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David Williamson Shaffer

A Case Study of Learning Through Professional Practices

This paper is part of a small but growing literature on how professional practices and professional training can inform the creation of learning environments for younger students (Cossentino, 2002; Cossentino & Shaffer, 1999; Erickson & Lehrer, 1998; Hmelo, Holton, & Kolodner, 2000; Jacobson & Lehrer, 2000; Kafai, 1996; Kolodner, Crismond, Gray, Holbrook, & Puntambekar, 1998; Penner, Schauble, & Lehrer, 1998; Resnick & Ocko, 1991; Schon, 1985; Shaffer, 1997c, 2002a, 2002b, 2003, 2004b; Stevens, 2000). The paper describes one particular project in this genre, Escher's World, as an occasion to explore one of the theories that underlies such work, the theory of *pedagogical praxis* (Shaffer, 2004a). Pedagogical praxis suggests that new technologies make it possible to take pedagogies developed in the context of professional training—pedagogies that typically emphasize participation in meaningful projects in epistemologically rich contexts—and adapt them for younger students. That is, pedagogical praxis suggests that the practices through which professionals are trained can provide constructive models for helping students learn from participation in personally relevant projects using computational microworlds.

Escher's World explored this theory by examining how practices from the architectural design studio helped middle school students understand transformational geometry through design activities in a computationally rich learning environment. In the Escher's World summer program, 12 middle school students participated in 56 hours of design activity over a one-month period, leading to the installation of an exhibit of their work at the Massachusetts Institute of Technology (MIT) Museum. Students worked mostly (though not exclusively) with Geometer's Sketchpad, a commercially available microworld designed for use in geometry classrooms.

A key component of the theory of pedagogical praxis is that using microworlds to support learning through professional practices depends on the *thickly authentic* (Shaffer & Resnick, 1999) adaptation of such practices: that is, pedagogical praxis suggests that students can learn effectively by engaging in computer-supported activities that preserve the linkages between pedagogy and epistemology in processes by which professionals become members of their community of practice. In particular, pedagogical praxis argues that a critical component of such adaptations is using a computational tool to make connections from the pedagogy and epistemology of a profession to interests of students, on the one hand, and to significant skills, habits, and associations from a domain of inquiry such as mathematics, on the other (Shaffer, 2004a). The learning environment in Escher's World was thus explicitly modeled on an architectural design studio course studied in depth before creating the Escher's World program (Shaffer, 2003). The summer program was deliberately studio-like, attempting to use the Geometer's Sketchpad to adapt the pedagogical and epistemological underpinnings of design

[1] I would like to thank the students and program leaders of Escher's World, and the students, faculty, and staff of the MIT Media Laboratory where this work began. I would also like to express my gratitude to the many colleagues who provided comments and criticisms on the study in the episodes of its own iterative process.

practices as faithfully as possible to the intellectual and social life of adolescents and to the cognitive demands of the domain of mathematics.

In the discussion that follows, my goal is to use Escher's World as an occasion to examine some of the mechanisms that underlie learning through pedagogical praxis. My intent is not to prove that Escher's World was a "successful intervention." It would hardly come as a surprise that students would learn *some* mathematics after participating in 56 hours of design activity in a mathematical microworld, and previous work (Shaffer, 1997b, 1997c, 2002b) has already shown that students can and do learn mathematics through microworld-based design activities similar to those in Escher's World. Rather, my aim here is to uncover some of the processes through which middle school students developed mathematical understanding in a computational microworld while engaging in activities based on the practices through which designers are trained. I focus particularly on the interactions among three precursors to the development of mathematical understanding in Escher's World: (a) enactment of specific participant frameworks from the design studio, (b) the autoexpressive properties of the computational tool being used, and (c) the articulation and transformation of students' own interests through their design work.

The approach I use to address these interactions is somewhat unusual in design research in the sense that I look for these precursors neither at the microgenetic level nor at the level of the intervention as a whole. This study takes a *mesogenetic* perspective on the analysis of learning, in which participant frameworks, tools, and interests interact over extended periods of time through the design process.[2] I look at this level because students' understanding of (and adoption of) participant frameworks, fluency with a tool, and development of interest took place over extended explorations in Escher's World. Thus, following Lemke's (2000) argument that different processes operate at different time scales in ecosocial systems such as learning environments, I hypothesize that there may be important interactions that are difficult to observe at the local level—or that may be observable at that level only once it is clear what to look for from observation at this broader horizon.

I ground this level of analysis using *design histories*: accounts of students' design work based on extensive field notes collected during the Escher's World project. Although this data set does not afford detailed descriptions of local interactions as might be expected in a microgenetic account, it does make possible thick descriptions (Geertz, 1973) of design activity over time. The analysis is presented here in an extended case study of the work of one student in Escher's World. From this case study, I draw a grounded theory (Glaser & Strauss, 1967; Strauss & Corbin, 1998) about the processes through which participant frameworks from the design studio, the microworld being used, and students' interests interacted in Escher's World. I then use *intra-sample statistical analysis* (Shaffer & Serlin, 2004) as a means to demonstrate theoretical saturation for the grounded theory and to suggest that the pattern described in this case held more generally in the work of the 12 students in Escher's World. Of course, the small sample size and methods used mean that these statistical techniques support what is fundamentally a qualitative

[2] I describe this approach as mesogenetic because it analyzes an intermediate time scale between macro-level analyses of the effects of an intervention as a whole and microgenetic accounts of local interactions and the process of learning.

analysis. The claims here are about the experience of the 12 students who participated in Escher's World.

For rhetorical clarity, I depart somewhat in what follows from the usual organization of an empirical study. I begin by discussing the theories and prior work upon which the Escher's World environment was designed and within which I frame this analysis. Part I of the Methods section describes the structure of the intervention and the methods for data collection and qualitative analysis. Part I of the Results section then presents the first case study and the grounded theory the results support. Following this, Part II of the Methods section describes the segmentation and coding of the data based on the grounded theory, and Part II of the Results section describes the statistical analyses.

I conclude with a discussion of the possible implications of this study for the development of learning environments based on design practices. Although much work has already been done on this issue, the results presented here suggest that design learning is a complex system—perhaps more complex and more richly interdependent than has previously been acknowledged in adaptations of design practices for K–12 students in traditional subjects. The experience of the students in Escher's World reaffirms that the practices of design can be powerful tools in supporting learning through computational microworlds. However, as the theory of pedagogical praxis argues, these results also suggest that developers of learning environments based on design activities may need to pay careful attention to how those participant frameworks are linked to the underlying epistemology of design practices.

Background

Microworlds, Mathematics, and the Design Studio

In Escher's World, students learned mathematics through technology-based design activities explicitly modeled on training in an architectural design studio. As such, Escher's World addresses the intersection of two domains that have been studied in some depth: (a) the role of microworlds as tools in the development of mathematical understanding and (b) the use of design practices as models for the creation of learning environments for K–12 students. Each of these domains has a rich literature associated with it: a large body of research since Papert's *Mindstorms* (1980) in the case of microworlds as learning tools (Bassingthwaighte, 1985; Clements & Battista, 1992; Gargarian, 1996; Goldman-Segall, 1991; Harel & Papert, 1991a, 1991b; Kafai, 1996; Kafai & Harel, 1991a, 1991b; Kaput, 1992; Lehrer, Lee, & Jeong, 1999; Noss & Hoyles, 1996; Papert, 1993, 1996a, 1996b; Resnick, 1994, 2001; Resnick & Ocko, 1991; Resnick & Rusk, 1996; Shaffer, 1995, 1997a, 1997b, 1997c, 1998, 2002a, 2002b, 2003; Tinker, 1985; Wilensky, 1995; Yelland & Masters, 1997), and a smaller but growing body of work in the case of design practices (Erickson & Lehrer, 1998; Hmelo et al., 2000; Jacobson & Lehrer, 2000; Kolodner et al., 1998; Penner et al., 1998; Schon, 1985). There is, of course, also a significant but still smaller body of work looking at microworld-supported design activity (Greeno, 1997; Kafai, 1996; Resnick & Ocko, 1991; Stevens, 2000), including earlier studies in the Escher's World project showing that students can and do learn about transformational geometry through design activities using Geometer's Sketchpad (Cossentino & Shaffer, 1999; Shaffer, 1997a, 1997b, 1997c, 2002a, 2002b). The goal in this paper is thus to investigate some of the mechanisms through which students developed mathematical understanding through one set of design studio activities using one computational microworld as a means to shed light on this larger body of research through the lens of pedagogical praxis.

Mathematical microworlds. More than 3 decades of research on microworlds has documented the processes at work in a wide range of computational tools and a variety of subjects: mathematics and science in symbolic microworlds such as LOGO (Harel & Papert, 1991b; Papert, 1980), StarLogo (Resnick, 1994), and Boxer (diSessa, 2000), or direct manipulation environments such as the Geometer's Sketchpad (Goldenberg & Cuoco, 1998; Serra, 1997; Shaffer, 1997b, 1997c, 2002b); civics, economics and urban planning in simulations such as SimCity (Adams, 1998); history in games such as the Oregon Trail (Smith-Gratto & Fisher, 1999) and Civilization (Frye & Frager, 1996). Along the way, Papert's (1980) theory of Constructionism—that students can build understanding by building computational models—has been extended to a more general theory that microworlds allow students to learn by engaging in open-ended projects that ask them to interact with a model of a complex domain in service of some meaningful end.

One of the seminal concepts that has emerged from this body of research in the context of mathematical microworlds is that of *autoexpressivity* (Noss & Hoyles, 1996). Microworlds such as LOGO create virtual worlds in which the behavior of objects depends on explicit articulation of mathematical relationships (Papert, 1980). Students come to autoexpressive microworlds with a set of beliefs (usually implicit) about the mathematical relationships underlying a domain of interest. As students act in the microworld, the responses of the tool depend on the way in which the students articulate those relationships. Acting in the microworld thus helps students surface, challenge, and ultimately refine their understanding of fundamental mathematical principles. Noss and Hoyles (1996), for example, describe how one student came to understand ratio as a multiplicative (rather than an additive) relationship by developing a LOGO program to construct a BIGHOUSE. The program was an extension of an initial program, HOUSE. In BIGHOUSE, the student enlarged each of the dimensions of HOUSE by a constant amount, but BIGHOUSE did not behave correctly until the student was consistent in using multiplication rather than addition to increase the dimensions of the structure. In the process, the student came to understand that multiplication by a constant preserves proportion whereas addition by a constant does not. Noss and Hoyles coined the term *autoexpressive* to refer to tools whose behavior reflects the extent to which an idea has been represented explicitly. They argue that autoexpressivity plays a significant role in transforming creative activity into mathematical understanding.

Mathematical understanding. Mathematical understanding is, of course, not a singularly defined concept. Some scholars argue that mathematical understanding is inextricably connected to the use of particular mathematical tools—that is, rather than learning about mathematical objects and operations in some general sense, students learn how to solve particular kinds of mathematical problems using particular mediational means, whether graphing calculators, computational microworlds, or algorithms instantiated with traditional pencil and paper (Engestrom, 1999; Pea, 1993; Wertsch, 1998). Work on the situated nature of mathematical understanding and the difficulties in transferring mathematical understanding developed in the classroom to other settings supports this view of mathematical understanding as highly contextual and tool-dependent (Boaler, 1993, 1996; Lave, 1988; Light & Butterworth, 1992; Nunes, Schlieman, & Carraher, 1993).

Despite these caveats, work on mathematical microworlds typically focuses not only on mathematical understanding that students manifest using the microworld, but also on forms of mathematical thinking students are able to accomplish in other contexts after working in the microworld—what Salomon, Perkins, and Globerson (1991) describe as the *effects of* the tool as distinct from the *effects with* the tool. For example, prior work has shown that there are *effects of* activities such as those used in this study: in previous studies, students were able to define symmetry relations, notice symmetry in other contexts, and use visual problem-solving strategies when responding to word problems using paper and pencil more often and more effectively after participating in Escher's World–like activities (Shaffer, 1997b, 1997c, 2002b).

The mechanisms through which such transferable mathematical knowledge develops are not entirely clear. It may be that mathematical understanding consists of abstract representations encoded in formal schema that describe classes of mathematical objects and operations, and the relations that exist among them (Bassok & Holyoak, 1989; Novick & Holyoak, 1991). Alternatively, mathematical understanding may be represented by a web of connections between ideas rather than abstract schema—a view of mathematical understanding as concrete instances connected by a rich set of associations that make them broadly applicable (Noss, 1994; Noss, Healy, & Hoyles, 1996; Noss & Hoyles, 1996; Wilensky, 1991).

