the geometric figure by using one, two, three rows to make it the same as the one on the opposite side. Do you see how important a basis is now?

K tried to make his students understand axis of symmetry to a certain degree by introducing the basis concept that is never mentioned in the textbook. When he explained about symmetric transposition, K only turned around section paper in the air. However, he deliberately used material with a basis marked on it to show his students how to describe the result of geometric figure movement. This is a didactic device for using textbook activity as the first artifact and utilizing language and picture to move toward the second artefact (Bartolini Bussi et al., 2005).

K emphasized on the procedural knowledge of describing the result of textbook activities mathematically, whereas textbook focuses on the procedural knowledge of describing the activities in terms of the use of concrete materials including transparent paper. Thus his students participated in the different activities under K's direction. In the discussion on K's classes, L and P said that K was, in a way, teaching a fixed scope of mathematical knowledge when using activities. This, they concluded, prevented a variety of approaches from taking place while making his classes stable and poised despite the use of activities.

Dilemma from visual representation of activity

The chapter on rotational displacement in the textbook did not specify rotation angle like 90° or 180°. Instead, it depicted the degree of the rotation on a round-shaped figure. K explained in detail how to describe this depiction.

K:	Do you see the clock-like thing in the middle? What time is it? I mean, it is not a real clock, but it looks like 3 o'clock, right? To 3 o'clock, until it reaches 3 o'clock. No, not 3 o'clock, but the angle that represents 3 o'clock. You know what a right angle is, don't you? Three o'clock is a right angle, right?
Students:	Yes.

K: As much as the right angle to the right. That is··· (demonstrating a rotation) to move, no, to rotate up to this point until it forms a right angle. How many times has a rotation been done to the right? Let's call it a half of the half rotation from now on.

K believes in socio-cultural perspective in mathematics learning (Cobb et al., 1992). However according to his analysis, the textbook didn't offer him suitable tools to communicate with students on how much figure is rotated. When he decided that the visual depiction in the textbook was not enough for proper communication, K developed a linguistic depiction such as a half of the half and taught it to his students. They practiced and internalized the expressions presented by K and put them together to form mathematical knowledge.

In K's classes, the students thought and learned based not on visual depiction in the textbook, but on additional expressions that K created. There were few instances where a student devised a new expression or depended on his/her own interpretation during activities. C, a discussant, argued K deprived his students of an opportunity to express and internalize their own thought processes while P said that the activities were implemented in a significant way because K skilfully distinguished different and confusing concepts, which can cause so called Topaz Effect (Brousseau, 1997). Interviews with the students found they clearly remembered detailed guidelines that K presented and saw the guidelines as important. This gave an impression that the activities were regarded somewhat as an established piece of knowledge.

CONCLUSION

Activities implemented by K were not identical to those in the textbook by his own didactic transposition (Chevallard, 1985). After his classes was over, K said, "I agree with the idea behind activity-based mathematics classes but believe we should guard against a situation where students engage in activities without much thought. If detailed procedures are not presented, students will be lost in activities without gaining mathematical knowledge. The teachers who participated in the discussion also share a view that "focusing on the activities themselves weakens attainment of mathematical knowledge and vice versa." This is the dilemma of activity-based mathematics classes recognised by K, which represents the gap between the knowledge to be taught and the knowledge taught (Chevallard, 1985).

As K did, many Korean mathematics teachers reconstruct textbook activities into different detailed activities and ensure that the approaches taken by students are not overly varied. C described this as an indispensable approach because they often find dilemmas in their activity-based classes. P expressed this as a compromise shared among teachers. K argued that it is necessary for teachers to discuss on the dilemmas and the compromise by themselves since teachers have a very different view of the uses and purposes of activity. Activities are essential in teaching mathematics, but their effective use is not always straightforward and they can work against the didactic intent. Identifying dilemmas and compromise in activity-based classes recognised by teachers can be a significant way of professional development in teacher education and bridging the gap between the theoretical and the empirical approaches in mathematics education.

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INDUCTION, ANALOGY, AND IMAGERY IN GEOMETRIC REASONING[1]

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Mathematical thinking that advances into generalization through induction, analogy and imagery is an important tool with which mathematicians find mathematical principles. Mathematically gifted students, also, need to experience this thinking process. This research is focused on followings: how mathematically gifted 6^{th} and 8^{th} grade students utilize induction, analogy and imagery in their geometric reasoning; how the problems that were developed to give impetus to the diverse thinking of students are solved using what strategy of the gifted students in actuality; and whether they are solved through the thinking types and paths predicted by the researchers through thought experiment are observed.

INTRODUCTION

According to the studies that observed and analysed the thinking characteristics of mathematically gifted students (Heid, 1983; Presmeg, 1986; Sriraman, 2003, 2004; Lee, 2005), mathematically gifted students efficiently utilize the problem-solving strategies including generalization, simplification, visualization as necessity arise, grasp the meaning and structure of a problem in a very short time and solve it progressively. Sriraman (2003) reported that gifted students invest considerable time in understanding the meaning and condition of a problem and their thinking behaviour including creative problem solving, generalization, formalization, etc. correspond to those of mathematicians. Lee (2005) found that gifted students have the tendency to advance into the higher-level reasoning through the reflective thinking about their early reasoning.

Polya (1954, 1962) emphasized induction and analogy as very important mathematical reasoning faculties. A study on the meaning and development of induction (Holland et al., 1986) and those on the meaning and development of analogical reasoning (English, 1997; Alaxander et al., 1997), show how much important elements induction and analogy are in the development of logical thinking. Studies have been made also on the important role of the mathematical imagery in mathematical learning (Wheatley, 1991; 1997; Presmeg, 1992). Apparently, induction, analogy and imagery seem to be great tools that make one feel the beauty and strength of mathematical thinking. However, studies into which tasks and which teaching methods can stimulate such reasoning and

[1] This work was supported by Korea Research Foundation Grant funded by Korea Government(MOEHRD, Basic Research Promotion Fund) (KRF-2005-079-BS0123)

those into in what way mathematically gifted students combine or uniquely utilize those reasoning elements are insufficient. The objective of this research is to obtain detailed information on the way mathematically gifted students utilize induction, analogy and imagery in their geometric reasoning. To achieve this objective, this research was conducted focusing on the following two questions:

- How do the mathematically gifted students utilize induction, analogy and imagery in their task solving process?
- What role do induction, analogy and imagery play in making mathematical discoveries?

THEORETICAL FRAMEWORK

Polya (1954, 1962) insisted that exploration into a polyhedron can be made briskly through induction by suggesting the question, "Is it generally true that the number of faces increases when the number of vertices increases?" In this research, too, to induce the students to try generalization through induction, a question of the similar structure will be used while teaching.

As assumed in the experiment of Alaxander et al. (1997), students might fail to utilize analogy in a desirable way as they pay attention to surface-level similarities. The task used in this research needs relational or structural analogy; so it is expected that students might pay attention to surface-level similarities even though the students were identified as gifted in mathematics. If so, we will closely observe in what type of similarity they take notice of and what role induction and imagery play at those times.

Presmeg (1992) argued that image exists in diverse forms including concrete, dynamic, pattern, abstract, etc., playing diverse roles. In this research, also, in anticipation that students will form diverse shapes of image in accordance with individual experience, habit and preceding knowledge, will look into how they are utilized in problem solving, particularly in the discovery of mathematical idea.

The results of the studies on induction, those on analogy and those on imagery (Holland et al., 1986; English, 1997; Wheatley, 1991; 1997) suggested specific process of mathematical thinking a learner might experience. In this research, based on the results of preceding studies, attention will be paid more to the relations between the three thinking elements.

RESEARCH METHOD

Participants

To find out how mathematically gifted students utilize induction, analogy and imagery, this research was performed following the research method that intentionally conducts the sampling of proper cases, observes and makes an in-depth analysis (Strauss & Corbin, 1990). The subjects of this research are three 6^{th} graders (age 12) (E1, E2, E3) in elementary school, three 8^{th} graders (age 14) (M1, M2, M3) – all of them are receiving education for the gifted in an academy for the gifted attached to a university.

Tasks

The tasks were prepared by the researchers either by reviewing, of the existing studies on the fields of geometry, algebra, probability and statistics, those that paid attention to the improvement of mathematical thinking ability and then partially revising them or through new development. Teaching materials with which a polyhedron can be easily made were provided for the subjects to use in the task-solving process. The tasks that will be mainly analysed in this paper fall under the field of geometry and are as follows:

- [Task 1] The sum of all the interior angles of a triangle is 180°. Can you find a similar property in a tetrahedron? Make tetrahedrons with the tools in front of you as needed; observe them; and find the similar statement or property.
- [Task 2] As to a solid that has n-number of faces, can the sum of the internal angles of the polygons that compose each face be generally worked out? Why do you think so?

Both the two tasks were developed for the objective of making students discover mathematical ideas and justify them using induction and analogy. The intention was to make them experience induction and analogy while solving task 1, and apply the experience to solving task 2. So task 1 was provided with a view to have the students learn the thinking pattern required to solve task 2. In the case of imagery, since it is hard to make specific anticipation about it in advance, it will be arranged by classifying the students' responses.

Procedures

Nine units in three education programs for each field of geometry, algebra, probability and statistics were provided to them. Elementary school students and middle school students participated in the research being separated from each other; and for each student one research assistant was assigned to conduct concentrative observation and interview. Each teaching unit program lasted for three hours; all the responses of the students were audio-/video-taped; and background information including their activity record, home background, etc. was also collected.

Considering that the students' mathematical thinking cannot be completely expressed in language, data was analysed taking heed of such non-linguistic responses as facial expression, behaviour, etc. The interaction between the students was allowed only if necessary: the students were to solve most of the tasks in accordance with the individual habit or strategy. Their thinking characteristics were analysed inductively focused on the scene where induction, analogy and imagery are specifically linked to problem solving, particularly on how the three kinds of reasoning are linked to each other.

RESULTS

The mathematically gifted students were identified as very promising in mathematics by the selection process run by the university professors of the gifted centre. However their performance in this study was not so good unlike the researchers' expectation which proposed the changes in the selection process. In the participants' geometrical reasoning, how induction, analogy, and imagery were revealed and connected in some way is explained in the following table:

Task	E1	E2	E3	M1	M2	M3
1	Imagery ↓ Analogy	Induction	Imaging	Imaging	Induction	Imagery ↓ Analogy
2	Imagery ↓ Induction	Imagery ↓ Induction	Imagery ↓ Induction	Analogy ↓ Induction	Induction	Induction ↓ Imagery ↓ Analogy

Table 1: Emerged Reasoning Patterns

In case of trying just induction without utilizing analogy and imagery, appropriate geometrical reasoning was not completed. As a result, the focus of the task was missed (M2). In spite of the fact that the task was asking for induction, a student who succeeded in analogy based on suitable imagery without using induction could resolve the task (E1, M3). Compared with elementary students, middle school students try not to reason based on imagery. When imagery is combined with induction or analogy, the conversion of thinking was realized rapidly. When induction and analogy was combined or at least utilized at the same time, inductive analogy and analogical induction were not found.

Surface-level analogy and blind imaging

Task 1 is making and observing various kinds of tetrahedron and to find out its specific features on the sum of some angles. In spite of the fact that the task was asking to observe various tetrahedrons, some students just made one tetrahedron and didn't make other tetrahedrons any more. Focusing on just one tetrahedron, they changed the imagery about angle and tetrahedron. Based on such imagery, they tried to induce. The following is a part of conversation made between M1 and a researcher.

M1: Here, this angle is made of three dimensions!

Interviewer: (pointing at the angle of the picture drawn by M1) Are you talking about this interior angle?

M1: (pointing at the angle of the tetrahedron she made) Yes, that angle made of three dimensions is what I am talking about.

Interviewer: How do you know the degree of that angle?

M1: That is the problem. I can't find it. But I think someone can.

M1 didn't care about the relation or structure in the given statement about the sum of interior angles of a triangle. She spent most of the time thinking how to measure an interior angle of a tetrahedron that she defined as the counterpart of an interior angle of a triangle. The 'three dimension angle' within the tetrahedron is not approachable by the existing tetrahedron image that the student has. The student just noticed the new image form such as the angle that one edge meets another edge or one face meets another face. The image of angle is in the process of expanding from a plane figure to a solid figure but the student failed to exactly capture the meaning. She couldn't explain why she had to know it. Also how the change of angle is connected to the kinds of tetrahedron was not clear to her.

Surface-level analogy and blind induction

E2 tried to observe and analogize various kinds of tetrahedron including regular tetrahedron according to the guidelines of task 1. However, he couldn't find the common feature by noticing each character that various cases have. Away from the original point, 'the sum of the interior angles' that should be analogized, he showed interest in the changes of number such as an edge, a vertex, etc. So he failed to draw a meaningful conclusion on the sum of some angles, although he continuously tried induction of what he observed. In case of M2, he made various kinds of polyhedron to resolve task 2, but the analysis on such polyhedrons was not done systematically. So he couldn't draw any conclusion.

If induction for various kinds of tetrahedron and polyhedron is not combined with proper imageries, it can't be connected to the proper reasoning, losing the direction. In other words, trying induction without direction strays from the essential.

Relational analogy fuelled by proper imagery

Student E1 also focused on the triangle of each face after making just one regular tetrahedron using the teaching manipulative while resolving task 1. He also didn't show any interests in making various kinds of tetrahedron. However the proper imagery he developed led to analogy. The following is a part of conversation made between E1 and a researcher.

E1: A regular tetrahedron has 4 regular triangles. So the sum of angles times 4 is 720.

Interviewer: What about a general tetrahedron?

E1: A tetrahedron may have a triangle and a quadrangle.

Interviewer: A triangle and a quadrangle?

(After a long pause for imaging the development figure of a tetrahedron)

E1: No, no. A quadrangle can't be fit in. If a quadrangle enters, there are not enough edges. We need more edges to connect this part (draw some figures in the air). If you connect one, it already becomes 5.

The above conversation shows that the student doesn't make other kind of tetrahedron and imagine the scene making any tetrahedron to resolve the task. Especially to explain each face of the optional tetrahedron is a triangle, he suggested a reasonable logic by imaging the solid figure and the development figure of it in his thought including a quadrangle without using induction. Unlike M1, E1 is focusing on each face's polygon rather than an interior angle. This means that the development figure of the tetrahedron is being used as an image.

The crucial meaning included in the first statement of task 1 is that the interior angles of a triangle can be different, but the total of three angles is consistent. By relational or structural analogy, he discovered the fact that the kinds of triangle compose the tetrahedron can be changed, but with only 4 triangles one can compose any tetrahedron. He exactly analogized with not just isolated element about an interior angle itself but the relation or structure included in the statement. Based on such relational or structural analogy, he resolved the task. The development figure and the shape of each face in tetrahedron seem to play a key role in the relational analogy.

Relational analogy fuelled by essential induction and proper imagery

M3 also didn't feel any necessity to check various kinds of tetrahedron when he resolved task 1. Like E1, he analogized the character of the tetrahedron, using the development figure in a tetrahedron. As for task 2, he observed various kinds of polyhedron. He suggested an example that although the number of face is the same, the sum of interior angles in a polyhedron can be different. At this point, he noticed a quadrangular pyramid and a triangular prism with 5 faces. And then he checked whether he could get the general features of each pyramid and prism. He checked that the number of vertexes and faces in the n-pyramid is n+1. Also he checked that the number of vertexes in n-prism is 2n and the number of faces in n-prism is n+2. For each case, he found the formula to get the sum of interior angles of a polygon in each face.

The development figure which was utilized in resolving task 1 played a key role in advancing into the analogy from induction upon various polyhedrons for M3 to resolve task 2. M3 drew the development figure of any polyhedron and focused on each plane figure that composes the development figure. He made efforts to get the formula to total the interior angles of a polygon in each face. He analogized the changes of each face by disassembling special cases including a regular polyhedron, a triangular prism, etc., with the development figure and each plane figure. The following is a part of conversation made between M3 and a researcher.

M3: Let me explain it with a regular tetrahedron. The total edges are 6. We have to get the angle of each face. So if we separate each face, we get 4 triangles. There are 12 edges, so it's two times of the total edges. The total of the interior angles of each face is n-2 times π , so if we put the number of edges as n.

Interviewer: Do you put the number of edges as n? Why do you think about the number of edges?

M3: Imagine that we remove all the faces.

(He tries to draw the process of dismantling faces in the air.)

M3: Then the number of edges doubles because we remove faces from here and there.

M3 found out that if we dismantle an ordinary polyhedron by polygon, the total of the interior angles for each polygon located in each face can be calculated. Also he found out that it can be expressed as one formula by connecting the total number of faces and edges. Analogizing from task 1 and systematically inducting and utilizing imagery properly played the key role in resolving the task 2.

DISCUSSION AND CONCLUSION

The suggestion of Polya (1954, 1962) is that the character of a solid figure can be found by analogizing the character of a plane figure and reach to the generalization by induction. After applying this suggestion to the mathematically gifted students, only 2 among 6 students showed the expected response (E1, M3). Some students spent most of the time finding out the meaning of the interior angle of a tetrahedron (M1), or just observing various cases without a system (E2, M2). Those students couldn't draw out any new character about a solid figure. Without relying on induction, rather than trying the deep analysis on some example or trying the inference based on image reached the successful analogy (E1, M3). Especially changing the imagery dynamically and utilizing it in induction and analogy played a key role in discovering the mathematical idea and its generalization (M3). In the study of Alaxander et al.(1997), many students could utilize analogy by choosing a very similar calculation question in terms of structure. However, this study tried the analogy between very different structures such as a solid figure and a plane figure, which made even the mathematically gifted students get lost in analogy. Thus more experience of tackling this kind of tasks is suggested to be introduced in gifted education.

Imagery seems to provide a very important base for developing structural analogy to resolve the tasks in geometry. Because the structure among objects should be grasped to enable analogy, so some specific imagery among various features of geometric objects plays a key role. Activities with imagery, such as drawing, writing down, or describing verbally the spatial imageries students used when solving tasks (Wheatley, 1991) also needs to be heavily considered in gifted education.

Induction itself seemed to be difficult to be utilized properly to resolve the tasks in this study. E2 and M2 tried to find many cases, but they had difficulties in generalizing the results. They couldn't grasp the relation between the objects that observed. Just drawing out the isolated features, they couldn't systemize them and utilize the image. As a result, they couldn't draw a desirable induction or analogy. Like the preceding studies on the role of image (Wheatley, 1997; Presmeg, 1992), some students in this study utilized or changed existing images to utilize or apply induction or analogy. However many students are needed to develop tendency or ability to use imageries.

The relation between induction and analogy is not clear. We predicted that induction about many cases could be the base of analogy, but in reality, the deep analysis on one case developed a strong tendency to lead to analogy in this research. There was a case that tried to start from analogy and reach induction (M2), but the student failed in resolving the task because his analogy and induction were both incomplete. Induction can be the first tool when solving geometric problems, but it needs other reasoning skills such as analogy or imaging simultaneously. If we analyse the responses of students about algebra, probability and statistics, the relation among induction, analogy and imagery will be clearer.

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THE ANALYSIS OF ACTIVITY THAT GIFTED STUDENTS CONSTRUCT DEFINITION OF REGULAR POLYHEDRA[1]

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This study was conducted with the focus on the process of constructing a definition and produced definitions rather than gifted students' conceptions of a mathematical definition. Accordingly, instead of a mathematical subject that students would come into contact with as part of the curriculum or in their ordinary lives, this study used regular polyhedron as its subject matter which students are not familiar with even if they may have encountered it in their ordinary lives. In this study, students were asked to make platonic polyhedra, observe them and then construct a definition of regular polyhedron based on their observations. We sought to gain various suggestions through the analysis of the observations and definition laid down by the students and through the characteristics shown by the students in the process of defining the concept.

INTRODUCTION

There have been many different opinions regarding the definition of giftedness and gifted children and many scholars have had different perspectives about them, as the criteria for giftedness and gifted children have differed with the changes in the times, cultural and social values (Song, 1998). For the purpose of this study, gifted children are restrictively defined as children who was selected as a gifted children by experts of institute for science gifted education supported by the government.

Up until now, studies on characteristics on the way of thinking of mathematically gifted students have focused on generalization, abstraction, justification, reasoning ability, etc. that are at play during the process of problem-solving and proving (Krutetskii, 1976; Lee, 2005; Sriraman, 2003; 2004).

Definition accounts for the important part of mathematics and mathematics education (Harel, Selden, & Selden, 2006; Ouvrier-Buffet, 2002; Shir & Zaslavsky, 2001) and the significant roles that definition plays in grasping mathematical concepts, solving problems and proving have been emphasized by numerous researchers (Shir & Zaslavsky, 2002; Skemp, 1971; Vinner, 1991). In mathematical learning, construction of definition rather than the provision of constructed definition is regarded as

[1] This work was supported by Korea Research Foundation Grant funded by Korea Government(MOEHRD, Basic Research Promotion Fund) (KRF-2005-079-BS0123)

important as problem solving, guessing, generalization, and proving (De Villiers, 1998; Mariotti & Fischbein, 1997; Ouvrier-Buffet, 2004; 2006).

This study seeks to deal with another aspect of mathematically gifted students by analysing the characteristics displayed by students in the process of making five types of regular polyhedra, observing them and constructing the definition of a regular polyhedron. This study is also intended to give some suggestions as to the selection of mathematically gifted students and the curriculum for their education.

BACKGROUND

Kang & Cho (2002) have identified 5 definition-methods that are used in the geometry of school mathematics - *synonymous, denotative, implicative, constructive, analytic*. And they categorized them into practical and scientific methods. The former 3 definition-methods are classified as practical whereas the latter 2 definition-methods are classified as scientific. In making definitions, practical methods select directly perceived attributes and directly useful characteristics while scientific methods select 'causality', 'generation' or 'relationship,' which show how things are mutually dependent on one another and how they interact mutually. Accordingly, the latter methods enable us to identify connectivity between the discrete pieces of information. In this study, we used Kang & Cho (2002)'s study to examine whether students depended on directly perceived attributes or took notice of the relationship between the components that make up a regular polyhedron.

Fischbein (1987) claimed that examples play a core role in intellectual activities and emphasized the importance of denotative method. According to him, 'paradigmatic model' is basically an example but is beyond a mere example. An example of a concept refers to the object that carries all the attributes of the concept. Example as paradigmatic model not only carries all the attributes of a concept but also plays a core role in intellectual activities. Skemp (1971) attached a particular significance to conceptual learning as it lays foundation for higher level of mathematical principles and problem solving. According to him, the best way to administer conceptual learning in mathematics is through inductive reasoning whereby proper examples related to the concept at hand are presented to help students identify commonalities of the examples and construct the concept from the commonalities. This study is based on Fischbein (1987) and Skemp (1971) and determines that platonic polyhedra give a proper situation for constructing the definition of a regular polyhedron.

In this study, students participating in this experiment were asked to make platonic polyhedra using materials (Znodome system), observe them, and construct a definition of each regular polyhedron based on their observations. That is, they were asked construct a definition through inductive reasoning using examples.

Abstracting is one of the most important things and generalizing and synthesizing form a prerequisite basis to abstracting. Generalization is to derive or induce from particulars, to expand familiar processes, and abstracting is constructive process building mental structures from properties of and relationships between mathematical objects (Dreyfus, 1991). According to Dreyfus (1991), this process depends on the isolation of proper properties and relationships. Such constructive mental activity on the part of a student depends on the student's attention being focused on those structures which are to form part of the abstract concept and drawn away from those which are not relevant in the intended context. Synthesizing means to combine or compose parts in such a way that they form a whole, an entity. Unrelated facts hopefully merge into a single picture, within which they are all composed and interrelated. This process of merging into a single picture is a synthesis.

METHODOLOGY

Participants

Participants in this study are 21 intellectually gifted elementary school students in the 5^{th} grade (11 years old) - 14 boys and 7 girls - who are being instructed under the program of institute for science gifted education attached to a National University and supported by Korean government. But they had no This institute selects gifted students through 3-stage steps : (1) Recommendation by a principal, (2) Testing of students by experts on high level of mathematics problem solving, and (3) Testing of students by experts on abilities on a solve problems requiring ingenuity. The students in this institute educated 60 hours in science and 42-hours in mathematics for one year. Math programs, which are 3-hour long each, deal with various fields such as algebra, geometry, probability, etc. with the focus on improving students' abilities on problem-solving, reasoning, and justification. We have confirmed that these students did not experience any class previously on how to construct definition on a certain concept through the regular curriculum of education either at their schools or at the institute.

Activities

The teaching experiment designed for this study was part of the regular curriculum of this institute and was administered to the students for three hours on end after dividing the students into 3 groups. The experiment consisted of 4 steps and the details on these steps are as follows.

Step 1: Making regular polyhedra (in group). For starters, pictures of platonic polyhedra were presented to the students. Each group was asked to make the regular polyhedra using the materials based on these pictures.

Step 2: Observing regular polyhedra (in individual). Students were asked to observe the regular polyhedra that their group had made and record their observations (characteristics, attributes, etc.) about each type of regular polyhedron in the activity sheet. They were asked to record as many observations as possible and a mutual discussion within each group was permitted.

Step 3: Defining regular polyhedra (in individual) Students were asked to construct a definition and record it on the activity sheet based on the observations they have made in step 2.

Step 4: Further develop their definition (in group) Students of each group engaged in a group discussion to refine the definition of regular polyhedron constructed by individual members of each group in step 3.

RESULTS

In this study, analysis was conducted with the focus on step 2 and 3. The main data to analysis was the activity sheet that students prepared and one-on-one interviews were conducted in case the need arose to clarify certain terms and expressions used by students. The discussions in this study were not classified and organized by a conventional framework but, rather, are the results that were derived through an inductive method based on the responses of the students (Denzin & Lincoln, 1994; Goetz & LeCompte, 1984).

According to analysis results, students were categorized into three groups based on the relationship between the responses of students in step 2 of observing five platonic polyhedra and recording their observations and the responses of students in step 3 of defining a regular polyhedron based on these observations. Part of the responses shown by students of each group is as follows:

<table>
<tr><th colspan="2" rowspan="2">Group (sample students)</th><th colspan="5">The critical components of the figures by students in step 2</th><th rowspan="2">The definitions by students in step 3</th></tr>
<tr><th>Regular 4-hedron</th><th>Regular 6-hedron</th><th>Regular 8-hedron</th><th>Regular 12-hedron</th><th>Regular 20-hedron</th></tr>
<tr><td rowspan="2">Group 1</td><td>S1</td><td colspan="5">The shape of face. The number of vertex and edge.</td><td>A regular polyhedron has the same area of faces and the same length of sides. The angles formed by adjacent edges are the same.</td></tr>
<tr><td>S2</td><td colspan="5">The length and number of edge. The angle formed by adjacent edges. The number of vertex. The shape and number of face.</td><td>A regular polyhedron has sides of the same length, angles of the same size and faces of the same area.</td></tr>
<tr><td>Group 2</td><td>S3</td><td colspan="5">The length and number of edge. The number of vertex and face. V+F-E=2.</td><td>A regular polyhedron is a solid that has edges of the same length and faces of the same area. V-E+F=2. The numbers of edges, vertices and faces are all even.</td></tr>
</table>

Group 3	S4	The number of face and diagonal.	The number and shape of face. The number of vertex.	Making method. The number and shape of face. Looks like a top.	The number and shape of face. The number of vertex. The number of vertex to meet on each vertex.	Making method. It has many edges and skewed parts.	It should have faces. The numbers of vertices are four, eight and twelve. That is, the numbers are multiplied by 2. The numbers of the edges are multiples of six. It should consist of plane figures.

Table 1: The critical components of the figures & definitions by students

[On group 1] Students in Group 1 constructed a definition that is logically congruent with the mathematical definition. Not only they took notice of the critical components but also the components observed were consistent in observing the characteristics of each regular polyhedron in step 2. For examples, student 1 recorded their observations with a consistent focus on the number of edges and vertices, the shape and number of faces that make up the regular polyhedra while student 2 recorded his observations with a focus on the size of angles formed by two adjacent edges, area of faces, the number of vertices and edges, and the number and shape of faces.

Definition by student 1 satisfies the mathematical definition of a regular polyhedron, “A regular polyhedron is composed of congruent regular polygons”. The statement that “All the faces are the same area while all the edges are the same length” means that all the plane figures that make up a regular polyhedron are congruent. An addition of the statement that “the angles of the adjacent edges are the same” indicates that all the plane figures are congruent regular polygons. Though this definition an enumeration of the factual observations that represent the characteristics of regular polyhedra, this satisfies the mathematical definition of a regular polyhedron. That is, they tried to present sufficient conditions that are needed to make a mathematical definition. The definitions of student 2 also satisfy the mathematical definition of a regular polyhedron.

It was confirmed, through student 1 and 2 that students in group 1 grasped the relationship between the components that make up a regular polyhedron in observing five types of regular polyhedra, and they perceived regular polyhedra’ structures so that recognized the fundamental attributes of regular polyhedron. They recognized the critical components that make up a regular polyhedron and defined with the relationship between the components. In addition, in defining a regular polyhedron, student 1 recognized the fact- angles formed by two adjacent edges are the same size - that were not expressed in step 2, and used the fact to define a regular polyhedron. It shows that the student recognized the relationship between the components that make up a regular polyhedron and the fundamental attributes.

[On group 2] Students in Group 2 constructed a imperfect definition. They took notice of the critical components and the components observed were consistent in observing the characteristics of each regular polyhedron as the students from Group 1 did. Foe example, student 3 observed the regular polyhedra with a consistent focus on lengths of sides, number of vertices, number of faces and edges, the relationship (vertices + faces –edges = 2) between the numbers of vertices, faces and edges. Even though the definition of student 3 used the results of observations appropriately, it simply enumerates observations in a superficial fashion while failing to identify the relationship between the components that make up a regular polyhedron. Thus, this definition includes part of Catalan polyhedron in addition to a regular polyhedron.

[On group 3] Students in Group 3 had difficulty defining a regular polyhedron. Student 4 was not able to put a consistent focus when observing platonic polyhedra in step 2. That is, he was not taking a systematic approach. Though student 4 observes regular polyhedra with a focus on number and shapes of faces, number of vertices, number of edges that meet on a vertex, number of diagonals, how to make regular polyhedra using materials and overall shape, etc., the student's focus changes depending on the type of the regular polyhedron - number of faces and diagonals in the case of a regular tetrahedron and number & shape of faces and number of vertices in the case of a regular hexahedron. Accordingly, this student presents the number of components instead of defining a regular polyhedron by identifying the relationship between the components that make up a regular polyhedron.

CONCLUSION

According to the studies of Shir & Zaslavsky (2002) and Zaslavsky & Shir (2005), students showed a tendency not to adopt the definition that uses something other than critical components that make up the figure (faces, edges and vertices in the case of a regular polyhedron). However, students of Group 1 voluntarily used things other than critical components of a regular polyhedron to construct definition. It shows that students recognized the relationship between the components that make up a regular polyhedron and the fundamental attributes. Students of Group 1 could not only generalize and synthesize the facts that were observed through platonic polyhedra but also abstract a regular polyhedron by grasping the relationship between the components that make up a regular polyhedron and capturing the critical characteristics of a regular polyhedron. Thus they were able to construct a definition that is logically congruent with the mathematical definition.

Students from Group 2 succeeded in generalizing the facts that were observed through platonic polyhedra but they failed to synthesize the facts that were observed and abstract a regular polyhedron by capturing the critical characteristics of a regular polyhedron so that they were not able to construct a complete definition. Kang & Cho (2002) argued that for a learner to be able to define things through examples, attentiveness and comprehensive faculty are prerequisites and Dreyfus (1991) said that abstracting is possible when the student's attention is focused on those structures which are to form part of the abstract concept, and drawn away from those which are not relevant in the intended context. In constructing definition of a regular polyhedron, students of Group 2 seems to be lacking in understanding needed to recognize the relationship between the components and in attentiveness needed to distinguish between different types of polyhedra.

According to the study of Mariotti & Fischbein (1997), students who tend to view geometrical figure as visual gestalts have a tendency to rely on unnecessary characteristics while overlooking decisive characteristics of the figure. Students from Group 3 had difficulty defining definition of a regular polyhedron as they failed to generalize and overlooked important characteristics by perceiving the solids as visual gestalts only.

It was confirmed, through students in Group 1, defining a mathematical concept was a useful activity through which the abilities of generalization, synthesizing and abstraction that are characteristics of gifted students as confirmed by various studies (Krutetskii, 1976; Sriraman, 2003) could be verified and participating students could exercise the abilities. It was confirmed, through students from Group 2, that in order to define a mathematical concept, the ability to recognize the relationship between the components that make up the concept and to capture fundamental characteristics are needed in addition to the abilities of generalization. With this experiment, it was confirmed that the activity of defining concepts could be used for selecting gifted students and developing programs for gifted students. It was also found out that the ability to recognize the relationship between the components that make up the concept and to capture fundamental characteristics should be taken into consideration in addition to the abilities of problem-solving, generalization, and justification. It was confirmed, through students in Group 2 and 3, defining a mathematical concept only with examples are difficult even to gifted students. It is important to give counter examples in considering the level of learners.

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MULTIPLE SOLUTION TASKS AS A MAGNIFYING GLASS FOR OBSERVATION OF MATHEMATICAL CREATIVITY

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In this paper we introduce multiple solution tasks as a tool for examination of mathematical creativity in school children. Students from three ability groups – gifted, (non-gifted) proficient, and regular – were asked to solve problems in different ways. We present in detail the criteria for the analysis of mathematical creativity addressing the novelty of solutions, students' flexibility and fluency when producing multiple solutions. Two tasks – conventional and non-conventional – are under discussion in this paper. We also outline the findings: Non-gifted proficient students and their gifted peers differed in solutions of the non-conventional task but manifested similar results when dealing with the conventional one. Students from these two groups differed meaningfully in all the parameters from regular students. Based on the findings we hypothesize that non-conventional multiple–solution tasks are an effective tool for examinations of mathematical creativity in school children.

BACKGROUND

Multiple-solution problem solving in mathematics education

It is commonly accepted by mathematics educators that linking mathematical ideas and deepening understanding of how more than one approach to the same problem can lead to equivalent results are essential elements of the developing of mathematical reasoning (NCTM, 2000; Polya, 1973, Schoenfeld, 1985; Charles & Lester, 1982). On the other hand, Polya (1973) claims that problem solving in different ways characterizes experienced mathematicians since solving problems in different ways requires a great deal of mathematical knowledge. Additionally, Krutetskii (1976) argued that problems with several solutions allow examining flexibility of individual's mathematical thinking through investigating the switches from one mental operation to another. Polya (1973), Krutetskii (1976) and later Ervynck (1991) and Silver (1997) stressed that solving problems in different ways characterizes creativity of mathematical thought while some solutions may be more creative (more elegant/short/effective) than others. However, we did not find a systematic study that demonstrated that multiple solution tasks indeed may be used to show differences in mathematical creativity in groups of students with different ability levels.

Creativity and giftedness

Usually creativity is considered as one of the main components of giftedness (e.g. Renzulli, 2002). Research literature distinguishes between general and specific giftedness, and general and specific creativity (e.g., Piirto, 1999). *Specific giftedness* refers to clear and distinct intellectual ability in a given area, for example, mathematics. It is usually reflected in socially recognized performance and accomplishment.

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Specific creativity is expressed in clear and distinct ability to create in one area, for example, mathematics.

Gifted students "are those identified by professionally qualified persons who by virtue of outstanding abilities are capable of high performance" (Davis & Rimm, 2004: p. 18). Definitions of giftedness vary in different research sources; however several criteria are broadly accepted: high IQ scoring (for the general giftedness), high performance in a particular field (for the specific giftedness), above average ability (usually related to the first two criteria), task commitment and creativity.

Torrance (1974) defined *fluency*, *flexibility* and *novelty* as main components of creativity. Krutetskii (1976), Ervynck (1991), and Silver (1997) connected the concept of creativity in mathematics with multiple-solution tasks. In this context (Silver, 1997, Ervynck, 1991, Leikin, accepted), *flexibility* refers to the number of solutions generated by a solver, *novelty* refers to the conventionality of suggested solutions (see later in this paper a more precise definition), and *fluency* refers to the pace of solving procedure and switches between different solutions.

Solution spaces of multiple-solution tasks

Leikin (accepted) suggested a notion of *solution spaces* that allows researchers to examine mathematical creativity when solving problems with multiple solution approaches: ***Expert solution spaces*** are spaces of solutions that expert mathematicians can suggest to the problem. With respect to school mathematics expert spaces include *conventional solution space*s that are generally recommended by the curriculum and displayed in textbooks, and *unconventional solution spaces* that include solutions to problems, which are usually not prescribed by school curriculum. ***Individual solution spaces*** are also of two kinds. The distinction is related to the ability of a person to find solutions independently. *Personal (available) solution spaces* include solutions that individuals may present on the spot or after some attempt without help of others. These solutions are triggered by a problem and may be performed by a solver independently. *Potential solution spaces* include solutions that solvers produce with help of others. The solutions correspond to personal ZPD (Vygotsky, 1978). ***Collective solution spaces*** characterize solutions produced by a group of individuals. Both individual and collective solution spaces are subsets of expert solution spaces. Collective solution spaces are usually broader than individual solution spaces within a particular community and are one of the main sources for the development of individual spaces.

