Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education

Prague, Czech Republic
July 16-21, 2006

Volume 3

Research Reports

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The Proceedings are also available on CD-ROM

ISSN 0771-100X

Logo: Ivan Špirk
Cover Design: Jarmila Novotná & Ivan Špirk
Printing: Atelier Guimaec, s.r.o

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EXPLORING REALISTIC MATHEMATICS EDUCATION IN ENGLISH SCHOOLS

Frank Eade, Paul Dickinson

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This paper provides an account of the initial phases of a project involving the trialing of a new approach to teaching in lower secondary schools in England. The method being trialed is based on Realistic Mathematics Education (RME), originally developed in the Netherlands. The paper describes background concerns and research methods, initial findings and explores emerging issues for the project.

INTRODUCTION

The TIMSS (1995) study stimulated a great deal of research into international comparisons of approaches to teaching mathematics in schools. Indeed the National Numeracy Strategy (NNS), developed as a direct response to the perceived underachievement of pupils in England, is now part of the framework for teaching mathematics in all primary and secondary schools in England.

In exploring the use of an approach to teaching from one country in another we need to be aware of a range of issues including the desired outcomes of teaching mathematics, social and cultural norms, curriculum compatibility and pedagogical concerns. This paper discusses teaching in a variety of countries and outlines the initiation of the use of Realistic Mathematics Education, originally developed in the Netherlands, in lower secondary schools in England.

BACKGROUND

Mathematics in England

In response to TIMSS (1995) The National Numeracy Strategy was developed to raise the standard of mathematics in primary schools, borrowing techniques and strategies from a range of relatively successful countries whilst leaving unchanged much of the existing content and approaches to teaching (Reynolds *et al.* (1999)). Some specific and welcome changes included delaying the introduction of formal algorithms, the use of informal and mental approaches, and the use of “models” such as the empty number line and the multiplication grid. It is difficult, however, to detect any significant influence on the development of pedagogy at KS3 (international grade 6 – 8) except, perhaps, that teachers have responded to a perceived improvement in mental skills. Teachers have been given little guidance as to how they might delay formal algorithms when, for example, teaching algebra. For most teachers, explanation followed by exercises remains the dominant approach to teaching.

It is worth noting that at the macro level, the strategy suggests sophisticated approaches to teaching but it is difficult to see these being implemented at the micro

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 1-8. Prague: PME.

level, where the examples and structure provided do not seem to explicate such approaches. In the video and other support material provided through the NNS, we see examples which appear to be based on the assumption that the teaching and learning of mathematics are relatively straightforward and unproblematic. This is particularly significant for secondary teaching, where few structural changes were required to implement the strategy. Although most teachers at KS3 now use the NNS three-part lesson structure, for many, the approach used in the main body of the lesson remains unchanged. Teachers can now justify telling and explaining through the Strategy by citing "demonstrating" and "modelling". It is also worth noting that the "little and often" approach (sometimes justified in terms of the spiral curriculum) has, if anything, been intensified under the NNS without any hard evidence to suggest that this should be the way forward.

Clearly the strategy has been effective in changing some of the patterns of behaviour of teachers and in shifting the emphases on different parts of the mathematics curriculum. However, the work of Anghileri *et al.* (2002) and Brown *et al.* (2003) would suggest that there may be grounds to doubt that these changes have been as effective as the government would have us believe. Despite apparent short-term improvements as measured by the end of KS2 (international grades 3-5) and KS3 assessment, Smith (2004), Brown (2003) and Anghileri (2002) all highlight worrying concerns about longer-term conceptual understanding and achievement in terms of understanding and applying mathematics.

The Smith Report (2004) suggests the need for "… greater challenges… harder problem solving in non-standard situations, a greater understanding of mathematical interconnectedness …" The report also indicated that the mathematical skills developed by pupils age 16 are not concerned with " the growing mathematical needs of the workplace… mathematical modelling or ... problems set in the real world contexts." Smith also suggested that in comparative terms "England seriously lags behind its European competitors" in terms of the number of pupils achieving an appropriate level 2 qualification.

The above would suggest a clear need to develop a pedagogy of mathematics education that supports pupils' conceptual understanding and problem-solving skills and their use in real world situations.

Mathematics in the Netherlands

The Freudenthal Institute, University of Utrecht was set up in 1971 in response to a perceived need to improve the quality of mathematics teaching in Dutch schools. This led to the development of a research strategy and to a theory of mathematics pedagogy called Realistic Mathematics Education (RME). RME uses realistic contexts to help pupils develop mathematically. Pupils engage with problems using common sense/intuitions, collaboration with other pupils, well judged activities and appropriate teacher and textbook interventions. (See Treffers (1991) and Treffers *et al*. (1999) for further discussion of RME.)

At a surface level, RME resonates strongly with progressive approaches used in England where investigative and problem-solving strategies are utilised and where pupils are encouraged, as a whole class, to discuss their work to resolve important issues. One difficulty with this approach to teaching in England is that pupils tend to stay with naïve mathematical strategies and are often unwilling to move to more sophisticated ones. Through intensive research, trialing and re-evaluating materials and approaches, Dutch mathematics educators have developed a variety of ways of encouraging and supporting pupils' mathematical progress. So, for example, pupils remain in context throughout and stay with a topic for a much longer period of time than would be usual in England. One of the essential features of fostering development using RME is the use of "models" as scaffolding devices (see van den Heuvel-Panhuizen (2003) for a thorough analysis of the use of models under RME).

Mathematically, the Netherlands is now considered to be one of the highest achieving countries in the world (TIMSS (1999), PISA (2000)).

Mathematics in the USA

Stigler *et al*. (2001) have provided a comprehensive analysis of mathematics teaching in the USA which for our purposes can be summarised as concentrating on knowledge, isolated skills and algorithms. The USA has performed relatively poorly in both TIMSS and PISA. As part of the Reform Movement in the USA motivated by TIMSS and PISA and guided by principles initiated by NCTM, a number of curriculum development projects were initiated, one of which, Mathematics in Context (MiC), involved the development and trialing of materials based on RME.

Mathematics in Context (MiC)

In 1991, The University of Wisconsin, funded by the National Science Foundation (USA), in collaboration with the Freudenthal Institute, started to develop the MiC approach based on RME. The initial materials were drafted by staff from FI on the basis of 20 years of experience of curriculum development. After revision by staff from UW, the material was trialed, revised and retrialed over a period of five years. Trialing involved checking a variety of versions of questions for effectiveness and also the careful examination of teacher needs, beliefs and expectations. (See Romberg T. A. and Pedro J. D. (1996) for a detailed account of the developmental process and van Reeuwijk M. (2001) for an account of the care taken in developing one aspect of the scheme.) The first version of MiC was published in 1996/7 and has undergone several revisions since then. The teacher material, which supports the pupil books, provides a comprehensive analysis of issues pertaining to the topic and provides the teacher with insights into teaching and learning trajectories. Webb, D. *et al.* (2002) provide a useful summary of research into the effectiveness of MiC.

Considered together, the research suggests that the scheme has the potential to allow access to a challenging mathematics curriculum for the full range of middle years (international Y5 –Y8) pupils. Romberg T. (2001) provides a valuable summary of design and research features including the influence of MiC on teacher behaviours and beliefs about teaching and on their perception of pupil capability.

Although transfer of a pedagogy of mathematics education to parts of the USA from the Netherlands has been achieved, we must not assume that this will transfer to England without a thorough consideration of cultural similarities and differences together with appropriate trialing in England.

PROJECT

After positive results from an initial study using RME in a secondary school in the academic year 2003-2004, the Gatsby Foundation agreed to fund a project based around trialing RME (utilizing MiC) over a three year period. The Economic and Social Research Council (ESRC) has also agreed to fund an examination of how teachers' beliefs and behaviours change as a result of engagement in the project. The project is conceived in three phases:

Year 1 (2004-05)	Trialing materials with Year 7 pupils (international Grade 6) in six schools with two teachers in each school. The schools chosen have reasonably close links to MMU. In addition, in the pilot school, six teachers trialed materials with Year 7 and two teachers with Year 8.
Year 2 (2005-06)	Whilst continuing with the six original schools using materials with Year 7 and 8 pupils, six new schools with two teachers in each school became involved. The six new schools with two teachers in each school were chosen to be more representative of the national population of schools and teachers.
Year 3 (2006-07)	Whilst continuing with the 12 schools, the intention is to expand the project to four universities in other parts of England, each supporting four schools in their respective regions.

RESEARCH ISSUES

The research methodology has been strongly influenced by Boaler (1996), Anghileri (2002), Romberg (2001) and Ball, S.J. (1996). The research methodology is based on a number of assumptions: that we require a mixture of qualitative and quantitative methods for data collection and analysis; that as awareness develops, then the distinctive elements of the study will emerge; that where possible, triangulation is required for some degree of security in our findings, and that elements of research, curriculum development and training inform and support each other. For convenience the research variables are identified in terms of:

Pupils

We are examining changes in attitudes to and beliefs about mathematics; willingness to engage in mathematics; development of problem solving strategies (horizontal mathematisation) and strategies for engaging in a more sophisticated way with maths (vertical mathematisation) – the intention is to describe both forms of mathematisation in terms of "learning trajectories". We are also examining the development of proportional reasoning skills and strategies and change in content attainment as measured by Standard Assessment Tests (SATs).

Teachers

We are examining changes in beliefs about mathematics and pedagogy, changes in understanding of pupil progress and mathematical development, changes in approaches to teaching including what strategies are now being used in project classes and what are being transferred to other classes and development of understanding of the pedagogy of RME.

Data Collection

Data are collected through control and project groups' Standard Assessment Test results as measures of added value; problems solving tests to gain insights as to how particular skills and strategies develop; attitudinal questionnaire, and questions on proportional reasoning to examine how a crucial aspect of development for lower secondary pupils changes. We will also observe and interview pupils.

We will examine changes in teachers' vocabulary in describing learning and teaching, beliefs and inclinations. We are also observing lessons to examine what strategies and techniques are being utilised by pupils and teachers.

RESULTS FROM END OF FIRST YEAR OF PROJECT

In this report we are going to focus on two aspects of our data collection:

End of Year written test

The content of this test was the Year 7 (international grade 6) curriculum as designated by the National Strategy for mathematics at Key Stage 3. As such, it reflected the work done by our schools with their non-project classes.

The MiC curriculum on the other hand differs in the stress that it places on certain items, and significantly in its approach to Algebra at this level. Pupils were identified in three groups, each containing 100 pupils, with matched project and control pupils. Matching was achieved using the precise level data from the end of KS2 SATs exam (taken by pupils at the end of year 6 (international 5). The results from this test are given in the Table 1 below.

Further analysis of the pupils' scripts shows that the difference in marks for the highest ability pupils can be accounted for entirely in the algebra questions. In particular, a question on solving equations with x on both sides, worth 2 marks, was simply not accessible to project pupils. At the other ability levels, although control pupils still did slightly better on the algebra, this was nullified by project pupils gaining more marks on questions involving number.

KS 2 Level	Programme pupils Average score/60	Control pupils Average score/60
Levels 3 and 4c (bottom 25% of national population)	25.0	25.0
Levels 4a & 5c (average & average + 40% of national population)	38.6	38.3
Levels 5b & 5a (top 20% of national population)	45.6	47.8

Table 1

The difference between how lower and higher ability pupils are performing under RME will be considered in more detail in the longer version of this report.

The style of this test allowed little scope for consideration of pupil methods but politically and educationally it was important to have some measure of confidence that pupils are not being disadvantaged in terms of content attainment through studying mathematics under RME.

Assessment of Problem Solving Ability

Question Number	**Pupils' attainment measured using KS2 SATS**					
	Low attaining		Middle attaining		High attaining	
	Project	Control	Project	Control	Project	Control
1(a)	22%	6%	52%	33%	69%	73%
1(b)	15%	0%	18%	16%	43%	55%
2(a)	37%	6%	64%	36%	76%	82%
2(b)	51%	19%	70%	61%	88%	93%
4(J)	61%	29%	70%	63%	72%	58%
4(D)	52%	38%	64%	46%	60%	58%
4(a)	45%	37%	68%	56%	68%	69%
4(b)	34%	16%	53%	54%	72%	82%
4(c)	0%	0%	26%	13%	49%	47%
Average	35%	17%	54%	42%	66%	69%

Table 2

In this test, pupils were given ten minutes to complete each of five questions. Consequently, much more detailed solutions were produced, and it was possible to analyse methods and approaches in addition to whether or not pupils had arrived at a correct answer. The percentage of pupils, as sampled above, getting fully correct answers at each level is shown in Table 2 above.

Clearly there will be inter-school and inter-teacher variability, and where scores are close it is difficult to attribute differences to anything other than these. In some cases differences are striking however and clearly warrant further analysis in the longer report. For the first year of the project we believe it to be more useful to focus on evidence of the project influencing the approaches and strategies of the pupils whilst problem solving rather than absolute achievement. The longer report will provide analysis and examples of pupils' solutions to individual problems.

SOME EARLY ISSUES:

Currently in England, teachers tend to describe pupils' development purely in terms of content attainment. This is accentuated by the current national framework which has goals such as recognition of equivalence between fractions, without giving any sense of what progress towards that goal might look like, except in terms of progression from, say, halves and quarters to "harder" fractions. Under MiC, teachers are finding it increasingly difficult to view progress purely in this manner. Our current focus is on generating a vocabulary to describe development in terms of learning-teaching trajectories (van den Heuvel-Panhuizen (2003)) which consider not only content aims but also the stratified nature of the learning process.

The project team have noted the early reluctance of pupils to work with the context or image, tending to move more towards the manipulation of symbols without, in many instances, having any underlying sense of how the symbols are related to the original context. The project team conjecture that teachers tend to support inappropriate movement to the abstract. We are supporting teachers in developing strategies to stay with the image/context and increasingly to recognise the significance of visualisation, imagery and sense-making in promoting mathematical development (see Misallidou, C. et al. (2003) for discussion of related issues). The team is also currently working with the conjecture that utilizing context and staying in context supports lower-attaining pupils in making sense of their engagement in mathematics and supports their mathematical development. We also recognise that we may need to develop a different account for the highest attaining pupils.

We are starting to recognise some of the support needs of teachers moving from transmission and discovery learning approaches and approaches led by content objectives towards guided reinvention where closure is not the norm and where pupils' sense of engaging in mathematics is very different. We are also noting how teachers are negotiating meanings and justifying actions through current practices, beliefs and structures. In supporting development, we recognise the need to come to terms with this dialectic. This is a significant aspect of the ESRC part of the research.

The longer report will provide an up-to-date summary of the data collection and analysis, emerging theory, examples of pupils' engagement in solving problems and significant issues near the end of the second year of the project.

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REASONING ABOUT NON-LINEARITY IN 6- TO 9-YEAR-OLDS: THE ROLE OF TASK PRESENTATION

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Several studies by Ebersbach et al. revealed, simultaneously to a linearity concept, the existence of intuitive knowledge about non-linearity in children. Main objective of the current study was to facilitate tasks by presenting them in an inductive reasoning format to assess both the linearity and non-linearity concept even in 6- to 9-year-olds. Therefore, children forecasted linear and exponential growth processes. Linear growth was estimated appropriately by all age groups. Exponential growth was estimated distinctly different with regard to both the magnitude and curve shape of the estimations, whereas 6- and 9-year-olds performed better than 7-year-olds indicating an interference with the linearity concept at this age. However, findings suggest that even preschoolers have intuitive knowledge about non-linear processes.

THEORETICAL AND EMPIRICAL BACKGROUND

Non-linear functions are introduced relatively late in formal maths education, i.e. usually to children older than 12 years, and even then, adolescents and adults show striking difficulties in understanding and forecasting non-linear processes (De Bock, Verschaffel, & Janssens, 1998; Mullet & Cheminat, 1995; Wagenaar, 1982). One possible explanation of these difficulties lies in the assumption of a dominant linearity concept that is applied to almost each numerical relation and also used to solve tasks requiring thinking in a non-linear manner. This inappropriate behaviour, also called “illusion of linearity” (Freudenthal, 1983), implies using proportional reasoning by assuming a constant rate of change on one variable due to a constant rate of change on an associated variable also in tasks where the rate of change actually increases or decreases.

Results of prior research suggest that the concept of non-linearity is both difficult to acquire and to understand and, furthermore, it seems to be applied hardly as primary approach in problem solving (Van Dooren, De Bock, De Bolle, Janssens, & Verschaffel, 2003; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2004; Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2002).

However, it might be helpful to examine whether an intuitive understanding of non-linearity exists developing independently of formal maths education. Knowing at which age and under which conditions the concept of non-linearity develops might provide important hints for the development of appropriate teaching methods and help to improve people’s deficient performance in tasks involving non-linear processes.

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 9-16. Prague: PME.

A study by Ebersbach and Wilkening (2005) using tasks involving the estimation of linearly and exponentially growing amounts of plants revealed that, indeed, 9-year-old children already showed intuitive knowledge about non-linear processes by assuming an increasing rate of change in exponential but a constant rate of change in linear growth and, accordingly, by estimating the exponentially growing quantity to be significantly more than the linearly growing one. This early non-linearity concept was also found in beliefs of 7-year-olds about psychophysics, i.e. beliefs about the relationship between sugar or salt concentration in a solution and its perceived intensity (Ebersbach & Resing, 2005), but not even in adults with regard to the estimation of shadow sizes where a non-linear relation between light-object- distance and shadow size exists (Ebersbach & Resing, 2006), implying a domain-dependent emergence of the non-linearity concept.

A shortcoming of the study by Ebersbach and Wilkening (2005) was that information about growth behaviour of plants was given mainly verbally, which might have caused an overload of the working memory particularly in younger children resulting in a weaker performance. The study presented here aimed to avoid this potential memory effect by using inductive reasoning tasks to investigate the estimation of linear and exponential growth.

RESEARCH QUESTIONS AND HYPOTHESES

The current study addressed the following questions. First: At what age do children develop a concept of non-linearity allowing them to differentiate between linear and exponential growth? We assumed that presenting tasks in an inductive reasoning format without extensive verbal instruction might improve children’s performance in estimating exponential growth so that the non-linearity concept might become apparent even in children younger than nine years of age. And second: Is there an effect of task sequence? Based on previous findings, we hypothesized that in particular younger children exhibit difficulties in switching between a linear and a non-linear estimation strategy affecting their overall performance. Systematic manipulation of task sequence was expected to reveal children’s estimation performance controlled for task switching ability.

METHOD

Children of three age levels (each n = 30) were tested individually: 6-year-old preschoolers (13 boys and 17 girls; age: 58 - 81 months, M = 72 months), 7-year-old first graders (14 boys and 16 girls; age: 79 - 93 months, M = 86 months), and 9-year-old third graders (15 boys and 15 girls; age: 104 - 123 months, M = 110 months), all living in a medium sized town in Germany.

In order to illustrate linear and exponential growth, two stories were presented. Linear growth was embedded in a story about a dwarf who lived in a house with mushrooms growing in his garden. Likewise, exponential growth was introduced by a story about a fairy living in a house with flowers growing in her garden. Children saw a total of seven plastic boxes arranged in a line. Boxes were placed on a yellow paper strip symbolizing a time line, on which seven fields separated by black bars represented seven days. The first three boxes were filled in accordance to the growth type with red wooden beads, i.e. 3, 4, and 5 beads for linear, and 3, 6, and 12 beads for exponential growth. Identical boxes, each containing 250 red beads that should be used for estimating further growth, were placed behind the fourth, fifth, sixth and seventh empty plastic boxes. Children were told that the story character saw on the first day 3 plants in his garden, on the second day 4 or 6, respectively, and on the third day 5 or 12, respectively, depending on growth type. They were then asked to estimate the quantity of plants the character would see on the fourth, fifth, sixth and seventh day by filling as much beads into the empty boxes. This required that children themselves had to detect the underlying growth rule in the default of three sample days and to apply it in order to estimate future quantities. Each child estimated each growth type twice successively starting either with linear or exponential growth.

RESULTS

Children's mean estimations of linear and exponential growth are displayed in Figure 1, separate for growth type and task sequence. In order to test the effects of our variables, a 3 (age group: preschoolers, first graders, third graders) x 2 (task sequence: linear growth first, exponential growth first) x 4 (growth duration: 4, 5, 6, 7 days) x 2 (time of measurement: first, second time) ANOVA with repeated measures was conducted separately for each growth type. Dependent variable was the estimated amount resulting from linear or exponential growth, respectively.

Linear growth was estimated more appropriately by older children, $F(2, 84) = 5.92$, $p = 0.004$, $\eta^2 = 0.12$, and, as effect of task sequence, more appropriately when it had to be estimated at first, $F(1, 84) = 19.94$, $p < 0.001$, $\eta^2 = 0.19$. This sequence effect appeared to be stronger for younger children, $F(2, 84) = 5.10$, $p = 0.008$, $\eta^2 = 0.11$. The ANOVA over estimations of exponential growth yielded also an effect of age, $F(2, 84) = 3.90$, $p = 0.024$, $\eta^2 = 0.09$, which, however, differed from the one concerning linear growth. Repeated contrasts yielded higher and, thus, more appropriate estimations by preschoolers than by first graders, $p = 0.007$, whereas first graders and third graders did not differ on the 5%-level. However, third graders tended to estimate exponential growth higher than first graders, $p = 0.086$. These results are indicative for a U-shaped progression over age as far as the estimation of exponential growth is concerned. Furthermore, exponential growth was estimated more appropriately as it was estimated at first, $F(1,84) = 8.22$, $p = 0.005$, $\eta^2 = 0.09$. This sequence effect was for all age groups equally strong, with no interaction between both variables.

In order to inspect the effect of task sequence on estimations of linear growth separate for each age group, additional 2 (task sequence: linear growth first, exponential growth first) x 4 (growth duration: 4, 5, 6, 7 days) x 2 (time of measurement: first, second time) repeated measures ANOVA's were conducted. Results confirmed a significant sequence effect in preschoolers, $F(1, 28) = 13.03$, $p =$

0.001, η^2 = 0.32, a weaker sequence effect in first graders, $F(1, 28) = 5.52$, $p = 0.026$, η^2 = .17 (not significant after Bonferroni correction) and no effect in third graders, $F(1, 28) = 1.40$, $p > 0.05$. With regard to estimations of exponential growth, an effect of task sequence can be assumed to exist in all age groups suggested by a main effect of task sequence and, simultaneously, the absence of an interaction between age group and task sequence in the already reported ANOVA results.

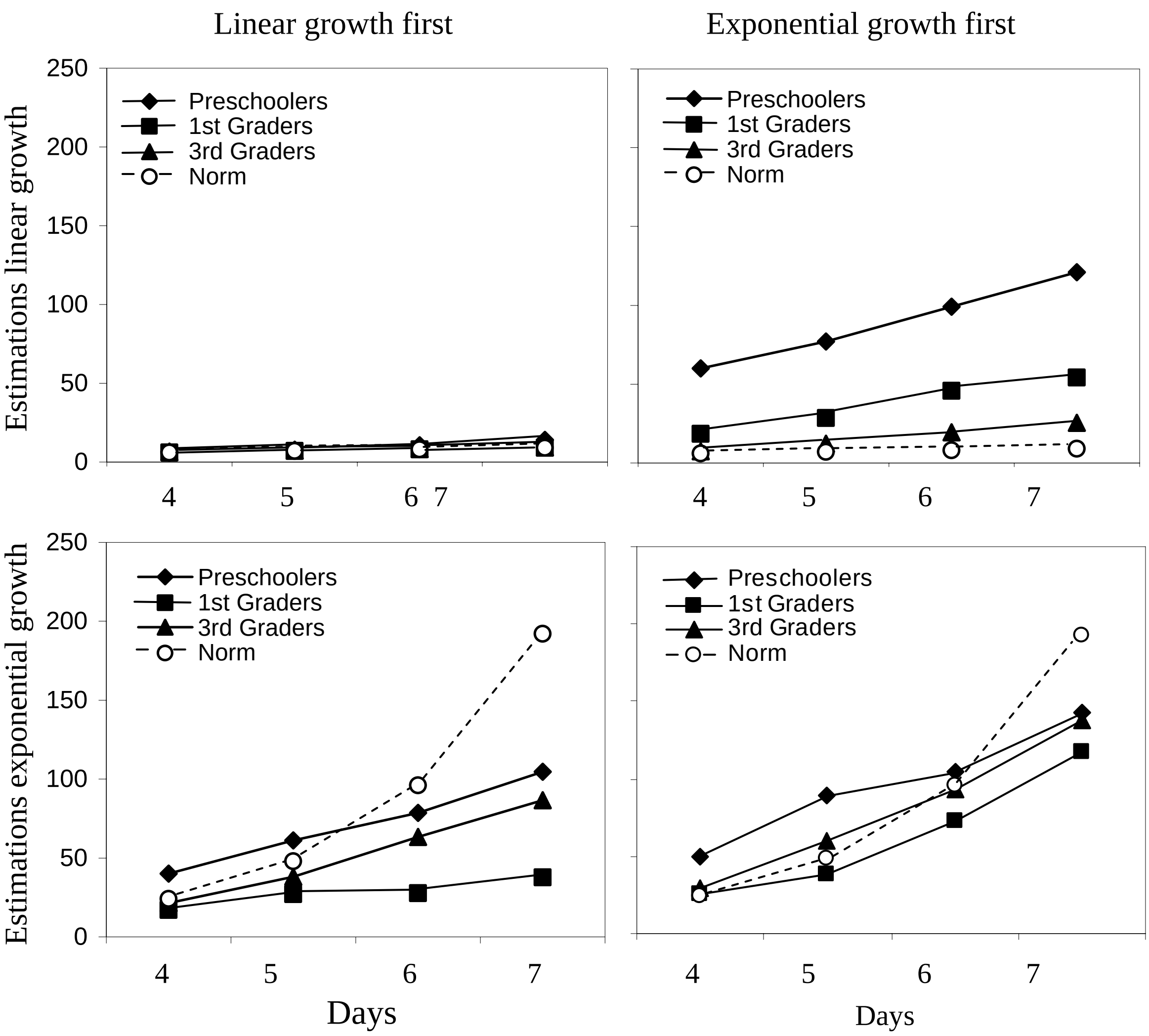

Figure 1: Mean estimations of linear (top) and exponential (below) growth per age group, separate for task sequence.

In order to examine whether children differed by means of their estimations between linear and exponential growth, a repeated-measures ANOVA with the within-subjects factor growth type (linear vs. exponential) was conducted per age group. Results (Bonferroni corrected) confirmed that children of all age groups estimated linear growth significantly lower than exponential growth (preschoolers: $F(1, 29) = 8.42$, $p < 0.05$, $\eta^2 = 0.23$; first graders: $F(1, 29) = 8.81$, $p < 0.05$, $\eta^2 = 0.23$; and third graders: $F(1, 29) = 60.26$, $p < 0.01$, $\eta^2 = .68$), pointing even in preschoolers to the ability to discriminate between linear and exponential growth.

Whereas analyses so far were concerned with the absolute magnitude of estimations, their mathematical appropriateness will be focused on in the following. One question in this concern was whether children estimated linear and exponential growth according to the norm. Therefore, only deviations between estimations for the seventh day, i.e. the day with the farthest temporary distance to the default amounts, and mathematically correct values, i.e. 9 for linear and 192 for exponential growth, were calculated. Using multiple t-tests separate for each age group, task sequence and growth type with Bonferroni corrected alpha level we examined whether these deviations differed significantly from zero.

As a result, linear growth was estimated according to the mathematical correct solution by children of all age groups as the task started with the estimation of linear growth (preschoolers: $t(15) = 1.29$; first graders: $t(14) = 1.00$; third graders: $t(14) = 1.00$; all p's > 0.05). As linear growth was estimated subsequently to exponential growth, only estimations of preschoolers deviated significantly from the norm, $t(13) = 3.85$, $p < 0.05$, whereas first and third graders made correct estimations, $t(14) = 2.42$ and $t(14) = 1.17$, p's > 0.05. On the other hand, exponential growth on the seventh day was, as it was presented first, appropriately estimated by preschoolers, $t(13) = 2.11$, and third graders, $t(14) = 2.82$, p's > 0.05, whereas estimations of first graders exhibited significant deviations from the norm, $t(14) = 3.61$, $p < 0.05$. However, estimations for exponential growth deviated in all age groups significantly from the norm as linear growth had to be estimated first (preschoolers: $t(15) = 3.99$; first graders: $t(14) = 21.03$; third graders: $t(14) = 5.69$; all p's < 0.05).

A further aspect of forecasting linear and exponential processes is the shape of the estimation functions. Thus, the question was whether a linear or an exponential model would fit better to the estimations of linear and exponential growth. Individual curve fittings per child, separate for growth type and task sequence, were calculated on the basis of repeated measures. The fit of a linear model of the general form $y = \beta$ x a was compared with the fit of an exponential model of the general form $y = \beta^{a}$. The comparison of both R^2 values served as indicator of which model fitted better. On basis of these R^2 values participants were assigned to either a linear or exponential estimation strategy.

The category "other" included children for which both models fitted equally well or in which the concurrent model fitted better. The frequency pattern of these categories is presented in Figure 2 separate for each growth type. Concerning estimations of linear growth it became obvious that the better fit of the linear model increased with higher age. A chi-square test confirmed a higher frequency of the linear compared with the concurrent models regarding the estimations of linear growth only in third graders, Fisher exact test $p < 0.001$. Concerning exponential growth, a better fit of an exponential model was revealed more frequently in preschoolers and third graders, Fisher exact test $p < .05$ and $p < 0.01$.

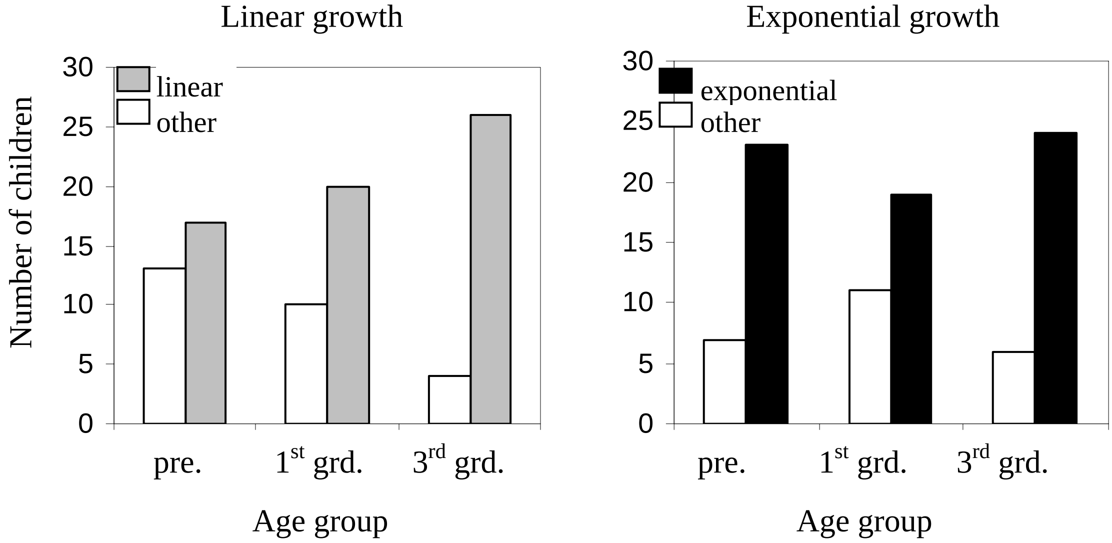

Figure 2: Number of children for which a linear model fitted better to their estimations of linear growth (left) and for which an exponential model fitted better to their estimations of exponential growth (right).

However, results reported so far showed also a significant effect of task sequence. Therefore, curve fit data were also inspected separately for each task sequence. It became apparent that, if the task started with the estimation of linear growth, this growth estimation fitted best a linear model in 87.5% of preschoolers, 100% of first graders, and 93.3% of third graders. In contrast, if linear growth had to be estimated subsequently to the estimation of exponential growth, data of only 21.4% of preschoolers, 33.3% of first graders, and 80% of third graders fitted best a linear model. The effect of task sequence was weaker concerning the estimations of exponential growth. If exponential growth had to be estimated first, data of 78.6% of preschoolers, 73.3% of first graders and 86.7% of third graders fitted best an exponential model. On the other hand, as exponential growth was estimated subsequently to the estimation of linear growth, data of a similar proportion of preschoolers, i.e. 75%, but only of 53.3% of first graders and 73.3% of third graders fitted best an exponential model. A further result of this analysis was that the effect of task sequence seems to diminish with higher age.

CONCLUSION AND DISCUSSION

Main motivation of our study was to find an intuitive non-linearity concept also in children younger than nine years old by using tasks in an inductive reasoning format. Results showed that even 6-year-old preschoolers discriminate between linear and exponential growth processes by means of their estimations, i.e. exponential growth was estimated correctly higher than linear growth suggesting that both a linearity and a non-linearity concept existed in these children. That assumption is also supported by analyses of the mathematical appropriateness of estimations. Linear growth on the seventh day was estimated correctly by 6-, 7-, and 9-year-olds, however, 6-year-olds overestimated linear growth only as exponential growth had to be estimated beforehand. Thus, the linearity concept appeared to be affected by task sequence. In contrast, exponential growth on the seventh day was estimated correctly by 6-, and 9-year-olds, but not by 7-year-old first graders. This was true only for the condition where exponential growth was estimated first. Children of all three age groups underestimated exponential growth as it had to be estimated subsequently to linear growth. Finally, analyses of the curve shapes revealed that a high proportion of children in all age groups estimated linear and exponential growth according to the appropriate function type.

Our study yielded two further interesting results. First, children's estimations were highly affected by task sequence, thus, they underestimated exponential growth as they estimated it subsequently to linear growth and vice versa, which was in particular true for younger children. This finding supports the assumption of a weaker ability to switch between tasks and strategies in young children also reported in other studies (e.g. Rosselli & Ardila, 1993). The sequence effect can also be discussed in terms of negative transfer or set forming, in that younger children, although possessing alternative strategies, apply their first chosen strategy inflexibly to different tasks whereas older children are able to choose flexibly the appropriate strategy for each task type (Chen & Daehler, 1989; Kuhn, 1995).

A second finding worth to discuss concerns the fact that first graders exhibited weaker performance compared to preschoolers in estimating exponential growth whereas third graders again performed better suggesting rather an U-shaped developmental pattern than a continuous one. This might be explained by assuming an interference of the linearity concept with exponential growth tasks especially in this age group. First graders normally undergo a massive development of their numerical knowledge by extending their number line from 10 to 100 or more in the first year of schooling. This focus on counting might reinforce the linearity concept resulting in an overgeneralized use of linear models to solve different kinds of estimation tasks. This hypothesis was also supported by observations of the experimenter reporting that first graders sometimes focused on counting the beads while ignoring the original task that required estimating growth.

To resume, our results supported the assumption that both an intuitive linearity and non-linearity concept exist even in preschoolers. However, children's performance was highly affected by antecedent tasks implying relative difficulties to switch between tasks and strategies. Finally, presenting tasks in an inductive reasoning format facilitated access to children's knowledge and might be a promising method to investigate further knowledge about functions.

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TEACHERS, CLASSROOM, STUDENTS – A HOLISTIC VIEW ON BELIEFS CONCERNING THE EDUCATIONAL PRACTICE

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This report focuses on a research project that combines three aspects of a curriculum concerning teachers' beliefs, their classroom practice, and their students' beliefs. Firstly, the theoretical framework and methodology will be outlined. The results of this project will be used to discuss the relations between one teacher's individual curriculum and the beliefs of five of his students.

INTRODUCTION

> If we accept that teacher thinking determines how the curriculum gets interpreted and delivered to students, then the nature of mathematics teachers' thinking becomes a key factor in any movement to reform the teaching of mathematics. (Chapman, 1999, 185)

An important aspect of the increasing research into teachers' beliefs is the acceptance of the central role of teachers in changing or reforming mathematics education (see Wilson & Cooney, 2002). Nevertheless, in Germany, in particular, there still is an attempt to reform mathematics education only by publishing didactical curricula, or by decreeing administrative curricula. Furthermore, there are still few research results providing insights into teachers' beliefs concerning a specific mathematical subject like stochastics, as opposed to teachers' beliefs on mathematics in general.

Another aspect of the research into teachers' beliefs is their conviction that their own impact on students' beliefs is high (see Chapman, 2001, p. 233). Research, however, has as yet yielded few results which facilitate understanding the relations between teachers' and students' beliefs.

This report focuses on a research project intended to fill in gaps of research and links three aspects of the educational practice in mathematics and in particular in stochastics (statistics and probability theory), its approach being inspired by a curriculum model of Vollstädt et al. (1999) that includes:

- teachers' planning: teachers' conscious choices of mathematical contents and their reasons for theses choices, the teachers' *individual curricula*,
- teachers' classroom practice: the realisation of the intended individual curricula, the teachers' *factual curricula*, and
- students' attained beliefs and knowledge, the *implemented curricula*.

Here, the curriculum means the selection of mathematical contents, and the reasons for these choices. The objective of this research project is to provide insights into the educational practice in a holistic manner, including teachers' beliefs, classroom practice, and students' beliefs. This report will discuss some aspects concerning the

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 17-24. Prague: PME.

structure of one teacher's individual stochastics curriculum, and the relations between this individual curriculum and the implemented curricula of five students.

THEORETICAL FRAMEWORK

The approach of this line of research is inspired by the well-elaborated psychological approach of subjective theories (see Groeben et al., 1988).

The construct of subjective theories is based on the psychological construct of action. Action is understood as "the physical behavior plus the meaning interpretations held by the actor" (Erickson, 1986, p. 126). For this reason, action is understood as an internal and subjective process that depends on situation, and on how individuals interpret a situation.

Subjective theories are defined as a complex system of cognitions (and emotions), which contains an at least implicit rationale. Hence, individual cognitions are connected in an argumentative mode. Furthermore, the definition of subjective theories includes the assumption that subjective theories are constructed in much the same way 'objective' theories are. Subjective theories contain

- subjective concepts,
- subjective definitions of these concepts, and
- relations between these concepts that constitute the argumentative structure of the system of cognitions.

With regard to the teachers' individual curricula, subjective concepts are goals of the curriculum. The relations between the goals can be described as goal-method-argumentations, a construct proposed by König (1975). How teachers' goal-method-argumentations concerning their individual stochastics curricula are reconstructed has been outlined elsewhere (Eichler, 2004; Eichler, 2005). In the psychological sense, the individual curricula are non-observable intentions of action, which need to be reconstructed qualitatively by interpretation.

In contrast, the *intentions in action* constitute the teachers' classroom practice, or his/her factual curriculum as it can be observed. The results of the observation can be seen as evidence that the teachers' individual curricula have been adequately reconstructed.

With regard to the curricula students attain (students' implemented curricula) the subjective concepts are about the students' major concepts concerning a) stochastics, b) teaching stochastics or mathematics, and c) about learning stochastics or mathematics. The argumentative mode is constituted by a system of major concepts, of sub-concepts, of central examples (i.e. manifestations), or of representations.

METHODOLOGY

Reconstructing both the teachers' individual curricula and students' implemented curricula adheres to a five-step-methodology (see figure 1).

The methodology is based on case studies. Cases are defined as individual teachers, or as individual students. These cases are selected according to the theoretical sampling (see Charmaz, 2000).

Individual cases	→ Interview transcripts	→ Belief system	→ Second interview	→ Types/modal structures
Data collection	Hermeneutics, interpretation	Reconstruction	Validation (teachers)	Theory building

Figure 1: The five-step-methodology

Data were collected with semi-structured interviews comprising several clusters of questions concerning following subjects (see table 1). Within these obligatory clusters, the teachers or the students determine the interviews.

Interview with the teachers	**Interview with the students**
Contents of instruction	Stochastic concepts
Goals of mathematics instruction	Uses of mathematics instruction
The nature of mathematics and school	The nature of mathematics
Teaching and learning mathematics	Teaching and learning mathematics
Institutional boundaries	Students' self-efficacy

Table 1: Clusters of the semi-structured interview

Interpreting transcribed interviews adheres to principles of classical hermeneutics (Danner, 1998). The objective of this first phase of reconstruction is to identify subjective concepts, and to see how they are defined. The second phase concerns the construction of belief systems, i.e. teachers' individual curricula or rather students' implemented curricula. The structure of the teachers' individual curricula is described by using goal-method-argumentations (see Eichler, 2004). The description of the structure of the students' implemented curricula is inspired by a method named "Heidelberger-Struktur-Lege-Technik" (see Scheele & Groeben, 1988). The objective of this method is to structure the students' subjective concepts into a system of major concepts, of sub-concepts, or of manifestations (central examples).

Only for purposes of reconstructing teachers' *individual* curricula, there is a third stage named communicative validation. This stage includes a second interview, the objective of which was to reach consensus on the adequacy of the reconstructed and formalised teachers' individual curricula.

The fifth step of the methodology concerns the theory building, which includes the construction of types of teachers' individual curricula, or students' implemented curricula. Besides, this step serves to identify connections and relations between the three aspects of a curriculum outlined above, i.e. between the teachers' individual curricula, the teachers' factual curricula, and the students' implemented curricula.

Observing the factual curricula facilitates linking the teachers' individual curricula to the students' implemented curricula. The classroom practice of the teachers was observed during the time they taught stochastics. In this report, the results of observing the one factual curriculum will be given in detail, but serve as foundation for the data collection concerning students' implemented curricula.

RESULTS

This report focuses on one aspect of the larger study, i.e. the relation between the individual curricula of the teacher Ian (47), and sets of beliefs five of his students show. Other results will only be sketched when necessary.

Ian's individual curriculum

Ian teaches stochastics in a special course of mathematics in grade 12 at a gymnasium (a secondary high school). Here, his individual curriculum will be outlined in a brief version. This version will be structured by five aspects of a teacher's individual curriculum: by instructional contents, by goals concerning stochastics, by goals concerning mathematics, by goals concerning students' views concerning the usefulness of mathematics, and by goals concerning efficient teaching. The goal-method-argumentations will be not described in detail here.

<u>Instructional contents (aspect 1)</u>: Ian's curriculum concerning the instructional contents is traditional (in Germany). His emphasis is on probability theory only. His curriculum includes the concepts of chance, random experiments, probability, combinatorics, and binomial and hypergeometric distribution. Ian primarily uses the statistical approach to probability, and examines Laplacean probability secondarily. Besides this traditional curriculum, Ian examines Bayesian statistics as an aside to conditional probabilities, and thus Bayesian approach to probability.

<u>Goals of the stochastics and mathematics curriculum (aspect 2 and aspect 3)</u>: Ian's first principal goal in both in the stochastics curriculum, and in the mathematics curriculum in general, is to convey an understanding of mathematics as a process. For him, the process is more important than its results, e.g. mathematical concepts, or mathematical methods. While he has begun with pure mathematical problems in the past, he now starts with realistic applications of mathematics. In his view, however, it is not possible to qualify students' for coping with real problems on the basis of school mathematics alone. He never examines realistic applications with the purpose of solving a real problem, but rather intents to use them for developing mathematical concepts, or mathematical methods.

<u>Students' use of mathematics instruction (aspect 4)</u>: Where his students are concerned, Ian's goals are twofold. On the one hand he has pragmatic goals: the students must learn to identify, and to adequately solve basic mathematical problems in order to graduate from school to gain access to university (Abitur). For this reason, they must have acquired algorithmic skills, the ability to formulate, and to argue appropriately. On the other hand, Ian desires his students to learn problem solving in the following sense: whenever his former students encounter a (mathematical)

problem after school, he wants them to get up the nerve to sit down, to reflect on the problem, to use basic heuristics, like graphs, and to try to solve the problem just as they did as students in his class. Beside these goals, Ian wants his students to find out that doing mathematics can be fun.

Effective teaching (aspect 5): One of Ian's central goals, which in his view facilitates efficient teaching, is to show his own engagement in mathematics. His own involvement is the prerequisite for getting students involved in mathematics, and in particular in problem solving. Finally, a further principal goal of his is to make allowance both for students with high mathematical performance, and for students with poor mathematical performance.

The implemented curricula of five of Ian's students

All the students are 17 years old. Brenda is a student with very poor mathematical performance. Chris's and Amanda's mathematical performances are average. Dave shows a high mathematical performance, but sometimes does not make an effort. Eric is one of the best students Ian has. The five students were interviewed one week after Ian finished his stochastics curriculum. What follows will discuss the relation between the five aspects of Ian's individual curriculum on the one hand, and his students' beliefs concerning theses aspects on the other.

Instructional contents (aspect 1): From the contents of the curriculum, Brenda remembers only the probability trees. Furthermore she referred to the HIV-test as to an example (manifestation) of Bayesian statistics. Confronted with some instructional contents, she has inappropriate or no beliefs about the central stochastic concepts of the curriculum. Amanda and Chris remember more instructional contents. While Amanda remembers in particular stochastical concepts like Bayes's theorem, or binomial distribution, Chris remembers in particular manifestations of the concepts used in specific applications like the HIV-Test, or the problem of production errors. Chris has few and often inappropriate beliefs about the stochastic concepts he was asked for. By contrast, Amanda explains most stochastic concepts appropriately, but is unable to give examples (manifestations) for them. Dave and Eric remember all major concepts. They are often able to explain the concepts in a very sophisticated way, and to give one or more examples besides.

Concerning all five students, it is remarkable that, except for one case, none of them is able to reconstruct a formula, e.g. the formula for the binomial distribution, or for the Bayesian formula. Finally, the central concept of Ian's curriculum, the concept of probability, will be discussed. Brenda and Chris have a static belief concerning probability, which is perhaps based on Laplacean probability. For example, Brenda, when asked for an example of evaluating a probability in an arbitrary context, says:

> Perhaps, how the chances are, when I am strolling in town, to meet a certain person. Phew, to be honest, I don't have any idea how this ought to be computed, but I would relate that somehow to the number of persons in town, and to the single case, the positive case.

In contrast, Amanda, Dave and Erik have a dynamic belief concerning probability, which agrees with the statistical approach to probabilities. Eric says:

> What are the chances that a Mercedes will drive by here within the next hour. This, however, would amount to a test series [...] I would position myself here and count all the cars to begin with, and then also note which make [...] I'll say, 100 cars will drive by here, and five among them are Mercs [...] and then I can also tell if there are 50 cars in one day, there will be two to three Mercs.

<u>The meaning and the uses of stochastics and of mathematics (aspects 2, 3, and 4)</u>: Although Brenda mentions that stochastics has real applications, she is unable to provide an example. She does not expect to benefit from stochastics. The uses of mathematics in her everyday life are confined to perhaps teaching it as a tradition to the next generation of students, her answer to the question whether mathematics has any value for her life:

> Well, it has relatively little. If I wanted to become a teacher, for instance, I could of course add mathematics to my subjects, because I already have a little idea of it. But for many vocations, I think it is totally unnecessary.

Amanda's and Chris's beliefs in real applications of stochastics are restricted to the examples they have examined in the curriculum, like the HIV-test. While Amanda does not expect any uses of stochastics for her life, Chris expects uses in risk assessment in his everyday life, but is unable to give any example. Concerning the uses of mathematics, he mentions basic arithmetic skills, e.g. for comparing prices when buying a car, and that he might need mathematical knowledge he needs for his potential profession. While Amanda agrees with the latter point, she adds that mathematics helps her think logically:

> A lot of logical thinking, as it were. Well, it explains itself quite well. There is no having to swallow it just so, rather, you can explain things to yourself quite well. That's not like in other subjects, where you are confronted with something, and well, that's how it is.

Whether this 'logical thinking' has an application in her life she is unable to explain.

Dave and Eric explain the benefits of stochastics and of mathematics very broadly quoting a number of situations taken from beyond Ian's stochastics instruction. Dave, in particular, shows a sophisticated belief about the uses of stochastics:

> Basically, this (probability) is a model to somehow describe the world, albeit in a rather idealized way, that is rather imprecise if applied to the real world. And you try to handle not how things occur. You look at something, yes, precisely, you try to make forecasts.

In contrast to this general belief concerning the uses of stochastics, Eric emphasises the uses of stochastics in economics, a field where he wants to become active as a professional. Both students say that for them, mathematics is fun. They expect to apply mathematics in their future professions, i.e. in economics for Eric, and in sociology or philosophy for Dave. Eric explains the uses of mathematics by quoting some basic algorithmic skills and the ability to criticise to be a requirement in his future profession. Dave adds, just as he did for the uses of stochastics, a sophisticated belief concerning the uses of mathematics:

> Mathematics is simply the model with which we human beings just try to describe what we want [...]. And I think that his can be modified. Well, models will always describe things only approximately. This entire field of science is one, which I find incredibly fascinating, and it is in any case far from being exhausted.

<u>Teaching mathematics (aspect 5)</u>: Just as for the other aspects of the realised curricula the five students of the sample can be classified into showing three attitudes. Brenda does not like Ian's teaching style. She argues that he goes too fast, integrating only a group of students with high mathematical performance. By contrast, she and other poor students do not get any opportunity to take part in classroom discussions. Besides, she states that Ian does not give enough room to algorithmic exercises, an activity where she has a chance to achieve.

Amanda and Chris only complain that Ian rushes through the subject matter while emphasising problem solving. While they are unconvinced that his style of teaching is correct, they are unable to criticise it.

Dave and Eric confess that they like Ian's teaching style. They emphasise his broad knowledge in mathematics, and his ambition to set for few, but difficult tasks:

> Yes, he entices you to think. That's what I find good. And it's not only this drudgery (solving exercises). And you really see that he knows about things. You see that also from the problems he poses. That's not only some meaningless exercise from the book, but there is content conveyed which makes you to sit up.

DISCUSSION

This report discusses only three aspects of the relations between Ian's individual curriculum and the implemented curricula of his five students.

Firstly, there are the contents of Ian's curriculum. Although it is not new, and even Ian is aware that students can not remember the totality of mathematical subject matter treated, it is surprising that three of the five students of Ian remember but little contents, and often incorrectly. Besides, it is remarkable that two students have a static belief concerning probabilities, although Ian has primarily taught the (dynamic) statistical approach to probability.

Furthermore, there is a difference between the goals of Ian's stochastics curriculum and his students' beliefs concerning both this special mathematical discipline and mathematics in general. To Amanda, Brenda, and Chris, mathematics appears to be at best a static system, comprising rules and formulas for algorithms, and both a logical, and a closed structure of knowledge. Only for Eric, and in particular for Dave, stochastics and mathematics are models to describe the world derived from a process of thinking. Only these two students have internalised the principal goals Ian originally set with his curriculum.

Finally, it is still an open question how students' beliefs of self-efficacy, mathematical performance, and beliefs about the actual teacher's teaching style are related. Here, however, a correlation between self-efficacy, mathematical performance, and level of agreement with Ian's teaching style seems to exist.

Certainly, more sets of teachers' individual curricula and related students' implemented curricula must be analysed to obtain strong evidence about the relation between these two aspects of instructional practice. In particular, other types of teachers' individual curricula (see Eichler, 2005), and their relations to teachers' factual curricula and students' implemented curricula must be examined. From the present status of research, however, we may already conclude: As had been said in the introduction, we may accept that teachers will have a central role in any movement to reform the teaching of mathematics. We must, however, also accept that changing teachers' individual curricula (if this is possible) does not automatically lead to changes in students' implemented curricula. For this reason, it is indispensable for any attempt at reforming mathematics education to step up research into the relations between teachers' beliefs, their classroom practice, and the beliefs and knowledge their students' attain.

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THE EFFECTS OF DIFFERENT MODES OF REPRESENTATION ON PROBLEM SOLVING: TWO EXPERIMENTAL PROGRAMS

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This paper explores the effects of two experimental programs on the development of arithmetic problem solving (APS) ability by 6-9 year-old pupils. The programs stimulated flexible interpretation and use of a plurality of representations in the context of APS with emphasis on a particular mode: informational picture or number line. An a priori model has been validated for all the pupils, suggesting that different modes of representation of the problems influence APS performance, irrespective of the kind of instruction they have received. Data analysis also revealed the beneficial effects of both programs applied not only to the tasks in the representation that was emphasized in each program, but also to the tasks represented in other modes, indicating their success in the general development of APS ability.