In this study, the focus is not on demonstrating *that* students developed mathematical understanding that they could apply in a new context—in this case, a paper-and-pencil test of transformational geometry—although the results will show that students' scores did improve on such a test following the workshop. Nor is the focus on the *kind* of mathematical understanding that they developed—whether abstract schema or a richly elaborated network of embodied experiences—except to say that whatever the nature of the understanding they developed, it was understanding that could transfer to the relatively distinct task of answering questions on a traditional geometry test. Rather, my aim here is to explore the linkages among tool, practice, and interest in this particular environment that led to the development of mathematical understanding that students were able to apply elsewhere.

Uncovering those linkages, in turn, requires identifying a local proxy measure for the development of tool-independent mathematical understanding: evidence that working in the microworld has led to some form of mathematical thinking that students will be able to apply in contexts beyond the microworld. To that end, in what follows I look at students' ability to formulate mathematical conjectures during, and as a consequence of, their design activity. Conjectures are, of course, an important form of mathematical thinking. Although often discussed as a first (and perhaps necessary) step to experimentation and ultimately formal proof (Cox, 2004; Lindquist & Clements, 2001), conjectures are also significant in their own right because one important part of mathematical understanding is the ability (and motivation) to form inferences about general principles from specific observations (Cantlon, 1998; Davis & Hersh, 1982; Fitzgerald, 1996; Gruber & Voneche, 1995; National Council of Teachers of Mathematics, 1989, 1991). Here, however, I am not looking at conjectures as ends in themselves or as precursors to more formal mathematical deduction; rather, I use them as a proxy measure for mathematical understanding that students will be able to mobilize in other situations independent of the microworld and practices within which those understandings developed.

Because conjectures in this sense are *indicators* of mathematical thinking rather than *measures* of mathematical thinking, I do not focus in what follows on conjectures as elements of formal mathematical discourse (Gee, 1990, 1992)—that is, as statements judged within the framework of formal mathematical literacy (Sfard, 2002). Rather, I posit a broader category of mathematical talk that includes any form of claim about general properties of mathematical objects, operations, or relations separate from their instantiation in the particular tool being used. Such claims might be descriptions of some abstract principle derived from practical experience or descriptions of an intuition about the nature of some mathematical relationship: for example, if—as happened in Escher's World—a student, as a result of his or her design activity, explains that rotating things means putting them into a circle, or that dilation changes an object's size without changing its shape or position, or that repeated rotation of an object creates a blurred circle whereas repeated translation of an object produces a blurred line. That is, I posit a class of mathematical conjectures, formal or informal, that are independent of the microworld. In what follows, I refer to this kind of mathematical thinking as *conceptual insights about mathematics*, and I argue that understanding when and how students develop such insights is one important part (though certainly not the only important part) of understanding how appropriately constructed microworlds might be useful tools for supporting development of mathematical thinking through design activities.

Design and the design studio. Schon (1985) and Simon (1996) describe design as an iterative process. The designer is asked (or chooses) to address a particular issue. A solution is proposed. Strengths and weaknesses of the solution are analyzed, and based on this analysis, the designer refines the original approach. The new solution is again analyzed, and so the process continues until the analysis of one of the iterations suggests that it is a satisfactory way to resolve the issue for the intended audience (typically the client, in the case of a practicing architect, or a jury of experts, in the case of an architect in training). The iterations or episodes in the design cycle are thus characterized by *design goals*, *design moves*, and *satisficing* (Akin, 1986; Schon, 1985; Simon, 1996). Design goals are the recognition of some problem, sub-problem, condition, or issue that needs to be addressed—a desired outcome for a given design episode, which may or may not be explicitly articulated and may or may not include a plan for addressing the particular issue at hand. Design moves refer to the series of steps taken to understand and or resolve the initial issue—to express the design goal through design activity. Satisficing refers to the development of some insight into the nature of the initial problem that suggests a conclusion, an end to the exploration, and/or a more interesting issue to pursue in subsequent work. Satisficing may or may not include resolving the initial design problem; for example, the designer may decide that the problem is intractable under the given conditions and decide to take a radically different course of action. Thus, satisficing is a more generic resolution to design activity than solving a problem in the traditional sense.

Design students are trained to carry out this iterative process in the design studio. The design studio can trace its roots back more than a century to the École des Beaux-Arts in France (Chafee, 1977). The focus of a designer's training today still follows the Beaux-Arts tradition of open-ended projects and a variety of structured conversations that culminate in a public presentation of work (Akin, 1986; Anthony, 1987; Coyne & Snodgrass, 1993; Crowe & Hurtt, 1986; Frederickson & Anderton, 1990; Gaines & Cole, 1980; Mitchell, 1994; Rowe, 1987; Schon, 1985, 1987, 1988). In the design studio, students work on individualized projects with expressive goals. They are asked to convey a personally meaningful idea through the product of their design work—to “say something” with their designs to a particular audience (Shaffer, 2003). In this sense, the studio enacts a principle of learning discussed by Dewey (1915; 1958) and Parker (1894/1969) more than a century ago. Parker described how *attention* and *expression* are inseparably connected: creating an expressive product is an integral part of understanding any new experience—a principle rearticulated by Papert (1980; 1991; 1996a) in his theory of Constructionism. Dewey argued that understanding develops during expressive activity when students overcome systematic resistances in the medium of expression—what in more recent years might be described as constraints (Norman, 1993)—if those obstacles are relevant to the expressive goal. As students solve recurring problems they encounter in a particular medium, they learn something about the nature of the medium and about the problems they are solving. DiSessa (2000) and Lehrer, Lee, and Jeong (1999) add that working successfully in a medium also implies developing a sense of audience. Having a sense of audience requires understanding and accepting a set of standards against which work is judged—or developing such standards (Erickson & Lehrer, 1998).

In the design studio, students are supported in the process of progressively coordinating their expressive intentions with the standards of their profession through a participant framework (Goffman, 1981; O'Connor & Michaels, 1996) known as the *desk crit*, or simply *crit*. Crits are extended conversations between designer and critic (expert or peer) over a design-in-progress: a loose but nevertheless structured interaction in which the critic works to understand what the student is trying to do with his or her design by asking probing questions; the critic then helps the student develop the design idea by offering suggestions, validating existing design choices, pointing out potential problems, or working on a piece of the design together. Schon (1985) studied the desk crit process in some depth and argued that the crit is central to the development of a student’s ability to design thoughtfully, functioning as a *zone of proximal development* (Vygotsky, 1978) in which students progressively internalize a design process they can first carry out only with the help of others.

Pedagogical Praxis

Prior work thus suggests there is good reason to expect that the development of conceptual insights about mathematics through design activity in a computational microworld would involve interaction among the autoexpressive properties of a tool, crits between students and peers or experts, and the ideas students were interested in and able to express through the design process. But even after having identified such key practices and principles, orchestrating these elements into a coherent whole remains a challenge.

Brown and Campione (1996) argued that educators need to develop coherent environments—contexts for learning that not only include sound pedagogical practices, but also align those practices with a set of underlying principles. Successful curricula, they suggested, function as coherent systems, and many failed attempts to implement curricula have been the result of seeing what should be a coherent whole as a set of isolated parts—the “Chinese menu . . . one from Group A, one from Group B” approach to curriculum design and implementation (p. 314). Any successful implementation of a curriculum depends, they argued, on a clear articulation not only of *surface procedures*, but also of the underlying *principles of learning* that lead to the creation of the particular pedagogical strategies (p. 291).

In past decades, we have learned a great deal about principles of learning such as the value of autoexpressivity, the effects of participant frameworks such as the desk crit, and the role of students' own expressive desires in the development of understanding. One approach to creating pedagogical coherence, explored by many designers of thoughtfully innovative learning environments, is to articulate a set of learning principles and then design activities and assessments based on those principles (Bransford, 1994; Brown & Campione, 1996; Enyedy, Vahey, & Gifford, 1997; Goldenberg, 1996; Goldman-Segall, 1997; Hmelo et al., 2000; Jackson, Stratford, Krajcik, & Soloway, 1996; Jacobson & Lehrer, 2000; Kafai, 1996; Kolodner et al., 1998; Resnick, 1994; Scardamalia & Bereiter, 1996). A challenge of this approach is that the number of principles and practical constraints multiplies quickly. Designing an environment that simultaneously addresses a complex system of requirements can be daunting.

Pedagogical praxis takes a different and potentially powerful approach to creating coherent contexts for learning. Pedagogical praxis suggests creating learning environments for K–12 students by adapting existing learning practices rather than designing from first principles. Building on the work of Lave and Wenger (Lave, 1991; Lave & Wenger, 1991; Wenger, 1998), pedagogical praxis argues that different communities of practice (for example, different professions) have different epistemologies: different ways of knowing, of deciding what is worth knowing, and of adding to the collective body of knowledge and understanding. In the context of professional activities, these ways of knowing are constituted in practice, and the processes of professional training are designed to link praxis and epistemology through pedagogical activity. Pedagogical praxis thus takes a *learning practices perspective* (Hall & Stevens, 1996; Schwartz & Sherin, 2002), using the ways in which professionals are trained as a model for learning environments.

As Brown and Campione (1996) might argue, this adaptation cannot merely be ad hoc borrowing. Rather, the adaptation has to preserve the essential epistemological principles of the original. Resnick and I have suggested the concept of *thick authenticity* to describe such environments (Shaffer & Resnick, 1999). We argue that authenticity is a form of alignment between student activities and some combination of (a) goals that matter to the community outside the classroom, (b) goals that are personally meaningful to the student, (c) ways of thinking within an established domain, and (d) the means of assessment. Thickly authentic learning environments create all of these alignments simultaneously—for example, in the design studio when personally meaningful projects are produced and assessed according to the epistemological and procedural norms of an external community of practice.

The model of pedagogical praxis is to create such environments by (a) uncovering the principles embedded in existing learning practices of a profession (a problem of cognitive anthropology and descriptive ethnography), (b) developing or adapting technologies to help students participate in these practices (a problem of engineering and technology development), and then (c) creating experimental learning environments designed to develop skills and understanding through participation in the pedagogy of a community of practice (a problem of program design and action research).

In Escher's World, this meant studying in some depth the learning processes in an architectural design studio and then using the Geometer's Sketchpad to adapt those practices for middle school students. I close this section of the paper by describing briefly the ethnographic study of a design studio that guided the development of Escher's World (Shaffer, 2003). Part I of the Methods section of the paper then describes the way in which this model was adapted for middle school students and the tool that was used to make the adaptation possible.

Ethnography of a Design Studio

The pedagogy of Escher's World was based on a study of the Oxford Studio, a mid-level architecture studio for undergraduate and graduate students taught at the MIT School of Architecture and Planning.

The Oxford Studio was quite unlike a typical K–12 class room. Eleven students had more space for their own individual drafting areas than most K–12 schools provide for a class of 25–30 students. The studio had access to an additional meeting space the size of a typical seminar room and a large open space for formal presentations of student work. The pace of work in the Oxford Studio was also quite unlike that in a traditional class. Class met from 2:00 to 6:00 p.m. 3 days a week, but this was more of a rough guideline than a fixed schedule. Students and teaching staff routinely came to studio before or after 2:00 p.m. depending on the work they had to do on a particular day, and the studio was busy through the night and on weekends as project deadlines approached. At any given time in the studio, students and teachers might be discussing projects around a seminar table, or students might be working individually—or checking email, or stepping out for a cup of coffee, or meeting with faculty or peers in a desk crit.

By the standards of a traditional K–12 classroom in the United States, the Oxford Studio was thus an extreme example of an open learning environment. One might even be tempted to call it chaotic. But the large blocks of relatively unscheduled time and the flexibility of the routine made it possible to hold extended conversations about and around students' emerging design projects. The seeming chaos of the Oxford Studio made possible the participant framework of the desk crit.

The focus of the Oxford Studio was the design of a new business school for Oxford University in Britain. Each student was asked, over the course of the semester, to develop, present, and defend his or her solution to this broad design challenge. To guide students in accomplishing this end, the semester was organized around a series of six assignments, each of which focused on a particular aspect of the overall design goal. A typical assignment included a summary of the assignment's requirements, an explanation of the reason for the particular assignment, a description of the professor's expectations, and almost always discussion of examples of work for students to use as models. After this initial introduction, students began working on their response to the assignment. As questions came up, as students ran into problems in their emerging designs, or as students finished some stage of their design process, they would sign up for desk crits with the professor or with a teaching assistant.

Crits usually lasted somewhere between 20 and 40 minutes in the Oxford Studio and had a loose but clear and quite consistent structure. In the desk crit, the student (a) explained his or her design goal to the critic, (b) described the design moves undertaken to that point, and then (c) presented a particular issue or decision that needed to be satisficed. The critic typically (a) asked clarifying questions about the design goal and design moves, (b) discussed with the student problem areas in the existing design (which may or may not have been the issues identified by the student), and then in some cases (c) co-designed possible next steps to the problem for the student to consider.