In this study we use solution spaces as a tool that allows exploring students' mathematical creativity. By comparing individual and collective solution spaces of students from different groups with expert solution spaces we evaluate students' mathematical knowledge and creativity.

THE STUDY

Research purpose

In the light of definitions presented above, and the lack of systematic research in the field, we were interested in examining whether and how performance on multiple solution tasks demonstrates mathematical creativity. We suggested and examined ways in which multiple solution tasks allow analysing novelty, flexibility and fluency of the solutions.

Population and procedure

By using multiple solution tasks we explored mathematical creativity of students from three groups of school students, each including 6 students: *Group G:* generally gifted students, those identified with high IQ scores and having high achievements in mathematics. *Group P:* proficient students in mathematics, those who were not identified as G but showed high performance in high level mathematics; *Group R:* regular students who have high scores in mathematics at medium level. In order to reduce the knowledge differences resulting from mathematical curricula of different levels, we examined 10^{th} grade students in groups G and P and 11^{th} grade students in group R.

Task 1:

Solve the system: $\begin{cases} 3x + 2y = 10 \\ 2x + 3y = 10 \end{cases}$

Solutions:

(1.1) Linear combination, (1.2) Substitution, (1.3) Equalizing the algebraic expressions in the equations, (1.4) Equalizing algebraic expressions for x (for y), (1.5) Symmetry considerations, (1.6) Graphing, (1.7) Trial and error strategy – substituting numbers, (1.8): Matrices.

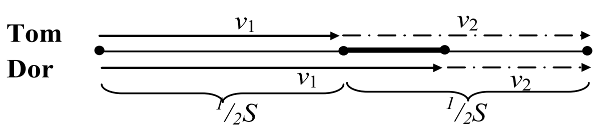
Task 2 (From Leikin, 2006)**:**

Dor and Tom walk from the train station to the hotel. They start out at the same time. Dor walks half the time at speed $v_1$and half the time at speed v_2. Tom walks half way at speed $v_1$and half way at speed v_2. Who gets to the hotel first: Dor or Tom?

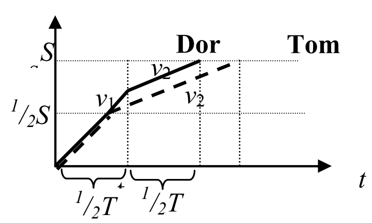
Solution 2.1 *- Logical considerations*:

If Dor walks half the time at speed v_1 and half the time at speed v_2 and v1>v2 then during the first half of the time he walks a longer distance that during the second half of the time. Thus he walks at the faster speed v_1 a longer distance than Tom. **Dor** gets to the hotel **first**.

Solution 2.2 *– Illustration of logical considerations*:



Solution 2.3 *– Graphing:*



Solution 2.4 *– Table-based inequality*

Solution 2.5 *– Experimental modelling* (walking around the classroom)

Figure 1: Two tasks in the study

All the students were presented with three tasks during individual interviews and were asked to solve the tasks in as many ways as they can. In this paper we report on two algebra tasks only (Figure 1). Task 1 is a conventional task borrowed from a school textbook. It fitted school curricular of the three groups of students. Its expert solution space includes 8 solutions (Figure 1) most of which are prescribed by the curriculum. Task 2 is an unconventional task taken from an Olympiad textbook (Babinskaya, 1975). Its expert solution space includes 5 solutions (Figure 1). Regular school (table-based) solution of this task is complex, based on algebraic manipulations and solving inequalities, whereas other (alternative) solutions are short and elegant but require insight. Mathematical knowledge of students from the three groups allowed them (at least) understand the solutions.

We developed a system of hints that allowed us stimulate students' search for new solutions. For example to guide students towards the symmetry based solution of Task 1 the interviewer asked whether "something special may be seen in this system of equation".

Data collection and analysis

All the students were individually interviewed. The interviews were video-recorded and transcribed. The transcripts and videotapes were analysed from the perspective of students' creativity. The following criteria were used to evaluate students' mathematical flexibility, novelty and fluency:

Novelty was evaluated according to the conventionality of solutions, their availability and repetitions. *Conventionality* of the solution was evaluated with respect to its belonging to the school curriculum of a particular group of students. Conventional solutions (e.g. 1.1, 2.4) are those solutions that are recommended for a task by the curriculum and are included in school mathematics textbooks. Solutions were considered as *partly non-conventional* if the solution strategy belongs to the curriculum but in a different situation or topic, and thus its use requires original thinking. The solution was accepted as non-conventional in the cases when it was not a part of the curriculum. For example for students from Groups R and P solutions (1.8) was non-conventional. The same solution was considered as partly non-conventional for students from Group G, since they had learned the topic of matrices in school. *Availability* of the solutions indicated students' independent thinking. When students were able to produce a solution with the researcher's hint we considered the solution as belonging to the potential solutions space. Production of *repeating solutions* indicates that students are less creative and critical and do not evaluate differences between the solutions thus this was an additional criteria for the evaluation of novelty.

We developed a *scoring scheme* as presented in Table 1: Students individual (without hints) solutions were scored with 2, 4, and 6 respectively to the level of their conventionality, the same solutions produced with hints got half scores.

Flexibility was evaluated by the number of solutions (a) in individual solution spaces of the available solutions and (b) in individual potential solution spaces. Each student was given a final score for each task independently. We also analysed the number of solutions in the group solution spaces. The score was calculated as a linear combination of the number of solutions with different scores and the scores given to the solutions. ***Fluency*** was evaluated with respect to the time spent by the students for performing the solutions. Finally we compared Novelty, Flexibility and Fluency of the problem-solving performance by the students from different groups (G, P, R).

Table 1: Types of the solutions for two tasks in the study and the scoring scheme for evaluating the novelty of the solutions

	Solutions	Conventional	Partly non-conventional	Non-conventional
Distribution of solutions	For Task 1	1.1*, 1.2, 1.3, 1.4, 1.7	1.6, 1.8 (for G)	1.5, 1.8 (for P and R)
	For Task 2	2.4	2.3	2.1, 2.2, 2.5
Scoring Scheme**	Available	2	4	6
	Potential (with hint)	1	2	3

* Description of the solutions presented in Figure 1

FINDINGS

The following analysis is reflected in Table 2 that presents the summary of the data collected in the study. Note that within the space limit of this paper we do not provide an example of the in-depth transcript analysis that we plan to present at the conference. We also do not report herein about the individual differences between the students in the different study groups. We just outline the findings of the study related to the groups differences in order to explain how multiple-solution tasks allow analysing students' mathematical creativity, and why we consider them as effective tool for identification of mathematical creativity.

Novelty

There were clear differences between the novelty of solutions of G-students and those from groups R and P on the two problems. The differences between the students from groups P and R were less significant then those between groups G and P.

For the system of equations novelty of the solutions varied from 2 to 6 in group G, from 2 to 4 in group P and from 2 to 3 in group R. Four of six G-students suggested unconventional solution --using symmetry -- (scored with 6). No students from groups R and P realized individually this regularity of the system of equations and used it when solving the task. However, 5 students from group P performed this solution based on a hint (scored with 3) whereas only two of the R-student solved the system of equation using symmetry even though the hint was provided. There were minor differences with respect to the conventional solutions of the system of equations: all the students from the three groups solve the system at least in two conventional ways (as studies in school).

When solving word problem all the students from group G produced unconventional solutions. Three P-students and one R-student solved the task in an unconventional way individually and 3 of P-students and 4 of R-students solved it unconventionally with a hint. The difference appeared in the number of unconventional solutions produced by the students from different groups. Four of six G-students solved Task 2 in at least two unconventional ways whereas each of P-students and 5 of R students produced one non-conventional solution either individually or with a hint. Partly-conventional solution (Graphing: Solution 2.3) was performed by five G-students, five P-students and two R-students with a hint only. The conventional solution (Solution 2.4) was performed by 3 of 6 gifted and 2 of 6 proficient students only. It appeared to be too complicated for the performance of regular students.

Table 2: Summary of numerical data in the study

Group		**Novelty** No of solutions of a particular **level**							**Flexibility** No of solutions in a space			**Final score**[4] novelty×flexibility	**Fluency** (time per solution)
		Repeat	**1** after hint	**2**	**2** after hint	**3** after hint	**4**	**6**	Personal	Potential	Collective		
	Task 1: System of linear equations – conventional task												
G	No of students[1]	2		6	3	1	3	4					
	No of solutions[2]	(2)		18	3	1	4	4					
	Mean[3]			**3**	**0.5**	**0.15**	**0.7**	**0.7**	**4**	**0.5**	**8**	**13.23**	**93.04**
P	No of students	3		6	5	5	1						
	No of solutions	(6)		18	5	5	1						
	Mean			**3**	**0.8**	**0.8**	**0.15**		**3.2**	**1.7**	**7**	**10.98**	**111.99**
R	No of students	5		6	3	2							
	No of solutions	(10)		20	3	2							
	Mean			**3.3**	**0.5**	**0.33**			**3.3**	**0.8**	**7**	**9.03**	**117.41**
	Task 2: Word problem – unconventional task												
G	No of students			3	5			6					
	No of solutions			3	5			11					
	Mean			**0.5**	**0.8**			**1.8**	**2.7**	**0.5**	**5**	**13.67**	**171**
P	No of students		1	1	5	3		3					
	No of solutions		1	1	5	3		3					
	Mean		**0.15**	**0.15**	**0.8**	**0.5**		**0.5**	**0.7**	**1.5**	**4**	**6.67**	**213**
R	No of students				2	2		1					
	No of solutions				2	4		1					
	Mean				**0.33**	**0.7**		**0.15**	**0.2**	**1**	**3**	**3.67**	**204.6**

(1) No of students who suggested solutions of a particular level.
(2) The total no. of solutions of a particular level produced by the students in a group.
(3) Mean: per student in a group (the total quantity for the group divided by 6)
(4) Final score was calculated for each student as the sum of all the scores received by a student for each particular solution. Thus we considered it as an indication of the combination of novelty and flexibility

Flexibility

Students' flexibility when solving multiple solution tasks was analyzed by addressing the number of solutions in individual available and individual potential solution spaces. Flexibility of students from the three groups differed meaningfully especially on Task 2. For the conventional task (Task 1) we found that individual solution spaces of all the students in three groups included more than 3 solutions. For the unconventional task (Task 2) personal solution spaces differed meaningfully: whereas all personal solution spaces of G-student included more than 2 solutions, for P and R students some of these spaces included one solution only and some of them were empty. The potential solution spaces of P-students included 1 or 2 solutions. In other words hints helped them in producing multiple solutions; however, alone they were not able to perform the solutions. They were less flexible than G-students in their ability to change the direction of their mathematical thought without help.

Not less important for the analysis of the students' flexibility was consideration of the solutions in collective solution spaces. On the two problems collective solution spaces of G-students covered expert solution spaces whereas collective solution spaces of P and R students were less complete (Table 2).

Fluency

Time per solution presented in Table 2 demonstrates time spent by the students on successful solutions only. Overall G-students were more fluent in their successful solutions; they performed them and switched the directions more quickly than students in other groups did. Note that the fact that P-students spent more time on the solutions than R-students when solving Task 1 demonstrates that these students are more persistent in their attempts to solve the problem. In this paper we did not considered time spent on the ineffective attempts the students made when solving the problems that is included in larger study. Thus this table does not reflect the actual time the students spent with the researcher during the interviews.

FINAL HYPOTHESIS

By observing the differences between the groups of students in their novelty, flexibility, and fluency we used definitions of flexibility suggested by Torrance (1974) for general flexibility, and by Ervynck (1991) and Silver (1997) for mathematical creativity. Consequently we asked how we can combine novelty, flexibility, and fluency as analysed in this study to evaluate mathematical creativity.

We found that the differences between the groups are task-dependent. The final score demonstrated the differences between the groups in the combination of novelty and flexibility. Not surprisingly gifted received higher score than proficient students and they received higher scores than regular students. However we were surprised by the differences in the gaps between the different groups for the two problems (Table 2).

After several trials that we are planning to present at the conference, we suggest here the following criterion for the analysis of mathematical creativity by means of multiple-solution tasks: *final score (novelty by flexibility) divided by the product of time per solution and the number of solutions in the expert solution space of the task.*

Table 3 presents the criterion calculated for the two tasks presented in this paper for the three groups pf students. By observing this table we hypothesise that Task 2 may allow identification of students' mathematical creativity and (probably) giftedness.

Table 3: Suggested creativity criterion

	Conventional task	Unconventional task
Generally Gifted students	1.8	1.6
Students Proficient in mathematics	1.2	0.6
Regular mathematics students	0.96	0.3

Now, we start the next stage of the study which focuses on refining the criteria with larger population and quantitative research tools.

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INTERACTIVE WHITEBOARDS AS MEDIATING TOOLS FOR TEACHING MATHEMATICS: RHETORIC OR REALITY?

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Interactive whiteboards (IWB) are an innovation that is gaining considerable presence in many contemporary classrooms. This paper examines the use of IWBs in mathematics classrooms. Using a productive pedagogies framework to analyse classroom videos, it is proposed that the classrooms observed used a restricted approach in their use of IWBs. It was found that they were used for quick introductions to lessons, were teacher directed, whole class teaching and fostered shallow learning. Through interviews with the teachers, it was found that the approaches observed were based on assumptions about learners and technology.

In this paper, we explore the ways in which teachers use Interactive Whiteboards (IWBs) in mathematics classrooms. There is a sense that this tool may offer considerable potential to enhance student learning. Promoters of the tool provide case studies of the novelty and support that can be achieved through the clever use of the tool for example, (Edwards, Hartnell, & Martin, 2002). How this is enacted in classrooms is the focus of the analysis in this paper. In exploring computer-mediated learning, Waycott, Jones and Scanlon (2005, p.107) reported that there is a reciprocity between the tools and the learner where "the user adapts the tools they use according to their everyday practice and preferences in order to carry out their activities; and how, in turn, the tools themselves also modify the activities that the user is engaged in." We argue that this is the same for teachers.

BACKGROUND

The introduction of interactive whiteboards into schools in the UK has been strongly supported by the government (Beauchamp, 2004), with over £50m being spent on their implementation in primary and secondary schools (Armstrong et al., 2005). However, it has not received the same fiscal support in Australian schools. Many schools are supporting the intorduction of these devices through various means but without systematic support. In most cases, the implementation of IWBs is a school-based decision and as such is supported by funds raised by the schools. How the IWBs are implemented within a given school is dependent upon the resources of the school to provide the equipment and the beliefs of the teaching staff as to the value of the tool. As such, there is considerable variation across Australia as to their uptake and implementation.

Drawing from socio-cultural perspectives on the use and uptake of mediating technologies – in this case IWBs, Armstrong et al (2005) suggest that there is a tendency for teachers to use IWBs as "an extension of the non-digital whiteboard" (p. 458). Beauchamp (2004) argued that the transition from traditional modes of teaching

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 169-176. Seoul: PME.

to the totally integrated use of IWBs in classrooms demands a shift in pedagogical style of the teacher. For teachers to realize the potential of IWBs, Glover and Millar (2002) contend that teachers need to recognize that there is considerable interactivity associated with their use. They argue that the IWB can engender an approach that fails to radicalize pedagogy and where the IWB is used to enhance students' motivation rather than become a catalyst for changing pedagogy.

The extension of the computer through the use of IWBs creates new opportunities and obstacles to learning. In studying the use of IWBs in English classrooms, it was reported that

> "IWB can facilitate and initiate learning and impact on preferred approaches to learning. The pupils describe how different elements of software and hardware can motivate, aid concentration, and keep their attention. On the negative side, pupils candidly describe their frustration when there are technical difficulties, their desire to use the board themselves and their perceptions of teacher and pupil effects (Wall, Higgins, & Smith, 2005 p. 851).

Greiffenhagen (2000) argued that the availability of IWBs as a teaching aid is only of value where it becomes part of the regular pattern of classroom life.

In their study of the uptake of IWBs in a secondary school, Glover and Miller (2001) proposed that IWBs offered considerable benefits to learning. They reported that students were more likely to engage in learning due to the surprise element that was offered through the IWB, the large visual cues offered through the IWB presentation format, and the quicker pace of lessons.

As a teaching tool, IWBs have considerable potential to change interaction patterns. In their study of classrooms – both literacy and numeracy in IWB and non-IWB classrooms – Smith, Hardman and Higgins (2006) found that there is a faster pace in lessons using IWBs than non-IWB lessons; that answers took up considerably more of the overall duration of a lesson; and that pauses in lessons were briefer in IWB lessons compared with non-IWB lessons. They also reported a faster pace in numeracy lessons than in literacy lessons. While they reported some support for the potential of IWBs, they concluded that overall the use of IWBs was not significantly changing teachers' underlying pedagogy. The majority of teacher time was still spent on explanation and that recitation-type scripts was even more evident in IWB lessons. They found that while the pace of the lessons increased, there had been a decline in protracted answers from students and that there were fewer episodes of teachers making connections or extensions to students' responses. They also claim that there is a faster pace in lessons but less time is being spent in group work. There is a tendency for teachers to assume a position at the front of the class when using IWBs (Maor, 2003). Similarly Latane (2002) suggested that there needs to be a move from teacher-pupil interaction to one of pupil-pupil interaction. In studying mathematics classrooms, Jones and Tanner (2002) reported that interactivity can be enhanced through quality questioning where the quality of the questions posed and the breadth of questioning needs to be developed to ensure interactivity in mathematics teaching when using IWBs.

DATA COLLECTION

The data reported here compares data collected as part of a larger study (Lerman & Zevenbergen, 2006) with subsequently collected data where teachers have been using IWBs. In this paper, we present the analysis of classroom lessons using a particular framework. A total of nine schools participated in the study. Over the three years of data collection, some schools dropped out of the study, and others came in. Five schools remained in the study for its duration. Purposive sampling techniques were used in the selection of schools. The schools were selected on their representativeness of the diversity found in Australian schools in terms of social groupings being served (high, medium and low SES), geographical location (city, rural, remote); technology implementation (high or medium; integrated into classroom, computing laboratories); and school structure (single age classes, multi-age classes). Classrooms from the upper primary sector were involved in the data collection.

Video data were collected in two classrooms where the teachers had access to IWBs. The teachers video-taped their lessons which were than analysed using the productive pedagogies framework.

DESCRIPTIVE ANALYSIS OF PEDAGOGY

Two analyses were conducted on the video data. In the first instance, a running record was taken of the lesson with a transcript developed of the lesson. This record consisted of both description of the lesson and the interactions between teacher and students. Our data confirm that of Smith, Hardman and Higgins (2006) where we could observe the level of questioning being used by teachers in these lessons. It was of a lower level format where teachers were asking more recall questions than those requiring deeper levels of understanding. This type of questioning also allowed for a quicker pacing of the lesson since teachers were able to ask quick fire questions where there was little depth in the responses required.

The predominant approach used by teachers when using the IWBs was that of whole class teaching. In these settings, the teacher controlled the lesson, inviting students to participate in manipulating the objects. In all cases, only one child was involved in such manipulations at any one time. The remaining students sat on the floor or in their desks. However, in observing the students, there were very few behavioural issues one would expect to see when children are seated for such lessons, and that they were predominantly focused on the teacher talk and actions. This observation was consistent across the lessons and schools suggesting that even though the lessons were whole class and teacher lead, the students appeared to be engaged with the lesson.

In all cases, the teachers used the IWBs as the introduction to the lesson. Typically, the orientations with the IWBs were between 5-15mins and were used to orientate the students to the topic that would then be followed. The introduction was whole class and quick pacing. In some cases the teachers used pre-existing lessons that had been developed by other teachers and were available through the resources. In other cases, they used the tools (such as fractions, calculator or clocks) that came with the IWBs. In all cases, they used the resources that were part of the packages supplied with the board. Once the students had been involved in the introductory component of the lesson, they returned to their desks to work on activities related to the topic being introduced.

Depending on the resources used by the teacher, there were instances where the IWB made possible a rich introduction to aspects of mathematical language. For example, in one lesson the teacher was using the fraction tool in which a shape (chosen by the teacher - circle, rectangle and square) were used to represent various fractions. These could be shown in a variety of ways such as pies in the case of circles or through horizontal, vertical or grids on the rectangles and squares. Through the ease with which the shapes could be selected and how they were represented, the teacher was then able to draw on a repertoire of language to discuss the shapes, representations and fractions. The ease and speed at which shapes and denominators could be selected enabled a lot of talk/questions about the numbers being represented. As with other lessons, the speed of questions and delivery meant for fast pacing. However, there was little to no evidence of deeper probing of concepts or for mathematical thinking in terms of drawing patterns across the experiences. In the lesson on fractions, for example, while students saw a range of fractions (halves, quarters, thirds, sixths, eighths, tenths), these were simply representations of denominators and with different numerators being used. In some cases, equivalence was discussed - such as 4/8 was talked about as being equivalent to ½. However, this discussion was only undertaken when the 4 shaded pieces were adjacent so that it was clearly ½. The discussion did not occur when it was possible for the 4 segments to be scattered. Similarly, there was no discussion about the relationship between the size of the segments and the number in the denominator – that is, the inverse relationship between the segment and number. So while the students were exposed to a range of experiences, the richness of the mathematics was not being drawn out of the lessons.

PRODUCTIVE PEDAGOGIES ANALYSIS

While the observations provided us with some indicators of how the IWBs were being used in the classroom, we also employed a quantitative measure to document the use of IWBs. This measure allows us to more rigorously analyse the lessons. We have used this approach in analyzing the use of ICTs in classrooms (Lerman & Zevenbergen, 2006) so are able to compare those data against the use of IWBs. The process involves three observers observing the lessons which had been videotaped. Each observer rates the lesson against nominated criteria on a scale of 0-5 where 0 indicates that there was no evidence of that criterion in the lesson and 5 indicates that it was a strong feature that was consistent throughout the lesson. The ratings are made at the completion of the lesson and the score is for the overall lesson. If there is some evidence of a criterion in the opening phase of the lesson but does not appear again, then this means that it was not a strong feature of the overall lesson. The three observers rate their observations independently and then come together to come up with a common score. This involves a process of negotiation to arrive at the common outcome. In most cases, there was usually a difference of 1 between the ratings and the ensuing discussion meant that the observers needed to negotiate their ratings with the other two.

The framework we have used draws on the work of the Queensland Schools Longitudinal Reform Study (Education Queensland, 2001) in which the researchers analysed one thousand lessons in terms of the pedagogies being used by teachers. The method was that described above and where the criterion for each rating was based on the Productive Pedagogies. There are four dimensions within the framework – Intellectual Quality, Relevance, Supportive School Environment and Recognition of Difference – in which there are a number of items that are evidence of that theme.

Within the Productive Pedagogy approach, there is a strong emphasis on raising the quality of teaching in terms of the intellectual experiences and social learning. The outcomes of the Queensland study (Education Queensland, 2001) indicated that teachers were very good at providing a supportive learning environment but that the intellectual quality was quite poor. When the analysis was undertaken across key learning areas, it was reported that the learning environments in mathematics scored the least favourably suggesting that the intellectual quality in mathematics (across all years of schooling) was poor.

In seeking to explore the use of IWBs in mathematics classroom, we undertook the same analysis of the classroom videos. As can be seen in Table Two, the scores are low in most areas. We have included the analysis of classroom data were ICTs were used in mathematics classrooms as a comparison.

	ICTs		IWBs	
Dimension of Productive Pedagogy	Mean	SD	Mean	SD
Depth of knowledge	1.64	1.36	1.5	1.46
Problem based curriculum	2.19	1.38	*.92*	0.83
Meta language	1.69	1.07	1.25	1.87
Background knowledge	1.76	1.16	1.67	1.63
Knowledge integration	1.48	1.27	*0.42*	0.45
Connectedness to the world	1.38	1.44	*0.42*	0.45
Exposition	1.19	1.64	0.83	0.82
Narrative	0.31	0.78	0.17	0.18
Description	2.24	1.02	*1.42*	1.25
Deep understanding	1.43	1.47	1.25	1.19
Knowledge as Problematic	1.14	1.47	1.33	1.36
Substantive conversation	1.26	1.40	*0.5*	0.46
Higher order thinking	1.31	1.55	1.33	1.36
Academic engagement	2.23	1.38	*1.5*	1.46
Student direction	0.79	0.92	0.33	0.28
Self regulation	3.24	1.12	*2.5*	2.45
Active citizenship	0.30	0.78	0	0
Explicit criteria	2.83	1.17	*1.33*	1.28
Inclusivity	0.33	0.75	0	0
Social support	2.51	0.25	*1.25*	0.62

Table Two: Productive Pedagogy Analysis of IWB use in Upper Primary Classrooms

We have reported the data for when teachers used ICTs to support numeracy learning elsewhere (Lerman & Zevenbergen, 2006) and this showed very low levels of quality learning potential. However, when using the same framework to analyse the use of

IWBs, the results were even lower. Nine out of the twenty pedagogies (those in italics) scored substantially lower when using IWBs. Most of the lower scores were in those two dimensions that relate to the intellectual aspects of mathematics learning. From these data we can conclude that the use of IWBs actually reduces the quality of mathematical learning opportunities; provides fewer opportunities for connecting to the world beyond schools; and offers little autonomous/independent learning opportunities for students. After these scores were obtained and analysed, we returned to the schools and interviewed teachers to seek some explanation of the findings.

TEACHER INTERVIEWS

In this section for reasons of space, we provide commentary on just 3 aspects of teachers' pedagogy that appear to us to be the most salient in their responses.

Motivation

One of the observations in the earlier findings was that the IWBs seemed to be used for the introduction to the lessons. In following this observation, teachers were asked if this were the case and if so, why. In the interviews, it was confirmed that the teachers tended to use the IWB to orientate the lesson and to motivate the students.

> Marcie: When the kids are all sitting and we are doing with the whiteboard, there are very few behaviour problems. They seem keen to be involved, and very eager to be the one to come to the board. You can see that they are all really wanting to get up the front and have a go. Some of my quiet kids getting really animated when we do the whiteboards whereas in the normal work, you hardly know they are there.

> Heidi: I use it to get the lesson started. The kids are all together, there are all on the one task, they know what we are doing. That is a good way to start the lesson. It is also good as the kids are very motivated by the boards so they are keen to get into the lesson.

Pacing

When using the IWBs, it would appear that the teachers were aware of the faster pace of the lessons. They articulated that they posed a lot more questions and the students had greater opportunities for participating in the lessons due to the increased questioning.

> Maxine: One of the things that I like about the whiteboards is that I can ask a lot more questions. You just have to click on the menu and there is the lesson or the things you need so you are not wasting a lot of time putting up overheads or drawing things on the board. I can ask more questions to the kids to see what they know and to get them to think about things. Like when we did the lesson with the clocks. You just click on the clock and there it is. You can just move the time around as quick as they kids respond. I think they like the quicker speed. They seem to enjoy the race of the lesson. If they answer quickly, then we can do another one or something a bit different.

Saves Time

Most of the teachers had some comment about the time factor in the use of IWBs. It was seen to save preparation time in two different ways. In the first instance, one teacher commented on how he drew on the resources that had been made by other teachers as these were 'tried and proven' examples of lessons that worked. In observing his lessons, he would select from the databank and then implement the lesson.

> Shane: I find that there are a whole lot of really good lessons that I can just use. If I am doing something on area for example, there are lessons already made up. Some other teachers have developed them so they have to be good ones. I am sure that the company only puts up the best examples. I have found these to be very handy and they save me doing the preparation work. I guess I change them a bit to suit me and the kids but they are pretty much there.

Another teacher commented on how, when using the IWB, the toolkit meant that the resources were all in the one place so she did not have to hunt around for them. Knowing that the protractor, ruler, clock, calculator were all on the screen and at the touch of the board, was seen to be a considerable timesaver.

> Jemima: I think the whiteboard is a great resource. You have the tools there on the board, you just need to click and they are there.

CONCLUSION

There is little doubt that IWBs have the potential to enhance learners' opportunities to experience mathematical representations and develop their mathematical thinking. As with all resources, mathematical or other, internalising a tool, be it the number line or a calculator, LOGO, dynamic geometry or Graphic Calculus, or presentation tools such as overhead projectors or IWBs, transforms the world, in this case of mathematical pedagogy for the teacher. That transformation is always mediated by other experiences, however. However, by themselves they will not transform pedagogy, no matter what their potential. Indeed, as we have reported in this paper, the technologically impressive features of the IWB can lead to it being used to close down further the possibility of rich communications and interactions in the classroom as teachers are seduced by the IWB's ability to capture pupils' attention. We suspect, also, that teachers' advance preparation for using the IWB, often via the ubiquitous powerpoint package or pre-prepared lessons for the IWB, are leading to a decreased likelihood that teachers will deviate in response to pupils' needs and indeed might notice pupils' needs less frequently through the possibility to increase the pacing of mathematics lessons. Elsewhere (Zevenbergen and Lerman, in preparation) we apply an activity theory framework to try to understand the tensions and contradictions in teachers' use of the IWB and to identify possible developmental trajectories for realising some of their potential to change pedagogy for the better.

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FROM CONSTRUCTION TO PROOF: EXPLANATIONS IN DYNAMIC GEOMETRY ENVIRONMENT

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Abstract *This report discusses an oral explanation and a written proof given by a Hong Kong secondary school student for a construction task in Sketchpad. The analysis gives hints on what spoken and written discourses in DGE might look like and sheds light on bridging the empirical-theoretical gap in DGE research. In particular, the idea of diachronic objects in DGE is proposed to be useful in studying reasoning and discourse in DGE.*

Introduction

Dynamic geometry environment (DGE) has been a widely studied computer platform for the learning and teaching of geometry in the past two decades (see for example, Educational Studies in Mathematics 44:1-161, 2000; International Journal of Computers for Mathematical Learning 6:229-333, 2001; Math ZDM 34(3), 2002; Laborde, 2003). A major agenda in this research area has been how students understand and perceive geometry in this dynamic environment. DGE vitalizes the arena of experimental mathematics and opens up mathematics classroom into scientific-like laboratory. Much fruitful work has been done on studying students' dragging strategies and the process of conjecture forming in DGE (see for example, Hölzl, 1996; Arzarello et al, 2002). However, researchers in DGE have been uncertain about the effect of DGE in "promoting" students' cognitive realm from empirical experiences to theoretical understanding; in particular, from making geometrical conjectures to production of formal axiomatic proof. Henderson stressed that proof means "a convincing communication that answers-Why?" (Henderson and Taimiņa, 2005, p.40). What does a convincing communication in DGE look like? This concerns how one *talks about* and *writes about* geometry in the context of DGE. Studying the spoken and written discourses in DGE might be a key to bridge the empirical-theoretical gap, since language is a vehicle that carries internal thoughts and in turn brings about the conceptualization of the world around us. Clearly these DGE discourses must be rooted in visualization. Rodd (2002) contended that visualization can be regarded as an a-linguistic mathematical warrant ("that which secures knowledge", Rodd, 2002, p.222) and she illustrated her argument using Giaquinto's (1992) "criteria for a geometrical proposition to have discovered by visualization" in a dynamic geometry context. This visual warranting involves "a syntheses of tacitly comprehended properties" of what is being looked at and "the intention to make obvious and requires a threshold grasp of the concepts in the visual presentation" (Rodd, 2002, p.238). Furthermore, Giaquinto (2005) proposed that visual experiences and imagining can indeed trigger belief-forming dispositions leading to acquisition of geometrical beliefs which constitute knowledge. Hence,

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visualization (a-linguistic) can have an epistemic function in mathematics in parallel with symbolic formal axiomatic proof (linguistic). How do these a-linguistic and linguistic warrants relate to each other? Halliday (2004) regarded language as a vehicle to re-shape human experience. Experience "was being transformed into meaning; and this transformation was effected by the grammar" where "grammar is a theory of human experience" (Halliday, 2004, p.9). "A verb means happening, a noun means and entity – a thing; and both typically have some correlates in the world of perceptions. We call this mode of meaning the ***congruent*** mode of the grammar" (ibid., p.14). Thus we could study how dynamic visual experiences and perceptions in DEG can be re-shaped in the congruent mode. In particular, how does one explain geometrical beliefs acquired by dynamic visualization in DGE. The purpose of this report is to initiate an exploration in this direction.

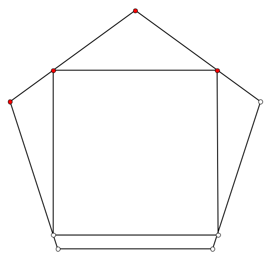
Background

Or (2005) studied the construction-experimentation-conjecturing cycle in DGE with a group of secondary students in Hong Kong. He observed that in successful DGE student exploration episodes, the construction-experimentation-conjecturing cycle acts as a fundamental recursive cycle that generates a cognitive vertical spiral with each level situated on a higher cognitive plane than the previous one. This spiraling process eventually stops when an explanation is reached. We will discuss in depth an oral explanation and a written proof given by one of the students in Or's study for a successful DGE construction task. This student's work will offer us a glimpse of what a discourse in DGE might look like.

Morris was an intelligent Form 4 (Grade 10) student in a Hong Kong secondary school. He was very keen in mathematics and a high achiever in the subject. Since Form 1 (Grade 7), Morris had been the recipient of the Form Mathematics Subject Prize. He was a core member of the school team to participate in mathematics competitions in Hong Kong and showed outstanding performance. Morris was chosen by his school to enroll in a training program for selecting secondary school students to represent Hong Kong to participate in the International Mathematics Olympiad. He was an experienced Sketchpad user, often using Sketchpad to solve geometric problems on his own. Morris attended a Sketchpad workshop organized by the school's Math Club supervised by the second author. During the workshop, the participants were taught the technique of relaxing a condition (Straesser, 2001) when solving construction problems in Sketchpad and using the TRACE function to visualize locus while dragging. Afterwards, the following problem was given to the participants as an exploration task:

Square inscribed in a regular pentagon

Investigate how to construct a square with 4 vertices lying on the sides of a regular pentagon as show in the figure on the right. Write down your method of construction and explain why your construction works.



Morris solved the problem without much difficulty and he was asked afterwards to explain his construction.

The Interview

The following is a transcript of the interview conversation between Morris and the second author. The spoken language used was Cantonese (the local Chinese dialect used in Hong Kong). The English translation is faithful to the Cantonese original, capturing critical moods and word meanings. During the interview, Morris was accessible to Sketchpad.

(I: Interviewer, M: Morris)

1. M: Let a movable point (G) on this side (AB) of the pentagon. Construct a square (FGHI) like this. Then this point (I) would lie on this line (*l*) (see Figure 1).

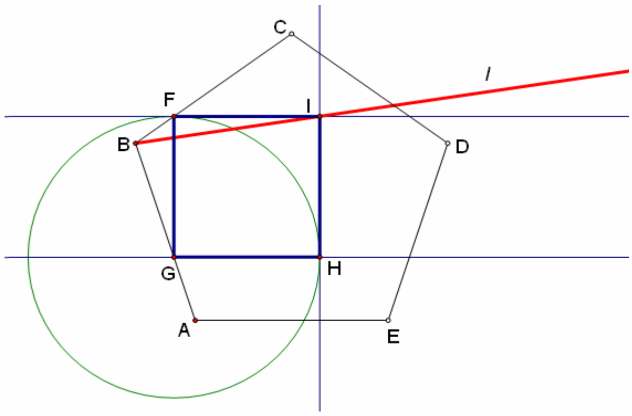


Figure 1

2. I: Why? Do you mean when G varies, the locus of I is this straight line (*l*)?
3. M: Ym…. Yes. (Morris traced the locus of I to show that the locus of I is the straight line *l*).
4. I: So what are you going to do next?
5. M: So I project this line (*l*) and take the intersection (of *l* and CD) and draw a square … (Morris used the intersection R to draw a square PQRS as shown in Figure 2).