INTRODUCTION

The role of representations in mathematical understanding and learning is a central issue of the teaching of mathematics. The most important aspect of this issue refers to the diversity of representations for the same mathematical object, the connection between them and the conversion from one mode of representation to others. This is because unlike other scientific domains, a construct in mathematics is accessible only through its semiotic representations and in addition one semiotic representation by itself cannot lead to the understanding of the mathematical object it represents (Duval, 2002). Kaput (1992) found that the use of multiple representations help students to illustrate a better picture of a mathematical concept and provide diverse concretizations of a concept. Students' ability to link different modes of representation of a common mathematical situation or concept is thus of fundamental importance in mathematical understanding (Griffin & Case, 1997). The conversion of a mathematical object from one representation to another is a presupposition for successful problem solving (Duval, 2002). Yet, many studies have shown that students tend to have difficulties in transferring information gained in one context to another (e.g., Gagatsis, Shiakalli, & Panaoura, 2003; Meltzer, 2005; Yang & Huang, 2004). Lack of competence in coordinating multiple representations of the same concept may result in inconsistencies and delay in mathematics learning at school.

Pupils' difficulties in solving verbal arithmetic problems have been studied extensively (e.g., Nesher & Hershkovitz, 1991) since the early days of 20th century. Previous studies on additive problems have identified three main categories of semantic structures, i.e. the meaning of the text of which the problem is stated: change or transformation linking two measures, combine or composition of two measures and compare or a static relationship linking two measures (Nesher, Greeno,

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 25-32. Prague: PME.

& Riley, 1982; Vergnaud, 1982). Previous empirical evidence supports that problems within the same semantic category vary in difficulty, since the placement of the unknown influence students' performance (Nesher et al., 1982; Adetula, 1989). In the present study we focus on one class of problems: one-step change (measure-transformation-measure) problems, which describe a transformation or a change (an increase or a decrease) in a starting situation, resulting in a final situation.

Although a vast amount of research have illustrated the important role of the semantic content of the problems on students' APS performance, only a limited systematic attempt has been undertaken to investigate APS from the perspective of representations. This paper reports on the second phase of a large-scale research project, which aims to investigate separately the influence of the mode of representation and the semantic content of a problem on pupils' APS performance. A report on the initial research phase was provided by Gagatsis and Elia (2004), who proposed a model involving four first-order representation-specific factors influencing APS ability. The present study attempts on one hand to validate the model of the previous phase indicating the significant effects of different modes of representation on APS and on the other hand to identify if the aforementioned model is altered by the implementation of two experimental programs.

In the light of the above, the purpose of this study is threefold: (a) to investigate whether the structure of the model proposed and validated by Gagatsis and Elia (2004) in the initial research phase needs to be modified due to the experimental teaching's outcomes, (b) to examine the effects of the experimental programs on pupils' APS performance, and (c) to explore the contribution of teaching so that different representations have a positive impact on pupils' APS performance. Correspondingly, three hypotheses are tested: (a) the general structure of the model in APS ability at the second research phase is expected to remain the same with the initial research phase's one. No differences are to be expected in the general structure of the effects of the different representations in APS ability between the pupils who were exposed to the experimental programs and the pupils who were exposed to instruction according to Cyprus curriculum; b) pupils who participated in the experimental programs are expected to exhibit significantly higher achievement levels in APS than the other pupils; and (c) pupils who received experimental teaching are likely to have a significantly higher performance at the problems, involving the mode of representation that was emphasized at the program they received, than the other pupils.

METHOD

The results of the initial research phase, as well as Duval's (2002) semiotic theory formed a basis to initiate the development of the two experimental programs at the second research phase. Experimental Program 1 promoted the flexible use of the informational picture (i.e. a picture which provides information that is essential for the solution of the problem) in APS, while Experimental Program 2 focused on the flexible use of number line in APS. Pupils who were exposed to Experimental Program 1 comprised Experimental Group 1 (EG1) and pupils who were exposed to

Experimental Program 2 formed Experimental Group 2 (EG2). These groups were compared on the basis of a test with pupils receiving a more classic instruction according to Cyprus curriculum, who comprised the Control Group (CG).

Participants

In total, 1491 pupils in Grades 1, 2 and 3 (6 to 9 years of age) of primary schools in Cyprus participated in the study. Particularly, 778 pupils (275 in Grade 1, 251 in Grade 2 and 252 in Grade 3) comprised the CG; 357 pupils (114 in Grade 1, 116 in Grade 2 and 127 in Grade 3) comprised the EG1 and 356 pupils (112 in Grade 1, 119 in Grade 2 and 125 in Grade 3) formed the EG2. The distribution of the pupils is presented at Table 1.

	Grade 1	Grade 2	Grade 3	Total
CG	275	251	252	778
EG1	114	116	127	357
EG2	112	119	125	356
Total	501	486	504	1491

Table 1: The distribution of the participants with respect to age and group

Procedure

The main activities that were involved in the experimental programs aimed at developing the following abilities: (a) Recognizing, understanding, solving and posing problems with the same mathematical structure in different modes of representation, i.e. manipulatives, informational picture (within Experimental Program 1) or number line (within Experimental Program 2) and written text, (b) analyzing and interpreting the informational picture or the number line in connection with the content of the problem, (c) coordinating different representations of the same mathematical situation, (d) comparing different representations with respect to their components, structure and use in APS, (e) transferring from one mode of representation to another, e.g., from informational picture or number line to written text and vice versa, for the same problem. Each program was comprised by four to five 40-minute instructional sessions over a week period. The classes that participated were taught by their teachers in a normal school environment. Before the instruction, teachers had received a series of seminars on the aims of the research project, epistemological background, the findings of the initial research phase, and the ways of implementing the experimental programs and the instructional material.

A test, consisting of 18 one-step change problems with additive structures (a+b=c) in different modes of representations was developed and administered to all the groups, after the instruction. The classification of the problems in the test and the symbolization used for them in terms of the data analysis are provided in Table 2.

Type of representation	Join situation (b>0) Placement of the unknown *Initial amount (α)*	*Transform-ation (b)*	*Final amount(c)*	Separate situation (b<0) Placement of the unknown *Initial amount (α)*	*Transform-ation (b)*	*Final amount(c)*
Text (verbal description)	VJa*	VJb	VJc	VSa	VSb	VSc
Text with informational picture	IJa	IJb	IJc	ISa	ISb	ISc
Text with number line	LJa	LJb	LJc	LSa	LSb	LSc

Table 2: Specification table of the problems included in the test

*Explanation of the symbolization: Symbols V, I and L at the first position stand for the mode of representation of the problem: V→verbal (written text), I→text with informational picture and L→text with number line; symbols J and S at the second position stand for the mathematical relationship of the problem: J→join situation and S→separate situation; symbols a, b and c at the third position represent the placement of the unknown: a→initial amount, b→transformation and c→final amount.

Each correct solution procedure (equation or description in words) and correct numerical answer were marked as 2, each correct answer or solution procedure as 1, and each incorrect answer and solution procedure as 0.

Data analysis

Structural equation modeling (SEM) was employed to test the first hypothesis of the study and more specifically the assessment of fit of the a priori model to the data of the second phase of the study. In particular, data were analysed by using Confirmatory Factor Analysis for the different groups of pupils, i.e. EG1, EG2 and CG. A SEM computer program, namely MPlus, was used to test the proposed model. In order to evaluate the extent to which the data fit the model tested, the chi-square to its degree of freedom ratio (x^2/df), the Comparative Fit Index (CFI), and the Root Mean Square Error of Approximation (RMSEA) were examined (Marcoulides & Schumacker, 1996). It is generally recognized that observed values for $x^2/df < 2.5$, for the CFI > .9 and for the RMSEA < .06 are needed to support model fit. To specify the possible influence of the experimental programs on pupils' APS performance a multivariate analysis of variance (MANOVA) was employed.

RESULTS

A multiple-group analysis was applied so that the structure of the model, analogous with the one of the initial research phase, could be fitted separately on each group (EG1, EG2 and CG). The model consists of three first order factors which are hypothesized to construct a second order factor. The first order factors stand for the three types of representational assistance used here, i.e. pupils' abilities in solving problems represented verbally (V), as an informational picture (I), and text accompanied by a number line (L). The second order factor represents pupils' general ability to solve one-step change problems of additive structure (APSAb). First order factors were measured by six tasks in the corresponding mode having the placement of the unknown at the final state, the transformation and the initial state. Figure 1 (see Table 2 for information about the tasks for each factor) presents the results of the elaborated model, which was found to fit the data reasonably well [$x^2(426)=1108.003$, CFI=0.942, RMSEA=0.057]. Thus, the first hypothesis of the study is verified since the structure of the second order model is considered appropriate for interpreting the APS ability for all the groups of the second phase, irrespective of their participation in an experimental program or not. Further evidence is provided for the assertion that was held at the first phase of the research supporting that apart from the semantics of the problem, the different modes of representation within the problem have a major role in APS.

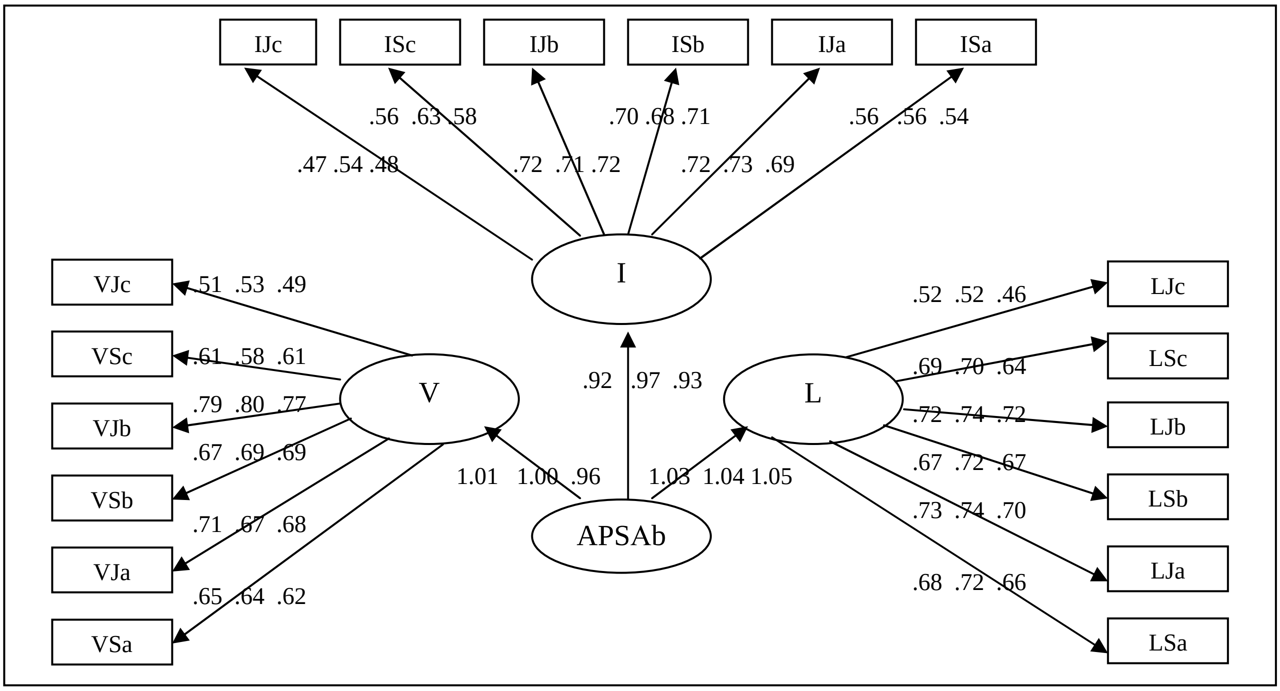

Figure 1: The elaborated model for the CG, EG1 and EG2

Note: VJc, VSc, VJb, VSb VJa, VSa, IJc, ISc, IJb, ISb, IJa, ISa, LJc, LSc, LJb, LSb, LJa, LSa = the items of the test, V=solution of verbal problems, I=solution of problems with informational picture, L=solution of problems with number line, APSAb=APS ability

The results of MANOVA verified the latter two assumptions of the study. In line with the second hypothesis the effect of the two experimental programs was significant {$F(2,1472) = 48.817$, $p<0.0001$, $\eta^2=0.062$} on pupils' APS ability. As Scheffe's analysis showed, the pupils of the EGs achieved better outcomes than the pupils of the CG ($\overline{X}_{CG}=1.394$, $\overline{X}_{EG1}=1.552$, $\overline{X}_{EG2}=1.524$), indicating a general improvement of pupils' performance in arithmetic problems in different representations and positions of the unknown, as a consequence of their participation in the experimental programs. The particular analysis demonstrated significant interactions between the position of the unknown and the group of the pupils (EG1, EG2 and CG) {$F(4,2944) = 11.413$ $p<0.0001$, $\eta^2=0.015$}, suggesting that the differences between pupils' groups varied with the different positions of the unknown. Next, employing one-way ANOVA revealed statistically significant differences between the EGs and the CG at all the problem categories with respect to the position of the unknown, i.e. problems with the unknown at the initial amount {F (2,1473) = 21.971, p<0.001}, at the transformation {F (2,1473)= 23.047, p<0.001} and at the final amount {F (2,1473) = 7.110, p<0.001}. These findings are illustrated in Figure 2 and verify in an analytic way the second hypothesis of the study, suggesting that the two experimental programs contribute to the improvement of pupils' performance at the problems irrespective of the placement of the unknown.

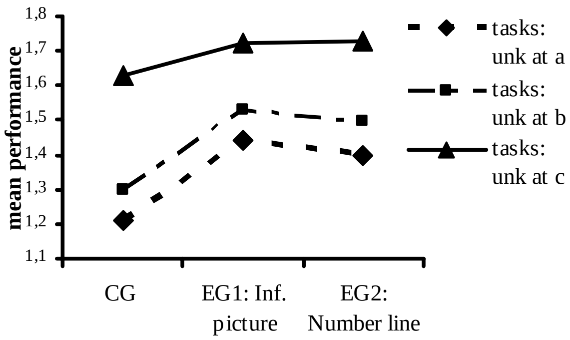

Figure 2: Mean performance of the CG, EG1 and EG2 at the problems with different positions of the unknown

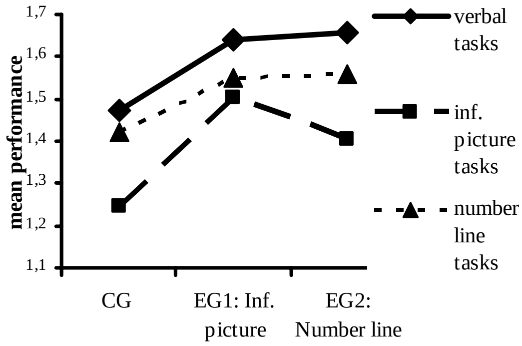

Figure 3: Mean performance of the CG, EG1 and EG2 at the problems with different modes of representation

Significant interactions between the mode of representation of the problems and the group of the pupils (EG1, EG2 and CG) {F (4,2944) = 9.217, p<0.0001, η^2=0.012} occurred, indicating that the differences between pupils' groups varied with the problems' representational mode. Subsequently, we employed one-way ANOVA, which revealed statistically significant differences between the EGs and the CG at all the problem categories with respect to the representational mode, i.e. verbal problems {F (2,1473) = 21.669, p=0.001}, problems with informational picture {F (2,1473) = 26.997, p<0.001} and problems with number line {F (2,1473) = 9.049, p=0.001}. The above findings are illustrated in Figure 3 and provide support to the third hypothesis of the paper, since pupils of the EG1 exhibited significantly higher outcomes at the solution of problems with informational picture, while pupils of the EG2 showed higher performance at the problems with number line, than the pupils of the CG. It is noteworthy that pupils of the EGs had significantly greater scores not only at the problems in the representational mode that each experimental program promoted, but also at the problems in other modes, providing further evidence to the second hypothesis of the study.

DISCUSSION

The findings of the study confirmed the strong effects of the representational modes of arithmetic problems on pupils' problem solving performance. A model, similar to the model that was elaborated in a previous research phase (Gagatsis & Elia, 2004), has been proved to be consistent with the data collected at the second phase of the project, a part of which is reported here. According to this model the abilities to solve problems in multiple representations, i.e. verbal description, text with number line and informational picture, influence APS ability. The particular model was validated even for two groups of pupils that participated respectively in two experimental programs, aiming at developing the abilities involved in the model and specifically emphasizing the use of the number line and the informational picture, respectively, and their coordination with other representations in the context of APS. This consistency of the model can be seen as a validation of the assumption that the ability to solve problems in various representations is an important component of APS.

Despite the invariance of the structure of the model for the EGs and the CG, some significant differences occurred as regards pupils' achievement levels. The pupils of the experimental group that participated in the program emphasizing the use of the informational picture in APS illustrated a higher performance at the problems with the corresponding representation than the CG. The analogous finding holds for the pupils of the group that participated in the experimental program focusing on the use of the number line in APS. Thus, support is provided to the importance of appropriate teaching so that representations have a positive role on pupils' APS performance. In this case, pupils need to develop the skills required for "reading" the informational picture or the number line and using them effectively in the solution process. These conclusions are in line with Klein's (2003) view that teachers should teach pupils how to use representations for learning, as well as the assertion of Dreyfus and Eisenberg (1990) that reading and thinking visually are so important that should not be left to chance, but learnt systematically.

This study illustrated the general success of both experimental programs. The superiority of the EGs relative to the CG applied not only to problems in the representation that was emphasized in each program, but to the solution of all the problems irrespective of their representation or position of the unknown. This finding indicates that the programs contributed to the general development of pupils' APS ability, justifying that developing the ability to link different modes of representation of a common mathematical situation is of fundamental importance in mathematical understanding and problem solving (Griffin & Case, 1997; Duval, 2002). Further evidence is provided to the significance of the aforementioned model, which formed a basis for designing the experimental instructions, for mathematics educators. It may offer a means to teach APS by using the semantic approach integrated with the perspective which refers to the diversity of representations and transferring information from one representational mode to another.

Finally, it is important for future research to take into account pupils' individual differences, i.e. having different thinking styles, either visual or verbal, and cognitive abilities, i.e. working memory, speed and control of processing, for validating the invariance of the model of the present study, and investigating the effects of the two experimental programs on APS performance.

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CAN MODERATE HERMENEUTICS HELP US TO UNDERSTAND LEARNING AND TEACHING IN THE MATHEMATICS CLASSROOM?

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This theoretical paper proposes that moderate hermeneutics, a theory about human understanding which sees interaction as the exchange of interpretations, has a contribution to make to research in mathematics education. Moderate hermeneutics is situated within a tradition of thought which often remains at an abstract level. This paper argues that the principles and processes suggested by moderate hermeneutics are useful in considering classroom data, and give us additional insights to those provided by theories currently in use.

TAKING THEORY TO THE CLASSROOM

As soon as research turns its attention to the classroom it begins to be overwhelmed by what it sees. The sheer number of 'players', interactions and influences makes the task of insightful analysis difficult. We must focus on some things to the exclusion of others, selecting and prioritising to build a coherent picture from the chaos of exchange that we observe. To do this we apply our theoretical framework, often characterised as a lens (Lerman, 2001) or viewer. The lens metaphor permits the idea of zooming in and out, focusing on the bigger picture or the detail. It also implies the framing of the subject – something always escapes the scope of the lens you select. The periphery remains un-photographed. However broadly we set the lens, or describe the context, there are always a multiplicity of factors left unconsidered and a multiplicity of factors left invisible, unseen and unrecognised. Our personal history as researchers frames our seeing and sets the agenda for the selection of a framework before we even take our lens to the classroom. When we try to see something as complicated as a primary classroom this has important implications.

Kieran, Forman and Sfard (2001) trace the history of the recognition of some of these factors in mathematics education research. Their editorial describes the move away from a focus on the concepts of the individual, to recognition of the essentially social nature of learning. They use the term 'discursive approach' to describe work which looks at the social construction of understanding in the classroom. Techniques such as discourse analysis and following concepts as they emerge through classroom interaction are tools that allow researchers to analyse changes in thinking. While this analysis of communication has brought about useful insights it does not acknowledge the layers of interpretation present when we 'look into' a classroom. Extracts of dialogue are analysed to show certain patterns of participation or the emergence of new language or ideas in the group. The researcher's interpretation of these is given, but the basis from which this interpretation arises is rarely explicated. Even our

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 33-40. Prague: PME.

selection of the classroom as a useful place to see learning indicates our theoretical tendencies.

Cobb and his colleagues (Cobb, 1989; Cobb, Wood, Yackel & McNeal, 1992; Yackel, 2001) use Blumer's (1969) sociological theory to look at sociomathematical norms, the rules by which reform classrooms operate. The norms of sharing ideas, having ideas open to constructive critique, making sure everyone understands and being accountable for group work make the discourse of the classroom a rich source of material for analysis. These norms themselves are also the product of interpretation by both child and teacher – what does the teacher want / what should I get the children saying? This fundamental 'coming together' of teacher expectations and child interpretations occurs in all classrooms.

With these concerns in mind we need a theoretical framework which acknowledges the complex interaction between what we think and what we do. It needs to account for factors relating to the children and teacher under observation and also for factors relating to the researcher themselves. We need to account for the individual nature and the social nature of human learning experience, while respecting the difference between what we can see and infer and what might be 'really' happening. In a setting with one teacher and many children, we need to account for miscommunication and failure to achieve shared understanding with everyone in a class or group. Children learn despite these misunderstandings. We need to have a theory which explains the creativity in children's responses.

MODERATE HERMENEUTICS

Hermeneutics is a way of viewing discourse and activity which has its origins in the analysis of written text. It is the study of interpretation. It is not, however, a consistent and united theory. Gallagher (1992) identifies four approaches associated with hermeneutics. He labels them conservative, moderate, radical and critical. Each has its own origin and implications, and while they are all concerned with interpretation they differ widely in the conclusions they draw and the methods they use. Statements made within one tradition of hermeneutics would be nonsense within another. Gallagher (1992) expands the theory of moderate hermeneutics (Gadamer, 1989; Heidegger, 1996; Ricoeur, 1981) towards education, and it is this theory that will guide the discussion here.

Beginning with the reader and the writer, hermeneutics considers the gap between the intention of the writer and the experience of the reader. Hermeneutics is concerned with narrative, with plot, with structure and with metaphor. It is interested in time and in the temporal and historical constraints and liberations allowed by the writing and reading of texts (Gadamer, 1989). Hermeneutics explores what it is to understand and how interpretation relates to understanding. For Heidegger (1996) and Gadamer (1989) interpretation arises from an understanding of the world developed over time. New concepts or objects are not considered in a void of understanding. Elements of understandings developed from our past experience project forward into new situations, creating a circularity between understanding, interpretation and the world.

This circularity draws past understandings forward. Our past understandings, which Heidegger (1996) and Gadamer (1989) term fore-conceptions, contain elements of the history of our ideas. Gadamer (1989) sees all interpretation as conditioned by this history, which he terms tradition. The hermeneutical circle of understanding and interpretation itself preserves and evolves traditions. These traditions both constrain our interpretation and provide a framework for seeing which enables us to make sense of what we observe. The hermeneutic task of interpretation exists in a tension between the familiar and the new.

This distance leaves a gap between our interpretation and the object we are trying to understand. Moderate hermeneutics takes an optimistic view of this gap and regards it as an opportunity for the projection of possibilities. These possibilities are constrained by tradition, but there is an opportunity for exploration in trying to reach for understanding.

Ricoeur (1981) takes this a step further and argues that spoken language is also ‘text’ and is subject to hermeneutic analysis. He then extends the idea of ‘text’ to ‘meaningful action’. Ricoeur (1981) explores action according to the criteria he lays out for a hermeneutic analysis of written text and concludes that hermeneutics is useful to the ‘human sciences’. He explores the traditional dichotomy of ‘explanation’ and ‘understanding’, where explanation pertains to non-human sciences where explanation is possible and understanding pertains to human sciences where we can interpret, but not explain. Hermeneutics is thus freed from reader-writer considerations alone, to the realm of understanding and explaining human actions and oral discourse. This makes it useful to education, where we are concerned with the discourse and activity of people in learning situations.

Gallagher (1992) begins the task of applying the philosophical tradition of moderate hermeneutics to education. He argues that the key principles of moderate hermeneutics also underlie education’s concerns and interests.

Three of the principles of moderate hermeneutics drawn forward by Gallagher (1992) make important links with the research traditions of mathematics education. The first is the hermeneutic circle, the recursive process of interpreting and explaining which ensures that we use what we already know to interpret what we see. Gallagher (1992) uses Piaget’s term ‘schema’ alongside ‘fore-conception’ to discuss the importance of prior knowledge and experience in conditioning our interpretations. This fits in with Piagetian notions of schema conditioning action and learning. Gallagher also uses the terms ‘assimilation’ and ‘accommodation’ in talking about how these fore-conceptions become altered by interpreting the world.

The second is the emphasis on language, which links to the discursive tradition outlined by Kieran et al (2001). Gadamer (1989) describes the primacy of language in a hermeneutic framework: “Language is the medium in which substantive understanding and agreement take place between two people” (Gadamer, 1989, p. 84). Gallagher (1992) states that in hermeneutics “…all interpretation is linguistic.”(p. 83). Language in discourse constrains and empowers interpretation.

The discursive approach focuses on the social nature of thought, and in doing so must emphasise language as the means by which this social sharing occurs. Kieran et al (2001) describe this as "…learning mathematics conceptualized as developing a discourse."(p. 6). Language is of key importance in both accounts.

The third synergy is that both moderate hermeneutics and the discursive approach see the classroom as culturally and historically situated. Interactions in the classroom are a product of time and place in both their content and the way in which they occur. For the discursive approach, this provides a background to the observation of teachers and children. In hermeneutics this idea extends to the creation and development of social and cultural traditions through interpretation. The element of time receives more emphasis in the hermeneutic account. Fore-conceptions used in interpretation ensure the carrying-forward of ideas and processes, and they project into the future through interaction. Classroom interaction not only exists within a cultural and historical context, it continues to create the cultural and historical context as it proceeds. This aspect of hermeneutics is helpful when we consider the historical development of mathematical ideas. The history of a concept gives us important information about how individual students learn it today, in a distant time and place. Hermeneutics provides an interesting account of why this is so.

In addition to sharing concerns and interests with themes of thought in mathematics education, moderate hermeneutics also adds some additional tools for thinking about mathematics learning. Firstly it helps in considering the personal and social dimensions of learning and secondly it helps to account for creativity and the emergence of individual creative thought through seemingly unrelated discourse. The notions of the hermeneutic circle and a productive gap between the interpreter and the interpreted have been introduced above. Gallagher (1992) moves these ideas from their text-based origins to the classroom.

Lerman (2005) contrasts constructivist and socio-cultural analyses describing them as 'parallel discourses' (p. 180). This personal/social dichotomy is linked to the Piagetian/Vygotskian research traditions respectively. Attempts to break down this dichotomy have been made by the use of sociological theory (Blumer, 1969) and multiple lenses (Lerman, 2001). The notion of foregrounding one perspective while holding the other 'in mind' as a significant background is common. However this conceals many important decisions – how was the 'foregrounded' perspective selected? What is happening in the background where we can't see? How can we allow 'the background' into our reported data when it becomes momentarily significant and still keep our 'lens' focused? These and many other questions of methodology and explanatory value are raised by the attempt to simultaneously account for the personal and the social. Clearly both types of information yield fascinating patterns and tell interesting stories. In moderate hermeneutics both these stories can be explored by considering learning as the exchange of interpretations – both of the situation and the subject. Ricoeur's (1981) extension of hermeneutics from written text to meaningful action and oral discourse allows Gallagher (1992) to say that "…the interchange of learning in the classroom situation is an interchange of interpretations" (p. 35)

The interpretations exchanged in the classroom are multifaceted. Consider a child's response to a question from the teacher. The response is an interpretation in several senses. It includes the child's interpretation of what the teacher has asked, what counts as an answer to these types of questions, the sort of language that would be appropriate, and the depth of answer required as well as the child's interpretation of the content of the question – what the question means, what they know about the answer, their current explanation for this concept and so on. The response is neither a complete match for what may be 'in the child's head' nor a totally social construction. As an interpretation, located in time and place, it contains both the personal and the social. The hermeneutic circle, which operates at several levels within classroom exchange, ensures this.

The hermeneutic circle is the recurring pathway from experience to personal explanation, formed by interpretation. Although it is circular it does not imply a static state, where ideas are 'going around' but not changing. We interpret experience through the framework of our 'fore-structures', and then alter our fore-structures in the light of our interpretations. Both the teacher and the child enter an exchange with fore-structures in place which condition the interpretive process from the beginning. The hermeneutic circle is operating within each person. In addition, the cycle of interpreting – developing or altering an explanation – acting in the world – interpreting occurs inter-personally between the teacher and students. The teacher's presentation and the students' understanding never coincide completely, keeping the hermeneutic circle open as each seeks to understand the other (Gallagher, 1992). Discourse and the social sharing of ideas are essential here, bringing the social into play as constituted by, and constituting, the personal interpretation of the event.

The second way in which a moderate hermeneutic analysis can add to what we see in the classroom is by providing an account of creative and original thought arising from the interpretive process. A Piagetian view of learning sees children making new discoveries, but along a predictable track. They discover what we already know; come to think as we do. The Vygotskian view considers the child as an apprentice to an expert, developing performance in socially approved ways, rather than creating new ways. Discourse analysis of classroom transcripts allows researchers to map the 'history' of an idea through a discussion, charting its emergence and development. Individual children's understandings may not match this socially-generated explanation, however. Moderate hermeneutics uses the idea of 'distanciation' to account for this.

> Interpretations…never simply repeat, copy, reproduce, reconstruct or restore the interpreted in its originality. Interpretation produces something new.
>
> Gallagher, 1992, p. 128

There is a distance between the interpreter and the interpreted. This distance creates a productive gap, a gap which produces something new. It is a space where the fore-conceptions of the interpreter interact with interpretations or objects, and in trying to understand, produce changes in conceptions and new ideas. We are operating within a tradition as we experience distanciation, and not all of the aspects of the tradition are transparent to us, so we cannot project all possibilities, only those which appear available. This has explanatory value in the classroom as we see the possibilities constrained by the presentation of the material and by the ‘class rules’ about how things are learned. As children interpret these traditions differently, their perception of possibilities differs.

Gallagher (1992) illustrates persuasively that hermeneutics adds to our understanding of education and learning. Brown (1994, 2001) considers hermeneutics more specifically, in the context of mathematics education. His key concern is with ‘school mathematics’ and how it can be understood. Brown (2001) tends towards a more radical hermeneutic analysis than Gallagher (1992), but his classroom examples utilise some of Gallagher’s principles. Brown’s (2001) interest in language use in the classroom leads him to consider the ‘spaces’ that open up during communication or activity and in which individual interpretations are made. This analysis of spaces extends Gallagher’s (1992) discussion of the productive gap produced through the principle of distanciation.

HERMENEUTICS AND THE CLASSROOM

How can we use hermeneutics to understand what is happening for learners in a classroom? How does it change what we see and what we emphasise? Can we apply hermeneutics to specific instances in a useful way?

Five elements of moderate hermeneutics can add to our understanding of the teaching and learning of mathematics. Within the framework of moderate hermeneutics we are looking at classroom interaction and processes as the exchange of interpretations. Interaction is not giving us a blueprint of invisible cognition; it is necessarily mediated by the social and cultural context in which the interpretations are exchanged. Thus moderate hermeneutics can help us to consider the personal and social dimensions of learning at the nexus of interaction and activity.

Secondly the notion of the hermeneutic circle fits with and extends our knowledge of schema/prior knowledge a significant contributor to future learning. Each interaction with the world is a selected interpretation of the situation and the fore-conceptions held by the participants.

Tradition processes emphasise the history of ideas and help us to explain and understand how ideas persist and resist change as we work in classrooms. Identifying and investigating these traditions can focus our efforts in altering thinking or practice.

Distanciation, and the idea of a productive gap between interpreter and interpreted, gives us a mechanism for looking at idiosyncratic or creative responses and for understanding how learners may develop understandings about concepts that were not the target of instruction through their participation in classroom processes.

Moderate hermeneutics shares an emphasis on the significance of language with discursive approaches. Extension of moderate hermeneutics from text to oral language and 'meaningful action' (Ricoeur, 1981) includes activity and interaction with objects in a broad consideration of what 'language' is.

CONCLUSION

Moderate hermeneutics provides us with an additional view of teaching and learning; one which incorporates valued ideas and methodologies while challenging us to consider what we see in a different light. Lester (2005) suggests we act as *bricoleurs* in using theoretical sources to inform our view of teaching and learning. Moderate hermeneutics has a role to play in such a process. As a theory it suggests a methodology which produces a rich view of classroom processes. It also accounts for the 'common sense' experiences of teaching and learning which underlie many research questions – that we learn in a context, but carry our learning with us, that sometimes we keep quiet even though we know the answer, that sometimes we struggle to interpret a situation or an idea and so forth. Moderate hermeneutics can help us to delve more deeply into the experience of learning and teaching in mathematics classrooms and enhance our understanding of how people learn.

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SOCIOCULTURAL APPROACHES TO EMOTIONS IN MATHEMATICS EDUCATION: INITIAL COMPARISONS

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Given the growing interest of sociocultural approaches in the area of affect and emotion in maths education, the time is ripe for a consideration of the main positions developed – and also for initial comparisons aiming to highlight differences and commonalities, and to explore possibilities for the different approaches to challenge, and support, each other's development. Our motivation is to help avoid the simple accumulation of theoretical frameworks in mathematics education research, and to suggest common directions for research. To this end, we compare an 'exemplar' of recent work from each of three main sociocultural approaches.

AFFECT AND EMOTION IN MATHEMATICS EDUCATION

Since the mid-1990s, there have been a number of key developments in research on mathematical affect, since McLeod's key contributions, (1992; McLeod & Adams, 1989). He argued for the importance of *transitory* emotions, experienced during the *process* of problem-solving, rather than being restricted to measures of 'durable' attitudes and beliefs, the focus of most previous research in mathematics education.

Recently, several broadly 'sociocultural' approaches have developed, to challenge the psychological emphasis on individual characteristics, and stress the *social organisation* of affect. Three can be distinguished: (a) Socio-constructivist: see e.g. Cobb et al. (1989 in McLeod and Adams), Op 't Eynde et al. (forthcoming); (b) Discursive practice approaches: see e.g. Walkerdine (1988); Evans (2000); Evans, Morgan, Tsatsaroni (forthcoming); (c) Cultural - Historical Activity Theory: see Roth (2004, forthcoming).

Overall, a trend can be discerned towards an emphasis on emotion, rather than beliefs and attitudes, as in previous periods. There are a number of reasons for this. It allows description of any affect-laden activity as a *dynamic process*. The activity can be described *in context*. And the more volatile emotions can be seen as the basis for more durable attitudes and beliefs. This trend has been reinforced by the widespread interest in the biological bases of emotion, in the light of the neuroscientific work of Damasio (1996) and others.

A broad range of social scientists, psychologists, sociologists and psychoanalysts agree on three key aspects of emotional states: (a) *bodily processes*, not just the brain, but also nerves and organs (e.g. heart, stomach); (b) *behavioural* (including verbal) *expression*; and (c) *subjective experience* or 'feeling' (Evans, 2000, pp. 112-3).

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 41-48. Prague: PME.

Despite the initial appearance of these aspects of emotion as individually based, on reflection, (b) and (c) at least can be seen as learned in social settings by human beings. Certainly, there much evidence that they are lived very differently, in different cultures and social groups. Thus it is reasonable to conclude that emotional expression and experience are embedded in a social context, and thus can be seen as *socially organised* (by prevailing beliefs, norms, etc.) − just like thinking or learning.

Thus it seems useful at this point to examine more closely the sociocultural frameworks that are developing in mathematics education, with these aims:

1. to highlight differences and commonalities, exploring possibilities for the three approaches to challenge (and possibly support) each other;

2. to explore ways of avoiding the simple proliferation of additional 'segments' (theoretical approaches) (cf. Lerman et al., 2002);

3. to suggest (or to note) some further directions for research.

SOCIOCULTURAL APPROACHES TO EMOTIONS

In the following sections we will illustrate recent sociocultural work on emotions, grouped under three heads: socio-constructivism (SC); discursive practice (DP) approaches; and cultural-historical activity theory (CHAT). Given our aims, in order to examine the three approaches to emotion in mathematics education, we will use specified categories and indicators for these categories, drawing on Schoenfeld's (2002) and Lerman et al.'s (2002), despite our somewhat different aims. Given space limits, we compare the three sociocultural traditions by referring mainly (but not exclusively) to one 'exemplar' from each. The articles are:

DP: Evans, Morgan and Tsatsaroni, 'Discursive Positioning and emotion in School Mathematics Practices' (forthcoming)

SC: Op 't Eynde et al., 'Accepting emotional complexity: a socio-constructivist perspective on the role of emotions in the mathematics classroom' (forthcoming)

CHAT: Roth, 'Motive, Emotion and Identity at Work: a Contribution to Third-Generation Cultural Historical Activity Theory' (forthcoming)

Our categories are:

1. *Conceptual framework*: We will use as indicators: a) the *key concepts* of the basic conceptual framework of the approach, (b) the characterisation of *emotion*.

2. *Problems addressed*: (c) the *problems* motivating the research; (d) *outcomes* (Lerman et al., p. 37), or use of approach: applying theory only / revisiting theory and expressing support / dissatisfaction for theory; revisiting theory and revising it.

3. *Methodology*: (e) *key phases* in the research; (f) *preferred research methods* of data collection and analysis.

DISCURSIVE APPROACHES

Research report: Evans, Morgan and Tsatsaroni (forthcoming; see also Evans, 2000; Morgan et al., 2002).

Key Concepts: discourse, practices, positioning in practice, identity. Discursive approaches focus on specific social / institutional *practices*, which are recurrent forms of behaviour / action. A *discourse* then is the system of ideas / signs organising and regulating the related practices. Discourse defines how certain things are represented and thought about, and helps to construct identities and subjectivities (which include affective characteristics and processes).

A key concept is that of *positioning,* a process whereby an individual subject takes up and/or is put into one of the *positions* which are *made available* by the discourse(s) at play in the setting. Thus the approach allows for a mutual influencing of social and individual: the social setting makes available specific practices, and individuals retain agency, to strive to position themselves in available (or 'created') positions. The social produces other effects: different positions are associated with membership of different social groups (class, gender, ethnicity), and with different degrees of power. In this approach, a person's *identity* comes from repetitions of positionings, and the related emotional experiences. Here, the authors show how (more durable, less context-specific) *identities* are produced in this way: one boy in a small group, Mario, from repeated positionings becomes 'identified' as weak in problem solving.

Emotion is visualised as a charge attached to ideas and terms in which they are expressed. This charge has a physiological, behavioural (including verbal), and a subjective aspect (see above). This allows emotion to be seen as attached to ideas (cognition), but in ways that are fluid, not fixed. Some of this fluidity can be seen as related to psychic processes of 'displacement', where feelings flow along a chain of ideas (or signifiers) and 'condensation' (Evans, 2000).

Problems addressed: This paper aims to 'show emotions as socially organised', within a structure of social relations where power is exerted (including that of the media and policy-makers). In practical terms, it aims to *sensitise* teachers, teacher educators and policy makers to the (often neglected) importance of emotions in the learning (and use) of mathematics.

Outcomes: The authors apply their theory to a 'critical case': classroom data, not originally collected for studying emotion, and involving several students, and argue that the results support a wider scope of use for the theory than originally thought.

Methodology: Two *key phases of analysis* draw on the interdisciplinary approach:

1. Structural: uses Bernstein (2000)'s sociology of education to show how pedagogic discourse(s) *make available* particular positions; for example, the discourses at play in school invariably include evaluation practices, which *make available* positions of *evaluator* and *evaluated*. The 'official discourse' (often 'traditional') is contrasted with 'local pedagogy' (in this classroom, relatively 'progressive'), where students may be encouraged to evaluate each other's work. Of course, other discourses from

'outside' or from the peer-group are also at play in the classroom. Conflicts between expectations of two practices may elicit emotion.

2. Textual: 2A shows how positions are *actually taken up by* subjects as positionings, in social interaction, itself represented as *text* (e.g. transcripts); e.g. in formal education, claims to know or to understand are powerful.

2B1 points to indicators of emotional experience understood within the subculture: e.g. verbal expression of feeling; behavioural indicators (tone / pitch of voice);

2B2 points to indicators of emotional experience suggested by psychoanalytic insights, mainly indicators of *defences* against strong emotion (anxiety, conflicts between positionings (see above; Evans 2000)), such as 'Freudian *slips*', e.g. a 'surprising' error in problem solving, *denial* (e.g. of anxiety ('protesting too much')).

Preferred research methods: This study analysed transcripts of classroom interaction (see also Walkerdine, 1988). Other studies have used interviews with teachers or students (Morgan et al., 2002; Evans, 2000) or questionnaires (Evans, 2000).

SOCIO-CONSTRUCTIVIST APPROACHES

Research Report: Op't Eynde et al. (forthcoming, see also Op't Eynde, pp 2004).

Key Concepts: practice; participation; context; situated; beliefs; motivation. By engaging in the *practices* of a community people discover meaning. Students' learning is perceived as a form of *engagement* that enables them to realise their identity through *participation* in activities *situated* in a specific *context*. Students' mathematics-related *beliefs*, together with mathematical knowledge, underlie students' understanding of and functioning in the mathematics classroom.

Emotions are conceptualized as consisting of multiple processes (Scherer, 2000), which mutually regulate each other over time in a particular context. These processes are characteristic of five different systems: the cognitive system, i.e. appraisal processes; the autonomic nervous system, i.e. arousal (affect); the monitor system, i.e. feeling (affect); the motor system, i.e. expression (action); the motivational system, i.e. action tendencies (action). Emotions are seen as social in nature and situated in a specific socio-historical context, because of the social nature of an individual's knowledge and beliefs (which play a role in appraisal).

Problems addressed: Analysing the relation between students' mathematics-related beliefs, their emotions, and their problem-solving behaviour in the mathematics classroom.

Outcomes: The research highlights methodological implications of the theoretical framework presented, e.g. the need to study learning and problem solving in the classroom, and to take into account the different component systems constituting an emotion. Given the close relation between emotions and beliefs, investigation of students' emotions can enhance understanding of their beliefs and therefore behaviour. This methodological approach is applied and illustrated with the data set.

Methodology: Two *key phases* in data collection: First, a beliefs assessment, where the students were presented the Mathematics-Related Beliefs Questionnaire (MRBQ). Second, problem-solving behaviour and interviews: students are asked to solve mathematical problems, and a sequence of measures is taken: 2A) On-line Motivation Questionnaire (OMQ), after the students had skimmed the problem, before actually starting work; 2B) Videotaped “Thinking aloud” during problem solving; 2C) Immediately after finishing, an interview procedure using a Video-Based Stimulated Recall Interview.

The analysis of the data can be divided into three *key phases*. First, for each student, researchers used the different data sources to describe the different experiences and activities the problem-solving process, thereby producing narratives. Next, the narratives were content analysed. Third, the data were reanalysed (cyclic procedure) to unravel and explicate relations between students' task-specific perceptions (OMQ), students' mathematics-related beliefs, and their problem-solving behaviour.

Preferred research methods: As indicated above, the study adopted a multiple approach to collect data, involving protocols and video tapes of problem solving episodes, questionnaires, interviews. The analysis of these data includes coding emotions through existing systematic coding systems.

CULTURAL - HISTORICAL ACTIVITY THEORY (CHAT)

Research Report: Roth (forthcoming; see also 2004).

Key Concepts: socially organised activity, action, operations, tools, motivation, identity. The *context* for any action is the *activity* in which the subject is engaged; this has inevitably a social aspect. The basic elements of an activity include *subject*, *object*, *tools*, *community*, *rules* and *division of labour*. Activities are oriented toward collective motives, which have arisen in the course of cultural historical development; they are organised in a trilogy of activity / action / operation (the latter ‘unconscious’).

Emotions in this approach come from the body, as described by Damasio (1996), whose findings on the integral role of emotions in decision making are referred to. Emotion is seen as ‘integral to practical action’ in two ways: first, ‘the general emotional state of a person shapes practical reasoning and practical actions’; second, practical action is generally directed ‘toward increases in emotional valence’ (Roth, forthcoming). ‘Emotional valence’ appears to equate to levels of pleasure, rather than pain; however, an increase in emotional valence is sometimes meant to indicate an increase in ‘room for manoeuvre’ (a greater choice of actions to choose from) or to being ‘better off in the long run’ (*ibid*.).

Emotion is seen as a crucial basis for *motivation* and *identity*, which derive from it: 'motivation arises from the difference between the emotional valence of any present moment and the higher emotional valence at a later moment to be attained as a consequence of practical action.' Identity is related to an individual's participation in collective activity, and to the ‘recognition’ received as a member of the community;

this relates to individual and collective emotional valences arising from face-to-face interaction with others.

Problems addressed: This paper aims to extend the relatively cognitive approach of 'CHAT' to encompass emotion, motivation and identity – and to provide evidence of the need for that. This is to provide the basis for a fuller explanation of performance, notably mathematical thinking and modelling, at work (Roth, 2004).

Outcome: This study revisits CHAT theory, and contributes to a significant revision.

Methodology: The first *key phase* of this study was Roth's full-scale ethnography of a salmon fish hatchery in Western Canada. When the author decided his claims about emotions required more convincing indicators, this was supplemented by systematic work on speech intensity and pitch. The *preferred methods* are thus participant observation and systematic behavioural measurement.

CONCLUSIONS AND DIRECTIONS FOR FURTHER RESEARCH

1. All three sociocultural approaches considered, using exemplary studies, view mathematical thinking as 'hot', infused with emotion. In terms of *key concepts*, the socio-constructivists (SC) understand emotions as related to two of the organism's 'component systems', and highlight the role of appraisal in emotional production, and the effect of knowledge and beliefs on this appraisal. The discursive practice (DP) approach sees emotion as an affective charge which may be attached to ideas (carried by signifiers), and shows how a range of emotions are associated with each subject's positioning in practices, and especially conflicts in positioning. The cultural - historical activity theorists (CHAT) see the person's emotional state as dependent on the physiological, and as reciprocally related to practical reasoning and action.

1(a) Despite differences in key concepts, all three approaches stress the importance of social, the 'context' of learning. The SC conceptualisation captures this via careful measurement of knowledge and especially beliefs. The DP researchers see a person's positioning within discursive practices as constituting the context. CHAT researchers see activity within a community (located culturally and historically) as the context.

1(b) These accounts no longer see emotions towards mathematics as largely 'negative', but often show them as 'positive' (or even ambivalent, e.g. due to positioning conflicts in DP). As is needed to deal with dynamic processes, all have methods for capturing the fluidity of emotion (e.g. "Thinking aloud" while problem solving in SC, detailed semiotic analysis in DP, speech intensity and pitch in CHAT).

2. Comparison on *problems addressed* reveals similar motives for including emotions in the theoretical framework, such as the need of a better understanding of an individual's mathematical behaviour, and its relation to social factors. Comparison about *outcomes* (as defined above) shows somewhat different uses of the study for theory development.

3. Comparisons on *methodology* reveals multi-phase, multi-method procedures, which differ in specific ways (described briefly above) among approaches. A range

of methods has been used, including self-completion questionnaires; systematic behavioural measurement; several types of interview; participant observation.

4. Do the commonalities and differences among approaches suggest any ways of avoiding, or attenuating, a proliferation of approaches to the study of emotion in mathematics education? Here, we can only aim to pose several provocative questions.

4(a) Is there any overlap in the key concepts used, that might allow 'fruitful mutual challenges'? For example, in what essential ways do 'activities' (CHAT) and 'discursive practices' (DP) differ as a context of thinking? Does CHAT have an analogue of 'positioning'?

4(b) The term 'unconscious' is used in three distinct senses here: (i) routinised, not needing conscious attention, as with *operations* (in CHAT); (ii) 'autonomic' as for physiological processes, such as the heartbeat; and (iii) repressed via defence mechanisms into the (Freudian) unconscious (in DP). These need distinguishing.

4(c) Psychoanalytic insights pose a challenge to any strongly cognitivist point of view, that emphasises thinking as largely 'conscious' and normally bound by rationality. This is because many emotional reactions, and even beliefs, including those relating to mathematics (etc.) are often not conscious, much less rationally arrived at. Feelings like anxiety can be *displaced* to mathematical objects from others, via movement of emotional charge along a *chain of signifiers*: so what seems to be 'mathematics anxiety' may relate to anxiety from other practices. Thus emotion may *transfer* across practices (Evans, 2000), like ideas, perhaps having originated in early relationships, or in images in popular culture (e.g. films).

5. This discussion opens several areas for further research. First, both 'motivation' and 'identity' have been marked here as of interest in the affective area; the first in particular has been neglected in mathematics education research until recently. Second, each of these studies offers suggestions as to how to rethink the links between beliefs and attitudes seen as durable aspects of individual 'identity', and transitory emotions. Third, the DP approach especially suggests studies of the ways that popular culture has effects on emotions, e.g. using representations of mathematic(ian)s in films. Fourth, Roth's study of working adults raises the issue of child vs. adult differences in affective patterns and emotional experience. Finally, the sociocultural approaches together raise questions like: (i) When should there be an emphasis on "enjoying maths" in class – and when not? (ii) Should educational policy makers try to control emotions in schools, or require teachers to develop students' 'emotional literacy'? In an 'emotionally literate' classroom, which students (gender, class) stand to gain / lose?

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MORAL EDUCATION IN THE TEACHING OF MATHEMATICS

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This paper investigates the question how the curricular teaching of mathematics can provide an opportunity for intentional and explicit moral education. The linking of moral education to the teaching of mathematics is motivated, and it will be suggested that human morality is imaginative in nature and moral education is to be about developing students' moral understanding and moral imagination. Drawing on three approaches to humanized mathematics education, possibilities within those approaches are explored for providing students with the opportunities to develop their moral understanding and moral imagination and, thus, provide the opportunity for moral education in the curricular teaching of mathematics.

THE ISSUE

Classroom teaching has a moral dimension, whether it is recognized and intentionally addressed or not (Buzelli and Johnston 2002, Hansen 2001). Jackson et al. (1993, part 1), for instance, distinguish between eight different ways in which classroom teaching influences the moral life in classrooms, at least five of which are subject matter independent. In this sense, teaching of mathematics as other subject matter teaching is (intended or not) moral education. The question to which the answer is less clear, and which is at the centre of this paper, is whether the *curricular* aspects of teaching mathematics can provide a basis for intentional and explicit moral education. The challenge of systematically connecting the teaching of the *curricular* aspects of mathematics education and moral education is that mathematics education does not seem to allow for the consideration of moral content. Literature studies and social studies provide a fertile curricular ground for the inclusion of moral content and, hence, moral education, however, mathematics is often understood and experienced "as a depersonalized, uncontextualized, non-controversial and asocial form of knowledge" (Brown, 1996, p. 1289).

Underlying the suggestions made in this paper is the central assumption that moral education (at least in the way suggested here) is a worthwhile endeavor in schooling in general and in classroom teaching in particular, and that it, thus, should influence curricular subject area teaching. This assumption is sensible for at least the following two reasons. First, if general schooling in general and classroom teaching in particular have a moral dimension, implying that it has an impact on the moral development of students, then purposefully influencing that development through intentional moral education seems a sensible thing to do. That applies to schooling and teaching in general as well as to subject matter teaching. Second, saturated in intentionality, human action and experience is inevitably moral in character,

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 49-56. Prague: PME.

because "intentions include desires and motives that go beyond needs, to encompass notions of what it is good to be (in character) and do (in conduct)" (Martin & Sugarman 1999, p. 44). Education that has as at least one of its purposes to engage students in and prepare for the different aspects of life, then, needs to address this moral aspect of being human. Considering the dominance of subject matter teaching in students' formal education, subject matter teaching should contribute to this purpose of education.