The goal of the critic was to understand what the student was trying to do with his or her design and then to help him or her develop that design idea. To this end, crits in the Oxford Studio were characterized by a number of specific interactional forms, including validation of work done and suggestions for further avenues to explore, all orchestrated by the most prominent interaction of asking questions about the design in progress. The professor in the Oxford Studio described having a crit with students as “trying to get into their head” and “help them flesh out their own ideas, their own perceptions,” with the role of the critic being to help students “focus on a generative idea” and “unlock the door to make the whole thing better” by asking them “to start anticipating [problems] now before they complete their design.” In other words, probing questions about the design goal, design moves, and future directions were a key component of the crit process in the Oxford Studio. Crits thus formed a conventional configuration of communicative activities and roles—a participant framework (Goffman, 1981; O'Connor & Michaels, 1996)—within which student and critic interacted in the Oxford Studio.

Based on feedback from crits, students returned to their projects, perhaps signing up for a desk crit again before presenting the assignment, perhaps asking for a desk crit with a teaching assistant. Periodically within each assignment, work was discussed publicly in *pin-ups*, with students literally pinning their work up on the wall of the presentation space and presenting it to the group for question, comment, and suggestion—in effect, a crit with the class acting collectively as critic. The results of this iterative process were assessed in the Oxford Studio primarily through *reviews* or *juries*: formal group discussions in which students displayed their work, presented their plans, and received feedback from professionals outside the studio.

It is worth highlighting here that in the Oxford Studio (as in most design studios) the iterative process of design guided by desk crits and public presentations of work was aligned with and orchestrated by a critical feature of the epistemology of architecture: that design ideas reflect an individual interpretation of an architectural problem. Students in the Oxford Studio were presented with design challenges that had an infinite number of potential resolutions. Their task in the semester was to develop a unique solution, to understand that solution, and to convey in words, diagrams, and models how the solution they chose met the demands of the original problem. The idea they developed was of their own choosing—as long as they could, with help from desk crits, develop a design based on that idea in a coherent way and defend their rationale to the larger community of architects in practice.

The Oxford Studio thus suggested a number of key features that Escher’s World would need to preserve to create a thickly authentic adaptation of practices from the design studio for middle school students. The most central were that in the Oxford Studio students worked on projects (a) in depth, (b) in a series of iterative design episodes, (c) over an extended period of time, (d) in an open environment, (e) for public presentation, (f) which were motivated by compelling exemplars, (g) discussed in desk crits, and (h) over which students exercised significant creative control. A long list of guidelines, to be sure. However, it is important to note that these guidelines for Escher’s World were not “principles” for the development of a learning environment in the usual sense. Rather, they were attempts to describe explicitly some key aspects of the Oxford Studio that helped align pedagogy and epistemology, and therefore critical structures we needed to preserve in adapting the practices of the design studio for younger students. Following the theory of pedagogical praxis, we were using a computational microworld to create a thickly authentic simulation of existing professional learning practices using these features as explicit markers, rather than designing *de novo* based on a set of abstract principles or isolated and reassembled pedagogical forms.

Methods, Part I

This section of the paper describes the adaptation of a design studio for middle school students using the Geometer's Sketchpad, a commercially available mathematical microworld designed for use in geometry classrooms. For rhetorical clarity, I begin with a description of Sketchpad, as it is difficult to describe the activity structure without referring to the technology that anchored the work students were doing. Following this explanation of the computational microworld, I provide an overview of the curriculum and daily schedule of the Escher's World summer program, including an overview of one day of design activities. I explain the key differences between practices in the Oxford Studio (and design studios more generally) and those in Escher's World, and end this first part of the Methods section of the paper by explaining the processes through which we collected data and compiled the design histories that form the bases of this qualitative analysis. Later in the paper (in Part II of the Methods section), I describe a formal coding scheme for the data that I used as the basis for quantitative analysis.

Geometer's Sketchpad and the Medium of Dynamic Geometry

My explanation of Escher's World as a design environment starts with a description of the software tool students used to create mathematical images. The Geometer's Sketchpad is an example of a *dynamic geometry* environment (Goldenberg & Cuoco, 1998; Hoyles & Noss, 1994) in which Euclidean objects such as points, lines, circles, arcs, and polygons can be manipulated. In Sketchpad, these objects can be created as independent entities (e.g., a "free point"), or they can be *constructed* using mathematical relationships. One can construct a point at the intersection of two lines, for example, or construct Line *b* perpendicular to Line *a* and passing through Point *C*. When any object in the Sketchpad environment is moved (by clicking with a mouse and dragging), *the other objects in the Sketch move as needed to preserve the constructed mathematical relationships,* and the image is updated continuously in real time to reflect those changes. In this sense, the geometry is dynamic: Sketchpad preserves mathematical relationships under dynamic transformation. However, independent objects do not react in this way. Only mathematical relationships that were explicitly constructed are preserved.

Sketchpad is thus autoexpressive in the sense that Noss and Hoyles (1996) describe: the behavior of objects depends on the explicit articulation of mathematical relationships because Sketchpad distinguishes between objects that are *drawn* and those that are *constructed*. Figure 1 shows two shapes made using the Sketchpad program. The first (*a1*) was *drawn* as a square. It looks like a square, but when its vertices are moved (*a2*, *a3*), it undergoes continuous deformation, revealing that it was *constructed* as an arbitrary quadrilateral whose vertices were moved into position to make it look square. The second shape (*b1*) was *constructed* as a square using perpendicular lines and a circle (see Figure 2 for construction details). As a result, when its vertices are moved (*b2*, *b3*), it changes size and orientation but remains a square because of the mathematical relationships among its parts. Put in simpler terms, if you construct a square in Sketchpad, it will always remain a square, no matter what else happens in the design. If you just draw a square, it may or may not keep the shape you intended as you change the design.

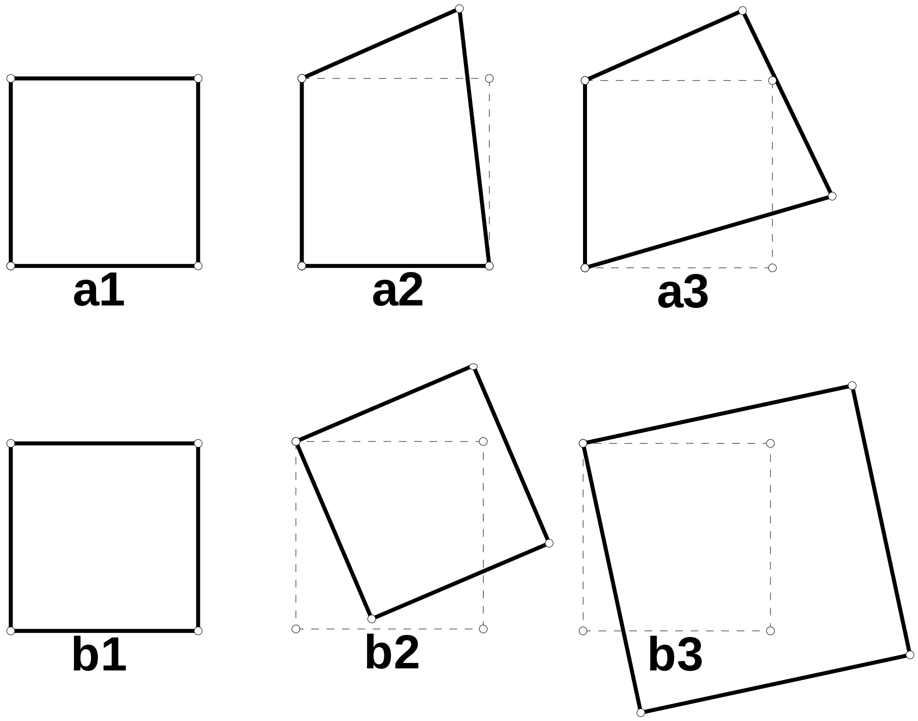

Figure 1. Constructing versus drawing in Sketchpad.

Square a1 *was drawn in Sketchpad: it looks like a square, but when vertices are moved (*a2, a3*), the square does not hold its shape. Square* b1 *was constructed using parallel and perpendicular lines and a circle (see Figure 2 for details of the construction). When the vertices are moved (*b2, b3*), the figure can change size and orientation but is mathematically constrained to remain a square.*

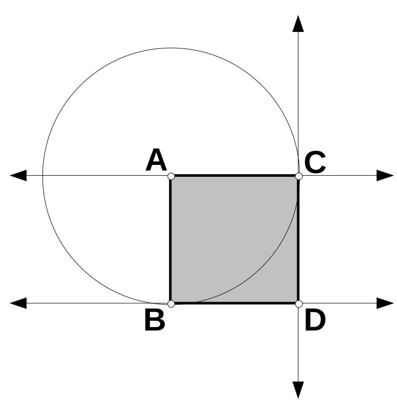

Figure 2. Constructing a square in Sketchpad.

One method for constructing a square in Sketchpad is to make segment AB*, then to construct lines perpendicular to* AB *through points* A *and* B*. The circle with center at* A *and radius of length* AB *gives point* C*. Constructing a line perpendicular to* AC *through point* C *gives point* D*, the final vertex of the square.*

Adapting the Design Studio Using the Geometer's Sketchpad

Content. Among the many mathematical relationships built into the Geometer's Sketchpad are the fundamental mappings of transformational geometry: reflection, rotation, translation, and dilation. The properties of these mappings are a part of the traditional geometry curriculum, although they are typically given less emphasis than the process of formal proof, congruence theorems, and constructions (Schoenfeld, 1989). Escher's World introduced middle school students who had not yet taken a formal course in geometry to (a) the definitions and properties of the fundamental Euclidean objects (point, line, segment, ray, circle, arc); (b) concepts of curvature and angle; (c) the Euclidean properties of parallel and perpendicular lines; (d) the fundamental transformations of translation, rotation, dilation, and reflection; and (e) the process of fractal recursion.

In the Geometer's Sketchpad, these mathematical concepts are instantiated in dynamic imagery: mathematical ideas appear as relationships that determine the behavior of images on the computer screen. Thus, students could explore mathematical principles through the creation of compelling designs. The work of M. C. Escher (1971; 1982; Schattschneider, 1990), famous for its playful use of deep mathematical structure to achieve powerful artistic effect, inspired both the name and the activities of Escher's World. Since freehand curves are not part of the traditional cannon of Euclidean objects (and therefore not primitives in the Sketchpad environment), designs made in Sketchpad tend towards abstract compositions, typically with clean lines and idealized forms, reminiscent of work by artists such as Calder (1989; 1993), Kandinsky (1992), Klee (Geelhaar, 1973; Klee, 1961), Lewitt (1975; 1978; 1992a; 1992b), Miró (1972), and Mondrian (Blotkamp, 1994; Fauchereau, 1994). Thus, in addition to mathematical principles, students in Escher's World were also introduced to design principles of form, negative space, color, depth, and balance suitable to the analysis and production of such works (Arnheim, 1974).

Activities. In developing activities to let students explore these concepts, we tried to create a structure that would preserve the essential features of the design studio as articulated in the Oxford Studio. Following the Oxford Studio model, activities were organized into a cumulative sequence of design explorations (see Table 1 for a list of mathematics and design topics and activities). As in the Oxford Studio, each exploration began with a set of exemplars. Exemplars were followed by an activity using traditional materials, and discussion in a pin-up of this work in traditional materials highlighted the mathematical concepts students would be working with in a particular exploration. Students were then shown how to use the software to instantiate the mathematical relationship that had come up in their work with traditional materials, and were given a design challenge. A pin-up of work on the first challenge focused on emergent design properties, and a revised challenge asked students to work simultaneously with mathematical and aesthetic criteria. Each assignment ended with a public discussion of work in a final pin-up. The schedule for a typical day of design work is shown in Table 2.