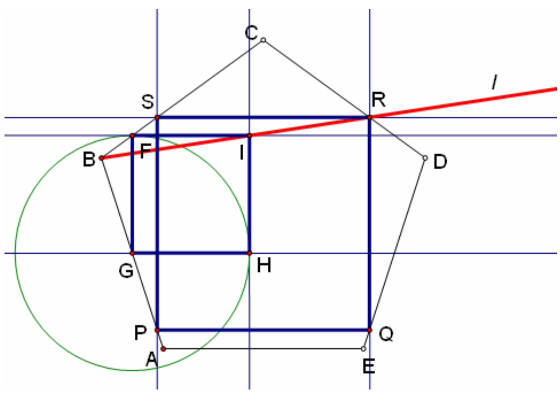
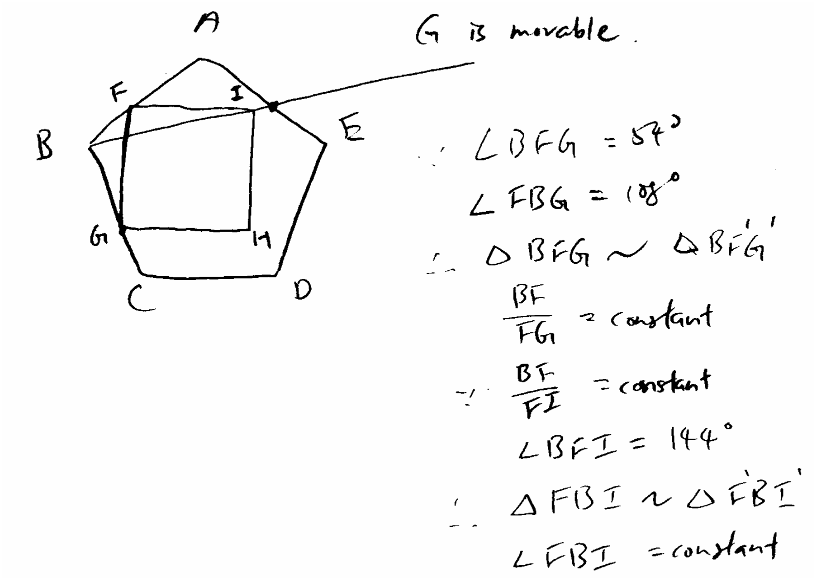


Figure 2

6. (Morris then hid all the subsidiary lines and circles leaving only the pentagon and the square PQRS. Subsequently he dragged E arbitrarily to show that the constructed square is robust.)
7. I: Why? Why is the locus a straight line? Can you explain?
8. M: This line (FI) and this line (AE) are parallel. This side (BC) is fixed. Therefore this angle ($\angle$BFG) is fixed (when G moves). This triangle (ΔBGF) is similar. No matter how you move G this triangle is always similar. That is, the ratio of this side (BF) and this side (FG) is always a constant. Since this side (FG) and this side (FI) are equal, that means the ratio of BF and FI is always a constant. Since the included angle ($\angle$BFI) is the same, the triangle (ΔBFI) becomes similar, and hence the angle in the left ($\angle$FBI) is always the same. Therefore it comes out to be a straight line.
9. I: Why is this angle ($\angle$FBI) always the same?
10. M: Because of similar triangle!
11. I: Which triangle is similar to which?
12. M: Every triangle is similar! (Morris dragged G when he said so.)
13. I: You feel that they are similar?
14. M: They are always similar!
15. I: O.K.! How did you come up to this?
16. I don't know …. I just use an arbitrary method to prove this … the angle ($\angle$FBI) is a constant.
17. I: How do you know that you should prove this angle ($\angle$FBI) to be a constant?
18. M: Mm… it moves back and forth (Morris dragged G back and forth when he said so). This point (I) lies on the line (*l*) and we should look at this angle ($\angle$FBI).
19. I: Have you seen this problem before? Is this the first time you work on this problem?
20. M: I think so.
21. I: When you move (G) back and forth it reminds you that you should look at this angle ($\angle$FBI). Can I say so?
22. (Morris nodded his head slightly.)
23. I: Do you consider yourself solved the problem?
24. M: Ym … I think so.
25. I: Can you write down what you have already said? You can just write it down briefly.
26. M: Ym … I can try.
27. (After ten to fifteen minutes Morris presented the following written explanation.)



(*A scanned image of Morris' written proof*)

Analysis

The construction

The key idea in Morris' construction of the robust square is the "movablilty" of point G which induced the construction of line *l*. The description given by Morris in Line 1 was a possible "if…then" type statement in DGE:

If an object O_1 is *movable* (that is, draggable), then certain object O_2 dependent on O_1 *would behave* in a certain way.

In such a statement, action words or phrases are used to state the potentiality of a dynamic situation, rather than merely ascertaining a rigid outcome. However, Morris' description was not completely transparent, plausibly because he was engaged in a mixed cognitive mode in which he tried to express DGE phenomena using the spoken language. Thus the interviewer pressed on, prompting Morris to "clarify" his description using the formal mathematical term "locus" (Line 2). Interestingly, Morris responded to the word "locus" by actually tracing it in DGE (Line 3), indicating that his cognition indeed was immersed in DGE. Morris then went about constructing the required robust square (Lines 5, 6).

The oral explanation

The interviewer probed deeper and asked Morris why the locus was a straight line (Line 7). This required Morris to explain a phenomenon (or certain behaviour) in DGE. The explanation that Morris gave (Line 8) was quite illuminating and the descriptive language he used might shed light on possible discourse in DGE. It is not difficult that after reading Line 8, one can observe immediately the concept of invariance under variation. This is reflected by the repeated usage of the words "fixed", "always", "constant" and "same". And these words were qualified most of the time explicitly or implicitly by phrases that referred to the movability of point G: "when G moves", "No matter how you move G", "the triangle ΔBFI *becomes* similar". Again, Morris' description was not immediately sensible. In particular, a key idea in the explanation was the curious statement "this triangle (ΔBGF) is similar". The use of the singular "this" is intriguing since "this" here actually referred to a (continuous) sequence of ΔBGF under the dragged movement of point G. The interviewer indeed pursued this apparent ambiguity afterwards. Morris responded to this by dragging point G and proclaiming "Every triangle is similar!" (Lines 11 to 14). A probable reason for using the signifier "this" is when Morris dragged G, the labeling of ΔBGF did not change though ΔBGF was varying; it was always ΔBGF on the screen! This could induce a special type of reasoning (or explaining) in DGE in which a signified object in DGE could have a diachronic nature. That is, one has to conceptualize a draggable object in DGE as it varies (over time) under dragging. Hence, a whole object in DGE should be *understood* as a (continuous) sequence of the "same" object under variation. Morris adopted this mode of thinking when he talked about ΔBGF, $\angle$BFG, $\angle$BFI and $\angle$FBI in Line 8.

The written proof

The interviewer requested Morris at the end of the interview to write down his oral explanation. Morris produced an intriguing "formal" proof explaining that the locus of I under the movement of G is a straight line. The proof was written up in the format of a proof in Euclidean deductive geometry with a few DGE twists in it. There was a diagram depicting a static instant of the sequence of squares and the straight lines that passed through G and I. Beside the diagram was a statement "G is movable". Together the diagram and the statement formed a premise upon which subsequent arguments could be derived. However, any "logic" used hereafter must be one that could reflect the movability of G. Corresponding to the phrase "This triangle (ΔBGF) is similar" (Line 8) that Morris used in his oral explanation, he wrote in the proof "$\Delta BFG \sim \Delta BF'G'$. Apparently, $\Delta BF'G'$ was not in the diagram. The primes that accentuated F and G seemed to symbolize the varying F and G under dragging. This was consistent with Morris' diachronic understanding of objects in DGE discussed above. Another type of such diachronic expression that appeared in Morris's proof was "BF/FG = constant". The word "constant" had a deeper meaning than just being a numerical value; it meant invariant under variation via dragging. Thus the juxtaposition of a symbolic deductive proof formalism and a DGE-interpreted usage of symbols/signs seems to make Morris' written proof into a bridge that transverses the domains of experimental geometry (DGE) and deductive geometry (axiomatic Euclidean).

Discussion

From the above analysis there emerged a few ideas that might become significant when studying possible discourses in DGE.

(1) Words like “movable”, “become”, “always”, “constant” that connote (or is congruent to) motion, transition, invariance should be prominent in a DGE discourse. These words should be interpreted under the drag-mode or any other function in DGE that induces variations.

(2) Drag-sensitive objects in DGE are diachronic in nature. The concept of a whole could be a concept of continuous sequence of instances under dragging or variation. Consequently, the denotation (or congruency) of such objects may transcend the usual semantic of the spoken languages. For example, a singular “this” may actually mean many.

(3) Writing up “formal” DGE proofs may involve using mathematical symbols or expressions that transcend the usual semantic of a traditional mathematical symbolic representation. For example, a DGE ΔABC may not point to a particular triangle; rather it represents all potential triangles ABC under dragging. In traditional axiomatic proof, one would say “for an arbitrary ΔABC”. The diachronic nature of objects in DGE replaces the imaginary arbitrariness assumption in traditional mathematical proof.

These observations may serve as good sign posts pointing to paths that could lead to the conceptualization of written or spoken discourses in DGE. Diachronic simultaneity and invariant under variation are key concepts in understanding discernment in the theory of variation (Marton and Booth, 1997). Leung (2003) discussed how these features could be used to interpret phenomena experienced in DGE. Variation plays a core role in DGE reasoning and discourse. Morris’s success was largely due to his talent in mathematics and his familiarity with Sketchpad. He was able to cognitively immerse in and to integrate his mathematical knowledge with what he experienced in DGE. This enabled him to speak and write creatively using his prior knowledge to reflect his reasoning in DGE. Hence prior mathematical knowledge and experiences are essential for a learner to construct deep meanings in DGE (another type of diachronic simultaneity). What should be the appropriate semantics and syntax that carry mathematical meanings in a DGE discourse, consequently forming a basis for the type of reasoning in DGE? We hope this report will stimulate interest in this research direction.

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PROSPECTIVE MIDDLE SCHOOL TEACHERS' KNOWLEDGE IN MATHEMATICS AND PEDAGOGY FOR TEACHING - THE CASE OF FRACTION DIVISION

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In this paper, we investigated the extent of knowledge in mathematics and pedagogy that prospective middle school teachers have learned and what else they may need to know for developing effective classroom instruction. We focused on both prospective teachers' (PT) own perceptions about their knowledge in mathematics and pedagogy and the extent of their knowledge on the topic of fraction division. The results reveal a wide gap between these PT's general perceptions/confidence and their limited knowledge in mathematics and pedagogy for teaching, as an example, fraction division. The results also suggest that PTs need to master specific knowledge in mathematics and pedagogy for teaching in order to build their confidence for classroom instruction.

Accumulated research findings in past decades have led to the understanding that teachers' knowing mathematics for teaching is essential to effective classroom instruction (e.g., Ma, 1999; RAND Mathematics Study Panel, 2003). Corresponding efforts have also been reflected in teacher preparation programs that call for more emphasis on prospective teachers' learning of mathematics for teaching (CBMS, 2001; NCTM, 2000). Such efforts can presumably increase the quality of pre-service teacher preparation and prospective teachers' confidence and ultimate success in future teaching careers. Yet, much remains to be learned about the extent of knowledge in mathematics and pedagogy that prospective teachers acquire and what else they may need to know for developing effective classroom instruction. As a part of a large research study of prospective middle school teachers' knowledge development in mathematics and pedagogy, this paper focuses on a group of prospective middle school teachers' knowledge of mathematics and pedagogy for teaching in general and on the topic of fraction division, in particular.

The topic of fraction division is difficult in school mathematics not only for school students (Carpenter et al., 1988), but also for prospective teachers (Ball, 1990; Simon, 1993). Mathematically, fraction division can be presented as an algorithmic procedure that can be easily taught and learned as "invert and multiply." However, the topic is conceptually rich and difficult, as its meaning requires explanation through connections with other mathematical knowledge, various representations, and/or real world contexts (Greer, 1992; Ma, 1999). The selection of the topic of fraction division, as a special case, can present a rich context for exploring possible depth and limitations in prospective teachers' knowledge in mathematics and pedagogy. Specifically, this study focuses on the following two research questions:

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 185-192. Seoul: PME.

(1) What are the perceptions of prospective middle school teachers regarding their knowledge in mathematics and pedagogy for teaching?
(2) What is the extent of prospective middle school teachers' knowledge in mathematics and pedagogy for teaching fraction division?

CONCEPTUAL FRAMEWORK

The conceptual complexity of the topic of fraction division is evidenced in a number of studies that documented relevant difficulties pre-service and in-service teachers have experienced (e.g., Ball, 1990; Borko et al., 1992; Contreras, 1997; Simon, 1993; Tirosh, 2000; Tzur & Timmerman, 1997). Although both pre-service and in-service teachers can perform the computation for fraction division, it is difficult for teachers, at least in the United States, to explain the computation of fraction division conceptually with appropriate representations or connections with other mathematical knowledge (Ma, 1999; Simon, 1993). Teachers' knowledge of fraction division is often limited to the invert-and-multiply procedure (e.g., Ball, 1990), which restricts teachers' ability to provide a conceptual explanation of the procedure in classrooms (e.g., Borko et al., 1992; Contreras, 1997). Because the meaning of division alone is not easy for pre-service teachers (e.g., Ball, 1990; Simon, 1993), fraction division is even more difficult (Ma, 1999). Based on the findings from studies on teachers' knowledge and difficulties with division and fraction division, it can be summarized that teachers often have the following five types of difficulties:

(a) How to explain the computational procedure for "division of fraction" with different representations (e.g., Contreras, 1997; Ma, 1999)
(b) How to explain why "invert and multiply" (e.g., Borko et al., 1992; Tzur & Timmerman, 1997)
(c) Mathematical relationships between fraction division and other mathematical knowledge (e.g., fraction concept; addition, subtraction, and multiplication of fractions) (e.g., Ma, 1999; Tirosh, 2000)
(d) Related misconceptions (e.g., can not divide a small number by a big number, division always makes a number smaller) (e.g., Greer, 1992)
(e) Solving problems involving fraction division (e.g., Greer, 1992)

The identification of these five types of difficulties provided a general framework for the current study and served as a guideline for examining the nature of prospective middle school teachers' possible difficulties with fraction division.

METHODOLOGY

Subjects

The participants were prospective middle school teachers enrolled in a mathematics and science interdisciplinary teacher education program at a southwestern U.S. university. These prospective teachers were in their last stage of study in the program. They had already taken all of the required mathematics courses and were completing mathematics methods course at the time of their participation in this study. A majority of the participants were seniors with only a few juniors. A total of 46 prospective teachers participated in the study for data collection.

Instruments and data collection

Two instruments were developed for this study. The first instrument was a survey of prospective teachers' general knowledge in mathematics and pedagogy. Many items were adapted from TIMSS 2003 background questionnaires (TIMSS 2003).

The second instrument was a math test that focused on prospective teachers' content knowledge and pedagogical content knowledge of fraction division. It contained items targeted to prospective teachers' possible difficulties as specified in the conceptual framework. While some items were adapted from school mathematics textbooks and previous studies (e.g., Hill, Schilling, & Ball, 2004; Tirosh, 2000), others were developed by the researchers of the current study.

All 46 prospective teachers enrolled in the mathematics methods course were invited to participate in this study. The participants were notified that both the survey and the test were for research purposes only and should be completed anonymously. The survey and the tests were administrated at the last class of the senior methods course. Participants were requested to complete the survey first, then the mathematics test.

Data analysis

Both quantitative and qualitative methods were used in the analysis of the participants' responses. Specifically, responses to the survey questions were directly recorded and summarized to calculate the frequencies and percentages of participants' choices for each category. To analyze participants' solutions to the problems in the mathematics test, specific rubrics were first developed for coding each item, and subsequently, the participants' responses were coded and analyzed to examine their use of specific concepts and/or procedures.

RESULTS AND DISCUSSION

In general, the results presented a two-sided picture that illustrated the importance of examining and understanding prospective teachers' knowledge in mathematics and pedagogy for teaching.

On one side, the results from the survey indicated that (1) participating prospective middle school teachers in this sample knew about their state curriculum framework in general; (2) they were confident in the preparation they received in mathematics and pedagogy for future teaching careers; and (3) they had developed general pedagogical understanding for mathematics classroom instruction.

On the other side, however, this group's performance on the mathematics test revealed that their knowledge in mathematics and pedagogy for teaching fraction division was procedurally sound but conceptually weak. The apparent inconsistent patterns in their responses suggested that these prospective teachers did not know what they would be expected in order to develop effective teaching. Their confidence was built upon their limited knowledge in mathematics and pedagogy. The following sections are organized to present more detailed findings corresponding to the two research questions.

Prospective teachers' perception of their knowledge preparation in mathematics and pedagogy for teaching

Prospective teachers' responses to the survey were quite positive. The following items were selected from the survey to illustrate prospective teachers' perception of their knowledge preparation needed for teaching, as related to fraction division.

For item 1: How would you rate yourself in terms of the degree of your understanding of the Mathematics Curriculum Framework in your state? On a scale of four choices (High, Proficient, Limited, Low), 9% and 91% of the participants chose "High" and "Proficient", respectively. None of the prospective teachers perceived themselves to have limited or low understanding of their state mathematics curriculum framework.

For item 4-(6): Choose the response that best describes whether students (grades 5-8) in your state have been taught the topic - *Computations with fractions.* In a scale of five choices (Mostly taught before grade 5, Mostly taught during grades 5-8, Not yet taught or just introduced during grades 5-8, Not included in the state math framework, Not sure), 96% participants indicated that the topic of fraction division is "mostly taught during grades 5-8", while the remaining 4% chose the first response ("Mostly taught before grade 5"). This result, in conjunction with the participants' response to question 1, suggested that this group of prospective teachers had general knowledge about their state's mathematics curriculum.

For item 5-(2): Considering your training and experience in both mathematics and instruction, how ready do you feel you are to teach the topic of "Number – Representing and explaining computations with fractions using words, numbers, or models?" On a scale of three (Very ready; Ready; Not ready), 60% of the participants thought they were "ready", while 38% chose "very ready" and 2% "not ready." The results indicated that this group of prospective teachers was confident in their preparation for teaching fraction computations, including fraction division. This result was further supported by their general pedagogical understanding for teaching mathematics.

In particular, item 6 on the survey contained several sub-items that examined participants' attitudes toward mathematics teaching. For example:

> To what extent do you agree or disagree with the following statement?
>
> (1) More than one representation (picture, concrete material, symbols, etc.) should be used in teaching a mathematics topic.
>
> (3) A teacher needs to know students' common misconception/difficulty when teaching a mathematics topic.
>
> (9) Modeling real-world problems is essential to teaching mathematics.

With a scale of four choices (Agree a lot; Agree; Disagree; Disagree a lot), the following table summarizes the responses.

Item	Agree a lot	Agree	Disagree	Disagree a lot
6-(1)	89%	11%	0%	0%
6-(3)	69%	27%	4%	0%
6-(9)	78%	22%	0%	0%

Table 1: Percentages of participating prospective teachers' responses

The results showed that this group of participating prospective teachers had developed a positive attitude toward mathematics teaching.

In general, prospective middle school teachers' responses to the survey suggested that they were confident in their preparation and they were ready to teach. In fact, these results are similar to what has been found from the US eighth-grade mathematics teachers in the TIMSS 2003 study (Mullis, Martin, Gonzalez, & Chrostowski, 2004). The consistency in responses between prospective teachers in the current study and US eighth-grade mathematics teachers in the TIMSS 2003 study suggests that US teachers develop their confidence quite early and hold their confidence for what they can do in teaching mathematics.

The nature of prospective middle school teachers' knowledge in mathematics and pedagogy for teaching fraction division

The prospective teachers' responses to the mathematics test allowed a closer look at the participants' knowledge in mathematics and pedagogy for teaching, especially on the topic of fraction division. Results indicated these prospective teachers did very well in computing fraction division. For example, for the problem "find the value of $\frac{7}{9} \div \frac{2}{3} =$", 93% of the participants solved the problem correctly. However, when the problem was changed slightly with a conceptual requirement, their performance decreased. As an example, to find "How many $\frac{1}{2}$'s are in $\frac{1}{3}$?", only 52% answered correctly. Many gave an answer as either "none" or "0". In fact, this problem is a typical problem in mathematics textbooks for middle school students. These prospective teachers' responses revealed their possible misconception related to division and their weakness in understanding fraction division in the verbal format.

Moreover, these prospective teachers experienced difficulty in solving problems that involved fraction division, especially for some of the multi-step problems. For example, only 39% participants solved the following problem correctly.

> Johnny's Pizza Express sells several different flavors of pizza. One day, it sold 24 large-size pepperoni pizzas. The number of large-size plain cheese pizzas sold on that day was $\frac{3}{4}$ of the number of large-size pepperoni pizzas sold, and was $\frac{2}{3}$ of the number of large-

size deluxe pizzas sold. How many large-size deluxe pizzas did the pizza express sell on that day?

The prospective teachers were also asked to explain given computations of fraction division. In particular, the problem of "How would you explain to your students why $\frac{2}{3} \div 2 = \frac{1}{3}$?; Why $\frac{2}{3} \div \frac{1}{6} = 4$?" (adapted from Tirosh, 2000) was included in the test. It was found that about 26% participants drew and used pictorial representations (e.g., fraction bar, pie chart) to explain the division procedure (e.g., how to divide 2/3 by 2 to get the answer 1/3), and 22% explained with "flip and multiply." Most (46%) other participants failed to provide a complete explanation to both computations. Surprisingly, none of these prospective teachers tried to explain the computations as why you can flip and multiply (e.g., why you can transform "divide 2/3 by 2" to "multiply 2/3 by 1/2").

These participating prospective teachers seemed to have even more difficulty when the computation procedure for fraction division was presented in a different way. In solving the following problem (adopted from Tirosh, 2000):

> You are discussing operations with fractions in your class. During this discussion, John says
>
> > It is easy to multiply fractions; you just multiply the numerators and the denominators. I think that we should define the other operations on fractions in a similar way:
> >
> > Addition $\frac{a}{b} + \frac{c}{d} = \frac{(a+c)}{(b+d)}$
> >
> > Subtraction $\frac{a}{b} - \frac{c}{d} = \frac{(a-c)}{(b-d)}$
> >
> > Division $\frac{a}{b} \div \frac{c}{d} = \frac{(a \div c)}{(b \div d)}$
>
> How will you respond to John's suggestions? (Deal with each separately.)

About 90% of the participants indicated that the given computations for fraction addition and subtraction were not correct, and only 2 out of the 46 prospective teachers stated that the given computation for fraction division was correct. The majority of others stated that the fraction division should be "flip and multiply" or "KFC" (i.e., keep the first, flip the second, and change the sign). The results suggested that these prospective teachers actually had very limited procedural understanding of fraction division, especially when related to other mathematical knowledge.

The results from these prospective teachers' responses on the mathematics test revealed their difficulties in all five types, as specified in the framework. However, prospective teachers' difficulties across these five types varied to a certain degree. It appeared that these prospective teachers can do a relatively better job when their thinking and explanation are aided by drawing pictorial representation, a result that is consistent with existing findings about US students' preference in using visual representation (e.g., Cai, 1995). However, performance became much less satisfactory when multiple mathematical relationships or mathematical ideas in an abstract format were presented.

CONCLUSION

The findings from this study show two different and seemingly contradictory sides of prospective teachers' knowledge in mathematics and pedagogy for teaching. The more positive perspective is revealed by the prospective teachers' responses to the selected survey questions. Certainly, these positive perceptions and attitudes can possibly help drive prospective teachers in their future efforts in developing effective classroom instruction. At the same time, however, their positive perceptions and attitudes are likely built upon insufficient (or limited) mathematical knowledge and pedagogical knowledge in mathematics. As revealed by their performance on the mathematics test, these prospective teachers had many difficulties with fraction division that they may not realize or recognize as deficiencies in their knowledge base. It is not realistic to expect prospective teachers to determine by themselves what they need to learn for future teaching career. Instead, as teaching mathematics requires a special set of skills (Viadero, 2004), it becomes necessary and important for teacher educators to identify what knowledge in mathematics and pedagogy prospective teachers need to learn through their program study. Ideally, prospective teachers would build their positive perceptions and attitudes upon their solid understanding of specific knowledge in mathematics and pedagogy for teaching.

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IMPROVING STUDENTS' ALGEBRAIC THINKING: THE CASE OF TALIA

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This paper presents the case of an 11th grader, Talia, who demonstrated improvement in her algebraic thinking after five one-hour sessions of solving problems involving inequalities and equations. She improved from association-based to coordination-based predictions, from impulsive to analytic anticipations, and from inequality-as-a-signal-for-a-procedure to inequality-as-a-comparison-of-functions conceptions. In the one-on-one teaching intervention, she progressed from the sub-context of manipulating symbols, to working with specific numbers, to reasoning with "general" numbers, and eventually to reasoning with symbols. Three features were identified to account for her improvement: (a) attention to meaning, (b) opportunity to repeat similar reasoning, and (c) opportunity to explore.

INTRODUCTION

Research has shown that some students will spontaneously apply a procedure or algorithm as soon as they are given a mathematics problem. For example, Cramer, Post, and Currier (1993) observed 32 of 33 students in a mathematics methods class apply the proportion algorithm to solve this problem: *Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?* Without appearing to understand the underlying structures, many students inappropriately apply procedures taught to them; Fischbein and Barash (1993) call this *improper application of algorithmic models*. Errors such as thinking that $\frac{3a+5b}{2a+5c} = \frac{1}{2} \Rightarrow \frac{3+b}{2+c} = \frac{1}{2}$ and $(x-6)(x-9) < 0 \Rightarrow x < 6$ or $x < 9$ are rather common among algebra students (Matz, 1980). Many students fail to make the connection between *structural conception* and *operational conception* (Sfard, 1991). For example, they interpret an equation as an object to be transformed into "x = ___". The only source of meaning is the rules for solving the equation (Sfard & Linchevski, 1994). Without conceiving an equation as a relation, high school and college students may interpret the equal sign as a signal to do something—for example, to solve for a variable or to find its derivative (Kieran, 1981). Consequently, students exhibit *non-referential symbolic reasoning* (Harel, in press) when they operate on symbols as if "the symbols possess a life of their own" without attending to referential meaning.

The research that this paper reports sought to characterize the way students *anticipate* as they solve non-routine problems involving algebraic inequalities and equations. I define *anticipating* as a mental act of conceiving a certain expectation without performing a sequence of detailed operations to arrive at the expectation. This research

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 193-200. Seoul: PME.

had three objectives: (a) to identify and characterize students' anticipations, (b) to identify the relationship between the characteristics of students' anticipations and students' interpretation of inequalities/equations, and (c) to explore the potential for advancing the way students anticipate. Results related to the first two objectives were presented in the 28th PME-NA conference (Lim, 2006). This paper presents results related to the third objective.

THEORETICAL FRAMEWORK

This research was based on several theoretical constructs: Piaget's (1967/1971) notion of anticipation, von Glasersfeld's (1998) three general kinds of anticipation, and Cobb's (1985) three hierarchical levels of anticipation. In addition, Harel's (in press) notions of way of understanding and way of thinking were employed to analyse students' act of anticipating.

According to Piaget (1967/1971), anticipation is one of the two functions of knowing; the other function being conservation-of-information, an instrument of which is a scheme. The anticipation function deals with the application of a scheme to a new situation. It allows us to strategize and plan, have foresight, make predictions, formulate conjectures, engage in thought experiments, etc. Foresights and predictions are possible because of our ability to assimilate situations into our existing scheme(s); "anticipation is nothing other than a transfer or application of the scheme … to a new situation before it actually happens" (p. 195).

von Glasersfeld (1998) elaborated on Piaget's notion of anticipation by pointing to three general kinds of anticipation: (a) implicit expectations that are present in our actions, e.g., the preparation and control of our movements when we grope in the dark; (b) prediction of an outcome, e.g., predicting that it will soon rain upon noticing that the sky is being covered by dark clouds; and (c) foresight of a desired event and the means for attaining it, e.g., a child's anticipation of the capitulation of his parent if he were to throw a temper tantrum in public. In this research, I focused on the latter two kinds of anticipation. I define *predicting* as the act of conceiving an expectation for the result of an event without actually performing the operations associated with the event, and *foreseeing* as the act of conceiving an expectation that leads to an action, prior to performing the operations associated with the action.

Cobb (1985) identifies three hierarchical levels of anticipation: beliefs, problem-solving heuristics, and conceptual structures. At the global level, students' beliefs about mathematics influence their anticipations. At the intermediate level, a child anticipates a *heuristic*—"a metacognitive prompt which delimits a subcontext within which the child anticipates she can elaborate and solve the problem" (Cobb, 1985, p. 124). For example, anticipation of a guess-and-check strategy may lead a student to operate in the sub-context of plugging in numbers. At the most specific level, a child's expressed conceptual structures (i.e., evoked schemes) dictate the child's anticipations. According to Cobb, higher-level anticipations constrain lower-level anticipations, i.e., students' specific anticipations are confined both by their beliefs and by the sub-context in which they operate.

I used Harel's (in press) MA-WoU-WoT framework to analyse students' *mental acts* (MA_s) of predicting and foreseeing. Predicting and foreseeing are among the many mental acts that one might carry out while solving a mathematics problem. Other mental acts include interpreting, symbolizing, generalizing, justifying, and inferring. A *way of understanding* (WoU) refers to the product of a particular mental act, and a *way of thinking* (WoT) refers to a character of this act. Taking the act of predicting as an example, a WoU refers to the result a student actually predicts whereas a WoT characterizes the manner in which the student predicts. Likewise, students' interpretations of inequalities/equations can be viewed as W_soU associated with the act of interpreting inequalities/equations.

METHOD

This research was conducted in a university-based charter school in Southern California. Fourteen 11th graders were interviewed, each for about 60 minutes. Four of these interviewees participated in a one-on-one teaching intervention, which involved five problem-solving sessions followed by a post-interview.

Tasks used in the clinical interviews include: (a) Is there a value for x that will make $(2x - 6)(x - 3) < 0$ true? (b) Given that $5a = b + 5$, which is larger: a or b? And (c) Given that m is greater than n, can $m - 14$ ever be equal to $7 - n$? These tasks differ from typical tasks in textbooks in that they do not direct students to perform a specific task such as "solve for x" or "simplify." I found this non-directive feature effective at eliciting a variety of anticipatory behaviours. All the tasks were phrased in the form of a question so that students could predict the answer, if they chose to, prior to performing any actions. Tasks used in the teaching intervention involved only one variable. This way, participants' responses to two-variable tasks in the post-interview allowed me to see whether the improvements in their W_soT and W_soU went beyond the context in which these W_soT and W_soU were learned.

The designing, sequencing, and assigning of tasks in the teaching intervention were guided by the three primary pedagogical principles in Harel's DNR-based instruction (2001, in press). The *Duality Principle* asserts that the W_soT students possess influence the W_soU they produce, which in turn influences the development of their W_soT. The *Necessity Principle* stipulates that for students to learn a particular concept, they must have an intellectual need for it. The *Repeated-reasoning Principle* asserts that "students must practice reasoning in order to internalize, organize, and retain" what they learn.

All the interviews and problem-solving sessions were videotaped and transcribed. *Observation concepts* (Clement, 2000) for students' W_soT associated with predicting/foreseeing and students' W_soU inequalities/equations were identified. Categories for W_soT and W_soU were derived from the data using a *constant* *comparative approach* (Glaser & Strauss, 1967), in which categories were constantly revised by comparing current data with previously analysed data. The analysis involved identifying instances of the mental acts of predicting and foreseeing (inferred from student's actions and statements), generating, comparing, and refining categories for W_soT and W_soU, and consolidating and collapsing some of the categories. For each of the four learners, a table of codes was created to track the changes from the pre-interview to the post-interview in: (a) the learner's W_soT associated with predicting/foreseeing, (b) W_soU inequalities/equations, (c) *sub-context* in which the learner was operating, and (d) quality/correctness of solutions. Episodes of all five problem-solving sessions for the learner Talia were analysed to gain a general sense of her ways of thinking and ways of understanding. I later revisited the data to account for significant transitions as well as to account for the change in her ways of thinking and ways of understanding.

RESULTS AND DISCUSSION

In this study, three ways of thinking associated with predicting were identified: *association-based prediction*, *comparison-based prediction*, and *coordination-based prediction*. Five ways of thinking associated with foreseeing were identified: *impulsive anticipation*, *tenacious anticipation*, *explorative anticipation*, *analytic anticipation*, and *interiorized anticipation*. In addition, five ways of understanding inequalities/equations (I/E) were identified: *I/E-as-a-signal-for-procedure*, *I/E-as-a-static-comparison*, *I/E-as-a-proposition*, *I/E-as-a-constraint*, and *I/E-as-a-comparison-of-functions*. Students' W_soT associated with predicting/foreseeing were found to be related to the quality of their solutions as well as to the sophistication of their W_soU inequalities/equations. These results are presented in PME-NA (Lim, 2006). In this paper, I focus on Talia's improvement from pre-interview to post-interview, her trajectory from the sub-context of manipulating symbols to the sub-context of reasoning with symbols, and some features of the teaching intervention that might account for her improvement.

Talia's Pre-to-post-interview Improvement

In the pre-interview, Talia was operating in the *sub-context of manipulating symbols* for single-variable tasks. While operating in this sub-context, she tended to be procedure-oriented and thus exhibited *impulsive anticipation*. For the task, "Is there a value for x that will make the following statement true? $(2x - 6)(x - 3) < 0$", she spontaneously expanded the expression without studying the inequality, used the quadratic formula, obtained $\frac{6\pm\sqrt{6^2-4(1)(9)}}{2}$, and commented "that reduces to 3, which is less than 0 (wrote $3 < 0$). That's not true." Her not attending to the meaning of the symbols contributed to her exhibiting *association-based prediction*. She predicted that 3 was not a solution because she saw $3 < 0$ was false. Her prediction was based on her associating the result of $\frac{6\pm\sqrt{6^2-4(1)(9)}}{2}$ with the output of $x^2 - 6x + 9$; i.e., she conflated the root of a quadratic function with the function itself. When she plugged 3 into the inequality and obtained 0 < 0, she predicted that 6 might be a solution: "Maybe I'm supposed to multiply by 2." Because of her association between the value of $\frac{6 \pm \sqrt{6^2 - 4(1)(9)}}{2}$ and the function $x^2 - 6x + 9$, she thought she should double the resulting value of 3 so as to compensate for halving $2(x^2 - 6x + 9)$ to get $x^2 - 6x + 9$.

In the post-interview, Talia was operating in the *sub-context of reasoning with symbols*. While operating in this sub-context, she tended to be goal-oriented, and thus exhibited *analytic anticipation*.

Talia: Um, $2x$ minus 6 times x minus 3 is less than 0. So … this [side] has to give me a negative number. I can get a negative number from here $(2x - 6)$, oh, but there is also a negative times negative is positive. So I have to make one of these negative and one of these positive. In order to get this, so this will be negative if it is less than 6, but then if I want to make this one positive, it has to be greater than 3. So, or I could go the other way around. … This side could be, umm, greater than 6, x could be greater than 6, makes this positive, $2x$, I'm sorry, $2x$ [could be greater than 6]. And x could be less than 3, which will make this negative, and so these two conditions will make this statement true.

Talia analysed the inequality with the goal of making the function $(2x - 6)(x - 3)$ less than zero, and she foresaw the sub-goal of making one factor positive and the other negative. Talia's pre-to-post-interview improvement, as depicted in Figure 6.1, is considered significant because only 2 out of the 16 inequalities/equations used in the teaching intervention involved quadratic functions in factored form. Moreover, both inequalities, $x(6x + 8) < 0$ and $3x(500 - 2x) < 30(500 - 2x)$, do not involve repeated roots.

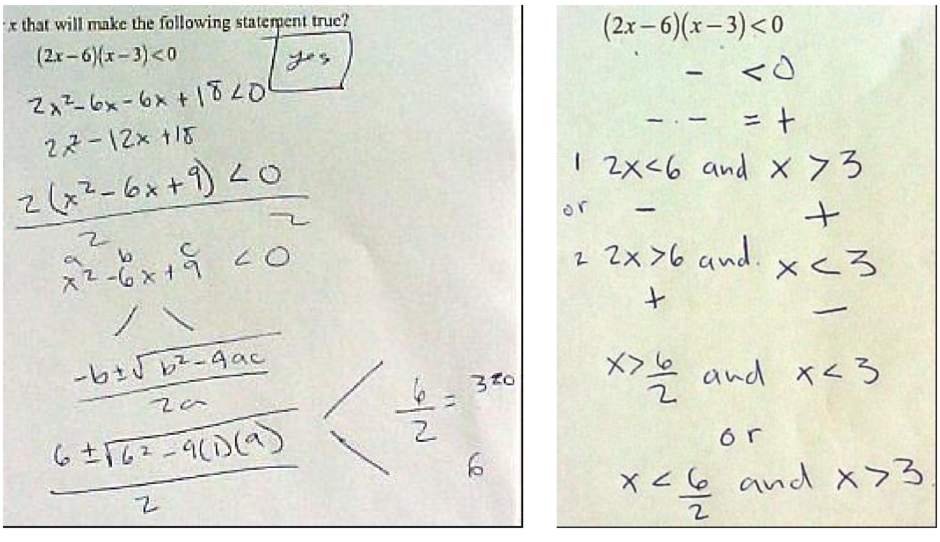

Figure 6.1: Pre-and-post-interview comparison of Talia's work

When working on two-variable tasks, Talia demonstrated more instances of coordination-based prediction and analytic anticipation in the post-interview than in the pre-interview. For example, she exhibited only one instance of comparison-based prediction in the pre-interview, but two instances of coordination-based prediction and one instance of comparison-based prediction in the post-interview for this task: "Given that $5a = b + 5$, which is larger: a or b?" In the pre-interview, her prediction appeared to be based on a comparison between the two sides in terms of their arithmetic operations: "if a and b were equal, then a would be larger because, I mean this ($5a$) value would be larger." In the post-interview, her prediction, though still incorrect, incorporates change and compensation: "b will have to be larger, just because you need more adding than you do multiplying in order to get [$b + 5$] large."