What is suggested here is an integration of the larger purpose of schooling (moral education) into subject matter teaching, which includes the teaching of mathematics. However, this does not imply neglecting or ignoring other, subject matter specific purposes of the teaching of mathematics like the development of mathematical literacy (see, for instance, NCTM, 2000).

In the following moral education is linked to the curricular teaching of mathematics in two steps. First, a view of morality is adopted according to which moral understanding and moral imagination are front and centre in human moral functioning, and according to which moral education is centrally about the imaginative exploration of one's prototypical moral concepts and moral metaphors. In the second step, approaches within the tradition of humanized mathematics education are used to suggest opportunities for students in the curricular teaching of mathematics to engage in imaginative explorations of their moral concepts and metaphors.

MORAL EDUCATION: DEVELOPING MORAL UNDERSTANDING AND MORAL IMAGINATION

There are two dominant approaches to moral education in schooling in at least North America. One understands itself as being in the tradition of Ancient virtue theory (Aristotle, trans. 1976) and is generally referred to as character education (Lickona, 1991), while the other one is in the tradition of Piaget's (1997) and Kohlberg's (1971) cognitive approach to the development of moral judgment and reasoning (Edelstein et al., 2001). The first group of moral education approaches focuses on the development of particular 'virtues' like honesty, responsibility, fairness and ranges from a view of moral education as social control to the view of moral education as the development of virtue-based moral agency. The second group, on the other hand, focuses more or less exclusively on influencing the development of moral reasoning of students, assuming that a person's moral reasoning capacity impact accordingly on the person's conduct.

A quite different approach to morality has been recently pursued by Mark Johnson (1993, 1996, 1998), who, based on work in cognitive science (Lakoff & Johnson, 1999), has proposed a theory of morality that puts *moral understanding* and *moral imagination* at its centre. Johnson (1993, p. 198) summarizes his approach as follows:

> A theory of morality should be a theory of moral understanding. Moral understanding is in large measure imaginatively structured. The primary forms of moral imagination are concepts with prototype structure, semantic frames, conceptual metaphors, and narratives. To be morally insightful and sensitive thus requires of us two things: (1) We must have knowledge of the imaginative nature of human conceptual systems and reasoning. This means that we must know what those imaginative structures are, how they work, and what they entail about the nature of our moral understanding. (2) We must cultivate moral imagination by sharpening our powers of discrimination, exercising our capacity for envisioning new possibilities, and imaginatively tracing out the implications of our metaphors, prototypes, and narratives.

As theories of physics provide us with an understanding of gravity, force, etc. rather than tell us how to build bridges, theories of morality – Johnson (1993, p. 188) argues – should provide us with an understanding of human moral functioning rather than tell us how to live a virtuous life. It is the *imaginative exploration* of our prototypical moral concepts (like fairness) and our moral metaphors (‘an eye for an eye’) in specific moral situations that is the basis of our moral functioning as human beings. At the same time, it is these imaginative explorations that help us *develop further* our understanding of human moral functioning in general and our own idiosyncratic moral functioning in particular. Moral education – the intentional and purposeful influencing of moral development – within this framework focuses, then, on the development of moral understanding through the imaginative exploration of one’s prototypical moral concepts and one’s moral metaphors.

Adopting Johnson’s approach to morality and moral education, the question, then, is whether the *curricular* aspects of teaching mathematics can help develop students’ moral understanding and moral imagination by providing opportunities for the imaginative exploration of their prototypical moral concepts and their moral metaphors. The rest of the paper addresses this very question.

THE FRAMEWORK: HUMANIZED MATHEMATICS EDUCATION

As long as mathematics is understood, practiced and experienced “as a depersonalized, uncontextualized, non-controversial and asocial form of knowledge” (Brown 1996: 1289), it is hard to see how the curricular teaching of mathematics can provide an opportunity for developing moral understanding and moral imagination. Even if this is by far the most dominant view of mathematics underlying – often unarticulated – the teaching of mathematics, there are calls in the academic and professional mathematics education community to give consideration to the ‘human element’ in mathematics. This ‘humanizing of mathematics education’ has taken different forms with different foci. For instance, for some the focus is on the ‘human’ nature of mathematics (Hersh, 1997; Lakoff & Núñez, 2000), some suggest a focus on the social responsibility of mathematics and mathematics teaching (Skovsmose & Valero, 2001) and others, again, suggest different ways of ‘humanizing’ the curricular teaching of mathematics (Brown, 1996; Freudenthal, 1968; Katsap, 2002; Wheeler, 1975). But what all these approaches to mathematics education have in common is the view that mathematics is (to at least one part) something *humans do*, *a human activity* and, thus, that mathematics has a social context.

Within this broader class of approaches to humanized mathematics education it is in particular these latter approaches to humanizing the curricular teaching of mathematics that promise the most in terms of providing opportunities in the teaching of mathematics for the development of moral understanding and moral imagination. The next section explicates some of those opportunities.

DEVELOPING MORAL UNDERSTANDING AND MORAL IMAGINATION IN THE TEACHING OF MATHEMATICS

In the following, I extract out of the class of the above given approaches three particular ones and explicate the possibilities within those approaches for students to imaginatively explore their prototypical moral concepts and their moral metaphors, and, thus, help developing their moral understanding and moral imagination within the curricular teaching of mathematics.

History of Mathematics

The first approach in the humanizing of the curricular teaching of mathematics suggests incorporating historical aspects of the development of mathematics with particular focus on the life and contributions of mathematicians into the teaching of mathematics (Wheeler 1975, Katsap, 2002). Here, the 'human element in mathematics' is the mathematicians as the doers of mathematics and contributors to the mathematical science. Wheeler (1975, p. 6) characterizes the purpose of this form of humanization as follows: "These questions [the questions dealt with in a history of mathematics approach to the humanization] are concerned with an enlargement of our experience and our understanding through vicariously sharing the experience and understanding of others."

Suggestions to include historical aspects into the curricular teaching of mathematics are generally limited to linking those aspects to respective mathematical content (Eves, 1969; NCTM, 1969; Fauvel, 1991). For the purpose of providing students an opportunity to imaginatively explore their prototypical moral concepts and metaphors I suggest to expand in the following way on what Wheeler in the above quote has already hinted at. I suggest including the 'human aspects' that can be extracted from the historical accounts. For instance, the legend about the Pythagoreans killing one of their own because he proved that irrational numbers exist (Pappas 1997) provides an opportunity to engage in imaginative explorations of our understanding of human intentions, fears, and emotions, and our own prototypical understanding of moral concepts like motivation and justifications for killing, etc. The moral deliberation would include the imaginative exploration of how far we would go to protect an idea that is as important to us as the idea that all numbers are rational was to the Pythagoreans. Human emotions and their functioning are very central to our moral understanding (Johnson, 1993). Here, the teaching of the concept of irrational numbers as part of the mathematics high school curriculum through historical references provides an opportunity for students to engage in moral deliberation through imaginative exploration of moral concepts and understanding of human moral functioning.

Mathematization of Life Experiences

The second approach in the humanizing of the curricular teaching of mathematics suggests moving away from understanding mathematics education as the transmission of mathematical procedural and factual knowledge and to understand mathematics education in the way that students are 'doing mathematics' with the goal of developing skills and habits of mind for students to '*mathematize*' their life experiences (Wheeler, 1975, Freudenthal, 1968). Here, the 'human element' is the students' intellectual capacity to use mathematics to make sense of their life experiences. Wheeler (1975, p. 6) explicates his specific idea of the 'mathematizing' capacity as follows: "In a crude attempt to make explicit the nature of mathematisation, I would include the following ingredients: the ability to perceive relationships, to idealise them into purely mental material, and to operate on them mentally to produce new relationships" (see also Gravemeijer & Terwel, 2000 on Freudenthal's view of mathematization). Accordingly, the purpose of this kind of humanized mathematics education is the development of students' mathematical competency in a way that allows them to use mathematical conceptualizations to understand at least part of their world, to see this part of their world through the eyes of mathematical relationships.

In this approach to the humanization of mathematics education, areas and issues from the world experiences of students are chosen to illustrate, develop with or practice the mathematization of their world around them. But rather than choosing areas or issues that are neutral with respect to human moral functioning, opportunities can be created for students to engage in an imaginative exploration of their prototypical moral concepts and moral metaphors within this teaching of the mathematization of life experiences. For instance, statistical concepts are used for the mathematization of relationships between quantifications of certain qualities that capture a certain area of our world experience. This provides an opportunity to engage in moral deliberation with students about fear as one of the human qualities relevant to our moral functioning by, for instance, considering how the documented subjective feeling of a more dangerous environment in our cities is in opposition to statistically documented declining crime in those very cities. Here students have an opportunity to explore their understanding of their own fear of being a victim of crime, how this fear is affected by the information and the understanding they have, and how this fear can be influenced and manipulated. Students have an opportunity to imaginatively explore their moral concepts and metaphors around 'crime and fear', even around 'crime and punishment'.

Developing General Human (Meta-Cognitive) Capacities

The third approach in the humanizing of the curricular teaching of mathematics is guided by the more general goal of helping students develop general human (cognitive) capacities and life-relevant skills. In this approach the ‘human element’ is those very capacities and life-relevant skills. For Wheeler (1975, p. 9) the teaching of mathematics can be utilized to educate children’s awareness, which he explicates as “the act of attention that preserves the significant parts of experience, that pegs and holds them in the self so that they are available for future use.” Güting (1980, p. 420) includes into the teaching of mathematics the teaching of skills like “how . . . to learn as effectively as possible”, “to make the best use of textbooks and other resources”, “to plan . . . time”, and “to check . . . results”. Katsap (2002, p. 14) suggests that teaching the history of mathematics can aid in “celebrating cultural diversity” and “intensifying a humanistic world view”. Compared to the other two approaches to the humanizing of the curricular teaching of mathematics, this approach is less mathematics-specific in the sense that it could take the same form in other subject matter teaching.

Developing meta-cognitive skills – for instance, learning how to learn – could be considered being part of this approach to develop general human (cognitive) skills through the curricular teaching of mathematics. Here, then, the teaching of meta-skills of learning to learn (mathematics) can provide opportunities to engage in an imaginative exploration of the conditions under which humans function cognitively, which is an important factor in human moral functioning, since morality (as understood here) is about *understanding* human (moral) functioning. For instance, questions like ‘What motivates me to learn (mathematics), what blocks my learning (mathematics)?’ can guide explorations of one’s metaphor of oneself as a learner, which is directly linked to self-guided moral development. Or questions like ‘Who determines the mathematics curriculum in the first place?’ allows exploring the political aspects of mathematics education (Noddings, 1993) which can guide imaginative explorations of the moral notions of norms and expectations.

This type of exploration provides also the opportunity to imaginatively explore students’ understanding of human vulnerabilities (cognitive and others) and individual differences as part of our human condition. The former can happen by making explicit students’ experiences about their sensitivity to and dealing with failure and success in their learning of mathematics, the latter can happen by making explicit students’ different ways of learning mathematics. Here the moral notions of empathy, tolerance, equality and equitability and metaphors around those notions can be explored.

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THE ABDUCTIVE SYSTEM

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The purpose of the present study was to build a cognitive model which would help to recognize creative processes of an abductive nature. To this end, Peirce's theory of Abduction and Harel's Theory of Transformational Proof Scheme have been used. The result has been the construction of the Abductive System whose elements are {facts, conjectures, statements, actions}. The definition of Abductive System allows the researcher to analyse a broader spectrum of creative processes, and it gives the opportunity to name and recognize the abductive creative components present in the protocols. An example of that kind of analysis will be provided.

INTRODUCTION

Research in mathematics education has long acknowledged the importance of autonomous cognitive activity in mathematics learning, with particular emphasis on the learner's ability to initiate and sustain productive patterns of reasoning in problem solving situations. Nevertheless, most accounts of problem solving performance have been explained in terms of inductive and deductive reasoning, paying little attention to those novel actions solvers often perform prior to their engagement in the actual justification process. For example, cognitive models of problem solving seldom address the solver's idiosyncratic activities such as: the generation of novel hypotheses, intuitions, and conjectures, even though these processes are seen as crucial steps through which mathematicians ply their craft (Anderson, 1995; Burton, 1984).

The purpose of the present study was to build a cognitive model that would help to recognize creative processes of an abductive nature. The issue of creativity in the hypothesis creation process has enhanced the idea of reading Charles S. Peirce's works and his definition of *Abduction*: abduction is any creation hypothesis process aimed at explaining a fact:

> The surprising fact C is observed. However if A were true, C would be a matter of course. Hence, there is reason to suspect that A is true (CP. 5.188-189, 7.202)

Taking into account Peirce's definition of abduction and Magnani's elaboration on it (see Magnani, 2001), one of the first steps of the research was to give two different problems at two different periods of the semester to a group of students attending freshman year of an engineering degree.

> **Problem 1:** let f be a function continuous from [0,1] onto [0,1]. Does this function have fixed points? (Note: c is a fixed point if f(c) = c).
>
> **Problem 2:** given f differentiable function in R, what can you say about the following limit? $\lim_{h\to 0}(f(x_0+h) - f(x_0-h))/2h$

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 57-64. Prague: PME.

A first attempt of an a-priori analysis of the aforementioned problems quickly unearthed some difficulties in predicting possible student creative mechanisms according to Peirce's theory of abduction. In fact, Peirce's abduction refers to a hypothesis that could explain an observed fact, (which is deemed to be true); on the contrary, problem 1 and 2 present a direct question, which means the solver not only has to find hypotheses justifying a fact, but also has to look for a fact to be justified.

The tenet of abduction has also been confronted by Cifarelli part of whose research is concerned with the relationships between abductive approaches and problem-solving strategies. The basic idea is that an abductive inference may serve to organize, re-organize, and transform a problem solver's actions (Cifarelli, 1999). This new point of view gave me the impetus to reflect on a broader typology of abductive processes, where the fact is also represented by a strategy/procedure or regularity. As a consequence of these new considerations about abductive processes, the research questions were: 1. Is a broader definition of abductive process needed to describe some creative students' processes in mathematics proving? If so, what is that definition? 2. How much is important the level of confidence of the built answer to guide an abductive approach? 3. Which elements convey an abductive process? In particular, does transformational reasoning (Harel, 1998. p. 258) facilitate an abductive process?

THE ABDUCTIVE SYSTEM

According to the initial difficulties of analysing the problems using only Peirce's definition of abduction, and the new considerations made about tasks requiring not only the construction of a hypothesis but also of the answer, I have constructed new definitions and tools which have been employed in the analysis of the protocols. I define the *Abductive System* as being a set whose elements are: *facts*, *conjectures*, *statements*, and *actions:* **AS** = {facts, conjectures, statements, actions}. For *fact* I adopt the definitions of Collins' Dictionary:

> (1) referring to something as a **fact** means to think it is true or correct; (2) **facts** are pieces of information that can be discovered.

For *conjectures* I adopt the definition given by the Webster's dictionary:

> **conjecture** is an opinion or judgement, formed on defective or presumptive evidence; probable inference; surmise; guess; suspicion.

The *conjectures* assume a double role of: (1) *Hypothesis*: an idea that is suggested as a possible explanation for a particular situation or condition. (2) *C-Fact (conjectured fact):* final answer to the problem, or answer to certain steps of the solving process.

Facts and Conjectures are expressed by *statements* that can be *stable* or *unstable*. A *stable statement* is a proposition whose truthfulness and reliability are guaranteed, according to the individual, by the tools used to build or consider the fact or conjecture described by the proposition itself. An *unstable statement* is a proposition whose truthfulness and reliability are not guaranteed, according to the individual, by the tools used to build or consider the conjecture described by the proposition itself.

The consequence of this is the search of a hypothesis and/or an argumentation that might validate the aforementioned statement. *Abductive statements* are of special interest for us. An *abductive statement* is a proposition describing a hypothesis built in order to corroborate or to explain a conjecture. The abductive statements too, may be divided into stable and unstable abductive statements. The former, according to the solver, state hypotheses that do not need further proof; the latter require a proof to be validated, that means a process that brings back and forward.

It is important to clarify that the definitions of *stable* and *unstable statement* are student-centered, namely, the condition of stable and unstable is related to the subject; for example, what can be stable for one student may represent an unstable statement for another student and vice-versa; or the same subject may believe stable a particular statement and this may become unstable later on when his/her structured mathematical knowledge increases (e.g.; he or she learns new mathematical systems; new axioms and theorems). Another situation leading the student to reconsider a statement from stable to unstable is the "didactical contract"; the subject might believe the visual evidence to be sufficient in order to justify a conjecture, but the intervention of the teacher could underline its insufficiency and therefore the students would find themselves looking for new tools. Furthermore, the statement may transform from unstable to stable inside a similar process because the subject follows the mathematicians' path: they start browsing just to look for any idea in order to become sufficiently convinced of the truth of their observation, then they turn to the *formal-theoretical world* in order to give to their idea a character of reliability for all the community (Thurston, 1994).

Behind any statement there is an action. *Actions* are divided into *phenomenic actions* and *abductive actions*. A *phenomenic action* represents the creation, or the "taking into consideration" of a fact or a c-fact: such a process may use any kind of tools; for example, visual analogies evoking already observed facts, a simple guess, or a feeling, "that it could be in that way"; a phenomenic action may be guided, for example, by a didactical contract or by a transformational reasoning (Harel, 1998). An *abductive action* represents the creation, or the "taking into account" a justifying hypothesis or a cause; like the phenomenic actions, they may be conveyed by a process of interiorization (Harel, 1998), by transformational reasoning (ibid) and so on. The abductive actions may look for: 1. *A hypothesis*, to legitimate or justify the previous met or built conjecture. 2. *A procedure*, to legitimate or justify the previous built conjecture. 3. *Tools* to legitimate the adaptation of an already known strategy to a novel situation.

After a broad description, the Abductive System could be schematised in the following way: *conjectures* and *facts* are 'acts of reasoning' (Boero et al., 1995) generated by phenomenic or abductive actions, and expressed by 'act of speech' (ibid) which are the statements. The adjectives *stable*, *unstable* and *abductive* are not related to the words of the statements but to the acts of reasoning of which they are the expression. Hence, the only tangible thing is the act of speech, but from there we may go back to a judgement concerning the act of reasoning expressed through the adjectives given to the statement. For further details on the Abductive System, its framing and its use in the analysis of the protocols, see Ferrando (2005).

METHODOLOGY

<u>Site and Participants</u>: the data has been collected at the Production Engineering Department of the University of Genova (Italy) during the academic year 2001-2002, and the participants are freshmen enrolled in required calculus classes for engineers. The courses cover differentiation and integration of one-variable functions as well as differential equations. The student participants are 18 or 19 years old. At the beginning of the Calculus course the teacher introduced me to the students as a Teacher Assistant, working once a week with them in class for a session of three hours, during which the students would solve problems proposed by me, and they would be able to discuss possible problems raised by them. During the week, the students would be able to come to my office for further explanations about topics discussed in class, or about exercises solved autonomously.

<u>Data Collection</u>: the data (audio-recordings, videos and written texts) was collected through two different exercises given, at two different periods of the semester, to the participants in the project (twenty students took part in the project, according to a decision that was left to them). In the problem solving phase the participants were asked to work in pairs (leaving to them the decision about whom to work with); the choice was motivated by the conviction that the necessity of “thinking aloud” to communicate their own ideas gives the opportunity to bring to light guessing processes, creations of conjectures and their confutations, namely those creative processes which in great part remain “inside the mind” of the individual when one works alone, and very often only the final product is communicated to the others (cf. Thurston, 1994; Lakatos, 1976; Harel, 1998). The participants were not asked to produce any particular “structured” solution, my aim being to leave the students completely free to decide their solution process and to autonomously evaluate the acceptability of their solution for the learning community.

ANALYSIS OF THE DATA

In the wide research (see Ferrando, 2005), and only partially presented in this paper, the analysis of the protocols was divided into two phases. The first phase showed a comprehensive description of students’ behaviours in tackling the problem; in the second phase the creative processes were detected and interpreted through the elements of the Abductive System. The following analysis is related only to the second phase; the excerpt of one protocol is followed by a table divided into two columns where the left column is used to write the excerpts considered relevant to the creative processes (while my own interpretation of the statements are in brackets); the right column has been used to write the interpretation of the excerpts through the tools of the Abductive System; furthermore, the vertical arrows linking one excerpt to another describe the possible cognitive movement leading from one statement to another one.

Transcript of Marco and Matteo (fixed point problem)

1 M1: this function is in the middle; I would say…I mean…it goes from here to there

2 M1: if the function starts from 0 and goes up and goes down, it takes all the values one time…and we have two fixed points.

3 M2: the fixed are these then?

4 M1: I suppose that if the problem asks, the function will have a fixed point.

5 M2: how can we find this fixed point?

6 M2: a fixed point is here, another one is here…

7 M2: therefore, the fixed points are those that have y=x?

8 M1: I would say yes…I would say that the fixed points are on…y=x and if our function must assume all the values of the image in such a way if it is continuous it must through this line…there will be a point for sure…

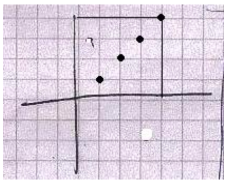
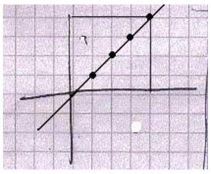
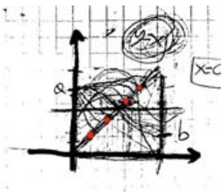
9 M1: supposing that it does not have to intersect this thing, and given the fact that it must take all the values from 0 to 1, the value with x=0 must exist, if for this x=0 y were equal to 0 we would have a fixed point, therefore it does not work, then y must be different to 0 and at this point we would have one of these points here. When we want to go to x=1 or y=1 and we don't want to, therefore y≠1, then we have one of these points here and one of these points here to go from here to there in any way we have to go through here and therefore any function which brings one of these points here to a point there must intersect the bisector line, for sure…

10 M1: in my opinion we should think of a counterexample, somebody saying that it is possible to pass, I have to find the way to prove that we can't pass without intersecting the line

11 M1: we have to prove that f(x) intersected with y=x is not empty, different to the empty set. We have to prove that it is possible to go from here to there without intersecting the bisector line, but if a>b taking a as the point where x =0 and that lies on the upper side of the bisector line, b the point where y=1 and b lies on the lower side of the bisector line there must be a point between the two where the x=y; there must be for sure and I can do the same thing changing the position of the two points respectively …I have to write it down in formal way

12 M1: by contradiction we take 'a' that is greater and ≠ 0 and 'b' minor, now we say by absurd it does not go to, at this point 'a' will take in this point here any point in the middle and that a ≠ y, therefore a point in which y>x always because in a first moment we said that it was greater therefore y must be greater than x and in this other little point here and here and here it will always be greater strictly greater we arrive here where it must be greater than x, at this point we have to take all these points here; its value in 1 cannot be less than 1, equal 1 or more than 1, because it must stay in this interval here, therefore it is absurd.

Analysis through the tools of the Abductive System

Excerpt	Interpretation through the tools of AS
4: I suppose that if the problem... (**f probably has fixed points**) *Search of a justifying hypothesis. It needs the construction of a theory; i.e.: to identify and explicate the properties of the fixed points. The need to broaden the cultural background in order to be*	Conjecture with role of answer to the problem, therefore it is a c-fact. The c-fact is created by a phenomenic action guided by a didactical contract: "if the problem asks…" the statement describing the c-fact is an unstable statement because M1 and M2 don't believe the didactical contract sufficient to validate the statement.
6: a fixed point is here, another… (**the vertex of the squares on the paper sheet represents a fixed point**)  *The graphic exploration continues*	Fact created by a phenomenic action. It is expressed by a stable statement, in fact M1 and M2 justify it through a visual impact that seems to be sufficient
7/8:the fixed points are on y=x… (**the set of the fixed points is the bisector line**)  *Now they have a new property in their cultural*	Fact created by a phenomenic action guided by the visual impact and an unconscious consideration of the density of R^2. The fact is expressed by a stable statement, justified by: 1) the vertexes of the squares represent the fixed points; 2) cognitive jump: between two squares there are infinitely many others. The visual impact seems to be sufficient.
9:[…] in any way we have to go… (**continuous functions in [0,1] intersect the bisector line**)  *Choice of a proving strategy: "proof by contradiction". Probably guided by a didactical contract, because they recently saw such kind of procedure*	Two different stages. 1st stage: the act of reasoning is created by a phenomenic action guided by a visual impact; and it is expressed by an unstable statement based on: 1) continuous function in [0,1] onto [0,1] (given of the problem). 2) bisector line as set of the fixed points (built by the student). 3) continuous function in [0,1] means no gaps in the interval (student's elaborated conception). At this point an abductive action is accomplished: the c-fact is reinterpreted as possible hypothesis corroborating the initial c-fact ("the function has probably a fixed point"). The statement becomes an unstable abductive statement, unstable because M2 and M1 do not believe the three aforementioned conditions

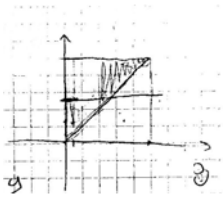
	sufficient to validate the hypothesis expressed by the statement. **Obs**.: in the first phase it has been said that the act of reasoning represents a c-fact, relating its “instability” to that one considered when the act of reasoning takes the role of hypothesis. Nevertheless, we do not have to ignore the hypothesis that if the act of reasoning had been stopped at the first step, the visual impact could have been enough for M1 and M2, and then the act of reasoning would have been expressed by a stable statement.
10: we should think of a counterexample… (**there exists a function continuous on [0,1] such that it does not intersect the bisector line: gr(f) $\cap$ b = $\varnothing$**)	Phenomenic action guided by the structure of the proof by contradiction; this action creates a c-fact expressed by an unstable statement.
12: […] y must be greater than x… (**gr(f) $\in$ to the upper triangle**) 	Creation of a hypothesis through an abductive action guided by a visual impact. The hypothesis is stated by an unstable abductive statement in the sense that M1 and M2 believe the visual impact to be insufficient to validate the hypothesis

Table 1: Analysis through the tools of the AS

CONCLUSIONS

The definition of Abductive System allows the researcher to analyse a broader spectrum of creative processes than those covered by the already given definitions of abduction, and the experimental phase revealed to show the presence of those components I have given a name inside the Abductive System. The analysis of the data through the tools of the Abductive System allowed answering to the questions stated in the Introduction. Indeed the Abductive System, in general, is a possible answer to the first question, got by broadening the definition of abduction; and the distinction between stable and unstable statements opens the way to a possible answer for the second question. Indeed, when an act of reasoning is expressed by an unstable statement, the subject needs to find a hypothesis that could validate or confute it. The last question brings to light the issue of the role of the transformational reasoning (Harel, 1998) in facilitating a possible abductive process; the research has confirmed that perceptual and transformational reasoning have played a fundamental role in the construction of both conjectures (c-facts and hypotheses) and facts.

There is a further element we need to take into consideration, which is the typology of the sample; it cannot be defined as a random sample, since the students voluntarily offered to participate in the project, and probably were those who positively accepted a didactical contract that encourages an approach promoting the understanding how things work, the making of connections among mathematical ideas, creating conjectures and validations of mathematical ideas, rather than a formal deductive approach. Nevertheless, regarding what concerns the didactical implications, I hypothesize that, since the creative abductive processes do not seem to be an attitude of a particular elite of subjects, what has happened with a particular sample of students may be extended to a larger population of students, if the same previously mentioned conditions are created on the side of the students. Furthermore, the creative abductive attitude met in the students, cannot be considered only an inclination of human nature, but it also probably depends on the scholastic and extra-scholastic experience of the student, and certain kinds of didactical contract may positively influence such creative processes.

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REMEMBERING AND IMAGINING: MOVING BACK AND FORTH BETWEEN MOTION AND ITS REPRESENTATION

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This paper considers the activity of 13th grade students who work in group to track a 3D uniform circular motion. A device called Motion Visualizer 3D is used to display in real time the trajectory of motion on a computer screen. The students are asked to draw the 2D temporal representations of motion along the three directions (width, height, depth). The analysis of their speech and gestures shows moments in which they make present past knowledge or actions (remembering), and moments when they anticipate features of the not yet known graphs (imagining). In so doing, the research points out that understanding motion in mathematical terms grows out of a complex dynamics between recollections and expectations.

INTRODUCTION

A quite recent study in Mathematics Education has showed the significance of making present something absent in the process of symbolising (Monk & Nemirovsky, 2000). Somehow, the idea of making present reminds the phenomenological enquiry on memory and phantasy (Husserl, 1893-1917). These studies are starting points for the research presented here. Their integration allows defining remembering and imagining in a fresh way, in terms of making present. It gives insights on the ways remembering and imagining take place in the students' processes of understanding, in activities of symbolising. Students who are asked to model motion through graphing face activities of this kind. Graphs are symbolic representations of the real phenomenon; they need some level of abstraction to be understood in relation to motion. Experiences in which technology is used have been shown to help learners to understand the mathematics connected with the phenomena (Nemirovsky *et al*., 1998; Ferrara & Robutti, 2002).

This paper concerns the activity of grade 13 students required to model a 3D uniform circular motion through its 2D temporal representations. The use of a technological device (a Motion Visualizer 3D) allows having in real time the trajectory of motion on a computer screen. Starting from the perspective discussed above, the analysis traces the dynamics between remembering and imagining that takes place in understanding the mathematics related to motion. To this aim, attention is on students' speech and gestures from a semiotic stance (see Radford, 2003; Ferrara, 2004). That is, as semiotic means of objectification that "individuals intentionally use in social meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities" (Radford, ibid.; p. 41).

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THEORETICAL BACKGROUND

> I remember the illuminated theater – that cannot mean: I remember having perceived the theater. Otherwise the latter would mean: I remember having perceived that I perceived the theater, and so on. I remember the illuminated theater means: "in my interior" I see the illuminated theater as having been. […] I remember yesterday's illuminated theater; that is, I bring about a "reproduction" of the perception of the theater. The theater then hovers before me in the representation as something present. I mean this present theater, but in meaning it I apprehend this present as situated in the past in relation to the actual present of the perceptions occurring right now. Naturally, it is now evident that the perception of the theater did exist, that I did perceive the theater. What is remembered appears as having been present, doing so immediately and intuitively; and it appears in this way thanks to the fact that a present that has a distance from the present of the actual now appears intuitively. The latter present becomes constituted in actual perception; the former intuitively appearing present, the intuitive representation of the not-now, becomes constituted in a replica of perception, in a "re-presentation of the earlier perception" in which the theater comes to be given "as if it were now". This re-presentation of the perception of the theater must not be understood to imply that, living in the re-presentation, I mean the act of perceiving; on the contrary, I mean the being-present of the perceived object. (Husserl, 1893-1917; pp. 60-61)

Using the argument above, E. Husserl, the founder of Phenomenology, has discussed the difference between the processes of *perceiving* and *remembering*. This difference resides in the temporal character of the objects of perception and memory. Perception is constituted as presentation on the basis of sensations; instead, memory is the re-presentation of something in the sense of the past. When the re-presentation occurs immediately joined to perception, memory has to be intended as retention (primary memory). But the re-presentation can occur independently, without being attached to perception: this is *recollection* (secondary memory). Memory is similar to perception. They have in common the appearance of the object, although the appearance itself has a modified character: "the object does not stand before me as present but as having been present" (ibid.; p. 61). Other than perception and memory, Husserl also speaks of *phantasy*. Phantasy has the temporal character of expectation; it constitutes imagination. The major difference between recollection and phantasy, that is, between remembering and *imagining*, then consists in the fact that the former is embedded in a sense of "having been", whereas the latter is not.

The Husserlian idea of re-presentation (or making present) is evoked by a recent study in Mathematics Education on the nature of symbolising (Monk & Nemirovsky, 2000). The image of a child who plays using a stick as a horse is discussed: "the child jumps around his friends, goes places, feeds the horse, claims that the horse is lazy, and so forth" (ibid.; p. 177). Thus, the child is *making present* a horse that otherwise would be absent in his life. Besides, he is doing things with it. The horse is not simply present but also *ready at hand*: it is made to participate in the child's activity. It is an example of symbolising, seen as the "creation of a space in which the absent is made present and ready at hand" (ibid.; p. 177). The way the creation occurs is related to the notion of *fusion*. Fusion has been at first introduced in the context of studying motion graphs: "understanding a graph of position versus time would be grounded in the ability to establish links between points on the graph and positions of the moving object at a given time or between the slope of the graph at a certain time and the velocity of the object" (Nemirovsky *et al.*, 1998; pp. 140-141). Fusion consists then in "merging qualities of symbols with qualities of the signified events or situations, that is, talking, gesturing, and envisioning in ways that do not distinguish between symbols and referents" (ibid.; p. 141). Concerning the child playing with the horse-stick, fusion is the child's acting, talking, and gesturing without distinguishing the horse and the stick, or treating the stick *as if* it were a horse. The play has a second feature: trail-making. It is "an ongoing creation in which actions and words, rather than stemming from a planned sequence, emerge from the activity itself in open-ended ways" (Monk & Nemirovsky, ibid.; p. 178).

Remembering and Imagining in terms of Making Present

The play of the child with the horse-stick is an activity of symbolising as modelling motion. Fusion and trail-making are then qualities of activities of symbol-use. They give insights on the ways imagining and remembering, as discussed by Husserl, can intervene in activities of such sort, especially when unfamiliar. In this perspective, imagining and remembering can be expressed in terms of making present as follows:

- *Remembering* is *making present the past*. The past is meant as past or everyday experience, acquired knowledge, classroom culture and practices[1].
- *Imagining* is *making present the not yet known*. The not yet known is meant as everything that has not been yet experienced, seen or learnt, as the goal of an activity, and that appears for the first time in the course of an activity.

It is my contention that analysing the dynamics between remembering and imagining that take place in a segment of the students' mathematical activity can shed light on the way the understanding of a motion graph occurs. In carrying out the analysis, students' speech and gestures will be useful to interpret this process.

THE CONTEXT

Methodology. The activity is part of a teaching experiment carried out in February and March 2005. The experiment took place during after-school time, involving grade 13 volunteers in a series of activities. A researcher (myself) and an observer (a pre-service teacher) were present. The students spend a substantial period working together in small groups of 3 or 4. Tasks were given on paper, and the students were asked to write down solutions and results. At some point, I conducted a general discussion that allowed learners to expose and compare their group solutions and results. If needed to overcome troubles, brief discussions began in the middle. All the written materials (texts, drawings, sketches, etc.) were collected. In addition, a moving camera filmed one group work and the final discussions. Transcriptions of the videotapes were then produced. The set of resources obtained in this way has

[1] The past mathematical activity and the act of remembering, meant in terms of mathematical experiences or didactical memories of the students, are the subject of a very recent research study (see Assude *et al.*, 2005). This study considers the way remembering (intended as above) relates to mathematics learning and to changes occurring in learning.

been used to analyse passages that showed to be salient in terms of understanding processes.

Technology and Mathematics. The experiment was aimed at introducing students to the relations between derivative and primitive, by the trigonometric functions that are characteristic of uniform motion. To this aim, a graphical approach was realised with the support of a computer and of a three-dimensional motion detector working with colours, called Motion Visualizer 3D (MV 3D). The students had never used this tool before, although they participated in two teaching experiments on motion (in grade 9 and 11) where they used 2D motion detectors. At the moment of the experiment, the classroom background did not include wide knowledge of primitive. The students saw just a general definition of integral and were learning rules to integrate polynomials.

The activity. The activities had to do with the uniform circular motion of an orange ball attached to a rod turning vertically at a constant speed (motion occurs in a counter-clockwise direction; fig. 1). We will focus here on a segment of the first activity that asked learners to track the motion of the orange ball, and then to sketch the functions of position versus time along the directions of motion (the motion of the ball can in fact be broken up along the coordinates *x*, *y* and *z* of the room). The device was set to allow students to see the origin in real time of the trajectory on a computer screen. The segment that will be discussed is about the drawing of the 2D graph of *x* versus time.

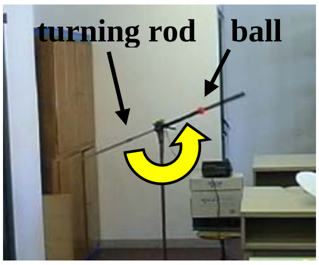


Figure 1

DISCUSSION

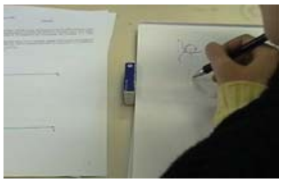
The segment analysed here is relative to the work of three students: Alberto, Chiara and Silvia. After having discriminated between the three coordinates, and recognised *x* as width and time as the independent variable, the students begin to discuss on the behaviour of *x* over time. On paper (where the axes for the graph are already drawn), they have inserted labels on the axes. On a new paper, Chiara has also sketched the trajectory of the ball (a circumference) and the axes of the room, showing the need for a space where recreating the context of motion. The problem is to understand the starting point of the graph in relation to the starting point of the motion of the ball (for ease, the space where the graph has to be drawn will be called *graph space*, the paper where the circumference is drawn will be called *drawing paper*):

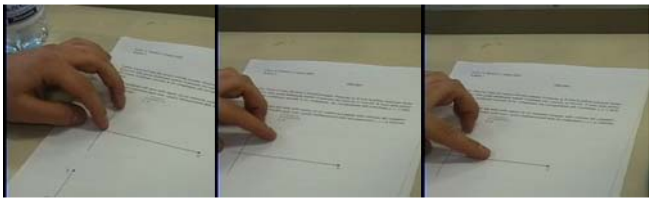
Silvia: *We have time going on… And x… x goes, starts here* [pointing to the east on the drawn circumference]*… hence* [Silvia is shifting to the axes, and Chiara is locating her pen on the circumference] *it starts from a certain value* [Chiara is miming half a circumference in correspondence with the circumference], *no? And then it returns to that value* [Chiara keeps her pen fixed on the circumference]*… it has a value* [Silvia is miming over the graph space the same half a circumference as Chiara did] *as if we started from a value* [shifting to the drawing paper, sketching two orthogonal axes and pointing to a value on the vertical axis: Fig. 2]

Alberto: *But, isn't it circular?* [miming a counter-clockwise circumference with his left index finger on the graph space]

Silvia: *To me it is a wave* [keeping the pen fixed on the previous point on the vertical axis]
Alberto: *Right*
Silvia: *Because then this value* [*that of the starting point she was referring to*. Alberto is tracing with his left index finger a piece of wave on the graph space: Fig. 2] *varies* [drawing a small arc on the drawing paper: Fig. 2]
Alberto: *Yeah, right*
Silvia: *It returns here again, but it is as…* [following from left to right the small arc] *the opposite* [pointing to the north on the circumference]


miming half a circle (Silvia)


from a value (Silvia)


tracing a piece of wave (Alberto)

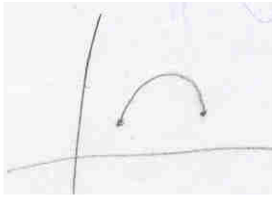
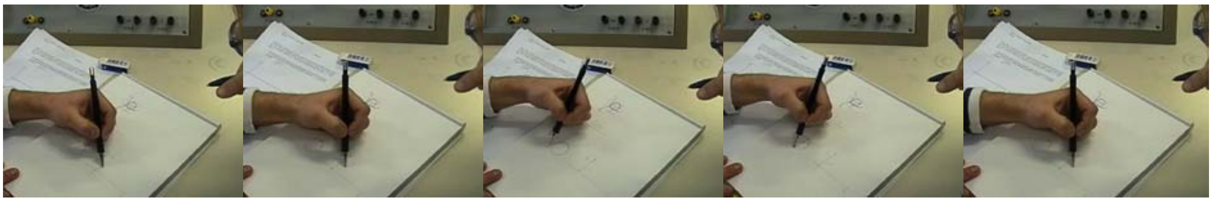

a small arc (Silvia)

Figure 2: Some gestures occurring in the dialogue

Once the feature of time going on for temporal graphs is shared within the group, Silvia starts to reason on the different positions of the ball over time. It is clear that *x* changes, and its changes are expressed through verbs of motions: *x* ‘goes’, ‘starts’, and ‘returns’. The behaviour of *x* is explored in relation to the movement of the ball from a *local* point of view. As the initial pointing gesture shows in fact, most attention is drawn to a specific position on the circumference. This position appears to be an initial *pivot* to begin to discriminate between some moments of motion, and to try linking them with what happens to *x* over time. To Silvia, it represents the points in space where *x* reaches the same value, as marked by the use of deictic terms (‘here’, ‘from a certain value’, ‘to that value’). It is interesting to observe the way Chiara accompanies Silvia’s speech with a gesture miming the shape of half a circumference. She is back to motion, *remembering* the ball turning in that part of the circumference and thinking of the changes of *x* occurring during it that are not yet clear. Silvia mirrors the same gesture (see Fig. 2) *to imagine* what happens of the values of *x* (‘it has a value’). It is not the most powerful gesture, since along such half a circumference *x* does not keep the same value at the ends of the diameter. Yet, the gesture is useful to shift the situation in space to two dimensions (the use of ‘we’ marks a change of perspective, as if Silvia recognised the task: ‘we started from a value’), as the pointing gesture on the sketched vertical axis highlights (Fig. 2). Alberto is still confusing the trajectory in space and the position versus time, when he *remembers* the circularity of motion in words, and mimes the trajectory itself with a gesture, which also recalls the motion of the ball moving in a counter-clockwise direction. Silvia helps him explicitly referring in speech to the shape of the graph as that of a wave, starting from some value (referred to by pointing). The fact that only pointing is being performed seems to show her effort *in imagining* the wave. Alberto shares and early mirrors, without speaking, this image just expressed in words by Silvia, in a gesture (Fig. 2) that *makes present* on the graph space the representation of $x(t)$ (the representation is *still physically absent* but becomes *imaginary present* through the gesture). Silvia explains this shape through a single verb, referring to the changes she was thinking before in terms of values reached by x: the initial value 'varies'. Her gesture drawing the initial shape of the graph is significant (Fig. 2): it produces a *new sign*, a sketch of the graph, on the drawing paper; it *makes* a part of the graph *actually present*, making fixed Alberto's gesture. But the trajectory and the behaviour of x in time are still confusing students, as Silvia highlights when she performs a gesture to follow the small arc just drawn, and then points on the circumference to the location 'opposite' to the starting one. In fact, on the circular path the chosen points (the origin of motion and its 'opposite') have the same height (the same vertical position) but different widths (different horizontal positions). On the contrary, on the arc representing $x(t)$ it is as to return 'here again' (at the initial value) since the marked points have the same value for x, although being different points at different times.

Alberto faces the conflict through a narrative that tries to reconstruct the relationships between motion and one of its representations, between the trajectory followed by the ball in space and the function of the horizontal position versus time:

Alberto: *Just a second* [taking Silvia's pencil]. *So, the ball turns in this way, doesn't it?* [sketching on the drawing paper a circumference following a clockwise direction] *And this is the graph* [drawing two orthogonal axes] *Let's take this one* [pointing to the north as the point where motion begins] *as the zero position* [pointing to the origin of the axes, and then returning to the circumference] *with time passing, in this way* [following the horizontal axis from left to right] *this one* [pointing to the point on the circumference] *let's say that it goes up and arrives here* [following the circumference from north to west], *hence it will be in this way* [tracing the first increasing piece of curve: Fig. 3]. *Then it begins to go down* [following the circumference from west to south: Fig. 3] *but time* [pointing to the maximum of the drawn piece: Fig. 3] *increases*



hence it will be in this way *Then it begins to go down* *but time*

Figure 3: Explaining the shape of the graph

Alberto's explanation of the shape of the graph of $x(t)$ lays in an interplay between gestures and utterances. At first, he needs to focus on the movement of the ball, and on the path followed in space, which is produced by the drawing of the circumference. Alberto *is remembering* the motion of the ball when saying 'the ball turns in this way', but he is also thinking of its trajectory as marked by the circumference drawn in a clockwise direction (the ball actually did not turn in that way but counter-clockwise). Attention is then suddenly shifted to the drawing of the orthogonal axes, recalling the difference between motion and one of its representations: the turning motion is a thing, but the graph of x versus time is another thing (‘this is the graph’). The following pointing gestures are significant in linking trajectory and $x(t)$. The first gesture features the choice (‘let’s take’) of a specific position on the circumference, the north, as the starting position for the motion of the ball; it well matches the deictic words referring to such a position (‘this one’). The second pointing gesture marks the correspondence of the point on the circumference with a specific point on the graph, the origin, where the x has value zero and time is zero (since the ball is not moving); it also matches speech that considers the origin as the point where the horizontal position is zero (‘as the zero position’). Thus, the north on the circumference acts as a *cognitive pivot* that allows determining a particular correspondence. Once this correspondence has been established, Alberto starts looking at the behaviour of x from a *more global* point of view. Time is an essential component of his interpretation. The nature of having time ‘passing’ (feature of the temporal graphs) is highlighted not only in speech, but even through a gesture that traces the horizontal axis just following the direction along which time goes on. As time goes on, the ball moves on the circumference and its position changes with respect to the initial one (‘this’ in speech, and pointed to in gesture). The following gesture that resembles a quarter of circumference is performed while Alberto *is remembering* the motion of the ball turning counter-clockwise (as shown by the direction of the gesture), and already *imagining* the curve of x over time (as the words ‘it goes up and arrives here’ seem to suggest, since actually the hand is moving in the opposite direction, going down along the circumference). In speech ‘hence’ has a logic function. It marks the occurrence of a consequence (of the fact that time passes while the ball moves from north to west), which is expressed on the graph of $x(t)$: the shape of the curve results in the sketch Alberto draws on paper (Fig. 3). The *memory* of the next part of motion is *made present* again by a gesture (Fig. 3), which follows just the second quarter of circumference where the ball ‘goes down’. But Alberto seems to want to say that ‘time’ is passing (‘increases’) and thus as the ball goes down during its motion, the curve also goes down starting from the top of the first increasing piece, fixed through pointing (Fig. 3). Keeping reference to this point on the graph while thinking of time going on appears relevant to understand how the part of the curve, corresponding to the motion along the second quarter of circumference, is to be sketched. It is also interesting to note the great attention Chiara (on the right side of the frames in figure) is paying to Alberto, gazing all the time on his gestures with the same orientation. Her reflection entails the following conclusion:

Chiara: *Of course, it is a wave* [she is miming over paper the motion of the ball, turning counter-clockwise three times] *because it is a periodic thing*

Chiara has suddenly no doubts on the shape of the curve as that of a wave. Her gesture brings her *back to motion* miming the ball turning, but in the same time Chiara *is imagining* to have a wave as her words highlight. She seems to look at the situation in a *global* manner, giving account of this shape linking in speech the wave with the case of having a ‘periodic thing’, meaning a periodic motion. Although the gesture differs from words in that it does not resemble a wave, it is significant in anticipating what comes up in the following words: the feature of motion of being periodic is given by the repetition of the circular trajectory in gesture. At this point, the students share this view, and Alberto can draw the curve up to complete a period.

CONCLUDING REMARKS

The discussion above has shown that the process of drawing the graph of $x(t)$ grows out of an intricate dynamics between remembering and imagining. Students' speech and gestures, through which this dynamics is studied, reveal a transition from a local to a global interpretation of the behaviour of x. In an early stage in fact, most attention is drawn to some positions on the trajectory of motion. These positions work as pivots to discriminate between moments of motion that begin to be recollected. The recollection allows the students to link the particular positions on the circumference to specific points reached by x in time. Attention is shifted to what happens of the values of x, resulting in efforts to imagine the shape of $x(t)$ as a wave, and then to make present the wave through producing a sign on paper: a sketch of it. At this point, the relations between the trajectory travelled in space by the ball and the shape of the graph need to be accepted within the group. To this aim, they have to be reconstructed, through a new recollection, that looks at the behaviour of x from a global point of view. Time is essential component of this interpretation, having the feature of "passing". It is as if, by recollecting, the motion of the ball were 'stretched and flattened' (mathematically, projected) along the horizontal direction. The shape of the wave, other than being understood, is thus linked to the fact that the projected motion is a periodic motion between the ends of the diameter of the circumference.

Acknowledgments. This paper is a result of a research program funded by MIUR and by Università di Torino and Università di Modena e Reggio Emilia (PRIN n. 2005019721).

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FROM VERBAL TEXTS TO SYMBOLIC EXPRESSIONS: A SEMIOTIC APPROACH TO EARLY ALGEBRA

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This paper describes some outcomes of a long-term project designed to combine the teaching of mathematics and language from grade 1 to 5. The focus is on the activity of inventing notations in order to communicate mathematics and to solve problems. To outline the work done and the opportunities it provides, we present two episodes which occurred at different school levels (third and fourth) and focus on the development of the use of symbols by pupils. The examination of the outcomes shows a wide range of teaching and learning opportunities as concerns the transition from arithmetic to algebra, above all in the crucial step from mathematical stories to symbolic expressions.

THEORETICAL FRAME

A pragmatic perspective on the role of language in mathematics education has been described by Ferrari (2002, 2004) along with examples and data from middle school and college level respectively. The main goal of this paper is to discuss some outcomes of a long term project for primary school based on the perspective mentioned above. In this section the basic assumptions of the framework are concisely sketched.

First of all, the project recognizes the discursive approach to mathematics learning[1] as an appropriate starting point in order to both explain students' behaviors and produce new teaching ideas. This means that language is regarded not just as a carrier of pre-existing meanings, but as a designer of the meanings themselves. So the linguistic means adopted in communicating mathematics are critical in the development of mathematical thinking, and poor linguistic resources are expected to produce poor development of thinking. A deeper investigation of the linguistic resources required to understand mathematics shows that in mathematical activities at any school level language has got to play at least two functions: communicating among people and describing mathematical knowledge. To fulfil the former function, the linguistic means usually adopted in everyday life communication are quite enough. The latter function needs more advanced linguistic resources, which we refer to as 'mathematical registers'[2]. This often causes the same words to take different meaning according to the function actually played by the text. For example, the ordinary meaning of 'rectangle' is quite different from the 'mathematical' one. Naming 'rectangle' a quadrilateral with four congruent sides and angles is quite appropriate to

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[1] see for example Sfard (2001)

[2] a register is a use-oriented linguistic variety

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 73-80. Prague: PME.

describe the properties of geometrical figures, but it might prove confusing from the standpoint of communication, as most people would expect to hear the word ‘square’.

On the other hand, mathematical registers share plenty of properties with written literate registers. This means that understanding mathematics requires both the metalinguistic awareness needed to switch between different registers and some familiarity with literate ones. So, the colloquial use of language is not enough to fully understand mathematics, and the linguistic resources appropriate for mathematics learning are not innate but are to be developed as early as possible. In other words, here it is not assumed that just talking or writing would produce understanding, but that well designed activities can support the development of linguistic skills appropriate for the growth of mathematical thinking. So the basic goal of language education should be the control of the text, and in particolar of the relationship between text, context and goals. Of course, pupils, when producing a text, should be not just aware of but must share its goals as well. This means that pupils must be allowed to decide themselves the goals of their activities.

THE PROJECT

Within this frame a long term experiment has been designed, based on the close coordination of the teaching of mathematics and of language at primary school level. A class of about 20 italian pupils has been taught by the same couple of teachers for 5 years, from 2000 to 2005, from the age of 6 to 11. As common at that time in Italy, one teacher was to focus on language, the other on scientific concepts. Anyway, their agreement on both the whole plan and the daily activities was full.

Since grade 2 the pupils spent a considerable amount of time in writing texts aimed at the description of their work and achievements. They have progressively agreed on goals and features of the texts they were developing. The teacher has helped them in organizing the discussions, focusing their arguments and summing up. Although she played a central role in the management of all activities and in the attainment of all the specific goals, she has been successful in getting the pupils to feel responsible of their decisions and to be engaged in fulfilling them. These are the goals and the reasons that induced them to work on writing texts also with the mathematics teacher:

- Analogy with the work on language.
- Help for the pupils who missed some lesson.
- Common memorandum, to reconstruct resolution procedures, to get track of their progress, to collect results and methods useful for future activities.