In the 3rd day of the 3rd week, for example, activities focused on the mathematical concept of dilation and the design principle of balance. The day began with students' cutting out a shape of their own choosing from construction paper and figuring out how to make the same shape at one-half size. Designs were shown in a pin-up, and the various methods students used were discussed. Students were shown how to construct dilations using Sketchpad. The group

		Warm-up	Activity 1	Activity 2	Math concepts	Design concepts
Week 1	Monday	Use compass make circle and square	Draw square with circles	Draw square with as few curves as possible	Circles, arcs, curvature	Marks, shapes, perceptual elements
	Tuesday	Use straight lines to draw curve	Draw ball using lines	Clarify salient feature of ball	Lines	Interpretation
	Wednesday	Field trip				
	Thursday	Make composition using squares	Make a connected series of images	Refine images to highlight design progression	Parallel and perpendicular lines	Iterative process of design
	Friday	Charette: refine one design from the week for pin-up and discussion				
Week 2	Monday	Trace repeated motif on acetate squares	Create motif and translate by 2 vectors	Use translated motif to create negative space	Translation	Negative space
	Tuesday	Make motif that looks the same in a mirror	Create a design using 1 motif and 2 mirror lines	Use color to reveal structure of mirror design	Mirror symmerty	Effects of color on design
	Wednesday	Field trip tp museum				
	Thursday	Create 2- and 4-fold rotation using acetate	Create a design with *n* -fold rotation	Create static design w/multiple rotocenters	Rotational symmetry, angles	Movement and dramatic tension
	Friday	Prepare design for review				
Week 3	Monday	Design review with external reviewers				
	Tuesday	Watch film				
	Wednesday	Duplicate motif at 1/2, 1/4 size	Duplicate motif at several scales	Static, balanced image from previous design	Dilation	Balance
	Thursday	Arrange duplicate motifs into composition	Scaled motifs with other transformation	Show same relation in motif and whole	Composition of functions, recursion	Structure
	Friday	Begin work on final project				
Week 4	Monday	Plan installation with curator				
	Tuesday	Continue work on final project				
	Wednesday	Complete final project				
	Thursday	Install gallery				
	Friday	Final exhibition				

Table 1
Schedule of Topics and Activities for the Escher's World Summer Program

9am	Students arrive, check-in discussion
	Warm-up activity
	Group discussion of warm-up, and intro to GSP functions
10am	Work on activity 1
	Break with snack
11am	Pin-up and discussion of activity 1
	Work on activity 2
12noon	Pin-up and discussion of activity 2
	Check-in discussion, students depart

Table 2
Schedule of a Typical Day of Design Work in the Escher's World Summer Program

discussed "master works" relevant to the design challenge, including three Calder (1993) mobiles and an installation by Lewitt (1992b). Discussion of master works was followed by presentation of the first design challenge: students were asked to make an interesting composition using multiple dilations of one or more shapes and a single vanishing point (see Figure 3).

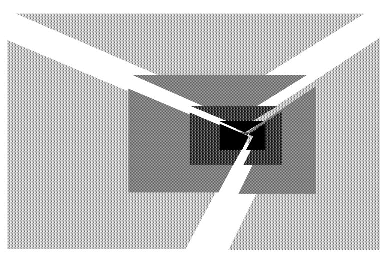

Figure 3. Student work in Escher's World: A design using multiple dilations.

The initial design phase lasted approximately 50 minutes. Students consulted periodically with program leaders or peers for technical help with the software or for desk crits about their emerging designs. After the initial design explorations, each student presented his or her work-in-progress in a pin-up. During the pin-up, students looked at additional exemplars, including works by Klee (Geelhaar, 1973) and Escher (1982). The design challenge was revised to include additional criteria—in this case, the idea of balanced composition. Students were asked to use the same set of dilated objects to make one balanced and one unbalanced composition. Students worked on the revised challenge for just over an hour, with desk crits continuing throughout. The day ended with a pin-up of students' final products.

As in the Oxford model, explorations in Escher's World built upon one another, leading to increasingly public presentations of students' designs (see Table 1). Activities in the 1st week led to a *charette*—a rapid exploration of a complete design project—in which students took a day to produce a design using the concepts and techniques they had encountered up to that point. At the end of the 2nd week, students prepared a design for review by professors of architecture and graduate design students who came to the studio for a day; for many students, these designs became the basis for their final museum piece. By the end of the 3rd week, work was focused on developing a final design for exhibition in the museum.

Because students in a design studio typically work on individual projects in their own personal workspace, students were given their own desks with individual computers—the same desks, in fact, that were used by students in the Oxford Studio. We also used the meeting, pin-up, and review spaces from the Oxford Studio, and preserved the high expert/learner ratio of the Oxford Studio, with three program leaders (two graduate students and one undergraduate) for 12 students.

Constraints. Although we were most fortunate to be able to use the space of an authentic design studio, we were obviously forced to make compromises in designing this activity structure to accommodate the practices of the Oxford Studio to the needs of middle school students in a summer program. One immediate issue was that the temporal structure of the studio—usually used with undergraduate and graduate students in architecture—needed some adaptation for middle school students. Students in the Oxford Studio worked at various hours of the day (and night), particularly in the days leading up to a final review. This was not practical with middle school students. Students came to Escher's World as a summer program from 9:00 a.m. to 1:00 p.m. every weekday for 4 weeks. Six days were spent on field trips, on installing the final museum exhibit, or on other non-design activities. As shown in Table 1, a total of 14 days was modeled as described above on the practices of the Oxford Studio, though with the addition of morning and afternoon check-in times and a snack break.

A second issue of timing was that although 56 hours of design activity is quite a substantial amount of time by the standards of a traditional middle school class, it is brief compared to the 300–400 hours of design work that take place over the course of a semester in a typical design studio course. Design explorations in Escher's World were thus compressed relative to an architectural design studio, and the number of iterative cycles between initial conception and final project was necessarily abbreviated. This posed challenges in the design of activities, but it also worked well for middle school students who had no previous design experience, and thus needed help organizing their work in extended blocks of time.

Participant structures. In light of these adaptations, we took particular care to preserve as faithfully as possible two key participant structures of the Oxford Studio: desk crits and public presentations of work in pin-ups and design reviews. All of the project leaders had either participated in design studio courses as students or teaching assistants or participated in the ethnographic study of the Oxford Studio. The project staff practiced giving desk crits before the beginning of the program, observing and critiquing performances with the aim of faithfully rendering the roles and interactional structure of desk crits as enacted in the design studio tradition. Similarly, we developed a standard format for pin-ups, modeled on the practices of the Oxford Studio, in which students presented their work and others were invited to comment using the same general interactional forms of the desk crit. External reviews were conducted by architects, professors, and advanced graduate students, as was the case in the Oxford Studio.

Participants. We recruited students by mailing flyers to local youth organizations and school district offices. advertising a summer workshop in computers, mathematics, and design leading up to an exhibit at the MIT Museum. Three hundred flyers were distributed, and responses were screened for age (12–14 years) and prior mathematics coursework (we admitted only students who had not yet taken a formal course in Euclidean geometry as typically taught to eighth- or ninth-grade mathematics students). The first eight applications received from males and seven applications received from females that met the screening criteria were accepted. (There were considerably more male than female applicants.) In the end, 12 students (7 males and 5 females) attended the entire program. Three students dropped out in the first few days, one due to health reasons and two due to family travel plans. (They had apparently never intended to complete the program.) Although the applications were not purposefully selected for diversity, the program wound up with a mix of students from urban (6/12) and suburban (6/12) areas in and around Boston. There were two African American students, two Asian students, two Latino students, and six White students. The majority of the students (11/12) had recently completed Grade 6, 7, or 8 in a local public school. One student was home-schooled. The program was designed and run by the author, with assistance from a graduate student in the MIT Department of Architecture and an MIT undergraduate student majoring in design. Two of these program leaders were male, and the other female. All three workshop leaders were present throughout the program.

Data Collection

Data for Escher's World was collected in two ways. The main source of data was field notes documenting student work and student thinking during the program, compiled into design histories: illustrated records of students' design activity over the course of the program. Students also completed pre-workshop, post-workshop, and follow-up interviews to document mathematics learning, providing an outcome measure for work in the program.

Design histories. Understanding how students developed conceptual insights about mathematics through design activity required a fairly comprehensive record of students' work and thinking through extended design activity. It was not practical to use think-aloud protocols for 12 students through 56 hours of work; nor did we feel it would be reliable to interview students only at the end of the program—or even at the end of each 4-hour day—about what they did and what they were thinking. Instead, the principal source of data for this study was extensive field notes based on *in situ* clinical interviews (Denzin & Lincoln, 1998; Ginsburg, 1997; Gruber & Voneche, 1995) conducted during desk crits.

As described above, desk crits begin with a description of the history of a design: the student describes his or her design goal and the design moves that he or she has taken to reach that goal. Next, the student and the critic talk about problems in the current state of the design and discuss possible solutions. In an ideal desk crit, the critic also has a chance to ask about the design understanding that the student developed and incorporated into the design-in-progress. In practice, desk crits take place at critical points in students' design work, and desk crits in Escher's World thus provided an opportunity to gather information about and insight into students' design processes and design thinking in an ongoing way. As part of the training in the conduct of desk crits described in the Ethnography of a Design Studio section above, program leaders practiced using desk crits as clinical interviews in this sense. During the workshop, program leaders wore tie-clip microphones to record their desk crit conversations. At the end of each day, each of the program leaders was responsible for using these recordings to write up a detailed account of all of his or her desk crit conversations—usually a summary of the comments of critic and student in each of the exchanges in the discussion, with verbatim excerpts for key points. (Unless specifically described otherwise, all excerpts quoted in the results sections that follow were from these verbatim quotations of student comments.) These summaries were supplemented by images from the student's designs, illustrating the critical moments in the student's work.

At the conclusion of the program, these field notes were combined into a single design history for each student: that is, the three different (and often overlapping) accounts of a student's design work from the field notes of the program leaders were combined into a single illustrated description of the work of each student in the program. These design histories are not comprehensive in the sense that they do not represent a complete recording of students' design work. However, the design histories do give a detailed account of students' work over an extended period of time. As such, they provide the foundations for thick descriptions (Geertz, 1973) of students' design work that shed light on how students learned mathematical ideas through design activity in Escher's World.

Interviews. Each student had a pre-interview immediately before the program, a post-interview immediately after the program, and a follow-up interview 3 months after the conclusion of the program. The protocol for these interviews was based on previous work documenting change in students' mathematical thinking as a result of design activities similar to those used in the Escher's World summer program (Shaffer, 1997b, 1997c). As part of these interviews, students took a test of 18 questions about transformational geometry appropriate to the topics covered in the Escher's World program and compiled from geometry textbooks (Aichele et al., 1998; Manfre, Moser, Lobato, & Morrow, 1994; Moise & Downs, 1971; Rubenstein et al., 1995a; Rubenstein et al., 1995b; Serra, 1997). Topics for the test included identification of parallel and perpendicular lines, mirror lines, rotational symmetry and rotocenters, and questions about translation and dilation, composition of functions, congruence and similarity, proportion, and angle measure. (See Figure 4 for a sample question, and Appendix A for a copy of one form of the test.). Three control questions on fractions and algebra (topics not addressed during the program) were also included. Three forms of the test were produced with isomorphic problems. Forms were randomized for pre-, post-, and follow-up-interviews.

Data Analysis, Part 1

Design histories. As described in the Background section above, design is an iterative process that takes place through a series of design episodes characterized by a design goal (recognition of some initial problem, sub-problem, condition, or issue that needs to be addressed), design moves (steps taken to understand and or resolve the initial issue), and

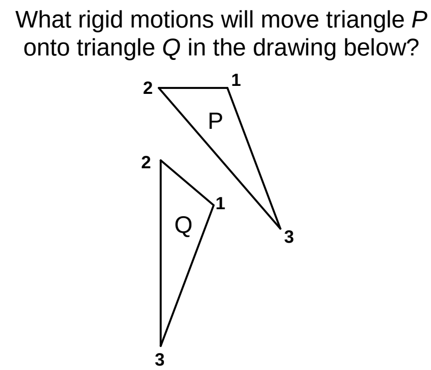

Figure 4. Sample geometry question used in student interviews.

satisficing (development of some insight into the nature of the initial problem that suggests a conclusion, an end to the exploration, and/or a more interesting issue to pursue).

Design histories from students in Escher's World were thus segmented into design episodes, and those episodes were analyzed within a grounded theory framework (Glaser & Strauss, 1967; Strauss & Corbin, 1998). Using the constant comparative method, design episodes were examined for emergent themes and patterns relevant to the questions framing the study. In particular, analysis focused on the relationships among students' interest in expressing particular ideas, the autoexpressive properties of the tool, interactions between students and program leaders in desk crits, and the development of conceptual insights about mathematics.

Test scores. Results of paper-and-pencil math tests in pre-, post-, and follow-up interviews were compared using paired *t*-tests.

Quantitative analysis. As described in the introduction, design histories were also coded and analyzed using intra-sample statistical analysis (Shaffer & Serlin, 2004). The coding scheme for and results from that analysis are described in the second part of the Methods and Results sections below.

Results, Part I

The results from this study of Escher's World are described in two parts. In this first part of the Results section, I briefly present test scores from pre-, post-, and follow-up interviews to show that students did learn mathematics in Escher's World. I then present an extended case study of the work of one student in Escher's World, Hallie,[3] which I use to describe a grounded theory about the processes through which participant frameworks from the design studio, the autoexpressive properties of the Geometer's Sketchpad, and students' interests interacted in Escher's World to support the development of conceptual insights about mathematics. Following this extended case study, in the second part of the Methods and Results sections, I describe a coding scheme based on this grounded theory and use statistical analysis to further support the theory developed in the case studies.

[3] All names used in this paper are pseudonyms.

Test Scores

Previous work has shown that students learned concepts in transformational geometry when they used the Geometer's Sketchpad to engage in design activities similar to those in Escher's World (Shaffer, 1997b, 1997c, 2002b). That students would learn some mathematics during 4 weeks of work in Escher's World is thus hardly a surprising result—although it is one worth confirming, since understanding *how* students developed mathematical understanding is reasonably predicated on knowing *that* they did so.