Talia's Trajectory from Manipulating-symbols to Reasoning-with-Symbols

Talia's transition from the sub-context of manipulating symbols to the sub-context of reasoning with numbers involved two intermediate stages: working with specific numbers and reasoning with general numbers (e.g., large positive numbers, small positive numbers, and negative numbers). In the first problem-solving session, when Talia was presented with the inequality $\frac{x-5}{x-10} < 0$ with no accompanying instruction, she interpreted the inequality as a signal to solve for x.

Lim:	Alright, this ($\frac{x-5}{x-10} < 0$) is the first problem.
Talia:	OK. So I just solve it? Alright, arrr, so I'm trying to solve for x. So I'm just going to multiply both sides by x minus 10, x minus 10, and it's x minus 5 is less than 0. And then you just add 5 to both sides. x is less than 5. Um, I think that's my answer.
Lim:	What does this answer ($x < 5$) mean?
Talia:	Um, that, this equation is true for any value of x that are less than 5, so, let me just try that out. So, 4 minus 5 over 4 minus 10.

Having found that $x = 4$ did not make the inequality true, Talia continued to think of alternative means for manipulating the inequality: "How am I supposed to solve this? Um, maybe I can factor something out." It was only when she was asked, "What does solve for x mean?" that she attended to meaning and responded, "To find the values for this problem where the statement is true." She then foresaw plugging in numbers.

Talia:	So, umm, I'm just going to try some random values for this, 2. 2 minus 5 is -3. 2 minus 10 [is] -8. Umm, it has to be a number that is positive on the top and negative on the bottom, so I can get a negative number, and then the statement will be true. So, something that will give me positive is 6 minus 5, and 6 minus 10. This is…positive 1, over negative, um, 10, 3 4 5 6 (finger counting), 4 and that's less than 0. So one-fourth is a value that makes this statement true. … I'm sorry, 6.

Within the context of working with specific numbers, Talia could reason in a goal-oriented manner and foresaw plugging in 6 to make the numerator positive and the denominator negative. She even extended her reasoning to obtain all the values that would make the inequality true: "So x can be anything that is, um, bigger than 5, but less than 10. So 6 7, 6 7 8, 9."

The change in sub-context from manipulating symbols to plugging in numbers was probably initiated by questions such as "What does this answer mean?" and "What does solve for x mean?" This implies that mathematics teachers should help students *attend to meaning*.

The transition from working with specific numbers to reasoning with general numbers occurred in Talia's initial response to the second task: "Is $x(6x + 8) < 0$ always true, sometimes true, or never true?"

Talia: Is x [times] quantity of $6x$ plus 8 less than 0 always true, sometimes true, or never true? Mmm, I'm thinking if I make x into a negative number so that, um, so that the whole function will be negative. So if there is an answer, it will probably have to be negative because if I make x positive, it's going to be greater than 0 all the time. Right? … OK. Um, so let me just try a negative number, -1.

In this task, Talia began to reason with general numbers: x being positive would make the inequality "greater than 0 all the time." The transition from working with specific numbers to working with general numbers might be due to the inequality having x as a factor. An implication for teaching is that instructional tasks should be designed to allow students to apply, and then extend, their ways of understanding. The quadratic inequality $x(6x + 8) < 0$ is considered a good follow-up to the rational inequality $\frac{x-5}{x-10} < 0$ because the two functions are structurally different, yet they both foster the same way of thinking: being goal-oriented so as to make one factor positive and one factor negative. Hence, assigning $x(6x + 8) < 0$ as the second task is consistent with the *Repeated-reasoning Principle* (Harel, 2001).

The transition from reasoning with general numbers to reasoning with symbols began with the above task and continued through the entire teaching intervention. Reasoning with symbols involves certain ways of understanding, emergence of which required Talia to explore the problem situations by plugging in specific numbers and/or reasoning with general numbers, such as a number in the interval [-1, 0]. This observation suggests that mathematics educators should use reasoning with numbers as a platform for students to explore algebraic structures. I contend that Talia's undesirable ways of thinking—such as non-referential symbolic way of thinking and association-based prediction—probably resulted from her working with algebraic symbols without the support of numbers. A lack of numerical support for algebraic reasoning is a plausible cause for why some students perceive the world of algebra and the world of arithmetic to be disconnected, a phenomenon observed by Lee and Wheeler (1989).

CONCLUSION

The case of Talia demonstrates the feasibility of helping students improve their algebraic thinking—in particular, moving from manipulating symbols in a non-referential symbolic manner to reasoning with symbols in a goal-oriented manner, from association-based prediction to coordination-based prediction, and from impulsive anticipation to analytic anticipation. This research underscores the importance of helping students attend to meaning, creating opportunities for students to repeat certain reasoning, and using numbers as a platform for students to investigate algebraic expressions and structures.

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THE EFFECT OF A MENTORING DEVELOPMENT PROGRAM ON MENTORS' CONCEPTUALIZING MATHEMATICS TEACHING AND MENTORING

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ABSTRACT

The study was to develop a mentoring program and examine its effect on mentoring mathematics teaching. A collaborative mentor study group consisting of four mentors and the researcher was set up. The course with 78 hours to develop mentors' theoretical and professional knowledge in which underpins mentoring practice was carried out in the half-year internship. Two surveys, pre- and post-test of pedagogy, self-assessment in mentoring, interview, classroom observation, and reflective journal were the data collected for the study. The satisfaction with the initiates, improvement of mentoring knowledge, and the transfer from the program to support interns on questioning, problem-posing, and anticipating students' solutions were as a result of the mentoring program.

INTRODUCTION

Mentoring as a reform is increasingly used in both preservice programs. Socioculturalists agree that mentoring has greater potential to support teacher learning since knowledge is situated in and grow out of the contexts of mentors use. With support of a mentor working intern's zones of proximal development, the intern can learn to perform beyond his/her independent performance level. This is relevant to the issue of effective mentoring. In recent years, teacher education programs have been encouraged to devote more attention to developing partnerships with schools and helping teachers become equipped to mentor interns (Sutherland, Scanlon, Sperring, 2005). The reformers regard the university-school partnership with the mentor-novice relationship in the context of teaching as one of the important strategies to support novices' learning to teach, thus, to improve the quality of teaching (Odell, Huling, & Sweeny, 1999). A great deal of research on mentoring has identified a wide range of qualities needed (Wang & Odell, 2002). For instance, essential prerequisites include supporting the development of effective classroom practice. This indicates that mentors should be supported to meet the quality of mentoring. Thus, there is a need for a teacher education associated with a school to develop a mentoring development program to support mentors on interns' learning.

Due to the rapidly increased number of preservice teachers has overloaded the teacher education system in Taiwan. One of the shortcomings of the system is the lack of specific subject mentoring by faculty during the internships in schools. With a heavy mentoring load and widespread geographical locations of interns mentored,

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much of the responsibility for mentoring interns lies with the mentors in the schools rather than with teacher educators. Moreover, most of the elementary school teachers teach several subjects, it leads to require interns to have teaching in all the subject areas. However, this requirement might not have been fulfilled completely because the mentors had little knowledge or different ideas in their minds about the roles of the interns in teaching mathematics (Lo, Hung, & Liu, 2002). To improve the quality of mentoring, a mentor development joint research project including mathematics, language, and science, as an innovative approach, was initiated at a university associated with a school. The aim of the joint project is to train professional subject mentors and to improve interns' teaching in all subjects. This study involving mathematics as one of the three sub-projects contained in the joint project was intended to develop a mentoring development program that was designed to develop both mentors conceptualizing mathematics teaching and practicing for mentoring interns. The effect of the program on mentors' conceptualizing and practicing in teaching and mentoring will be examined in the study.

Theoretical Framework of the Mentoring Development Program

The theoretical framework of mentoring to support interns learning to teach is based on three models of mentor preparation in which are widely used in both preservice programs. The knowledge transmission model assumes that knowledge of mentoring comes from research rather than from mentors' own experiences and practices, so that such knowledge can be transmitted to mentors in the form of discrete concepts and skills. Although this model helps mentors to develop many mentoring skills and techniques, there is no evidence that mentors are able to apply such learning in their practice with interns (Evertson & Smithey, 2000).

The theory-and-practice connection model assumes that knowledge of mentoring comes both from research and mentors' practical knowledge, so that mentoring skills and knowledge should be actively constructed by mentors and then through integration of their practical knowledge of teaching and learning, with the support of teacher educators. This model, unlike the transmission model influences mentors' sensitivity the needs and problems of interns. This model, teacher educators are still distant from actual mentoring practice (Wang & Odell, 2002).

The collaborative inquiry model stresses mentors' active construction of mentoring knowledge through the integration of their practical knowledge of teaching, the application of what they have learned in practice and constant dialogue with teacher educators. The distinction of this model from the theory-and-practice connection model in that teacher educators work with mentors and interns residing in the context of teaching and mentored learning to teach. Through this process, the teacher educator helped to deepen the mentor's understanding of practice. This model not only values mentors as learners who actively inquiry into teaching and mentoring but also views teacher educators as learners who examine and develop the knowledge and skills of mentoring in the context of teaching and mentoring. In this model, mentors, interns, and teacher educators are all researchers, learners, contributors of knowledge related to teaching and mentoring.

The effects of each model on mentoring are likely to be different. The mentoring development program with university-school partnership developed in this study takes the assumption of collaborative inquiry model that knowledge and skills of mentoring are constructed through practice-centered conversation and collaborative inquiry with a community of mentors in the contexts of teaching and mentoring.

METHOD

Study Context

In developing the program to enhance mentor development, the main consideration was dependent on the willing of the mentors and interns. There was no agreement that enabled them to be remunerated for their participation. When developing the activities involved in the study were conscious of the need to maximize the interns' involvement in the internship while at the same time minimizing the disruption this participation might cause the mentors and the school.

Four teachers (Yeu, Lang, Ju, and Zue) from an elementary school met regularly with a teacher educator (the researcher) from a university supervising interns (Huei, Ting, Jong, and Jun) to investigate collective experiences with mentoring. The half-year placement plan enabled four interns to be placed to the school during the study. The researcher was assigned for four hours per week as the university program liaison responsible for interns' supervision at the school.

The Mentoring Development Program

The goal of the program was to create the opportunity of support mentors learning in mentoring such that enhancing the quality of teacher education offered at the university by providing the interns with greater involvement with mentors. The course of the program was designed to develop mentors' theoretical and professional knowledge in which underpins mentoring practice. The theoretical knowledge provides mentors with an understanding of designing principles and gives rise to the professional knowledge when mentors enact it in implementing tasks in classrooms (Shulman, 1998). The two components integrated into the course containing 78 hours with twenty six 3-hour units were implemented at two stages: (i) a summer workshop with 36 hours containing twelve 3-hour sessions (ii) school-year initiates with 42 hours containing fourteen 3-hour sessions in a half year. Table 1 introduces the contents, units, and periods of the workshop of the course.

During the half-year internship, a collaborative mentor study group (CMSG) consisting of the research and four school teachers was set up. To provide the support of learner-oriented teaching and mentoring to the mentors, the structure of the activities was developed. The lessons of the mentors were scheduled to be observed twice in turn. The first of which was to enable each mentor to watch a pair of mentor-intern preparing a lesson, how to carried out the lesson, and de-briefing conversation between them on the lesson. The CMSG met routine weekly with 3 hours allowing mentors mutually support to learn from one another's mentoring in preparing a lesson, observing, and reflecting on a lesson. The intern was asked by the observed mentor to identify areas which she felt more problematic in implementation and reflection on the lesson. Each weekly meeting was audio and video taped.

Contents of the course	# of units	# of units in workshops	
		Summer	School-year
Pedagogy of mathematics contents	7	3	4
Assessment of mathematics	2	1	1
Design of mathematics activities	3	0	3
Classroom observation	1	1	0
Use of teaching aids	2	1	1
Management of mathematics classroom	1	1	0
Mentoring practices	9	1	8
Professional development and reflection	1	0	1
Totally	26	8	18

Table 1: Contents, units, and period of workshops of the course of the mentoring program

The researcher, as a learning partner of the mentors, played the roles in facilitating, probing, and giving feedback to the mentors and created the opportunity for the mentors to discuss how to maximize the opportunities of the interns' learning. The researcher believes that mentors' knowledge of mentoring is actively constructed through the integration of practical knowledge of teaching and experience of learning via social interaction.

Data Collected and Analyzed

Kirkpatrick and Kirkpatrick's (2006) model was the basis of the study to examine the effect of the mentoring program. At the reaction level, the mentors were interviewed on the feedback of summer workshop and half-year school mentoring activities for measuring what they thought and felt about the program. At the learning level, pre-test and post-test were conducted aligned with self-assessment 5-scale questionnaire professional standards, which was built in previous year of the study (Lin & Tsai, 2007), to assess the extent to which mentors change attitudes, improve knowledge and skill. The instrument with 15 items consisted of 5 items for assessing knowledge of content, 5 items of pedagogical knowledge, and 5 items of knowledge of students' cognition. The constructed validity and reliability of the questionnaires has been examined in previous study (Lin & Tsai, 2007). At the behavior level, classroom observation, interview, and mentors' mathematics journal were measured how mentors transferred their knowledge and skill in mentoring as a resulted of the mentoring program. Each mentor was also conducted individually with a semi-structure interview. The interview included questions about their views of teaching and mentoring, their mentoring practices as well. For the purpose of the study, only some parts of the data in the interview were used.

RESULT

Reaction Level: Satisfaction of the Course of the Program

Table 2 describes the mentors' satisfaction of the activities of the half school-year initiates. They not only had a consistent agreement on the importance but also had a satisfaction with the topics including pedagogy, observation, and sharing practice of mentoring conducted in summer workshop. Nevertheless, for them, classroom arrangement is not as important as other topics. Zue commented that

…I have already developed my own way of classroom management through years of teaching experience. General classroom management is not needed at this moment. Instead, to create the norms of students' discourse in mathematics classroom is my weakness so that it is hard for me to start. I finally realized the significance of instructor's role of playing students' discussion from this summer workshop. … (Zue, Interview).

Contents of the course	Importance		Satisfaction	
	Mean	SD	Mean	SD
School-year workshop on mathematics teaching				
Understanding the logic sequence of activities	4.0	0	4.5	0.5
Enhancing teachers' ability in questioning	4.0	0	4.25	0.83
Enhancing teachers ability in problem posing	4.0	0	4.25	0.83
Designing lesson plan	4.0	0	3.75	0.83
Social mathematics norms	4.0	0	3.5	0.87
Diagnosing students' misconception and remedy	3.75	0.43	3.75	0.83
School-year workshop on mentoring teaching				
Observing peers' lessons	4.0	0	4.0	1.0
Working with intern on lesson plan	3.75	0.43	4.5	0.5
Observing mentor-intern mentoring on lesson plan	3.75	0.43	4.25	0.83
Working with other mentors on lesson plan	3.75	0.43	4.5	0.5
Discussing with intern in post-lesson	3.25	0.43	3.5	0.5
Discussing with the CTSG after their own teaching	4.5	0.5	4.5	0.5
Discussion with the CTSG after peer's teaching	3.75	0.43	4.0	1.0

Table 2 Mentors' Satisfaction of the Course of Mentoring Program

Two kinds of activities engaged in the CMSG were to foster mentors' knowledge of teaching and mentoring in teaching, as described in Table 2. Of the knowledge of mathematics pedagogy, the mentors had a consistent agreement on the importance of mentoring activities including the logic structure of activities, enhancing skills of questioning, and problem posing. Ju reflected on fraction teaching as follows.

…Although having 14 years of teaching, I have never realized various meanings of fraction. In the workshop, I know the various meanings of fraction including part-whole model, iteration of unit, a value of number line, operator, and ratio. There is a different degree of difficulty for students learning among them. …(Ju, Interview).

The mentors had the least satisfaction with the work of social mathematics norm (Mean=3.5), although they agreed the importance of the social mathematics norm of a learner-oriented approach. The data of Table 2 shows that the mentors were satisfied with the mentoring in teaching, in addition to lesson plan working with intern (Mean=3.5). Lang expressed her opinion on lesson plan working with intern as follows.

…prior to my teaching, I worked a lesson plan with Jong. Initially, I asked him to read textbook and searching for relevant resources of teaching in advance. In the lesson plan meeting, he grasped the objective of each activity, but he was not aware of the need of adaptation of the activities covered in the textbook. While planning a lesson, Jong did not attend to the importance of anticipating students' possible strategies and posing a contextual problem.…. (Ju, Interview).

Learning level: The Extent to Which Mentors Improved Knowledge and Skill.

Comparing the pretest, the mentors had a better performance than on posttest of knowledge. The percentages of pre- and post-test four mentors Yeu, Lang, Ju, and Zue performed were from 40% to 73%, from 53% to 86%, from 40% to 80%, from 40% to 67%, respectively. The result indicates that the mentors improved their content knowledge, pedagogical knowledge, and had better understanding of students' learning, but they still had a space to improve continually their knowing about teaching. The data also shows that the mentors had very poor understanding of the logic structure of activities performed in the pre-test. With the mutual support of the CMSG, they gradually constructed the structure of teaching in specific topic.

Self-assessment of confidence in mentoring is the indicator of improving mentors' knowledge and skill in mentoring. Before entering the program, the mentors had no confidence in performing 9 items out of 16 items of professional literacy, 20 items out of 34 items of mathematics teaching, and 24 items out of 36 items of mentoring practice, respectively. With the help of the program, they gained more confidence in mathematics teaching and mentoring their interns. The items they had no confidence in performing were from 53 decreasing to 15 items including 9 items of teaching and 6 items of mentoring teaching. Their improved knowledge and skill of questioning and problem-posing was also supported by their reflective journals.

…in terms of asking key question, I know it is important to clarify students' thought but I had an anxiety with this after summer workshop. With the help of classroom observation, I know that it is important to give students longer time to think about the question I asked. Before entering the program, most of the questions I asked were too closed to stimulating students' various solutions (Lang, Journal).

Behavior Level: Transfer occurred in Interns' Teaching Mentored by Mentors

Classroom observation was measured how mentors transferred their knowledge and skill in mentoring interns' teaching as a result of the program. As observed, the mentors not only improved their pedagogical knowledge but also enhanced their ability in transferring to guide interns' on problem-posing, asking key questions, and anticipating students' possible solutions. Yeu reminded Huei of connecting to students' daily context while giving students a problem to solve.

…I worked with my intern on reviewing the structure to examine whether the problems given in teacher guide are relevant to students' daily life. I suggested them keep the objective of the lesson but using more attractive problems to replace the de-contextualized problems if needed. … (Yeu, CMSG meeting).

The four interns' performing in teaching is one of the indicators of examining the effect of the mentoring program. Each intern's lesson was assessed by eight assessors consisting of the researcher, four mentors, and three interns. The average score of each of the 10 items of classroom behavior for each intern was listed in Table 3. The data shows that excepting the item of encouraging students to figure out various solutions and comparing them, they performed well in other items of teaching behavior. The average score of each item has greater than 4.0. The result indicates that mentors' mentoring in interns' knowledge and skill dealing with encouraging students to figure out solution is not so easy as other teaching skills.

Teaching behaviors	Huei	Ting	Jong	Jun	Mean
1. Drawing students' attention by using various strategies	3.7	4.8	4.5	4.6	4.4
2. Appropriateness of using teaching aids (e.g. technology).	3.6	4.9	4.9	4.6	4.5
3. Asking key questions to support students' thinking.	3.8	4.4	4.1	4.5	4.2
4. Posing problems relevant to students' daily life.	4.3	4.4	3.5	4.5	4.2
5. High interaction with students.	3.6	4.9	4.5	4.0	4.3
6. Encouraging students' to figure out various solutions	4.1	3.6	4.1	3.9	3.9
7. Enable to diagnose students' difficulty and	3.6	4.3	4.3	3.8	4.0
8. Giving students' feedback at a right time.	3.9	4.7	4.4	4.1	4.3
9. Giving a creative teaching.	3.3	4.9	4.8	4.8	4.5
10. Reaching the objective of the lesson.	3.4	4.1	4.3	4.4	4.1

Table 3: The average score of each item of teaching behavior for each intern

DISCUSSION

One of the challenges of mentoring was to provide mentors with opportunities for authentic experiences. Through the university-school partnership, the mentors and the teacher educator created more opportunities of the dialectics and justification between theory and practice of mentoring in mathematics teaching. This study supporting mentors to engage with the practice provided an example of innovative method of mentoring on conceptualizing mentoring and practicing, and then enhanced the interns learning to teach. For the mentoring development program to be successful, it was required the willing participation of the mentors who were asked to accept an extra load by assisting the interns. Besides, the course structure of the program provided opportunities for the mentors to relate the theoretical knowledge and the practical mentoring. In these experiences, the mentors were not focused on the technical skills of mentoring. Instead, they were engaged in meaningful professional-related tasks. Engaging in meaningful tasks appeared to facilitate the development of the mentors professional knowledge and skill in teaching and mentoring. They were able to relate the theory offered by the teacher educator to the practical needs of mentoring. In this way, the theory or model of mentoring became more meaning for the mentors.

This study also suggests that collaborative mentor study group with offered one promising avenue for supporting mentors learning in teaching and mentoring. Although focused on developing mentors' understating about and practice of mentoring in mathematics teaching, this study also provided an example of the kind of professional learning to promote inquiry-oriented practice more generally. The teacher educator and the mentors jointly constructed understanding of mentoring practice by fostering interactive conversation around artifacts of mentoring practices, developed out of the practice of CMSG participants,

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USES OF EXAMPLES IN GEOMETRIC CONJECTURING

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We present 24 sixth graders' conjecturing when given geometric conditions. We have provided simultaneously three different graphs: typical, conjunctive, and extreme examples. The students were asked to guess what other geometric invariance has to exist. We discovered that the students would generate more related conjectures if they looked at one example instead of two or three at the same time in the first minute after each question begins. They would generate more conjectures if the example were conjunctive example. However, the percentage of related-correct conjunctions is less than observing typical examples. The quantity and quality of conjectures decreases when students look at extreme examples that have bizarre shapes or scales. We also discovered that the activating of relational schema is related to the conjecturing.

INTRODUCTION

Mathematics is regarded by many as a demonstrative science, but Polya thinks that the formation of mathematics has no differences from any other knowledge. He pointed out all knowledge include mathematics consisting of conjectures. You have to guess a mathematical theorem before you prove it. If the learning of mathematics reflects to any degree the invention of mathematics, it must have a place for guessing and plausible reasoning (1954, Vol. I, pp. v –vi). Lakatos (1976) and Boero, Garuti, and Mariotti (1996) also share the same opinion. They believe that the first step of "discovering" is to propose conjectures. Conjectures allow correlations to be formed between concepts or properties. Mathematical thinking is also expanded with the follow-up processes of justifying, proving, and improving. Therefore, the forming of conjectures is the premise for solving problems and proving geometry.

Mason, Burton and Stacey (1982) define conjecture as a plausible proposition of which the truth has not yet been established. In other words, this is a conclusion that has not yet been refuted by analyzing any relevant examples. Polya (1954) has provided many examples to describe that the process of conjecture is an induction that includes generalization, specialization, and analogy. Generalization is passing from the consideration of a given set of objects to that of a larger set. Specialization is passing from the consideration of a given set of objects to that of a smaller set. Analogy is a sort of similarity, on a more conceptual level.

The teaching of geometry often targets at helping students to learn how to proof. Before learning deductive geometry, students learn about geometrical objects through the "intuitive" method. It is difficult to understand why they need to put so much effort into proving a known fact (Mariotti, 2000). Therefore, many instructional experiments on argumentation involve having debates over a topic that is waiting to be justified in a mathematics classroom. Obviously, the need for proving since a proposition is plausible rather than completely true or false. Conjecture provides the kind of intellectual need that fits right into this need.

Using a fifth grade math class activity, Reid (2002) points out that mathematical reasoning contains many characteristics of the so-called "scientific investigation" proposed by Karl Popper, in which students observe the pattern shown in an example, generate conjectures, test the conjectures, and they either return to the process of observing the case or prove the conjecture and continue to look for a more generalized process of deduction. The last part of which is the difference between mathematical deduction and scientific deduction since mathematical deduction enters the stage of determining how a conjecture can be proven after it has been justified. This also explains that conjectures serve the purpose of promoting the importance of proving.

This study is targeted at exploring how sixth graders generate geometrical conjectures when geometrical conditions and graphs are given-especially how they observe the graphs and use measurement information. This allows us to understand how the students who are still in the process of intuitive geometry generate conjectures from visual-based geometric materials.

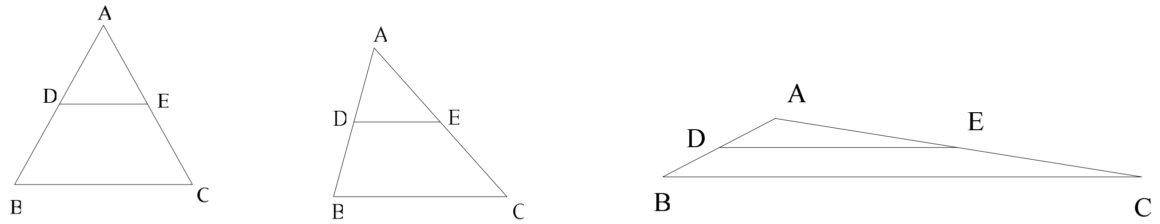
METHOD

Participant

The participants are 24 sixth graders from five classes of two elementary schools in Taipei City. Based on the performance of eight pilot-study students, we have decided to choose students who have medium mathematic performance and enjoy expressing their ideas and do not overly rely on their teachers' reactions. We thus asked their teachers to give us a list of the students who match these conditions and have the interviewer choose four to six of them from each class.

Material

There is a total of six questions including transversal of parallels, circumferential angle and central angle, triangle congruence theorems, the exterior angle theorem application of elements of a triangle (see Appendix), and triangle median theorems (such as Fig.1). In the question, there are given geometric conditions with three figures as examples. The students are asked to think about what other geometric invariance would surely exist under the given conditions. The graphs of the three examples were specially-designed. A "typical example" is a graph that exactly matches all the conditions given in a question, and this is the graph that is usually given in textbooks or by teachers (see the central graph in Fig. 1). The second one is the conjunctive example which is the result of the conjunction of the conditions given in the question with other condition(s). Take the left graph in Fig. 1 for example, it is not just a triangle but an isosceles triangle. The third one is an extreme example, which is a graph that matches the conditions but has unusually shape or small angles or line segment ratios. The graph from the right in Fig. 1, for example, is a rarer triangle since it has angle ratios and sides which are drastically different.



conjunctive example typical example extreme example

There is a triangle ΔABC. Point D is the middle of $\overline{AB}$ and E is the middle of $\overline{AC}$. Connecting points D and E forms $\overline{DE}$.

Fig. 1: The Three Examples in Question 6

Procedure

Interviews are given individually. When dictating a question, the interviewer also shows the given conditions on a piece of paper which is only taken away until that question is completed. Three 21cm×21cm examples that are separated by 4cm are shown to allow us to see which of the graphs can be clearly observed by the students. Most of the conjectures in the questions can be done via visual observations except for the two questions (fig.1 and circumferential angle & central angles) that were given last since they involve conjectures on ratios and require more precise data. Measurement tools such as rulers, protractors, and triangles were also provided. The four questions that do not involve measurement tools are given randomly in order to balance the practice or fatigue effect. The locations of the three example graphs are also random. The conjectures proposed by the students were converted into question forms by the interviewer via mathematical symbols. For example, for "this angle is as large as the other one," the interviewer would record what the student points out as "$\angle$ADE = $\angle$ABC". The interviewer would also report her every recording to acquire the participant's confirmation. There are systematic recording principles for standardization procedural.

RESULTS

Description of conjectures

Since a conjecture is plausible to the one who proposes it, thus whether the conjectures are correct or not, the lowest number of conjectures (total in the six questions) given by each participant was 14, the highest was 47, averaging 29.5 conjecturers (SD = 9.8) per person. In other words, 709 propositions emerged among the 24 students. However, after a conjecture was proposed, we evaluated whether this conjecture was natural based on the given conditions. We then discovered that there were 574 propositions that were correct conjectures, which was 81% of the total amount, averaging 23.9 per person. This kind of conjecture includes correct questions that are not important in future geometrical proving. For example, in the question in Fig. 1, the students pointed out that ADE must be a triangle. The related-correct conjecture refers to the propositions that are often used in geometric prove. There are 261 related-correct conjectures, which are 37% of the total amount of conjectures, averaging 10.9 per person. The so-called conjecture accuracy in the following refers to the number of related-correct conjectures/the number of conjectures per questions.

By analyzing related-correct conjectures, we can see that most of the conjectures involve an equal relationship between two variables (such as AB = CD, or $\angle ADE = \angle ABC$) and linear function (such as $\angle ABC + \angle ABE = 180^o$). The variables concentrated on the length of segments, angles, and area. Correct but not related conjectures include "this is a triangle" or "there are three sides and three angles"; or adding some segments and describing the graphs, including "drawing a line from point A that is perpendicular to BD gives us the height of ΔABC".

Categorization of preliminary observation behavior

The average amount of time for a participant to look at a question was 12 minutes. It was impossible for the interviewer to completely observe and record each participant's behavior in observing the examples. However, if we assume how the participant observes an example in the first minute tells us the important source of conjectures, then that preliminary observation behavior (referred to as "POB" below) should be correlated to the quantity or quality of conjectures. The research has divided POB into three types. In the first type, a participant would observe only one graph. His visual focus would shift between the three graphs constantly after reading the question. Yet, he would quickly choose an example as the target of observation and mostly ignore the other two graphs. Or, although he compares two graphs, his attention is obviously focused on only one of them. In the second type, the participant observes two graphs. He would scan all the graphs and pick two of them and spend a roughly equal amount of time on comparing them. In the third type, the participant observes three graphs, meaning he spends similar amounts of time on all three examples.

Each of the 24 participants answered 6 questions, and a total of 144 questions were answered, thus there are 144 POBs that can be classified. Table 1 list the number of conjectures for the above three types of POBs, the number of related-correct conjectures, and their accuracy. Since there are many POBs in which only one graph was observed, we had to classify which of the graphs was actually observed.

The relationship between conjectures and POB

We had combined and averaged the data on observing 2 and 3 graphs. The number of conjectures for each question is 3.58 ((3.67+3.38)/2), and comparing this to the data on observing only one graph, 5.23 ((6.17+5.15+4.35)/3), the result of point-biserial correlation is $r_{pb} = -0.225$ ($p<0.05$), showing that more conjectures are generated by observing only one example. If we analyze the accuracy rate, the result also reaches the level of significance ($r_{pb} = -0.191$, $p<.05$), showing that it is easier to generate more related-correct conjectures by observing one example instead of other kinds of observations. This explains that POB is indeed correlated to the number of conjectures and the ratio of valid conjectures.

	Observe 1 Graph (122[a])			Observe 2 Graphs	Observe 3 Graphs
	conjunctive(35)	typical (53)	extreme(34)	(9)	(13)
Average No. of conj.	6.17[b]	5.15	4.35	3.67	3.38
Average No. of related-correct conj.	2.23[b]	2.29	1.24	0.67	0.69
Accuracy (%)	36.11	44.40	28.38	18.18	20.45

a. The numbers inside the ( ) are the number of questions for a category's preliminary observation. The total number was 6×24 = 144.

b. The number is the average number of conjectures or related-correct conjectures for a question.

Table 1: Number of Conjectures and Related-correct Conjectures and the Accuracy under Three Observation Behaviors

The behavior of observing only one graph can also be divided into three scenarios. Using conjunctive examples yields more propositions (F (1, 67) = 8.11, p <.01, contrast to extreme example), but the accuracy rate is slightly lower than that yielded by using typical examples. Participants who observed typical examples showed a marginally better accuracy than observing extreme example (F (1, 67) = 3.33, p = .07). Participants who observed extreme examples had a more difficult time in generating conjectures, and the ratio of good conjectures was also lower.

The relationship between conjectures and measurement

Through the behavioral observation of individual interviews, we discovered that measurement tools possibly play the following three roles in the participants' conjecture generating and reporting. In the first situation, a participant has already made a guess. He then uses measurements to verify that guess and only reports it if he does not see contradictions in the result. Following this logic, the measurement may have appropriately eliminated an incorrect guess, or the erroneous measurement has eliminated the correct guess. In this study, we have observed two participants who gave correct conjectures before they used measurement tools. However, they were not sure about their conjectures so they used the tools, and they actually rejected their correct conjectures due to measurement errors.

In the second situation, a student does not yet have a correct guess, and he/she then gathers a large amount of data through measurements and attempts to combine the data together. This kind of student's behavior is constantly measuring the line segments, angles, or certain pieces of information, thus spending much time and effort on the measurements. This kind of effort usually collects a lot of information. But plausible conjectures could not be yielded from this kind of strategy, because one must have enough working memory in order to organize and calculate the information. Nevertheless, there are successful cases. For example, when a participant tried to solve the question in Fig. 1 and was measuring the lengths and altitudes, he suddenly discovered that the height in the smaller triangle ADE is the same as that in the trapezoid DECB, thus reaching a correct conjecture by accident.

In the third situation, a participant may not yet have a correct guess but has a certain type of relational schema, for example, equality schema ($x_1 = x_2$). He would then choose a type of variable and substitute the measured data into the relational structures to verify whether the data match. For example, he would use a ruler to measure the length of the sides that appear identical and look for the two segments that match the relational structure. If he still could not find the matching data despite constant effort, the participant would then change the variables (e.g., look for identical angles rather than length) or change the relational structure (e.g., change it to $x_1 + x_2$ = constant).

The above three situations often emerge within a participant or even a question. We also discovered that the third situation and the conjectures yielded from schema activating are most commonly observed. In other words, a participant may begin with a vague relationship ("maybe this and that are equal," "maybe something will come up if I add this with that"), and he then acquired numbers that can be used for substitution through visual estimation or actual measurements. When this becomes successful, he/she then makes a plausible guess.

Conjecturing and schema activating

Among the 24 participants, one of them has especially demonstrated the phenomenon of schema activation. He proposed 19 conjectures for the 6 questions. 18 of them were equality relationships such as $X = Y$ and $X + Y = Z$, and most of which were related to angles (e.g., $\angle EOA = \angle OPC$, $\angle ACD = \angle CAB + \angle ABC$, $\angle EOB + \angle OPC = 180°$). Only one of them was related to area (area ΔABD = area ΔACD); four of them were related to length (segment BD = segment BC). Although 16 of the 19 conjectures were accurate, showing that the activated equality schema indeed helped him to generate many related-correct conjectures, the pattern of this activation was very limited, resulting in a limited content of conjectures. For example, he was unable to discover that 2DE = AB or DE//AB in Fig. 1.

CONCLUSION

Most teachers do not want their students to make uneducated guesses, but they also want them to be brave enough to make guesses that are meaningful in mathematics. This kind of expectation is self-contradicting. How are students expected to be able to conjecture well if they never dare to make them at the first place?

In advanced mathematics, conjectures often involve deducing one element(s) from another element(s) and involve less induction. However, to elementary students who do not understand geometry greatly or are not very good at deductive reasoning, their experiences of conjectures can be enriched through visual-based geometric materials such as observing examples, analyzing properties, and conjecturing by induction.

In this study, we discovered that when the students begin observing the examples, they tend to make guesses from a single example, and they make more conjectures by observing one example instead of two or three examples. This phenomenon is related to the "generalization" discussed by Polya (1954, p.12-17). By looking at just one example, the participant receives less information or restrictions, thus he has more capacity to generalize more conjectures. Moreover, comparing the accuracy of conjectures, we see that it is higher in those who observe only one example instead of two or three. We argue that if a person generalizes conjectures from an example and verify them through other examples, it will be better than making conjectures that match all the examples. This still has to do with our cognitive loading; when a person exceeds this loading, he/she either is unable to make conjectures or makes faulty conjectures.

When observing only one example, a student who observes the conjunctive example generates more conjectures than he would do with other two types of examples, and this result matches with the nature of conjunctive examples-they are examples generated by the given conditions and other conditions. Thus, additional conditions may also lead to some conjectures, but this is not a necessary feature of the given conditions. Therefore, the number of accurate conjectures is roughly the same as typical examples, and the accuracy after division becomes slightly lower. Just by observing the typical examples, we see that since they are the graphs that exactly match with the given conditions, the quantity of the students' conjectures is not as high, but the accuracy is higher. The students who observed extreme examples did poorly on all three indicators, and this is probably because these examples' visual bizarreness prevented them from making guesses. For example, when two segments that are of the same length could not be observed easily due to their location, the students completely ignored the possibility that these two segments were of the same length, thus no follow-up confirmations were made.