The following writing criteria have been explicitly agreed by pupils:

- Texts had to be simple and easily understandable.
- Anyone was encouraged not to use improper or outmoded words, nor words whose meaning was not clear.
- Verbal tenses had to correspond to the actions described.
- Anyone was asked to minimize the use of generic words (like ‘do’, ‘thing’).

Pupils were committed in mathematical activities as well. They would invent problems related to the topics they were dealing with, and sometimes they proposed games. For example, they would split into two teams and each one had to propose the other some problems. Of course, the fun was to invent problems as difficult as possible. To ban unfair play, a set of rules had to be agreed: each problem had to have a solution, all the data needed were to be given and so on. Problem solving has been widely practised through the years, and special care has been paid to the representation of solutions strategies. In the next sections two episodes will be described. The problems involved are different with regard to subject and complexity but are appropriate to describe relevant steps in the representation of strategies.

METHODOLOGY

Since the class has followed an innovative curriculum through primary school, we were mainly interested in testing the opportunities of the methods adopted. So our goal was to investigate the results achieved thoroughly, in order to understand how they could be transferred to other classes. For these reasons qualitative research methods have been adopted. For both the episodes there are the reports written by small group of pupils and alla the minutes of their work and most of the interactions have been audio-recorded. Our first step was to compare the reports with the theoretical frame and develop hypotheses apt to explicate the relationships between language and mathematics learning, as concerns problem solving and the transition arithmetic-algebra. The reports have been written in Italian, and a thorough linguistic investigation requires to deal with the original texts, as the english translation, if it may properly convey some basic aspects of the text, may fail in preserving some linguistic features, like register. The discussion is based on the english translations of pupils' writings, but some expressions are quoted in Italian when appropriate.

EPISODE 1

At the end of grade 3 some of problems have been proposed (in many cases by the pupils themselves) with the shared goal of improving the representations of the solution procedures. The following problem elicited some interesting behaviors.

> In the library of our class there were 58 books. The teacher has bought 26 more. Last night some thieves broke into the school and stole 19 books. How many books were left in the library?

The pupils quickly solved the problem, and the first representation of the procedure most of them proposed was:

(libri precedenti + libri nuovi) – libri rubati = libri rimasti[3]

Andrea remarked that this way of writing is too long and proposed to write only the initials, as follows:

(l.p.+l.n.) – l.r. = l.r.

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[3] (existing books + new books) – books stolen = books left

Marco remarked that the expressions ‘libri rubati’ e ‘libri rimasti’ have the same initials but different meanings. He suggested to use the first two letters when needed:

(l.p. + l.n.) – l.ru. = l.ri.

Biagio claimed that this system does not work in all cases, since there are words that coincide for more letters, such as

rimasto – rimorchio[4]

Then he suggested to add a symbol to the initials. Andrea said that one could use symbols only, and write a legend such as those reported in many books, as follows.

$$(* + \Delta) - \text{O} = \square$$

Legend:	*	$\longrightarrow$	libri precedenti
	Δ	$\longrightarrow$	libri nuovi
	O	$\longrightarrow$	libri rubati
	□	$\longrightarrow$	libri rimasti

Davide remarked that this looks like a writing from an extraterrestrial people. Then Biagio suggested not to use these symbols but standard letters:

$$(a + b) - c = d$$

Legend:	a	$\longrightarrow$	libri precedenti
	b	$\longrightarrow$	libri nuovi
	c	$\longrightarrow$	libri rubati
	d	$\longrightarrow$	libri rimasti

The pupils wondered which letter they could use to denote the unknown number in any problem. Andrea proposed ‘tot’, which in Italian is used in some idiomatic expressions to mean an unknown amount, generally not a small one. Biagio proposed x, because it is widely used to mean something unknown (‘Mister x’ and so on). So the expression became:

$$(a+b) - c = x$$

and

$x \longrightarrow$ libri rimasti

was added to the legend.

Comments to episode 1

The pupils seem aware of the functions of the representation of the resolution procedure, which is expected to help people not taking part in the discussion and to keep track of methods they could apply to other problems. These functions have not been proposed from someone other but are shared: the pupils regard them as ways to achieve goals they are overtly committed to. So the text should allow the reader to reconstruct the meanings and result understandable and concise. It is relevant, in this

[4] ‘rimorchio’ means ‘trailer’

context, pupils' unchallenged preference for verbal texts compared to numerical expressions: a verbal description allows the reader a much easier reconstruction of meanings than a sequence of computations. Concision is much appreciated as well. Pupils do their best to achieve it without damaging clarity and significance too much. The 'legend' is the key that allows them to go on. The legend allows them to freely arrange a portion of language according to their goals without damaging its accessibility. The adoption of legends provides evidence for pupils' mastery of literate registers, with the explicit definition of the meanings of words, which means a high degree of lexicalization. The legend is a cultural object (as socially accepted) which allows a smooth transition from the everyday-life meaning of words to the conventional one of symbols. At the beginning expressions like 'libri rubati' carry a natural meaning. The abbreviations proposed (such as 'l.r.') are aimed at keeping some link with that meaning. Notice that the episode could have ended here: the polisemy of 'l.r.' could have been neglected, or Marco's idea of using 'l.ru.' and 'l.ri.' could have been accepted. But pupils do not want an *ad hoc* solution: they want a widely accessible text, and a uniform solution, i.e. a solution that can be adopted in other occasions without the need of arranging it each time. Biagio's remark is quite abstract: his argument is more related to the general goal to develop an appropriate notation system than to the specific problem situation. His argument is accepted, notwithstanding its abstractness and the lost of the link to the natural meaning of letters, which is only partly counterbalanced by the legend. The last step, from non-alphabetical symbols to standard letters completes the process: symbols like '∆', 'O', '□' are new and cannot carry natural meanings, then they need a legend. The letters, that are preferred because more practical and accessible, might still carry a natural meaning, but by now explicit definitions (and the process of lexicalization) have got the upper hand: from letters as abbreviations, through iconic signs, pupils get to letters as symbols, with a totally defined meaning.

EPISODE 2

This problem has been dealt with at the end of grade 4.

> A class of 28 pupils spends a week on the mountains. The daily charge for accomodation (7 days) is 65€ per person. The skilift costs 15€ per day. The first and the second day it snows and skiing is not possible. The total cost of transportation (for all the class) is 560€. One pupil, Luca, gets ill and can ski for 3 days only. What is the total cost for each pupil? What is the total cost for Luca?

This time each pupil solved the problem and represented the solution on her/his own. The methods adopted were slightly different each other. Here I focus on Francesca's and Biagio's solutions.

Francesca chose to use 2 unknowns, *x* and *y* and produced the following legend:

a = 28 pupils
b = 7 days
c = 65 € daily cost for accomodation for each pupil
d = 15 € daily cost for the ski-lift

e = 2 snowy days (when nobody can ski)
f = 560 € total charge for transportation
g = 3 days in which Luca can ski
x = ? total cost for each of the 27 pupils
y = ? total cost for Luca

Francesca performed the calculations as follows:

$(b \times c) + [d \times (b - e)] + (f \div a) = x$

then

$x - (d \times e) = y$

then

$(7 \times 65) + [15 \times (7 - 2)] + (560 \div 28) = x$
$455 + 75 + 20 = x$
$x = 550$€ cost for each of the 27 pupils
$x - (d \times e) = y$
$550 - (15 \times 2) = y$
$550 - 30 = y$
$y = 520$€ cost for Luca.

Other pupils, among which Anna, splitted the problem into two cases, the general one and Luca's, and wrote a legend and introduced one unknown (x) for each one. Biagio applied a similar method with one difference: he replaced the occurrence of $b - e$ with 5 at once and organized his computation as follows:

$(b \times c) + (d \times 5) + (f : a) = x$
$(7 \times 65) + (15 \times 5) + (560 \div 28) = x$
$455 + 75 + 20 = x$
$x = 550$ cost for each of the 27 pupils.

Although Biagio's method was only slightly different from the others, it raised some critical remarks. Almost all the pupils conceded that Biagio's method is correct, but they claimed that it does not fit the text of the problem. They argued that if someone should read Biagio's report, he or she could fail to understand where the number '5' came from, whereas the meaning of '$b - e$' could be easily detected through the legend. Francesca's preference for 2 unknowns raised much less criticism.

Comments to episode 2

All methods are equivalent as to the use of letters[5], although the solution procedures described are slightly different. The pupils seem to use a letter to represent a well-defined number that sometimes is given in the legend (e.g., "g = 3 days in which Luca can ski" in place of "g = number of days in which Luca can ski".) The introduction of the second unknown by Francesca is accepted as a normal step, although most pupils agreed that Anna's method was simpler. Biagio's procedure worried the class much more. For most pupils the use of letters was critical in order

[5] See for example Booth (1988) or Usiskin (1988).

to keep some link with the original story. Until all the values are represented by letters, the legend allows anybody to get the original meanings. The replacement of numbers causes the lost of that link.

FROM VERBAL TEXTS TO ALGEBRAIC EXPRESSIONS

The transition from verbal texts to the representation of solution procedures by means of algebraic expressions has widely been recognised as a challenging one. The troubles mostly originate from the semiotic features of verbal language and algebraic notation. Verbal language allows people to design texts that iconically match the actions described (such as texts reflecting the chronological order of the actions) and is equipped with a range of opportunities to make meanings clear and well marked. The same does not hold for algebraic notation.

An empirical study on the difficulties in the transition from story to formulas have been carried out by Radford (2002), even though with much older pupils. Radford argues that such transition is smoother if pupils can manage the links between signs and meanings in a flexible way, developing, changing and abandoning their previous interpretations if necessary. There is evidence that the pupils involved in my study can do this, both when solving a specific problem and in their overall learning process. In episode 1 pupils pass from letters as abbreviations with natural meaning to symbols external to language to new letters with conventional meaning. The unknown '*x*' is introduced which initially is related to the specific context ('books left') but soon is functionally characterised as an unknown to be used in any problem. In both episodes 1 and 2 the signs adopted (letters, abbreviations, icons) seem initially be related to specific numbers, although pupils deal with them as somewhat independent from numbers. If we focus on the transition from a specific problem to similar ones, the independence of the signs adopted from numbers is not directly related to the (mathematical) need for generalisation, through the application of the expressions to other problem situations, but rather to the (semiotic) need for communication, through the explanation, by means of the legends, of the links between the original problem and the new ones. In other words, pupils use letters not just to apply some expression to more problems, but to preserve and communicate the meanings of the expression. For these pupils, the critical point (what Radford names 'the collapse of narratives') seems not to be placed in the transition from a mathematical story verbally expressed to the 'symbolic narrative', but rather in the transition from the algebraic expression with letters to the result of the replacement of letters with numbers. If this process could be generalised, it would allow a smoother transition from arithmetic to algebra. On the other hand, it is necessary to be aware of the difference between the two situations. Radford's problems are significantly more complex, from both the mathematical and the semiotic standpoint. Behaviors as those described in this paper should be tested with more complex problems too, and with problems involving a less smooth transition from verbal text to symbolic expression.

FINAL REMARKS

Within the boundaries mentioned above, the study anyway suggests some new ideas for introducing expressions with letters. The fact that the use of letters can be induced by explicitly shared communication needs, places the teacher's management of the activities at the very heart of the educational process. Communication and generalisation requirements of course are not incompatible each other from the epistemological perspective nor from the actual development and management of teaching units. This means that the current historic-epistemological interpretations many teaching ideas are based on could be profitably combined with a careful investigation of the semiotic functions of the notations. The outcomes discussed here strongly depend on some of the features described above, such as:

- Pupils' linguistic competence, which allows them to profitably discuss and to be aware of the requirements of the transition to the algebraic notation.
- Pupils' full and active involvement in the goals of their activity, including non-mathematical goals.

Needless to say, the teacher's role is critical, as he or she has to run a complex process without restraining pupils' opportunities for taking decisions, which first of all means the opportunity of sharing the goals of their activity.

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VERBAL MEMORY SPAN LIMITATIONS AS A FACTOR IN EARLY MATHEMATICAL LEARNING DIFFICULTIES

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Researchers exploring the underlying causes of mathematical learning difficulties have demonstrated an association between poor working memory resources and low mathematical performance. This paper examines the verbal memory span of a sample of Australian children, with a focus on those children who were identified by State-wide school-based assessment (Queensland Year 2 Diagnostic Net) as at risk of developing early mathematical learning difficulties. Results confirm previous findings, and are discussed in relation to implications for intervention, and possible mechanisms by which verbal memory span limitations might constrain the development of advanced counting strategies in these children at a critical stage.

INTRODUCTION

Although prevalence rates of mathematical learning disability are cited between 5 – 8% (Geary et al., 2004), until recently the cognitive bases of mathematical learning difficulties have been little researched. This is in spite of the well-established finding that a significant number of students will show persisting difficulties in learning mathematics that will seriously limit their capacity to think and communicate mathematically.

Indeed, Russell and Ginsburg's (1984) now classic finding of difficulties with arithmetic fact retrieval and word problems has proven surprisingly robust. In their comprehensive study of 4th grade students whom they described as mathematically disabled, Russell and Ginsburg found that while the students had average abilities in magnitude judgements, and were reasonably adept at mental addition tasks involving 2 and 3 digit additions:

> Paradoxically... they had unusual difficulty with the simplest number facts.
>
> (Russell & Ginsburg, 1984, p. 241)

This outstanding difficulty in remembering basic arithmetic facts has been demonstrated consistently (e.g. Cumming & Elkins, 1999; Geary, Brown & Samaranayake, 1991; Geary, Hamson, & Hoard, 2000; Jordan, Hanich & Kaplan, 2003; Ostad, 1997; Russell & Ginsburg, 1984) and “seems to be the major feature differentiating children with and without learning disabilities” (Ginsburg, 1997).

At the same time, students with mathematical learning difficulties typically remain reliant on slow and ineffective strategies for computing basic additions and subtractions, including finger counting and counting from one. As well as being time-consuming and drawing on attentional resources that could be directed to more

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 81-88. Prague: PME.

complex aspects of operations or problem-solving (Cumming & Elkins, 1999), these strategies are more error prone than those used by normally achieving students.

At the PME conference in 2005, Pegg, Graham and Ballert (2005) reported the successful results of an exploratory teaching program, QuickSmart, focussed on improving the basic skills of pupils aged 11 to 13 years, with similarly low patterns of fact retrieval and achievement to those just described. Following a 25-week intervention the students were able to significantly reduce their response times needed to recall number facts, and these gains were maintained 12 months later. Pegg attributed the additional improvements on standardised tests of mathematics to the students' greater use of retrieval and decreased use of effortful strategies, hence freeing up the demands made on the students' working memory. Pegg (2005) concluded his presentation by calling for research which further explored the relationship between automaticity of basic mathematics skills and working memory capacity.

This paper reports data on the verbal memory span of a sample of 6 to 7 year old Australian students considered at risk of developing a mathematical learning difficulty. The results are part of a larger study aimed at identifying early indicators of mathematical learning difficulties.

THEORETICAL BACKGROUND

The persisting ineffective strategy use of MD students is puzzling because studies of normal mathematical development have suggested that the developmental shift from object counting, through to verbal thinking strategies and eventually retrieval strategies to solve basic additions is based on an adaptive drive to save mental effort (Siegler & Jenkins, 1989). In his strategy choice model, Siegler has demonstrated convincingly that while children use a variety of strategies to solve addition problems, their choices of strategy are generally determined by the efficiency of problem solution (Siegler & Jenkins, 1989; Siegler & Shipley, 1995). Students with mathematical learning difficulties initially use the same strategies as their normally achieving peers (Jordan & Montani, 1997; Geary et al., 2004). What has now been demonstrated clearly is that mathematically disabled students continue to use the less efficient sum strategy, and rely on finger counting for much longer than other students, and they make many more procedural errors (Geary et al., 1991, 2004).

Research exploring the verbal memory span of students with low mathematical achievement offers a promising framework for consideration of how this impasse may occur. Using the Digit Span task Geary, Brown and Samaranayake (1991) found a significant difference between the highest forwards span of mathematically disabled and normally achieving students, with a mean highest forwards span of 4.2 and 5.2 respectively at the age of 8 years. Koontz and Berch (1996) found a similar delay in working memory capacity of 10 year old students with specific arithmetic difficulties, with a mean highest forwards span of 4.98 on Digit Span tasks compared to a mean of 6.0 for the normally achieving students. This consistent finding of a 1 digit delay across studies seems significant in the light of Geary's cross cultural comparison of the digit span and strategy use of Chinese and American 6 year old students (Geary, Bow-Thomas, Fan & Siegler, 1993). Geary's observation that a digit span of 5 was associated with a transition to verbal counting strategies for solving simple additions, whether the students were Chinese or American, suggests that the maturation of digit span from a span of 4 to a span of 5 may be a critical factor in enabling children to develop efficient verbal strategies. The current study was undertaken to investigate the working memory capacity and mathematical performance of Australian students of comparable age.

METHODOLOGY

The results reported in this paper are part of a larger study designed to explore the mathematical skills and cognitive characteristics of students at risk of a mathematical learning difficulty.

Participants and Procedure

A comprehensive range of mathematics and processing tasks was administered to a subset of 60 students in three classes of Year 2 students in two metropolitan schools in Brisbane, Queensland. The mean age of the students was 7.1 years. This report presents the findings of students' performance on one measure of verbal memory span, the Digit Span task, distinguished by whether the students were caught in the State-wide school-based Queensland Year 2 Diagnostic Net (Numeracy) as at risk of falling behind their peers in their mathematical development.

Queensland Year 2 Diagnostic Net (Numeracy)

The Year 2 Diagnostic Net (Numeracy) is a process of assessment and intervention carried out during the first three years of schooling in Queensland (Education Queensland, 1997). Children’s mathematical development in key areas of Counting and Patterning, Number Concepts and Numeration, Operations and Computations, and Working Mathematically with Numbers is mapped onto a developmental continuum listing key indicators of expected progress during Years 1 to. 3. Students are expected to be operating in Phase C (Beginning Number Study) in the middle of Year 2, distinguished by use of the count-on strategy for solving basic additions, and beginning mastery of some addition facts. Any student who has not reached this stage is offered learning assistance at one of 3 possible levels – classroom, small group, or individual – to support their numeracy development.

Digit Span

The Digit Span subtest of the WISC-III (Wechsler, 1991) is administered in two parts: Digits Forward, and Digits Backward. Digit Span includes series of orally presented number sequences which the child repeats verbatim for Digits Forward, and in reverse order for Digits Backwards. The sequences vary in length from two to nine digits, and each item includes two trials. The test is discontinued when the child fails both items of a trial.

RESULTS

Year 2 Diagnostic Net (Numeracy)

Of the 68 students who were available for school-based assessment with the Year 2 Diagnostic Net, 17 students failed at least one of the Key indicators, and were described as “caught” in the Year 2 Net. These students will be referred to in the following analyses as Net students. While some students failed only one Key indicator and could be assisted at the classroom level, others failed several of the Validation tasks. The class teachers and learning support teachers identified 10 of the 17 students who needed more intensive learning assistance at the one-to-one or small group level. These students will be described as Intensive Net students.

Digit Span

Highest forwards and highest backwards span refer to the longest span a student was able to reproduce correctly on a single trial. The range of highest forwards span scores was from span 3 to span 7, while the highest backwards span scores ranged from backwards span 2 to backwards span 4. Overall, almost half, or 27 students out of the 60 students (45%) assessed on Digit Span had a highest forward span of 4. However, a disproportionate number of Net students had a highest forward span of 4 (12/17 or 70.6%), compared with 34.9% (15/43) of the normally achieving students. This group of 12 Net students included nine out of the ten students recommended for intensive intervention by their teachers. One of these students was the only student in the sample to have a highest forwards span of 3.

The mean scores for the longest forward and backward span of the Net students and Normally achieving students are shown in Table 1. The mean highest forward span for the Net students was 4.29 and for the Normally achieving group was 4.95. This difference was significant, $t(58) = 2.659$, $p < 0.05$. If we consider separately the results for the ten Net students who were considered to need one-to-one or small group learning assistance (Intensive Net), the difference in forward span is even clearer: mean Intensive Net = 4.0 compared to mean Normally achieving = 4.95. This difference is significant $t(51) = -3.150$, $p < 0.01$.

Digit Span	*Intensive Net (n=10)	Net (n=17)	Normally achieving (n = 43)
Mean Highest Forward Span	4.00	4.29	4.95
SD	(.471)	(.686)	(.925)
Mean Highest Backward Span	2.90	3.00	3.09
SD	(.738)	(.612)	(.648)
Mean Digit Span SS	8.50	9.00	10.6
SD	(1.841)	(1.837)	(2.546)

Table 1: Mean highest forward and highest backward span on the Digit Span tasks as a function of Net status (*Intensive Net students are a subset of the Net students)

The mean highest backward span results were more comparable with the Intensive Net, Net and Normally achieving students having mean results of 2.90, 3.00 and 3.09 respectively (Table 1). None of the comparisons between groups on the highest backwards span task were significant.

Overall scores on the Digit Span subtest are also reported in Table 1. The Digit Span scaled scores ranged between 6 and 17, and the mean for the total sample was 10.17 (*SD* = 2.478). Comparisons between the performances of Net students (mean Digit Span = 9.0) and normally achieving students (mean Digit Span = 10.6) were significant: $t\ (58) = -.2.361$, $p < 0.05$. The comparison between the Intensive Net group (mean Digit Span = 8.50) and the normally achieving students was significant at the same level $t\ (51) = -2.460$, $p < 0.05$. The results above suggest that this difference on total Digit Span score reflects significant differences between the groups on their forward verbal memory span.

The longest forwards and backwards Digit Span data from the WISC-III standardisation sample (Wechsler, 1991) for 6 to 10 year olds is presented in Table 2. If we compare the Year 2 results with the WISC-III standardisation sample of 6 and 7 year olds, we can see that the mean forwards span for the Year 2 normally achieving students (mean = 4.95, *SD* .925) is consistent with the mean forwards span of WISC-III 7 year olds (mean = 4.98, *SD* 1.03). In contrast, the results for the Year 2 Intensive Net and Net students, with a mean of 4.0 and 4.29 respectively, are well below the mean highest forwards span of the WISC-III 6 year olds (mean = 4.73).

The backwards span performance is of the three Year 2 groups is comparable to that of the WISC-III 7 year olds.

	6 years	7 years	8 years	9 years	10 years
Mean Highest Forwards span	4.73	4.98	5.28	5.67	5.67
SD	(0.94)	(1.03)	(1.09)	(1.09)	(1.11)
Mean Highest Backwards span	2.49	3.05	3.38	3.72	3.89
SD	(1.00)	(0.90)	(0.95)	(0.91)	(0.93)

Table 2: Mean highest forwards and highest backwards span on Digit Span tasks for the WISC-III standardisation sample (6 years to 10 years)

Correlations were computed between performance on the Digits Forward subtest, and other processing and performance measures. Only the results relevant to performance on the Diagnostic Year 2 Net are reported in this paper. The highest forward span was correlated negatively with being caught in the Year 2 Diagnostic Net (Numeracy) $r = -.330$, $p = 0.01$.

DISCUSSION

This paper reports a comparison of the verbal memory span performance of three groups of Queensland 6 and 7 year old students, distinguished by teacher evaluations and whether they were caught in the State-wide Queensland Year 2 Diagnostic Net as at risk of falling behind their peers in mathematical development. The results suggest that the forwards Digit Span task is a sensitive measure of verbal memory span which distinguished on a group level between those students who were facing significant early mathematical learning difficulties, and those who were achieving normally. Moreover, the highest forwards span measure was negatively correlated with being caught in the Year Two Diagnostic Net (Numeracy). This finding confirms previous studies suggesting that relatively poor working memory capacity is a significant factor in the development of early mathematical learning difficulties (Geary et al., 1991, 2004; Siegel & Ryan, 1989), and challenges researchers and educators to find ways of addressing limited verbal memory span in young students.

While the noted differences in highest forwards span between the groups may sound small, consideration of the WISC-III longest span data across the age ranges from 6 to 10 years (Wechsler, 1991) demonstrates that the ability to encode and successfully recall digits in correct sequence is a slowly developing skill, with an average increase of approximately .3 digit per year between the ages of 6 and 9 years. In this context, the demonstrated delay in verbal memory span for the Net/Intensive Net students (well below the expected development for 6 year olds) could be expected to have a meaningful impact on their ability to develop efficient verbal recall, and counting based strategies. We need to acknowledge that the lower verbal span may be constraining students from developing fluent memory for small combinations, that can form the basis for efficient derived fact strategies (Gray, 1991; Gray & Tall, 1994).

Geary has drawn attention to the possible transitional nature of a forward span of 5 in facilitating the development of efficient verbal counting strategies (Geary, Bow-Thomas, Fan & Siegler, 1993). In line with Geary's predictions, the results showed that 9 out of the 10 students whom were considered by their teachers to need intensive learning assistance had a highest forwards span of 4 or less. These students were variously showing difficulties in mastering the counting sequence, including teen/ty confusions, and in reliably carrying out counting strategies, especially with larger addends. Students with relatively low verbal memory spans may need to be given early assistance in alternative means of mastering arithmetic facts by encouraging them to visualise fact combinations (Finnane, 2003). Early attention to promoting partitioning and grouping skills may be a particular issue for students with specific language impairment, where poor counting skills and low verbal memory span may combine to inhibit the development of retrieval based strategies (Donlan, 2003).

As raised by Pegg et al. (2005), students with low verbal memory span are in particular need of automatised number facts, in order to apply their working memory resources to more complex calculation and problem solving. The author (Finnane, 2005) reported the immediate benefits to an 8 year old student of mastering the ten facts through a game based intervention, which gave him the confidence to teach himself the nine times tables through an active process of monitoring his errors. It was argued that automatisation of the ten facts was instrumental in creating the working memory capacity for the student to begin to attend to and correct his errors.

It will be noted that there were other students in the sample with a forwards span of 4 who were not caught in the Year 2 Net. Some of these students showed low performance on other measures of strategy use not reported on in this paper, but this finding indicates that a highest forwards span of 4 is not a sufficient cause for developing a learning difficulty in mathematics. Future research should explore other skills, knowledge and processing variables which may assist a student to compensate for a low verbal memory span.

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CONNECTING ALGEBRAIC DEVELOPMENT TO MATHEMATICAL PATTERNING IN EARLY CHILDHOOD

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Pattern exploration is advocated as an essential element of young children's mathematical development. However, past research has shed little light on the effect that mathematical patterning experiences can have on the development of children's understanding of specific mathematical concepts. This paper explores the content of mathematical patterning experiences that were observed in a multi-site case study conducted in Australian preparatory and preschool classrooms with similarly aged children. These experiences were analysed to ascertain the potential contribution they make to algebraic development. From the results, it appears that the content of these mathematical patterning experiences in prior-to-school environments, provide limited connections to algebraic thinking.

PATTERNING AND MATHEMATICAL LEARNING IN THE EARLY YEARS

Mathematics and patterning are closely interrelated. Mathematics has been described as "the science of patterns" and "the search for patterns" (National Research Council, 1989), while pattern exploration has been identified as a central construct of mathematical inquiry (Heddens & Speer, 2001; NCTM, 2000). Patterning involves "observing, representing and investigating patterns and relationships in social, and physical phenomena, and between mathematical objects themselves" (Australian Education Council, 1991, p. 4).

Globally, there has been increased interest in two research arenas where children's pattern exploration features significantly – early childhood education and algebraic thinking in the early years. Algebra is "a generalization of the ideas of arithmetic where unknown values and variables can be found to solve problems" (Taylor-Cox, 2003, p. 14).

The value of patterning in the early years has been endorsed by many researchers. Owen (1995) suggests that an affinity with and understanding of repeating patterns offers younger children access to "elements of mathematical thought which are not available to them through any other medium in mathematics" (p. 126). Williams and Shuard (1982) also endorse the mathematical value of patterning for young children: "The search for order and pattern … is one of the driving forces of all mathematical work with young children" (p. 330). Hence, from children's earliest years, patterning is foundational to learning because it assists children to make sense of their everyday world. Prior to attending school, children recognise,

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 89-96. Prague: PME.

compare, and analyse patterns in daily events, chants, nursery rhymes, movement, and physical objects.

Research from the past twenty years has concluded that young children are capable of mathematical insights and inventions which exceed our expectations and necessary groundwork and foundations are laid for future mathematics learning (Ferrini-Mundy & Lappan, 1997). Additionally, research has increased our expectations of young children's learning. In the early years, the study of patterns is a productive way of developing algebraic reasoning (Ferrini-Mundy & Lappan, 1997). Steen (1988) has suggested that observations of patterns and relationships lie at the heart of acquiring deep understanding of mathematics – algebra and function in particular.

There is an interrelationship between patterns and algebra in content groupings of curriculum and in research agendas. Patterns, functions and algebra comprise one of the strands in the Principles and Standards for School Mathematics (National Council of Teachers of Mathematics [NCTM], 2000). Members of the algebra working party, which was established at PME27 (2003), include the study of patterning in their research agenda (e.g., Warren, 2005).

The NCTM (2000) also suggests that students need to be prepared for success in algebra by teaching them to think algebraically in the early years. However, if the study of patterns is an effective way of developing foundations in algebraic reasoning, it needs to be accommodated in programming students' early educational opportunities in prior-to-school settings. The development of appropriate curricula to support mathematical learning in the early years environment has received much attention (Clements, Sarama, & DiBiase, 2000; National Association for the Education of Young Children [NAEYC]; & NCTM). However, it is essential that these curricula provide adequate guidance for teachers to support the development of patterning knowledge and algebraic reasoning in young children.

THE STUDY

This paper reports on one aspect of a multi-site case study (Yin, 2003) that investigated the nature of patterning in the pre-compulsory years of schooling. This paper examines the mathematical patterning activities designed and implemented by two teachers in prior-to-school settings and the possible opportunities that these activities provide for developing the foundations of algebraic reasoning.

Setting and participants

This study was conducted in a preschool and preparatory setting because these sites are typical examples of Queensland children's learning environments in the year prior to the commencement of compulsory schooling. The two schools chosen for involvement in the study were located in the inner city suburbs of Brisbane. These schools were geographically close and shared similar socio-economic clientele. The preschool was in a state school and operated a five day per fortnight program. The preparatory class was in a private school and conducted a full-time program of five days per week.

By coincidence both settings had 13 female and 12 male students. The students in each of these programs were required to turn five by 31 December of the preceding year to be eligible to attend. The preparatory class teacher, Mrs Jones, had 12 years experience in early primary classes and was experiencing her first year in a preparatory setting. The preschool class was staffed by Mrs Smith, a four-year trained early childhood teacher, who was experienced in teaching in preschool.

Data Collection and Analysis

A case study was undertaken to gain an understanding of the nature and occurrence of mathematical patterning in pre-compulsory settings. Briefly, this study involved ongoing observations of the pre-compulsory settings until a full day of activities had been observed. This data collection period spanned 4 weeks. Typical of a case study, multiple sources of data were collected. These data comprised a semi-structured interview with each teacher (outside of class times), copies of their programs and video-taped observations of the classes. Analysis of a total of approximately 80 hours of video observations collected in the two classrooms revealed ten mathematical patterning episodes. These comprised of three teacher-planned, four teacher-initiated, two child-initiated and one teacher intervention episode. This paper focuses on the three episodes identified as teacher-planned. Teacher-planned episodes were events containing mathematical patterning, which the teacher planned for the children. The activity appeared in the teacher's daily plans and may have been confined to a verbal dialogue or required the creation of an end product. A discussion of the child-initiated episodes is reported elsewhere (Fox, 2004, 2005).

FINDINGS

Three teacher-planned episodes were analysed to identify the nature of mathematical patterning within the activities. The first episode occurred in the preschool site and involved tessellations. The children created tessellations (the arrangement of shapes to form spatial patterns) using *pattern blocks* on the carpet as one of their small group rotational activities. Mrs Smith questioned the children. "How do you make tessellating patterns?" to which children variously responded "make it grow". The teacher further probed the children's understanding, "What is the difference between a tessellating pattern and one you make in a line?" Children made different responses, such as "It goes by itself", "It goes out" and "It goes round." After the children gave their ideas, Mrs Smith shared her definition of a tessellating pattern by stating "You do the same on both sides."

The children began creating tessellating patterns on the carpet. Four of the five children began their designs with a central shape and then added shapes around the centre point (see Figure 1). One child, Sam, was the only child to create a random linear design (see Figure 2). He made a line of hexagons and red rectangles which were placed on either side of a central shape. When he ran out of hexagons he added shapes in a second layer on top of the first line. Mrs Smith identified this design to the group as a symmetrical pattern. Sam had randomly placed the pattern blocks onto his design and as he ran out of one shape or colour he substituted it for another.

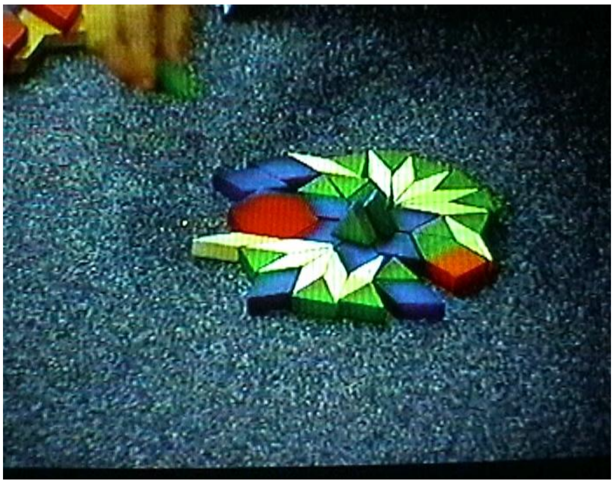

Figure 1. Example of tessellating pattern.

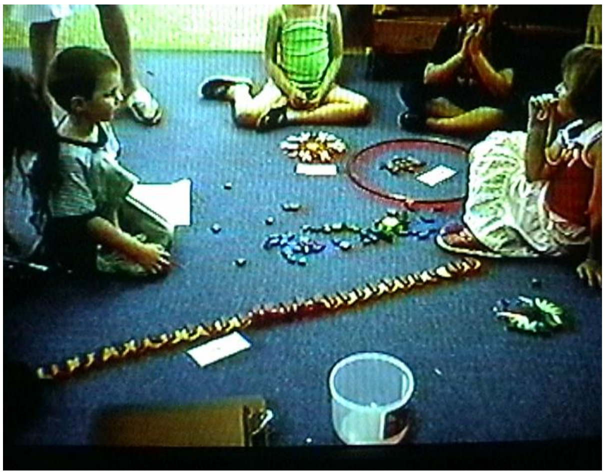

Figure 2. Example of a random linear pattern.

Tessellations involve patterning skills but also knowledge of shape, space and angle. However, this is not the definition that was articulated to the children. The teacher's knowledge of tessellating patterns was not clear and no connection was made to relationships, generalisations, or any other algebraic notions. Working with patterns should encourage children to identify relationships and form generalisations (NCTM, 2000); however this episode incorporated limited references to repeating cycles of shapes or recurring segments.

The second teacher-planned episode required the students in the preparatory setting to create a pattern on a school uniform for a paper doll. The wearing of school uniforms is often regarded by children of this age in Australia as a rite of passage to school. To introduce the patterning activity, Mrs Jones showed the students various items of clothing to demonstrate patterns. The designs on the clothing were a mixture of shapes, colours, flowers, stripes, checks, hearts, and stars. These examples demonstrated random designs and it was very difficult to identify any regularities. It is the repetitive nature of pattern that distinguishes it from random arrangement or design. Mrs Jones mentioned the need for repetition when discussing a floral dress, when she observed the "same pattern over and over again in lines" (Figure 3). However, Mrs Jones did not focus on the identification of repeating elements. She suggested the children could also use "lovely patterns" like stripes, flower patterns, different shapes, or checked patterns on their doll uniforms.

Figure 3. Floral dress.

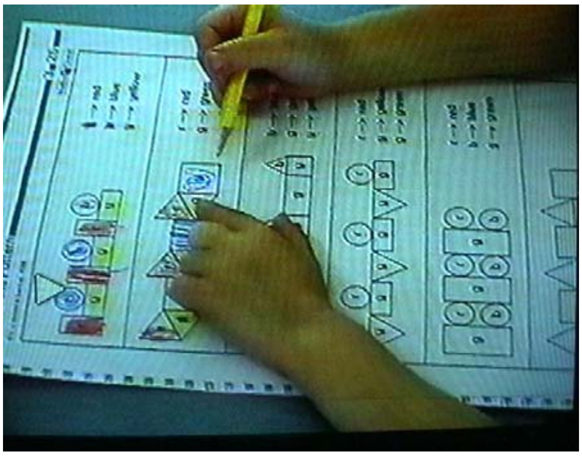

Figure 4. Patterning worksheet.

During the teacher's introduction, three children discovered patterns on their own clothes. The examples shown by both the teacher and the children were a combination of patterns (i.e., repeating designs), line symmetry and random designs. However, all were labelled by the teacher (or children) as patterns. At the end of the activity, only one of the children's doll uniforms depicted a repeating design. The child had drawn stripes on the uniform using an ABC pattern and another child had copied this pattern. The other 11 children who participated drew uniforms of random designs with no identifiable repeating elements. The teacher had not provided the students with consistent examples of pattern features. Thus, children might have been operating from a variety of interpretations of the term 'pattern'.

The third teacher-planned episode, which occurred in the preparatory setting, was to complete a patterning worksheet. The worksheet indicated via a colour code, which colour was to be used in which space, and when completed correctly it would create a pattern (Figure 4). Mrs Jones did not discuss with the children what a pattern was or what she actually expected them to create. The children were unable to decipher the colour code and their attempts to create patterns largely failed. The children did not identify the repetitious nature of the shapes nor did the teacher suggest any prediction strategies. An identification of regularity makes it possible to predict what lies ahead, however these strategies were not identified by the teacher. Essential components of linear patterns were neither verbalised to the children, nor were examples given.

DISCUSSION

The three episodes planned by the teachers had the potential to be meaningful learning opportunities for the students. However, apparent weaknesses in the teachers' knowledge together with the nature of the activities chosen reduced the learning opportunities within the episodes.

Mrs Smith (preschool) designed episodes that explored the concept of tessellation. Tessellations follow the principles of shape and space and incorporate the use of inquiry, discussion and reflection. Students developing tessellation knowledge also require experience with pattern and angle. The guidelines given by Mrs Smith were “make it grow…on all sides.” However, these directions did not fully describe the concept of tessellations.

The episodes developed by Mrs Jones (preparatory) required the students to create their own repeating linear patterns. The objective of the second episode *(uniform activity)* encouraged children to make their own patterns, whilst the third episode *(pattern worksheet)* was to use colour to create a pattern. The teacher’s instructions to the students did not provide consistent information on the development of patterns. Whilst Mrs Jones used pattern terminology such as ‘repeat’, ‘over and over’ and ‘over and over again’, she did not discuss key components of patterns. Furthermore, the teacher did not offer consistent definitions or examples to the student or make explicit features of mathematical patterning. Mrs Jones’ restricted knowledge of patterning or the limited knowledge she shared with the children effectively contributed to the limited opportunities for learning. The promotion of mathematical patterning in pre compulsory settings relies heavily on the teacher’s ability to identify concepts and convey them to the students (Fox, 2005).

CONCLUSIONS

Various forms of patterns, from basic repetition through to spatial surface patterns were documented in the observed patterning experiences. Warren’s (2005) work also showed that children in Australian early childhood classrooms explore simple repeating and growing patterns using shapes, colours, and movement. These forms of patterning activities have the potential to expose children to the beginning notions of algebraic thinking. It was evident however in this study, that both the teachers and the children had limited understanding of the types, levels and complexity of patterns. Experiences with identifying, creating, extending and generalising patterns, recognising relationships, making predictions, and abstracting rules provide foundations for future algebraic development. However, the powerful contribution patterning can have to both mathematical development and algebraic foundations, appears to be largely unrealised in pre-compulsory years classrooms. The NAEYC and NCTM joint statement (2002) clearly stated that patterning, as a component of algebra “merits special mention because it is accessible and interesting to young children” (p. 9) and most importantly patterning “grows to undergird all algebraic thinking” (p .9).

Whilst it is believed that young children are capable of thinking both algebraically and functionally (Blanton & Kaput, 2004) and that work with patterns is valuable in “fostering logical reasoning and algebraic thinking” (Ginsburg, Cannon, Eisenband, & Pappas, in press, p. 12), teachers play an important role in drawing connections and creating explicit learning opportunities. NAEYC and NCTM (2000) claimed that making connections needs special attention: “teaching concepts and skills in a connected integrated fashion tends to be particularly effective” (p. 8). Teachers who are better informed and more knowledgeable about mathematical patterning and algebraic development can provide children with appropriate, meaningful and powerful mathematical foundations. This study demonstrates that opportunities for children to explore mathematical patterning do occur in pre-compulsory settings. However, there is a need for teachers to have a deep understanding of the nature and power of mathematical patterning. Understanding what to teach, when to teach, and how to teach will provide the opportunity for children to engage in rich patterning experiences and to promote meaningful algebraic foundations.

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ALINE'S AND JULIA'S STORIES: RECONCEPTUALIZING TRANSFER FROM A SITUATED POINT OF VIEW

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We find disagreement over the question of transfer of knowledge in the mathematics education literature relating to situated learning. We discuss some theoretical and methodological difficulties raised by this disagreement. We tell Aline's and Julia's learning stories as a basis for contribution to the debate. We conclude by using these stories as a basis for elaborating a reconceptualization of transfer in terms of what we call super-ordinate or overarching communities of practice.

INTRODUCTION

Transfer of knowledge refers in general to the use or application of knowledge learned in one context in another. The question of transfer is quite controversial and, for this reason, has been central in much debate on educational research, particularly in mathematics education. It is suggested by some researchers (e.g. Lave, 1993; Anderson, Reder and Simon, 1996; Greeno, 1997) that those choosing a cognitive perspective would agree that the answer to the question 'Is knowledge transferred?' is 'yes', for knowledge is an individual property. Such a conception would be based on the assumption that knowledge is something relatively stable, generalizable to different situations and characterized by personal attributes in the sense that once acquired, the subject carries it with her from one place to another. As Boaler (2002a) points out, situated learning perspectives offer an interpretation of knowledge that is radically different: a representation of knowledge as activity, as something that is shared or distributed by persons; something that is located between persons, the environments in which they are inserted and in developing activities. From the situated perspective it is not that cognitive structures are not considered, but they can not be detached or abstracted from learning contexts. However, we still find that some mathematics educators taking a situated perspective are evasive in their approach to transfer and that there is also much variation between their approaches. This raises theoretical and methodological difficulties for those who aim to research the issue of transfer from this perspective. We do not provide a wide review of the transfer literature here. Instead we explore how transfer has been approached by some researchers whom we judge broadly to share our theoretical perspective. We discuss the discrepancies and difficulties we find with the aim of encouraging further theoretical and empirical studies concerning the question of transfer of knowledge or, better, knowing. We tell some stories about the learning of two fifteen year-old students - Aline and Julia - as a contribution to the debate. We conclude by using these stories as a basis for elaborating a reconceptualization of transfer in terms of what we call super-ordinate or overarching communities of practices.

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 97-104. Prague: PME.

TRANSFER OF KNOWLEDGE UNDER SOME SITUATED VIEWS

Though some mathematics educators and researchers have been contributing to the debate in innovative ways, there is clear disagreement over transfer in the mathematics education literature relating to situated learning. For example, Greeno (1997) proposes that it is more appropriate to treat the issue in terms of generality of knowing than transfer of knowledge. His intention is twofold: firstly to say that the word *generality* is better than the word *transfer* to express how much the apprehension of aspects of a specific kind of practice and interaction depends on the resources available within that practice or interaction, and how much such *apprehension*[1] depends on resources available in quite different kinds of practice; secondly to say that the participation of individuals in interaction with others as well as with material and representational systems is better represented by the expression *process of knowing* than the word *knowledge*. Lerman (1998) refers to Bernstein's sociological perspective and that of Dowling to support his suggestion that *transfer-ability* in mathematics is a specific activity that can be learned. Such ability would be related to the potential to read texts with mathematical eyes no matter in what form they are presented. And this would only be possible if the subjects were appropriately positioned within the discursive domain of mathematics. Later, Lerman (1999) returns to the problem of transfer adding to these perspectives an examination of the problem from some anthropological (e.g. Lave, 1988) and linguistic/discursive perspectives (e.g. Walkerdine, 1988). Lave treats the issue in terms of meanings within practices. Lerman notes that Lave has recently suggested a more flexible notion of boundaries between communities of practice. This notion sees 'the range of practices in which any individual engages to be overlapping, mutually constituting and related' and 'offers the possibility for conceptualizing transfer across…boundaries, where practices have family resemblances to each other' (Lerman, 1999, p. 96). Lerman observes that, although Walkerdine approaches the issue in terms of a 'disjuncture between practices and of discontinuities of meanings across boundaries' (Lerman, 1999, p. 97), she is concerned to bridge these gaps by identifying areas where there might be overlaps and to show how the teacher can provide a structure in the school discourse so as to allow transfer. Boaler (2002b) uses inverted commas to say that students are able to 'transfer' mathematics under certain circumstances. Her research with secondary and calculus course students led her to conclude that these students were able to 'transfer' mathematics partly because of their knowledge, partly because of the practices they have engaged in and partly because they had developed a productive and active relationship with the discipline. In this way transfer of knowledge is a practice that can be learned, as Lerman points out, but it depends, essentially, on a development by the students of a relationship with the discipline, which articulates knowledge, practice and identity. Winbourne (2002) also shares the idea that transfer and predisposition to learn are strongly

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[1] Our italics. *Apprehension*, in a sense to which we return in the concluding discussion, is a central notion for us

linked. He proposes that the issue of transfer may be better approached in terms of the types of knowledge the students carry with them from one context to another; rather, "as participants in mathematical practices, they carry with them identities that predispose them [or not] towards looking for and making use of mathematical knowledge in a range of contexts" (p. 16). Taking a philosophical perspective Ernest (1998) proposes a classification for mathematical knowledge that takes into account the nature of the *explicit* and *tacit* features or components of mathematics practice (see Frade 2005, Frade & Borges, 2005), and discusses the circulation of these types of knowledge between practices in general. He says that while explicit knowledge from different cultures is easily intertranslated, tacit knowledge, by definition, is not. To be translated, tacit knowledge would have to become explicit first. But, as long as this can be done only partially there will always be residues of tacit knowledge that will remain bounded in the practice that gives its meaning (Ernest 1998, p. 250).

METHODOLOGICAL POSSIBILITIES

The degree of variation in the theorizing of the problem of transfer from perspectives of situated learning raises methodological difficulties for those who aim to research the issue from these perspectives. We agree with Lerman (1999) that 'we would want to ask how can children learn to be conscious of contexts… and cross the boundaries of practices successfully' (p. 94). In this sense there is still much work to be done in the study of crossing boundaries between practices - between school practices and practices out of school. Greeno (1997) recognizes the need to develop a learning theory with broad scope within the situated framework. According to Greeno, situated learning perspectives should focus on the consistency or inconsistency of the patterns of participative processes within situations. These patterns have contents and structures of information, which are important aspects of social practice. Methodologically speaking, empirical research from a situated perspective relating to crossing boundaries should adopt as the unit of analysis interactive systems which include individuals as participants, interacting with each other and with material and representational systems. Lerman (1999) also points to the emergence of studies on crossing boundaries not only to point to directions for the teaching and learning of mathematics, but also to contribute to the development of theories on socially and culturally situated knowledge. Lerman also draws on Vygotsky's psychology to help to conceptualise transfer. In particular, he identifies four elements of Vygostky's psychology that can be used to address the question of crossing boundaries between different contexts of socially and culturally situated knowledge. These elements are: the social origin of consciousness; affect; symbolic mediation as cultural tools; and the notion of zone of proximal development - ZPD (p. 102). Evans (2000) adopts in part a situated position in approaching the problem of transfer. He does not abandon entirely the term 'transfer', and refers to three main forms of transfer; we prefer to interpret these in terms of crossing boundaries between practices: 1) from pedagogic contexts to work or everyday activities; 2) from out-of-school activities to the learning of school subjects; 3) from a specific school subject to another. However, he goes further in relation to two aspects we have only touched upon so far: the first is the importance of emotions in the process of transfer; the second concerns 'the ways in which meanings are carried by semiotic chains, in particular, the capacity of a signifier to provide unexpected links between a mathematical term or problem, and a non-mathematical practice.' (p. 232). These aspects, he says, make transfer difficult to predict or control, and this implies methodological difficulties. We are particularly interested in interdisciplinary school research and, for this reason we will devote the rest of this paper to reflection on the third form of crossing boundaries mentioned above: crossing boundaries between school subjects, and crossing boundaries between what might appear as insulated, non intersecting school practices.

ALINE'S AND JULIA'S STORIES

We will first explore the idea of subject boundaries a little further. Bernstein's (2004) theoretical perspective accounts for the social production of such subject boundaries and their associated pedagogies. From this perspective and that of Wenger (1998) - that practice means 'doing' something not just in itself, but in a historical and social context, which gives a structure and meaning to what is being done - we can agree with Evans (2000) on at least, the following: 1) curricular subject contents co-constitute school disciplines and practices; 2) the structural differences between the language codes used in such practices may be used to define the boundaries between them (though we want also to focus on identity within and, indeed, across such practices); 3) for learners to cross these boundaries they need to grasp or apprehend – and this does not need to be a conscious, explicit or articulated action – something of the nature of what Bernstein (1996) calls 'recontextualisation'. By recontextualisation Bernstein means the process in which the instructional discourses of subject disciplines are inevitably shaped by the regulative discourse operating in the institutional context. The notion of transfer reconceptualized in terms of Bernstein's perspective of boundary crossing, brings with it pedagogic baggage that may well play a significant part in how teachers and students seek to bring it about; from the perspective of situated cognition (Lave, 1988, 1993; Lave and Wenger, 1991; Winbourne and Watson, 1998; Winbourne, 2002) transfer of knowledge is an unhelpful idea. However, given that some people do appear, within the school context or between school practices, to be able to act as if they are transferring knowledge or crossing boundaries between subject contents, how might we account for this from a situated cognition perspective? Bernstein can help us to explain why it is that a major challenge for teachers and students in schools is to do what looks like transfer; what looks like boundary crossing. Boundaries may be socially produced, but they are no less real for this in the experience of teachers and students. So, why is it that some people are so disposed to do what looks like boundary crossing that, for them, boundaries appear completely permeable? Why is it that, for others, boundaries have a solidity which makes the very thought of crossing impossible? We think that it is helpful to account for this kind of boundary crossing in terms of what we will call super-ordinate or overarching communities of practices. We will tell Aline's and Julia's stories as a basis for developing this idea. First we describe briefly what we take to be the contexts of these stories. Cristina and Selma set out to do research with a year-9 secondary school class (28 students) that they both taught that would underpin the development of mathematics and science collaborative work. The objective of the research was to investigate how and under what circumstances such collaborative work might encourage their students to cross the boundaries between these disciplines. The subject matter chosen by the teacher-researchers was proportionality in mathematics, and density in science (the same concept of course from the point of view mathematics, but probably not recognized as such by the students). Cristina and Selma spent a lot of time planning and organizing the materials and activities for their class, and discussing how and when bridges could be built between their disciplines. It was agreed that proportionality lessons would be given first. Cristina made an interactive text about direct proportionality which, among other things, invited the students to discuss some ‘special ratios’: speed, demographic density, energy expenditure during a period of time, and Pi (3.1416…). The students were divided into small groups to work on the text/exercises. When the groups finished this work Cristina encouraged them to talk about it. The proportionality activity took 4 class-hours. Data were collected in the form of 1) students' written exercises, 2) video recording of group discussion, and 2) video recording of interviews conducted by two undergraduate students who were doing their teaching practice in Cristina's class. Selma gave 8 class-hours to the topic of density. Here the students worked through activities from their science textbook and carried out laboratory activities in small groups in which they calculated the density of materials and did some experiments to check the relationship between density and the buoyancy of these materials in water. The science activity was recorded on video and the students’ individual written exercises were collected. Before starting the data collection both teachers talked with the class about the research, its objectives and its procedures. The proportionality activities began in August 2005; the density activities began two months later (November 2005). Aline’s story begins after the work on proportionality has been collectively and carefully corrected. The class had been discussing ‘special ratios’ and Cristina had made careful notes on the blackboard. Cristina asked the students if they knew any other special ratios. Aline said, ‘density’ (and so did a number of other students). Cristina asked what they meant by density and these students said ‘mass divided by volume’. Cristina asked Aline to talk further about the connection between proportionality and density. Both the teacher and the students discussed the densities of water, iron, oil, and other physical materials. At the end of the class Cristina asked Aline and two more students to talk with her. Among other things, she asked them when they had first identified ‘density’ as a special ratio; had it been on the day they had been working on the text, or on the day when they had corrected their work together? All of the students said that they had made the connection when they had corrected their work. Aline said, ‘I’ve already studied this in chemistry. Then I saw the ratio. Then when I compared this with the ratios that were on the blackboard yesterday, acceleration [for example], then I remembered density’. The boy said, ‘I think it was more because the discussion was more open [democratic]. I think it helped. Everybody giving an opinion, saying something, so remembering a little bit here, a little bit there…’. The other girl said, ‘It’s because I saw her [Aline] speaking, I only remembered when she spoke.’ Later in this conversation, Cristina asked if they had worked on density before. They all had - in two different contexts: when they were 11 in a science class (taught, coincidently, by Selma); and recently in an additional private 'cramming' course they were doing for entry to high school. They all commented that Selma didn't like formulas, that she preferred to work on understanding; they voluntarily contrasted her approach with the 'straight' (procedural) methods of the cramming course: 'this is for this, that is for that and you have to use this (formula for density) in this way'. More interestingly, these students suggested an awareness that they had learned different things, albeit with the same label, in the two settings. Julia’s story starts during a technical outing to a hydroelectric plant, by coincidence planned by Selma at the time the class were studying density. The objective of this visit was to observe the phases of the process of energy transformation *in real life*. The first stop the group made was at a reservoir. Their task was actually to watch how water passed through sluices designed to regulate flow into a canal. Sadly, when the students got closer to the reservoir they came across the dead body of a dog floating in the water. The body floated amidst papers, plastic bottles, pieces of wood and other rubbish. The sight of the dead body unsettled the students. Some of them began to wonder how the dog – presumably a good swimmer – had drowned. Others were disturbed by the teacher's observation that the Brazilian attitude to the environment allowed the river to become a rubbish dump. And then Julia exclaimed: ‘floating and sinking!’ This surprised Selma whose agenda was no longer density, but the processes of transformation of energy. But, she used Julia’s observation to revive the talk about ‘floating and sinking’ and new problems arose: why were all those materials floating? The students had no difficulty recalling their studies about density. Some said, ‘because they were less dense than water’. Others said: ‘But, what about the dead body of the dog, why does it float?’ The story concludes with Selma and the students talking about the dead body of the dog and applying what they had learned about density in the science laboratory back at school. Selma also extended the talk to include other scientific relationships, including that between buoyancy of bodies (including human bodies) and the structure of lungs. This talk took over a good part of the visit, and all students engaged in the conversation. What could lead Aline and Julia to make these connections? Why did other students engage so quickly in the conversation that followed these connections?