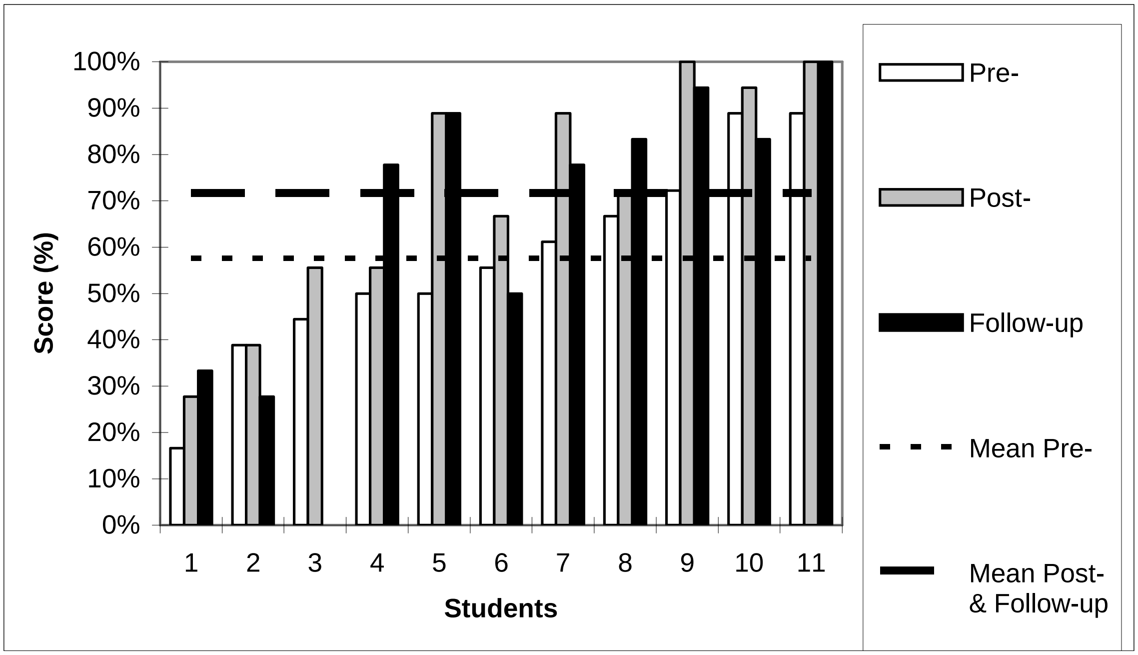

Figure 5. Pre-, post-. and follow-up test scores.

Scores on a paper-and-pencil test of transformational geometry knowledge rose significantly between pre- and post-interviews (mean pre = 58%, mean post = 72%; p < .01). Gains were stable in final interviews 3 months later (mean final = 72%).

As shown in Figure 5, students' scores on paper-and-pencil tests of transformational geometry knowledge rose significantly between pre- and post-interviews (mean pre = 58%, mean post = 72%; $p < .01$). These gains were stable in final interviews 3 months later (mean final = 72%). Scores on the three control problems about algebra were not different among pre-, post-, and follow-up interviews ($p > .50$ for all three comparisons). Students reported developing some kind of mathematical understanding in 17% of their designs (42/242), and all students described having conceptual insights about mathematics at some point in the program. Student designs clearly became more sophisticated in their use of design principles and mathematical relationships through the Escher's World program (see Figure 6).

Thus, it seems that, as we might have expected, students did learn about transformational geometry through their work in Escher's World. In the next part of the paper, I present a case study of one student's work to develop a grounded theory as to how that mathematical understanding developed.

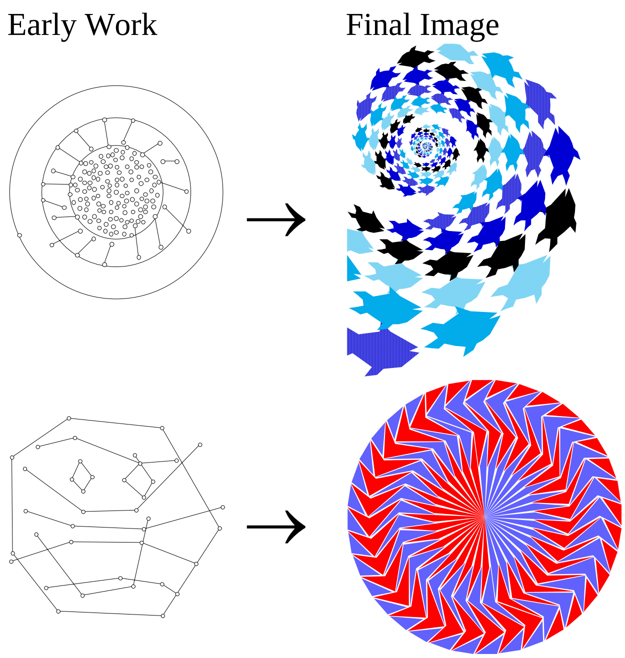

Figure 6. Two students' early work and their final drawings.

Hallie's Story

Hallie was a soft-spoken young woman, slight, shy, and somewhat awkward among her peers in the program. She appeared to be in transition: not quite comfortable with the students who talked about Legos and going to the movies with their parents, not quite comfortable with the students who were interested makeup and dating. What follows is a brief version of how Hallie—who spoke so softly that it was difficult at times to hear her—found a voice, both literally and mathematically, through the design activities of Escher's World.

In the process of describing Hallie's work in Escher's world, I hope to show that Hallie's interactions with program leaders in desk crits, her interactions with the autoexpressive features of Sketchpad, and the development and expression of her interests in design and mathematics were intimately intertwined with the process of transforming design activity in the computational microworld into conceptual insights about mathematics. That is, I hope to use Hallie's story to show how the enactment of design processes in service of expressing ideas that Hallie found interesting supported her development of mathematical understanding in a computational microworld.

Working without explicit goals: "I'm just fiddling around." The first day of the Escher's World program began with a design challenge to make a straight-edged figure (such as a square) out of curved lines (circles or arcs). Students looked at sample solutions (see Figure 7), and were also shown how to construct a circle using the software tool.

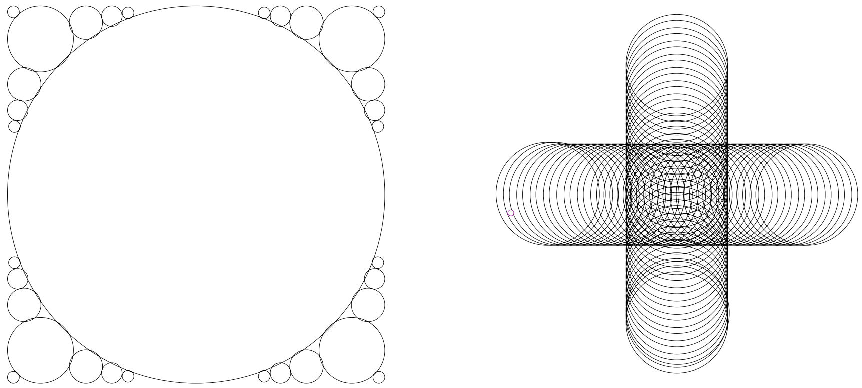

Figure 7. Two examples of drawing a square using circles that students were shown on the first day of Escher's World.

Hallie's response to the challenge was first to draw the outline of a square "by hand" and "by eye" (see Figure 8). [4] After talking with program leaders in desk crits and seeing the work of her peers in pin-ups, Hallie began to use straight lines as guides for her drawing (the rightmost image in Figure 8). Hallie's work on these first design challenges shows progressively more sophisticated use of the idea of curvature and properties of arcs. In her second image, for example, she uses a single circle with large radius to form one straight side of a square; in the final image she uses arcs to wrap around the bottom vertices of the triangle. But at this early stage, Hallie showed little evidence of thinking explicitly about these issues. About the second image she said only, "I just wanted to use little circles." When asked why she chose a triangle for the final design, she replied: "It was a simple shape. I'm just fiddling around."

Constructing an image to express an idea: "I want the bottom lighter." The next day, students were asked to make an image of a ball. The challenge was twofold. First, the images of this round object were to be made using only straight lines. Second, and perhaps more important, the designs were supposed to convey an interpretation of some aspect of the ball that was of interest to the designer: for example, its weight, the texture of its surface, or the way it bounces.

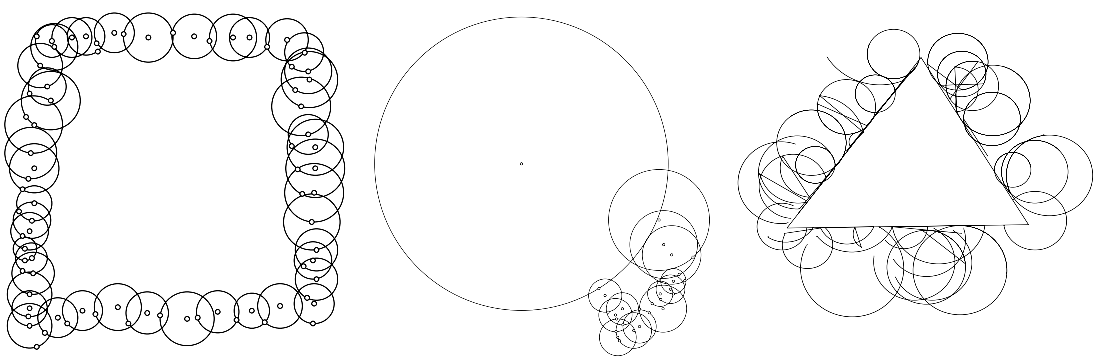

Figure 8. Hallie's responses to the challenge of drawing straight edges with curves.

[4] As described above in the Methods section, unless specifically indicated otherwise, all quotations are taken from student comments recorded verbatim in field notes. Thus, the description of Hallie's drawings as having been made "by hand" and "by eye" were her own.

Looking at Hallie's response to this second challenge, we can already begin to see a change in her work. After making one drawing of a ball in compression as it was "bouncing off of a ceiling" (see Figure 9), Hallie found herself struggling to get the artistic effect she wanted. She called over a program leader and in the course of the desk crit explained: "I want the bottom lighter . . . not lighter, more open, not really there. . . ." As Hallie and the critic explored the drawing, they talked about her design goal and also about the underlying construction of the image.

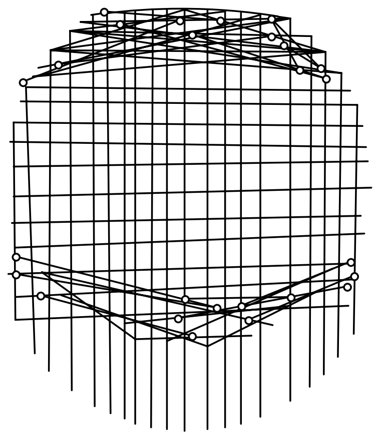

Figure 9. Hallie's original image of a ball bouncing off a ceiling.

What they discovered was that Hallie had used a circle as a guide for the line segments that made up the ball. Hallie had used the software to hide the circle, but its position was still determining the location of the line segments in the image (see Figure 10)—that is, each of the line segments that made up the ball had been constructed with a point on the circle as one of its endpoints. Hallie realized that by dragging the center point of the circle, she could change the radius of the circle and spread out the segments on the lower edge of the ball; having been constructed with an endpoint on the circle, the segments moved when the shape of the circle

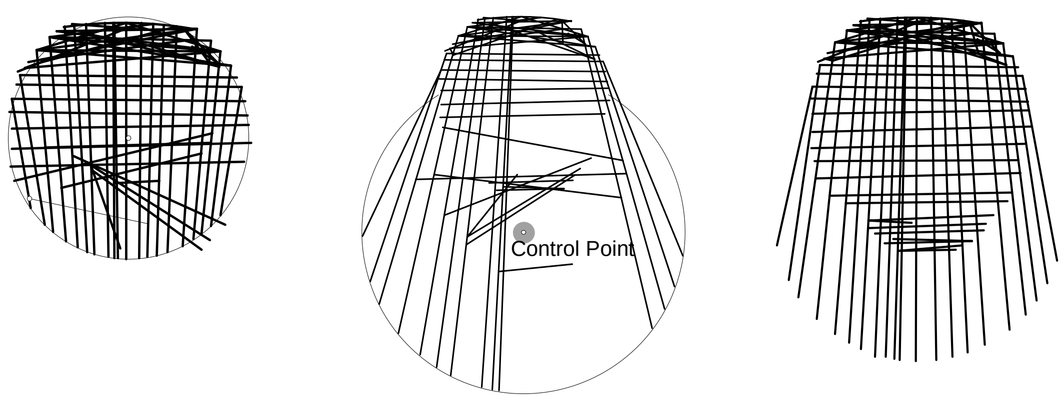

Figure 10. Hallie's exploration of a sketch

Hallie and a critic explored the underlying construction of a problematic image (left), enabling Hallie to realize her design goal in a final design (right). The middle sketch is a reconstruction—the labeling of the center point was added, and several line segments were removed for clarity.

changed. Armed with this discovery, Hallie returned to the design with both a clear goal and a means to manipulate the image. Hallie spent the remainder of her working time adjusting and reconstructing the design so that, as she said of her final image, "when it goes out, it's sort of . . . motion," representing the movement of the bouncing ball.