Our study also shows that this kind of visual-based conjecturing is closely related to the activation of relational-schema. For example, when the length of two segments in a graph look identical, equality schema would be activated, and the students would more carefully observe or verify this kind of relationship with tools. They would even use other examples to verify this, and if the two lines were indeed equally long, the students would then have successfully made a plausible conjecture. Afterwards, the students would continue to check whether other lines or angles were identical with each other.

Having this kind of activity of making conjectures encourages students to generalize a vague guessing from an example. If they make more observations, propose conjectures, and verify them, they would be more likely to spontaneously access this relational schema when they see other questions in the future. Moreover, teachers can ask students “are there other relationships other than being identical” to guide them to systematically look for other schema. Afterwards, students would again make simple verifications via visual observations to acquire more plausible conjectures. Students need more experience in order to develop better conjectures, and visual-based geometric materials are possibly a good starting point.

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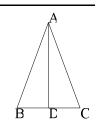
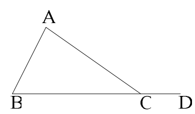
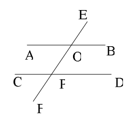
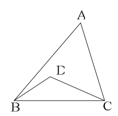
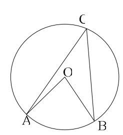
Appendix: Question 1 to 5 and their typical examples

Triangle Congruence Theorems There is a triangle ΔABC where $\overline{AB} = \overline{AC}$ and $\overline{AD} \perp \overline{BC}$.	
The Exterior Angle Theorem There is a triangle ΔABC with extended $\overline{BC}$.	
Transversal of Parallels $\overline{AB} // \overline{CD}$. $\overline{EF}$ intercepts $\overline{AB}$ in point O and $\overline{CD}$ in point P.	
Application of Elements of a Triangle There is a triangle ΔABC with internal point D forming ΔDBC.	
Circumferential Angle and Central Angle O is the center of a circle. On the circle, there are three points, A, B, and C, to form $\overline{OA}$, $\overline{OB}$, $\overline{AC}$, and $\overline{BC}$.	

ALGEBRIFICATION OF ARITHMETIC: DEVELOPING ALGEBRAIC STRUCTURE SENSE IN THE CONTEXT OF ARITHMETIC

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ABSTRACT

The research investigated whether direct intervention within specific numerical contexts will lead students to a better understanding of algebra. All seventh graders in two consecutive years in 4 schools in Israel participated in the study. The findings point to a greater transfer from the numerical context to the algebraic one after the intervention. Results support the assumption that "teaching arithmetic for algebraic purposes" can prevent beginning algebra students from making certain structural mistakes. However, when it came to the advanced algebra the achievements remained low although in terms of change a noticeable change was also found.

THE PROBLEM

The common perception of the algebraic structural rules as rules that draw their legitimization and meaning from rules valid in the world of numbers (Buxton, 1984; Davis, 1985; Smith, 1997), has generated the search for a model to describe the relation between students' understanding of the number system and of the algebraic one (Collis, 1971; Lee and Wheeler, 1989; Linchevski and Herscovics, 1996b). In the context of the school curriculum, it has motivated a teaching approach that may be described as "teaching arithmetic for algebraic purposes" (Davis, 1985; Arcavi, 1994; English and Sharry, 1996; Milton, 1999). The underlying assumption is that understanding of the structural rules in arithmetic is a key for understanding the corresponding parts in algebra (Kuchemann, 1981; Booth, 1984; Kieran, 1989; Lee and Wheeler, 1989; MacGregor, 1996; Tirosh, Even and Robinson, 1998; Milton, 1999; Philipp and Schappelle, 1999; Carraher, Schliemann and Brizuela, 2001; Chappell and Strutchens, 2001).

The literature on problems students experience in early algebra (e.g., Collis, 1975; Kieran, 1985, 1989; Steinberg, Sleeman and Ktorza, 1990; Sfard and Linchevski, 1994; Sfard, 1995) provides a detailed description of difficulties beginning algebra students have with the algebraic structure. What is lacking, however, is a sufficient theoretical definition as for what will be considered as "difficulties with algebraic structures" that stem from "difficulties with numerical structures", and empirical evidence, to establish the assumed isomorphism between students' difficulties in algebra and in arithmetic. Along these lines, Linchevski and Livneh (1999) designed numerical tasks that were structurally analogous to tasks that had been found to be problematic in the algebraic context and presented them to beginning algebra students. Their study confirms the assumption that obstacles detected in algebraic contexts exist also in corresponding numerical contexts, and that these obstacles are widespread.

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 217-224. Seoul: PME.

However, this in itself does not necessarily lead to the conclusion that "teaching arithmetic for algebraic purposes" (Davis, 1985; Arcavi, 1994; English and Sharry, 1996; Milton, 1999) guarantees transfer of the cognitive schemata constructed within the world of numbers to the world of algebra. As Linchevski and Livneh (2002) state: the question of whether systematic work on "structure sense" within the world of numbers will lead students to a better understanding of algebra in general, or at least to a better understanding of the algebraic corresponding parts, remains open. The purpose of the study presented here was to address this question. We investigated whether direct intervention within specific numerical contexts will lead students to better success in the corresponding algebraic contexts.

<u>Research question</u>: The major research question was whether direct intervention within specific numerical contexts leads students, which have been identified as having difficulties in these numerical contexts, to succeed better in the corresponding algebraic contexts.

In order to answer this question we first developed a screening tool that would identify, at the beginning of the 7th grade, students who have difficulties in specific numerical contexts and confirmed that: 1. The identified students would indeed experience difficulties in the corresponding algebraic contexts (Henceforth referred to as SAR – Students At Risk); 2. The other students, those who would not be identified by the screening tool (Henceforth referred to as SNR – Students Not at Risk) would succeed better in the corresponding algebraic contexts. The screening tool was designed and validated during Stage A of the research. Stage B of the research was designed to answer the major research question. In this paper we report only on Stage B.

STUDY DESIGN

Since a study of this kind requires a close cooperation with the schools for a few years, a large staff, resources, and budget, the research had to be limited to 4 junior highs. We chose, for stage B, the base-line study design in which each of the 4 junior high schools was compared with itself, This choice was based on the assumption that the nature of the population in each school – students and teachers - and the way of teaching mathematics do not radically change in two consecutive years, The research population included all 7th graders in 4 junior high schools in 2 consecutive years. The 4 schools were chosen randomly from 20 schools that agreed to participate in the study (2 out of 12 schools located in large cities and 2 out of 8 schools in smaller towns). Schools 1 and 2 are from large cities and school 3 and 4 from smaller towns.

<u>The first year of stage B – The Base line year</u>: In the first year of stage B, the identification of the SAR students was carried out by the researchers and three research assistants that were trained for this task. At the end of the school year a posttest was administrated to all 7th graders. During this year no intervention was implemented by the researchers. However, in order to put off a possible alternate conjecture, that what is really responsible for a probable change is the time devoted to the students working with a teacher in small groups, and not our planned intervention, each of the SAR students received extra teaching time from their classroom teachers

in small groups, as if they were going through the intervention process. Each SAR received teaching time in accordance with what he or she would have received if it were the intervention year, but the content of this extra help was determined solely by the teacher. At the end of this year, a posttest was administered to all 7th graders - SAR and SNR.

Second year – The Intervention year: In the second year, after in-depth training, the identification process was carried out by the school teachers, with the support of the research team. Also this year the intervention was carried out by the school teachers but, this year the intervention activities were the ones designed by the researchers (especially aimed at "teaching arithmetic for algebraic purposes"). The intervention sessions took place once or twice a week, in groups of 1 to 4 students. Each group was given well-defined activities for a pre-defined period of time. The average number of sessions per student or group was 14. At the end of the school year the posttest was administrated to all 7th graders - SAR and SNR.

Research tools

• *Teaching modules:* The Teaching modules were prepared on the basis of previous reported research and the analysis of students' work in Stage A of the study. These modules were in pure numerical contexts and were designed to address those arithmetical structural difficulties we planned to tackle during our teaching intervention (e.g., order of operations; "detachment from the minus sign," "grouping like (numerical) terms"; (Linchevski & Herscovics, 1996a); the equality sign, (Behr, Erlwanger & Nicholes, 1976). We considered the designed tasks as "algebra compatible" - tasks in the sense that they reflect algebraic competence albeit in a numerical context. For example, the item "Is 75 - 25 + 25 equal or not equal to 75 - 50?" was considered compatible to a possible future algebraic task: "Is 16 - 4x + 3x equal or not equal to 16 - 7x?". The activities were designed to elicit cognitive conflicts in a context that allowed a meaningful process of hypothesis testing, thus having the potential of leading to cognitive gains (Doise, 1978).

• *The screening tool – the Pretest:* A written pretest devoted solely to numerical contexts was administered to all seventh graders at the beginning of the school year. The pretest contained 18 items, 10 out of the 18 were taken into consideration in the process of identifying SAR. Of these 10 items, 3 dealt with order of operations and 7 were considered "algebra compatible". The remaining items were generalization tasks and translation from the spoken language to the mathematical language tasks (e.g. write a number that is bigger by 2 from 7) and computation with whole numbers. Students that were identified as having problems with computation were provided with a calculator (a simple calculator that does not follow the order of operations). A student was defined as SAR if he or she gave an incorrect answer for at least 2 out of the 3 first items (order of operations) or at least 4 out of the 7 "algebra compatible" items.

• *Individual interviews*: All students in the at-risk group (SAR) were interviewed individually with the aim of constructing an intervention plan based on the items answered incorrectly on the pretest.

• *Intervention plan:* Each SAR student had his or her own personal intervention plan. This plan included teaching modules directed solely at those topics the individual student had difficulty with in the pretest, thereby keeping the teaching intervention to the minimum. The teaching intervention was carried out in small groups of students having a similar profile.

• *Posttest:* The posttest included three types of items: numerical tasks; algebraic tasks compatible with the numerical tasks (henceforth COM-Algebra - compatible algebra); and algebraic tasks that were not compatible with the numerical tasks, such as generalization tasks in which the student is expected to construct a suitable algebraic sentence, and algebraic word problems and the like (henceforth ADVANCED-Algebra). Four scores were calculated for each student: a total score and one score for each sub-test.
As the aim of the research was to investigate whether intervention in a numerical context prevents difficulties in the algebraic one, we looked for a transfer of knowledge acquired within a numerical context to compatible and Advanced algebraic contexts. Thus, the intervention was in a pure numerical context, while the posttest included numerical tasks as well as COM-Algebra and ADVANCED-Algebra tasks.

RESULTS

By the end of the base line year, and by the end of the intervention year, we gave a posttest to all 7th graders in all 4 schools. The posttest was identical in the two years. However, in the base line year the posttest was administrated and marked by the research team. The schoolteachers were not part of the process and had no access to the test or to the results. In the intervention year the teachers administrated the tests and marked them with the guidance of the research team. The results are analyzed according to two lines: in each year (base and intervention) 1. The SAR and the SNR students' results are compared; 2. The SAR students' results in the two years are compared, and the same is done for the SNR students.
The results of the base line year and of the intervention year, for each of the 4 schools, are displayed in Table 1.

Table 1
Posttest marks of SAR and SNR students at the end of the Baseline year and the end of the Intervention year, and the Difference between populations in Sd. U

		total	numerical	Basic algebra	Advanced algebra
School 1					
baseline year	SAR	60.06	72.85	52.43	53.61
	SNR	73.40	85.74	69.76	64.02
	Dif. in Sd. U	1	0.99	0.81	0.78
	t (significance)	6.086 (0.00)	3.729 (0.00)	7.421 (0.00)	6.277 (0.00)
intervention year	SAR	67.9	79.33	68.79	57.38
	SNR	78.34	88.6	77.7	69.47
	Dif. in Sd. U	0.86	0.73	0.55	0.83
	t (significance)	5.263 (0.00)	4.330 (0.00)	3.188 (0.001)	5.061 (0.00)
School 2					
baseline year	SAR	49.69	70.35	31.38	37.68
	SNR	63.05	79.88	54.68	41.32
	Dif. in Sd. U	1.17	0.80	1.49	0.38
	t (significance)	5.937 (0.00)	4.959 (0.00)	5.283 (0.00)	3.751 (0.00)
intervention year	SAR	64.38	75.93	63.10	55.40
	SNR	74.60	86.82	72.87	64.04
	Dif. in Sd. U	0.52	0.50	0.34	0.60
	t (significance)	4.901 (0.00)	4.693 (0.00)	3.152 (0.001)	5.361 (0.00)
School 3					
baseline year	SAR	54.92	72.92	41.67	41.41
	SNR	67.32	80.36	57.52	48.66
	Dif. in Sd. U	0.98	0.60	0.99	0.61
	t (significance)	3.502 (0.001)	2.388 (0.011)	1.835 (0.036)	3.626 (0.01)
intervention year	SAR	72.74	84.40	76.03	60.24
	SNR	80.84	88.69	82.04	71.81
	Dif. in Sd. U	0.69	0.32	0.42	0.80
	t (significance)	3.146 (0.001)	1.386 (0.085)*	1.837 (0.035)	3.719 (0.00)
School 4					
baseline year	SAR	56.21	69.22	52.55	51.44
	SNR	72.55	86.04	74.14	65.02
	Dif. in Sd. U	0.94	0.88	0.79	0.98
	t (significance)	10.768 (0.00)	7.547 (0.00)	13.167 (0.00)	5.358 (0.00)
intervention year	SAR	66.75	82.35	62.62	62.63
	SNR	74.38	88.81	70.81	66.05
	Dif. in Sd. U	0.53	0.49	0.36	0.29
	t (significance)	5.095 (0.00)	4.122 (0.00)	4.278 (0.00)	5.1031 (0.00)

* not statistically significant

We first used the base line year results to re-confirm the validity of the pretest. As can be seen from Table 1, the performance of SAR students at the end of the base-line year was quite poor, by the end of the 7th grade the SAR students were the weak students in the algebra class. Their total mark was at the passing mark or below and their marks in Basic algebra (COM-Algebra) were in the range of 31% to 53%. Moreover, our conjecture that without any direct intervention the SNR students would experience fewer difficulties at the beginning of algebra was also confirmed. From Table 1 we can see that their total mark was in the range of 63% to 74%, and their achievement in Basic algebra was in the range of 55% to 74%. From Table 1 we can also see that by the end of the year the differences between the SAR and their SNR classmates were statistically significant.

What we can also learn from Table 1 is that the achievement of the entire population on the more Advanced-algebra items was low; both populations did poorly on the advanced items.

Thus our hypothesis was confirmed. The SAR students were not able to meet on their own the requirements of the Basic algebra course.

These results, of course, do not necessarily imply that our planned intervention will improve the SAR students' performance in Basic algebra. We would like to emphasize again that since we wanted to check a transfer of knowledge from numerical to algebraic contexts, the intervention was in a pure numerical context, while the posttest included numerical tasks as well as Basic algebra (COM-Algebra) and Advanced algebra tasks.

Table 1 also displays the posttest marks of both populations at the end of the baseline year and at the end of the intervention year for each of the schools.

Our assumption was that the structural knowledge acquired in our focused and controlled intervention, in a purely numerical context, would be transferred to the Basic algebraic context that is part of the regular 7th grade mathematics syllabus. The criteria for confirming our conjecture were: 1. the differences in achievements on the posttest between the SAR and the SNR students in the Intervention year, in comparison to the differences in achievements between the two populations in the baseline year; 2. the differences in achievements between the SAR students in the baseline year and in the Intervention year.

As can be seen in Table 1, the differences between the two populations, in Standard deviation units (Sd. U) were smaller in the Intervention year than in the baseline year in each of the 4 schools. These decreases were statistically significant. Baring in mind that the Basic algebra part was our major target we can conclude that according to our first criterion our conjecture was confirmed. The SAR students indeed, implemented the structural knowledge they were exposed to in the numerical context in the algebraic one, albeit the difference in performance between them and the SNR students remained in the range of 0.5 Sd. U. This difference is still statistically significant.

As for the raw marks, from Table 1 it can be seen that the achievements of the entire population, not only the SAR students, were improved in the Intervention year. This

change might be due to the fact that the Intervention units were presented to the SAR students by their mathematics class-room teachers who participated in our special workshops. These teachers were also the regular teachers of the entire class, thus it is reasonable to assume that some of the ideas that these teachers implemented in the SAR Intervention-groups were also implemented in the regular mathematics classes.
The second criterion for confirming our conjecture was the gap in achievements between the SAR students in the Baseline year and in the Intervention year.

Table 1 displays the results of the posttest of the SAR students at the end of the Baseline year and at the end of the Intervention year separately for each of the schools. The results show that in schools 1, 3, and 4 the change in the gap was statistically significant while in school 2 although an improved was detected the change was not statistically significant. From the Table we can also see that it is not only that the marks improved but also that the raw marks of the SAR students by the end of the Intervention year were in the norm and even above. The total average marks were in the range of 64% to 73%, and in Basic algebra they were in the range of 63% to 76%. The achievements in Advanced algebra, however, remained low although in terms of change (in Sd. U) a noticeable change was also found.

DISCUSSION

This study empirically tested the assumption that "teaching arithmetic for algebraic purposes" would prevent beginning algebra students from some structural mistakes in compatible algebraic contexts and help them in more advanced algebraic contexts. For this purpose, identification of the target population and the teaching intervention were carried out exclusively in numerical contexts, while evaluation of the process was done in numerical and algebraic contexts.

Results support the theoretical assumption to some extent. In the year that the intervention was specifically aimed at "teaching arithmetic for algebraic purposes," the progress of at-risk students in compatible-algebra tasks (those tasks that the teaching intervention aimed at improving) was statistically significant. However, progress in the other algebraic tasks was much smaller. It is interesting to note that the SNR scores in the second year were also higher than those in the first year. A possible explanation for this finding is that the teachers went through a change that influenced their regular classroom teaching to a certain extent.

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THE POTENTIAL OF PATTERNING ACTIVITIES TO GENERALIZATION

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This study presents partial results from the project of Ma (2002). It was conducted to obtain the deeper appreciation and knowledge of children approaches to quadratic sequence though patterning activities. The participants were 40 elementary school students in Taiwan. The conclusions drawn from this study were: (a) it was not to guarantee that students could benefit from geometry or number approach because three main obstacles and two students' individual reasons existed. Students with the main obstacles only did a pattern generalization, not a sequence generalization. (b) Students will have potential for developing generalization, if they apply the geometry approach.

INTRODUCTION

Mathematics can be thought of as a search for patterns and relationships (Biggs & Shaw, 1985); Mathematics is described as "a science of pattern and order (Van De Walle, 1998). Thus, a more apt definition of mathematics becomes fully apparent; that is, mathematics is the science of patterns. The mathematicians seek patterns in number, in space, in computers, in science, and even in imagination.

Patterns involve a progression from step to step, and in technical term these are called "sequence". Patterning activities develop directly a sense of pattern and regularity, and practice the skills of searching for pattern, extending patterns, and making pattern generalization. These processes will involve in variable and the concept of function. As a result, Usiskin (1995) states the view that algebra is the language through which we describe patterns. Patterning activities play a significant role for primary graders to establish the algebra foundation (Herbert & Brown, 1997;, 2002).

Algebra is a source of considerable confusion and negative attitudes among students (Cockcroft, 1982). Some students experience difficulty making the transition from arithmetic-based programs to the ideas of algebra (Greenes & Findell, 1999). "Expressing generality" is described as one of four different roots of algebra (Mason, Graham, Pimm, & Gowar, 1995). Thus, the use of patterns, leading to an improvement in this unhappy situation, has become an ordinary route into expressing generality within school mathematics curricula (A. Orten & J. Orten, 1999).

Hargreaves, et al. (1999) denotes that the need to generalize about the given terms might have two meanings. One is "a pattern generalization," it is to see more in the set of numbers than is given. The other is "a sequence generalization," it is to go beyond the set of numbers. Ma and Wu (2006) show that in the process of expressing generality, most fifth and sixth grades were unaware of linking patterns to algebraic

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 225-232. Seoul: PME.

concepts; what they did only was "a pattern generalization". Few could obtain the concept of a function that describes the relationship between any object and its position in a sequence; what they did was "a sequence generalization".

The school practice involving generalization in algebra often starts from figures or numeric sequences. Children need to realize that there are two representations of the same situation, and need to enable to switch from one to the other (Ursini, 1991). J. Orten, A. Orten, and Roper (1999) suggest that there are three purposes of setting patter tasks within pictorial context. One is for those who could possibly find support or their thinking from a more geometrical approach. Thus, it might be assumed that pictorial context adds meaning to the task. Second is pictorial context might be more elementary than purely symbolic context. Third is just to vary the format to create more of a problem to be solved. The students might use different method to convert pictorials to numbers sequence. Based on Orten et al. (1999), there are three methods of translating pictorial to number. One method is to count the dots for each shape presented in the task, then immediately converting the shapes into a number sequence. A second method is to look at how many more dots each new shape requires. A third method is based on seeing the shapes.

Quadratic sequences are those where the difference of the differences (i.e., the second difference) is constant, nevertheless, the majority of the sequences used in the textbooks are linear, where the difference between successive terms is constant. For a deeper appreciation and knowledge of children approaches to sequences they less met, this study was conducted focusing on pupils' generalization about and process approach to quadratic sequences. Especially the patterns in the sequence were set with pictorial and numerical contexts, which were two representations of the same situation. What processes would be involved when students work with quadratic sequences? What would be students' obstacles along the road to successful generalization while they observe and summarize patterns? Which method would they adopt while they convert pictures to numbers sequences? Which approach (geometry or number) could students benefit from? However, pupils were not expected to produce a formula for the general, or *n*th, term, while they did not yet receive formal algebra curricula. The purposes were as follows:

(1) To analyse the processes and how these might relate to generalizations

(2) To investigate the obstacles along the road to successful generalization;

(3) To explore the approaches students could benefit from while they work at patterning activities.

METHOD

An internet discussion board would be given an educational meaning while shifting the mathematical activities to it (Ma, 2001, 2004, 2005). The participants in this study operated on the pattern activities via an internet discussion board. Except for the traditional functions of word typing and recording, it also includes the functions of picture and chart pasting. Figure 1, for example, shows a screenshot on the internet discussion board. Each of eight problems, pattern question, was posed on the internet every two to three weeks. Among them problem 1, 3, 5, and 7 were with pictorial contexts, while problem 2, 4, 6, and 8 were with numerical contexts. Problem 1 and 6 are two representations of the same situation. They are quadratic sequences.

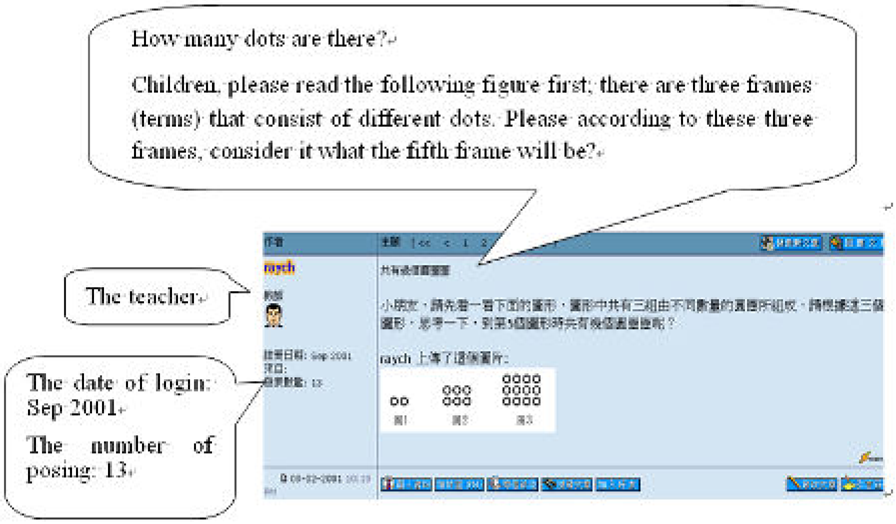


Figure 1: A screenshot on the internet discussion board

The participants in this study were 28 sixth graders and 12 fifth graders from Taiwan. They had basic computer skills and used the internet regularly. Each participant was anonymous but had fixed code. Three codes or four codes (adding “m” in front of three codes) represented the sixth and fifth graders respectively. Among three codes, the first symbolized the sex (b: boy, g: girl), the second symbolized the mathematics achievement (h: high, m: middle, l: low), and the last was only serial number. For example, bh1 represented the first (1) high-achieved (h) boy (b) from grade six, mgl2 represented the second (2) low-achieved (l) girl (g) from grade five (m).

For solving the problems the students were asked to search for pattern, extend patterns, and make pattern generalization. They were allowed to work on the problems at anytime and from anyplace. After having completed the activity on the internet, some students were interviewed to understand their thinking by teachers. Data relating to students’ understanding of pattern in the sequence were collected in both forms. One is written form with students’ responses to describing the rule for the patter on an internet discussion board, and another is oral form by interviewing students on a one-to-one basis. This activity lasted from September, 2001 to March, 2002.

DISCUSSION

The written responses of problem 1 and 6, two representations of the same situation, were examined to gain insights into the kinds of approach used, the processes applied and how these might relate to generalizations. Problem 1 with pictorial contexts was in figure 1 above, and problem 6 with numerical format was as 2, 6, 12, and 20. The pupils were asked to predict the number for the next, fifth, tenth, and hundredth in the sequence. Five students were chosen as examples in this study. For convenience, the researcher would use “P” and “N” to represent the pictorial and numerical context respectively, and use number (1, 2, and 3,) to arrange in students’ responses order. For example, “P-1 bl1” would represent the first response of bl1 to pictorial context.

Protocol 1

P-1 bl1: I add 2 on 4. 6 times 5 are 30. There will be 30 dots in the fifth.

N-1 bl1: 1×2=2, 2×3=6, 3×4=12, 4×5=20, 5×6=30,…, 10×11=110.

Student bl1 preferred to geometry approach, where he showed the analytic thinking. For example, he viewed “12”as “3×4” (N-1). Thus, he could see that both sequences were equivalent. By interviewing, bl1 expressed that the row and the column respectively required more one dot from shape 1 to shape 2, and it required more two dots from shape 3 to shape 5. Thus, His method was based on seeing the shapes before converting pictorial to number (P-1). However, he only could find more terms in a sequence such as the fifth and the tenth. He did not try to do far generalizing tasks (e.g., the hundredth), Stacey (1989) describes. The individual interviews conformed that his arithmetical incompetence could be his obstacles.

Protocol 2

P-1 gm4: 2, 6, 12. 3 multiplied by 2 are 6. 6 multiplied by 2 are 12. They go up in 3, 6, 9, and 12. 2×9=18. 2×12=24. There will be 24 dots in the fifth.

P-2 gm4: There will be 480 dots in the hundredth. There are 24 dots in the fifth, 100 divided by 5 equals 20, and then 24 times 20 equal 480.

N-1 gm4: Number 2 is 2 plus 4, number3 is 6 plus 6, number 4 is 12 plus 8,…, and number 10 is 90 plus20. It will be 110 in the tenth.

N-2 gm4: Number 10 is 110, and 100/10=10. It will be 1100 (110×10) in the hundredth..

The student gm4 preferred to numerical approach, where she immediately translated the shapes into a number sequence by counting the dots (P-1). She adopted idiosyncratic methods unpredictably to extent the pattern, such as 2×9=18, 2×12=24 (P-1). Unfortunately, the method was not useful to make generalization. She fixedly swapped to a short-cut method for far generalization after finding more terms in a sequence. For example, A100= A5×20= 24×20=480 (P-2) and A100=A10×10= 110×10=1100 (N-2). This suggested that gm4 did not perceive that both sequences were two representations of the same situation. Thus gm4 had obstacles along the road to successful far generalization.

Protocol 3

P-1 gh1: 2 dots plus 4 dots equals to 6 dots, and 6 dots plus 6 dots equals to 12 dots. They go up in 4, 6, 8, and 10. 12 dots plus 8 dots equals to 20 dots. 20 dots plus 10 dots equals to 30 dots. There will be 30 dots in the fifth.

P-2 gh1: There are 900 dots in the *99*th. I add 200 dots to the *99*th shape. There will be 1100 dots in the *100*th shape.

N-1 gh1: …..They go up in 2s, i.e., 4, 6, 8,..,18, 20. 30+12=42, 42+14=56,…., 90+20=110. Number 10 is 110.

N-2 gh1: 2, 6, 12, 20, 30, 6/2=3, 12/2=6, 20/2=10, 30/2=15. (3, 6, 10, 15) I worked it out like I counted in 3 to 6 is 3, 6 to 10 is 4, 10 to 15 is 5. (3, 4, 5) 3 to 4 is 1, and 4 to 5 is also 1.

Student gh1 was affected by different context. Her method of converting pictorial to number was to look at how many more dots each new shape requires (P-1). She could see how the addend was related to the position of the term, for example, adding 200 dots to the 99th shape (P-2). At last, she could not get the number of the 99th shape (900 dots was assumed) because she depended on a recursive approach. In addition, she described quadratic sequence by different way, that is, the constant difference (i.e., 1) of differences (i.e., 3, 4, 5) was between first quotients (i.e., 3, 6, 10, 15) (N-2). Unfortunately the quotients came from unpredictably personal methods of gh1, and it was not useful to make generalization. She had the fixation with a recursive approach in extending a pictorial or number sequence (P-1, N-1). She was used to looking for a local rule rather than a global rule. Thus gh1 had obstacles along the road to successful far generalization.

Protocol 4

P-1 mgl2: 2, 3+3, 4+4+4, 5+5+5+5, 6+6+6+6+6. Thus, there will be 30 dots in shape 5.

P-2 mgl2: It would be 101 dots. It requires more one dot from shape 99 to shape 100.

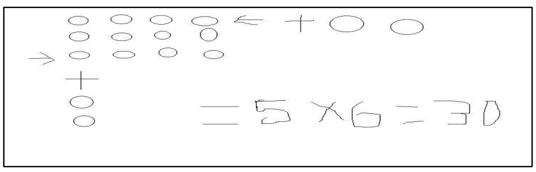
N-1 mgl2: Number 1 is 2, number 2 is 6, number 3 is 12, number 4 is 20, and number 5 is 30. 2+4+6+8+10+12+…+30+…. Keep adding them up and you will get the hundredth.

Student mgl2 preferred to geometry approach, where she showed the analytic thinking. For example, she viewed "12"as "4+4+4" (P-1) or "2+4+6" (N-1). Her method was based on seeing the shapes before converting pictorial to number. She focused on component parts of shapes. For example, the third contains three rows, and each row covers four dots, thus there is "4+4+4"(P-1). She made a correct verbal statement (101 dots), although she did not continue to work out the number of the hundredth shape (P-2). The individual interviews conformed that she could not handle the big number (i.e., 101+101+…+101). In addition, she made a creditable attempt at the approach to algebra, where she expressed the number in the form of "2+4+6++…+30+…"(N-1). In both contexts mgl2 noticed the method, not the answer. She had potential for developing far generalization although she only did near generalizing tasks.

Protocol 5

P-1 bh2: 3 dots in the left would be width and 4 dots in the bottom would be length, if we viewed shape 3 as the rectangular arrays of dots. 3 plus 2 equals 5, 4 plus 2 equals 6, and 5 times 6 equals 30.

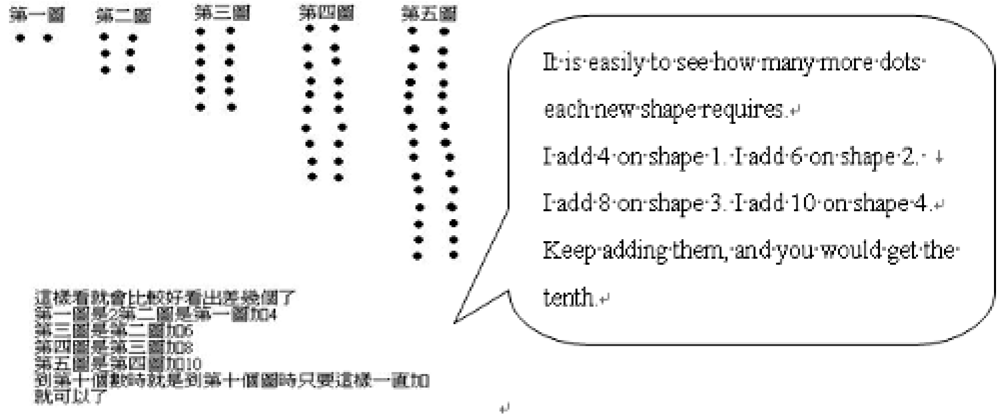
P-2 bh2:



P-3 bh2: Frame 1: 1×(1+1). Frame 2: 2 × (2+1). Frame 3: 3 × (3+1). Frame 4: 4 × (4+1). Frame 5: 5 × (5+1), 5×6=30 (dots), Frame 100: 100 × (100+1)= 10100 (dots) Frame n: n×(n+1).

N-1 bh2: Number 1 is 2. I add 4 on number1. 2+4=6. I add 6 on number2. 6+6=12. Keep adding even number and you will get the hundredth.

N-2 bh2: It will be easily to count, if we convert number to picture. The way is:



Student bh2 was affected by different context, but he could only benefit from geometry approach. His method was based on seeing the shapes before converting pictorial to number (P-1, P-2). He made far generalization because he could see how the numbers of width and length of rectangular were related to the position of the term in the sequence (P-3). It might prove the assumption of pictorial context adding meaning to the task, possibly enlivens it or simplifies it. In fact, Ma (2002) indicated that there were 84.4% students prefer to pattern with pictorial contexts, while there were only 15.6% like numerical contexts. The reasons of the former were easier, more creative and interesting, or giving extra hints.

Student bh2 used two methods to extend the number sequence. One is numerical way (N-1); another is to switch to figure (N-2). He illustrated the notion of equivalence with visual materials, in which each shape fell into two lines and new shapes required more 2×2, 2×3, 2×4, and 2×5 dots. He made a creditable attempt at the approach to figure, but what he did was just like the numerical way. He only recognized the recursive nature of the pattern, and thus only produced a local rule (N-2).

CONCLUSION

The results from this research were the following.

1. Few students perceived that both sequences are two representations of the same situation. Some students' preference was for geometry approaches (e.g., bl1, mgl2), and yet others students' preference was for number approaches (e.g., gm4).

It was not to guarantee from which approach students could benefit. There were three main obstacles along the road to successful "far generalization".1. Students merely produced a local rule (i.e., a recursive formula), not a global rule (e.g., bh2). 2. Students always thought about the numerical answer, not the method itself. The answer-driven approach leads student away from thinking about the methods of arithmetic and what they might mean. 3. Students applied a short-cut method, an inappropriate but simple method (e.g., gm4). As a result, these students with obstacles above could not complete far generalizing tasks. That is, they only did a pattern generalization, not a sequence generalization.

In addition, students' individual reasons, such as arithmetical incompetence (e.g., bl1, mgl2) and unpredictably idiosyncratic method (e.g., gm4, gh1), could influence the development for near or far generalization. The student gm4 and gh1 described the sequences by different way, that is, they divided each term by "the first term" and got the quotients (e.g., 6/2=3, 12/2=6).

2. Students will have potential for developing generalization, if they apply the approach based on the pictures, not on the equivalent number sequence. For example, bh2 could do far generalization (i.e., Frame 100: 100× (100+1) = 10100) with sequence presented in picture format. Even bl1 and mgl2, low achievers, might have opportunities to do far generalizing tasks. Their thinking was related to "analytic thinking". For example, "12" could be represented as "3×4" by bl1 and as "4+4+4"or "2+4+6" by mgl2. These methods with geometry approaches students applied were based on seeing the shapes of sequence. Bednarz, Kieran and Lee (1996) denote that certain geometry approaches appear to be a possible precursor to the emergence of analytic thinking in learning of algebra.

3. The students who had geometry approaches but were not yet aware of seeing relationship of patterns might easily progress from a recursive approach to an explicit approach after suggesting. For example, mgl2 might easily progress from seeing patterns as 2, 3+3, 4+4+4, 5+5+5+5,…or 2, 2+4, 2+4+6, 2+4+6+8,… to viewing them as 2×1, 3×2, 4×3, 5×4, or 2×1, 2+2×2, 2+2×2+2×3, 2+2×2+2×3+2×4,…. Therefore, mgl2 will be able to find an algebraically useful pattern such as frame 10: 11×10, or 2+2×2+2×3+…+2×10. Student bh2 switch number to figure sequence, and could see patterns within patterns such as 2×1, 2×3, 2×6, 2×10, 2×15 (N-2 bh2). He might easily progress from seeing patterns as 2, (shape 1) + 4, (shape 2) + 6, (shape 3) + 6,…to viewing it as 2, 2+2×2, (2+2×2)+2×3, (2+2×2+2×3)+2×4,…. Thus, bh2 will enable to obtain an algebraically useful pattern such as frame 10: (2+2×2+2×3+…+2×9) +2×10.

Thus, students need progress from a recursive approach to an explicit approach. In the process, students will focus attention on the method, not the answer. This suggests that there is potential development for "far generalization". Thus, in patterning activities teachers should encourage students to work at expressing their own generalization through geometry approaches, and then their algebraic thinking will take place.