CONCLUSION

In Bernstein’s terms we might say that Cristina and Selma, having translated for each other their specific discipline codes and worked together to prepare and organize their collaborative work and to build bridges had set up 'something' that enabled the crossing of the boundaries between their disciplines (seen as specialized symbolic systems within a *vertical discourse* (Bernstein, 1996)). Our theoretical construct is that the 'something' these teachers had set up can be seen as a mathematics and science ‘super-ordinate (or overarching) community of practice’ – SCoP, which had some durability and stability. We suggest that it was in large part the activity of

Cristina and Selma that led to the constitution of the SCoP. They planned together and shared their goals and purposes with the students; they worked with the students to develop their understanding of boundary crossing and how to plan for it in connection with specific activities in mathematics and science. In this way a community of practice was constituted which overarched practices that might otherwise have stayed unconnected within the two insulated subject disciplines. From this perspective, the comments and the connections that were made by Aline, Julia and others are signs of the students' participation in such a SCoP as well as evidence of the ZDPs that have emerged within it. Indeed, the students' comments suggest that they have caught on to their teachers' intentions that they learn to make such connections (see Lerman and Meira, 2001) Equally important in the constitution of the SCoP are the predispositions of the students, their readiness to become active participants in these practices. The students' apparent awareness of differences in teaching, of the ways in which these differences are produced and why, might be taken as evidence of their *apprehension* of something of the principles of recontextualisation. It may also be another facet of the students' predispositions and so important in the constitution of the SCoP. A question that must be of continuing concern to us is what schooling might have to do with the development of such dispositions and the understanding that follows from these. Ricoeur (1981, p. 56) notes that: 'The first function of understanding is to orientate us in a situation. So understanding is not concerned with grasping a fact but with apprehending a possibility of being.' Such 'apprehension of a possibility of being' may be a central, possibly a defining feature of our developing idea of SCoP; this idea is closely linked in our minds to Vygotsky's notion of ZPD and we look forward to the joint development of both as a way of planning for the powerful learning of students.

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INSIGHTS INTO STUDENTS' ALGEBRAIC REASONING

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This research examines the mathematical activity of a group of eight-grade students who participated in open-ended mathematical investigations of quadratic functions. In particular, the study traces the students' mathematical reasoning and notation as they work on determining rules of quadratic functions. The study is also an outgrowth of a 3-Year NSF-funded longitudinal research on the development of mathematical ideas and ways of reasoning involving students from an urban minority school district in New Jersey, USA, before they experience formal instruction. The results provide insights into the students' building of meaningful and powerful algebraic ideas and ways of reasoning in the context of the Guess My Rule approach.

INTRODUCTION

This paper describes patterns in the algebraic reasoning of three eight-grade students as they engaged in open-ended mathematical investigations of quadratic functions in the Guess My Rule approach (Alston & Davis, 1996). Typically, researchers make up a rule or equation for a function and ask the students to guess it. Then, they play a game, in which the students provide input values and the researchers return output values according to the rule. A table of ordered pairs is built and students use it to guess the rule. The students can construct the table from a problem situation or get it ready-made from the researchers. Boxes and triangles can also be used instead of the traditional *x* and *y* to denote input and output values, respectively. Students can also graph from the tables or rules.

Three research questions guided the present study: (1) what ideas or conceptions did the students build about functions, (2) how do they represent them, and (3) what connections did the students make between representations of functions? The study is an outgrowth of the *Informal Mathematical Learning* project (IML), an after-school 3-Year NSF-funded longitudinal study (Award REC-0309062) with support from the Rutgers University MetroMath Center for Learning. The project investigated the development of mathematical ideas and ways of reasoning in middle-grade students [6^{th}-8^{th} grades] in problem-solving investigations, involving challenging open-ended tasks in different mathematical domains, which include combinatorics, probability and algebra. In total, the IML project involved approximately fifty students from an urban minority community in New Jersey, USA. This study reports on the mathematical activity of three students in a 3-month algebra strand implemented in the last year of the project. The results provide insights into the students' building of meaningful ideas and ways of reasoning about functions.

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 105-112. Prague: PME.

THEORETICAL FRAMEWORK

There is a substantial amount of literature on students' ability to engage in algebraic reasoning at an early stage (Bellisio & Maher, 1998; Carraher & Earnest, 2003; Schliemann *et al*., 2003). The studies reflect particular views to Algebra. Davis (1985) distinguishes between two opposing views. In one, learning algebra involves the mere acquisition of rules and expertise in the manipulation of symbols. In the other, learning algebra involves the building, from experiences, of algebraic ideas and ways of reasoning about algebraic concepts such as a variable, function, and a graph. In the process, the students develop a mathematical language and notation, which help them describe their mathematical activity. The idea is to help students build meaningful and durable knowledge. Sfard and Linchevsky (1994) differentiate between the notions of symbols as unknown fixed numbers and symbols as variables, which correspond to *algebra of a fixed value* and *functional algebra*, respectively. They argue that algebraic objects may be conceived as series of operations (*operational conception*) or static entities (*structural conception*) and students should be allowed to engage in operational algebra before structural algebra. Confrey and Smith (1994) identify two approaches in the treatment of functions. The *correspondence* approach starts with the building of a rule of correspondence between *x*-values and *y*-values, usually as an equation of the form $y = f(x)$.The *covariational* approach starts with a problem situation and the students construct a table of (x, y) pairs, by first filling in *x*-values, which increase by 1, and then adding *y*-values through some operation constructed in the problem situation. They claim that covariational approach is "more powerful," as it enhances reasoning about rate-of-change.

The Guess My Rule approach is consistent with the covariational approach regarding the emphasis placed on the construction of tables of ordered pairs and problem situations. It enhances the students' building of the notion of a variable and a language to express their mathematical reasoning (Davis & Alston, 1996). The approach also enhances the students' ability to make connections between different representations of functions (Davis & Maher, 1996). In a study using the Guess My Rule approach with seventh grade students, Bellisio and Maher (1998) reported students' successful engagement in algebraic reasoning before the formal study of algebra. They also reported a movement whereby students first verbalized an idea before they attempted to write it in some symbolic form. Similarly, Stacey and MacGregor (1997) reported cases, where students predicted the value of *y*, usually for large values of *x*, but could not describe the relationship between *x* and *y*, or write it in algebraic symbols. This study aims at deepening our understanding of the use of the Guess My Rule in promoting students' algebraic reasoning. The results are consistent with Bellisio and Maher's movement from verbalization to rule writing, Sfard and Linchevsky's (1994) transition from operational to structural algebra, and provide insights into students' development of meaningful and durable reasoning.

METHODOLOGY

This study relies on videotapes of two consecutive 1.5-hour problem-solving sessions with three eight-grade students, in which they worked on determining rules that fit tables of functions $Y = (X-1)^2$ and $Y = (X+1)^2$ within the Guess My Rule approach. The students are referred to as Chris, Ian and Jerel and had previously played the Guess My Rule games with functions of the type $Y = AX + B$ and $Y = X^2 + A$. The data is part of an extensive database of the IML project housed at the Robert Davis Institute for Learning, at Rutgers University, in New Jersey USA. Data analysis built on six video-related treatment procedures (Powell, Francisco & Maher, 2003). These included, (1) watching all videotapes of the sessions to have a sense of the content as a whole, (2) partitioning the data into significant episodes, (3) describing the episodes, (4) characterizing the significance of the episodes, (5) transcribing the episodes and (6) engaging in a structural analysis across the episodes to identify emerging themes about the students' algebraic reasoning. Due to space limitation, this paper focuses only the mathematical activity and insights from four episodes.

RESULTS

This section describes the four episodes that illustrate the students' reasoning. The insights and their significance are presented as conclusions in the in the next section.

Episode 1: Symmetry and a recursive pattern

When working on the rule $Y = (X-1)^2$, Ian noticed a symmetry pattern in *y*- values, which helped him add the point (-3, 16). He had noticed that the numbers 1, 4, and 9 were below and above 0. Since an extra 16 was below 0, Ian guessed that a 16 had also to be above 0 and added the point [Fig. 1]:

Ian: I just noticed something. Look. Look, look, look. 9, 4, 1, 0, 1, 4, 9, 16 [Points at the Y-values column in the table]. It will be 16 up here and a negative three [Adds point (-3, 16) at the top of the table].

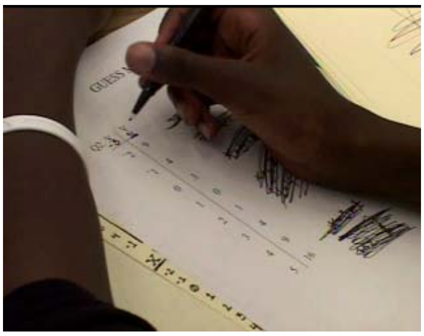

Figure 1: Ian notices symmetry in the table.

Ian then focused on changes in *y*-values in the table. He computed finite differences between *y*-values $(y_n - y_{n-1})$ and noticed another pattern. He plotted the ordered pairs and noticed the same pattern in the graph. The pattern was a recursive rule, whereby, starting at zero, *y*-values increased by 1, 3, 5, and 7 up and down the *y*-column in the table and the *Y*-axis in the graph [Fig. 2]:

Ian: [Pointing in the graph] One. One. One, two, three. One, two, three, four, five. I was right. All that is right there. It's odd, that's it. Look. [Inaudible] One, [Pause] one. One, two, three. One, two, three, four, five. It's right there.

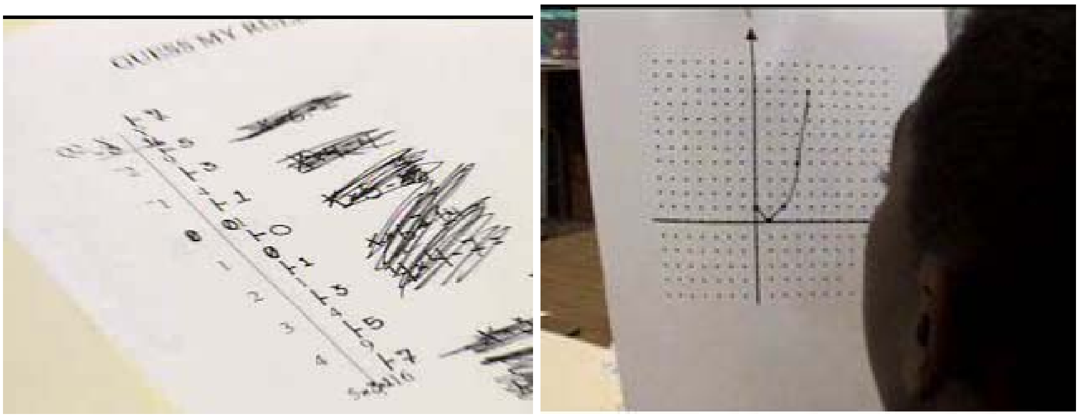

Figure 2: Ian notices a recursive pattern in the table and graph.

The recursive pattern was more powerful than the symmetry, as Ian could add more points.

Episode 2: Family of functions

Ian continued to work on determining the rule for the function $Y = (X-1)^2$. He wrote "5×5 – 9 = 16" and "4×4 – 7 = 9" and claimed to have the rule:

Ian: I got it but. I just got it. Look, four times four minus seven equals nine [writes 4×4 – 7 = 9]. Look. Then if you do the next three times three minus five [writes 3×3 – 5 = 4] I got the freaking answer, but it's not freaking coming up. [Pause] Look. I got it. It's right there! [Adds 2×2 – 3 = 1]. I just don't know what the rule is. It's *x* times *x* minus odd number.

Ian had noticed that he could change *x*-values into *y*-values by multiplying *x* by *x* and subtracting an odd number, which changed by two. This is the family of functions $X \times X - (2X - 1) = Y$, where $2X - 1$ is the odd number that changes systematically [Fig. 3].

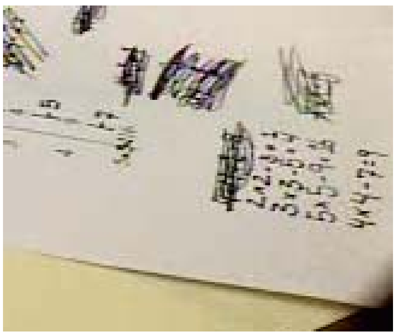

Figure 3: Ian's family of functions

The excerpt suggests that Ian may not have expected the rule to consist of a <u>family</u> of functions.

Episode 3: Identity rules

When working on the rule $Y = (X + 1)^2$, at some point Chris claimed to have a "rule." Chris had come up with an identity expression [Fig. 4]. The other students rejected it because Chris had subtracted and added the same quantity:

Ian: [Checking the rule] I still don't know how you did that. 'Cause you are subtracting and then putting it back [sic adding]. You're a cheater. It's like you ain't [sic aren't] adding nothing [sic anything]. You just put zero. You're a cheater.

Jerel: Oh yeah. He's just subtracting and then putting it back. That's cheating.

Chris: I am smart.

Ian even referred to the students' past experience to question how the rule was written. He suggested that it did not look like the ones they used to write:

Ian: Can you subtract *y* from *x*? Is that possible? Can you subtract *y* from *x*? Cause before we couldn't do that. We could never do that before. We could always put *x* plus *y*. We could never put *y* minus *x*.

The students subsequently dropped Chris' "rule". Moments later, Ian proposed another "rule". However, it was another identity expression [Fig. 4]. Chris immediately pointed out that it had a similar "mistake" as his "rule". He claimed that Ian had used multiplication and division, when he had used addition and subtraction, of the same quantity:

Chris: [Looks at Ian's rule] It is the same think I am doing. You are just dividing and multiplying. It's like this adding [Inaudible]. It is like him multiplying and dividing. It's like me [adding and] subtracting, right?

Jerel: Yeah, you're cheating.

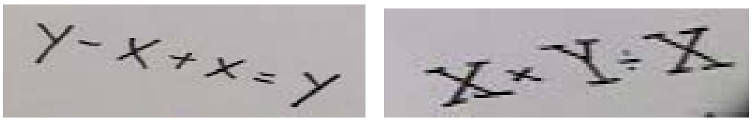

Figure 4: Chris's identity rule [left] and Ian's identity rule [right].

The students dropped Ian's "rule" and continued the search for another rule.

Episode 4: A composite functional representation

The students worked on determining the rule $Y = (X + 1)^2$, for long periods of time, without making much progress. They kept looking for rules of the type $Y = X^2 + A$. This prompted an intervention from the researchers in which the students revisited the function $Y = X^2$ and re-played the Guess My Rule game for the function $Y = (X + 1)^2$.They stopped considering algebraic expressions with A terms and, for the first time, they also started adding or subtracting numbers to *x* before multiplying it by itself. Eventually, Jerel was able to verbalize the rule:

Jerel: I got it. I got it before all of you [smiles in joy]. It is *x* plus one, and then you multiply, I mean, then, you time the sum. I mean then you get the sum, then you time the sum. Ah, then time the sum, then you time the sum by the sum.

The students agreed to write down the rule. Chris' used a notation that included the word "sum." Jerel called it "the new X" and noted as "nx" [Fig. 5]:

Chris: I get it. I get it. It's *x* plus one [pause] *x* plus one times *x* equal *y*.

Jerel: No. Equal the new *x*. Times <u>that</u> *x* equals *y*.

Chris: All right. I get it [writes his equation and show it to others].

Jerel: Yeah. It's the same. But, it's the same. The new *x* plus [sic times] *x* [sic *x*+1] equals *y* [the students writes their equation].

Figure 5: Chris' rule [left] and Jerel's rule [right].

The students used pair $(-2,1)$ to explain their rule. They first computed the sum $(i)\,-2+1=-1$, then the product $(ii)\,(-1)\times(-1)=1$. Interestingly, they argued that the "-1" in the product was not the same as "-2+1" in the sum because, according to Jerel, in the product, "They would not know that that I added first." So, the students had come up with a composite functional representation as $x+1=z$ and $z\times z=y$. In particular, *z* equals Jerel's "new x" and Chris' "sum". For a relatively long period of time, the students used this representation. However when the researchers suggested writing the "new *x*" using parentheses as $(x+1)$, the students were able to re-write their rule as $(x+1)(x+1)=y$.

CONCLUSIONS

There is evidence that the students focused extensively on how x and y-values [co]varied with each other up and down the table and the graph of functions. In episode 1, they attended to the distribution of y-values in the table and graph using finite differences to compute changes in y-values. In episode 2, they explored relationships between x -values and y-values in the table and came up with a family of functions. This is consistent with Confrey and Smith's (1994) claim regarding the advantages of the covariational approach. In episodes 1 and 2, the students came up with symmetry, a recursive pattern and a family of functions. In episode 1, it the students were able to make connections between a table and a graphical representation of the same function.

In episode 4, the students first verbalized a rule for the function before they were able to write it in symbolic form. This is consistent with Bellisio and Maher's (1998) findings about a movement in students' algebraic thinking from verbalizing an idea to writing it down in some symbolic form. In the same episode, the students verbalized their rule using a term such as "the new *x*" and initially wrote down their rule with non-formal notation such as "*nx*". This is consistent with the claim regarding the advantages of the Guess My Rule approach in helping students develop a personally meaningful language to express their mathematical activity and reasoning.

The students' rules of the type $Y = AX + B$ and $Y = X^2 + A$ and their rule in episode 4 seemed to be rules for operations that should be done to the numbers in left column of the table to get the numbers in the right column. Their language in verbalizing their rule included expressions such as "It is *x* plus one", "then you multiply", "then you get the sum", and "then you time the sum by the sum", which support the claim that the students were thinking operationally. The students were also initially reluctant to accept that "*z*" [a product] be replaced by *x+1* [a process]. Jerel's claim that "They would not know that I added first", further indicates that the distinction was based on their perceived roles as either product or process. Therefore, the students seemed to be thinking of algebraic expressions as series of commands or processes. Sfard and Linchevsky (1994) call this operational thinking. This may be one reason why the students had such difficulties to come up with a formal rule and notation of it in episode 4: if $x + 1$ is an operation to be executed, then how could this operation be multiplied by another operation. The students solved this difficulty by inventing a composite functional representation, which allowed them to create an operational rule. This is consistent with the claim that operational reasoning is natural in students and precedes a structural approach. Sfard and Linchevsky (1994) argue that to avoid functional algebra becoming a mere application of arbitrary operations without meaning, students should build an operational basis for the structural algebra by learning first algebra of a fixed value (unknown). This study further suggests that students can build the operational basis in functional algebra.

Students' difficulties in moving from their composite functional representation to the formula $(x+1)(x+1) = y$ are consistent with the challenge that Sfard and Linchevsky's (1994) claim students to experience when trying to make a transition from operational algebra to structural algebra. They claim that it requires the building of a "dual outlook" or "process-product duality" interpretation of algebra formulae, which involves being able to consider algebraic expression as representing both process <u>and</u> product, as opposed to <u>either</u> processes or products, which must be kept separate from each other, often on different sides of the equal sign. Sfard and Linchevsky (1994) add that, in eth absence of the dual outlook, "The equality symbol looses the basic notion of an equivalence predicate: it stops being symmetrical or transitive (p. 104)." They also warn that the transition from a purely operational to a dual process-product outlook is not "a smooth movement" and the reification process may require a "quantum leap". So, it is inconclusive whether the students succeed in building a [durable] dual outlook in rewriting their rule as $(x+1)(x+1) = y$ after the introduction of parentheses notation.

Finally, the students engaged in an interesting discussion in episode 2, regarding whether the identity expressions that they came up with were acceptable functional rules. This means that the search for a rule may also involve a parallel debate on what counts as an acceptable rule. These issues would be regarded elsewhere as being at a cognitive and epistemic level of reasoning, respectively (Kitchener, 1983). In particular, this shows that students are likely to naturally engage in a mathematical activity that emphasizes sense making and justification of ideas.

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PROOF, AUTHORITY, AND AGENCY: INTIMATIONS FROM AN 8^{TH} GRADE CLASSROOM

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Much research in mathematics education has looked at students' conceptions and misconceptions of proof. The attempt to characterize these conceptions sometimes clouds the fact that they are fluid and unsettled. By assuming from the start that students' views on proof are not fixed, one can alternatively try and identify the forces at work forming them. The present paper adopts this second approach. Relying on qualitative data from an 8th grade classroom, evidence is adduced suggesting that students' emerging views of proof may coincide with emerging relations of authority.

INTRODUCTION

In past work (Amit & Fried, 2002, 2005), it has been shown that a 'web of authority' is ever present in mathematics classrooms and that those relations of authority or agency may sometimes interfere with students' reflecting on mathematical ideas. However, "...by shifting the emphasis from domination and obedience to negotiation and consent..." (Amit & Fried, 2005, p.164) it was also stressed these relations are fluid and are, in fact, a *sine qua non* in the process of students' defining their place in a mathematical community. But can these fluid relations be operative also in the formation of specific mathematical ideas? In this paper, we suggest that they may at least coincide with students' thinking about one significant mathematical idea, namely, the idea of *proof*.

Proof has been examined extensively in mathematics education—and rightly so, for it is indeed, as has so often been emphasized, at the heart of mathematics. Research in mathematics education naturally aims, therefore, to uncover the competences and overcome the difficulties involved in students' actually constructing proofs (e.g. NCTM, 2000) and to find ways for promoting mathematical understanding by means of proof (Hanna, 2000). But it has also been long-recognized that these goals depend on how (and if) students understand *what proof is* (e.g. Bell, 1976; Galbraith, 1981; Vinner, 1983).

In this connection, it has been observed, for example, that students often view proving as a matter of producing evidence (Chazan, 1993; Fischbein, 1982). Although in mathematics this 'empirical' view of proof is placed at the bottom of a 'hierarchy of understandings' of proof (Hoyles, 1997), in science education literature the opposite tendency is noted, namely, that explanation is preferred to evidence (Kuhn, 2001) where the need for evidence is recognized as a hallmark of *scientific* thinking. Such contrasts suggest that our sense of students' understanding of proof and, more importantly, that students' own understandings of proof may be highly dependent on context—that in students' minds such understandings are much less fixed, defined, and univocal than characterizations as 'empirical' or 'explanatory' might imply, and, therefore, that these understandings are likely to be continually shaped by circumstances and interactions. Indeed, that students' understandings of what proof is depends on curricular or social contexts has received considerable attention in mathematics education research for many years now (e.g. Hoyles, 1997; Yackel & Cobb, 1996).

For these reasons, in examining our own data, we kept in mind, first of all, that students' ideas of proof were not necessarily well-formed but were in a process of formation; second, that the forces at work in forming them were likely to have a social component, justifying as reasonable our question at the outset concerning authority and proof. The analysis presented in this paper bears out these assumptions.

RESEARCH SETTING AND METHODOLOGY

The research presented here was done in connection to the *Learners' Perspective Study* (LPS). The LPS is an international effort involving twelve countries (Clarke, 2001). It expands on the work done in the TIMSS video study in that instead of examining exclusively teachers and only one lesson per teacher (see Stigler & Hiebert, 1999), the LPS focuses on *student actions* within the context of whole-class mathematics practice and adopts a methodology whereby student reconstructions and reflections are considered in a substantial number of videotaped mathematics lessons.

As specified in Clark (2001), classroom sessions were videotaped using an integrated system of three video cameras: one recording the whole class, one the teacher, and one a 'focus group' of two or three students. In general, every lesson over the course of three weeks was videotaped, a period comprising about fifteen consecutive lessons. The extended videotaping period allowed every student at one point of another to be a member of a focus group.

The researchers were present in every lesson, took field notes, collected relevant class material, and conducted interviews with each student focus group. Teachers were also interviewed once a week. Although a basic set of questions was constructed beforehand, in practice, the interview protocol was kept flexible (along the lines of Ginsburg (1997)) so that particular classroom events could be pursued. This also meant that the interviews often had a conversational character lending themselves to the kind of discourse analyses described by Roth (2005).

The specific classroom which we shall refer to in this paper was taught by a teacher, whom we call Sasha. Sasha is a new immigrant from the former Soviet Union with several years' experience teaching in Israeli schools and much experience teaching in Russian schools. Like many teachers from the former Soviet Union (and unlike most Israeli teachers), his mathematical background is particularly strong, having completed advanced studies in applied mathematics. His 8th grade class is a high-level class and comprises 30 students. The lessons observed in Sasha's class all concerned geometry and, therefore, were particularly appropriate for examining students' ideas of proof.

DATA

Given the page limitation of this paper, we shall present data from one especially telling interview segment only. The focus group for this particular interview consisted of two very bright, spirited, and talkative girls, whom we call Yana and Ronit. The two are very good friends: they are generally attentive to one another, but, as good friends do, they also allow one another the independence to disagree and qualify one another's remarks. This is evident in their discourse style, and it says much about the character of their own interactions.

The first five minutes of the interview concerned the students' notebooks and workbooks. (Note: This was the 2^{nd} interview that day: the time notations carry over, so that this interview begins with '33 min.'.) Because the workbooks contained proofs to be completed by the students, we were able to shift the conversation to the question of proof itself. The initial response of Ronit and Yana was one of incomprehension:

> Interviewer: [38 min] Tell me now, are there also proofs in the book [the workbook], things you have to prove?
>
> Ronit: To prove?
>
> Interviewer: Yes.
>
> Ronit: Umm.
>
> Interviewer: Did you meet up with something you had to prove yourself?
>
> Yana: There are exercises here, what do you mean? I don't understand, like, prove what...like what was on the board?

At this point, Ronit offers what might be called a first definition, which we coded, accordingly, d_0.

> Ronit: [Referring to interviewer's question above] Like correct and not correct.
>
> Interviewer: Yes?
>
> Ronit: But you have to write if it is correct and not correct and to prove why this is correct and why this is not correct.
>
> Yana: Explain what you say.

So, d_0 is that "*Proof* is saying whether something is correct or incorrect and explaining what you say." But, d_0 subtly introduces another characteristic of the discourse, which we playfully coded TW, 'They & We'. Ronit begins by telling us what 'you *have to* do', in other words, what the book or teacher, *i.e*. 'they' *expect* you to do; Yana's refinement, that you explain what you say, adds that part of the proof must come from you, *our* contribution. Therefore, we have here a first hint that the discussion of proof is connected with external authority and individual agency: what *they* require or do or expect and what *we* do and think. But this only becomes substantiated as we move on to the next 'definition' of proof, which is made in contradistinction to 'argument'.

Interviewer: Is 'to argue' and 'to prove' the same thing?

Yana: Uh...depends on the case

Ronit: No, 'to argue' is to say why you think this way.

Yana://No, it depends...

Interviewer: [~39.5 min.] Hold on, Yana. Roni [indicating to her to go on].

Ronit: 'To argue'...

Yana: All right [laughs]

Interviewer: No, it's ok, yes.

Ronit: 'To argue' is to say *why you think that way* [emphasis added], and 'to prove' is, umm, to find something to support what you say.

Yana: Something that [you] already...

Ronit: Something existing, *something you already learned* [emphasis added].

That Ronit is referring to something learned from an external source becomes clear when, restating her position, she adds to 'something already learned', 'something *written'*. Definition d_1, then, was this: "*To argue* is to say why you think something, while *to prove* is to show how something is supported by what you have already learned." This second definition makes it very clear that the distinction between 'they' and 'we' hinted at in d_0 coincides with that between 'proof' and 'argument'; what is proved rests on what has been learned (recall, in the first transcript quotation above, Yana related proof to what was written on the board by the teacher), that is, what came from an external authority, while what is argued rest on what you yourself think.

At this point, Yana questioned Ronit's definition, making a comment wonderfully reminiscent of the 'learning paradox':

Yana: [40 min.] But if you try to prove something new? Then that's not something that's written...

Ronit: Yes.

Interviewer: I don't understand.

Yana: No, if you want to prove something new, like, that no one's ever proven before, then that can't be written, so...I don't know

Interviewer: That is, what you are doing then is...?

Ronit: You [4 secs. pause] prove. [Ronit and Yana laugh]

Just as in Plato's *Meno*, where the 'learning paradox' first appears, this interchange signals a new turn in the conversation towards to one's own proving and concrete instances, that is, a move away from the TW distinction and towards 'we' alone.

In the examples we presented in this phase of the conversation, we had other motives beyond the students' understanding of what proof is—for example, we wanted to see how they understood the logical import of counterexamples and contrapositive statements and the place of diagrams in proofs—but in the course of the examples new definitions of proof arose.

An example (ex. 1) we discussed at length was the following: “If one of the angles in a triangle is right, then the two other angles are acute. Argue yes or no.”

> Interviewer: [44 min.] “If one of the angles in a triangle is right, then the two other angles are acute.” True?
>
> Yana: Yes
>
> Interviewer: You wrote yes.
>
> Yana: We didn’t argue [the point].
>
> Interview: What is the argument?
>
> Ronit: The argument is...
>
> Yana: Umm...
>
> Ronit: That...[Ronit and Yana, at this point, laugh]
>
> Yana: If, wait a minute, if one of the angles, one of the angles of the triangle is right...
>
> Ronit: Since, if one is right and the other is obtuse, then this will go over 180 and then it won’t make sense.
>
> Yana: Also it won’t come out a triangle, one angle is right//
>
> Ronit://It won’t come out a triangle, exactly, one angle is right.
>
> Yana: And one is obtuse, so if you join [the sides opposite the right and obtuse angles], it comes out a quadrilateral, because it comes like this, right.

Now, throughout this whole discussion (which continued beyond what is quoted here), Yana and Ronit referred only to their own thoughts and never once mentioned something ‘learned’, even when they *were* relying on things learned—for example, that the angle-sum of a triangle is 180°. But this was consistent with the distinction they made in d_1: they were ‘arguing’ the point here, not proving, so they were only setting out *their own* reasons for their conclusion. We pressed the issue, therefore, and asked for a *proof* of what they were saying, reviving in this way the questions as to what is a proof and what is the difference between a proof and an argument.

> Interviewer: [~45.5 min.] Suppose I be nasty and tell you *to prove* what you’ve been saying. What do you think?
>
> Yana: What do mean, ‘to prove’?
>
> Interviewer 2: Supposing that [‘to prove’] was written here instead of ‘argue’.
>
> Interviewer: ‘Argue’ [or] ‘Prove’ your words. Will your answer be any different?
>
> Yana: [4 secs.] Here, I proved it [referring, more or less, to what we quoted above]

So, for Yana, at least, the distinction between proof and argument seems to have dissolved, and, with that, also the concomitant distinction between ‘they’ and ‘we’. Eventually, Yana says explicitly that there is no difference between proof and argument:

> Yana: ...For me, I don’t know what the difference is between an argument and a proof.
>
> Interviewer: Any conjecture, then?

Yana: If you write for me ‘argue’ or ‘prove’, I will write the same thing.

Interviewer: [~51.5 min.] The same thing?

Yana: Yes

That proving and arguing are the same thing, we referred to as d_3. Between d_3 and d_1, there was another definition and yet another afterwards, both Yana’s: “If I explain, I think, that if I explain in words and with a diagram I prove [~47 min.]” (d_2); “The proof of a proposition is the claim facing [*sic*.] the argument [52 min.]” (d_4). ‘Definition’ d_4 recalls the two-column proofs that Yana has seen both in class and in her workbook. Yet, like the theorem concerning the triangle angle-sum, there is nothing in the way she frames these ‘definitions’ to suggest an exterior source. The ‘they’ has disappeared—or has it?

One might expect that having worked on their own, felt their own ability to think about a proof, and reflected on their thinking—for example, after the discussion of ex. 1—Yana and Ronit would no longer see proof as something done under another’s authority, that is, that definition d_1 would be discarded, or, alternatively, d_3 would now represent a true harmonizing of one’s own thoughts or agency (‘argument’ of d_1) and the authority of books, teachers, and mathematicians (‘proof’ of d_1). But it turns out that the situation is not so straightforward. For with Yana’s statement of d_3 there ensues a discussion between Yana and Ronit in which d_1 returns in force (and not just as the position of Ronit), with the TW distinction playing an explicit part:

Yana: The argument is your opinion, what you think, and the proof is...

Roni://That is what I think [what I do]

Yana: And the proof is what they write? Like, what others write?

Ronit: No, in fact when you are asked why you think that way, so, umm...

Yana: You are not asked why you think that way, they ask you, argue [!]

Ronit: Come on [Nu! in Hebrew] that’s the same thing. So in fact when you are asked you answer, umm, you think this way because of what you have learned, I think. So, it comes out the same thing since in proof you write what you’ve learned before. [54 min.]

Yana: No, for an argument you write, like, what you say [i.e., what you mean]—that for an argument, that you think this way because of what you have learned and in a proof you write what you have learned...that’s what I understood.

With this return of d_1 (and, in fact, d_3 as well, for recall there was another definition before this exchange), the time has come for us to sum up.

CONCLUSION

The last exchange quoted above was followed by Yana and Ronit’s laughing, partly, perhaps, because of Yana’s not altogether clear last remark. But although the conversation continued a few more minutes along the lines of that exchange, their laughter seemed also to mark some kind of conclusion or summary of the situation. It was a slightly embarrassed laughter. It seemed express a sense of going in circles and of not really understanding what is proof, what is argument, and whether or not they are the same thing. And in a way that's right: Yana and Ronit do *not* yet have a settled understanding of proof, and, yes, in a way, they *are* going round and round. What we have been looking at is one turn in their continually spiralling process of coming to understand proof. We have also seen that this process coincides with a debate about 'they' and 'we', about the authority of a discipline, of their teacher, of their textbook, and their own agency, their own legitimate authority, their own ability to say why they think what they do.

But is this not true also of the mathematical community itself, the mathematical community with which we hope Yana and Ronit might eventually feel some commonality? Israel Kleiner and Nitsa Movshovitz-Hadar (1997), for example, have called proof 'a many-splendored thing' and have emphasized how the nature of proof is unsettled both in the present mathematical world and from the perspective of the history of mathematics. As for going round and round, Jo Boaler (2003) in her PME plenary a few years ago, also reminded us, relying on the sociologist of science Andrew Pickering's work, that in their thinking about mathematical ideas, mathematicians are engaged in an 'dance of agency', balancing their own agency with the authority of the discipline.

The main difference we need to recognize with regards to Yana and Ronit is that they, in contrast with mathematicians, very likely think that a settled view of proof is genuinely to be had, and that mathematical authorities (unalterably separate from themselves) possess such a view. Our challenge, as mathematical educators, is to bring Yana and Ronit to see that their continual debate, defining, and self definition is a normal state of affairs in mathematics. We need, in other words, to take only the embarrassment out of their laughter.

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STUDENTS' THOUGHTS ABOUT ICT IN SCHOOL MATHEMATICS

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In this paper I will report from a survey of students' thoughts about mathematics, in particular their beliefs about and use of ICT-tools in school mathematics. The survey was conducted within an ongoing project ICT in Mathematics Learning that explore how ICT tools can support inquiry into mathematics in a learning community. The results reveal that most students like to use computers or calculators and feel comfortable using the technology, but a minority of students has problems. There were differences between genders on some issues and between levels in school. The survey is part of a longitudinal study.

BACKGROUND AND RATIONALE FOR THE SURVEY

Over the last decades the Norwegian Department for Education has made efforts to stimulate use of ICT in schools aiming to give students competence to utilise computers and calculators as tools in their learning and work. In a new curriculum plan for Norway, effective from august 2006, facility to use digital tools is one of five basic proficiencies in all subjects, including mathematics, alongside with ability to read mathematics, express mathematics orally and in writing and perform calculations. In this context digital tools mean computers, calculators and odigital equipment. At upper levels the aim is that students are able use digital tools for problem solving, simulations and modelling, and to judge when their use is appropriate. Similar recommendations were given in the previous curriculum and in other places relating to mathematics teaching, e.g the NCTM Principles and Standards (NCTM, 2000).

Recent surveys concerning use of Information and Communication Technology (ICT) in Norwegian schools, reveals ongoing activity using the Internet and communication, but with limited use and influence on learning in school subjects, and in mathematics particularly (Erstad, Kløvstad, Kristiansen, & Søbye, 2005). My impression is that work and research on ICT in Norwegian schools has been on a fairly general level, with limited effort to meet the challenge of ICT in specific school subjects, in particular for mathematics (Erstad, 2004).

In an ongoing project named ICT and mathematics learning, the ICTML-project, the aim is to research how ICT can support and improve mathematics learning in school using a developmental research design. The ICTML-project co-operates closely with the LCM-project in developing a learning community involving didacticians[i], teachers and students in school with a focus on inquiry into mathematics (Jaworski, 2004). How can ICT provide situations and support for inquiry into mathematics and mathematics teaching and learning? The research deals with implementation of ICT in mathematics learning and research at all levels of co-operation in the development process and on activities in schools.

A longitudinal study is on-going across the two projects, researching students' thoughts about mathematics and their performance in the subject. Information from the first survey provides a background for further development and later surveys will follow. In this paper I will report from the first survey of students' thoughts, in particular concerning the use of ICT in mathematics. In this context ICT comprises computers in the years 7 – 10 in schools, and use of graphic calculators in upper secondary schools. The classes in upper secondary schools do not participate in ICTML only in the LCM project.

What is going on in the mathematics classes concerning the use of ICT? The purpose of the survey is to describe the present position for a selection of classes in the two projects concerning students' use of ICT in the classroom and their thoughts about using computers. These results from the survey from the first year in the projects will form a background for understanding of students' thinking that might influence their work and later we might be able to describe possible changes in their thinking.

THOUGHTS ABOUT MATHEMATICS AND ICT

Students' thoughts about mathematics in school cover a wide range of questions connected to the affective domain and partly the cognitive domain. McLeod (1992) describes three components of affect: beliefs, attitudes and emotions. Attitudes deal with questions of liking, and enjoying certain topics. Emotions express joy, frustration, aesthetic responses and similar. Beliefs concern what students believe about mathematics, mathematics teaching, themselves and about their social context.

Belief is not precisely defined in mathematics education and we can find several definitions of the concept (Leder, Pehkonen, & Törner, 2002). In this context I will emphasise some considerations about beliefs given by Furinghetti & Pehkonen (2002) who points to belief as connected to the cognitive domain but is subjective knowledge. Beliefs to some extent also include affective factors. The questionnaire prepared for this research encompasses both affective and cognitive elements that can better be described as or formed by the students' experiences or observations from the classroom, such as which tools and methods are used in the teaching. The subjective element is important in this context; we ask for students' opinion not for what is the objective truth concerning the statements. Students' thoughts about computers have been researched also in other contexts and with other methods, for example by Colleen Vale (2005). She found that boys were more positive than girls and thought computers are useful for them.

METHODOLOGY AND DATA COLLECTION

A questionnaire was used for systematic collection of data concerning students' thoughts about mathematics in school, teaching, learning and use of various resources. This data was collected in order to supplements classroom observations and give background for further observations and issues to address in the developmental research.

The questionnaire was given to a selection of classes in the LCM and ICTML projects, with students from grades 7, 8, 9 in compulsory school, later named lower level, and some classes in the first year in upper secondary school.

The questionnaire covered a wide range of questions about the subject of mathematics, about teaching and learning mathematics, the students' competence and experiences in mathematics and so on. The questions were organised in sections with separate headings. A specific section was prepared concerning the use of computers and a few questions about calculators were also included in other sections where this seemed appropriate. I used a few questions from previous projects, including one I have directed relating to ICT in mathematics (Fuglestad, 2005) and some were inspired by other research e.g. Vale (2005).

The questions were presented as statements where the students had to tick on a scale 1 – 4 or 1 – 5 with descriptions according to character of the statement, i.e totally agree, agree a little, etc, or never, rarely, fairly often etc.. The questionnaire was administered via the web in order to give ready access for students and easy import of data into Excel and SPSS for further processing and analysis.

RESULTS

The results were checked for differences between genders and between levels in school, using the Mann-Whitney test for statistical significance. Some differences were found and are reported in the separate sections below. In order to provide an overview the questions are presented in five sections each containing related questions. For the questions concerning computers (101,103-119)[ii] given to students in grades 7 – 10, corresponding questions concerning graphic calculators (144 - 161) were given to upper secondary students. In this way I can compare their thoughts about use of ICT i.e. computers or calculators across levels in school. The questionnaire was given to 469 students and about 70% answered.

Students' attitudes towards use of computers in mathematics

In this group of statements the students were asked if they liked to use a computer (101), if it is useful for them (104) or if it is important to be able to use a computer in tasks (116). In this section I report only the results from lower grades since the upper secondary schools do not use computers. The answers reveal differences between genders on these questions, with boys expressing they agreed to the statement and girls on average closer to neutral. On all the questions except one the differences were statistically significant, with $p < 0.001$. The largest differences were found in response to the question asking if they wanted to use computers more in mathematics (107), where 54% of the boys totally agree and 31 % agree a little and for the girls 46% agree totally or a little.

The last question in this group asks students if they prefer to learn mathematics without computers (114). Here the boys disagree and the girls slightly agree, meaning again that boys express liking to use computers for learning mathematics.

Quest. number		totally agree	agree a little	disagree a little	totally disagree	Mean boys	Mean girls	Mean total	N=
101		31,5	41,8	11,5	15,2	2,1	2,6	2,4	165
104		26,8	41,5	21,3	10,4	2,2	2,8	2,5	164
116		24,2	41,2	26,1	8,5	2,2	2,9	2,5	164
107	Boys	53,6	31,0	9,5	6,0	1,8			84
	Girls	18,8	27,5	38,8	15,0		3,0		80
	Total	36,6	29,3	23,8	10,4			2,4	164
114	Boys	15,5	25,0	32,1	27,4	3,3			84
	Girls	24,1	32,9	32,9	10,1		2,7		79
	Total	19,6	28,8	32,5	19,0			3,0	163
101	I like well to use computers in mathematics								
104	It is useful for me to use computers								
116	I think it is important to be able use computers in tasks								
107	I wish to use computers more in mathematics								
114	I prefer to learn mathematics without computers								

Table 1: Results about attitudes in grades 7 - 9. Frequencies in per cent. 1 totally agree, 2 agree a little, 4 disagree a little, 5 totally disagree

Experiences with ICT in the classroom

Some questions dealt with use of calculators in the classroom (72, 73, 83, 91, 95, 97). Students in all classes were given these questions. There were no significant differences between genders, but significant differences between levels in school. A majority of students answered that they often used calculated answers, 99% in upper secondary schools and 67% at lower grades. There was less use of calculators for investigations and experiments, with 71% in upper secondary school and 57% at lower grades. A majority of students also thought it is important to be allowed to use calculators on all calculations, more important for upper secondary that lower grades, with 61% answering important or very important in upper secondary school. These differences between school grades were significant with $p < 0.001$.

What happens concerning use of computers in the classroom? The students were asked how often the teacher used a computer for introducing a new topic (83). According to the students, this never happens for 77% and rarely for 17% of the students in upper secondary school. It never happens for 42% and rarely for 41% of the students at lower grades. The students were also asked how often they use computers and calculators in the classroom (95, 97). At upper secondary they hardly used computers at all, 80% answered never and 19% rarely. At lower grades they use computers more, but only 30% fairly often or often, and 46% rarely. For calculators the situation is nearly opposite, 83% of the students use calculators often. The results confirm and clarify my observations and what I have learned from contact with schools. It is important to note that the data is not from a representative sample of schools in Norway, but from a selection of those participating in our projects. However, it is my impression from other sources that this picture is not far form the reality in general. Further research is needed to confirm this.

In connection with questions about frequency of use of different teaching aids, including computers and calculators, the students were also asked to state what other aids they used and what they would like to spend more time on. Very few aids not listed in the questionnaire were offered by students so the list gave a fair picture of what was used, but about 55 students of 184 that answered the question, stated they would like to use a computer and 23 suggested more use of games.

Beliefs about use of ICT in mathematics

These groups of statements deal with what the technology is used for and how it influences the tasks and learning of mathematics. For upper secondary schools technology here means graphic calculators and for lower grades computers.

Students agree that they need to know how to use calculators (37). This is strongly agreed by 74% and agreed by another 20% of the students in upper secondary schools. At this level the use of graphic calculators is compulsory according to the curriculum. In grades 7 – 10 the corresponding frequencies were 36% strongly agree and 37% agree.

The students disagree slightly with “Clever students do not need computers to understand” (103, 145). They answer on average fairly neutral to “Since I avoid many errors, I learn easier with computers” (106, 148), with upper secondary students slightly agreeing and lower grades disagreeing.

Students think they still need to do mental calculation when they use computers and calculators, but the results on average give only weak disagreement to the statement “I do not need to perform mental calculations since the computer/ calculator does the work” (105, 147).

For the statement “I prefer to learn mathematics without computers/calculators” (114, 156) the younger students give on average a neutral answer, but for upper secondary the answer is disagree a little or totally disagree for 72 %. The difference in responses here may be due to the different uses of technology, where a graphic calculator can be regarded as necessary for some of the tasks in upper secondary schools. The difference is also significant with $p < 0.001$.

Concerning the purposes for which ICT can be used, the students think it is not just for checking answers (118, 160), and students in lower grades have significantly stronger ($p < 0.001$) opinion than the upper secondary students concerning this question. The students agree that computers/calculators are used for investigations and exploring unknown tasks (117, 159) and they can experiment and test out of their own ideas (119, 161). The opinions are expressed slightly stronger at lower grades, but the main trends are the same with more than 60% of the students seeing ICT as a tool for investigations and experiments.

Question number		totally agree	agree a little	disagree a little	totally disagree	Mean	N=
118	low	6,1	12,1	40,6	41,2	4,0	165
	upper	7,8	30,1	34,6	27,5	3,4	153
	Total	6,9	20,8	37,7	34,6	3,7	318
117	low	20,7	45,7	19,5	14,0	2,6	164
	upper	18,3	35,5	27,5	18,3	2,9	153
	Total	19,6	41,0	23,3	16,1	2,8	317
119	low	31,3	43,6	22,1	3,1	2,2	163
	upper	20,4	41,6	23,7	14,5	2,7	152
	Total	26,9	42,5	22,9	8,6	2,5	315
118	Computers are used only to check answers						
117	We use computers to investigate and explore unknown tasks						
119	With computers I can experiment and find out of my own ideas						

Table 2: Results – kind of ICT use, lower level and upper secondary. Frequencies in per cent. 1 totally agree, 2 agree a little, 4 disagree a little, 5 totally disagree

Self assessment concerning ICT

What do students think of themselves in relationship to computers and calculators? Some statements dealt with their thoughts about their own ability and experiences concerning calculators. The answers clearly indicate that most students think they can manage well with 89 % indicating they totally agree or agree a little and 83 % stated they are rarely or never unable to use calculators correctly. For lower grades the answers are even stronger.

The students only slightly agree to understanding mathematics better with the use of ICT, i.e computers at lower grades or graphic calculators for upper secondary students (108, 150). Most students disagreed to needing a lot of help (109, 151), and slightly disagreed to having to think hard when they use ICT (111, 153). For the statement “I do not understand anything when we use computers/graphic calculators” (115, 157) their answer is even stronger disagreement, and with no difference between the school levels. When tested for gender differences I found significant difference only in response to the statement “I understand mathematics better when we use ICT”, where the boys indicated stronger agreement.

Attitudes towards collaboration

In our project we encourage collaboration between students, and hope that the students also will appreciate working with their peers using ICT. Three questions also take up this concern. The statements were: “I prefer to work alone on the computer” (110,152), “I do not like others to see on the screen what I work on” (112,154), and the third: “I prefer to work together with others at the computer” (113, 155). The results show that students in upper secondary school on average disagree slightly to the first two statements, and students at lower grades are closer to neutral. A majority of the students also agree a little with the third statement, about working with others.

COMPUTERS AT FREE TIME

Computers are commonly used amongst young people, and students seem to use computers more outside school than in the classroom. In addition to the questions concerning their thoughts about computers and calculators in mathematics the students were also asked about their use of computers outside school, at home or with friends. The results reveal that 71% of students in upper secondary schools, and 54% in lower secondary school use computers daily at home, and further 23% and 26% use computers on weekly basis. About 40% also use computers with friends weekly. Games and use of Internet dominate with 45% – 60%, word processing is used by around 26% and spreadsheets only 10%. The students also indicated other software in use; for playing and downloading music, chat and messenger programs. The results concerning the use of computers corresponds with the main trends in other recent survey of ICT use with Norwegian students (Erstad et al., 2005)

SUMMARY AND DISCUSSION

I found some statistical significant differences between genders in this study, mainly on some attitude questions. The boys liked computers and agreed that they are useful and clearly stated they want to use computers more in mathematics. In previous projects I have exposed similar differences between genders in response to questions about attitudes towards computers in mathematics teaching (Fuglestad, 2005). These differences can partly be explained by boys often expressing stronger opinions than girls, showing they master computers and like to work on them. I think this is not the full explanation since the boys also use computers more frequently outside school.

On other questions the differences were between the students in upper secondary school and lower grades. Some of the differences could be due to the use of different equipment and the fact that students are at different levels in their education.

The questions about use of computers in the classroom reveal that there is very little use of computers in upper secondary school and not very much in lower grades either. This confirms the informal impressions gained from visits to classrooms and talks with teachers in meetings and workshops. On the other hand use of graphic calculator is frequent in upper secondary schools, but this is expected because of the requirement in the curriculum plan as is also the need to know how to use them.