Developing explicit design goals: "Exploding negative space." The next week, Hallie was working on a design challenge involving rotation and created an image with vibrating white space at the center of the design that she and a peer called "explosive" (see Figure 11). Hallie was excited by the effect but was not sure what to do with the idea. In a desk crit, one of the program leaders suggested that she try to figure out how to make the effect more dramatic. Following the desk crit, Hallie began a series of carefully conducted explorations into the workings of "exploding negative spaces." She made a set of designs—some 20 in all—empirically determining the factors that make a rotated shape look explosive: a "nice pointy shape," "enough" points, a dark color (see Figure 12). The idea of exploding negative space was a topic that Hallie returned to consistently in the following days, and ultimately became the subject of her final project.

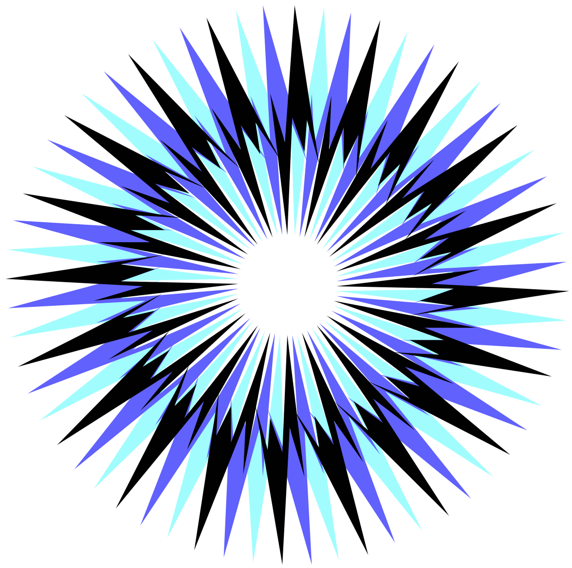

Figure 11. Hallie discovered that it was possible to create an "exploding negative space" (the vibrating white space) at the center of a rotated image.

Overcoming an obstacle: "You need to rotate it more times to keep it pointy." For her final project, Hallie decided to make "rings of explosions" using the same shape in each ring—as if a real object were exploding, she said, sending off shards in all directions. However, as she began to work on the final image, she ran into trouble. Hallie was able to make an explosion in the inner ring or the outer ring, but not both. When the points of the shapes in the inner ring made an exploding negative space, as in Figure 13, the points on the outer shape were too far apart to create the visual effect she wanted. When the points on the outer ring gave the right

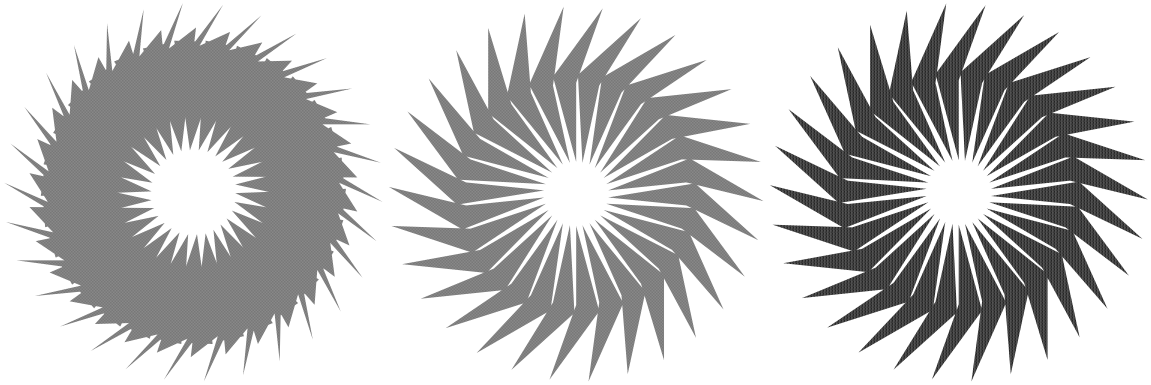

Figure 12. Hallie explored the factors that make an exploding negative space from a rotated shape.

effect, the points on the inner ring blurred into a continuous curve with no vibrating effect. As Hallie had done earlier in designing the bouncing ball, she called over a program leader when she had trouble getting her design to behave as she wanted it to in Sketchpad.

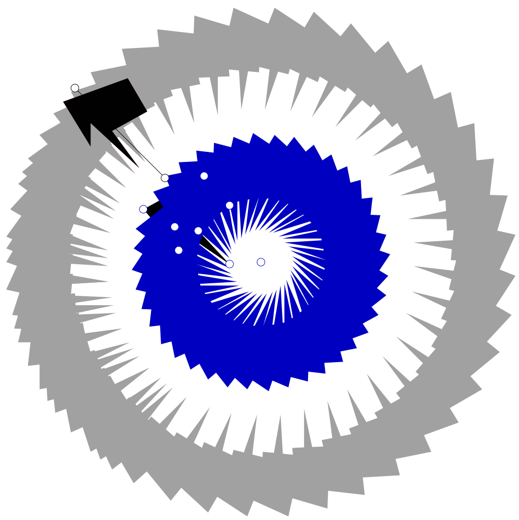

Figure 13. Hallie and a critic used this image to explore the nature of rotation.
Hallie was frustrated that she could not make an exploding negative space in both rings simultaneously. During a desk crit, she realized that she needed to rotate the underlying shape by a smaller angle in a large circle to keep the distance between objects the same (and thus create the vibrating effect) in both rings

Hallie's first explanation for the problem was that there was something wrong with the shape she had chosen—after all, she had conducted an extensive series of explorations, and knew that the shape of the polygon used in the rotation was one of the important factors in creating exploding negative space. The program leader asked Hallie how she had made the two rings. Hallie explained that she had taken a single shape and rotated it by 9 degrees repeatedly until it completed a circle. Then she had translated the original shape to where she wanted the second ring to be and rotated the image of the translation by the same amount around the original rotocenter to make the second ring. The program leader suggested that they look at whether the shape of the polygon was the problem by changing the color of one of the polygons on each ring of the exploding design so they could watch carefully as they changed its shape dynamically (see Figure 13). As they moved the points that determined the shape of the polygon Hallie had translated and rotated to create her design, she saw that the problem was not with the shape. Seeing the two polygons—one in the inner ring and one in the outer ring—move relative to their rotated images (the polygons next to them in the rings), Hallie realized that the problem was in rotating the polygon by the same amount in both the inner and outer rings. “Oh,” said Hallie, “you have to rotate the outside one more times—less degrees—[because] they’re farther apart. . . . It’s a bigger circle, [and] with a bigger circle you need to rotate it more times to keep it pointy.” With this mathematical insight about the relationships among the size (radius) of a circle, the angle of rotation (subtended angle), and the distance between object and image (an arc or chord length), Hallie was able to complete her project. The final image (see Figure 14) is of a negative space explosion sending shapes flying off in all directions.

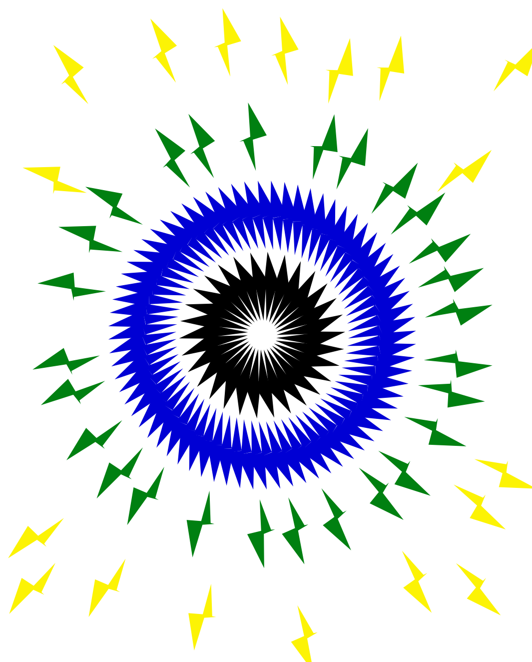

Figure 14: Hallie’s final image for the museum exhibit was a series of concentric explosions in the negative space of the image.

Learning Mathematics Through Design in a Microworld: A Grounded Hypothesis

Hallie’s story thus illustrates the importance of desk crits, the autoexpressive properties of Sketchpad, and the transformation and expression of Hallie’s own interests—and the linkages among these aspects of her design work—in her development of conceptual insights about mathematics. Questions about and co-exploration of emerging designs in desk crits helped clarify mathematical issues implicit in her work. The Geometer’s Sketchpad encouraged such conversations by providing a medium that created a dynamic representation of mathematical constraints and relations and gave autoexpressive feedback on the impact of decisions and actions within that structure. Hallie learned about rotation by trying to instantiate a particular design idea using the affordances of the Sketchpad environment—and by participating in a desk crit with a program leader when she couldn't get her design to work. The motive for pursuing such conversations came from the expressive nature of the endeavor; indeed, one way to read Hallie's story is as the progressive development of her interest and ability to master "exploding negative space." The development of that interest was, in turn, supported by interactions with program leaders in the participant framework of the desk crit. Hallie's design history, in other words, shows that (a) interactions within the participant frameworks of design helped Hallie turn design activity into mathematical understanding; (b) opportunities for such interactions arose when the tool did something unexpected and Hallie needed to figure out what went wrong; and (c) this ability to be surprised and desire to fix what went wrong came about when Hallie had developed a commitment to expressing a particular idea. Mathematical understanding in Escher's World thus appears to have arisen from complex interactions among elements of design practice during explorations in a mathematical microworld—a grounded theory I support in the next sections using statistical techniques.

Methods, Part II

The strength—and ultimately the validity—of this claim about how interactions among participant frameworks, autoexpressivity, and expressive goals supported conceptual insights about mathematics clearly depend on the richness of the qualitative data collected on the work of the 12 students in the Escher's World summer program. However, it is also possible to code the design histories and use intra-sample statistical analysis (Shaffer & Serlin, 2004) to support this grounded theory. Intra-sample statistical analysis is a technique for applying quantitative methods to qualitative data by using observations rather than individuals as the random effects unit in statistical analyses. Participants in the experiment are treated statistically as fixed effects, and the resulting conclusions generalize *within* the observations, rather than to a larger hypothetical set of "students like those in the experiment," as would be the case in a more traditional quantitative approach. That is, because students are treated as fixed effects in the model, the results of such an analysis apply only for the particular group of learners in the study. However, using observations as units of analysis makes it possible to identify statistically significant patterns of activity within sets of observations such as design histories. Intra-sample statistical analysis is thus useful for testing the hypothesis that a grounded theory describes a pattern that holds true more generally in the observed data, and thus for inferring that further observations would have shown the same pattern identified in the existing data. In this way, intra-sample statistical analysis can provide a warrant for theoretical saturation of a grounded theory.

Coding

As part of the qualitative analysis, design histories from Escher's World were divided into design episodes, with each episode characterized by a design goal, design moves, and satisficing. The grounded theory that emerged from qualitative analysis of the design histories was that students developed conceptual insights about mathematics during design explorations through a combination of explicit design goals, the autoexpressive affordances of Sketchpad during design activity, and interaction with program leaders during satisficing. Codes were developed to reflect each of these categories as follows:

- *Conceptual insights about mathematics* (CI). Episodes were coded for CI when in the satisficing phase of the design episode a student reported a conceptual insight about mathematics as a result of his or her design activity—specifically, when a student made a statement that was (a) about one of the mathematical concepts covered in the curriculum of Escher's World (see Table 1 above); (b) about a general property of that mathematical concept not specific to the instantiation of the concept in Sketchpad; and (c) a claim that the student had recognized this general property in the context of his or her current design exploration. So, for example, the following statements were coded as CI: "the mirror line creates symmetry, and symmetry creates equal parts on both sides, which makes it balanced"; "when you rotate something it's actually in a circle—you can rotate it a lot of [ways] and it will always make a circle"; "with a bigger circle you need to rotate it more times to keep it pointy"; "when you're closer to the center, if you rotate by the same amount, the same size shape is more cramped because there is not as much room closer to the center—it's just like if you go in space and you trace the area of your property out and trace it onto the atmosphere it gets a lot bigger because it's farther from the center point and everything gets more spread out and stretched out"; "if you need something repeated and repeated and repeated you can almost always . . . [make a] transformation for it."