Acknowledgements: We are grateful to National Science Council in Taiwan for funding the project (NSC 90-2521-S-275 -001). The opinions expressed do not necessarily reflect the views of NSC.

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INFINITE MAGNITUDE VS INFINITE REPRESENTATION: THE STORY OF Π

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This report explores students' naïve conceptions of infinity as they compared the number of points on line segments of different lengths. Their innovative (albeit incorrect) resolutions to tensions that arose between intuitions and properties of infinity are addressed. Attempting to make sense of such properties, students reduced the level of abstraction of tasks by analysing a single number rather than infinitely many. In particular, confusion between the infinite magnitude of points and the infinite amount of digits in the decimal representation of numbers was observed. Furthermore, misconceptions in students' understanding of real numbers and their representation on a number line were exposed.

The research presented in this paper is part of a broader study that investigates changes in students' conceptions of infinity as personal reflection, instruction, and intuitions are combined. It strives to uncover naïve interpretations of a concept that has puzzled and intrigued minds throughout history. By presenting a geometric representation of infinity, this study offers an occasion to observe how students respond to the contradictory or inconsistent results that they unearth. As students attempted to merge intuition with formal mathematics, some features of cardinal infinity that were at odds with their personal experiences and logical schemata exposed interesting ways to cope with the abstract concepts. Their attempts to make the material more comprehensible also revealed serious misconceptions regarding the magnitude of numbers that have an infinite decimal representation.

THEORETICAL BACKGROUND

Current research tends to focus on the counterintuitive nature of infinity, particularly on students' reasoning as they confront well-known paradoxes or issues of cardinality (Dreyfus and Tsamir 2004; Dubinsky, Weller, McDonald, and Brown, 2005; Fischbein 2001; Fischbein, Tirosh, and Hess, 1979; Tall 2001). To the best of my knowledge, only a few studies examine students' conceptions with regard to infinity in a geometrical context (see Dreyfus and Tsamir, 2004; Fischbein, Tirosh, and Melamed, 1981; Tall, 1980; Tirosh, 1999; Tsamir and Tirosh, 1996). Tsamir and Tirosh (1996), for instance, explored students' intuitive decisions when comparing geometrical objects such as squares of different sizes. In a similar study, Fischbein et al. (1979) observed students drawing a one-to-one correspondence between line segments of different lengths. Their conclusions supported the claim of Fischbein et al. (1981) that an intuitive leap is necessary to establish meaning about infinity.

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 233-240. Seoul: PME.

In my research, I build on several theoretical perspectives. The first framework introduced is Tall's (1980), which interprets intuitions that extrapolate experiences with finite measurements. In Hazzan's (1999) perspective, the use of familiar procedures to make sense of unfamiliar problems is an attempt to reduce the level of abstraction of certain concepts. She suggested that the tendency to apply familiar procedures – such as those of finite measurements – is indicative of a process conception. Process and object conceptions of infinity are characterized by APOS theory, another of the theoretical perspectives referred to. Dubinsky et al. (2005) proposed that process and object conceptions of infinity correspond, respectively, to an understanding of potential and actual infinity. Extending on these topics, my study examines students' naïve responses to tasks such as considering the number of points "missing" from the shorter of two line segments.

Tall's "Measuring Infinity"

As indicated, much of current research on infinity in mathematics education focuses on students' understanding of cardinal infinity. Tall (1980) suggested an alternative framework for interpreting intuitions of infinity that instead extrapolates measuring properties of numbers. Many of our everyday experiences with measurement and comparison associate "longer" with "more." For example, a longer inseam on a pair of pants corresponds to more material. Likewise, a longer distance to travel corresponds to more steps one must walk. Tall (1980) proposed extrapolating this notion can lead to an intuition of infinities of "different sizes," but one that is contrary to cardinal infinity.

A measuring intuition of infinity coincides with the notion that the longer of two line segments will have more points, though both have infinitely many. Tall (1980) called this notion "measuring infinity," and suggested it is a reasonable, and indeed natural, interpretation of infinite quantities, especially when dealing with measurable entities such as line segments and points. With this interpretation, if a line segment has $\aleph$ many points, then a segment twice as long has $2\aleph$ many points. This is different from cardinal infinity, which asserts any two line segments have the same number of points, $\aleph$, regardless of length. Certainly, cardinal infinity admits different infinite magnitudes – the natural numbers, for example, have cardinality less than that of the real numbers.

While at first, a measuring interpretation of infinity may seem at odds with cardinal infinity, it is consistent with properties of superreal numbers, a field extension of the real numbers that includes infinitely large and infinitesimal numbers. Moreover, infinite cardinalities and superreal numbers are linked via properties such as $\aleph 2 = \aleph$. Recognizing such properties may be a fundamental aspect of encapsulating infinity, as explained in APOS theory in the next section.

APOS Analysis of Infinity

Dubinsky et al. (2005) proposed an APOS analysis of conceptions of cardinal infinity. They suggested that interiorising infinity to a process corresponds to an understanding of potential infinity, while encapsulating to an object corresponds to actual infinity. For instance, potential infinity could be described by the process of, say, creating as many points as desired on a line segment to account for their infinite number. Whereas actual infinity would describe the infinite number of points on a line segment as a complete entity. Dubinsky et al. suggested encapsulation occurs once one is able to think of infinite quantities "as objects to which actions and processes (e.g., arithmetic operations, comparison of sets) could be applied" (2005, p.346). Dubinsky et al. also suggested that encapsulation of infinity entails "a radical shift in the nature of one's conceptualisation" (2005, p.347) and might be quite difficult to achieve. This theoretical perspective, as well as Tall's (1980) "measuring infinity," will be used throughout the study to interpret students' intuitions, and their attempts to reduce the level of abstraction of properties of infinity.

SETTING AND METHODOLOGY

The participants of this study were first year undergraduate university students enrolled in a mathematics foundation course. The 24 pre-Calculus students were unfamiliar with set theory and had no prior experience investigating properties of infinity in a mathematical context. None of the participants were mathematics majors.

Data collection relied primarily on a series of written questionnaires designed to elicit students' naïve conceptions of infinity. One of the aims of this study was to determine what sort of connection, if any, participants made between a geometrical representation of infinity and a numerical one. In other words, the question of whether students were associating points on a line with values on a number line was considered. The rationale behind administering a series of questionnaires throughout the span of several weeks was to determine if and in what ways students' ideas may change as a result of personal reflection. In order to avoid swaying students' responses, very little instruction was provided and it was made clear that there was no one "right" answer being sought.

The series of questionnaires was designed in such a way as to provide students with an opportunity to reflect on their ideas. Certain questions recalled students' previous responses and presented them with a slight twist. The rationale for this was to confront students with some of the counterintuitive properties of actual infinity that came up in their musings. Questions also took the form of presenting students with an argument that claimed to be from one of their peers, and asked them to evaluate and discuss the ideas involved. The basis for this style of question was to avoid presenting an authoritative position. It was imperative to this study that students' responses were not affected by seemingly correct solutions. The students addressed each issue based on its appeal to their own naïve ideas.

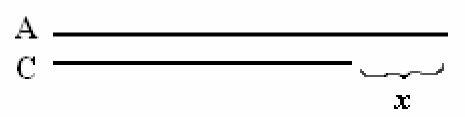
This paper focuses on students' responses to two particular questions. The first question (Q1) confronted students with an idiosyncrasy of infinite quantities and asked for an explanation. Of particular interest was the response of one participant, Lily. Her attempt to formulate an argument that was consistent with her experiences and intuitions prompted a follow up to this questionnaire. In Q2 students were asked to respond to Lily's argument as well as to a variation of it.

RESULTS AND ANALYSIS

The questions addressed here were posed towards the end of the course. By this time, there was a shared understanding that a line segment contains infinitely many points. Q1 addressed students' responses as they compared the infinite number of points on two line segments of different lengths.

Q1 and analysis of Lily's response

In an effort to explore conceptions of what it may mean for a line segment to have infinitely many points, students were asked to consider the number of "extra" points on the longer of two line segments.

> **Q1**.On a previous question, you reasoned that two line segments A and C both have infinitely many points.
>
> 
>
>
> Suppose that the length of A is equal to the length of C + x, where x is some number greater than zero. You also previously suggested that the segment with length x has infinitely many points. That is, the ∞ points on A minus the ∞ points on C leaves an ∞ number of points on the segment with length x. Put another way,
>
> $$\infty - \infty = \infty .$$
>
> Do you agree with this statement? Please explain.

Of the various responses to this question, Lily's stood out. Lily was a thoughtful student who was not particularly strong in mathematics. In one of the first questionnaires, she stated the length of the line segment was equal to its number of points. In her response to Q1, she disagreed with the possibility that $\infty - \infty = \infty$. She wrote:

> I disagree with this statement. For example, π is an infinite (on going) number. If we subtract $\pi - \pi$ the answer is 0, NOT ∞ . But, if there is a restriction that says we can't subtract by the same number it could still be an infinite number, but just a smaller value. For example, $\pi - 2\pi = -\pi$, is still an infinite number, only negative.

Lily reasoned that since π is an "infinite (on going) number" and $\pi - \pi = 0$, then the difference $\infty - \infty$ must also be 0. In Lily's conception, an "infinite number" appears to be a number that has an infinite decimal representation. Her objection to Q1 seems to stem from confusion between an infinite magnitude, such as the number of points on a line segment, and the infinite number of digits in the decimal representation of π. Her use of π to justify claims about infinite magnitudes suggests she has not made a connection between points on a line segment and values on a number line. Not only did she overlook the particular value of π itself, but she also failed to distinguish the differences between acting on one specific element as opposed to infinitely many.

Lily's generalization of properties of π to draw conclusions about the entire set of points is likely an attempt to reduce the level of abstraction of dealing with an infinite number of elements. The use of one number to explain properties of infinitely many coincides with Hazzan's (1999) observation that students will try to reduce the level of abstraction of a set by examining one of its elements rather than all of them. It is possible that addressing the entire set of points on a line segment as an entity itself may not be feasible at this stage of Lily's concept formation. Her use of the qualifier "on going" to describe her notion of an "infinite number" is evidence that she maintains a process conception of infinity.

Another interesting aspect to Lily's response was her use of "restrictions." She proposed that the difference of two "infinite numbers" might be another "infinite number" if there are appropriate restrictions placed on the quantities. By restricting the "value of infinities" she reasoned that it is possible to attain "an infinite number, it [will] just be a smaller value." For instance, she noted that a line segment with "missing points" may still have infinitely many points, just fewer than the longer segment. This idea is consistent with an intuition of measuring infinity (Tall, 1980). Also, Lily's responses are consistent with the observation that students' conceptions of infinity tend to arise by reflecting on their knowledge of finite concepts and extending these familiar properties to the infinite case (Dubinsky et al. 2005; Dreyfus and Tsamir 2004; Tall 2001; Fischbein 2001; Fischbein, Tirosh and Hess 1979). The use of familiar concepts and procedures to describe the unfamiliar properties of infinity is an example of Hazzan's (1999) "reducing abstraction". In this case, Lily applies the familiar procedure of subtraction not to the transfinite number ℵ, but to the real number π, thereby reducing the level of abstraction of working with the infinite number of points on a line segment.

Q2 and Lily's classmates

Lily's confusion between an infinite number of elements and an infinite number of digits in one particular element provoked my curiosity. The question of whether other students shared Lily's misconception naturally arose. Thus, a follow up questionnaire (Q2) recalled Q1, presented Lily's argument verbatim, as well as a similar one, and asked students to elaborate on whether or not they agreed with the arguments.

> **Q2**. Recall Q1.
>
> *Student X*: [Lily's response as quoted above]
>
> *Student Y*: I disagree with this statement. You can subtract two infinite numbers and NOT end up with ∞. For example, 1/3 is an infinite number, but 1/3 – 1/3 = 0, NOT ∞. Also, 4/6 and 1/6 are both infinite (on going) numbers, but if we subtract 4/6 – 1/6 = 3/6 = ½ = 0.5, which is not an infinite number. But sometimes it's possible to subtract two infinite numbers and get an infinite number. For example, 1/3 – 1/6 = 1/6, which is infinite and smaller than 1/3. So, sometimes ∞ - ∞ = ∞, but usually not.

Surprisingly, most participants agreed with at least one of the arguments above. The failure to distinguish between infinite magnitude and infinite decimal representation was shared by 22 of the 24 participants in this study. Two distinct interpretations of "infinite numbers" were observed. For the students who agreed with both arguments, the confusion between infinite magnitude and infinite decimal representation was broad: they ignored the finite magnitude of both rational and irrational numbers. For instance, Jack wrote:

> 4/6 and 1/6 are both infinite (on going) numbers but when subtracting them your result is ½ which is not infinite. This proves that an infinite number subtracting by another infinite number is not always another infinite number. As a result the statement ∞ - ∞ = ∞ is not true because sometimes the result is infinite but a different value and other times the result is not infinite.

Again it is clear that the differences between a particular (finite) value and an infinite quantity are being neglected. Also, this response highlights the common notion that infinity has no specific value. In particular, Jack seems to use the ∞ symbol to represent numbers of different magnitude. This and similar responses revealed that students were not only extrapolating their experiences with finite quantities, they were using them explicitly to justify their intuitions of infinity.

Conversely, there were students who recognized rational numbers as finite quantities but confused irrational numbers with infinite quantities. Students who agreed with Lily's argument but disagreed with Student Y associated rational numbers with points on a number line but did not make the same association with irrational numbers. This impression was exemplified in Rosemary's response to Q2.

> Rosemary: π – π = 0 that is correct because one is taking away the same amount of points from what they initially began with will give 0, but in the line segment question, the amount of points in *x* (which is ∞ amount) is much less than the amount of points in A and C. Which because of this, I agree with Student X's second statement of how there should be restrictions. In this case, points in *x* are less than points in A or C. Student Y states: 1/3 – 1/6 = 1/6 (which is an ∞ number) but 4/6 – 1/6 = 3/6 (which is only 0.5 and not an ∞ number). Well, when we represent these numbers on a number line *[drew two line segments, one from 0 to 1/6 and one from 0 to ½, and labelled the segments A and B, respectively]* then won't both line segments have ∞ points? (But of course segment B will have more than segment A)

Rosemary was a high-achiever who had consistently expressed the opinion that line segments had infinitely many points. She realized prior to Q1 that her arguments supported the counterintuitive ∞ - ∞ = ∞, and after reflecting, she rationalized the expression by invoking a measuring intuition. In her response to Q1, she claimed that while any line segment will have infinitely many points, a longer segment would have a larger infinite number of points. She also alleged that subtracting an infinite quantity from another (albeit "larger") infinite quantity would leave "a lot of points… extending into infinity," and "it will take forever" to count them. These last two statements pertain to a notion of potential infinity, and suggest a process conception.

In her response to Q2, Rosemary related Lily's notion of restrictions to her own measuring conception. Placing restrictions on the symbol used to represent the infinite number of points on each line segment accommodated the possibility that a longer line segment will have a greater number of points. Like Lily, Rosemary used π to reduce the level of abstraction of $\infty - \infty = \infty$. As she stated, "taking away the same amount of points […] will give 0" just as $\pi - \pi = 0$.

Rosemary also reiterated her thoughts regarding measuring infinity when she addressed Student Y's argument. In this case, however, she did not use the rational numbers analogously with infinite quantities, as she had used π. Although Rosemary stated that 1/6 was an "infinite number," she observed its specific value on the number line. Similarly, she remarked that though ½ was not infinite itself, when represented on a number line there were still infinitely many points between 0 and ½. This distinct handling of rational and irrational numbers suggests a serious misconception about real numbers: whereas rational numbers were associated with points, irrational numbers were not. Furthermore, Rosemary seemed to use the words "infinite number" in two different ways: to represent the number of (nonzero) digits in a decimal representation, as well as to represent the number of points in a line segment. It would be interesting to see if Rosemary's measuring conception would be so resilient had she not applied the same terminology to two distinct notions.

CONCLUSION

Confusion between the infinite magnitude of points on a line segment and the infinite decimal representation of particular numbers is a significant obstacle to students' understanding of several mathematical concepts. Not only does it hinder an appreciation or even recognition of properties of actual (cardinal) infinity, but it also demonstrates a shortcoming in the conception of number. The use of a finite quantity to explain phenomena of infinite ones misguides students' intuitions and ultimately their understanding. While "measuring infinity" may indeed have a distinguished place in mathematics research, intuitions that rely on numbers, or merely *a number*, are dangerous to the progress of mathematical reasoning about infinity. The various attempts to reduce the level of abstraction of infinitely many points by considering properties of a single point have, in the cases discussed here, revealed an intuition of infinity that may be at odds with future instruction on limits and set theory.

Certainly, the importance of establishing an apt understanding of number, magnitude, and infinite quantities is not trivial. It has been well established that when formal notions are counterintuitive, primary, inaccurate intuitions tend to persist (see among others Fischbein et al., 1979). Moreover, individuals may adapt their formal knowledge in order to maintain consistent intuitions (Fischbein, 1987). Fischbein et al. (1981) stressed that intuitive interpretations are active during our attempts to solve, understand, or create in mathematics, so it is clear that for the sake of advancing mathematical understanding, adequate intuitions must be developed.

This study opens the door for further investigation of some issues that may be over-looked or taken for granted, such as the relationship between magnitude and representation, and the connection between points on a number line and numbers.

Moreover, it provides insight into one naïve perception of infinity and its intuitive acceptance to pre-Calculus students.

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THE ABILITY OF SIXTH GRADE STUDENTS IN KOREA AND ISRAEL TO COPE WITH NUMBER SENSE TASKS

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This study compares the ability of sixth grade students in Korea and in Israel to cope with tasks which require the use of number sense. Results suggest that Korean students showed a preference of using exact calculations. The percentages seem to decrease dramatically, when students were explicitly asked not to use exact calculations, although it bounced back when there was no guidance. The Israeli students tended to use more of number sense considerations and less of calculations also in tasks in which there was no specific guidance on the strategy to be used.

INTRODUCTION

One of the mathematical issues which received increasing attentions in the last years in the field of mathematics education is the topic of number sense. Number sense is associated with the ability to understand the relationships between numbers; between numbers and operations; to work with numbers in a flexible way; to move from one representation to the other; to develop useful and efficient strategies for dealing with numbers and to judge the results to be reasonable within the mathematics and also in regard to everyday life (Markovits & Sowder, 1994; Reys et. al, 1999).

Despite the importance of number sense as being essential for life skills and its potential in developing mathematical thinking, it seems that emphasis of mathematics curriculum in elementary school is still mostly on computational algorithms and procedures. Reys and Yang (1998) investigated the relationship between computational performance and number sense among sixth and eighth grade students in Taiwan. They found that students' performance on tasks requiring written computation was significantly better than on similar tasks relying on number sense. Reys et.al (1999) compared number sense proficiency of students aged 8 to 14 years in Australia, Sweden, United States and Taiwan. They found, as expected, that the performance levels on the items varied across the countries, but also that regardless of country variable, students exhibited low performance on the number sense tasks.

Mathematics is being studied across the world. Almost in all countries children start the learning of mathematics in preschool and continue to learn mathematics in elementary school then in junior high school and in high school. Although mathematics is basically the same mathematics, textbooks differ from one country to the other, the ways teachers teach mathematics are not the same as well as are the ways in which teachers are prepared to teach mathematics. Ma (1999) found that American teachers who were mathematics majors no better coped with mathematical tasks than Chinese teachers who were not. Moreover, the Chinese teachers had better

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understanding of fundamental mathematics. Cai (2000) found that U.S. students and Chinese students performed differently on different tasks and used different strategies and representations to solve problems. The findings can be explained (Cai, 2004) due to different beliefs hold by the teachers.

In this study we report the ways sixth grade students from Korea and Israel cope with tasks which involve number sense. This report is a part of a larger study aiming to compare the ways in which sixth grade students from both countries cope with routine tasks, with number sense tasks and with questions regarding their beliefs about mathematics. The comparison turns to be interesting since Korean students keep doing very well on international tests, while Israeli students are ranked much lower on the list (e.g., Mullis, et al., 2000).

METHODOLOGY

Subjects

275 sixth grade students participated in the study, with 138 Israeli students from five elementary schools and 137 Korean students from four elementary schools. The schools in Israel and in Korea which were chosen for this study were classified as “typical schools" meaning average in the sense of students' abilities and in the sense of socio-economic background.

Questionnaire

A written questionnaire containing 30 open-ended tasks: 12 routine tasks, 12 number sense tasks and 6 belief questions, was developed and given to the students. The routine tasks (in which the students were asked to apply exact calculations in order to solve exercises involving whole numbers, fractions and decimals) were the first tasks on the questionnaire, followed by the number sense tasks and then by the belief questions.

The number sense tasks included in the questionnaire are listed below:

1. Fill in < or = or > and explain how did you decide which sign to fill in.
   4926 + 327 + 5909 + 3207 $\boxed{\quad}$ 327 + 5909 + 3207 + 4926
2. Fill in < or = or >. Is it possible to choose the sign without performing the exact calculation? Explain. 12,456 : 498 $\boxed{\quad}$ 12,456 : 499
3. Fill in < or = or > and explain how did you decide which sign to fill in.
   0.27 + ¾ $\boxed{\quad}$ 1
4. In the exercise 2 ¼ - 1 1/10 = 19/20 the result is incorrect.
   Do you need to use exact calculations in order to show that the result is incorrect, or is there another way? Explain.
5. Is it possible to choose 4 of the following numbers, such that their multiplication is exactly 4355? If so, find the numbers. If no, explain why.
   . 14 9 12 5 8
   15 28 10 4 2
6. Which of the following is closer to 0.52 X 809? Explain.
   a. 400 b. 1600 c. 430 d. 1700

7. What is the result of: 4 3/4 + 13.6 + 7 2/8 - 7.9 + 4.3 =

   Explain how you got the answer.

8. Fill in < or = or > and explain how did you decide which sign to fill in.

   17.014 – 3.948 $\boxed{}$ 17.013 – 3.984

9. Without using common denominator, arrange the following in increasing order, from least to greatest, and explain how you did it.

   $\frac{9}{10}$ $\frac{20}{39}$ $\frac{19}{40}$ 0.75 $\frac{15}{13}$ $\frac{1}{2}$

10. Given the exercise 1008 : 36
    Can you tell without doing the division, which of the followings have the same result as the given exercise? Explain your answer for each item:
    a. 2016 : 72 b. 3024 : 12 c. 504 : 18 same answer / not same answer

11. A bookshop ordered 600 new books. The owner sold 3/5 of the books for 25.25$ each. a. Did he sell more than 300 books or less? How did you decide?
    b. Can one change the number of books in this problem to 724, and keep the rest of the data? Explain you answer.

12. The height of a 10 years old boy is 1.5 meters. What do you think his height will be when he is 20?

As can be seen, the tasks include whole numbers, fractions and decimals, the four mathematical operations and two situations connected to real life. In almost all tasks there is a need to take into consideration the numbers involved and the operation or the situation but different tasks emphasize different aspects of number sense. In some of the tasks the student has to decide which part of the exercise is bigger (tasks 1, 2, 3, 8) by using number sense considerations. In others, the student can use estimation (6) and/or apply other number sense considerations (tasks 4, 5, 10), can use benchmarks when comparing fractions (task 9), apply an efficient way to add and subtract (task 7) and connect mathematics with real life (tasks 11 and 12).

Actually all number sense tasks can be solved by using direct computations. One of the aims in this research was to find out what kind of strategies will be used by the students when dealing with the tasks. Will they use number sense considerations or will they prefer to apply exact calculations? Thus, in the first task we simply asked the students to put the correct sign and to explain how they did it. In the second task, in order to "push" them toward the use of number sense, we explicitly asked, if possible, to decide about the correct sign without doing the exact calculations. In the third task we did not specify about the strategy to be used. As for the rest of the tasks, only in tasks 9 and 10 we explicitly asked not to use exact calculation.

RESULTS

We analyzed the answers given by the students as well as their solutions or explanations by establishing categories according to students' responses. Here we present the answers and the explanations of Korean and Israeli students for some of the numbers sense tasks included in the questionnaire.

Tasks 1, 2, 3, 8

In all these tasks the students had to choose the sign "<" or "=" or ">". Almost all students in both countries were able to put the correct sign in Tasks 1 and 2. Tasks 3 and 8 were more difficult for the Israeli students. As can be seen in Table 1 all Korean students answered the items (except of one student in Task 8) while some of the Israeli students did not answer items 3 and 8.

Category	Task 1		Task 2		Task 3		Task 8	
	Israeli students	Korean students	Israeli students	Korean students	Israeli students	Korean students	Israeli students	Korean students
No answer	0	0	2	0	13	0	14	1
Correct	97	90	90	93	63	86	74	86
Incorrect	3	10	8	7	24	14	12	13

Table 1: Answers for Tasks 1, 2, 3, 8 in percentages

When looking at the percentages of correct answer it is important to analyze what were the strategies used by the students, in order to realize whether the high percentages of correct answers were obtained by the use of numbers sense, or by the use of exact calculations, by which the task turns to be a routine one instead of a number sense task. In addition, it seems interesting to analyze the influence of the directions given to students on the strategy they used.

Students who based their answers on exact calculations showed the calculations on the questionnaire. Some of the calculations were correct leading to the choice of the correct sign, while others were incorrect leading to an incorrect answer. The same situation occurred with students who used number sense considerations. In most of the explanations reasonable number sense considerations were used, for example: "There is no need to calculate. The numbers appear on each side one time in different order and the operation is addition" (Task 1). In others, the students did use numbers sense considerations but incorrectly, for example: "You subtract from a smaller number so the answer will be bigger" (Task 8). This student did not calculate but looked at the numbers. He did realize the difference between 17.013 and 17.014 but thought that the numbers to be subtracted are the same. Table 2 presents the explanations given by the students. The category of "exact calculations" includes both correct and incorrect calculations and the category of "number sense" includes both reasonable and unreasonable considerations. The numbers in parenthesis stand for correct calculations or reasonable number sense considerations. The table shows that more students used number sense considerations in Tasks 1 and 2 than in Tasks 3 and 8, while in Task 3 the percentages are very low. One of the explanations might be the difference in numbers (whole numbers versus decimals and fractions).

Category	Task 1		Task 2		Task 3		Task 8	
	Israeli students	Korean students	Israeli students	Korean students	Israeli students	Korean students	Israeli students	Korean students
No answer	0	0	2	0	13	0	14	1
No Explanation	1	2	4	3	9	3	13	5
Number sense	91 (91)	68 (67)	86 (82)	86 (84)	23 (21)	14 (9)	62 (56)	50 (47)
Exact calculation	6 (6)	30 (23)	1 (0)	9 (6)	38 (30)	79 (75)	5 (4)	36 (29)
Something else	2	0	7	2	17	4	6	8

Table 2: Explanations for Tasks 1, 2, 3, 8 in percentages

Another possible interpretation for the massive use of calculations in Task 3 can be related to the use of benchmarks. Research shows that students have difficulty in employing appropriate benchmarks such as 1/2 or 1/4, among several essential components of number sense. In Task 3, students from both countries experienced difficulty using the benchmark of 1/4 or 0.25, which might lead them to rely only on computation.

The comparison among countries suggests that Israeli students tended more to use number sense than the Korean students did, while the Korean students tended more to perform exact calculations. It is interesting to compare tasks 1, 2 and 3 among the Korean students by looking at the percentages of students that used number sense and the percentages of students that used exact calculations. In Task 1, 68% used number sense and 30% used calculations. This was the first number sense task in the questionnaire and the students were not told how to approach the task. In the second task the students were asked if possible to answer without performing exact calculations. 86% of the students used number sense and only 9% used exact calculations. In Task 3, again we did not specify the way of coping with the task. In this task, 14% of the students used number sense while 79% used exact calculations, suggesting that when explicitly told not to use exact calculations the most of the Korean students are able to come up with number sense considerations, but they prefer to use exact calculations. It seems that the Israeli students prefer to use more numbers sense than exact calculations. In all items, with clear differences, less Israeli students compared with Korean students, used exact calculations. One of the explanations might be the awareness of Israeli students to number sense. Another reason has probably to do with the ability of Korean and Israeli students to perform exact calculations. In this research we have found that in all 12 routine tasks Korean students did better than the Israeli students did (Markovits & Pang, 2006). Probably the Korean students know that they are good in computations and use it whenever possible while Israeli students try to avoid exact calculations and to find other ways.

Task 6

The analysis of the answers show that the Korean students did much better than the Israeli students did: 82% of them chose the correct answer while only 42% of the Israeli students did so. (18% of the Israeli students and 2% of the Korean students did not answer, and other students chose one of the incorrect answers). The strategy used by the students in the two countries can explain the differences in the percentages of correct answers. Table 3 presents the explanations given by the students. The numbers in parenthesis stand for the percentages of students who gave the correct answer.

Category	Israeli students	Korean students
No answer or no explanation	43	8
Estimations	38 (28)	3 (1)
Calculations	9 (5)	86 (79)
Something else	10	3

Table 3: Explanations for Task 6 in percentages

In this task the students were not told how to reach the answer. As can be seen, almost all Korean students performed the exact calculation, which helped them to choose the correct answer. Only 3% of the students used estimation. As to the Israeli students, many of them did not answer or did not explain their way, but still about one third of them showed explicitly the use of estimation while only 9% performed exact calculations.

Task 4

In this task the students were explicitly asked whether they need exact calculations or if they can answer in some other way. Some of the students said that there is no need for calculations since " in 2 1/4 the 2 is bigger than the 1 and the 1/4 is bigger than the 1/10, so the result cannot be smaller than 1, and here it is less than 1". Answers like this we categorized as "complete number sense". Some of the students, who said there is no need for calculations looked at the whole numbers only. For example: "If you look just at the natural numbers 2-1 =1 and there is no natural number in 19/20". Answers like this we categorized as "partial number sense". Other students suggested that calculations are needed. For example: "I think that it will be better to calculate and then you really can see if the result is correct or not". Table 4 presents the results and shows that only 32% of the Israeli students and 57% of the Korean students said that there is need for calculations. Number sense was demonstrated by 35% of the Israeli students and by 28% of the Korean students. Once again it seems that the Israeli students tend to use less exact calculations

Category	Israeli students	Korean students
No answer	16	6
No need for calculations: complete number sense	13	10
No need for calculations: partial number sense	22	18
No need for calculations: something else	17	9
Need calculations and performed the calculations	32	57

Table 4: Results for Task 4 in percentages

DISCUSSION

The analysis of the number sense tasks (the analysis presented here and the analysis of rest of the tasks) suggest a different approach as demonstrated by Korean and by Israeli students. The Korean students showed a preference of using exact calculations. The percentages seem to decrease dramatically, when students were explicitly asked not to use exact calculations, although it bounced back when there was no guidance. This might suggest that Korean students have the ability of using number sense, but they are not used to activate it. The Israeli students tended to use more of number sense considerations and less of calculations also in tasks in which there was no specific guidance on the strategy to be used. This might suggest that Israeli students are more familiar with the use of number sense. This tendency might be explained by several factors. One explanation is probably related to the emphases of mathematics teaching and learning which have probably an influence on teachers' beliefs. In Korea, students are used to direct calculation in mathematics. Although the emphasis in computation in elementary mathematics becomes decreased in the current curriculum, skillfulness of computation is traditionally valued. This may explain why Korean students prefer direct calculation even with the tasks in which computation is not needed or ineffective. In Israel, at the time the questionnaire was delivered number sense was not a part of the curriculum or the textbooks (now it is a part of the new curriculum), but it was "in the air" in many in-service teachers' workshops. As a consequence, teachers could have addressed this issue in their mathematics lessons.

Another factor might have to do with students' ability to cope with exact calculations. This research indicates that Korean students performed much better than the Israeli students did all routine tasks (Markovits & Pang, 2006). Perhaps Korean students are aware of their ability to reach the correct answer when they use calculations, while Israeli students are aware of their limits and look for other ways.

Probably the use of exact calculations might also be explained by the difference in culture between the two countries. Another aspect which is probably strongly connected with the difference in culture is the tendency of Israeli students to skip items even when the items were not at the end of the questionnaire. Korean students answered almost all items, and rarely skip an item. It seems that Israeli students relate less seriously to tests and to the need of completing all tasks. If a task seems difficult or unfamiliar many of them tend to skip it and move to the next item. In Korea it is a norm for students to complete all tasks seriously.

This research raises other questions which needs more attention: the relation between the type of the number sense task and the ability of students to cope with it. On some of the number sense tasks in this research students exhibited good performance, this in contrary with the results described by Rey et. al (1999) in which poor performance on number sense tasks was exhibited by students in several countries. Thus, can we analyze what makes a number sense task easy to cope with and what makes another task difficult?

Yet another question has to do with the results of the international tests. How much of number sense is included in these tests? Are the students asked to explain their solutions so it can be seen whether they applied number sense considerations? This raises the issue of the difficulty in testing number sense.

In order to better understand the results of this study we plan to investigate Korean and Israeli teachers' content knowledge and pedagogical content knowledge regarding to routine tasks, number sense tasks, and beliefs toward several aspects of mathematics.

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CREATING YOUR OWN SYMBOLS: BEGINNING ALGEBRAIC THINKING WITH INDIGENOUS STUDENTS[1]

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Because mathematics education devalues Indigenous culture, Indigenous students continue to be the most mathematically disadvantaged group in Australia. Conventional wisdom with regard to Indigenous mathematics education is to utilise practical and visual teaching methods, yet the power of mathematics and the opportunities it brings for advancement lie in symbolic understanding. This paper reports on a Maths as Story Telling (MAST) teaching approach to assist Indigenous students understand algebra through creating and manipulating their own symbols for equations. It discusses effective Indigenous mathematics teaching, describes the MAST approach, analyses it in terms of Ernest's (2005) semiotic processes, discusses its applications, and draws implications for Indigenous mathematics learning.

For the last few years, we have been researching ways to reverse Indigenous mathematics underperformance. Because mathematics teaching in Australia is Eurocentric (Rothbaum, Weisz, Pott, Miyake, & Morelli, 2000) and does not take into account the models of the world Indigenous people have created to inform their knowledge, many Indigenous students perceive mathematics as a subject for which they must become 'white' to succeed (Matthews, Watego, Cooper, & Baturo, 2005) and which can challenge their Indigenous identity (Howard, 1998; Pearce, 2001). Teachers tend to have low mathematics expectations of Indigenous students, blaming underperformance on absenteeism, social background and culture rather than themselves and the education system (Bourke, Rigby, & Burden, 2000; Sarra, 2003). As a result, few Indigenous students complete advanced post-compulsory mathematics subjects that lead to tertiary study in disciplines with a mathematics basis (Queensland Studies Authority, 2006) and only one Indigenous person, the lead author, has graduated with a mathematics doctorate.

We have endeavoured to *contextualise* mathematics pedagogy with Indigenous culture and perspectives (Matthews et al., 2005) because this overcomes systemic issues of Indigenous marginalisation with respect to mathematics learning (Cronin, Sarra & Yelland 2002; NSW Board of Studies, 2000) and instils a strong sense of pride in students' Indigenous identity and culture (Sarra, 2003), both are prerequisites for mathematics improvement. However, although we can contextualise algebraic applications through modelling (Matthews, 2006), contextualisation is not so apparent for the teaching and learning of formal algebraic structure and symbol manipulation.

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[1]Project funded by Australian Research Council Discovery Indigenous Researchers Development grant DI0668328.

IMPROVING INDIGENOUS PERFORMANCE

We are aware that effective mathematics teaching is crucial for Indigenous students' futures as mathematics performance can determine employment and life chances (Louden et al., 2000). However, there is some ambivalence in the literature regarding the nature of effective Indigenous mathematics teaching. Indigenous students appear to learn best through contextualised concrete "hands-on" tasks (e.g., Day, 1996; Gool & Patton, 1998), "have greater sensitivity and success in dealing with visual and spatial information compared to verbal" (Barnes, 2000, p. 10), and "learn by observation and non-verbal communication" (South Australia DETE, 1999, p. 10). However, these findings may be an artefact of Indigenous students being taught in Standard English with which they may not have the words to describe many mathematical ideas (Roberts, 1998) and the words they have may be ambiguous (Durkin & Shire, 1991).

We are aware that school programs can dramatically improve Indigenous learning outcomes if they reinforce pride in Indigenous identity and culture, encourage attendance, highlight the capacity of Indigenous students to succeed in mathematics, challenge and expect students to perform, and provide a relevant educational context in which there is Indigenous leadership (Sarra, 2003). We recognise that non-Indigenous teachers with little understanding of Indigenous culture can have difficulties with contextualisation and reject it in favour of familiar Eurocentric approaches (Connelly, 2002; NSW Board of Studies, 2000). Thus, we believe in building productive partnerships between these teachers and the Indigenous teacher assistants (ITAs) employed from the community to assist them (Warren, Baturo & Cooper, 2004). We have also had success with educating ITAs by focusing on structural learning of mathematics (Baturo & Cooper, 2004) and we are aware that Indigenous students tend to be holistic, learners, a learning style that appreciates overviews of subjects and conscious linking of ideas (Christie, 1995, Grant, 1997) and should appreciate algebraic structure.