Considering the projects' aim to stimulate cooperation and inquiry in mathematics, it is promising to notice that more that 60% prefer to work together and more than 60% of the students regard ICT as a tool for investigations and experiments.

CONCLUSION

The overall impression is that computers have a motivating effect on students work although this is not strong. The students feel confident and think they can manage well, and do not think that they need a lot of help. The students in general want to use computers more, but expressed no strong expectations to learn mathematics from using computers/calculators.

The results indicate that there is not resistance, but rather a weak positive attitude towards use of computers/calculators in mathematics learning. The students seem not to have a strong opinion concerning collaboration in their work on computers. The results reveal that there is room for development concerning utilising computers for teaching and learning mathematics in the classes and with fairly neutral answers to many questions, there seems little resistance to overcome.

Further research in this area is needed in order to evaluate changes in students' attitudes and beliefs during the course of the projects. I see also need for a survey concerning teachers' attitudes and beliefs about computers/calculators and students learning and evaluate their influence on students learning. The teachers' beliefs may have stronger impact than the students' beliefs concerning use of ICT.

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[i] Didactician is used to mean researcher or doctoral student at the university college

[ii] Numbers in parenthesis refer to items in the questionnaire

PRIMARY TRAINEE TEACHERS' UNDERSTANDING OF BASIC GEOMETRICAL FIGURES IN SCOTLAND

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Whilst teachers' mathematics knowledge is known to play a significant role in shaping the quality of their teaching, much less is known about the nature and extent of that knowledge, how it develops, and how such development can be supported through initial teacher training and continuing professional development. Earlier research has indicated that pre-service (trainee) primary teachers' subject knowledge of geometry is amongst their weakest knowledge of mathematics. This paper reports on an analysis of geometry subject knowledge data gathered in Scotland from undergraduate pre-service primary teachers, focusing on their ability to define and classify quadrilaterals. The results indicate that many trainee primary teachers have relatively poor command of these aspects of mathematics.

INTRODUCTION

It is well known that teachers' mathematics knowledge plays a significant role in shaping the quality of their teaching (Ball, Hill & Bass, 2005). Yet as Ball *et al* (*ibid*, p16) explain, "although many studies demonstrate that teachers' mathematical knowledge helps support increased student achievement, the actual nature and extent of that knowledge—whether it is simply basic skills at the grades they teach, or complex and professionally-specific mathematical knowledge—is largely unknown". This is not to downplay the studies of teachers' mathematical knowledge that have been, and are being, carried out. More it points to the complexity of the issues involved, especially since the context in which teachers gain their own mathematical knowledge, and the form of teacher training they receive (both pre- and in-service), can be so varied, not only across countries, but also within particular countries.

The data reported in this paper are from one component of a larger study being carried out in the UK. The over-arching focus is on teachers' knowledge of geometry since, at this time in the UK, the nature of the school curriculum is under review (QCA, 2005) and there are recommendations that the geometry component of the mathematics curriculum requires special attention and strengthening (RS/JMC, 2001).

What is particularly interesting, when focusing on teachers' mathematical knowledge, is the context in which the teachers learn mathematics themselves, and the context in which they are trained. In Scotland, one of the constituent countries of the UK, there is no statutory national curriculum; rather there are national 'Guidelines' for the teaching and learning of mathematics for students aged 5-14 (Scottish Office Education Department, 1991). In these guidelines, geometry (in the

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 129-136. Prague: PME.

form of ‘Shape, position and movement’; *ibid*., 1991, p. 9) is one of four “attainment outcomes” (the others being ‘Problem-solving and enquiry’, ‘Information handling’, and ‘Number, money and measurement’). In contrast, in England, there *is* a statutory national curriculum, with geometry, in the form of “Shape, space and measures”, being part of the statutory specification for mathematics.

Preliminary analysis of data from a component of the wider study is finding that, in England, graduate pre-service (trainee) primary teachers’ subject knowledge of geometry is the area of mathematics in which they have the weakest knowledge (Jones, Mooney & Harries, 2002; Mooney, Fletcher & Jones, 2003). Their personal confidence in teaching geometry, gauged through a self-audit, is also low. This present paper reports on an analysis of geometry subject knowledge data gathered in Scotland from undergraduate pre-service (trainee) primary teachers. The chosen focus for this report is on their ability to define and classify quadrilaterals, partly because research studies have show that school students have difficulties with defining and classifying quadrilaterals (de Villers, 1994, p17; Jones, 2000), and partly because data from observing such trainee teachers has indicated that at least some of them cannot accept, for example, that ‘a square is a special type of a rectangle’.

THEORETICAL BACKGROUND

The terms ‘*concept image’* and ‘*concept definition*’ were introduced by Vinner and Hershkowitz (1980) in the context of the learning of some simple geometrical concepts and developed by Tall and Vinner (1981) in the context of more sophisticated mathematical ideas of limits and continuity. Given that *formal concept definitions* are definitions that are accepted as mathematical, Tall and Vinner (*ibid*, p. 152) defined a *concept definition* as ‘a form of words used to specify that concept’ and *concept image* as ‘the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and process’. In terms of geometrical figures a characteristic feature is their dual nature, in that both concept and image are closely inter-related. In this context, Fischbein (1993) proposed the notion of ‘*figural concept*’ in that, while a geometrical figure (such as a square) can be described as having intrinsic conceptual properties (in that it is controlled by geometrical theory), it is not solely a concept, it is an image too. (*ibid*, p. 141). Thus, when considering a square, it can be regarded as ‘a quadrilateral whose sides and angles are equal (a concept)’ as well as < □ > (an image) and not < □ >.

Taking this approach, on the one hand, individual students can be thought of as having their own *concept images* and their personal *concept definitions* of basic figures, all constructed through their own experiences of learning geometry. In this paper, for the purposes of analysis, we call examples of these a *personal figural concept*. On the other hand, there are formal concept images and definitions in geometry such that, when Euclidean definitions are used, a square, for example, is defined as a quadrilateral whose sides and angles are equal. We call such an example a *formal figural concept*. The research reported in this paper explores the nature of any gap between *personal figural concepts* and *formal figural concepts*.

Research on the teaching and learning of the classification of quadrilaterals illustrates these theoretical ideas. Following de Villers (1994), Heinze (2002) points out that mathematicians prefer a hierarchical classification for quadrilaterals (*ibid* pp. 83-4) and school curricula also follow this. One reason is its economical character, that is, if a statement is true for parallelograms, this means that it is also true for squares, rectangles and rhombuses. While this might seem straightforward to mathematicians, a number of studies have shown that many students have problems with a hierarchical classification of quadrilaterals (de Villers, 1994, p17; Jones, 2000), and this difficulty appears to persists with trainee teachers even though they are expected to have a sound knowledge of mathematics in order to teach this topic effectively. Kawasaki (1989), for example, found that only 5% could write a formal definition of a rectangle, and many of them defined it from their own image of rectangles, for example ‘a rectangle is a quadrilateral whose sides are different’.

All this suggests that a gap exists between *personal figural concepts* and *formal figural concepts* for trainee teachers who have themselves undergone education in mathematics and therefore are supposed to understand mathematical topics up to at least secondary school level. It also suggests that images in their *personal figural concepts* have a strong influence over how they define/classify figures.

METHODOLOGICAL DESIGN

In order to explore this possible gap between the *formal figural concepts* and *personal figural concepts*, trainee primary teachers on a four-year teacher training course in Scotland were selected because the curriculum guidelines for Scotland specify that most pupils are expected to be able to define quadrilaterals and classify them in accordance with their properties by the time they are aged 14-15 (see also Fujita and Jones, 2003a). What is more, the expected level of understanding of mathematics for trainees on the course is that, to be allowed to commence the course, trainee have to have a level of mathematics indicating that they are able to classify quadrilaterals according to their definitions and properties (in Scotland this is called ‘*Standard Grade Credit level*’).

Two sets of data are analysed below. One set of data comes from a survey of 158 trainee primary teachers in their first year of University study (most were 18 years old). After some taught input on the relationship between quadrilaterals, the following questions were presented to the trainee teachers:

Q1. Answer the following questions, and state your reasons briefly.

a. Is a square a trapezium?

b. Is a square rectangle?

c. Is a parallelogram a trapezium?

Q2. A kite is defined as ‘a quadrilateral, which has both pairs of adjacent sides equal’. Define the following quadrilaterals, and draw an image of each.

a. A parallelogram

b. A square

c. A rectangle

d. A trapezium

The design of this element of the study was informed by the research of Kawasaki referred to above.

The second set of data reported below is taken from a task used with 124 primary trainee teachers in their third year of University study (most were 20 years old). To show their understanding of hierarchical relationships in the classification of quadrilaterals, the trainees were asked to identify each quadrilateral in Figure 1 and draw arrows between particular pairs of quadrilaterals to show when one quadrilateral was a special case of another.

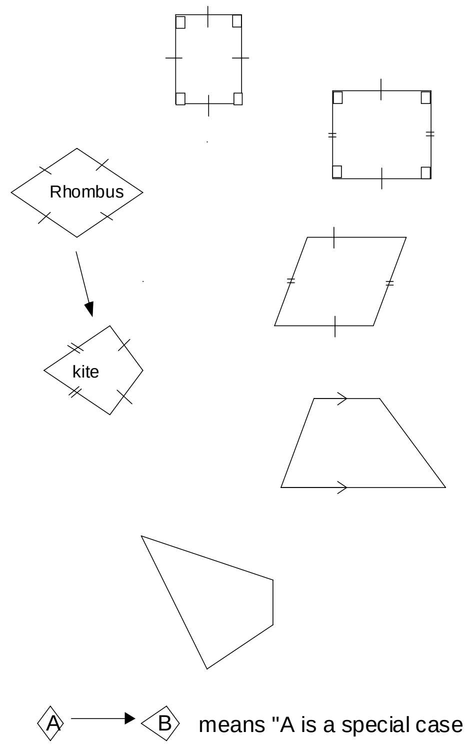


Figure 1: classifying quadrilaterals

For the analysis, we randomly selected 60 manuscripts, about 50%. Prior to the task, the trainees had a number of experiences of the teaching of simple geometrical shapes in primary school and had also studied ways of classifying quadrilaterals.

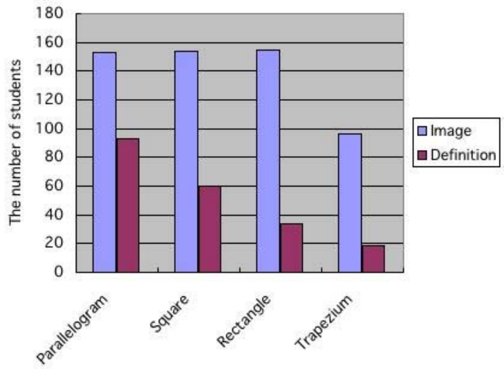
ANALYSIS

The results from the survey of the first year trainees are given in Figure 2 and Table 1 - the Table showing the results from the second question presented to the trainee teachers, and the Figure comparing the numbers of trainees providing the correct image compared to the number providing a correct definition.

This indicates that, for example, 14 trainees (8.9%) answered correctly the question about whether a square a trapezium, 20 trainees (12.7%) knew that a square is a rectangle, and 29 (18.4%) realised that a parallelogram is a trapezium. The latter result contrasts sharply with Kawasaki’s findings that 73% of Japanese trainee teachers can define a trapezium correctly.

Table 1 and Figure 2

Q2a Image Parallelogram	153 (96.8%)
Q2a Definition Parallelogram	93 (58.9%)
Q2b Image of a square	154 (97.5%)
Q2b Definition of a square	60 (38%)
Q2c Image of a rectangle	155 (98.1%)
Q2c Definition of a rectangle	34 (21.5%)
Q2d Image of a trapezium	96 (60.8%)
Q2d Definition of a trapezium	19 (12%)



Comparing image and definition in Figure 2, it can be seen that the majority of trainee teachers could at least draw a correct image of quadrilaterals (with the exception of a trapezium) but far less were able to provide their definitions. In the theoretical discussion in this paper, it was proposed that images in their *personal figural concepts* have strong influence when they define/classify figures, and this is appears to be bourn out in this study. For example, almost all trainees could draw a correct image of a square, while 62% (98 trainees) defined it incorrectly. Of these, 80 (about 82% of 98) wrote ‘a quadrilateral whose sides are equal’ and did not refer to ‘angles’. If they had fully considered their *figural concepts*, they should have noticed that a rhombus can also satisfy this condition, and therefore it would be necessary to include something about the angles as well.

However, it seems that the image < □ > is so strong for them that many do not recognise the need to mention the angles being equal. Similarly, while 155 (98%) could draw an image of a rectangle, only 34 (21.5%) could define it correctly. Almost 70% (86 out of 124) defined a rectangle as ‘a quadrilateral which has 2 longer sides and 2 shorter sides’. Again, they appear to be influenced by the image < ▭ >, and forgot to mention its angles. Moreover, 68 (43% of 158) defined both a square and a rectangle without mentioning angles. The results for parallelogram are slightly better, perhaps because the name ‘parallelogram’ is reminded them of ‘parallel lines’.

Table 2 summarise an analysis of the third year trainee teachers’ manuscripts, with the proportions obtained through counting the numbers of “correct” arrows from one quadrilateral to another (note that. some of the sample also drew additional “correct” arrows, such as, for example, from ‘a square to a parallelogram - such arrows were not counted given the focus is on the efficient characteristics of the hierarchical classification for quadrilaterals).

Table 2

Arrows	Correct answer (%, n=60)
square -> rectangle	65%
square -> rhombus	40%
rectangle -> parallelogram	70%
rhombus -> parallelogram	16.7%
parallelogram -> trapezium	48.3%
trapezium -> quadrilateral	40%
kite -> quadrilateral	28.3%

Incorrect arrows were also found. For example, 13 trainees (about 21%) drew an arrow from 'a rectangle' to 'a square'; that is, they regard that 'a rectangle' is a special case of 'a square'. Similarly, 12 drew an arrow from 'a rhombus' to 'a square'.

The weaker of the links shown in Table 2 occur in the relationships between 'a rhombus' and 'a parallelogram' (16.7%), and 'a kite' and 'a quadrilateral' (28.3%). The reason for these performances is uncertain, but it could be that trainee teachers persevere with their limited images of their *personal figural concepts* of, for example, parallelograms and rhombus and did not fully exercise their logical thinking skills. If they could flexibly 'examine' a rhombus, they might be able to notice that the opposite angles are equal in the rhombus and deduce that the rhombus has the pairs of parallel lines and therefore it is a parallelogram.

In summary, these results could be interpreted as relatively disappointing in that these trainee teachers do not seem to have a good understanding of the hierarchical relationship between quadrilaterals despite the entry requirements. Furthermore, even after two years or more years study on their course their understanding does not seem to improve. This suggests that a gap does exist between the *formal figural concepts* and their *personal figural concepts* such that their images are so influential in their personal figural concepts that they dominate their attempt to define basic quadrilaterals.

CONCLUDING COMMENTS

In Scotland there has been little study of the subject knowledge of trainee teachers. This paper presents in initial attempt to clarify what knowledge Scottish primary trainee teachers have. Further data is being collected of trainee teachers' *personal figural concepts* and their understanding of hierarchical relationship between quadrilaterals. Meanwhile, the data is also just one component of a wider study that brings in data from England. As Ball *et al* (2005, p16) recommend "What is needed are more programs of research that complete the cycle, linking teachers' mathematical preparation and knowledge to their students' achievement".

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AN EMPIRICAL FOUR-DIMENSIONAL MODEL FOR THE UNDERSTANDING OF FUNCTION

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Based on a synthesis of the relevant literature, this study explores students' abilities in four aspects of the understanding of function: problem solving, concept definition, examples of function, recognizing functions in graphic form and transferring functions from one mode of representation to another. A main concern is to examine problem solving in relation to the other abilities. Data were obtained from students in Grades 11 and 12. Findings indicated that students were more capable in giving examples of function rather than providing an appropriate definition of the concept. The lowest level of success was observed in problem solving on functions. Problem solving was found to have a predictive role in how students would apply the concept in various forms of representation, in giving a definition and examples of function.

INTRODUCTION

The concept of function is of fundamental importance in the learning of mathematics and has been a major focus of attention for the mathematics education research community over the past decades (e.g., Sfard, 1992; Sierpinska, 1992; Vinner & Dreyfus, 1989). Functions have a key place in the mathematics curriculum, at all levels of schooling, particularly in secondary and college levels where they get a wide range of expressions and representations. A vast number of studies have used different approaches to explore the concept of function in mathematics teaching and learning (e.g., Mousoulides & Gagatsis, 2004; Sfard, 1992; Sierpinska, 1992). In the present paper we will concentrate on three research domains that have a bearing in our study: The first domain refers to students' concept image for function; the second one concerns the different representations of the notion and the conversion from one to another; and the third one is related to function problem solving.

Concept image and concept definitions are two terms that have been discussed extensively in the literature concerning students' conceptions of function (Tall & Vinner, 1981; Vinner & Dreyfus, 1989). Although formal definitions of mathematical concepts are introduced to high school or college students, students do not essentially use them when asked to identify or construct a mathematical object concerning or not this concept. They are frequently based on a concept image which refers to "the set of all the mental pictures associated in the student's mind with the concept name, together with all the properties characterizing them" (Vinner & Dreyfus, 1989, p. 356). Consequently, students' responses to tasks or questions related to the concept depend on these conceptions and deviate from teachers' expectations.

A substantial number of research studies have examined the role of different representations on the understanding and interpretation of functions (Evangelidou, Spyrou, Elia, & Gagatsis, 2004; Hitt, 1998). The concept of function admits a variety of representations and consequently has the capability of being taught using diverse representations, each of which offers information about particular aspects of the concept without being able to describe it completely. The use of multiple representations (Kaput, 1992) and the conversions from one mode of representation to another have been strongly connected with the complex process of learning in mathematics, and more particularly, with the seeking of students' better understanding of important mathematical concepts, such as function (Duval, 2002; Romberg, Fennema, & Carpenter, 1993). Some researchers interpret students' errors as either a product of a deficient handling of representations or a lack of coordination between representations (Duval, 2002; Greeno & Hall, 1997).

Other researchers addressed the important role of connections between the different modes of representations in functions and in solving problems (Gagatsis & Shiakalli, 2004; Mousoulides & Gagatsis, 2004; Yamada, 2000). Gagatsis and Shiakalli (2004) found that university students' ability to translate from one representation of the concept of function to another is related to problem solving success.

This study attempts to synthesize most of the ideas discussed in the studies of the three aforementioned research domains, i.e. the different ways of constructing mental images of function, using a diversity of representations of the concept and function problem solving, in a four dimensional model, so as to investigate students' understanding of function in a more comprehensive manner. The present study aims to examine students' abilities in four different aspects of the understanding of function: (a) function problem solving, (b) providing a definition, (c) giving examples of function, (d) transferring from one representation of a function to another and recognizing functions in graphic forms of representation. A main concern of this paper is also to examine problem solving, which is considered central for the learning of the concept of function, in relation to the other three aspects of the understanding of the notion.

METHOD

Participants

In total 193 students from two high schools were recruited for this study, i.e. 109 students of Grade 11 (16 years of age) and 84 students of Grade 12 (17 years of age).

Research instrument

A test was developed and administered to the students at the beginning of the school year 2004-05. Below we give a brief description of the seven tasks of the test and the symbols that we used for coding students' responses at each task (most in parentheses): The first task asked students to explain what a function is and give an example of function. A correct definition was coded as "D1". Accurate set theoretical definition was included in this type of answers. An ambiguous definition was coded as "D2". Answers that made a correct reference to the relation between variables, but without the definition of the domain and range or answers that made reference to a particular type of function, such as a one-to-one function, were included in this group. An incorrect definition, which included an inappropriate relation that could not define a function, such as making a reference to relations of elements of sets, or not giving a definition, was coded as "D3".

The following five tasks were developed on the basis of the two types of transformation of semiotic representations proposed by Duval (2002): treatment and conversion. In particular, the second task requested the recognition of functions among four given graphs (G1, G2, G3, G4). The third task asked students to correspond each one of four algebraic equalities or inequalities (AL1, AL2, AL3, AL4) to one of the eight verbal expressions that were given. It is noteworthy that the verbal expressions described what the algebraic relations may represent in a Cartesian graph, i.e. "first and third quadrant". The fourth task requested students to select the algebraic relation among others that corresponded to each of the given two graphs of linear functions (GA1, GA2). The fifth task asked students to select the graph among others that corresponded to the algebraic expression of a function (AG).

Finally, a complex problem on function involving a situation of the real world, based on a problem used in a recent study by Mousoulides and Gagatsis (2004), was the sixth task of the test. It consisted of textual information about a tank containing an initial amount of petrol and a tank car filling the tank with petrol. Students were asked to use the given information to draw the graph of the amount of the petrol in the tank with respect to time and the graph of the amount of the petrol in the tank car with respect to time. Sketching correctly the graphs of the two linear functions were coded as "Pa" and "Pb", respectively. Incorrect sketching of the former graph due to the wrong conception that the two variables were proportional, or due to rough drawing were coded as "Ea1" and "Ea2", respectively. Rough drawing and thus incorrect response in relation to the latter graph was coded as "Eb". Next, students were asked to give the time that the tank needs to be filled (Pc1) and the amount of petrol in the tank car at that time (Pc2). Lastly, students were requested to find when the amounts of petrol in the tank and in the car would be equal (Pd1) and the amount of petrol in the tank or the car tank at that time (Pd2). Locating incorrectly the intersection point of the two lines in the latter questions due to rough sketching was coded as "Ed".

Data analysis

Primarily, the success percentages were accounted for the test and crosstabs analyses were performed by using SPSS. A similarity diagram (Lerman, 1981) was also constructed by using the statistical computer software CHIC (Bodin, Coutourier & Gras, 2000). This diagram, which is analogous to the results of the more common method of cluster analysis, allows for the arrangement of students' responses at the tasks of the test into groups according to their homogeneity.

RESULTS

Success percentages

For this study's needs only the results, which illustrate the strongest trends among the students, will be presented. A large percentage of students (60.6%) provided an appropriate definition of function. The highest success percentage (74.6%) in the test was observed at the question requiring an example of function. In the second task concerning the recognition of functions given in a graphic form, the disconnected graph of a function was recognized only by 46% of the students, indicating the idea of the majority of the students that the graph of a function must be connected or "continuous". In the third task, which required the correspondence of algebraic expressions of relations to the appropriate verbal descriptions, the easiest part was the equation $xy=0$, which was accomplished by 56% of the students. Lower success percentages were observed at the other parts of the same task (30.6%- 36.8%), which involved algebraic inequalities, i.e. $xy>0$.

In the fourth task, involving the selection of the correct algebraic expression for given functions in a Cartesian graph, the most difficult part was the decreasing linear function $y+3x=1$, which was carried out successfully only by 37% of the students. A high level of success (60.1%) was identified at choosing the line in a Cartesian graph that represents the function $y=-2x+1$, $-2 \leq x \leq 2$, as the construction of a graph on the basis of an algebraic expression by placing points is a standard activity of school mathematics (Duval, 2002).

Students' achievement reduced radically in solving a complex problem on functions, as only 15% of the students provided correct responses at all of the four parts of the sixth task. The solution of the problem required effective mathematical modelling, which involved the understanding of the connection between the real life situation that was presented and the corresponding mathematical relations.

Results based on the similarity diagram

Two distinct clusters, namely A and B, are identified in the similarity diagram of students' responses at the tasks of the test shown in Figure 1. Cluster A, which involves strong similarities, includes two groups, namely, GrA1 and GrA2. The strongest connections are formed in GrA2 by the variables Pa, Pb, Pc1, Pc2, Pd1 and Pd2. This means that students responded similarly at all the questions of the sixth task. This remark is consistent with the same percentages of students (15%) who succeeded at all of the parts of the problem. Group GrA2 is linked to GrA1, which involves the following variables: G2 and G3, representing the correct recognition of some non-conventional cases of relations in the form of Cartesian graphs; GA1 standing for the correct selection of an algebraic expression corresponding to a linear function in a graphic form; and AL1, AL3 and AL4 signifying the correct correspondence of algebraic expressions of inequalities to verbal descriptions. The two subgroups are connected with the correct response G4, i.e. recognizing whether a relation in a graphic form represents a function or not and the accurate correspondence of an algebraic equation to its verbal description (AL2). Cluster A is completed by the correct definition of function (D1), indicating that students who accomplished a correct solution to the problem of the test and responded successfully at the aforementioned tasks on functions by using graphic, algebraic and verbal representations provided also an appropriate definition of the notion.

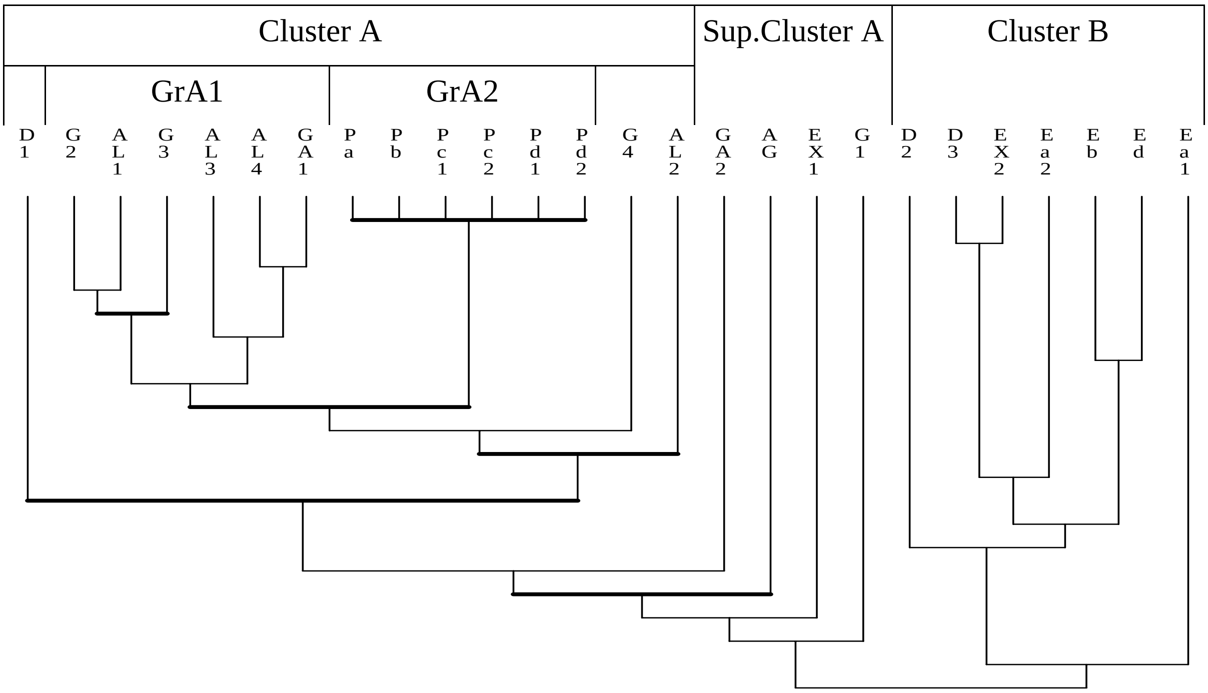

Figure 1: Similarity diagram of students' responses in the test

Similarities presented with bold lines are important at significant level 99%

Cluster A is supplemented by the following variables: GA2 signifying a successful recognition of the algebraic expression among others that corresponds to a linear function graph; AG standing for carrying out the reversible conversion, i.e. choosing the graph among others that corresponds to the given algebraic expression of a function; EX1 representing a correct example of function; and G1 involving the recognition whether a relation in a graphic form represents a function or not. Cluster A with its supplements entail a conceptual approach to function, integrating the four aspects of the understanding of function that are explored in this study: a) problem solving; b) recognizing whether a graph represents a function or not and transferring diverse types of function from one representation to another; c) defining the concept and d) giving an example of the notion. The results of the crosstabs analyses, which allowed us to investigate students' achievement in function problem solving in relation to their performance at the conversion and recognition tasks as well as students' constructed definitions and examples of function, provide further evidence to the above inference. All of the students (29 in number), who provided a correct solution to the problem of the test, gave a correct definition and an appropriate example of function. They also responded correctly at all of the recognition and the conversion tasks. These findings reveal the consistent and coherent behaviour of the successful problem solvers in all of the dimensions concerning the understanding of function that are examined in the present study.

Cluster B is formed by the variables D2 and D3, representing an ambiguous definition of function and an incorrect definition or not giving one, respectively; EX2 standing for an inappropriate example of function; Ea2 and Eb signifying incorrect sketching of the required graphs of the problem due to rough drawing; Ed which means locating an inappropriate intersection point of the two lines of the problem due to inaccurate sketching; and Ea1 involving the erroneous construction of the former graph of the problem by drawing the line passing through the point (0,0). As a whole, cluster B illustrates vagueness or a limited idea for the concept of function, regarding the definition and the examples of function, as well as, problem solving. Further support to this assertion is offered by the findings of the crosstabs analyses, which illustrated that a significantly larger number of students who gave correct definitions ($22.17 \leq x^2(2) \leq 58.48$, $p<0.01$) and examples ($11.61 \leq x^2(1) \leq 42.27$, $p<0.01$) succeeded in the recognition and conversion tasks involving diverse systems of representations as well as in problem solving, relative to the students who did not give correct definitions or examples. It is noteworthy that none of the students who gave ambiguous, incorrect or no definition or an inappropriate example of function succeeded in problem solving.

DISCUSSION

This study set out to investigate high school students' abilities in four different aspects of the understanding of function, i.e. problem solving, concept definition, examples of the notion, use of a diversity of representations of the concept, and how problem solving is associated with the other abilities. All four factors of mathematical thought examined in this study, described in their own unique way different aspects of students' progress of the acquisition of this complex concept. Findings showed that students were more able to give examples of function rather than providing an appropriate definition of the concept, probably because the formal definition is not discussed so systematically in an explicit manner as different examples of function in school mathematics. This finding is in line with the results of Evangelidou et al. (2004), who showed that the majority of university students (prospective teachers) did not give a correct definition, but made reference to an ambiguous relation. Moreover, students' constructed image of the function concept may deviate from the formal definition of the concept that is introduced in high school (Vinner & Dreyfus, 1989). Students' achievement in the different types of conversion of function among various modes of representation and recognition whether different relations represented a function or not varied with respect to the type of the relation involved and the direction of the conversion. The lowest level of success was observed in problem solving on function.

Findings showed that strong similarity connections existed between students' problem solving achievement, their abilities to handle different modes of representation of the concept in recognition and conversion tasks and to give a correct definition and examples of function. This indicates that problem solving, concept definitions, examples and ability to use different representations were not independent entities, but interrelated in the thought processes of students who accomplished a conceptual understanding of function. The relation between problem solving and ability to translate among different representations was revealed also by the findings of previous studies (Gagatsis & Shiakalli, 2004; Hitt, 1998).

The present study also provided support to the important role of problem solving in the understanding of function. It is evident that successful problem solving ensured the success in all the other dimensions of the understanding of function, examined here, revealing its predictive role in how students would apply the concept in various forms of representation, in giving an appropriate definition and examples of function. Students who accomplished problem solving, were successful in recognizing functions in a graphic form, in carrying out conversion tasks, in giving a definition or an example of function. In other words, students who demonstrated deficits in at least one of the three aforementioned abilities would surely fail in problem solving. An interpretation for this finding, which is in line with previous studies' findings (Gagatsis & Shiakalli, 2004) is that problem solving is a complex process that involves various abilities and in this case probably skills referring to the other three aspects of the understanding of function, examined here. For instance, the solution of the particular problem of the test required among other abilities the coherent articulation and coordination of various representations of function, i.e. verbal, algebraic and graphic, as well as, acquisition of what a function is (definition) and of different types of functions (examples).

The above assertions have direct implications for future research as regards the teaching practice on function. We believe that it is not adequate to describe what students know about a particular concept or how they use it on the basis of the particular dimensions that we proposed here, i.e. definition, examples of function, use of the concept in a diversity of representations and problem solving. It could be interesting to examine whether designing and implementing didactic activities that are not restricted in limited and separately taught aspects, but interconnected with each other on the basis of the above forms of understanding of the notion, may contribute to the development of a global and coherent understanding of function and successful problem solving. The results of such a research would be enlightening for mathematics educators about the importance of using a four-dimensional model constituted by these types of ability as a means not only to examine and explain how the function concept is understood by students, but to teach functions at secondary school.

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THE HIDDEN SIDE IN TAIWANESE CLASSROOMS - THROUGH THE LENS OF PLS IN GEOMETRY

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The paper presents a study aimed to inquire the factors and reasons causing a relatively big group of low achieving Taiwanese students to be left behind, while others have gained one of best results in the world (TIMSS, 2003). Using the methodology of Problematic Learning Situation (Gal & Linchevski, 2000), we characterised 9th grade geometry lessons. We identified situations of a double-focus difficulty - students' and their teacher's to help them, and went through multi-dimensional analysis to generalized class characteristics which were contributing to the gap. Results point at lack of learning opportunities for the lower achieving students.

BACKGROUND

International comparisons present excellent performances of some countries - e.g. Singapore, Japan, Korea, and Taiwan in TIMSS. However, a closer look reveals un-homogeneous results, which show that some students are left behind. Taiwanese 8th graders achieved very high average score in TIMSS (2003). Still, 15% of the students were "left behind", situated below intermediate or low benchmarks. This percentage is higher than other "best achieving" countries. One of the dimensions which could shed some light on the phenomenon and provide some explanation is the character of teacher-class interaction by means of teachers' awareness of their students' thinking processes and the way teachers cope with their students' difficulties. In this paper we wish to study the "left behind" phenomenon in Taiwan by zooming in geometry Problematic Learning Situations - PLS, after Gal & Linchevski (2000) - situations having a double-focus difficulty: student's difficulty and the teacher's difficulty to cope with it. The main "participants" in PLS are students below intermediate and their teachers. "Intermediates", 19% according to TIMSS, are also "candidates" for PLS.

The study should take into consideration cultural characteristics. Eastern, especially Chinese teaching approaches and learning habits, being different from western ones, were described in several studies in the last decade such as the wide-ranging study by Fan, Wong, Cai & Li (2004). As far as we know, Chinese-oriented or an east-west comparative study attending *specifically geometry* problematic learning situations have not been reported before.

This paper refers to the following questions: (a) What is the "pattern of learning" of low achievers in junior high school classes in Taiwan, i.e. what are the atmosphere, the reasons, the catalysers and the "partners" in which low achievements grow? (b) What is the nature of Problematic Learning Situations in geometry lessons in these classes, and

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 145-152. Prague: PME.

what are their appearance and their cognitive origin, as seen by the researchers? Further perspectives of the study will be reported elsewhere.

METHODOLOGY

Aiming to shed light on the lower achievers in Taiwan, and to get a deeper understanding of the situation and its roots, we used a new methodology according to which the delving into classroom occurrences was made by a careful capture of Problematic Learning Situations. The PLS were detected by observing actual class instruction. During the observing, or using class documentations later on, we used a double-focused search for moments where (a) *student*(s) had any difficulty, either expressed explicitly (e.g., announcing a difficulty, answering mistakenly, objecting other student's correct answer) or implicitly (e.g., being silent, answering mistakenly in a quiet voice that the teacher did not hear, writing a mistake in his/her note book), and (b) the *teacher* had difficulty to cope with the problem, including teachers' inability to even notice the difficulty, reacting in a way which did not satisfy the reason for the difficulty, or having no alternatives and just repeating a previous explanation.

The PLS were analysed by a multi-dimensional double-focused analysis (Gal, 2005) which uses various cognitive theories, such as van Hiele's theory about the development of geometrical thinking (e.g. Hoffer, 1983), conceptualisation (prototype, concept image etc.), and perception (e.g. Gestalt principles). The analysis of the PLS were generalised into class characteristics.

The research took place in Taiwan. We observed 5 teachers' 9th grade classes: two of them in Taipei City and three in regional schools of Taoyuan County. All schools were considered by local math educators as average schools. Lesson observations had manifold perspectives. In this paper we report only the first - the *Researchers-observing*. Each teacher was observed for 2-4 lessons. Altogether, 15 geometry lessons were observed, in which an "outsider" (Israeli) researcher was accompanied by an "insider" (Taiwanese) researcher. All lessons were videotaped, transcribed and translated into English. A short interview was held with the teacher after each lesson, asking for any difficulties he/she detected during the lesson. In order to expand our perspective on students' understanding and difficulties, we asked the teachers of each class to provide us the students' latest test papers. We used also a class-topics adjusted questionnaire, which we planned according to each class's relevant topics. The teachers also reported about their experience, professional background and alike.

FINDINGS

Results are based on whole-lesson observations, analysis of the PLS we detected (an average of 6-12 PLS in each lesson; each one around 2-8 minutes long), students' assignments and other data previously mentioned. Our first interest was to study the "pattern of learning" in which low achievements grew. Several observed characteristics - some are traditional Chinese - were probably contributing to reduce difficulties: (1) *Coherence* within and across lessons (Wang & Murphy, 2004). (2) The *"two basics" principle* (Zhang, Li & Tang, 2004), basic knowledge and basic skills, e.g. knowledge of quadrilaterals' properties or of properties which can be used for "backward reasoning" as well as skills of constructing quadrilaterals with different givens. (3) *Teaching with variation* (Zhang et al., 2004), emphasizing different ways and methods to solve the same problem.

Looking for characteristics which were catalysers of low achievements, we recall common beliefs, which usually blame some students' low achievements onto their low social-economical status, unwillingness to learn, or personal disabilities. However, the following results give some clues as to different reasons for low achievements.

I. High expectations. The tasks given by the teacher introduced high expectations from students. Some of the tasks which were presented to the class were sophisticated and required high competencies and mastering of properties, theorems, visual abilities, etc. Indeed, high competencies of high and average students were observed in reasoning, proof and computational geometry problems. For example, more than a few students could participate in class discourse when they were asked to compute the shaded area of a given parallelogram, whose area is 1 unit and whose two opposing vertices are connected to the mid-point of the "opposite" sides (see Fig. 1). Subjects which were studied in the 9th grade, such as sufficient conditions for specific quadrilaterals, constructions of quadrilateral and similarity of triangles were generally correctly demonstrated and answered by many (but not all) students in each class. This is consistent with the international comparisons (TIMSS, 2003).

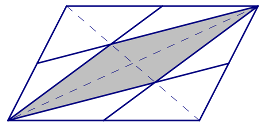

Fig. 1

Unfortunately, these expectations could not be satisfied by the lower achievers. The written data which we collected (class tests and researchers' questionnaires) introduces the other group as well: 20-35% of the students in each class failed to answer most of the given written questions (teacher's tests and researchers' assignments revealed similar results). For example we asked:

> In the given figure (Fig. 2) you are told that the **dotted sides** are equal. Is that a parallelogram? Why? Will any additional data make you change your mind?

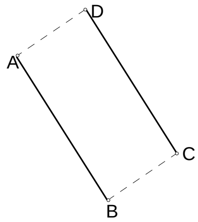

Fig. 2

In both classes in which that question was given, around 28% could not answer it properly. Analysis of their answers pointed at van Hiele's level 1-2 argumentations.

Observing the class through the "students' lens", we detected in each class several students who seemed to be "away" from the lesson, not attending any part of the class discourse, assignments and alike. Very often their teacher approved our impression, testifying that those students gave up their mathematics learning. Referring to one of the students who could not answer, his teacher said: "… because some children are more passive in learning… in teaching we see children just give up learning."

II. Whole class teaching - "the pace of classroom teaching is led by teacher's judgement based on most students' learning ability in the class" (Zhang et al., 2004). Unfortunately, since the teacher has the whole class to lead, this approach is frequently considered as conflicting taking care of the individual's difficulties. One of the teacher's words -

similar to other teachers' were: "Because we have the pressure of schedule, we really cannot help everyone to understand everything. In fact we have difficulties here."

III. Choral answers of properties, definitions and theorems took place with a plural number of participants and played a crucial role in class interaction. By choral answers we consider two main types of answerers: (1) an out loud *recall* of learnt phrases - theorems, properties etc. - by the whole class, reacting according to the teacher's instruction; (2) students' chorus (of a reasoning, theorem etc.) *conducted* by the teacher, as an *answer* to the teacher's question, while the teacher hints the first word(s). Following is an example of choral answers as well as of later issues.

PLS 1 *[We added our analysis in squared brackets]*

The students were asked to prove that a quadrilateral whose diagonals bisect each other is a parallelogram. Congruency of triangles AOB, COD (see Fig. 3) was proved.

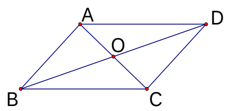

Fig. 3

Teacher: Then we have the corresponding angles are equal (<BAO, <DCO), right? OK, very well. After we proved they are parallel, do we need another pair?

Students: (several) NO *[Not every body responded!]*

Teacher: NO, because after we see the two triangles are congruent, these two angles (<BAO, <DCO) are equal. Then parallel, right? And because of congruency, this side (AB) and this side (CD) are equal. So what property am I using now?

[(i) The teacher summarised the process by herself. (ii) She was not aware of perceptual difficulties: Mental transformation of one of the triangles is needed for congruency. Decomposing the configuration into alternate angles contradicts Gestalt principles].

Students: (some responses…) *[Only some!]*

Teacher: Let's say it together

[Though understanding is doubted, the teacher went for chorusing. Students, not reaching van Hiele level 3 (ordering) might have not acquired the meaning of backward reasoning, i.e. the logic behind the use of sufficient conditions. Therefore they did not refer to the property which later they would choir, as the teacher will conduct towards]

Students: (choiring) A pair of opposing sides is equal and parallel, then it must be a parallelogram.

This PLS suggests that the students can easily choir the sufficient condition for parallelogram, but when they actually use it, the wording might be the only thing there: they do not realise that they have just used this condition to prove the proposition. The passive appearance of students' difficulties supports this claim.

Zooming in. These above described findings draw a "whole class" picture, whereas low achievers are hidden behind the many high performers of high expectations and in the back of the choir. It might be suggested that low achievers did not have appropriate learning opportunities. It seems that whole class teaching, which almost discouraged low achievers from posing their questions or raising their difficulties and did not consider their needs, as well as choral answers which masked students' difficulties, were both contributors and gradients of the phenomenon. For more evidence we took a closer look, focusing at the chance of the individual to overcome its difficulties.

Appearances of difficulties. A careful analysis of the observed PLS revealed various appearances of students' difficulties during the lesson. Here we refer to two of them:

(a) Explicit/Active appearance, i.e. when a student expressed a wrong answer or a wrong idea. In this case the teacher generally - though not always - explained the mistake and summarised by questioning if there are any further questions, generally as a rhetoric question, not expecting any answer. PLS 2 provides an example.

PLS 2

Teacher: …Now this case (see Fig. 4) . Eh? We can call it what?

(Small voice answered)

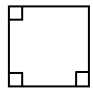

Fig. 4

Teacher: Anybody else? Eh? Ok. Somebody called it a square.

[Visual judgement points at van Hiele level 1, recognition, or level 2, analysis]

Let's do it slowly *[A difficulty was recognised]* Why do you call it a square?

(Small voice) *[Most students did not reply]*

Teacher: Come, you say!

Student: 4 sides are not necessarily equal.

Teacher: Oh! 4 sides are not necessarily equal! OK! So it can only be called what?

Student: Rectangle

Teacher: Good. Rectangle. Maybe some of you picked "square"!... .Remember, studying geometry we cannot be fooled by our eyes. We need proof

[The teacher refers to the van Hiele level 1-2 response, saying that a proof is a must, and relying on visual judgement is unacceptable. These ideas require level 3 of thinking].

This PLS presents an explicit, though quiet, appearance of difficulty. The teacher related to the mistake but not to the mistaken student: "4 sides are not necessarily equal" does not mean anything in level 1-2, since in the drawing (Fig. 4) they are equal.

An active appearance of difficulty, as in case of mistaken answers by volunteering students, was rare - students answered only if they were confident about the answer.

(b) Implicit/Passive appearance, i.e. when the student(s) did not express his/her difficulty out loud. This case was very frequent. Students almost never expressed their not understanding. Teacher's addressing a slow student was generally met with no reply. Passive appearance is seen in PLS 3, since only few responded.

PLS 3

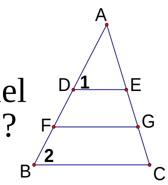

Fig. 5

Teacher: We start… from the similarity of triangles. If I say: DE is parallel to BC [See Fig. 5] … is the small triangle ADE similar to ABC?

[Decomposing the figure might be tackled perceptually]

Students (few): Yes *[Addressing the whole class, only a few responded]*

Teacher: What is the condition?

Student: Corresponding angles (in parallel lines) *[Only one responded]*

Teacher: Corresponding angles. Angle 1 and angle 2 are on the same position. And? Angle A equals angle A, right? Triangle ADE is similar to triangle ABC, right? AA similar. Good. *[The teacher gave the whole explanation, leaving no time for the students to think on their own.]* Side AD… will be equal to what? *[Small-step questions might encourage micro reasoning]*.

Student: AE to AC *[Only one student answered]*

Teacher: AE to who?

Students (few): AC *[Though one already gave a correct answer – only a few responded]*

Teacher: AC, right. AE to AC. And? Then which side? *[Small-step question]*

(Silence) *[(i) The difficulty could be originated in either perceptual reasons or conceptual ones. (ii) Teacher's coping was to narrow the step, giving a specific direction]*

This PLS suggests a difficulty which might be originated in not acquiring the concept of similarity and its derivatives. Also, it could be explained by means of perceptual difficulty: In order to focus at triangle ADE (see Fig. 4) one needs to decompose the configuration into subfigures, where ADE is one of them. But then, the mental figure of triangle ABC will be erased. Identifying corresponding parts of similar shapes is another perceptual difficulty. It seems that the teacher did not analyse these reasons.

The class discourse demonstrates an implicit and passive appearance of difficulties presented by the number of responders to the teacher's questions: 'few' (around 5 in a class of 37 students) was the maximal number of responders, some times it was 'one' and finally it came into 'none' (silence). This silence suggests that each of the low achievers grew in atmosphere which does not give them chances in their own class. PLS 1 could suggest another example of passive appearance.

Appearance⇨ Teacher⇩	Silence responds	Single responses	Choral recall+ conducted	Loud mistakes	Many responses
A	9	11	(2+2) 4	9	7
B	5	3	(3+9) 12	1	7
C	6	6	(1+1) 2	0	2
D	9	7	(1+1) 2	1	2
E	13	5	(0+1) 1	1	4

Table 1: Average number (in a lesson) of each appearance to the teacher's question

Response or responder, silence or many. Looking closer at the individual we found that though teachers were usually explaining and correcting mistaken *answers*, mostly they ignored the *answerer* and the process of thinking that led to the mistake. "Tight schedule" was a common excuse. Moreover, as seen in the examples, teachers were not always aware of their students' difficulties, especially when their appearance was implicit, and they were ignoring feedbacks such as silence or few responders. Data about "silence-responses", a single responder or mistaken loud answers, all as opposed to "many answerers", points at an atmosphere which in most teachers' classes prevented mistakes or difficulties to occur or to be presented (see Tab. 1). In these situations, the teachers generally answered or summarised by themselves, asked the students smaller-step questions or asked one of the best students to respond.

DISCUSSION

Comparing Taiwan to other best performers in mathematics such as Singapore, South Korea and Hong-Kong painted the big gap between the great success and the minority who were left behind in darker colours since the percentage of lower achievers in Taiwan was much bigger than those of the others (15% vs. 10% or less).

In order to look for reasons and catalysers in which low achievements grew, we found suitable the methodology of Problematic Learning Situations. Using these lenses, we could describe, characterise and get a deeper understanding of the low achievers' phenomenon in a high achieving population.

Not as in other educational studies, in which observations are mostly made by local researchers - "insiders", in our case the "outsider" researcher entered the class personally. Using such approach could help us have a different perspective of capturing the episode and configuration of PLS. It prevented un-avoided bias by the *content* of class discussion. Attention was free to note other parameters. Since we used the glasses of the "insider" as well - we could have the advantage of both perspectives.

Different from common beliefs as to reasons for low achievements, the findings suggest that the "left behind" had no sufficient and appropriate learning opportunities. "Learning pattern" - characterized by high expectations, whole class teaching and choiring - resulted in class atmosphere which was oriented towards high achievers and suggested that the group of low achievers did not get proper learning opportunities. "Silence-responds" and not addressing the mistaken student both supported our claim. Moreover, teachers were dominating their students' opportunities of thinking; students hardly had space to think individually.

Frequently teachers were coping with students' difficulties, by trying to hint and asking "directing" questions. These might be categorised as "small step" teaching (Zhang et al., 2004). Such an interpretation ignores potential pseudo-conceptual answers to "small step" questions, and leading towards micro level reasoning (Duval 1998).

Teachers encouraged a class loud rote. Memorising and chorusing of theorems, properties and definitions may contribute to the students' mathematical jargon, training and supporting the fluency of the "mathematical language". Nevertheless, we suggest being doubtful at a loud chorus as a "proof" of understanding; a property that was choired by the class is sometimes meaningless for some students.

We believe that the number of "left behind" students can be reduced. An appropriate process of instruction, considering the actual student's level of thinking, might help transfer it into a higher one (Hoffer, 1983).We recommend to consider the following:

1. Teacher training programs should provide teachers with relevant cognitive theories and pedagogic content knowledge in order to be able to identify and analyse difficulties. These are not required yet in most institutes in Taiwan. Attention should be

paid to make the knowledge retrievable during class instruction. Such an intervention, in which teachers' awareness to their students' way of thinking was enlarged, was successfully applied in Israel (Gal, 2005), and may be adopted into Taiwan.

2. Heterogeneous teaching approach and tools to support and promote low achievers should be introduced to Taiwanese teachers in order to consider a full spectrum of students. Moreover, teachers should be familiar with them. The norm of attention and awareness to the individual is recommended to be introduced to the teachers, and teachers should be encouraged to follow it and practice its applications.

Further analysis, including other perspectives of observations - **Post-observing** by the teachers themselves and **"External-observing"** by other teachers, as well as a detailed analysis of the number of various appearances (Tab. 1) will be reported elsewhere.

The research was funded by National Science Council, Taiwan (94-2521-S-003-001).

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THE ZERO AND NEGATIVITY AMONG SECONDARY SCHOOL STUDENTS

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CINVESTAV, Mexico

This article reports on the results of a Case Study in which a female student expresses five meanings of zero while solving arithmetic – algebraic tasks. The nil zero, the implicit zero, the total zero, the arithmetic zero and the algebraic zero all arose simultaneously at the levels of conceptualization of negatives found by Gallardo (2002). The fact that a student who is competent in algebra meets unavoidable difficulties when faced with an equation that contains negatives and zero is highly relevant.

The zero and negative numbers are presently topics contained in school study plans. Both are treated in general terms, without taking into consideration the important role they play in the extension of the natural number domain to the integers and in students' ability to become competent in the usage of algebraic language.

Piaget (1960) states that one of the great discoveries in the history of mathematics was the fact that the zero and negatives were converted into numbers. Some of the researchers who have studied negative numbers and the zero in the field of mathematical education include Freudenthal (1973), who used the inductive-exploratory method: an extrapolation to the other side of the zero; Glaeser (1981) considered the ambiguity of the two zeros: the absolute zero with nothing below it and zero as origin selected arbitrarily on a oriented axis; and Bell (1986), who analyzed the difficulties associated with crossing the zero on the number line.