- *Autoexpressive challenges* (AE). Episodes were coded for AE when during design moves a student expressed surprise or concern that designs did not behave as expected because he or she had drawn rather than constructed the design or part of the design. For example, the design episode in which Hallie was unable to adjust the lines on the bottom of the bouncing ball and the episode in which she realized she was having a problem keeping her "pointy shapes" close together in the inner and outer rings of her final design were both coded for AE.

- *Desk crits* (DC). Episodes were coded for DC when a student participated in a desk crit with a program leader during satisficing. A key element in determining which interactions with program leaders qualified as desk crits for the purposes of coding was evidence that as part of the interaction the critic asked probing questions about the student's design goals and design moves, characteristic of the desk crit participant framework.

- *Explicit design goals* (DG). Episodes were coded for DG when a student explicitly described the intent of his or her design activity. For example, statements such as "I wanted something to go in and out," "I wanted the [viewer] to focus on one of [the circles], so instead of just seeing a square, they'll see all the things that make it up," and "I wanted to express the negative space of . . . rotation" were all coded for DG.

Two additional codes were developed to account for possible alternative hypotheses. One code accounted for the possibility that it was not the autoexpressive properties of using Sketchpad that helped students develop conceptual insights about mathematics, but rather the process of encountering problems with the tool and seeking help more generally. A second code accounted for the possibility that it was not the participant framework of desk crits that helped students, but any form of help from program leaders. Specifically:

- *Learning the tool* (LT). Episodes were coded for LT if during their design moves students needed help using features of the tool when those problems were not related to the autoexpressive properties of Sketchpad—for example, if a student needed help figuring out the sequence of selections and menu commands to rotate a polygon around a point.

- *Other interactions* (OI). Episodes were coded for OI when they contained interactions with program leaders during the satisficing phase of students' design work that did not take on the framework of a desk crit—for example, if a program leader praised a design without engaging in questions about the design goal and design moves.

Each episode was coded by recording the presence (or absence) of activity representing each category above. Resource constraints made it impossible to have multiple coders examine the data, but during coding, design episodes were randomized, and identifying information suppressed. Each design episode was checked twice in the process of coding. Three additional descriptive codes were added to each episode. One indicated which student had participated in the design episode. A second recorded on which of the 13 days of design work the episode took place.[5] The third recorded which of the 242 student designs begun during Escher's World the particular design episode was a part of.

Intra-Sample Logistic Regression

Quantitative analysis focused on the grounded theory that explicit design goals (DG), the autoexpressive properties of Sketchpad (AE), and desk crits during satisficing (DC) interacted to support the development of conceptual insights about mathematics (CI) in Escher's World. Specifically, the hypothesis tested was that DG, AE, and DC—and their interactions—would be significant predictors of CI during students' work on a design.

For each design, episodes were summed to create a score for each code for the exploration as a whole. Because the outcome variable CI was discrete and had a very narrow distribution (most of the design explorations had 0, 1, or 2 episodes with insights), it was recoded into a categorical variable indicating the presence or absence of conceptual insights about mathematics during the exploration, and a logistic model was used for the analysis.

Coded design explorations were, of course, repeated observations of design activity by a small number of students over time. As described above, intra-sample statistical analysis uses a fixed-effects regression model to account for the systematic effects of students' unique skills, interests, abilities, and inclinations over multiple observations. Accordingly, individual designs were used as the units of analysis, with dummy variables for each student[6] in the regression model to control statistically for the intercorrelation due to repeated sampling.

Results, Part II

Over the 13 days of design work in Escher's World, students worked collectively on a total of 242 designs. Students worked on an average of 23.6 designs each (see Table 3), and designs had an average of 2.9 design episodes (min = 1, max = 13, S.D. = 2.2). We can thus infer

[5] The day of the design review, although part of the 14 days of activity explicitly modeled on the Oxford Studio, was removed from this analysis because students were not creating designs.
[6] Eleven dummy variables were actually used. The 12th would have been redundant in the model.

that the average length of time working on a design was approximately 50 minutes, and the average length of a design episode was 15–20 minutes, although some explorations were much shorter (as short as 1–2 minutes) and some were spread over considerably more time.[7] An average of 3.6 designs for each student led to conceptual insights about mathematics. As shown in Table 3, all of the actions described by potential predictors in the regression model were present in at least 33% of the design episodes, and all but two were present in more than 60% of the episodes, suggesting that no single type of action uniquely led to the conceptual insights, which occurred in 17% of all episodes.

	Designs per student			% of all
	Min	Max	Mean (S.D.)	designs
Total	15	31	23.7 *(4.46)*	100%
Concep. insights about math (CI)	1	9	3.6 *(2.43)*	17%
Explicit design goal (DG)	8	26	15.8 *(5.02)*	69%
Autoexpressive challenges (AE)	10	22	14.3 *(3.47)*	61%
Learning the tool (LT)	2	20	7.9 *(4.64)*	33%

Table 3
Descriptive Statistics for Codes of Student Designs
Maximum, minimum, mean, and standard deviations for the number of designs per student coded for outcome variable and predictors in the regression model in at least one episode; also, percentage of designs overall with at least one episode coded, by category.

A set of logistic regression models was used to test the hypothesis that explicit design goals (DG), autoexpressive properties of Sketchpad (AE), and desk crits during satisficing (DC) interacted to support the development of conceptual insights about mathematics (CI) in Escher's World. Model 0 in Table 4 shows that both the number of episodes in a design and the time in the program when the design occurred were statistically significant predictors of CI. Thus, in addition to dummy variables for each student, both the number of episodes in a design and the time in the program when the design occurred were included as controls in subsequent models.

Models 1 and 2 in Table 4 show that desk crits (DC) and explicit design goals (DG) were statistically significant predictors of conceptual insights about mathematics (CI). Models 3 and 4 show that the control variables for other interactions with program leaders (OI) and problems encountered while learning to use features of the tool (LT) were also statistically significant predictors, although encountering problems using non-autoexpressive features of the tool made it *less* likely that students would develop conceptual insights about mathematics.

Model 5 shows that, on their own, the autoexpressive properties of Sketchpad (AE) were not a statistically significant predictor of CI; however, Models 5a and 5b show that when either desk crits (DC) or explicit design goals (DG) and the relevant interaction are added to the model, AE becomes a statistically significant predictor.[8] This suggests that—as described in the

[7] Timing information was not recorded in the design histories. Given the method of the data collection, such numbers would have been only rough approximations.
[8] In Model 5b, the coefficient for AE is not statistically significant, but the statistical significance of the interaction term suggests that the predictor is significant as well when the interaction is included in the model.

qualitative analysis—the autoexpressive properties of Sketchpad were significant in conjunction with desk crits and/or explicit design intent.

Dependent variable = Concepual insights about mathematics (CI)

	Covariates									
Model	Number of episodes	Day in workshop	Desk crits (DC)	Explicit design goal (DG)	Other inter-action (OI)	Learning tool (LT)	Auto-exp. challenge (AE)	AE x DC	AE x DG	**Cox & Snell Pseudo R^2**
0	***1.2****	***1.4****								0.12
1	1.1	1.1	***1.5****							0.13
2	0.8	1.1		***1.7*****						0.15
3	1.1	***1.1****			***1.6****					0.13
4	***1.3****	***1.1****				***.6****				0.13
5	1.2	***1.2****					1.0			0.12
5a	0.8	1.1	***3.2*****				***1.9****	***.7*****		0.19
5b	1.0	1.1		***2.3*****			1.7		***.8****	0.16
6	0.8	1.1	***3.8*****	***1.5****	***2.1****	***.3*****	***2.3*****	***.6*****	**+**	0.26

***** *p* <.05 ** *p* <.01 + indicates term removed in stepwise regression

Table 4
Odds Ratios From Fixed-Effects Logistic Regression on Student Designs
Odds ratios show the increase (or decrease) in likelihood that a student would develop a conceptual insight about mathematics in a design based on predictors drawn from the qualitative analysis.

A final model was constructed using all of the predictors and including interactions of all predictors with AE. A stepwise logistic regression with backwards elimination for the predictors and interactions was used to remove redundant terms, and results are reported in Table 4 as Model 6.

The result of Model 6 is illustrated in Figure 15, which shows that autoexpressivity, desk crits, and explicit design goals are all statistically significant predictors of conceptual insights about mathematics. The significance of the interaction terms in the various models (and in the final model) suggests that, as shown in the qualitative analysis, it is not merely the *presence* of these elements of design activity, but the *combination* of these elements that contributed to the development of conceptual insights about mathematics during students' design work in Escher's World. The final model also suggests that participation in the participant framework of the desk crit was more likely to help students develop conceptual insights about mathematics than other forms of interaction with program leaders.

Intra-sample statistical analysis thus supports an inference that the grounded theory—developed through qualitative analysis and described in Hallie's case study—reflects a more general pattern in the design histories of students in Escher's World.

Discussion

My aim in this study has been to explore the mechanisms through which students developed conceptual insights about mathematics in Escher's World, an environment created

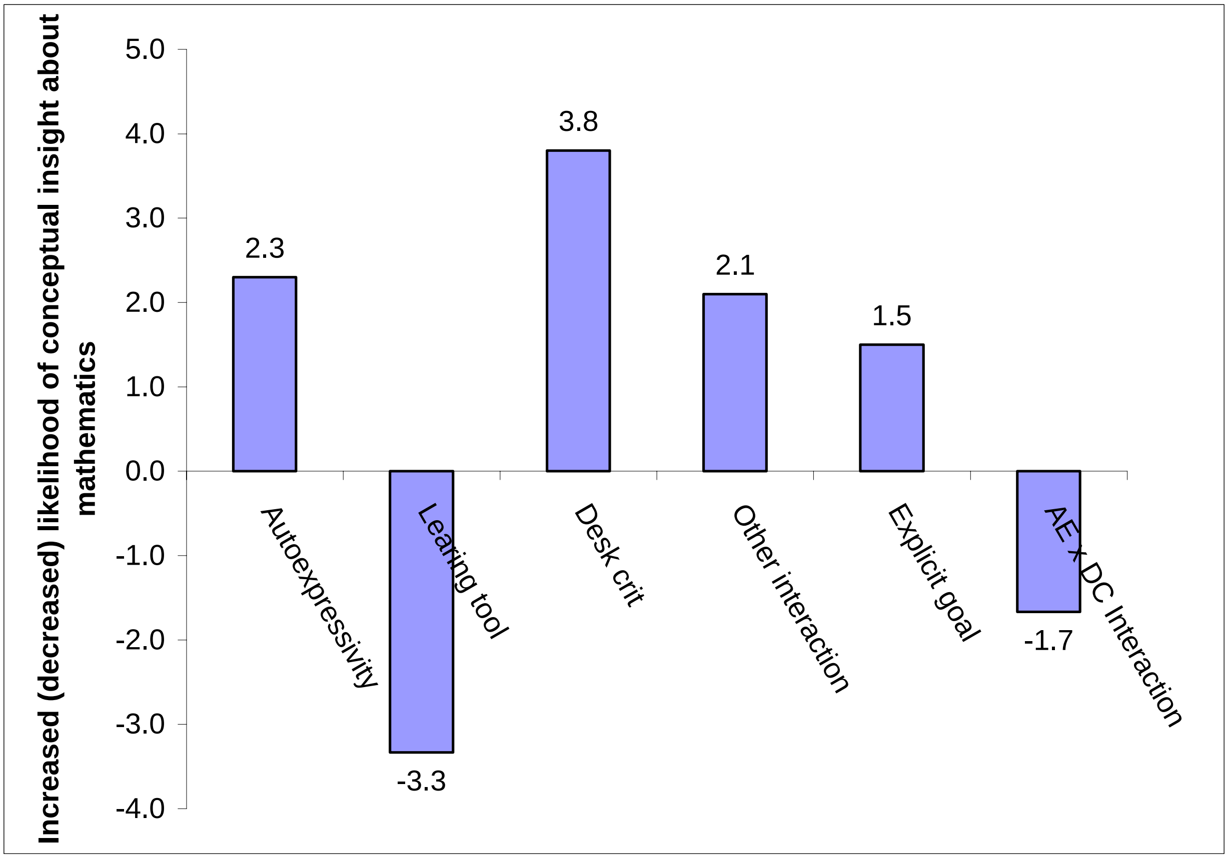

Figure 15. Graphical representation of results from the final logistic regression model of CI. Autoexpressivity, desk crits, and explicit design goals are all statistically significant predictors, and the graph shows the increased (or decreased) likelihood of CI for each predictor. For example, the model shows that students were 2.3 times more likely to develop CI if they encountered an autoexpressive challenge using Sketchpad in their design work compared to not encountering such a challenge; similarly, they were 3.3 times less likely to develop CI if they had problems learning to use features of the tool.

using the theory of pedagogical praxis (Shaffer, 2004a). In the discussion that follows, I briefly summarize the grounded theory developed from the design histories in Escher's World. I discuss two ways in which this theory extends previous work on learning through design activity in a computational microworld, and relate these findings to the theory of pedagogical praxis. After considering some of the limitations of Escher's World and of the analyses presented here, I conclude by suggesting some possible implications of this work for other efforts to incorporate design practices and computational microworlds into activities for younger students.