In our early Indigenous mathematics-education research, we focused on elementary mathematics and at-risk students. Our more recent projects have focused on assisting secondary school Indigenous students to use mathematics as a way of gaining high status employment. This has stimulated an interest on algebra for three reasons: (1) algebra is the basis of many high status professions; (2) algebra is based on generalising pattern and structure, skills with which Indigenous students may have an affinity because their culture contains components (e.g., kinship systems) that are pattern-based and which may lead to strong abilities to see pattern and structure (Grant, 1997; Jones, Kershaw, & Sparrow, 1996); and (3) algebra was the vehicle whereby the first author mastered mathematics. As he reminisced:

> When reflecting back on my education, my interest in mathematics started when I began to learn about algebra in my first year of high school. ⋯ For me, algebra made mathematics simple because I could see the pattern and structure or the generalisation of algebra much clearer than the detail of arithmetic.

SYMBOLS AND SEMIOTICS

Our answer to the dilemma of contextualisng the teaching and learning of algebra was to focus on representing mathematical equations as stories which leads to contextualising of mathematical symbols. Thus, we developed an approach to symbolisation based on students creating and using their own symbols, drawn from their socio-cultural background, to describe these stories as a precursor to working with the accepted mathematics symbols. We now describe the Maths as story telling (MAST) approach and analyse it in terms of Ernest's (2005) semiotic processes.

Maths as Story Telling (MAST). The approach utilises Indigenous knowledge of symbols within domains such as sport, driving, art and dance as a starting point for building understanding of arithmetic symbolism in a way that can be easily extended to algebraic symbolism. The approach has five steps.

Step 1. Students explore the meaning of *symbols* and how symbols can be assembled to tell and create a story. This is initially done by looking at symbols in Indigenous situations (e.g., exploring and understanding symbols in paintings) and then creating and interpreting symbols for simple actions (e.g. walking to and sitting in a desk).

Step 2. Students explore *simple addition story* by acting it out as a story (e.g. two groups of people joining each other). A discussion is then generated to identify the story elements such as the different groups of people and the action (the joining of the two groups) and the consequences of the action (the result of the joining).

Step 3. Students *create their own symbols* to represent the story. This step could be done in a freestyle manner; however, we have opted to take a more structured approach by using concrete materials (which are familiar to the students) to represent the objects (or people) in the story. The story is then created by allowing the students to construct the two groups of people with the concrete materials and construct their own symbol for "joining two groups" and lay this out to represent the action (or history) of the story. In a similar fashion, the students then construct their own symbol for "resulting in" or "same as" to tell the story of what happens after this action has taken place. Figure 1 gives an example of an addition story that was constructed by a student in Year 2.

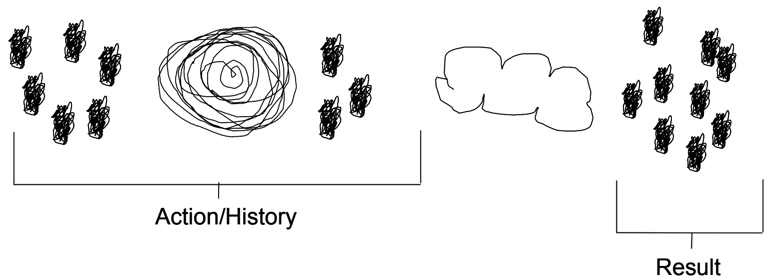

Figure 1. A Year 2 student's representation of the addition story 6 + 3 = 9.

Step 4. Students share their symbol systems with the group and any addition meanings their symbols may have. For example, in Figure 1, the student's "joining" symbol was a vortex that sucked the two groups together. The teacher then selects one of the symbol systems for all the students to use to represent a new addition story. This step is important to accustom students to writing within different symbol systems and to develop a standard classroom symbol system.

Step 5. Students *modify the story* (a key step in introducing algebraic ideas) under direction of the teacher. For example, the teacher takes an object from the action part of the story (see Figure 1), asks whether the story still makes sense (normally elicits a resounding "No"), and then challenges the students by asking them to find different strategies for the story to make sense again. There are four possibilities: (1) putting the object back in its original group, (2) putting the object in the other group on the action side, (3) adding another action (plus 1) to the action side, and (4) taking an object away from the result side. The first three strategies introduce the notion of compensation and equivalence of expression, while the fourth strategy introduces the balance rule (equivalence of equations). At this step, students should be encouraged to play with the story, guided by the teacher, to reinforce these algebraic notions.

Analysis in terms of Ernest's (2005) semiotic processes. Because the students create their own symbol system, the MAST experience bypasses the first process of Ernest's (2005) representation of Harre's (1983) semiotic model of "Vygotskian space", namely, *appropriation*. The MAST experience minimises the effect of the Ernest's fourth process (*conventionalisation)* so that students can freely express their creations and the meaning behind their symbol systems. The approach is designed to allow students to engage with Ernest's second and third processes (*transformation* and *publication* respectively*)* for symbols they create before being required to undertake the full four processes for the universally-accepted mathematical symbol system. Thus, the MAST steps could be considered as "twisting the Vygotskian space" to refocus on creativity and the expression of this creativity.

MAST Steps 3 and 4 are the essential steps that focus on transformation and publication. They enable students to: (1) create their symbols with personal meaning, by working backwards from meaning to symbol (and not forward from symbol to personal meeting as usually happens when learning the normal symbols); and (2) reinforce these personal meanings through sharing them with other students and sharing in the other students' symbols, to see the personal in relation to the collective (and not in the collective). As such, the steps are a powerful semiotic method for teaching and learning mathematics (in Ernest's, 2005, terms) because they are "driven by a primary focus on signs and sign use" (p. 23) and focused on how the students individually create, appropriate and openly express these symbol systems to a collective. Transformation and publication are important processes for MAST to encompass because they allow students to see: (1) beyond the "well-known pathological outcome of education in which learners only appropriate surface characteristics without managing to transform then into part of a larger system of personal meanings" (p. 25); and (2) a little of how a collective actively regulates and standardises symbols and their use. The variety of symbols experienced in the publication process in MAST Step 4 offers an opportunity for students to investigate commonalities across symbols systems, that is, to abstract at a high level. This develops the essence of the semiotic approach (i.e., the meaning of symbols, the relationships between symbols, and their underlying rules and applications).

MAST Steps 4 and 5 involve students discussing and critiquing each others' symbol systems (being proponents and critics for each other in Ernest's, 2005, terms) and, therefore, have the potential to develop high learning. As such, MAST introduces, very early on in the learning of symbols, the capacity to be creative and generate new expressions and possibly new meanings and structures within symbol systems.

APPLICATIONS OF MAST

MAST is the first product of the Minjerribah Maths Project which was set up to answer the following questions. *Can we improve achievement and retention in Indigenous mathematics by refocusing mathematics teaching onto the pattern and structure that underlies algebra? In doing this, are there Indigenous perspectives and knowledges we can use? Can we at the same time provide a positive self-image of Indigenous students?* MAST is our attempt to work from the story-telling world of the Indigenous student through to the formal world of algebra by experiences with the creation of symbols that have personal meaning. The story telling starts with simple arithmetic but moves quickly to algebraic thinking. It brings enables Indigenous students to bring their everyday world of symbols into mathematics.

The Minjerribah mathematics project. The project's focus is to put Indigenous contexts into mathematics teaching and learning (making Indigenous peoples and culture visible in mathematics instruction) and to integrate the teaching of arithmetic and algebra (developing the reasoning behind the rules of arithmetic while teaching arithmetic so that these can be extended to the rules of algebra). The overall aim is to improve Indigenous students' mathematics education so they can achieve in formal abstract algebra and move into high status mathematics subjects. This project is being undertaken through an action-research collaboration with teachers at a rural Indigenous Years P-10 school by putting into practice processes to improve and sustain these enhanced Indigenous mathematics outcomes. The research is qualitative and interpretive and adopts the "empowering outcomes" form of Smith's (1999) decolonising methodology which aims to address Indigenous questions in ways that give sustained beneficial outcomes for Indigenous people.

MAST in the classroom. MAST has been presented at professional development (PD) sessions for teachers within eight Queensland schools and has been used within Year 2 and Year 8 classrooms. Although results are preliminary, they appear to validate the potential we believe the approach has. Responses from teachers to the PD sessions have been overwhelmingly positive; no teacher has rejected the approach and most have been highly engaged in the activities. In particular, secondary teachers' responses to the PD activities have led us to add extra steps to the approach to introduce and solve for an unknown group of objects, thus reinforcing the balance rule. Interestingly, MAST experiences appear to provide teachers with a deeper understanding of algebra. Three teachers who were not mathematically trained jointly said: *This was the first* *time we understood algebra*. An English teacher said: *For the first time, I can see that mathematics is creative like poetry.*

The Year 2 trial was a revelation. The Year 2 students enthusiastically worked with the teacher to construct symbols to tell the story of one of their number walking into their classroom and sitting at her desk. They equally eagerly constructed symbols for three of their number joining another two. They were able to do all the work and all the MAST experiences were successfully completed. Some of their symbols were particularly creative and they were able to discuss and solve the equivalence activities. In fact, they were the first group that suggested the third strategy of adding another action; we had not thought of it. Interestingly, the teacher did not stipulate the use of materials to represent numbers and half the Year 2 students' first symbols were not linear (see Figure 1). For example, one student drew 2 circles and then drew 3 students in one circle, 2 students in the other and 5 where the circles overlapped, making the 5 <u>between</u> the 2 and the 3. Students who did non-linear drawings like this were able to change to linear, as in Figure 1, when the teacher stipulated this in the second part of the lesson.

For the Year 8 students, the MAST experiences provided a method for understanding more complicated equations as well as an introduction to symbols. This was shown later when a student asked why equation $2x = 8$ was divided by 2 to find x. The teacher directed the student to represent the equation in a quasi creative manner with two x's on one side of a line and 8 circles on the other. The student was then able to see that dividing both sides by 2 will give the value of x. The teacher argued that this could not have been done without the students' having previously experienced the MAST steps and created novel representations of equations.

IMPLICATIONS

The five MAST steps are an illustration of how the MAST approach could be used to introduce students to algebraic ideas, while the semiotic analysis indicates the implications of the approach for bridging the gap between arithmetic and algebra.

Creating one's own symbol system appears to be an effective way to introduce algebraic thinking to Indigenous students. In Ernest's (2005) semiotic terms, it meets all the requirements for relational and high level understanding. With Step 1, MAST contextualised algebraic symbolisation (Matthews et al., 2005), an experience for both teachers and students as they explore symbols in the Indigenous world view. Such contextualisation could be difficult for non-Indigenous teachers (Bourke, Rigby & Burden, 2000; Connelly, 2002; NSW Board of Studies, 2000) but it would certainly make learning two-way strong, from teacher to students and students to teacher, a positive outcome for Indigenous learning (Howard, 1998; Pearce, 2001). Seeing Indigenous knowledge underlying the most abstract of mathematics could well lead to growth in self confidence and development of positive self image for Indigenous students that, in turn, may well assist to reverse Indigenous mathematics underperformance (Sarra, 2003).

We believe that MAST has implications for all learners (Indigenous and non-Indigenous). It appears to be a powerful way to assist all students move from arithmetic to algebra. By taking emphasis away from foreign systems, it shifts the emphasis to algebraic pattern and structure within something that is familiar. Step 4 is designed so that, conversation "can be fluid and shifting in its actualisation" with "near spontaneous verbal responses as well as other modes of response ⋯ sought and encouraged" (Ernest, 2005, p. 30). This, along with each student creating their own symbolism, should provide a feeling of freedom within the MAST activity. In any case, MAST is a worthwhile activity for the way in which it utilises agency in initiating action.

However, it would be remiss of us not to mention an uncertainty in the approach; which is the process of translating from developed personal symbols to the conventionalised symbol system. This is a research question for this year: *Are there disadvantages of moving away from appropriation in the Vygotskian space?*

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EXPLORING STUDENTS' MATHEMATICS-RELATED SELF IMAGE AS LEARNERS

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In this study we investigate the role of affects in mathematics classroom, from a socio-cultural perspective. We present data from a case study that aims to describe Camila's different beliefs, feelings and attitudes related to mathematics, explaining the relation between them and mathematics learning in classroom. Research instruments included a written paper on a movie script about mathematics, worked out by students, which we found helpful in getting to know Camila. As a result, we could explain why, in spite of the agreeable environment created in classroom by her mathematics teacher, she remains stable in her negative mathematics-related self-image as a learner.

Introduction

This paper reports ongoing research investigating the role of affect in mathematics education. We intend to explore a notion of students' *mathematics-related self-image as learners* and its relation with their emotional processes in the mathematics classroom.

Here, students' *mathematics-related self-image as learners* refers to whom students believe they are, and to whom they would like others to believe they are, as out-of-school and school mathematics learners. In earlier study (Melo, 2003) supported by Charlot (2000) we investigate students' rapport on school mathematical knowledge, through stressing the dimensions he defined as social, epistemic, and identity. From data analysis, the *mathematics-related self-image* appeared as a component of student's relationship with mathematics which strongly interferes in their problem solving behavior.

This current project consists of data analysis from a multiple-case study of 8 students out of two secondary school classrooms, in the same learning institution. In this paper, we present the case study of Camila, selected for the issue it raises on the relation between affective factors in the classroom and the teaching and learning of mathematics.

In this paper, our research design is presented following the discussion of our theoretical perspective. Data analysis presentation starts by detailing Camila's mathematics classroom dynamics, followed by our description of her experience in such context. We then examine Camila's *self-image as a mathematics learner*, analyzing its' role in the emotional processes she experiences in the mathematics classroom.

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Research framework

We found in the research literature on affects in mathematics education, a theoretical ground to describe the notion described above. We build it as a configuration (or a system) of mathematics-related beliefs (De Corte and Opt't Eynde, 1996; Di Martino, 2004), feelings and attitudes related to earlier mathematical experience.
McLeod (1989) describes affects in mathematics education in terms of components he identified as beliefs, attitudes and emotions. His research has shown the importance of affect in the teaching and learning of mathematics. It represents an effort to overcome the Cartesian mind-body dichotomy, which isolate studies on cognition from those focusing on affective and emotional aspects of teaching and learning. Much has been done since then, although for some researchers, there still a need to respond *what* and *how* to observe when researching on affects (Di Martino, 2004).
As a consequence of the variety of theoretical constructs and approaches, there still so far diverse theoretical structures being used to conceptualize affect in mathematics education (Hannula et.al.2004). In recent publication, different approaches complement each other to analyze a same case study.
In our study, we explore the processes through which affective components develop and determine learning. This is done by attempting to describe the dynamics of a configuration of affective components (McLeod, 1989), relating it with students′ mathematical experience at school.
We found in DeCorte & Op'tEynde (1996) exploratory study on mathematics-related beliefs systems an identification of different kinds of mathematics-related beliefs. We use their main categories to undertake our systemic analysis.
Here we consider the notion of feeling as in Damasio (1994), accepting his distinction amongst feelings and emotions; considering the latter less stable and difficult to access. Feelings related to an object are seemed constructed from student's perception and experiences as a starting point; and beginning with individual's perception of the object.
Our perspective as researchers is built on our understanding that affect is being (re) constructed through students experience with mathematics, in a socio-cultural context. In our study, we made an attempt to consider the individual-in-classroom as the unit of analysis.

Research design

The entire project is a qualitative study developed in a public school within a working class neighborhood. Participants are in their final year of the secondary school. Many of them intend to continue their studies at the university level.
Two mathematics classes taught by the same mathematics teachers in this school were observed twice a week, during the whole academic year of 2006. Video recording was not allowed; so, field notes were taken during data collection. Yet procedures of data collected included an attitudinal questionnaire and a written task, both of which were

handed to all students to be completed in the classroom, as well as individual semi-structured interviews.

From classroom observation and learners' responses to the questionnaire and to the written task, 8 students were selected and invited for the interviews. Interviews were set to convey the meaning expressed in the written material and in classroom behaviour.

For the written task, students were handed two sheets of paper with the following instructions:

"Suppose you are a movie director, and you decided to make a movie about mathematics. In order to run this project, you first need to write a movie script and to submit it to a producer. In the script, you need to define the cast, the details of each character, the running time, the genre (adventure, thriller, romance, terror, drama, comedy, fiction, detective, etc), the soundtrack, rating, and the plot, along with the main scenes. Now that you have the main clues, good luck in your work!"

The context

João, the school teacher participating in this study, is a young professional. Nevertheless, he is highly praised by the school director, students and even their parents. He is known not only for his skills in mathematics and as a mathematics teacher, but also for his good relationship with students. For this reason, he was chosen by the school director to participate in the study. He was invited to collaborate and accepted. First, he was to select the two classrooms to be observed. He decided on classroom 301, to whom he "enjoys lecturing"; and classroom 305, where students are often disperse and he describes as "being difficult to work". Both classes are mixed (13 boys, 24 girls in classroom 301; 14 boys and 23 girls in classroom 305).

João provides opportunity for student interaction in his lessons. In spite of keeping a teacher centered style of learning, most of the time he encourages effective student participation. Students mainly ask questions about the homework. However, in many instances we could observe students guessing and formulating questions which were discussed by (nearly) the whole group, including the teacher. In another instance, when proving a theorem, João would invite: "…and now, what do you think I must do?"; most student responded by reasoning and building the mathematical argument. In each lesson, the affective relationship built by the students and the teacher during the presentation of each solution for the mathematical exercises seems to result in fun.

In this environment, and during the whole academic year, Camila always chose to sit opposite from the teachers desk, in the first or the second row. She never participates in classroom debates and never asks the teacher to clarify her doubts, which she often has. On the other hand, she is a very organized student. Her work-book and homework are neat; she makes notes of everything written on the blackboard during the lessons, and she always attempts to solve the tasks proposed by the teacher. The following traits best described Camila: her silence, her explicit absence in classroom debate, and her (physical) distance from the teacher contrasted to her responsible though mute and physically distant participation in the classroom. To get closer to her, we present the analysis of her film script combined with a triangulation of data from the interview.

Getting to know Camila

Camila is a 17 yea old girl who likes fashion and is intending to study fashion design or interior design at a university. She must pass school exams to be promoted to the next school grade. Although she ever had to repeat a grade in her entire academic life, she considers herself a student with difficulties in mathematics.

Camila wrote a short-movie script, with a 10 minute running time. According to her, her movie genre is "a mixture of thriller, terror and drama, which could be a comedy; depending on the perspective of the observer." She proposes a free rating film, where "all scenes are main scenes, given that the movie is short". Cast and characters are: Camila (same name as her), her father (no name) and João (same name as her mathematics teacher), conceived as:

"João seems to be a good guy. Camila is for sure a good girl, an ordinary girl, who unfortunately hates mathematics and who does not have any vocation to study it. (She is) the exact opposite of João. Ah, and she is very shy".

The script for the main scenes:

> One day, Camila lives a drama: she has just seen the bad side of mathematics. On this day, she went home crying and her father cuddled her:
>
> -My daughter, it (the mathematics) is necessary, you need it for everything. Try to enjoy it, just a little bit, put an effort into it.
>
> -But I can´t, I can´t.
>
> From then on things just got worse: reasoning took longer, calculations were longer and became more and more complicated.
>
> Then, Camila met João. Before that day, she thought she did not like math; but after meeting him, she found she hated it. João is a good guy, hard worker, responsible and he loves mathematics.
>
> On the day of the math test Camila arrives to the classroom and she sits at her desk. The bell rings, and the exams are handed to the students. Camila looks at the exam and sees a bomb. Her head explodes, she cannot feel her right leg and she starts shaking and sweating. Thank God the examination is over, and thanks to me my test was a disaster.

> But Camila does not give up. She studies, and studies, and studies, until achieving her dream: enjoying math a little bit, and she goes beyond that by becoming a mathematics professor at a Federal university.
>
> Well, I am Camila, and this is my story; [it is] nearly [my story], because I do not intend to become a mathematics professor.

Analysis

Camila is using her name and her teacher's name for the characters in the cast. At the end of the script, a paragraph indicates it is based on her actual experience in mathematics. Field note observations about João, her actual mathematics teacher, are aligned with the script references to the character João, a guy who enjoys mathematics, and who has "vocation" to study it. We will refer to the movie characters as Camila (c) and João (c).

Complementing our field notes, here we will get to know Camila through building her *mathematics-related self-image as a learner*. This is done as an affective configuration of *mathematics-related self-beliefs*, *feelings* and *attitudes.*

First, we identified she believes that some people are born to do mathematics, and that she was not. She also states the commonsense thesis that "mathematics is a necessary subject", and she believes that "enjoying it[mathematics] implies learning it". Unfortunately, she dislikes mathematics.

The "born to do mathematics" belief is suggested by Camila (c) who recalls she "does not have any vocation to study it [mathematics]". Later in the interview, we captured Camila saying: "I do agree that some people really love mathematics and really … to enjoy mathematics you must be born to do it, and I wasn´t, isn´t it? I agree completely with this statement".

Yet Camila described a good mathematics student as someone who must "…study a lot…sometimes the person was already born to do mathematics …however I think that if you compare a person who was born [she emphasizes the word born] to do mathematics and study, with a person who just studies mathematics, the one who was born [emphasis in the word born] to study mathematics, will do better than the one who just studies …"

There is no explicit reason why or for what "Mathematics as a necessary subject", when stated by the father during the first movie scene. Camila´s responsible attitude as a mathematics student indicates she might also believe this. The "enjoying implies learning" belief explains why she does not learn. In the last scene of the script, Camila (c) "studies, and studies, until she reaches her goal: to enjoy math a little bit".

Feelings are negative, and expressed in two dimensions. First, somatic feelings in the movie script are detailed; such as the tears in the first scene due to the recognition of her incapacity to overcome difficulties, and the description of exam day. Anxiety also permeates Camila's words during the interviews: "Well, ...I feel uncomfortable in the math class … you know .. it is a subject where I cannot find myself … I look at the watch … ai ...and specially on Mondays, when we have two lessons …then I become anxious, do you understand?" Second, there are feelings in relation to the others, such as those related to João. Attitudes in classroom are of being at the margins: "But I'm like that with everybody ... especially with João .. with João never … never…never… with other teachers I ask for help…".

Camila's explanation about her "never … never…never…" is as follows:

"I think it is... because of mathematics. Ah! I don´t know … I think it is even because, you know … I think it is also because of closer contact with the teacher … like João …but there are some people in the classroom who have more contact …but not with me, do you understand? He is open minded … but I don´t know … I think it is the (lack of?) contact …like the chemistry teacher … Pedro …I talk to him …I ask for help …"

Discussion

From the analysis above, we make an attempt to explore the affective processes through which Camila's self-image as a learner has developed and how it interferes in learning. Three issues are emerging: the cultural context shaping and being shaped by the mathematics-related beliefs, the relationship with others as one aspect of Camila's relationship with mathematics, and her mathematics-related feelings.

The configuration of Camila's mathematics-related beliefs includes beliefs *on mathematics as a social activity,* on the *significance of and competence in mathematics* (De Corte&Op't Eynde, 1996) and on an *inborn nature of mathematical competence*. The relevance of mathematics is explicit in Camila (c)'s father speech although there was no discussion on why and how mathematics is relevant. The father also suggests Camila(c) should enjoy mathematics, indicating an implicit belief *on the learning of mathematics*. Melo (2003) present data collected in our country, which are aligned with Charlot (2000) findings in France, where students claim they just enjoy mathematics if they understand it. Therefore, as commonsense statements, these beliefs are shaped by Camila (c) socio-cultural context, and thus, by Camila's context. Her socio-cultural context allows so far the belief on an inborn mathematical competence.

Beliefs and affective dimensions are constantly being reconstructed; although the processes are not homogeneously consolidated and do not shape identical individuals (Damásio, 1996). Individuals have a diverse re-elaboration of the same social influences; and even in the same socio-cultural context they build differently their relationships with others. Camila met João, and unlike her classmates, she dislikes mathematics even more. For her, João represents someone who she recognizes as good guy, hard worker, responsible. Unlike her, he is a competent mathematics teacher and loves mathematics- a subject domain she does not grasp as she believes she should. She feels less than João, and excludes herself from the group of classmates who get involved in the classroom debate.

This raises the important, though harmful in this case study, interference of social relationship (which includes relationship with others, values and expectancy) in mathematics-related feelings.

Final Considerations

In this paper we made an attempt to understand and describe Camila's *mathematics-related self-image as a learner*. We analyze data from classroom observations, the film script, and the interview. Thus we are able to identify beliefs which are grounded in her socio-cultural context and in her relations with others, as well as feelings built during her life experience with mathematics shaping her attitudes in the classroom. Through data analysis, we got to know Camila, who is a responsible secondary school student, though a silent (perhaps passive is a better word than silent as it seems she is not pro-active) mathematics learner.

What strikes us as mathematics teachers, is the complexity of getting students involved in mathematical learning, as seen in the example of Camila. In spite of recognizing João as a good teacher, Camila does not engage in the agreeable atmosphere he is able to create inside the classroom; nor does João's affective relationship with students during the lessons represent a positive emotional experience for her. She remains stable in her negative *mathematics-related self-image as a learner*, as if in a never-ending story which even an experienced teacher is not being able to change.

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DIFFICULTIES ON UNDERSTANDING THE INDEFINITE INTEGRAL

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Four first year students at a central Greek university are interviewed on basic facts concerning the abstract definition of indefinite integral to which they are exposed to at the 12th grade. Using different lenses we examine their content knowledge and at the same time exploit the complementary approaches in interpreting our findings.

INTRODUCTION

Over the last decades several theoretical frameworks have been proposed in order to explain student's ways of learning either by focusing on individual's mechanisms of cognition and knowledge or by taking under consideration mainly socio-cultural aspects of mathematics learning. How can competing theoretical lenses function when applied to the same phenomenon? There is an interdependence of facts and theories in the sense that the same facts can support different theoretical perspectives. According to Sfard (2001) in many situations it is reasonable to try to use different approaches to understand learning in an attempt to find which one would provide a more helpful solution of the problem at hand. In this article we exemplify how analyses of the responses of four undergraduate students to some problems on the indefinite integral by using different theoretical perspectives can lead to different interpretations of students understanding. The two basic frameworks we chose to operate in are the one that (based on the Piaget's notion of reflective abstraction) describe the cycle of mental construction using the "process – object encapsulation" (Tall, 2005) and the other is the "communicational approach to cognition" as described in Sfard (ibid.). For the first one we use two different descriptions of the cycle of mental construction: Sfard's (1991) *interiorization, condensation, reification* and Gray & Tall's (1994) *procedure, process, procept.* For the communicational approach, learning is viewed as participation in certain distinct activities thus shifting the attention to the activity itself and to its changing context-sensitive dimensions (Sfard, 2001). The goal of this paper is twofold. The immediate goal is to investigate by using different theoretical tools the difficulties students face when confronted with an abstract concept as the indefinite integral. The broader and more general goal is to explore the different (contradictory or complementary) perspectives that the above mentioned theoretical frameworks provide when applied on the same case. The article consists of three parts. The first part provides relevant background information about the problems posed and the methods used. In the second part we analyse the students' responses from both the cognitive science approach and communicational approach perspective. The last part confronts the two analyses and discusses the complementarity or incompatibility of the different approaches. We conclude by claiming that it is necessary to try both these approaches

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 265-272. Seoul: PME.

when analysing students' responses since they often provide useful insights depending on the questions asked.

BACKGROUND

In Greece 12th grade students are introduced to the concept of indefinite integral according to the definition: *indefinite integral of a function f continuous in an interval [a,b], is defined to be the set of all the anti-derivatives of f on [a,b]*. Then tables and methods of integration follow which are accompanied by more theoretical – conceptual exercises. As in the case of other calculus concepts, the Greek curriculum emphasizes abstract handling of the integral which makes interesting what the students really understand after a year of teaching. Seven first year students of the Math Department were interviewed using a questionnaire comprised of three parts. In the first part, 3 simple indefinite integrals were asked to be calculated just to make sure a basic algorithmic understanding of the integral had been attained. The second consisted of the following four questions:

Correct if necessary the following arguments Q.1 – Q.3 *:*

Q.1 $\int \frac{1}{x}dx = \int (x)' \frac{1}{x}dx = (x)\frac{1}{x} - \int x(\frac{1}{x})'dx = 1 + \int x\frac{1}{x^2}dx \Rightarrow \int \frac{1}{x}dx = 1 + \int \frac{1}{x}dx \Rightarrow 0 = 1$

Q.2 $2\int \cos x\, dx = \int \cos x dx + \int \cos x dx = \sin x + \int \cos x dx \Rightarrow 2\int \cos x\, dx - \int \cos x\, dx = \sin x \Rightarrow \int \cos x dx = \sin x$

Q.3 : since $0\int f(x)dx = \int 0 f(x)dx$ and $\int 0 f(x)dx = \int 0 dx = c$ and $0\int f(x)dx = 0$ it follows 0=c.

Q.4 : True or false: $\int f - \int f = 0$ Elaborate. Finally in the last part, they were asked to describe in their own words the definition and the role of the indefinite integral. The questions of part 2 were designed in order to create a learning situation involving elements of conflict and doubt. According to Fischbein (1987), the need for certitude is the basic component for learning. Doubt and confusion relate to cognitive conflict and conceptual change (Vosniadou, 2003). For Piaget (Piaget & Garcia, 1989) equilibration is a process that is stimulated by the disequilibrium that is caused by either inter or intra-personal conflicts. Following Zavlasky's (2005) categorization of the types of uncertainty entailed in certain mathematical tasks, we used the "competing claims" type of uncertainty in the questions posed to the students. That is we used outcomes that contradict well known mathematical truths (as in Q.1: 0=1) or contradicting statements with which the subject was confronted (as in Q.4 when the student had answered True the remark was made by the interviewer "in math isn't it always that A-A=0?")

RESULTS

Each interview was audio taped and the verbal data were analysed in conjunction with students' written answers. The goal was to describe patterns of interaction, and change in students' use of mathematical language and concepts and track their level of understanding through discussion and argument. We will present some characteristic points of the interviews from 4 students and analyse them through different theoretical lenses.

Student M

The student is asked whether the integrals, i.e. the sets $\int \frac{1}{x}dx,\ 1+\int \frac{1}{x}dx$ are equal.

M: So in order the sets to be equal, the elements of the first set should equal the elements of the other.

Interviewer: So the two sets contain the same functions?

M: no, no. They don't contain the same functions because the first is the integral of 1/x and the second is 1 plus the integral of 1/x,... they haven't the same formula.

Although student M remembers the set theoretic definition of integrals she is not capable of applying it in the right fashion, since she identifies the concept of function with its formula. This typical mistake, reveals a lack of the necessary encapsulation of the process of integration into a thinkable object in the sense of Tall (2004) or a non-reified object in the sense of Sfard (1991). The same student exhibits a similar procedural attitude towards the question B.2 : at first she answers that the result is true and when she is asked whether the fact that the derivative of sinx + c is also cosx would make her change her mind she argues as following:

Interviewer: So should we add a constant c here as in the previous case?

M: No, it's not a solution of an equation; it's just a simple implication. The fact that (sinx + c)′ also equals cosx is not relevant, I wouldn't add the constant c anyway.

Interviewer: the fact that sinx + c is also an anti-derivative makes you consider that we should add in the result a constant?

M: I believe no, the exercise is not asking for finding the set of the anti-derivatives but for proving something, I mean what we have here is the result of an implication, why consider a constant c ?

As Sfard has remarked, at the first stage of understanding a new symbol, its use is mainly templates-driven and only some time later it can be object-mediated. In this sense the student probably has formed a cluster with templates of the indefinite integral coming from former uses where the usual procedure was to apply some integration formula and find a result following well established implications: the exercises and examples in the Greek school-books are mainly of this type. It is procedural thinking, in the sense of Gray and Tall (1994), that the student exhibits here: the focus is on the procedure and the symbol of integral can not be seen as representation of a concept and its process at the same time. According to the theoretical model of dynamics of cognitive sensitivity developed by Merenluoto & Lehtinen (2004), the student which had no relevant perception of conflict followed the third path of this model demonstrating low certainty and a routine activity without any adequate relation to the cognitive demands of the task. This is what Piaget (1975) termed as alpha level of understanding (or stonewalling according to Chan et all (1997)). Obviously, what is presented as contradictory data from the point of view of the interviewer is not considered as such by the student so it's not enough to lead him to a meaningful conflict.

Student A

In dealing with the question Q3 student A changes his initial position after a lengthy conversation with the interviewer. Applying the definition he realizes the difference between the sets $0\int f(x)dx$ and $\int 0f(x)dx$:

A: ...so this set ($\int 0f(x)dx$) must be the set of constant functions

Interviewer: How many elements does it consist of?

A: infinite

....

A: ...and the set $0\int f(x)dx$ has only one element, the zero-function

Interviewer: What is the relationship between these two sets, are they equal?

A: Only if the constant equals zero

Interviewer: But the first is an infinite set...

A: Yes, so the first contains the second set

Interviewer: How does this solve the conflict regarding the equation 0=c ?

A: The zero belongs, so the equality holds for one case but not in any case, so we have a generalization failure?

In spite of his procedural conception, student A with the help of his interlocutor understands the inadequacy of his initial statements regarding the equality of these sets and goes on to a reorganization of his knowledge. We could say that he is not far from a conceptual restructuring, what Piaget named level beta (Piaget, 1975) or implicit knowledge building according to Chan et al. (1997).Nevertheless three are two questions that can not be explained in this frame: why does he repeat his initial position ("Only if the constant equals zero") even after his change of attitude and why is he concluding his statement with a question mark? If we interpret the change not as result of a cognitive but rather of a discursive conflict which contains as variables the use of words and the intended focus of discourse we could say that student's need for communication and the meta-discursive rule of the superiority of teacher-interviewer's word are the principal drives behind student's change. The student investigating the meaning of the expression $\int f-\int f=0$ is asked whether the equality A-A=0 holds for any mathematical objects:

A: When we have an object A then the equation A-A=0 holds but here we have a set no such an object Interviewer: The set is it an object? Can I see it as an object?

A: No, as an object, I don't think we can do

Interviewer: Can you regard the set of all natural numbers as an object?

A: ... no, we have an infinite number of natural numbers...

Interviewer: ... so in case we have a set containing an infinite number of elements, can I see it as an object?

A: ... no, definitely no...

Using the communicational approach as in Sfard (2001), A's failure of understanding the object status of the indefinite integral is understood as a failure to communicate and probably this is a dialogue failure more than A's failure. Student's intended focus of discourse is on the infinite cardinality of the set which determines the way he sees the integral. On the other hand, the interviewer's intention of creating a sense of ambiguity and conflict is not considered as such by the student. As Limon (2001) has remarked, students often fail to reach a stage of meaningful conflict, since what the teacher considers meaningful for his students cannot considered meaningful for them.

On another level, the 'met-before' idea (Tall, 2004) of an infinite set and the misconception that such a thing can not be treated as an object has serious consequences for the development of the concept of integral. The student can not see it as a reified object and this suggests that his understanding is still at the procedural level not being able to compress the set theoretic and algorithmic faces of the integral into an amalgam like the procept.

Student K

Student K dealing with Q.4 uses a specific function and gets $\int f - \int f = c$ which then argues that is the right answer:

K: ...this equality $\int f - \int f = c$ is too simple to hold for an indefinite integral˙ on the other hand this equality: $\int f - \int f = c$ is more logical, more mathematical

We see in accordance with the results of other studies like Healy & Hoyles (2000), that the more formalistic a formula is the more "valid" is in the eyes of many students. Although the student is not able to prove his claim, he insists on his belief even after questions by the interviewer which challenge her answer. The formal system of knowledge probably necessitates the existence of formalistic proofs and this often competes and prevails over the logic of the student himself. Afterwards, when asked if she ever felt the need to justify his claim by an example she emphatically answers:

K: ...no, this is not something I usually do... if it works with this way why bother search for an explanation?

Her orientation towards the pure algorithmic level of definition and the process it implies instead of a more general proceptual way of thinking is obvious. Also the meta-discursive rule of following the implementation of any routine that simply works without any regard to its understanding is apparent here.

Student T

In the same vein student T sees no apparent conflict between the definition of the integral and its use in computing specific values. In fact she states it, she computes "the values of every integral without bother thinking if any conflict arises. The following excerpt is characteristic:

T: ... adding two integrals, which amounts to adding two sets is ...according the definition it is the intersection of the corresponding sets...

Interviewer: ... so when you added these two integrals did you add them as two sets?

T: no, no, according to the tables

Interviewer: ... so when adding two integrals do you see them as two sets that add up or something else

T: ... no, I just add them up automatically using the formula-tables, I never saw them as sets although the definition describes sets ... actually it really crossed my mind once but I didn't know how to explain it

Interviewer: ... did you find it strange, I mean adding two integrals like adding two sets?