THE BEGINNING

We are working on a research theme that is part of a broader project currently in process[1]. Our theme addresses the integers in which the "simultaneous appearance" of negativity and zero is emphasized in problem and equation solving. We have based our work on Gallardo (2002), who found that five levels of acceptance of negative numbers were abstracted from an empirical study of 35 pupils aged 12 to 13 years. The following are the levels: Subtrahend, where the notion of number is subordinated to magnitude (in *a–b*, a is always greater than *b* where *a* and *b* are natural numbers); Signed number, where a plus or a minus sign is associated with the

[1] Work financed by CONACYT. Research Project: 44632. "Procesos de Abstracción y patrones de Comunicación en Aulas de Matemáticas y Ciencias con Entornos Tecnológicos de aprendizaje: Estudio Teórico – experimental con alumnos de 10 a 16 años de edad" ("Abstraction processes and communication patterns in mathematics and science classrooms where there is a technological learning setting: theoretical-experimental study with 10-16 year old students").

quantity and no additional meaning of the term is necessary; Relative number, where the idea of opposite quantities in relation to a quality arises in the discrete domain and the idea of symmetry becomes evident in the continuous domain; Isolated number that is the result of an operation or as the solution to a problem or equation; Formal negative number, a mathematical notion of negative number within an enlarged concept of number embracing both positive and negative numbers (today's integers). This level is usually not reached by 12–13 year old students.

The theoretical and methodological framework of our study is based on Filloy (1999). Therein Filloy states that the semiotic notion of Mathematical System of Signs (MSS) may be used to interpret empirical observations in mathematical education. The foregoing notion encompasses both the meaning of the sign at the formal level of mathematics, as well as the pragmatic meaning. Students often use intermediate sign systems or personal codes during the teaching/learning processes and are expected to become competent in the socially institutionalized MSS upon completion of those processes.

Our research questions are the following:

How does zero contribute to extending the numerical domain of natural numbers to integers?

Do students consider zero a number?

How does the zero relate to negative number conceptualization levels?

Do they understand addition, subtraction, multiplication and division by zero?

Does an historical-epistemological analysis of zero as a number contribute to an understanding of the conflicts presently faced by students?

The initial steps of our research theme were reported on in Gallardo and Hernández (2005), where it was concluded that the recognition of dualities as operator-equivalence of equations; unary-binary signs of integers and nullity-totality of zero, contribute as a possible means of extending the natural number domain to the integers. These first results partially answered above-mentioned research questions a) and b).

THE SECOND STEPS

In the article herein, we propose to respond to research question c). Our aim is to investigate how the different meanings of zero attributed by students actually coexist and how those meanings relate to the conceptualization levels found by Gallardo (2002) during the transition from arithmetic to algebra. Overall, 42 students aged 12 to 16 answered questionnaires and were the subjects of individual clinical interviews. The students were asked 1) to solve addition and subtraction operations; 2) to simplify open sentences; and 3) to solve linear equations.

Due to space constraints, in this article we shall only report on the results of the performance of the best student, Victoria (V). She had obtained the highest number of correct answers in the questionnaires and proved to be competent in her use of arithmetic and algebraic systems of signs.

The items that best exemplify the interview are expressed as follows: Statements made by **V** are expressed in quotation marks "...". Interpretations made by the interviewer, **I,** are expressed in brackets [...].

Item 1. Solve addition and subtraction operations: **67 + 34 + 29 – 34 =**

V: Writes:

$$\begin{array}{r} 67 \\ +\ 34 \\ \underline{29} \\ 130 \end{array} \qquad \begin{array}{r} 130 \\ \underline{-\ 34} \\ 096 \end{array}$$

…done.

I: Is there another way of solving it?

V: ..Yes, this one [67 + 34 + 29 – 34 =] and this other one are the same, well one is positive and the other is negative, by putting them together you get zero … it is "as though they were not there".

- *Resorts to the vertical MSS.*
- *Re-broaches the sentence for Item 1. In this horizontal MSS, she recovers the concept of relative number and identifies the zero as a nil element "it is as though it were not there", while also considering it an element made up of opposites "one is positive and the other is negative, by putting them together you get zero".*

Item 2. Solve addition and subtraction operations: 81– 39 +21 +16 – 79 =

V: Writes:

$$\begin{array}{r} 81 \\ 21 \\ \underline{16} \\ 118 \end{array} \qquad \begin{array}{r} 79 \\ \underline{+\ 39} \\ 118 \end{array}$$

118… then I add: 118 … the 118 is negative because [81– 39 +21 +16 – 79 =]…

This number and this other number are negative numbers… now you subtract …

$$\begin{array}{r} 118 \\ \underline{-\ 118} \\ 000 \end{array}$$

… the result is zero.

- *Uses the vertical MSS*
- *Recovers the signed numbers when she changes into the horizontal MSS.*
- *Does a subtraction of natural numbers whose result is zero in the vertical MSS.*
- *Acknowledges zero as the result of an arithmetic operation.*

Item 3. Solve addition and subtraction operations: **14+28–28+39–16+16=**

<table>
<tr><td>
V: Writes:
$\begin{array}{rrr} 14 & 12 & 35 \\ \underline{+\,28} & \underline{+\,39} & \underline{+\,16} \\ 42 & 51 & 51 \\ \underline{-\,28} & \underline{-\,16} & \\ 12 & 35 & \end{array}$
I: Is there another way of solving the expression?
V: … You could … by adding the 28 and the 16 that are negative and look for a number that could be added to give us zero … what’s left over is the result.
</td><td>
• Resorts to the vertical MSS.
• Appearance of the relative numbers and of the zero as total zero “look for a number to give us zero” by resorting to the horizontal MSS.
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</table>

Item 4. Solve addition and subtraction operations: **(+5) – (+17) =_____**

<table>
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V: I have to do a subtraction...
it would be...
$\begin{array}{r} 5 \\ \underline{-\,17} \\ -\,12 \end{array}$ ... it is minus 12.
I: Explain to me how you did it.
V: These two are positive …
[(+5) – (+17) =] five subtracted from seventeen is minus twelve …because the five does not have the positive anymore after taking the five away I’m left with minus twelve.
</td><td>
• To solve the operation, she uses a vertical MSS, obtaining isolated negative number –12.
• Resorts to horizontal MSS
(+5) – (+17) to explain her response, and recognized signed numbers.
She calculated the expression in her head and an implicit zero appeared, which she did not mention. Her explanation could have been written as:
5 – 17 =5 – 5 – 12 = 0 – 12 = – 12
</td></tr>
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Item 5. Simplify the open sentence: 9h – 5j – 4h + 3j – 5h

V: Writes: $\begin{array}{rr} 9h & -5j \\ \underline{-9h} & \underline{3j} \\ 0 & -2j \end{array}$ …it would be minus 2j I: What do you think the zero means? V: Well … the zero is as though it were not there, the zero doesn't count …that's why it isn't added …it isn't subtracted …or it can be added but it doesn't change the result, it is still zero.	• *Resorts to a vertical MSS, groups similar terms.* • *Appearance of the nil zero: "the zero is as though it were not there, the zero doesn't count"* • *Conceives of the algebraic zero: "it can be added but it doesn't change the result.*

Item 6. Simplify the open sentence: 2x + 7 = x + 7

V: Writes: 2x + 7 = x + 7 2x – x = 7 – 7 x = 0 …the result would be zero. I: Could you explain what it means for the x to be equal to zero? V: …Well…when proving it, instead of x we put in the zero … because two times zero is zero, zero plus seven is seven on this side (left) and on this other side (right), it would be zero plus seven, is equal to seven …the first gives seven and the second does too , so the expression is right.	• *Operates using the horizontal MSS.* • *Identifies that zero is the solution, justifying her answer by proving it.*

Item 7. Solve the linear equation: $4x - 8 = 3x - 8$

<table>
<tr>
<td>
V: Writes:
$4x - 8 = 3x - 8$
$4x - 3x = 8 - 8$
$x = 0$ …it would be zero.
</td>
<td>
• Resorts to the horizontal MSS.
• Identifies the algebraic zero as a nullifying element when multiplied and as a neutral element in a subtraction.
• Identifies the negative number as signed and relative numbers.
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I: Is there another way to solve the expression?
V: Yes… I'll do it … $-8 = 3x - 8 - 4x$
$-8 = 3x - 4x - 8$
$-8 = -x - 8$ ….
$-8 + 8 = -x$ …
$0 = -x$ … x is equal to zero.
I: But there it says that zero is equal to minus x …
V: You just turn it around and are left with x is equal to zero.
I: But wait, if you turn it around …look properly …how would it end up?
V: …Minus x is equal to zero …
I: Right, minus x is equal to zero and what we want to know is the value of x, not of minus x. How do we do it?
V: Well we just take it away, x is the same as minus x …
I: Then is x the same as minus x?
V: I don't remember …
</td>
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</table>

Different meanings for zero arose during this case study, meanings that were simultaneously expressed at varying levels for conceptualizing negatives. The student used arithmetic and algebraic systems of signs in vertical and horizontal formats to solve the tasks. The different meanings of zero that arose from the interview dialogues are named and interpreted below.

Nil zero: is that which "has no value", "it is as though it were not there" stated the student. In the vertical algebraic system of signs, the nil zero coexists with the negative number as a subtrahend. Only the binary sign is recognized (item 5).

Implicit zero: is that which does not appear in writing, but that is used during the process of solving the task. In the horizontal arithmetic system of signs, the implicit zero is put together with the relative number. The binary-unary signs are recognized (item 4).

Total zero: is that which is made up of opposite numbers ($+n$, $-n$ with $n \in N$). In the horizontal arithmetic system of signs, the total zero coexists with the relative number. The binary-unary signs are recognized (items 1 and 3).

Arithmetic zero: is that which arises as the result of an arithmetic operation. In the vertical arithmetic system of signs, the arithmetic zero is put together with the subtrahend. Only binary signs are accepted (items 1 and 2).

Algebraic zero: is that which emerges as a result of an algebraic operation or is the solution of an equation. In the vertical arithmetic system of signs, the algebraic zero is put together with the signed and insolated numbers (item 5). In the horizontal arithmetic system of signs, the algebraic zero is put together with the signed, relative and isolated numbers. Binary and unary signs are recognized (item 6 and 7).

DISCUSSION

Research question c) has been answered in this article for the case of one sole student. In a subsequent publication, we shall provide the results for the group of students that took part in the study.

The most noteworthy point to be highlighted in V's case is that her performance has led us to five meanings of zero that could be associated to the levels of conceptualization of negatives reported by Gallardo (2002). Said meanings of zero were expressed in four different mathematical systems of signs:

In the vertical arithmetic system, V associates the nil zero, the arithmetic zero with the subtrahends and relative numbers.

In the vertical algebraic system, V relates the nil zero and the algebraic zero to signed and isolated numbers.

In the horizontal arithmetic system, the nil zero, the implicit zero and the total zero are expressed simultaneously with the subtrahends signed and relative numbers.

In the horizontal algebraic system, the algebraic zero arises at the same time as signed, relative and isolated numbers.

It is moreover important to note that item 7 results in an unexpected conflict when V writes the equation: $-x = 0$. Simultaneously expressed in the latter equation are an implicit negative number, the coefficient minus 1, the arithmetic zero: "something equal to nothing" and the difficulty faced by students when interpreting the x variable during the transition from arithmetic to algebra. The conflict shows that although V is competent in the algebraic system of signs given that she simplifies open sentences, solves equations and verifies her solutions, when she is faced with the equation – x = 0, with both the zero and negatives, she says "I don't remember" (item 7), finding herself at the entrance to a dead end. In fact, Vlassis (2001) reported on how extremely difficult it is for students to solve the equation x = – a, with $a \in N$ and $a \neq 0$, an equation that is similar to our: – x = 0.

These findings must clearly be validated by an empirical study undertaken in greater depth, while we also continue our attempts to respond to research questions d), e) and f).

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STUDENTS' ACTIONS IN OPEN AND MULTIPLE-CHOICE QUESTIONS REGARDING UNDERSTANDING OF AVERAGES

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The debate about how to assess students' concepts of averages has given rise to different opinions about the suitability of each form of evaluation. In this work we analyse how students act when solving open-answer questions once they have selected the correct option in interrelated multiple-choice questions. Analysis has led us to note that many students who choose the correct answers in multiple-choice questions were completely unable to demonstrate any reasonable method of solving related open questions. This suggests that multiple-choice questions do not provide precise information about students' knowledge and reinforces the importance of open questions when assessing averages.

INTRODUCTION

Interest in research about how students reason when faced with questions involving arithmetic averages is gathering among researchers given the importance of this concept in everyday life. Some studies have shown the teaching-learning of this concept is apparently easy, but understanding of the concept gives rise to tremendous difficulties. Pollatsek, Lima and Well (1981) demonstrate that most students seem to know the average calculation rule or algorithm. According to Watson and Moritz (2000), for a large number of children, the average is simply a value in the centre of distribution. Study undertaken by Cai (1995) has shown that most students know the mechanism of "adding all together and dividing". However, only some of them were able to find an unknown value in a series of data where the average is known. Mokros and Russell (1995) demonstrate various difficulties faced by students in their understanding of averages. They have identified and analysed five different constructions of representation used by students. The debate on how to assess students' conceptions, has given rise to different points of view about the suitability of each form of assessment. Garfield (2003) describes a questionnaire for assessing statistical reasoning. She believes that most assessment instruments are centred more on the abilities of calculation or problem solving than on reasoning and understanding. Cobo and Batanero (2004) and Cai (1995) underline the importance of open questions for assessment and suggest that this type of questions be used to examine students' ideas about the concept of arithmetical average. Gal (1995) states that it is difficult to judge fully what a person knows about averages as an instrument for solving problems based on data unless a context is given that would motivate the use of that instrument.

In this work we set out to analyse how students act when faced with open answer questions that are closely related to multiple-choice questions. We wanted to find out if those students who choose the correct options in multiple-choice questions have

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 161-168. Prague: PME.

done so using clear criteria, basing our observations on the students' actions during the process of solving open questions. We asked ourselves the following questions: Do those students who correctly answer the multiple-choice questions carry out reasoned actions in open problems? Are their actions a real, convincing indication of having had a solid basis when answering the multiple-choice questions correctly?

Our belief is that most students who choose the correct answers in multiple-choice questions do not appear to do so on a solid basis. Our analysis could possibly highlight elements for reference to clear up various positions that are being constantly assumed in the debate on which forms of assessment are the most appropriate when examining the understanding of statistical concepts. In this respect we will cross reference results of pairs of interrelated problems.

METHODOLOGY

Sample

Our study was undertaken with 94 students in the final year of secondary education, their average age being 17 years old. Throughout their schooling they had received specific instruction in arithmetical averages and other topics concerning statistics and probability.

Questionnaire

In this work we put forward data regarding four questions that make up a questionnaire of seven questions and that form part of a wider study that we are carrying out on the assessment of the concept of average. The four questions, as shown below, are made up of both multiple-choice and open-question items, which we designate according to the context defining them.

Question "In One Class": A teacher decides to study how many questions her students do. The questions done by her 8 students during one class are shown below:

	Names of students							
	Juan	Lucía	Alberto	Ana	Pedro	María	Luis	Clara
Nº quest.	0	5	2	22	3	2	1	2

The teacher would like to summarise these data, calculating the typical number of questions done that day. Which of the following methods would you recommend?

(Mark one of the following answers)

_______a. Use the most common number, which is 2.

_______b. Add up the 8 numbers and divide by 8.

_______c. Discard the 22, add up the other 7 numbers and divide by 7.

_______d. Discard the 0, add up the other 7 numbers and divide by 7.

This question is taken from Garfield (2003) with some modifications in the text. It is used in order to try and examine students' knowledge of averages, the use of the average calculation algorithm, the effect of atypical values, as well the importance of the context.

Question "Children per Family": The school committee of a small town wishes to find out the average number of children per family in the town. They divided the total number of children by 50, this being the total number of families. If the average if 2.2 children per family, which of the following statement do you agree with?

_______a. Half of the families in the town have more than 2 children.

_______b. In the town there are more families with 3 children than with 2 children.

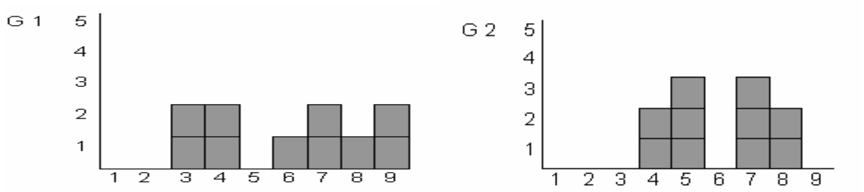
_______c. There is a total of 110 children in the town.

_______d. There are 2.2 children per adult in the town.

_______e. The most common number of children in one family is 2.

This question is also taken from Garfield (2003). Here the aim is essentially to evaluate understanding and proper use of the arithmetical average calculation algorithm.

Question "Marks Graph": Twenty high-school students take part in a mathematics competition. Ten of the students form Group 1 and the other ten Group 2. The marks they achieve in the competition are shown in the graphs below:



Each rectangle in the graph represents the marks achieved by each individual student. For example, in Group 1 the two rectangles appearing above Number 9 show that two students in this group achieved a score of 9.

5.A Group 1 has an average mark of 6.

a) Check that the average mark for Group 2 is also 6.

b) Which group seems better to you? Justify your choice.

5.B Which of the following statements is true?

__a Group 1 is better than Group 2 because the students who got higher marks are in this group.

__b Group 2 is better because there are no students with marks below 4.

__c There is no difference between the two groups because the average is the same.

__d Although the averages are the same for both groups, Group 2 is more homogeneous.

This question is of our own devising, although there is some similarity to one described by Garfield (2003). The aim of this question is to see how students interpret distributions shown in the form of a graph, find out if they know how to manipulate data graphically in order to calculate and examine what criteria they use when checking two samples based on their visual appearance.

Question "Family": The average family size in a given locality is 3.2 persons. Show 10 families that fulfil this average.

This question was adapted from Mokros and Russell (1995) and is designed to evaluate whether students are able to construct a distribution where the average is known. Also, we wish to check the strategies used by the students to find the distribution asked for, assess their understanding of the type of data to be used given the context, seeing that the average cannot form part of the distribution.

Categories

We set out a system of categories in order to codify students' answers, which would let us treat the information using statistical software. The categories were established taking into account the type of item. For multiple-choice answer types we related the category to the mathematical content of the distracter, while for open answers we put forward as a category the strategy used by the student when solving the problem. Below we give the codes that are directly related to the results given in this work:

Code	Code description
ADALG -	Adds up all the numbers and divides by the total of data.
DATIP -	Discards the value considered atypical, adds up the other 7 numbers and divides by 7.
DCERO -	Discards the 0, adds up the other 7 numbers and divides by 7.
MODA -	Uses the most common number.
ALG-SI -	Uses the simple arithmetical average algorithm.
ALG-SI-e -	Uses the simple arithmetical average algorithm, but incorrectly Uses the
ALGPOND -	weighted average algorithm correctly.
ALGPOND-	Uses the weighted average algorithm incorrectly
e -	No answer.
NC -	Undertakes incoherent transformations or puts forward confused
SC -	justifications or chooses more than one option.
	Gives a numerical value without showing operations.
SOPER -	Adds up the four values on the horizontal axis on which the rectangles are
SX -	built and divides by four.
	Indicate the option that says that there is a total of 110 children Gives a
STOTAL -	distribution that fails to fulfil the conditions given.
DINCOR -	Gives a correct distribution, but does not show operations.
DSOPER -	Gives a correct distribution, attaining the sum total.
DSTOTAL -	Justifies that the groups have the same averages or chooses the statement
IGUMEDIA	that indicates that there is no difference between the groups because the
-	averages are the same.
	Puts forwards a justification based on homogeneity or chooses the statement referring to this criteria.
MHOMO -	Uses as criteria for justification the greater mark factor or marks the option referring to this criteria.
MNOTA -	Uses as an argument the fact that there are no marks less than 4.
	Uses as justification the fact that there are more students who pass or
N<4 -	marks the option referring to this factor.
NAPROB -	Indicates that one group is better, but without justifying this idea.
NJUST -	

RESULTS AND DISCUSSION

We shall basically analyse the actions in open questions of those students who chose the right answer in the multiple-choice questions. Table 1 shows the data referring to the cross referencing of results for the question "Marks Graph, 5.A.a)" (rows) and "In One Class" (columns).

	ADALG	DATIP	DCERO	MODA	Total
ALG-SI	4	0	2	0	6
ALG-SI-e	1	0	0	0	1
ALGPOND	23	5	11	2	41
ALGPOND-e	1	1	0	0	2
NC	8	3	6	0	17
SC	1	0	3	2	6
SOPER	6	3	4	1	14
SX	2	1	3	1	7
Total	46	13	29	6	94

Table 1: Cross referencing of results for the **open question** "Marks Graph", 5.A.a (rows) and **multiple-choice question** "In One Class" (columns)

The data show that 46 (49%) students answered correctly (ADALG) the multiple-choice question while 47 (50%) also answered correctly the open question (ALG-SI and ALGPOND). The correct choice in the multiple-choice question is reached by adding up all the values and dividing the sum by the number of data, this being the average calculation algorithm. Of the 46 students who chose the correct option in the multiple-choice question, when given the open problem that required them to interpret the data from the graph before using the average calculation algorithm, 19 (41%) students were unable to attain the solution that was required of them. Of these, 8 (17%) students did not answer (NC), 6 (13%) supplied a numerical result without showing the pertinent operations (SOPER), 2 students added up the values of the variables without taking into account the frequencies, 2 used the average calculation algorithm incorrectly, and 1 used incoherent procedures. The right answer was reached by 27 (59%) students who used two different procedures: 4 (9%) students used the simple average algorithm. (ALG-SI) and 23 (50%) used the weighted average algorithm (ALGPOND). On the other hand, the data show that of the 47 students who answered the open question correctly 20 (43%) of them had indicated the wrong option in the multiple choice question (DATIP, DCERO and MODA).

These data seem to suggest that those students who marked the correct answer in the multiple-choice question merely considered that in those circumstances they could use the average calculation algorithm but when they were given a concrete situation they did not know how to use that instrument, perhaps because they did not know how to determine the various elements that make up the formula for calculating averages, a difficulty already underlined by Pollatsek, Lima and Well (1981).

Table 2 shows information taken after cross referencing the results for the questions "Family" (rows) and "Children per Family" (column).

	FAMADUL	MED	MODA	SC	STOTAL	Total
DINCOR	0	0	2	0	2	4
DSOPER	1	2	10	0	8	21
DSTOTAL	0	0	20	0	6	26
NC	1	2	14	0	14	31
SC	1	1	6	1	3	12
Total	3	5	52	1	33	94

Table 2: Cross referencing of the results for the **open question** "Family" (rows) and **multiple-choice question** "Children per Family" (columns)

A total of 33 (35%) students correctly answered the multiple-choice question (STOTAL), while for the open question 47 (50%) students correctly attained the solution required (DSOPER and DSTOTAL). Although attaining the correct solution for both these items depended on the same strategy (inversion of the average calculation average), the data show that of the 33 students who chose the correct answer in the multiple-choice question, only 14 (42%) were capable of solving the open question, while the other 19 (58%) had various difficulties: 14 (42%) did not answer (NC), 2 students showed a distribution that failed to fulfil the requirements stipulated in the question text (DINCOR) and 3 gave incoherent algebraic transformations (SC). Of the 47 students that answered the open question correctly, it is noteworthy that 30 (64%) appear to have confused the average with the mode, as they preferred the distracter related to this concept in the multiple-choice question.

We were greatly surprised by the fact that most students who correctly identified the solution to the multiple-choice question could not even demonstrate the initial steps in solving the open question, in spite of the fact that the methodology for solving the two questions had the same basis. This leads us to believe that these students failed to use any criteria when selecting the correct answer.

Use of the strategy of inverting the average calculation algorithm was a basic step in solving the two items. As Watson and Moritz (2000) point out, this procedure was the main way of successfully solving a similar problem when studying the intuitive meaning given by children to the term "average". However, for our open question in particular, it was crucial that students understood the ideas of distribution and average as an even sharing, as well as knowing the calculation algorithm, as explained by Cobo and Batanero (2004). It should also be pointed out that there was a further difficulty in that students were unable to use decimal numbers as data when constructing the distribution, due to the context. Possibly, then, most of these students found it difficult to solve the open question.

Finally, we show the cross referencing of results for the items "Marks Graph, 5.A.b)" (rows) and "Marks Graph 5.B" (columns).

	IGUMEDIA	MHOMO	MNOTA	N<4	NC	SC	Total

Table 3: Cross

IGUMEDIA	13	1	0	0	0	0	14
MHOMO	0	6	0	0	0	1	7
MNOTA	2	1	4	0	0	0	7
N<4	0	1	0	1	0	0	2
NAPROB	5	14	1	14	3	6	43
NC	1	6	0	0	0	1	8
NJUST	0	1	1	0	0	0	2
SC	2	6	2	1	0	0	11
Total	23	36	8	16	3	8	94

referencing of results for the **open question** "Marks Graph, 5.A.b)" (rows) and **multiple-choice question** "Marks Graph 5.B" (columns).

Table 3 shows that 36 (38%) students marked the correct justification for the multiple-choice question (MHOMO), where this option referred to homogeneity and indicated the need to take into account the dispersion of data when comparing groups with the same averages. With respect to the open question, we could only find 7 (7%) students who gave the correct justification (MHOMO) using the same criteria. Also, of the 36 students who chose the correct option in the multiple-choice question, 30 (83%) had used incorrect arguments for the open item, that is, only 6 (17%) students gave acceptable arguments. The baseless justification most in evidence was the one that took for its criteria of comparison the fact that there were more students passing (NAPROB), given by 39% (14) of these students. We also found that 6 (17%) students who failed to answer (NC), another 6 (17%) who put forward confused justifications (SC), 1 who said that there were no differences between the groups because the averages were the same (IGUMEDIA), and 2 who only looked at the maximums or minimums of the distribution (MNOTA and N<4). With regard to those students who correctly justified the open question, the data show that nearly all of the students also correctly got the answer to the multiple-choice question, except 1 (SC). These results lead us to suspect that those students who chose the correct answer for the multiple-choice question did so without taking into account a formal basis, since, after choosing the answer, they did not bother to rectify the wrong justifications they had put forward in the open question. This also shows that the students are not consistent in their affirmations. In the same situation they use completely different criteria! The difficulties arising when comparing samples in which students merely analyse only one part of the distribution of the maximum or minimum values were also found by Godino and Batanero (1997), and Estepa and Sánchez (1996). As interpreted by Estepa and Sánchez (1996), these difficulties are due to the fact that students have a local concept of association of variables and believe that this is the analysis which explains the differences between the two samples.

CONCLUSIONS

Our study has allowed us to see that many students who choose the correct answer for the multiple-choice questions are not able to demonstrate reasonable methods for solving open questions. The actions in open questions by those students who choose the correct answer in the multiple-choice questions, suggest that they choose these answers without any criteria. As can be seen in the results, especially in Table 2, most of these students are unable to follow the first steps in solving the open question, in spite of the fact that the solution to this question requires the same strategy that they should supposedly use before choosing the correct answer for the multiple-choice question. Another significant fact can be seen in Table 3, where students do not even remember to change the wrong justification they put forward for the open question after choosing the correct answer for the multiple-choice question.

The results also show incoherence in students' actions when they correctly mark the answer to the multiple-choice question and are unable to solve a related open question or vice versa. Furthermore, the results show that students are not consistent in their affirmations, given that in the same situation they use completely different criteria.

As can be seen in this study, the difficulties and incoherence evident in students' actions have been detected through the use of open questions, underlining the importance of this type of question when assessing the concepts held by students.

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PARADOXES: THE INTERPLAY OF GENDER, SOCIAL CLASS AND MATHEMATICS IN THE CARIBBEAN

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This paper explores a seemingly paradoxical relationship between earlier student outcomes in mathematics and current student views about mathematics using gender and social class as intervening factors in the analysis. The collected data shows that the relationship is not as straightforward as one might think, and no one general statement adequately describes the relationship across all students.

INTRODUCTION

This paper is based on a study which looked at Caribbean students' views of mathematics in light of outcomes for earlier students in school-leaving examinations. Data analysis has yielded a set of apparent paradoxes with respect to gender and social class influences on students' views and earlier outcomes. In particular, gender and social class interplay in quite different ways on what these views and outcomes are. This paper focuses on an exploration of this interplay, using selected data from the study to illustrate this interplay. The following outlines the context in which the study was set, the methods used for data collection, some selected results, and a discussion of the seeming paradox of these results.

CONTEXT OF THE STUDY

Academic literature and press reports in the Caribbean have recently highlighted a concern for a perceived low achievement of its students in mathematics, most of this based on the Caribbean Examinations Council (CXC) Caribbean Secondary Examinations Certificate (CSEC) results (Layne, 2002, p21; Williams, 2005). These examinations are taken by students reaching the end of secondary schooling in most of the English-speaking Caribbean territories, and in the early 1980s they replaced the British-based GCEs. The percentage of students passing mathematics in the CSECs has ranged between 25% and 42% over the 14-year period 1991-2004 (years for which data were obtained). These Caribbean results though should be interpreted with the following caveats in mind: as yet, secondary education is not universally available to all students across all Caribbean territories. Further, fieldwork conducted during the course of the study suggests that there is non-trivial drop-out of students during the secondary years so that a marked proportion of students do not reach the end of the 5th form, the point at which the CSECs are taken. Additionally, not all students reaching the 5th form write the examinations in this subject area. Finally, the examinations are 2-tiered, with the General proficiency level (the higher and more popular tier in terms of number of students writing, ratio being approximately 11:1 (CXC Statistics Bulletin, 2004)) being that which allows for college/university

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entrance, and it is this tier of the examinations which is considered here and upon which academic and press reports in the region are based.

Whilst there is a general awareness of a perceived ‘problem’ in school mathematics, much of what the ‘problem’ might be has been left up to un-researched theories or speculation at best. Note has been made of the influence of gender and social class issues in education generally, and of a need to encourage girls in mathematics (Berry, Poonwassie & Berry, 1999), but there appears to be limited understanding of the reasons or problems behind these issues. So the ways in which gender and social class may interplay in mathematics outcomes is still largely unknown. In effect, this was an area within Caribbean education that was in need of systematic research.

It is within this context that it was felt that a study which investigated current students’ views about mathematics might be instructive in providing some explanations as to the mathematics outcomes of earlier students, given the consistency of these earlier results. ‘Views’ here was seen as a catch-all word to include beliefs, opinions, attitudes, emotions, etc. There has long been the notion of a perceived link between what might be called students’ attitudes to mathematics and outcomes in the subject, as noted by Ma & Kishor (1997, p27). These authors in their review of the literature in this area cited gender (amongst others) as an intervening factor in this relationship, but go on to note that few studies have considered the link from a multi-factor level, so that not much is known about the combined interaction of say gender and social class on the relationship between attitudes and achievement in mathematics. This paper will attempt to address this issue via a consideration of selected data collected in the study.

METHODOLOGY AND METHODS

Most of the data for the study were collected in Antigua & Barbuda (A&B) where CSEC outcomes in mathematics over the 14-year period 1991-2004 have followed a pattern relatively similar to that of the wider Caribbean. The study used a mixed method approach (Creswell, 2003, p15), employing documentary evidence, student questionnaires, classroom observations and student interviews. It was hoped that this methodology would provide a better picture of how various factors interplay in student mathematics outcomes by drawing on the strengths of different research approaches. The main study participants were one 4th form class (penultimate year; mean age at time of data collection, 16 years) of 11 of the 13 main secondary schools in the islands. This yielded a questionnaire sample of 286 students (117 males, 169 females). This paper will provide data from the documentary analysis, some selected responses from the questionnaire with supporting data from classroom observations. In each case, an attempt is made to categorize the data in terms of gender and school type as this provides the basis for the seeming paradoxes. In this context, school type serves as a crude indicator of social class, with single-sex schools having proportionately more students of a higher social class than those in mixed schools.

RESULTS

This section has two further sub-sections, the first of which provides documentary evidence of earlier student outcomes in mathematics. This is followed by selected questionnaire data from current students of their views of mathematics, with some classroom observation data included to support the discussion which follows the sub-sections.

Earlier Student Outcomes – Documentary Data

The following graphs show CSEC statistics on the achievement of earlier students in mathematics in terms of the percentage of students passing. These results are broken down by gender and school type for students in A&B, and comparison graphs for English and All-subject areas are included.

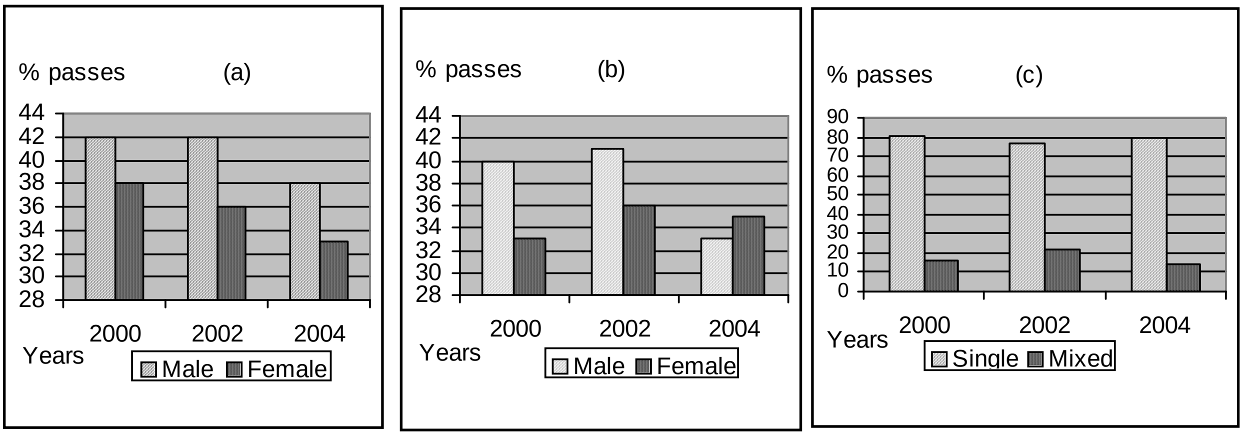


Figure 1(a, b, c): Percentage Passes in Mathematics.

(a) – Caribbean by gender; (b) A&B by gender; (c) – A&B by school type

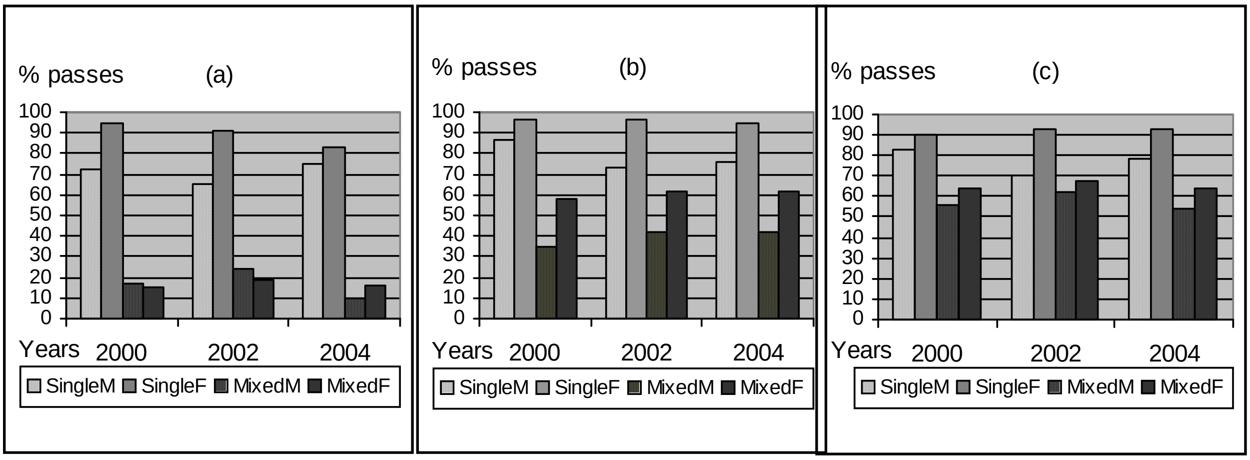


Figure 2(a, b, c): In A&B, Comparison of Percentage Passes by School type and Gender (a) – For Mathematics; (b) For English; (c) For All-subjects

Current Student Views and Behaviour – Questionnaire & Observation Data

The student questionnaire had four sections asking for, in order, personal information, family information, information about school in general, and then specifically about mathematics. In the section concerning general school information, students were asked to name their two favourite and two least liked subjects, and also to name the two subjects in which they performed best, and worst. The following table gives the student proportions (by gender and school type) mentioning mathematics, and where mathematics ranked in the subjects students named:

Questionnaire Items	Mixed Male (N = 63)	Mixed Female (N = 114)	Single-sex Male (N = 54)	Single-sex Female (N = 55)
Favourites	1^{st} – 32%	2^{nd} – 19%	1^{st} – 50%	3^{rd} – 18%
Do best	1^{st} – 29%	3^{rd} – 11%	1^{st} – 54%	5^{th} – 15%
Least liked	1^{st} – 27%	1^{st} – 48%	2^{nd} – 19%	1^{st} – 46%
Do worst	2^{nd} – 30%	1^{st} – 54%	2^{nd} – 20%	1^{st} – 49%

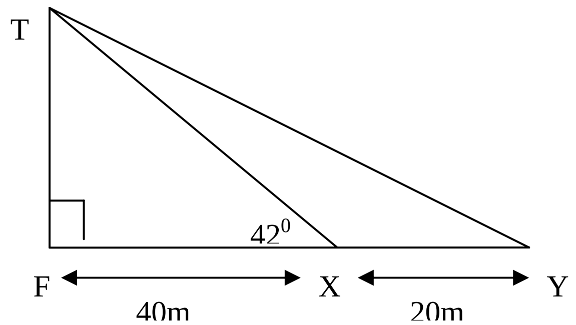
Table 1: Mathematics given as best and worst in relation to Liking and Performance with ranking and proportion of students (given as percentage of respondents)

Later in the questionnaire in the section dealing specifically with mathematics, the first question asked was 'Do you like maths?' with Yes and No categories provided, and an open adjunct inviting students to give a reason for their response. 77% of males and 55% of females replied 'Yes', a difference which was statistically significant ($\chi^2 = 15.166$, $\rho < 0.001$, df = 1). However, the most frequent reason given across all students for their response came from those replying 'No', this reason being that mathematics was hard. Table 2 provides a further analysis of some of these results by gender and school type, along with student response to a later 5-point Likert-scale type item asking students to state the extent of their agreement with the statement 'Maths is a difficult subject' (Strongly agree and Agree collapsed, Strongly disagree and Disagree collapsed for this item, Neutral category not shown here).

Questionnaire Items	Mixed M	Mixed F	Single-sex M	Single-sex F
'Yes' to 'Do you like maths?'	77%	53%	80%	59%
Agree: Maths is difficult	42%	53%	41%	65%
Disagree: Maths is difficult	42%	26%	39%	11%

Table 2: Affect for Mathematics

Classroom observations were carried out in a total of 3 schools, one Mixed and one Single-sex of either gender. In these schools, the following patterns of behaviour were observed: Girls were generally quiet and appeared to be listening to the teacher. They took notes regularly. When given work to do, girls tended to work individually, and were often seen referring to their notes. They only really interacted with other classmates as a means of checking their work. Boys, on the other hand talked more and therefore more often than girls appeared to be paying little attention to the teacher. Additionally they were often caught out on taking notes in that the board had been erased before they got around to taking the notes. When given work to do they were also more likely to be seen collaborating with other classmates about this work. The following excerpt from classroom observations in the mixed school will also be used as an illustrative example to support some of the points being made in the discussion to follow:



The teacher has written the following question on the board along with the sine and cosine rule and trigonometric ratios: **Calculate the length TF to two significant figures. Calculate the size of angle XTY.**

Students work individually on the first part of the question. A walk around the class looking at students' work shows that some have worked it correctly but others, notably the girls, appear to be referring to work they had done previously, turning back the pages of their exercise books, even though formulas are on the board. In the example that most are referring to the unknown length was in the denominator, so that a division was required in getting the answer, and even though students (mostly girls) had correctly written the first part of the solution as tan 42º = x/40, they then proceeded to do a division, 40/tan42º to arrive at a value for x, as this was what had been done in the example they were referring to (…) After corrections of the first part, the second part of the question is done as a whole class activity. The teacher asks the class what rule to use. A boy at the back eventually says the cosine rule. The teacher asks him why, and he says because the triangle is not a right-angled triangle. Some girls disagree with using the cosine rule because they say you need to have two sides and the included angle to use that rule, and you don't have that in this case. A boy at the front of the class says that you can find the angle at Y from the right-angled triangle TFY, and that would give two sides and the included angle, so that you could use the cosine rule. Various students (mainly the boys) then suggest some other methods of working the problem. These include: (1) Another boy at the back of class: Using ΔTFY, it is a right-angled triangle, and you know TF and FY and angle FTX is 48º because the angles in a triangle add up to 180º, so you could find… He is interrupted at this point, and when the teacher gets back to him he has lost his train of thought and doesn't remember. He says 'Let me think…' (2) The first boy at the back of the class referred to earlier says that the other side of the 42º is 138º, and you know XY and can find the length of TY from the right-angled triangle TFY, so can then use one of the rules – the cosine rule. A girl says to this that that method does not give you two sides and the included angle, so can't use the cosine rule, but maybe you can use the sine rule.

DISCUSSION AND CONCLUSION – A SET OF PARADOXES

The progression of the graphs of Figures 1 and 2 suggests that at least in A&B with respect to the outcomes of earlier students in mathematics, the 'problem' may be less one of gender than it might be one of social class (cf. Figures 1(b) and (c). Further, the 'problem' created by social class is more exaggerated in the case of mathematics compared to English or All-subject areas combined (cf. Figures 2 (a), (b), (c)).

That said, analysis of data from the present study provides a different perspective of the process side of the 'problem' which may influence outcomes. In student listings of their favourite, least liked, best, and worst performing subjects (Table 1) mathematics featured prominently, but there were marked gender differences (rather than social class) in the student proportions naming mathematics in any of the four categories. Mathematics appeared in the responses of all four student groups as one of the top five responses given in each of these four categories – a feature which was not 'true' of any other subject, including the other compulsory subject, English. This result compares in kind to that reported by Hoyles (1982) in a study which asked 14-year old pupils to give examples of good and bad learning experiences. Although not a mathematics-based study, the pupils gave proportionately more mathematics-related examples of both good and more so bad experiences than other subject areas. In the present study, for both male groups, mathematics was given by the greatest proportion of them in relation to what might be considered positive aspects of subjects at school, that is, a liking and their best performance. Both female groups are more diverse in their choice of favourites and whilst mathematics does make it to the list of their top five positive response subjects, the proportion of females giving it is decidedly smaller than it is for the males. In a direct reversal, mathematics was given by the greatest proportion of both female groups of negative aspects of subjects at school. Whilst mathematics also ranked 1st for the least liked subject of males in Mixed schools, the proportion of those males choosing it is markedly less than that of either female group. The results of Table 1 however do suggest a relationship between a like or dislike for mathematics and student perception of their best or worst performance in it based on student proportions.

The results in Tables 1 and 2 show that the differentials in affect for mathematics go beyond school type per se as gender patterns are more similar whatever the school type. However, from Figure 1 (b), (c) and 2(a) outcomes of earlier students in mathematics show marked differences based on school type and much less so by gender. This brings two interesting groups to the fore in relation to these last two statements, that is, males in Mixed schools and females in Single-sex schools. For both these groups, their affect responses are at odds with what seems relatively consistent outcomes in mathematics for earlier students in their school type, and it is this result which constitutes the first paradox. In particular, the quantitative analysis appears to point to boys and girls having qualitatively quite different experiences of their school mathematics, and this *even within the same classrooms* (e.g. cf. proportions of student responses in Tables 1 and 2 for males and females in Mixed schools). It is hoped that some resolve for this paradox might be provided from classroom observation data.

From the description of the general behaviour patterns observed, it might be concluded that girls were more on-task than boys in mathematics classes, and so were better placed for learning. An analysis of the observation excerpt given earlier shows that girls, because they took notes more regularly than did boys, relied on these notes more, engaging at times in what might be called 'matching strategies', trying to match given work to some previous example. This approach to doing mathematics allowed for girls to think less about the problem at hand, and generally engage less with the mathematics. Additionally, girls more so than boys showed an inclination to make use of all guidelines/rules given by the teacher, e.g. in the excerpt above, the discarding of the use of the cosine rule by girl(s) in solving the second part of the question because the triangle in question did not fit the two sides and included angle guideline. In effect the girls appeared to be trying to find a guideline or rule that would fit the question rather than thinking it through towards a solution.

On the other hand, because boys talked more and took notes less regularly, these, perversely, seemed to work for them in mathematics classes. Because they talked more, they were more likely to ask other classmates about given work, and were also more willing to make their answers public than girls. Because they took notes less regularly, they then tended to use other classmates to fill in the gaps or were forced to think more about the problem at hand as they might not have the notes to refer to, something that might not have happened if they then followed usual classroom norms of expected behaviour. This behaviour seemed to allow them to engage with and think about the mathematics more, e.g. in the observation excerpt given.

And herein lies another paradox. Although girls better fitted the profile of an ideal student based on classroom behaviour, these behaviours may be working against them in mathematics as it fostered their use of the matching, thinking less strategies outlined. In addition, being quiet and appearing to listen for some girls covered a multitude of sins, as it allowed them to appear ‘busy’ with the learning of mathematics. Alternatively boys, by talking more and taking fewer notes, gained more opportunities to make sense of the mathematics via each other, i.e. thinking and engaging more with the subject matter. There was more of a sense of ‘entitlement’ (Holland, Lachicotte, Skinner, & Cain, 1998, p. 125, 127) amongst boys than girls in both questionnaire and observation data, whatever the school type, that mathematics was something they could master, and a ‘positioning’ (ibid., 1998, p. 127) of themselves in relation to their sense of place with respect to mathematics.

With respect to the interplay of gender, social class and mathematics then, the following represents an emerging picture. Girls positioned themselves in relation to expected classroom norms of behaviour, but this positioning might in fact be constraining what and how mathematics is learned. A consideration of this, in tandem with the type of school girls were in and the associated implications for social class may also account for some of the variance in the outcomes in mathematics seen in Figures 1 and 2(a), i.e. that ‘more privileged’ girls had better outcomes, even though both female groups in the present study reported disaffection with mathematics in similar proportions (Tables 1 and 2). So proportionately fewer girls than boys liked mathematics and were finding it to be hard perhaps because the way in which they were learning the mathematics *was* hard. Girls were positioning themselves and were being positioned in mathematics classes in ways that did not provide a good fit for mathematics learning. In effect, they were ‘paying the price for sugar and spice’ (Boaler, 2002, p127) by conforming to the norm in their mathematics classes. Conversely, whilst boys positioned themselves in more deviant ways in relation to the expected norms of classroom behaviour, this positioning was providing a better fit for their engaging with the mathematics subject matter. Perhaps more boys were liking mathematics because they were learning it more in ways that facilitated sense making, i.e. ways that were ‘less hard’. But the enhanced perception of their ‘ability’ to do mathematics might partly be what gets in the way of their performance outcomes in the subject (e.g. Figures 1 and 2(a) compared to girls in Single-sex schools). Alternatively, what might be ‘in the way’ for boys may be other things not considered here, e.g. issues related to language (cf. Figure 2(b)).

In conclusion, Ma & Kishor (1997) noted that the research literature has not been consistent in providing research-based evidence of a strong positive correlation between attitudes and achievement in mathematics. The present analysis has shown that the relationship is further complicated by such intervening factors as gender and social class, as they interplay in complex and often unexpected ways. The ‘best answer’ for improving mathematics outcomes seems to lie in improving the social conditions of students, but this does not resolve the gender and affect issues which seem to be coming from classroom processes. There are no easy answers.

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INSIGHTS ABOUT THE ADDITION STRATEGIES USED BY GRADE 1 AND GRADE 2 CHILDREN WHO ARE VULNERABLE IN NUMBER LEARNING

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This paper explores the addition strategies used by Australian Grade 1 and Grade 2 children who participated in the Early Numeracy Research Project, and who were identified as vulnerable in their number learning. Prior to commencing an intervention program, the children's responses to a clinical interview were analysed so that any patterns in the strategies used could be identified. The findings indicate that some Grade 1 and Grade 2 children were unable to solve simple screened addition tasks, even after one or two years at school. Further, many students were not able to use count-on or reasoning strategies to calculate, and a notable number of students were reliant on count-all strategies.

INTRODUCTION

Despite the best endeavours of school systems, school communities and classroom teachers, some children experience difficulty learning mathematics and are at risk of poor learning outcomes (Wright, Martland, & Stafford, 2000). It seems that this is a perennial problem, and the onus is on those who work in education to address this situation by continuing to search for new insights about how to assist these children to learn mathematics successfully.

In response to this challenge, this paper presents an analysis of the learning needs and difficulties of 102 Australian Grade 1 and Grade 2 children who were all identified as vulnerable in number learning on the basis of a clinical interview and reference to a research based set of growth points used to describe the pathway of children's mathematics learning in nine domains (Clarke, Sullivan, & McDonough, 2002). The focus of this paper is restricted to an examination of children's addition strategies because these become significant as children first begin to calculate in order to solve problems (e.g., Fuson, 1992b).

However, it is important to note that the broader study examined children's difficulties in counting, place value, subtraction, multiplication and division also (see Gervasoni, 2004). It is anticipated that the analyses presented in this paper will provide insight about the type of difficulties such children experience in relation to simple addition tasks, and that the findings will form the basis of advice for classroom teachers and specialist intervention teachers about how to best customise instruction for vulnerable children.

THEORETICAL BACKGROUND

This research was based on the assumption that it is important for school communities to identify children who, as emerging school mathematicians and after one year at school, have not thrived in the school environment, and to provide these children with the type of learning opportunities and experiences that will enable them to thrive and extend their mathematical understanding. Further, the perspective that underpinned this research was that those children who have not thrived, have not yet received the type of experiences and opportunities necessary for them to construct the mathematical understandings needed to successfully engage with the school mathematics curriculum, or to make sense of the standard mathematics curriculum. As a result, these children are vulnerable and possibly at risk of poor learning outcomes. The term vulnerable is widely used in population studies (Hart, Brinkman, & Blackmore, 2003), and refers to children whose environments include risk factors that may lead to poor developmental outcomes. The challenge remains for teachers and school communities to create learning environments and design mathematics instruction that enables vulnerable children's mathematics learning to flourish.

A common theme expressed by researchers in the field of mathematics learning difficulties is the need for instruction and mathematics learning experiences to closely match children's individual learning needs (e.g., Ginsburg, 1997; Rivera, 1997; Wright et al. 2000). Ginsburg (1997) articulated a process for responding to children's learning needs that used Vygotsky's zone of proximal development (Vygotsky, 1978). Ginsburg's process requires that the teacher first analyses children's current mathematical understandings and identifies their learning potential within the zone of proximal development. For this purpose, the notion of a framework of growth points or stages of development is important for helping teachers to identify children's zones of proximal development in mathematics, and thus identify or create appropriate learning opportunities. This approach is aligned also with the instructional principles advocated by Wright et al. (2000) for the Mathematics Recovery program and Clarke et al. (2002) for the Early Numeracy Research Project (ENRP). Indeed, a feature of the ENRP was the use of a mathematics assessment interview that enabled teachers to identify children's current mathematical knowledge, and locate children's zones of proximal development within a framework of growth points. This use of a framework of growth points also enabled those children who were vulnerable in aspects of learning mathematics to be identified (see Gervasoni, 2004; Gervasoni, 2005).

Challenges For Developing Powerful Addition Strategies

The counting and reasoning strategies children use to solve addition and subtraction problems have been the focus of many studies (e.g., Clarke et al., 2002; Fuson, 1992a; Griffin, Case & Siegler, 1994; Steffe, Cobb & von Glasersfeld, 1988). The findings of these studies provide a basis for examining children's responses to assessment tasks and provide insight for curriculum and instructional design that aims to assist children to construct the more powerful reasoning strategies. Counting-based strategies identified in the research include *count-all* (including perceptual counting and counting by representing), and *count-on* (from largest and smallest addends). Reasoning strategies include doubles, near doubles, adding ten, adding nine, commutativity, combinations for ten, part-whole strategies, and retrieving answers from memory (e.g., Clarke, 2001; Fuson, 1992b; Griffin et al., 1994; Steffe et al., 1988; Steffe et al., 1983).