A Pattern of Activity in Escher's World

Both qualitative and quantitative analyses of students' design histories suggest that there was a pattern of activity that led to mathematical understanding in Escher's World: desk crits, autoexpressive features of the microworld, and students' own interests systematically contributed to the development of conceptual insights about mathematics. Hallie's design history exemplifies the pattern. We saw that significant learning took place when Hallie was trying to achieve particular design effects: apparent motion in an image of a ball bouncing off of the ceiling, or rings of exploding negative space made from rotations of isomorphic polygons. In each case, Hallie became frustrated when implementation of her design idea in the software did not behave as she expected. She called for help, and the desk crits that followed explored how mathematical relationships could be used to achieve her design goals. For example, when Hallie wanted to make her image of a ball bouncing off the ceiling look more open on the bottom, the resolution to the problem involved uncovering how her design had been constructed. When Hallie was working on her final image with progressive rings of exploding negative space, it was through uncovering and then making sense of the underlying construction of the drawing that Hallie came to understand something about the relationships among arc, radius, and angle.

Epistemology and Practice

That crits, autoexpressivity, and interest would play a role in mathematical learning in Escher's World is hardly surprising, as each of these elements of design activity in a computational microworld has been shown in prior work to influence learning. In the pedagogy of design, desk crits support reflective thinking, and their function as a zone of proximal development for young designers has been well established (Schon, 1985, 1987; Shaffer, 2003). Studies of microworlds have demonstrated that autoexpressive mathematical tools can support mathematical thinking through creative activity by encouraging students to think explicitly about mathematical issues in their work (Noss & Hoyles, 1996). Dewey (1915; 1938) argued over a century ago that the motivation to solve problems in the context of tasks such as design activities comes from the alignment of those tasks with students' own interests—an idea reiterated in the more specific context of activity in a computational microworld by Papert (1980).

Although the elements of the pattern of activity identified in Escher's World have thus already been described in the literature, the qualitative and quantitative results of this study make two important additions to research on learning through design activities in computational microworlds. First, Escher's World shows the affordances of design pedagogy and computational microworld intimately intertwined in the development of conceptual insights about mathematics, creating a complex system in which activity, motivation, and interaction led to the development of mathematical understanding in this particular context. In Escher's World, conceptual insights about mathematics emerged from the relationships among students, their program leaders, and a computational microworld—and from the ways in which those relationships came to systematically develop and incorporate desires for self-expression, feelings of frustration, and ritualized forms of critique, turning interactions between intent and obstacle in the context of expressive activities into opportunities for mathematical development. Hallie's story and the quantitative analysis suggest that all of these elements were important in the process of mathematical learning, and show that it was a complex interplay among—not merely the presence of—desk crit, autoexpressive tool, and interest that supported the development of conceptual insights about mathematics in Escher's World.

A second important point is that this integrated system of practice in Escher's World was orchestrated by and around a key epistemological principle of design—namely, that design ideas represent unique solutions to problems that express individual perspective through design activity. As in the Oxford Studio, desk crits in Escher's World supported reflective thinking by providing a structure of recurring questions about the relationship between design goals and design moves. In desk crits, critics repeatedly asked students: What are you trying to say (or show) here? How are you trying to show that? Do you think the result shows that? Why or why not? The impetus for students to engage with those questions in crits was the autoexpressive behavior of their dynamic designs in Sketchpad—an impetus that similarly arose in the context of explicit design goals. Because Sketchpad distinguishes in a dramatic way between designs that are drawn and those that are constructed, working effectively in Sketchpad meant asking: What mathematical relationships will produce the effect I want? That question, in turn, was motivated by one of the fundamental questions in the practice of design: What would I like this design to say? And, coming full circle, that question was an internalization of one of the central questions of the desk crit: What are you trying to say (or show) with this design? In Escher's World, the reciprocal interplay among participant framework, medium of expression, and activity structure was focused on the development of individual expression.

The result of this epistemologically focused interaction among participant frameworks, goals, and activities of Escher's World was a reflexive system of design activity—mediated by an autoexpressive tool and the participant framework of the desk crit—that iteratively clarified and made explicit students' expressive intentions and understanding of mathematical concepts relevant to those intentions: expressive intent led to drawing, which led to surprise and frustration, which led to discussions of intent and construction, which led to explicit mathematical insights and more explicit intent, which led to more sophisticated design work, and so on through the cumulative activity structure of the design studio.

In other words, understanding in Escher's World came from applying both the practices and the epistemology of design to student work in Sketchpad, and the results from Escher's World thus support a key claim of pedagogical praxis: that the effective use of microworlds to support learning through professional practices depends on preserving the linkages between practice and epistemology in professional pedagogies. In this sense, Escher's World was a thickly authentic adaptation of practice, and a demonstration of the potential power of such adaptations as a method for developing effective learning environments using computational microworlds.

Limitations

There are clearly significant limitations to this account of student learning in Escher's World. First and foremost, we were quite fortunate to have the opportunity to adapt design practices for middle school students in the same physical space used for graduate design studios; further adaptation (and further research on the effects of such adaptation) would be needed before this kind of program would be widely applicable. Second, both the qualitative and quantitative analyses presented here focus on a specific and limited operationalization of the concept of *mathematical understanding* as marked by conceptual insights about mathematics. That is, my choice to look at how students developed explicit conjectures about mathematical objects and relations limits the implications that might be drawn from this work. For example, it is possible (and even plausible) that tacit mathematical understandings, mathematical intuitions, and implicit (as opposed to declarative) understanding of mathematical procedures may depend less highly on explicitly reflective conversations such as those found in desk crits.

A third issue is that the use of design histories affords only a particular type and level of analysis for mathematics learning through design activities. As Lemke (2000) suggests, it is likely that other analytical techniques, looking at patterns of activity at larger or smaller scales, would add significantly to the account presented here. More generally, there were clearly processes at work in Escher's World that are not addressed in this study. The account here presents very little about the thinking or motivation of the program leaders in conducting desk crits, for example, or the role of public presentations of work in developing student interest. These (and other) issues played an important role in students' design activity, and in the development of their mathematical understanding, but are not well represented in the data collected for this study. Moreover, the particular method of data collection used—field notes based on recorded desk crits—may have overemphasized the importance of interactions with program leaders in the students' design work and mathematical thinking.

Finally, and perhaps most obviously, the qualitative grounding for both the case studies and the statistical techniques used means the results presented here do not generalize beyond these students in a formal sense.

Implications

Bearing these caveats in mind, the grounded theory developed from the work of these students may still address some significant issues in the design of learning environments. In Escher's World, design learning was a coherent system that linked participant frameworks from the practice of design and a mathematical microworld within an expressive activity structure. It mattered that students were working on projects that gave them significant autonomy to express their perspective on the design challenges at hand. Other studies have explored elements of design practice in the context of activities for K–12 students (Greeno, 1997; Hmelo et al., 2000; Kafai, 1996; Kolodner et al., 1998; Resnick & Ocko, 1991; Stevens, 2000). The work of the students in Escher's World suggests that adopting elements of design practice in isolation, and/or separated from the epistemological framework that organizes activity in the design studio, may be less than ideal. Desk crits may not work well for solving traditional mathematics problems—or even design-like challenges—that provide only one right answer, or a limited range of acceptable answers. Autoexpressive tools may not scaffold reflective thinking effectively without coherent participant frameworks such as the desk crit, aligned with the underlying epistemology of design, to help students turn obstacles encountered during design activity into opportunities for learning.

At the very least, this work shows that existing design pedagogy provides a useful image of what a design-centered computational learning environment might look like in other contexts, and that mesogenetic analysis of design histories based on field notes provides a window into the learning practices and processes at work in such environments. It shows the importance of—and the richness of—interactions among participant framework, autoexpressive tool, and expressive activities in the development of mathematical understanding through design activity. As the theory of pedagogical praxis suggests, this work shows that there may be critical linkages between epistemology and practice at the heart of effective computer-mediated design-based learning for younger students—and perhaps at the heart of effective computer-mediated learning activities based on other professions as well.

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Appendix A

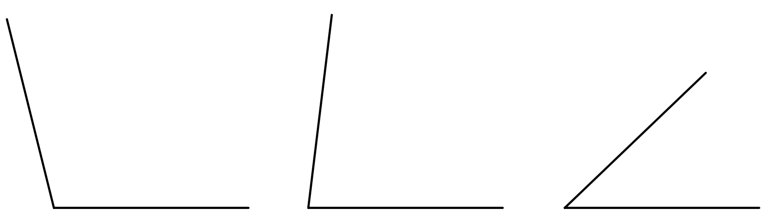
Form A

Geometry Questions

1. How many pairs of parallel lines are there in the figure below? ______.

How many pairs of perpendicular lines are there? _________.

2. Can you estimate the size of angles without using a protractor? Match the angles on the left side with the appropriate angle range (in degrees) on the right side.

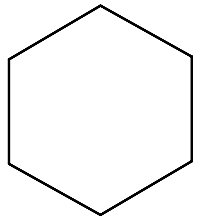


(1) (2) (3)

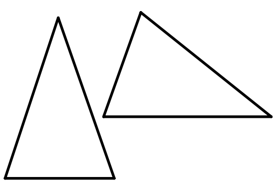
A. between 105 and 90
B. between 90 and 75
C. between 60 and 30
D. between 30 and 0

_________. _________. _________.

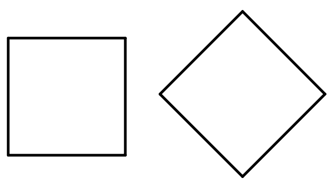
3. How many mirror lines does this hexagon have? ________.



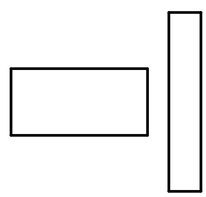
4. Which of the following pairs of figures are congruent?



Congruent?

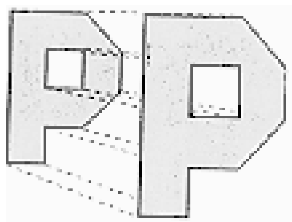


Congruent?



Congruent?

5. What is the scale of the dilation in the figure below?



________.

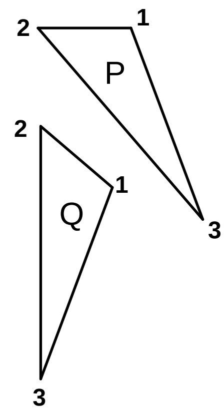
6. What type(s) of symmetry does the letter N have? ________.

A) Mirror symmetry with a vertical mirror line
B) Mirror symmetry with a horizontal mirror line
C) Rotation symmetry
D) Translation symmetry
E) No symmetry

7. What rigid motions will move triangle P onto triangle Q in the drawing below?

________.

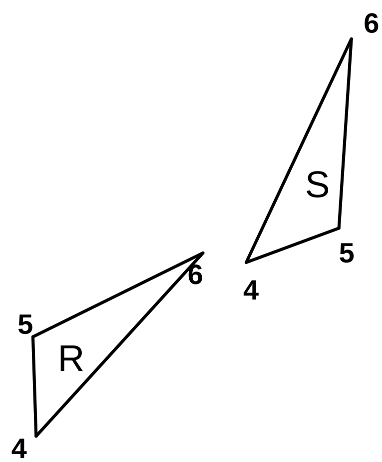
A) Rotation
B) Translation
C) Reflection
D) Reflection and Translation



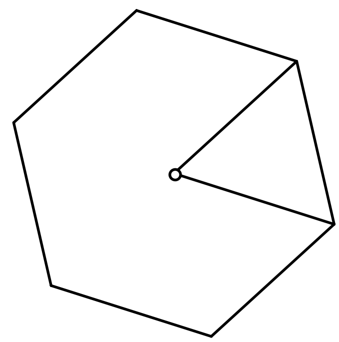
What rigid motions will move triangle R onto triangle S in the drawing below?

________.

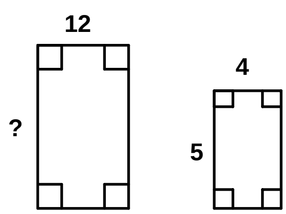
A) Rotation
B) Translation
C) Reflection
D) Reflection and Translation



8. What is the measure of the angle shown at the center of the circle in the regular hexagon below?________.



9. If the small rectangle on the right is similar to the large rectangle on the left, what is the length of the side marked with a "?"? ________.



10. Find the answer to the following problems:

$$1-\left(\frac{1}{2}+\frac{1}{3}\right)= \underline{\qquad\qquad}.$$

$$\left(1^2+2^2+3^2\right)-\left(1+2+3\right)= \underline{\qquad\qquad}.$$

if $3x+15=16$, then x= ________.