T: well yes, it's confusing but I just learnt what I needed to, I didn't examine these details

She exhibits low certainity to the conflict between her understanding of the definition (adding integrals as set intersection) and the way she solves integral problems (using the tables of integration) which unavoidably leads to a confusing and meaningless situation and to an implementation of a routine activity according to the Merenluoto & Lehtinen (2004) model. Also an existing gap between operational and structural conception (Sfard, 1991) of integral is evident here, since the student is not able to facilitate the dichotomy between the official definition and the algorithmic knowledge he uses in solving problems. In dealing with the question Q1 she quickly rejects any meaning for the subtraction of integral since it leads to a paradox (0=1) :

T: Although in usual mathematics adding two opposite elements we always get zero, this is not what happens to the ∞-∞ case so we have something similar here subtracting two integrals

Interviewer: And this explanation satisfies you or do you tend to seek it further ?

T: I am o.k.

Interviewer: Do you feel any need to have a more intuitive confirmation to this statement?

T: No, I don't think there should be something like this

Using the same model by Merenluoto & Lehtinen (ibid.) we could characterize his reaction of connecting the indefinability of ∞-∞ with the present case as superficial construction that is based on more primitive met-before concepts which is accompanied by high certainity (no need to check intuitively the claim).This is what Chan et al (1997) call patching: noticing surface discrepancies and patching differences by ad hoc rationalizations. Once more her procedural and algorithmic type of understanding is revealed. Furthermore, her denial of any need for further checking could mean the existence of a meta-discursive rule of no need for validating empirically – intuitively the formal methods and results. Her process like view of the integral is more clearly declared below:

Interviewer:	How would you best characterize the indefinite integral in a single word: object, process, tool, concept,..
T:	process
Interviewer:	Is it an object like a function?
T:	No, because the set of functions that constitute any integral is infinite, so it can't be an object

The concept of integral is not reified and probably not even condensed succesfully in the sense of Sfard (1991) and this is relevant to the insufficient reification of a lower level concept: the infinite element set. Inadequate reification of the concept of set, necessary for building higher level structures, hinders the successful interiorization of the new concept of integral.

CONCLUSION

This was a part of a larger study including the views of a number of in-service teachers and post graduate students in math education. Although all 4 of the students presented above were high graders in the national university entry exams and solved easily the typical exercises of our first part, none of them had reified the concept of the indefinite integral. Their understanding was mainly procedural and it was not easy at all for them (though with different grades of difficulty) to handle the integral as an object. This goes the same way with their statements in part three where they characterize the integral mainly as a tool or a "technique to solving efficiently differential equations". Considering the general emphasis of the Greek curriculum in formal ideas and the subsequent exercises on that style, this is even more astonishing. It seems that the definition the way it is given in the book is not "formally operable" in the sense of Bills & Tall (1998). Also, in some cases the inadequacy of lower level reification of the concept of infinite set was an obstacle to the higher-level interiorization of the integral concept. Using a different lens, the personal interpretation of words and the different intended focus of discourse are probably some of the factors that played crucial role in what could be seen as the failure to communicate. In this context the failure takes place between the textbook and the student and also between the teacher-interviewer and the student. As also is shown in other examples (Limon, 2001) in the efforts to teach conceptually difficult notions it is useful to use cognitive conflict techniques taking into consideration the cognitive distance between student's prior knowledge and the new phenomenon to be learned. Otherwise insufficient synthetic models arise. All in all, the different outlooks though incommensurable when applied to the same case simultaneously, they provided us useful insights in the abovementioned cases. And since a simple model of explaining the act of understanding is far from near, exploiting the strengths of competing frameworks is clearly more than useful.

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DETECTING THE EMERGENCE AND DEVELOPMENT OF MATHEMATICAL DISCOURSE: A NOVEL APPROACH

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*Six cases of development of pupils' mathematical discourse are examined in relation to a proportional reasoning task. A discourse analysis approach is used that was specifically designed for this study by combining principles from Toulmin's (1958) Gee's (1999) and Backtin's (1986) work and the use of Nu*dist software. Notable changes in the quality of argumentation are detected in all cases. The changes are attributed to the introduction during the discussion of a carefully designed pictorial model which was incorporated in the pupils' discourse and eventually transformed it.*

INTRODUCTION

The results reported here are part of a larger scale study with the aim to research and develop a proposal for effective teaching of the topic of 'ratio and proportion'. The first part of that study involved research on diagnostic assessment of children's proportional reasoning. The second part involved research on discussions within small groups of pupils. The pupils worked on selected ratio tasks and the discussions were coordinated by the author who also introduced tools that were hypothesized to facilitate the arguments within the group dialogue. The argumentation that stemmed from these discussions was analysed by using a discourse analysis approach that was designed specifically for this study. It combined principles from Toulmin's (1958) Gee's (1999) and Backtin's (1986) work and the use of Nu*dist software.

Selected products from that discourse analysis are presented in this paper. More specifically, the results come from six groups working collaboratively on an item called 'Paint'. This item comes from a diagnostic test that we have constructed and it has been found to produce a high frequency of the error strategy called 'constant difference' or 'additive strategy'(Misailidou and Williams, 2003). A pictorial representation of the item was used as a tool for facilitating discussion because previous analysis had shown it to make an unusually significant impact on the item difficulty. The quality of the pupils' argumentation is examined before and after the introduction of the pictorial representation in the discussion. It is claimed that the pupils' discourse was crucially affected by the pictorial tool.

METHODOLOGY AND DATA ANALYSIS

Groups of pupils were formed, whose responses to the 'Paint' item had been different on our previously administered and analysed diagnostic test, thus engineering conflict. Each group consisted of three pupils (aged 11-14 years old) and was involved in a researcher-guided discussion. The children were set the task of persuading each other by clear explanation and reasonable argument of their answer. The researcher,

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adopting the teacher's role, established rules for the children's argument in order to facilitate participation in discussion.

The 'Paint' item was adopted from Tourniaire (1986)

Sue and Jenny want to paint together.

They want to use each exactly the same colour.

Sue uses 3 cans of yellow paint and 6 cans of red paint.

Jenny uses 7 cans of yellow paint.

How much red paint does Jenny need?

A pictorial representation of the item (shown in Figure 1 together with work from pupils) was used as a tool for facilitating discussion: the pupils could refer to it while communicating their ideas or could work on it with the aim of developing further their strategies. The tool was referred throughout the discussions as the 'pictures-sheet'.

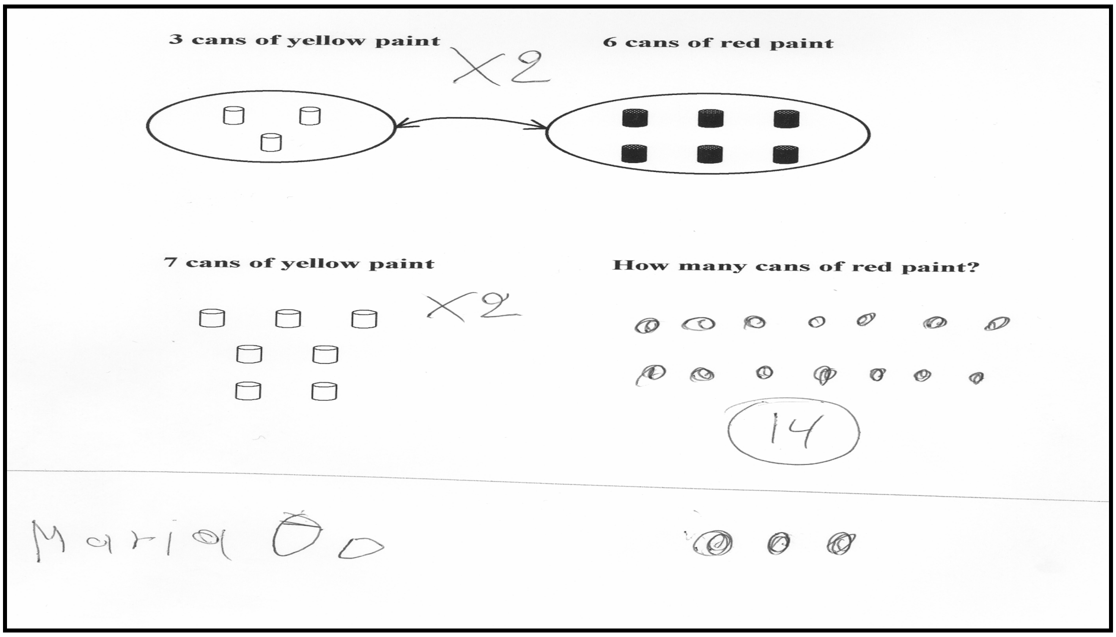

Figure 1: The 'pictures-sheet' used in the group discussions

The 'Paint' item provoked a high frequency of the error strategy called 'additive strategy'. This is the most commonly reported erroneous strategy in the research literature related to proportional reasoning (e.g. Hart, 1984). In this particular problem the result of the additive strategy is the incorrect answer '10' which can be obtained either by thinking 3+3=6 so 7+3=10 or 3+4=7 so 6+4=10.

This paper we will focus on the pupils that made this error in test conditions and then participated in the group discussions on 'Paint'. The pupils' discourse during the 'group discussion' is considered as an indication of the development of pupils' reasoning. Their discourse was analysed by using Toulmin's (1958) method. The utterances that made up the arguments were classified as 'data', 'conclusions', 'warrants' and 'backings'. Data are the facts that are requested as a foundation for the conclusion. Warrants are the utterances which demonstrate that 'taking the data as a starting point, the step to the…conclusion is an appropriate and legitimate one' (Toulmin 1958, p. 98). Finally, backings are defined as the assurances that strengthen the authority of the warrants. The children's arguments were schematically represented using Toulmin's categories and then coded in order to assess changes in their argumentation.

'Discursive paths' for each of the pupils that took part in the group discussions were composed and studied. A 'discursive path' is defined in this study as the evolution of the pupil's argumentation in the discussion. Each discursive path is comprised by one or more stages. A 'stage' in a pupil's discursive path is defined as a particular claim and possibly warrant and backing that a pupil offers at a certain point of the discussion. When the discursive features of either the claim, or the warrant or the backing change, a different stage 'emerges'. A stage can roughly be identified as a pupil's explanation for a particular answer for the task under discussion.

Pupils are not considered to reason proportionally when they just give an answer which is the result of a multiplicative calculational process but only when, additionally, their argumentation indicates that they have conceptualised the task multiplicatively. After applying to the context of proportional reasoning and refining Cobb's (2002) distinction of children's mathematical discourse, four categories were found relevant to classify pupils' explanations:

1. Numerical explanation: The pupil explains a numerical performance for finding an answer to the task without justifying it contextually.

2. Context-indexed, non multiplicative-indexed explanation. The explanation is defined as 'context-indexed' when the pupil's discourse indicates a connection of numerical relations or performances to relevant contextual data from the task. The explanation is defined as 'non multiplicative-indexed' when the pupils' discourse indicates a non multiplicative conceptualization of the task: a conceptualization of the task context that may prohibit the construction of proportional relations.

3. Context-indexed, pre multiplicative-indexed explanation. The explanations that belong in this category are context-indexed and also indicate a pre multiplicative conceptualisation of the task: the pupils have conceptualised the task 'pre-multiplicatively' when they can think relationally about quantities (for example in the Paint item they realize that more red paint than yellow is needed for a dark shade of orange), but not yet proportionally.

4. Context-indexed, multiplicative-indexed explanation. The pupils' discourse indicates that they conceptualized the task multiplicatively.

The pupils' utterances were studied as units of analysis and they were distinguished in the transcription text according to Bakhtin's (1986) definition: An utterance has 'an absolute beginning and an absolute end: its beginning is preceded by the utterances of others, and its end is followed by the responsive utterances of others' (p.71).

An important mediating element for the emergence of proportional reasoning is considered in this study the use by the pupils - during their dialogic activity- of a language that indicates a focus on the contextual data of the task. Such pupils' discourse is coded as context-indexed discourse throughout the group discussions and most importantly throughout the pupils' discursive paths. More specifically, a pupil's utterance was coded as 'context- indexed' when it contained words or sets of words that indicate (index) a consideration of the context of the task. In the group discussions concerning the task 'Paint' the pupils' discourse was coded as 'context-indexed' when it mentioned or indicated:

1. The importance of colour.
2. Different ways of using colour.
3. Mixing colours.
4. The importance of a 'shade' of a specific colour.
5. Evaluation of resulting shades.

For example utterances that contained phrases like 'Sue's got less yellow paint than Jenny', 'that one needs more red', 'it would be darker there' were all coded as 'context-indexed'.

In order to clarify the effect of the pictorial model on the flow of arguments two types of coding and analysis were performed. Firstly, each stage of each pupil's discursive path was coded and the analysis focused on the difference of the discourse quality between stages before and after the introduction of the pictorial model in the discussion. During the second type of analysis, the appropriate utterances were identified and coded as 'context-indexed'. Then with the help of Nu*dist, the amount of context-indexed discourse was calculated before and after the introduction of the model.

RESULTS

Six (out of 18) pupils that participated in the group discussions about 'Paint', gave the answer '10' when tested prior to the discussions. Their written explanations led to the assumption that all of them used the 'additive strategy' in order to obtain the answer. Thus they were coded as 'adders'. As expected, at the beginning of each discussion all of them insisted on their chosen method for solving the task. All of them though, 'changed their mind' during the discussions as indicated by their discourse in the transcription texts. All of them at the end of the discussions rejected their original answer '10' providing context-indexed justifications and most of them accepted the multiplicative answer '14' by providing again context-indexed explanations.

In order to examine the role of the pictorial model in the difference of the pupils' discourse quality, firstly, the adders' individual discursive paths were constructed and then studied carefully. (An example of a detailed discursive path can be seen in Misailidou and Williams, 2004.) It was found that there was a difference of quality in their explanations before and after the introduction of the 'pictures-sheet'. This is demonstrated in Table 1 by presenting for each pupil the immediate stage before and after the introduction of the model.

Change in argumentation		
	Before the introduction of the 'pictures-sheet'	After the introduction of the 'pictures-sheet'
Hannah	Stage 2 ▸ Numerical, non multiplicative-indexed explanation ▸ Numerical backing	Stage 3 ▸ Context-indexed, multiplicative-indexed explanation ▸ Context-indexed, image-indexed backing
Heather	Stage 1 ▸ Numerical, non multiplicative-indexed explanation ▸ Numerical backing	Stage 2 ▸ Context-indexed, multiplicative-indexed explanation ▸ Context-indexed, image-indexed backing
Addie	Stage 2 ▸ Context-indexed, non multiplicative-indexed ▸ Context-indexed backing	Stage 3 ▸ Context-indexed, pre multiplicative-indexed ▸ Context-indexed, image-indexed backing
Val	Stage 1 ▸ Numerical, non multiplicative - indexed explanation ▸ Numerical backing	Stage 2 ▸ Context-indexed, pre multiplicative-indexed explanation ▸ Context-indexed, image-indexed backing
Rachel	Stage 2 ▸ Numerical, (tentatively) multiplicative - indexed explanation ▸ No backing	Stage 3 ▸ Context-indexed, multiplicative-indexed explanation ▸ Context-indexed, image-indexed backing
Faye	Stage 1 ▸ Numerical, non multiplicative-indexed explanation ▸ Numerical backing	Stage 2 ▸ Context-indexed, pre multiplicative-indexed explanation ▸ Context-indexed, image-indexed backing

Table 1: Change in the adders' argumentation

In the table, it can be seen that after the introduction of the 'pictures-sheet' in the discussion, the quality of each of the adders' discourse has changed. Five (out of six) adders gave numerical explanations in their discursive stages just before the introduction of the pictures-sheet in order to justify their adopted strategies. All of them presented non multiplicative-indexed explanations except one whose explanation was coded as (tentatively) multiplicative – indexed. After the introduction of the pictures-sheet all of the adders presented a change in their discourse and moved to new stages in their discursive paths. All of them gave context-indexed backings, to support their chosen methods. Three of them gave pre multiplicative-indexed explanations and three of them gave multiplicative-indexed ones.

It is noticeable in the table, that after the introduction of the pictures-sheet all of the pupils' backings were coded as 'image-indexed' as well. In general, an individual's utterance was coded as 'image-indexed discourse' when it contained

1. words, or sets of words, or/and gestures that indicate (index) a consideration in the argument of the pictorial model (e.g. 'pointing' to and referring to the pictures of the model in order to explain their thinking)

2. and/or actions on the pictorial model (e.g. drawing on it).

The fact that after the introduction of the 'pictures-sheet' all the adders offered backings that were context-indexed as well as image-indexed is conceived as a further testimony of the influence of the model on the groups' discourse. It is hypothesized that the discourse became context-indexed *because* it was image-indexed, i.e. *because* it referred to and made use of the model. In addition, some of the pupils confirmed during the discussion that they changed their mind because of the pictures-sheet. Heather, for example, declared: 'I've changed because I got persuaded…[and later]…by a different method…by the pictures'.

By studying in detail the text of the group discussions transcriptions it was noticed that the adders' discourse in general (and not only their explanations as they appear in their discursive paths) appeared 'more' context-indexed after the introduction of the 'pictures-sheet'. To check this, the software Nud*ist was used to compare for each group discussion the amount of the group members' context-indexed discourse before and after the introduction of the pictures-sheet. This information is given in Table 2:

PAINT Percentage of the utterances that were coded as 'context-indexed discourse'						
Group discussion No	1	2	3	4	5	6
Percentage **Before** the introduction of the pictures-sheet	9.4%	23%	7.4%	20%	19%	5%
Percentage	38%	38%	39%	46%	24%	69%

After the introduction of the pictures-sheet						

Table 2: Context-indexed discourse before and after the 'pictures-sheet'

From a quantitative perspective, it is considered important the fact that there is a difference in the figures before and after the introduction of the pictures-sheet for each group discussion. As no research of this type has been reported in the relevant literature there are no results with which these differences can be compared. Nevertheless, the fact that there are differences shows that a larger amount of the pupils' discourse could be coded as context-indexed after the introduction of the pictures-sheet compared with their discourse before that. These figures indicate, in quantitative terms, that after the introduction of the pictures-sheet, the pupils' discourse was transformed in the sense that it appeared to be more context-orientated. These observations support further the hypothesis that the pictorial model might have facilitated the articulation of context-indexed discourse.

CONCLUSION

Karplus *et al.* (1983) stress that some students decide to use (or not) proportional reasoning influenced by the easy (or difficult) numerical structure of the problem rather than its context. Therefore, in this study, the emergence, within the pupils' dialogue, of the discourse that was coded as 'context-indexed' is considered essential for the development of their proportional reasoning.

The results presented above show that the pupils' context-indexed discourse increased after the introduction in the discussion of the pictorial model. In fact, the adders that participated in the discussions demonstrated a change in their discourse at the end, namely, context-indexed and multiplicative-indexed explanations. Furthermore, their backings became context-indexed as well as image-indexed. This is considered as a strong indication that the pictorial representation aided the pupils to discuss whether to use or not proportional reasoning based firmly on the problem context without being distracted by a possibly 'difficult' numerical structure.

Discourse analysis approaches have already been used in researching mathematical argumentation as reported in the relevant literature but not in the area of proportional reasoning. More specifically, the method of analysis proposed in this study has not been reported before in mathematics education. It is proposed as a practical method for researching the pupils' development of mathematical discourse in a context of group discussion where additionally a model is introduced because it is hypothesized as discussion facilitator.

Acknowledgement

This study was funded by the Economic and Social Research Council (ESRC, No: R42200034284)

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THE NATURE AND ROLE OF PROOF WHEN INSTALLING THEOREMS: THE PERSPECTIVE OF GEOMETRY TEACHERS

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We report preliminary results of research on the underlying rationality of geometry teaching, especially as regards to the role of proof in teaching theorems. Building on prior work on the classroom division of labor in situations of "doing proofs," we propose that the division of labor is different in situations when learning a theorem is at stake. In particular, the responsibility for producing a proof stays with the teacher, who may opt to produce the proof in a less stringent form than when students are doing proofs and who may do so for reasons other than conferring truth to the statement. We ground this claim on reactions from experienced geometry teachers to an animated representation of the teaching of theorems about medians in a triangle.

INTRODUCTION AND THEORETICAL FRAMEWORK

Classroom instruction relies on a basic division of labor vis-à-vis the knowledge at stake (the teacher teaches, the students study, and the teacher attests to their learning) that gets further specified (in terms of who does what and how) depending on the particular kind of symbolic goods that are at stake. With the expression "instructional situation" we refer to each of the systems of norms that organize usual transactions between work done and knowledge claimed as taught or learned. By a "norm" we mean a central tendency around which actions in instances of a situation tend to be distributed.[1] Earlier work has studied the situation of "doing proofs" in American geometry classrooms, in terms of its division of labor and its temporal entailments (Herbst & Brach, 2006; Herbst, 2002). We have further hypothesized that as teacher and students participate in an instructional situation, they hold themselves and each other accountable for responding to the presumption that they should abide by those norms. Teachers, in particular, make use of a practical rationality, a system of dispositions, categories of perception and appreciation that allows them to handle the presumption that they should abide by a norm (Herbst & Chazan, 2006). This paper explores the practical rationality that geometry teachers invest when handling the norm that a new theorem needs proof: What are the categories of perception and appreciation that matter on the decision that a proof is needed and on the appropriateness of the proof produced? The importance of this question lies in the fact that in spite of the fact that in mathematics every theorem has a proof, in American geometry classrooms some theorems are proved but others are not.

Our interest is the nature and role of proof in the geometry class. Herbst & Brach (2006) studied the division of labor between teacher and students in the instructional situation of "doing proofs" in which students' engagement in proving is exchanged for

2007. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 281-288. Seoul: PME.

claims on their knowledge of how to do a proof. In those situations teachers are responsible for identifying the task as one of doing a proof, for setting of initial conditions, and for providing the statement to be proved. Students are responsible for orderly producing the statements and reasons that constitute the proof. Each of those "statements" is often a description of a property of the figure represented in a diagram or a transformation of a prior statement, supported by a previously known "reason"—a theorem, postulate, or definition. Yet, the proposition proved—the claim that in the given conditions the conclusion is true—is often of little consequence: Being stated as a true statement about a specific diagram, neither the teacher is responsible to teach it nor students must demonstrate they know and can remember it.

A look at the work of mathematicians suggests that proofs might be present in other occasions of classroom life: when theorems, those general, consequential statements that collect knowledge for further use are installed in the public knowledge. Our inquiry as to the nature and role of proof in geometry instruction took us then to examine another instructional situation, which we call "installing a theorem" and which we define provisionally as the system of norms that regulate what teacher and students need to do to be able to claim that the class knows a new theorem.

METHODOLOGY WITH NOVEL TECHNIQUE

To pursue our interest on the normative aspects of teaching, rather than on the preferences and beliefs of particular individuals, our project gathers data from groups of experienced geometry teachers confronting together representations of teaching that showcase instances of the instructional situation we want to learn more about. We use a novel technique to gather data on teachers' practical rationality. We create stories of classroom interaction and represent those stories as animations of cartoon characters (Fig. 1). These characters interact in ways that might or might not be common in American geometry classes. They showcase instruction that straddles the boundaries between what we hypothesize to be normative and what we expect practitioners would consider odd. The representations of teaching are shown to the participants of monthly study groups composed of experienced teachers of high school geometry who meet for three hours every month to discuss one or two stories. The discussion among teachers, where they point to odd or intriguing moments in the story, suggest alternative stories, or bring concurrent stories of their own, is therefore a main source of our data. This paper focuses on discussions of the story called "Intersection of Medians" from two of those sessions. We describe the teaching represented in the animation to demonstrate how a story embodies hypotheses about the normative in teaching, and to ground anticipations of participants' reactions which we examine after.

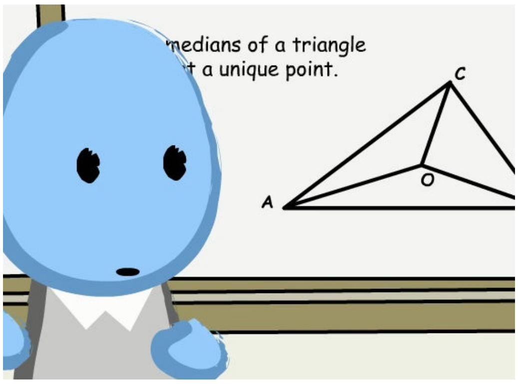

Fig. 1. A theorem on medians

ANALYSIS OF THE ANIMATION "INTERSECTION OF MEDIANS"

The animation "intersection of medians" deals with two theorems about the medians in a triangle. Theorem 1 states that "the medians of a triangle meet at a point." Theorem 2 states that "if O is the centroid of triangle ABC then the areas of triangles AOB, AOC, and BOC are equal." The following phases of the instruction represented in the animation describe the different ways in which those theorems are installed.

Phase 1: Defining median and conjecturing Theorem 1
Students are reminded that they know perpendicular bisectors, angle bisectors, and altitudes of a triangle. Medians are defined. The teacher invites a conjecture about medians and students propose that they meet at a point. Theorem 1 is introduced without proof. Centroid is defined.

Phase 2: Conjecturing
The teacher invites students to conjecture a property of the triangles made by the centroid of a triangle (Fig. 1). After some trials a student proposes that they are equal in area and the teacher writes Theorem 2.

Phase 3: Presentation of proof
Teacher presents a proof for Theorem 2 by subtraction of equal areas (see Fig. 2 and Herbst, 2006, p. 324). Students answer focused questions whereas the teacher steers the argument, writing it as a paragraph (omitting reasons and some statements "for the same reason").

Phase 4: Verification activity
The teacher hands a sheet with a triangle for students to find the centroid and measure the areas of the three triangles. They collect measures.

Phase 5: A new conjecture
Students are asked to look at the six triangles made by three medians inside the triangle and invited to make a conjecture. The conjecture that they are equal in area is disputed on account of perceptual differences.

How the theorems are installed in the animation

Conjecturing and corroborating the theorems. Theorem 1 is stated in analogy with other concurrency properties for segments in a triangle and admitted true without proof. In particular, the possibility that three medians might meet in three points is never entertained. This development conforms with customary practice in American geometry classes where all theorems are not usually proved. We anticipated whereas this oddness might prompt a mathematically educated observer to comment, experienced geometry teachers would not react to the installation of Theorem 1.

The installation of Theorem 2 is also odd but in a different way. The teacher invites students to conjecture the theorem providing only a diagram that students can look at (see Fig. 1). Whereas the mathematically educated observer might be able to use the definition of medians to enrich the representation, creating other resources with which eventually produce a reasoned conjecture, the resources provided by the animated teacher are at odds with what geometry teachers customarily afford when they give students opportunities to make a conjecture about metric properties in a diagram. The animated students are not given the chance to measure dimensions and calculate areas until after the proof has been done. When they produce the conjecture they grope from claim to claim until they succeed, leaving the impression that they are using the teacher's responses to prior guesses as resources to make a conjecture. We anticipated that this would frustrate our participants, and that they would suggest an earlier exploration focused on measuring areas of particular triangles. The empirical verification of the theorem conducted after proving would not only be unhelpful for conjecturing but would also risk casting doubt on the proof. (In the animation the area found for one triangle is slightly less than the other two and the animated teacher brushes that off as a measurement error.)

Characteristics of the proof produced. Fig. 2 contains the finished proof of Theorem 2 as shown on the board. Several characteristics are odd for those who are familiar with how proofs are usually written in geometry classes. Notably, the proof is written in a paragraph rather than in two columns of statements and reasons (Sekiguchi, 1991). The proof does not restate the given or spell out all statements, and it states none of the reasons. Additionally, the proof does not build on congruent triangles, which is the customary way in which proofs are done in geometry classrooms (Herbst, 2006). We want to understand what elements of the teachers' rationality are brought up by our study participants to notice and appraise the decisions made by the animated teacher in proving this theorem. The question was particularly important to ask apropos of this animation since, as the proof did not stage work similar than what students would do when "doing a proof," we anticipated teachers would not necessarily warrant this proof as an example of "how to do a proof."

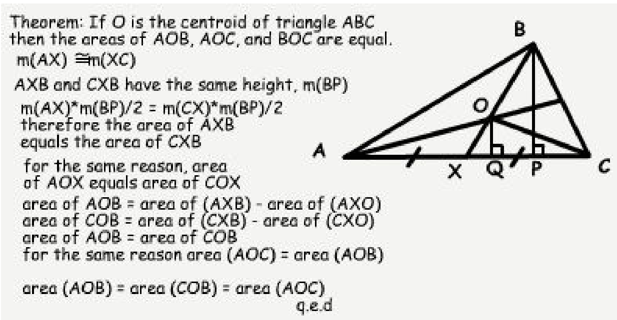

Fig. 2. Proof given in the animation

The division of labor in "intersection of medians." In the animation one can note several actions by teacher or students that straddle the boundaries of what is normative in geometry classrooms when students are engaged in "doing a proof." Not only the teacher writes each statement but also he authors what he writes, which appears often as something that students could not have produced. Unlike in a situation of "doing proofs," key elements to be used in the proof (such as the altitudes *BP* and *OQ*; Fig. 2) are not provided at the onset but called up as needed. Likewise, students use the diagram to produce statements that would ordinarily feature in the proofs they are used to do (e.g., *OB* is congruent to itself), but these are not taken into the argument. Whereas students partake of the production of the proof by uttering responses to the teacher's questions, those utterances serve at best as indication that some students are following the teacher (and others are not), not that they are producing the content of the proof the teacher writes. We thus conjectured that if study group participants were to positively appraise the animated teacher's decision to prove Theorem 2, they would invest categories of appreciation and perception different than those invested in managing the situation of doing proofs. We expected that those comments might help explain on what grounds some theorems are proved whereas others are not proved in high school geometry classes.

ANALYSIS OF STUDY GROUP SESSIONS

The data gathered consist of videos of study group sessions and their transcripts. By identifying odd aspects of the animation, our prior section identifies particular moments in the story where participants' comments could be expected. We provide illustrative data from two sessions (121405 and 040406) in which the animation was viewed, discussed as a whole, and commented after stops requested by participants.

All theorems need not be proved. Our participants confirmed that, from their perspective, proofs need not always be given, which obviously sets the situation of "installing theorems" apart from that of "doing proofs" in which a proof is required. Participants didn't object to the statement of Theorem 1 without proof and actually manifested that they had never proved Theorem 1 in their classes. Furthermore, some of them expressed that the proof for Theorem 2 might not be needed either. When "installing theorems", proving has less priority than introducing the theorem itself. Some of participants expressed this clearly for Theorem 2:

Tina	Yeah if you were choosing to do the other one, I mean—you have a limited amount of time for every theorem they're gonna be given. We don't have time to prove every single one of them. (121405)
Denise	If you know the theorem you would be able to use it. Sometimes—well all the time—you don't really have to know how to prove everything to be able to use it. So as long as they can use it. (121405)
Glen	It would ultimately be more important to go through that discovery process you just talked through than to go through that proof that none of them are gonna understand. […] (121405)

The reasons given stress that students need to be able to use a theorem (e.g., in calculations), that students may instead benefit by knowing where a theorem comes from (which presumably the proof does not provide), that understanding proofs is hard for students, and that proofs are optional investments of time for the teacher.

Corroborating a theorem. From those comments, we can see that, unlike in the usual practices of research mathematicians, the truth of a theorem is not really established in a geometry class by showing the existence of a proof. This may not surprise those who have characterized traditional mathematics teaching in broad strokes as relying on teacher and textbook authority (Smith, 1996). However, teachers in our study groups did allocate value to raising students' degree of conviction (or epistemic value; Duval, 1991) of a statement. They reacted to the animation on the point that the conjecturing process did not allow students to believe the truth of the theorem:

Megan	Before they start that theorem, they should be believers. Why would the areas be the same? Why does that make sense? Why would I believe that? […] (121405)
Lynne	But just can we even have an example that we think the conjecture is even true? Because the teacher just said 'yup the area one is going to work.' And I think the kid wanted a little verification there, like are the areas really equal? (040406)
Ester	[At] the end he has them measure them. Maybe, say, why don't they measure them first. (040406)

If teachers may not consider that the proof of a theorem is always needed but they value students' thinking to the point that they may spend time on (possibly empirical) work to build conviction of its truth, the question still remains as to what roles a proof may play the proof is provided. One reason may be to enlighten students in regard to the form a proof may have.

Form. The animated teacher called the proof he did a "paragraph proof" and took to do it deliberately, as an alternative to the "two-column proof" that students were used to. Participants' reactions to the form of the produced proof were positive:

Carl	I got the sense from his paragraph proof, which I liked actually […] one of the things I like about proof is that it's really just, um, documenting information that we're trying to keep track of. […] well not by the end of the proof chapter, but certainly by the time we've done a whole bunch of proofs, […] I'm really just more inclined to just say hey, just, you know, get that, get that down, don't forget about that triangle. […] (040406)
Megan	I guess I think I'm agreeing with him that now at this point of the year, I guess I'm relaxing on that and saying I want you to just write out a logical argument. Because some of them, I think the proof format ends up —they get so bogged down in the format. […] (040406)

Hence, a proof without all the detail might be acceptable as the year progresses. Students need to experience getting the important (as opposed to all) information needed for the argument, relaxing the strictures of how to write it and yet keeping it a logical argument. A proof of a theorem could thus be an opportunity for students to learn the difference between a logical and a detailed argument, but in that case the shares of work producing the proof might be different than when "doing a proof."

Proof production. In the animation, the teacher presented the proof. Students intervened little, and rarely introduced elements for the argument. Participants were ambivalent on this matter. Some positive appraisals included:

Karen	[…] there's this sort of thing where math as a spectator sport is absolutely wonderful. […] And it's like going to art appreciation. (040406)
Lynne	It's like coaching volleyball, you have to take them to a varsity game or a college game so they see the flow of how the whole thing works and then they really want to get back and try those smaller basic things (040406)

In some negative appraisals participants noted that the teacher didn't take into account students' comments while doing the proof. For example:

Denise [...] Because those kids were not doing the proof with him. I mean they were, but they wasn't. [...] people were just calling out the answers and he [the teacher] was just waiting on the right answer. (121405)

Marvin [...] you might say to the students, [...] how are we going to find areas in this figure, or what triangles do we know the area of? [...] do you students see any triangles in here that we could show [...] are equal area. Or, what do you [...] know about the figure in general? (040406)

Participants' comments note that the class didn't do what is expected of them when they "do a proof." Yet students' "appreciation" or understanding of the proof could also be a legitimate goal for the teacher. However, on that point the animated teacher is found wanting: a teacher should access and support students' understanding better.

A proof connects ideas. A participant also identified another role of proof in the proof given for Theorem 2. Whereas Megan downplayed the usefulness of the theorem itself she valued the proof because it connects some mathematical ideas:

Megan That particular theorem [is] not very useful on it's own after that. But the proof is actually very useful. I think that's a perfect example of a theorem where the proof's a lot more useful than the theorem is, in the end. Because you're talking to kids about the area of a triangle, and what is the crucial thing to know? I need to know the base and the height, and the fact that the—you have different triangles that have the same base and height and they all have the same area (121405)

Thus teachers' rationality allows for proof as a way to explain and connect ideas, agreeing with what scholars have proposed as desirable (e.g., Hanna, 1990).

CONCLUSION

From the teachers' observations as to the need for corroboration of a theorem, we infer a new element that distinguishes situations of "installing theorems" from situations of "doing proofs." In the latter the truth of the conclusion is not at stake: the problem often asserts that a proof can be done and the work is to show that the givens imply the conclusion. Herbst (2004) has argued that usual proof exercises confront students with a diagram that not only represents the objects in the statement but also is sufficiently accurate for students to be able to produce statements that also describe apparent diagrammatic truths. Theorem 2 illustrates that all geometric propositions cannot be warranted on such descriptive interactions with a diagram. Teachers' reactions suggest that it is important for teachers to achieve, in a different way than by visual apperception, students' conviction of the truth of the statement.

We have located differences in the division of labor between "installing theorems" and "doing proofs": in the former, statements can be unimaginable by students proof, details may be excluded, and a theorem may be established without proof. We may also see, from these results, some inconsistency which might happens in teaching practice across the situations of "installing theorems" and "doing proofs." When "doing proofs" the conclusion cannot be used until proved. However, when "installing theorems," the teacher makes it usable for students who need conviction but may be better convinced by arguments other than proof. Thus, sometimes a theorem is introduced without proof, and other times its truth is corroborated empirically. Our data suggests that such practices, albeit problematic in that they foster misconceptions well documented in the learning literature (e.g., Fishbein & Kedem, 1982) may respond to teachers' perception of what will convince their students at the moment when they need to appropriate the theorem.

Notes and acknowledgements

[*] This work is supported by NSF grant ESI-0353285 to P. Herbst. Opinions expressed are the authors' sole responsibility and don't necessarily reflect the views of the Foundation.

[1] The word "norm" is used with similar meaning as Tsai (2004) but with a meaning different than that of Yackel & Cobb (1996).

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