Counting-on and counting-back (Steffe et al., 1983) are important strategies for children to develop because they are based on children operating on mental and abstract images of numbers rather than needing to count, from one, all of the objects in collections, to determine a total. However, some children become over-reliant on physical modelling and counting-all, and experience difficulty developing more powerful reasoning strategies for solving addition problems.

Once children have developed a range of counting and reasoning strategies for solving addition calculations, it becomes important that they are able to choose wisely among these strategies to fit the characteristics of a strategy to the demands of a task (Griffin et al., 1994). However, not all children choose wisely or have each strategy available. Also, Gervasoni & McDonough (2000) identified typical difficulties that children experience when calculating. These include: forming mental images of numbers; counting-on accurately; counting-on from the larger number (commutativity); seeing relationships between numbers (part/whole relationships); and understanding addition and subtraction as inverse operations.

In summary, it is clear that researchers have identified key challenges that some young children face in the general course of developing strategies to solve addition calculations. To further explore these challenges, the study reported in this paper sought to provide insight about the addition strategies used by a group of 102 Grade 1 and Grade 2 children who were all identified as vulnerable in aspects of number learning and recommended for a mathematics intervention program. It is anticipated that any issues identified will have implications for curriculum and instruction.

Framework For Exploring the Addition Strategies of Children Who Are Vulnerable in Number Learning

As part of the ENRP (Clarke et al., 2002) that took place in Australia, ‘trial’ schools were invited to implement mathematics intervention programs for any Grade 1 (6-year-olds) and Grade 2 children (7-year-olds) who were identified as vulnerable in number learning. In 2000, 42 Grade 1 children and 60 Grade 2 children were selected. Grade 1 is children’s second year of school, and Grade 2 is children’s third year of school. The process for selection involved examining children’s growth point profiles that were established following a clinical interview, and then prioritising children’s need for an intervention program according to these profiles (Gervasoni, 2004). For example, the ENRP Growth Points for Addition and Subtraction Strategies are:

0. Not apparent in this context
Not yet able to combine and count two collections of objects.
1. Count all (two collections)
Counts all to find the total of two collections.
2. Count on
Counts on from one number to find the total of two collections.
3. Count back/count down to/count up from
Given a subtraction situation, chooses appropriately from strategies including count back, count down to and count up from.
4. Basic strategies (doubles, commutativity, adding 10, tens facts, other known facts)
Given an addition or subtraction problem, strategies such as doubles, commutativity, adding 10, tens facts, and other known facts are evident.
5. Derived strategies (near doubles, add 9, build to 10, fact families, intuitive strategies)
Given an addition or subtraction problem, strategies such as near doubles, adding 9, build to next ten, fact families and intuitive strategies are evident.

For the purposes of this study, Grade 1 children who had not reached Growth Point 1 (using count-all strategies) at the beginning of the school year were considered vulnerable in this domain because it was *not* apparent that these children had a successful strategy available to solve simple addition problems, and that this could preclude them from engaging in typical classroom experiences. Similarly, Grade 2 children who had not yet reached Growth Point 2 (using count-on strategies) were considered vulnerable (see Gervasoni, 2004) in this domain.

Before children commenced the intervention program, *Extending Mathematical Understanding* (EMU), the specialist teachers assessed each child's current knowledge using the EMU clinical assessment interview (Gervasoni, 2004). The EMU assessment interview enables teachers to gain detailed information about each child's current understandings, any specific difficulties that may be impeding their learning, and to determine the particular instructional focus for each child. The interview focuses on Counting, Place Value, Addition and Subtraction, and Multiplication and Division, and the assessment tasks are organised under growth point headings with children continuing in each section for as long as they experience success. Teachers record children's responses and strategies on a detailed record sheet. For the purpose of this study, these responses and strategies for the addition tasks were analysed to determine any patterns in responses.

INSIGHTS ABOUT VULNERABLE CHILDREN`S ADDITION STRATEGIES

Two tasks in the EMU assessment interview provide insight about the children's strategy use in addition situations. Both tasks involved the physical modelling of two collections and screening one or both collections to prompt children's use of more powerful strategies than count-all. Teachers determined the strategies children used through observation and questioning, and recorded children's responses on a detailed record sheet.

The first task involved one addend being screened. The teacher placed 5 small plastic teddies on the table, asked the child to count the teddies and then screened the teddies by placing a cover over them. The teacher next placed three more teddies on the table beside the screened collection and said, “Here are 3 more teddies. There are 5 teddies hiding and three more here. How many teddies are there all together?” Next the teacher said, “Please explain how you worked it out”. With this task the total is small enough to facilitate modelling using fingers. However, if children were not successful with this task, then the screen was removed and the question repeated to enable the children to perceive all items and to enable children to use a count-all strategy.

In the second task, 8 + 4, both numbers were screened, and the total is greater than ten. The task involves the teacher placing 4 teddies on the table and saying, “Here are 4 teddies.” The four teddies are screened and 8 more teddies placed beside them. The teacher says, “Here are eight more teddies” and also screens these, then continues, “There are four teddies hiding here and eight more hiding here. How many teddies are there all together?…. Please explain how you worked it out.” If children were not successful, then the screens were removed and the question repeated to enable the children to perceive all items and to enable them to use a count-all strategy. Table 1 shows the strategies children used to solve the two screened tasks.

Addition Strategies	Gr 1 Children (%)		Gr 2 Children (%)	
	5 + 3 task *n*=42	8 + 4 task *n*=41	5 + 3 task *n*=55	8 + 4 task *n*=56
Incorrect	21	54	18	38
Count-all - re-present	17	10	14	18
Count-on	50	27	59	45
Basic Strategies	7	5	9	0
Other	5	5	0	0

Table 1: Percentage Frequency of Children’s Strategies for Screened Addition Tasks.

Examination of these data raises some important issues. First, there was a notable proportion of both Grade 1 and Grade 2 children who were not successful with the 5 + 3 task when the ‘five’ was screened (21% and 18% respectively).

Second, for the children who were successful with the screened presentation of the task, there was little difference in the strategies used by the Grade 1 and Grade 2 children. Both groups were more likely to use the more powerful *count-on* strategy than the *count-all* strategy, and more children used *count-on* in the simpler task than in the more complex task. This latter point was particularly noticeable for the younger Grade 1 children. This suggests that experiences that include reflection on the use of the count-on strategy and using the count-on strategy with larger numbers may be important for children’s learning.

Third, for the more complex 8 + 4 task, teachers were asked to record whether children counted-on from the larger or smaller number. Slightly more children in both grades counted on from ‘4’ rather than counting on from ‘8.’ These children were most likely prompted by the first number stated in the problem, rather than by working out which number would be the most efficient from which to count-on. This corresponds to the finding of Gervasoni & McDonough (2000) and may be an issue to draw to children's attention.

When children were *not* successful with the screened presentation of the task, teachers removed the screens and repeated the question. The responses for this sub-group of children are presented in Table 2.

Addition Strategies	Gr 1 Children		Gr 2 Children	
	5 + 3 task *n*=9	8 + 4 task *n*=14	5 + 3 task *n*=13	8 + 4 task *n*=17
Incorrect	2	2	5	6
Count-all -perceptual	6	6	8	8
Re-present	1	5	0	1
Other	0	1	0	2

Table 2: Frequency of Gr 1 and 2 Children's Strategies for 2 Unscreened Tasks.

When the screen was removed in the 5 + 3 task, seven of nine Grade 1 children were able to solve the problem using a count-all strategy. However, only eight of thirteen Grade 2 children were successful, and a similar finding was apparent for the more complex 8 + 4 task. The fact that a higher proportion of the older Grade 2 children were *not* successful with these tasks was surprising because these children had been at school for one more year. To investigate this further, the Intervention Program Priority levels for these children were identified. Priority levels are used to rank children in terms of their need for an Intervention Program. This 'unsuccessful' group of Grade 2 children were all found to be classified as Priority 1, the highest level of vulnerability. In comparison, only half of the Grade 1s were rated Priority 1. This suggests that is important for Grade 1 teachers to identify children who are not successful with screened addition tasks and provide opportunities for them to construct successful strategies before they reach Grade 2, and may be an argument for implementing intervention programs earlier than Grade 2, before children's difficulties become broader in scope.

When the screens were removed during the 8 + 4 task, teachers were asked to provide more information about the count-all strategies used by this originally unsuccessful group of children. The teachers were asked to indicate on the record sheet whether the children used a *perceptual counting* strategy or a counting by *re-presenting* strategy (Steffe et al., 1983). Steffe et al. (1983) identified two count-all strategies: perceptual counting when all items must be able to be directly perceived; and counting by re-presenting when all items need *not* be directly perceived, but children must physically re-create or re-present the items in order to find the total. The data indicate an interesting difference between the strategies used by the Grade 1s and Grade 2s (see Table 2). More Grade 1s used the higher level counting by representing strategy than did the Grade 2s. It seemed that the Grade 2s relied almost exclusively on counting by ones the items that they could both see and touch. It is possible that the older children may become reluctant to try and have faith in more abstract strategies. Thus it may be important to identify these children and assist them to become aware of and use other strategies before the perceptual counting strategy becomes an entrenched strategy.

IMPLICATIONS AND CONCLUSION

The findings of this research indicate that some Grade 1 and Grade 2 children were unable to solve simple screened addition tasks, even after one or two years at school. Further, many students were not able to use count-on or reasoning strategies to calculate, and a notable number of students were reliant on count-all strategies. This was particularly true for the most vulnerable and older Grade 2 children.

It is important that school systems strategically deal with this situation through the provision of clear advice for teachers. Classroom teachers who are aware of the Grade 1 and Grade 2 children who are reliant on count-all or count-on strategies can design curriculum and instruction aimed at assisting them to construct reasoning-based strategies. The findings of this research suggest that it is important that this occurs before children's use of count-all strategies become entrenched, as was obvious with the more vulnerable Grade 2 students.

It is also recommended that school communities introduce assessment and monitoring strategies to identify any children who do *not* use reasoning-based strategies and provide more intensive instruction to assist them. This may involve the support of a specialist intervention teacher who is able to offer advice for classroom teachers, in-class support for children in need of more intensive instruction, or intervention programs aimed at accelerating children's learning. Finally, school systems can use the findings to inform the provision of advice about curriculum, instruction and assessment aimed at enhancing learning opportunities for children who are vulnerable, and associated professional learning programs for teachers. This may prevent children from becoming reliant on count-all and count-on strategies in the first place.

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A GENERIC ORGANIZER FOR THE ENRICHMENT OF THE CONCEPT IMAGE OF DERIVATIVE

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In this paper, we focus the case the approach to the concept of derivative based on the notion of local straightness. We briefly recall the theory of local straightness as well as our previous work concerning conflict situations in undergraduate calculus teaching. We argue that such situations may play a decisive role in the development of learners' concept images of derivatives. Our argument is supported by a qualitative study conducted with a Brazilian student.

INTRODUCTION

The use of technology in the teaching of mathematics may lead students to face situations in which graphical and numerical outputs provided by computers and graphical calculators conflict with their prior conceptions. However, such situations do not necessarily have negative effects on learners' cognitive development. Recent literature has pointed out that, within a suitably designed pedagogical approach, those effects may be very positive – especially on prompting deductive reasoning.

Hadas et al. (2000) present a set of activities designed on a dynamic geometry environment to motivate the need to prove, by causing surprise or uncertainty from situations in which the possibility of a construction was against students' intuition. The number of deductive explanations increased considerably in situations involving uncertainty. The authors conclude proofs were brought into the realm of students' actual arguments, and they naturally engaged into the mathematical activity of proofing. Similarly, Doerr & Zangor (2000) report pre-calculus classroom observation on the use of graphing calculators. The authors claim that, contrary to previous concerns, the device did not become a source of mathematical authority. They remark that that perspective was a consequence of the approach adopted by the teacher, particularly by her awareness to limitations of the calculator and her belief that conjectures are proved on the basis of mathematical reasoning.

In the particular case of calculus, many apparent contradictions may arise from the confrontation between the finite structure of computers' algorithms and the intrinsically infinite nature of the main concepts (limits, derivatives, integrals).

In this paper, we address the case of a computational generic organizer, *BestLine* (Giraldo, 2001), designed for the learning of derivatives, grounded on the notion of local straightness as a cognitive root (Tall, 1989). We briefly review the theoretical framework and resume the discussion about computational conflicts, established in our previous work (Giraldo et al. 2003a, 2003b; Giraldo et al. 2004; Giraldo 2004). We analyse reactions of a student, Antônio (pseudonym) using *BestLine* as a learning

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 185-192. Prague: PME.

environment and argue that, within a suitable approach, conflict situations may act as important factors for the enrichment of students' concept images in early calculus.

THEORETICAL FRAMEWORK

Tall (1986a) defined a generic organizer to be a microworld or environment that enables learners to manipulate examples and (if possible) non-examples of a specific mathematical concept or a related system of concepts. The environment *A Graphic Approach to Calculus* (Tall et al., 1990), an improved version of earlier software, includes various generic organizers for learning the main concepts of Calculus. The generic organizer for derivatives, labeled *Magnify*, consists of a microworld where users can zoom in a portion of a graph and observe that it looks straight if the function is differentiable, or wrinkled if it is not. The environment design prompted a theoretical reformulation of the original notion of generic organizer. In Tall (1989), the author claims that the structure of a generic organizer must be grounded on an anchoring concept, which may bridge learners' previous knowledge with more sophisticated theory to be built. Such concept is named a *cognitive root* by the author and defined as a cognitive unit holding two fundamental features: (1) be meaningful for the students in the beginning of the learning sequence and (2) allow cognitive expansion towards further theoretical development.

In general, a formal definition is not suitable as a cognitive root. This is the case of the derivative, as the definition is grounded on the concept of limit – which is deeply unfamiliar for students in the beginning of calculus learning. Many authors have reported learning obstacles related to conflicts between the theoretical formulation for limits and students' previous intuitions (eg. Cornu, 1981; Sierpinska, 1987). Therefore, the formal definition of derivative is not a cognitive root for the concept, since the first feature above does not apply (although the second one certainly does).

On the other hand, Tall (1989) claims that the notion of local straightness is a suitable cognitive root for derivatives, as it is based on the human perception that a curve looks straight if closely observed. According to the author, local straightness has global implications, as the individual looks along the graph and sees the changes in gradient, so the gradient is seen as a global entity. Thus, in an approach based on local straightness, the derivative is introduced through the primitive perception of global changes in a graph and the gradient at a given point as the slope of the line which the graph mingles with, when highly magnified. Therefore, the associated generic organizer is a computer environment allowing users to sketch a graph, change graphic window ranges and observe consequent changes in the graph's appearance.

In Giraldo (2004), a *description* is defined to be any reference to a mathematical concept, employed in a pedagogical context, which does not exhaust the referred concept, that is, which comprises limitations, in the sense that it stresses certain aspects and overshadows others. If such limitations do not match with learners' prior concept images, they may lead to situations of apparent contradiction, when theory seems to flaw or not to apply. We have used the term *conflict* to refer to a situation like that.

Theoretically speaking, one would expect that, when a differentiable function is highly magnified on the neighbourhood of a point, it should gain the aspect of its tangent line at that ponit. However, unexpected results (usually due to of algorithms' constraints) may show up as the magnification process is carried on. The software *Maple V*, for instance, displays a polygon, rather than a straight line (figure 1), for small graphic window ranges (smaller than 10^{-5}). This outcome is a consequence of the algorithm's structure, which carelessly interpolates a finite set of points. As a result, decimal approximations lead to the polygon-like image.

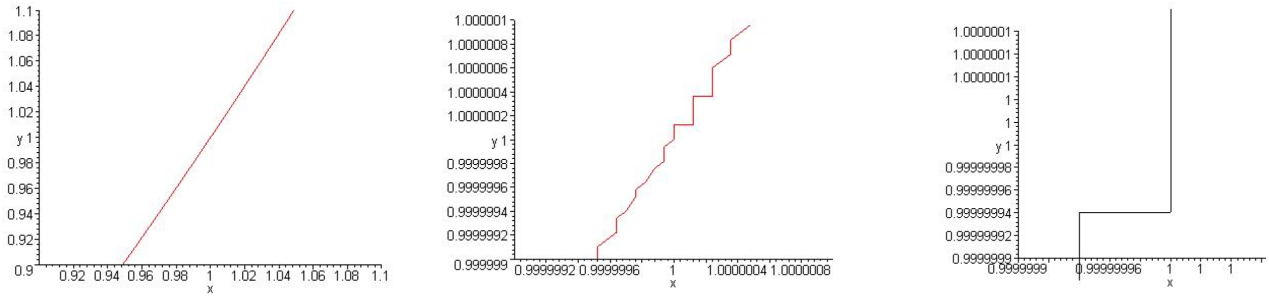

Figure 1: Limitations of a computer's graph sketching algorithm.

Therefore, the notion of local straightness is an instance of a description for the concept of derivative (in fact, figure 1 shows one of its limitations). At least until a limiting barrier (namely, the 10^{-5} window range, in the case of the example above), the outcome occurs as predicted by the theory. But if the process is continued the description's limitation pops up. More generally, we may conceive a description as been constituted by two facets: sometimes it matches learners' concept images, and sometimes contradicts it. We will refer to these facets as *comfort zone* and *conflict zone*, respectively. Although we have defined a description as a reference comprising intrinsic limitations, these limitations may be actualized as conflicts in very diverse manners (if at all). In other words, the boundaries between comfort and conflict zones depend on the whole pedagogical context: students' previous concept images, attitudes and beliefs; tutors' strategies and decisions; and so on. In our previous work, we have observed one same computational description acting distinctly in different situations (Giraldo et al. 2003a, 2003b; Giraldo et al. 2004; Giraldo 2004).

We hypothesize that the boundary between comfort and conflict zone has a crucial influence on learners' concept image. In fact, we believe that the way a description is dealt with may convert the associated conflicts into enriching or narrowing factors. Mathematics education literature provides evidence for this hypothesis.

A GENERIC ORGANIZER FOR THE LEARNING OF DERIVATIVES

The generic organizer *BestLine* (Giraldo, 2001) was originally designed to prompt learners to build on connections between cognitive units throughout the local magnification process (based on local straightness as cognitive root), by comparing graphic and algebraic descriptions. It consists of is a *Maple* routine with inputs and outputs described on table 1 below.

INPUTS	**OUTPUTS**
f – a real function	the graphs of f in the given domain
x_0 – a point in the function's domain	the straight line $r(x) = ax + f(x_0)$ in the interval $[x_0 - h, x_0 + h]$
a – a numeric value for the slope of a straight line passing through $(x_0, f(x_0))$	a vertical segment linking the graph and the straight line (representing the difference $\rho(h) = \lvert f(x_0 + h) - r(h) \rvert$)
h – a numeric value for h	the numeric values of $\rho(h)$ and $\frac{\rho(h)}{h}$.

Table 1: *BestLine*'s inputs and outputs.

The main idea is to compare graphically and algebraically the local behaviours of the curve $y = f(x)$ and the line $y = r(x)$ for $a = f'(x_0)$ and $a \neq f'(x_0)$ (that is, for tangent and non-tangent lines). Figures 2 and 3 reproduce examples of screens generated by *BestLine* for $f(x) = x^2$, $x_0 = 1$, with $a = 2 = f'(1)$ and $a = 2.5 \neq f'(x_0)$, respectively. By displaying both the graphic and algebraic representations, we aim to provide a broader view to the fact that, among all the straight lines passing through $(x_0, f(x_0))$, the tangent is the one which *best approximates* the curve, in the precise sense that not only the difference $\rho(h)$ tends to zero, but so does the ratio $\frac{\rho(h)}{h}$. The picture of the graphs provides a geometrical interpretation to the approximation: as the user zooms in, by decreasing the value of h, this value acts as a reference unit to the picture. If the straight line displayed *is not* the tangent, $\rho(h)$ vanishes, $\frac{\rho(h)}{h}$ does not, and the vertical segment is always visible (figure 3). On the other hand, if it *is* the tangent, both $\rho(h)$ and $\frac{\rho(h)}{h}$ vanish, and the vertical segment quickly disappears from sight (figure 2). In the case of the non-tangent straight line, $\rho(h)$ and h decrease in a balanced rate (since $\frac{\rho(h)}{h}$ does not tend to zero), whilst in the case of the tangent $\rho(h)$ decreases in a higher rate (since $\frac{\rho(h)}{h}$ tends to zero). Since h is the horizontal dimension of the graphic window, and $\rho(h)$ is the vertical segment, when the window is zoomed in, the segment disappears in the case of the tangent and does not in the case of the non-tangent.

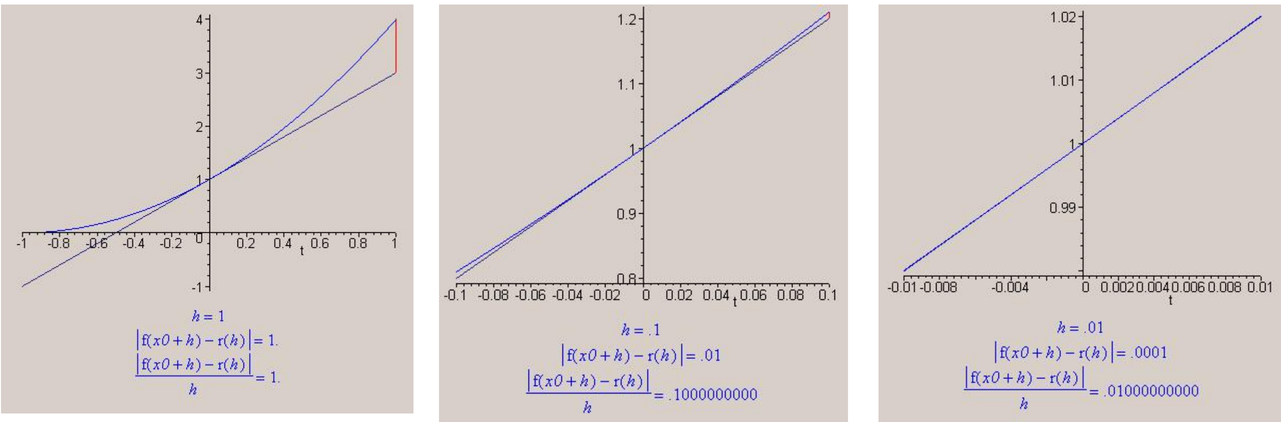

Figure 2: Screens of the generic organizer *Best Line* for a tangent straight line.

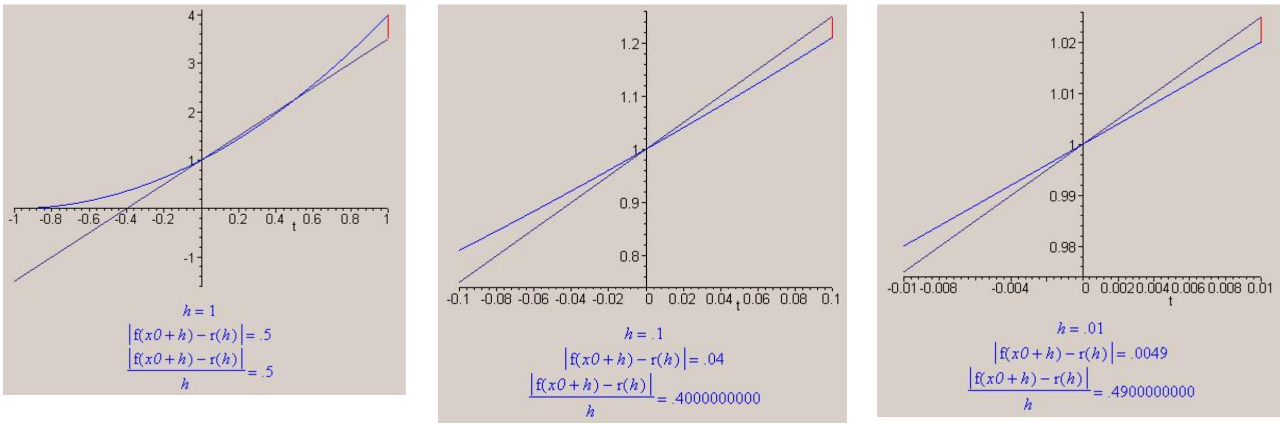

Figure 3: Screens of the generic organizer *Best Line* for a non-tangent straight line.

There are at least two main descriptions for derivatives involved with the design of *BestLine*. The first one, obviously, is the notion of local straightness. The second one is the notion of local approximation, usually expressed by the sentence: 'the tangent straight line to a function graph at a point approximates the function in the neighbourhood of the point'. One potential limitation of this second description is the fact that the meaning of the term *approximate* is mathematically inaccurate. In fact, this term has a precise meaning in the context of infinitesimal calculus: the tangent *approximates* the curve in the sense that the ratio, $\frac{\rho(h)}{h}$ tend to zero. Using a general meaning of the term, one might say that every straight line crossing a function's graph at a point would *approximate* the function, in the sense that the difference $\rho(h)$ between the function and the line tends to zero, whichever is the line (provided that the function is continuous).

In the whole experience illustrated by figures 2 and 3, one is likely to remain in the comfort zone of the local straightness description. However, the first screen (on the left) displayed on each figure might push us into the local approximation conflict zone. There is no clear distinction (nor graphic or algebraic) between the tangent and non-tangent straight lines, hence the precise idea of approximation is unclear. As we carry on the zooming process, we are pushed back to the local approximation comfort zone, since we may then grasp the mathematical meaning of approximation from both graph and algebraic expressions. Therefore, the generic organizer *BestLine* may prompt comfort zone and conflict zone of two different descriptions to overlap.

A QUALITATIVE STUDY

The approach described above has been tested in a calculus undergraduate course in a Brazilian university. A sample of six participants was selected for a qualitative study, among the students who took that course. Sampling was based on the responses to a written questionnaire and an interview, taken by the students who volunteered to participate. We aimed to test the approach with participants with backgrounds as different as possible. The empirical study was based on a series of individual structured and semi-structured interviews, in which they dealt with tasks involving computational descriptions for differentiable and non-differentiable functions, based on the notion of local straightness. All interviews were tape recorded and transcribed.

In this paper we report the reactions of a participant, Antônio, on interview 5, in which he was asked to perform local magnification processes with *BestLine* and verbally describe his impressions. The preceding interviews mainly involved limitations of the local straightness description (as the one displayed on figure 1). Throughout those interviews, Antônio spelled out very clearly his awareness of computer's limitations and of the possibility of 'mistaken' outcomes – this was a key aspect on his behaviour. We observed different effects of conflict situations on Antônio's concept image. In some cases, his attitudes towards the device gave him means to quickly grasp the theoretical issues related with the conflict. In those situations, the conflicts acted as *reinforcement factors,* strengthening his previous knowledge. In other situations, his knowledge was not enough to comprehend what was going on. The conflicts then triggered new linkages between cognitive units, acting as *expanding factors*. In other occasions yet, his previous beliefs constituted obstacles to perform the given task. The conflicts then served as *reconstructing factors*, prompting him to rethink and restructure concept image.

We transcribe below excerpts from Antônio's responses to interview 5 (translated from Portuguese). After observing the magnification for a non-tangent straight line, he explains *BestLine*'s outcome for the tangent line, comparing the two cases:

Antônio: In the beginning, they both look alike. […] That is, these two pictures that the computer shows look alike. But that's not what's really going on, I mean, mathematically speaking. If one trusts the computer, they'd probably just see these similar picture and think: yeah, that's the same. But, in my case, I do not trust it, so I need to go deeper. I mean, to try to understand what's going on. That's why I zoom in, to can see closer, I pay attention on these numbers [points $\rho(h)$ and $\frac{\rho(h)}{h}$ on the screen], I can really see the limits behaviour.

Interviewer: Could you explain what's really going on here, mathematically, as you said.

Antônio: So, this guy [points $\rho(h)$ on the screen] will have to decrease faster than h. Because if h went faster, than this guy would ever remain, and this length here [points the vertical segment on the screen] would remain too. […] But in the case of the derivative it doesn't happen. We see it's a very special characteristic. […] There exist infinitely many straight lines that are really close to the curve at the point. At the point, they all coincide anyway. […] But the derivative is that one which glues to the curve. Actually, I think it's even more than that.

Interviewer: In which sense is it even more?

Antônio: We actually have found a way of rewriting the definition. But it's more […] things aren't so hidden, you know? It's on one's face. You can feel what's going on. Actually we did write the definition, but this way is much stronger than the way we all know, with the limit and so on. Here it really shows the relation of those two magnitudes, if we divide one by the other, we compare them, and realize what goes on when it tends to zero.

We see that Antônio's awareness of computer's limitations play an important role on his reasoning and conclusions. He comments further this attitude:

Antônio: When you see that graph, you can't see what's actually happening. […] But, I'm very curious, you know. Sometimes the computer takes you where you can't go by other means. […] Sometimes the computer shows something that seems to be but is not. […] I used to dislike those computer mistakes. I still do, I guess, but I see the other side of it, what you can learn from its mistakes. […] They remind you that the computer is not always right. It makes you question, makes you learn. So it's good too, even its mistakes are good.

These results suggest that the experience with *BestLine* had an enriching effect on the participant's concept image: he formulated a deeper interpretation for the formal definition of derivative. As Antônio himself states, such effect is related with his familiarity with conflict situations, which he had acquired on his previous experiences with local magnification processes. Therefore, previous conflict situations furnished Antônio with conditions to go beyond the conflict zone of the local approximation description (when tangent and non-tangent lines look alike) and seek further theoretical understanding.

FINAL REMARKS

Quoted research shows that descriptions (particularly computational ones) may have opposite roles: they may act as narrowing factors as well as enriching factors on learners' concept images. Our investigation suggests that if conflicts are exploited within a suitable pedagogical approach – rather than merely avoided – they may trigger an enriching process. Hadas et al. (2000) instance experiences in which students benefit from conflicts, under careful guidance of the tutors. Doerr & Zangor (2000) confirm that a crucial condition for the success of such strategy was the teacher's posture: her awareness not only of the device's limitations, but also of the importance of deductive reasoning in mathematics. Similarly, in our own work, Antônio's critical standpoint towards the computer outcomes was decisive. Therefore, when designing curricula for pre-service and in-service teachers' courses, focusing on potentialities and limitations (and limitations *as* potentialities) of technological tools is needed.

Pedagogical potentialities of a description for a mathematical concept do not reside only in how faithfully it describes the concept, but also in what it lacks. That is, the effectiveness of a description arises from the judicious application of both comfort zone and conflict zone. Furthermore, (as instanced by Antônio's episode with local straightness and local approximation descriptions) comfort and conflict zones of different descriptions may complement each other to constitute a powerful pedagogical resource.

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PUPILS' ATTITUDES TO INTEGRATED LEARNING SYSTEMS IN MATHEMATICS

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This paper reports on the findings of a questionnaire survey of pupils' attitudes to their use of an Integrated Learning System (ILS) in Key Stage 3 (11-13 year old pupils) mathematics and examines the relationship between their attitudes and their performance in the Key Stage 3 (KS3) mathematics SATs. Their overall attitude was found to be largely positive but their opinions of the ILS varied considerably with regard to particular features of the software. Pupils' attitude was not found to be associated with their attainment as measured in standardised tests and did not vary according to frequency and duration of use. However, there was a statistically significant difference in attitude according to gender.

CONTEXT

A previous paper by the authors (Gkolia & Jervis, 2005) has presented the results of the statistical treatment of KS3 mathematics SATs data from 239 pupils in six schools who formed a test group and an equivalent number of pupils in the same number of schools that formed a control group. The SAT data were treated based on the value-added measure as described by the DfES (2003). ILSs were found to have a statistically significant negative impact on KS3 mathematics attainment. However, the effect size was found to be moderate to small (ES=0.2) and thus the effect of ILSs on mathematics was deemed educationally unimportant.

This paper reports on the results of a questionnaire survey of the attitudes of the pupils who formed the test group. Attitude in this paper refers to the way pupils/users of an ILS view their experience with the ILS in terms of helping them to learn, making their learning more enjoyable, motivating them to learn more and increasing their confidence in their academic performance in mathematics.

RESEARCH BACKGROUND

Few studies have explored the attitudes of pupils towards ILSs. Those which have done so have used either survey questionnaires or interviews with pupils as part of studies into the effect of ILSs on attainment. The rationale for using a questionnaire was to test the suggestion that achievement on the system was related to the attitude of the users towards it (see e.g. Becta, 1998). However, the instrument used in those studies was often an already established questionnaire seeking attitudes towards computers in general and not specifically targeted at ILSs.

Some of the early studies of ILSs also examined how those systems affected pupils' self-esteem and their self-perception of computer skills, and focused very little on

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their perceptions of learning in relation to the ILS. One example is the study by Gilman (1991) who investigated the effect of ILSs on pupils' affective traits in elementary education in a U.S. state. The study looked at pupils' self-esteem, attitudes towards school and computers, and their self-perception of computer skills, reporting a positive overall effect. However, careful examination of the results reveals signs of inconsistency across grades (Years) and across the different strands of the attitude questionnaire; the most surprising being the apparently negative effect of the ILS on pupils' self-perception of computer handling skills.

In the U.K., as part of their evaluation series, NCET (NCET 1996; Becta, 1998) examined pupils' attitudes towards ILSs. In the second phase of the evaluation the research team gathered attitudinal data based on interviews with pupils as part of case studies that involved some of the schools in the original sample. NFER (as cited in Becta, 1998) commented,

> "if there is one finding that did emerge consistently from the evidence available from the case studies, it is that pupils found integrated learning systems engaging and motivating." (p. 22).

This finding disagrees markedly with Kidman et al.'s (2000) study where pupils disliked the ILS so much that they ceased using it after a short time.

The third phase of the Becta (1998) evaluations approached the attitudes issue with a quantitative methodology. Researchers used a survey questionnaire to explore pupils' attitudes towards their experience of the system, which was not specifically linked to the ILS intervention per se. Their argument for it was that respondents, when involved in attitudinal interviews and questionnaires, "are generally biased against articulating negative opinions." (ibid, p. 23). This method too produced evidence of positive feelings of pupils towards the ILS that were in agreement with the findings of the previous phase.

Australian researchers (McRobbie, Baturo & Cooper, 2000) carried out a study that produced ambivalent results on the issue. As part of their large-scale multi-method study they looked into low achieving pupils' attitudinal changes as a result of using ILSs in primary and secondary schools, using a pre- and post-test computer attitude survey along with an ILS evaluation questionnaire that asked pupils' opinions on ILSs and required them to justify their answers. The investigation found that, although the majority of pupils liked using the ILS and believed that it helped them engage and focus on their learning, their overall attitudes towards computers post-test were significantly worse than their pre-test ones. The researchers attributed the paradox to the diversity of opinions between individuals as to what constitutes improved learning.

Presland & Wishart (2004) researched, in a small case study, how and why the use of an ILS motivated a group of Year 8 pupils in numeracy and literacy. This is one of the very few studies that used an ILS-specific questionnaire to do so and attempted to relate the 'motivational power' of ILSs to specific characteristics of the system. The authors found that use of ILSs had a motivational effect on pupils and was linked to raised self-esteem. It was also found that pupils' increased motivation was brought about by their ability to get high scores on the ILS's internal scoring system, their awareness of making progress due to continuous feedback, the linked incentives and rewards for their high performance on the ILS and, finally, their perceptions of associated benefits in the numeracy and literacy work outside the classroom (ibid).

Hativa (1994) in his six years of qualitative and quantitative studies on the effect of ILSs on mathematics also looked at the affective impacts of learning with an ILS. He, too, used an ILS specific questionnaire, as well as observations of ILS sessions, to look in detail at pupils' attitudes towards specific features of the ILS. Overall, he found that the vast majority of pupils (percentages ranging from 70% to 75%) liked their work on the ILS. When he focused, however, only on high achieving pupils, that percentage rose to 92%, while it dropped to 59% for low achieving ones. He did not find a statistically significant difference according to gender. In regard to specific characteristics of ILSs, Hativa found that the features that students rated the highest were the system's scoring system and regular feedback, something that agrees with Presland & Wishart (2004) and Jervis & Gkolia (2005). Hativa also found that pupils disliked the time limit on competitive tasks, the repetitive nature of some of the work and the system's tendency to provide them, sometimes, with work that was either too easy or too difficult for them. Finally, he reported that, during his observations, pupils at the low end of the ability range tended to respond very negatively to failure to complete tasks and were discouraged by the 'clear-cut' negative feedback of the ILS in such cases.

THE INSTRUMENT

A questionnaire was administered to all test sample pupils still present in the schools to examine their attitudes to an ILS. It consisted of six items seeking factual information about the time, length and frequency of ILS use for mathematics, 41 rating scale items of five levels (scored such that the fifth level always indicated the most favourable response to the ILS) and an open-ended question where respondents were free to add anything they wished relevant to their experience of the ILS. The rating scale items were initially developed in subgroups under category headings that represented the main strands of pupils' experiences with ILSs as indicated in previous work (Gkolia & Jervis, 2001).

The questionnaire was pilot tested with a class of Year 7 pupils in a secondary comprehensive school that was, at the time, making use of ILSs. The pilot aimed to test the questionnaire in terms of accuracy of expression, appropriateness of language (in terms of difficulty), and clarity of statements.

The completed questionnaires from the pilot were examined for consistently missing responses as well as contradictory responses to similar questions. A few amendments were made to the questionnaire following the results of the pilot phase.

Shortly after the pilot testing, the questionnaire was used as part of a case study. reported elsewhere (Jervis & Gkolia, 2005). As part of the analysis, the validity of the questionnaire's attitude scale items was tested based on the Partial Credit Model (PCM). The responses to the pupil questionnaire were subjected to Rasch analysis using the QUEST program (Adams & Khoo, 1996) in order to test the internal validity of the questionnaire. The majority of the questionnaire items fell within the suggested range; for the analysis of rating scale items, the infit mean square range suggested for validity is 0.6-1.4 (Wright & Linacre, 1994). Only two items had an infit mean square greater than 1.4 equating to a 5% level of misfit, which is an acceptable value in PCM analysis. Thus the analysis confirmed that the questionnaire validly measures one property, namely pupils' attitude to the use of the ILS. The two misfitting items were subsequently altered in the light of the PCM analysis (Gkolia & Jervis, 2004).

In the present study, 91 valid questionnaire responses were returned from five secondary schools. Questionnaires were subjected to statistical treatment in order to answer the following questions:

- What is the overall attitude of pupils who have used an ILS in KS3 mathematics?
- What is the attitude of pupils towards particular features of the ILS (e.g. feedback, task difficulty) and towards the ILS's effect on their mathematics achievement?
- Are there any differences in the overall attitude of pupils according to gender and patterns of use?
- Is pupil attitude towards ILSs associated with their performance in KS3 mathematics SATs?

ANALYSIS AND FINDINGS

Overall Attitude

Each pupil's total attitude score was calculated by adding all scores across all attitude scale items and the total percentage was calculated based on their 'relative' maximum score – that is the maximum total score a pupil could obtain if scale items with invalid answers were excluded from the calculation in each case as opposed to the 'absolute' maximum score which is calculated including all scale items in the questionnaire, which is the same in all cases.

The score that pupils could obtain on the attitude scale items was divided into five ranges that broadly categorise their attitude towards their experience of the ILS. The 0-21% range represented a 'strongly negative' attitude, the 22-45% a 'fairly negative' attitude while the scores ranging from 46-54% were categorised as 'neutral'. A 'fairly positive' attitude was represented by the 55-78% range and any score above 78% signified a 'strongly positive' attitude.

The vast majority (79%) of respondents found their experience of ILSs fairly positive while 8% thought it was strongly positive. Only 4% of respondents found their experience fairly negative and no respondents characterised it as strongly negative. 9% of pupils had no particular feelings towards it. That generally positive attitude agrees with much of the literature published in that area. The Becta evaluations (NCET, 1996; Becta, 1998), Hativa (1991, 1994), McRobbie et al. (2000) and Jervis & Gkolia (2005) found positive attitudes for the majority of their samples.

Attitude and Gender

Attitude data were also examined for differences according to gender using *t*-tests. Female pupils scored higher than their male counterparts and the difference between their mean attitude scores was statistically significant (p=0.01). This is an unusual finding compared to studies (Cooper & Weaver, 2003; Colley & Comber, 2003; Lynn, Raphael, Olefsky & Bachen, 2003) of the attitude of pupils to computers which that have found that, generally, boys are more positively predisposed towards computers. In ILS-specific research, however, Hativa (1991; 1994) did not find significant differences in the overall attitude of pupils towards work on an ILS according to gender.

Attitude and Patterns of Use

Multiple comparisons between groups were made using Tukey's HSD post-hoc analysis in order to examine whether there were any differences in the attitude of respondents depending on the number of KS3 Years that the ILS was used in and the total period they used it for during those Years. For both independent variables there were no significant differences between groups. There were no significant differences in attitude according to number of KS3 Years in which an ILS was used.

The Pearson correlation coefficient was used to test the existence of a relationship between attitude towards ILSs and frequency of ILS use per week and between attitude and the total time spent on an ILS per week. The Pearson coefficient value was returned as non-significant in both cases. Scatter plots confirmed that there is no apparent relationship between attitude and the two independent variables.

Attitude towards ILS Characteristics

The feature of the ILS that was rated highest of all (78% of respondents) was its feedback system, based on percentage scores, and its accessibility at any point during a session. This finding agrees entirely with the results of the case study reported by Jervis & Gkolia (2005) where the same questionnaire was used. Hativa (1991; 1994) in both of his studies reports precisely the same. In his analysis of what pupils/users of the ILS appreciated most, the system's feedback system came first on the list with a similar percentage (70-90% - depending on the brand of the ILS – thought so).

Respondents had mixed opinions about the presence of Americanisms in the software's subject content (both visual and audio). Percentages of 36% and 33% agreed and disagreed respectively with the possible Americanisation of the software while 31% had no opinion about it. That can be easily explained by the fact that the schools which provided the test sample had different brands of ILSs, which, although they were all initially manufactured in the U.S., have been anglicised by distributors based in the U.K. to different degrees. Additionally, in some of the test schools, ILSs were used for English, as well as mathematics instruction, where language and accent divergence is more noticeable and more closely related to the scope of the subject. Thus, there were valid explanations for the variability of pupils' opinion on this matter.

45% of respondents found the level of difficulty of the tasks presented to them satisfactory with 27% thinking otherwise and 28% having no strong feelings about it. This varied picture is similar to the one reported by Hativa (1991; 1994) in both of his studies.

Respondents had mixed feelings towards the ILS's screen presentation and graphics. Almost equal proportions of students, 36% and 35% scored positively and negatively respectively on the related items. The study by Jervis & Gkolia (2005) revealed similar attitudes. A possible explanation, provided in the same paper, for pupils' lukewarm attitudes towards the graphic presentation, was that their own extended experience of high quality screen displays and graphics at home affected their opinion of the ILS, which may constitute a weakness on the ILS manufacturers' side.

The feature that was rated lowest was the tendency of the ILS to repeat tasks for reinforcement. The vast majority of pupils (81%) found this process repetitive and boring. This is in agreement with Hativa (1991; 1994) who found that the characteristic of ILSs that pupils disliked the most was being presented with work in a repetitive manner.

One of the most interesting findings of the analysis of the individual questionnaire items has to do with pupils' comparisons of the ILS to the 'normal' non-ILS lesson and their teacher. While 53% of respondents preferred working on the ILS to being in the classroom, only 36% preferred the ILS to their teacher. This discrepancy, which is present to a higher degree in the case study by Jervis & Gkolia (2005), may indicate that, although pupils favour the ILS as a form of diversion from their everyday school reality, they do not prefer it to their teacher as a tutor.

More than half of the pupils (54%) believed that the use of ILSs had a positive effect on their performance in the classroom and their understanding of the subject, with only 16% believing otherwise. This proportion is very close to the one found by Jervis & Gkolia (2005). This finding agrees with McRobbie et al. (2000) and Presland & Wishart (2004) who also found that pupils felt that the ILS helped them to learn.

Attitude and Achievement

Finally, the relationship between the same pupils' achievement (Gkolia & Jervis, 2005) and attitude towards the ILS, as measured by the scale items of the pupil questionnaire, was examined. Value-added scores and attitude percentages were plotted against each other to allow examination of the relationship between the two variables. The scatter plot did not point to any particular relationship between value-added and attitude towards ILSs ($r = 0.02$). This is in agreement with the findings of the Durham team (Becta, 1998) where there was a marked conflict between the results for learning outcomes and the picture of a generally positive attitude towards the ILS in pupils.

CONCLUSIONS

Pupils appear to have well-formed attitudes regarding their experience with the ILS in mathematics. Their overall attitude is positive, but varies according to different aspects of the software whilst it is not associated with the frequency and duration of use. Female pupils held significantly more positive views about the use of an ILS than their male counterparts.

What is particularly interesting is that attitude does not appear to be related to pupils' performance in national examinations. This lends weight to the views of Wood, Underwood & Avis (1999) who, after critiquing the methodology and the results of the Becta series of evaluations, concluded that,

> "any exclusive reliance on 'user satisfaction' as an index of the effectiveness of technology should not be taken at face value and treated with considerable caution in the absence of converging evidence for the effects on performance and learning." (p. 95).

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THE DISCOURSE OF LOGICAL NECESSITY: RULES FOR ACTION IN PRE-SCHOOL MATHEMATICS

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Discursive Psychology, as suggested by S. Lerman in his 1998 plenary address to PME, has offered us very many insights on the practices of teaching-and-learning mathematics in school. Drawing on the discursive approach of D. Edwards (1997; Edwards and Potter, 1992), this paper aims at contributing to the study of language and communication in the mathematics classroom by focusing on: (1) how teachers formulate discursive rules in mathematics instruction so as to establish the relevance and visibility of mathematical "concepts" to/with the pupils; (2) how such discursive moves creates the logical necessity for actions to be performed by pupils in the mathematics classroom; (3) how this process impact the activity of young 5yo children who are just being introduced to school mathematics. In this presentation, we hope to join in the theoretical reflections and empirical analyses of language use in mathematics teaching-and-learning already made tradition in the PME community by such researchers as L. Radford (2004) or N. Presmeg (1999).

The local production of a set of activities *in* and *as* a specific domain of practices (e.g. mathematics), so as to portray those same activities as describable and explainable, in a word, as 'rational', is a common feature of many educational, professional and scientific practices (Amerine and Bilmes, 1990; Lynch, 1993; Suchman, 1987; Livingston, 2001). The translation between setting-defining 'rules' for action and their application is one of the central concerns of the analysis of practices since Wittgenstein's *Philosophical Investigations*. It relates to the ways in which states of affairs in a practice are put together and how they can be 'understood' in so many words, or, to put it simply, to how a formal recipe or instruction can be performed. Wittgenstein's contribution was to show that the relation between a certain rule and its application is socially established, an 'impressed technique', the rule standing as the best account of its practical context, but in no way capable of causing it from the outset (Collins, 1985). Contrarily, in traditional cognitive studies it is thought that such states of affairs are produced as effects of rule-caused or rule-governed processes, such as 'logic' generating 'reasoning' and 'grammar' generating 'speech' (Edwards, 1997).

The statement of abstract rules in so many words fulfils several functions in the sequential organization of thinking-in-action and describes a property of social interaction known in the ethnomethodological and discourse analytical literature as *reflexivity* (Garfinkel, 1967; Edwards, 1997). In simple terms, reflexivity refers to the fact that members constantly document what they are doing as such-and-such, as a fundamental part of the ongoing constitution of social order. The work of rendering order visible as a describable state of things has also been called elsewhere 'formulating' (Heritage and Watson, 1979). Formulating, or 'accounting for', in instructional actions is to make inspectable, to establish relevance and visibility, to 'make sense' in/of a setting; it constitutes the 'intension' of a given class of objects and practices so assembled (Law and Lodge, 1984). The ways instructed actions and formulations are interwoven are multiple and complex, and are open to empirical investigation. For example, generally formulated instructions can be a built-in aspect of an interactant's (e.g. a programmed machine) performance in a way that interpelates its counterpart's performance as 'structured' or 'planned' (Suchman, 1987). Rule-oriented accounts of behavior argue that general instructions, accounts, recipes, determine what is to be considered their proper extensions. In practice, things are more complex than that. Very frequently in instruction sequences, what count as 'abstract' categories are inserted at the end of a sequence of practical action, as a gist on what has been previously done and said (Heritage and Watson, 1979). As Garfinkel, Lynch, Livingston and others have shown, 'abstract' propositions – particularly written inscriptions – while taking their sense from the local conditions of their production, seem to elude them completely and to become intelligible objects of their own (Garfinkel, et. al, 1981; Latour and Woolgar, 1979; Livingston, 1987; Lynch, 1993). In a word, formulating is related to naming, selecting, classifying, theorizing, etc. (Garfinkel and Sacks, 1990; Goodwin, 1997; Lynch, 1993). For example:

Extract 1, pre-school:

Teacher: The number eleven has one ten and one unit

Extract 2, pre-school:

Teacher: Ten tens are one hundred units

The propositions above are hardly of the kind we are likely to hear in most (non-school) everyday contexts. In our daily life, the use of numbers rarely, if ever, has the level of thematic awareness and sophistication observed in extracts 1 and 2. Our common uses of number and number words display 'reason' and adequacy throughout, but seldom 'justification' (Wittgenstein, 1967). In extract 1, 'eleven' is qualified as a 'number' and dismembered into a given quantity of 'tens' and 'units'. This reflexive relation between quantities and analytical categories can be seen again in extract 2, and goes on to establish the 'place value' of a number in two digit-plus (written) numerals, where the knowledge of the relative positions of units, tens, hundreds, etc., are to be built into the skills required to perform operations.

In the following extract, the students (5 year-olds) are sitting down on the floor so as to form a 'circle', inside of which the next task's 'materials' (cards depicting diverse signs, e.g. = , ≠; numerals, e.g. 2, 7; and diverse objects, e.g. 'stars', 'fruits', 'matches') are made available for the activity to take place:

Extract 3, pre-school, T = teacher; P = pupil; M = Mateus:

1.	T:	for us to say that one thing equals another we do
2.		not always need to use the word equal (.) we can use a sign (.)
3.		I am going to show (.) who knows the
4.		sign for equality?
5.	P:	I know.
6.	P:	( )
7.	T:	I am asking when I mean this word equals that one
8.		(.) I am going to show you (.) look (.) the little sign we
9.		use to say that one thing equals another ((shows a
10.		card with the sign =)) (.) this is the little sign
11.		of equal (.) I mean (.) Daniel, look (.) house
12.		equals house ((simultaneously shows three cards
13.		that make [CASA] [=] [CASA] together)) (.) this is
14.		the little sign that says that things are equal (.)
15.		each one of you is going to get now a little sign (…)
		(a few seconds later)
16.	T:	everybody now have the little sign of equality (.)
17.		I'm going to place it here in the centre of the
18.		circle ( ) (.) you're going to see that here is
19.		this kind of material with various objects in
20.		various shapes, various colours (.) here are the numerals (.)
21.		I want you to form two sets of equal
22.		things, or two equal sets
23.	M:	I've already formed
24.	T:	have you formed a set?
25.	M:	((shows [7] [=] [7] with the cards))
26.	T:	seven equals seven (.) is this a set? (.) what's
27.		lacking to form a set there, Mateus?
28.	M:	one more
29.	T:	look (.) this is a numeral, the numeral seven (.)
30.		where are the things to mean that there are

31. seven things here? (.) I'm not seeing anything look
32. I'm only seeing numerals (.) a numeral seven and
33. another numeral seven (.)
34. I want a set.

It can be said that this event immediately precedes the work of producing a state of affairs that complies with, and is accountable in terms of, the formulations it represents. It delivers a task for which the appropriate courses of action are not yet specified. In lines 1-2 an investment in the worldly sense of the task is made by stating that 'equal', also a vernacular term that 'we say', can be represented otherwise. 'Equal' is a 'word' that 'we' 'use', an arbitrary token, as semioticians would argue, since we do not have 'always' to display it to convey what we 'mean'. The word 'equal' is less than its meaning, for which the 'equal sign' can be another representation, equally able to 'say' it (lines 8-9). The meaningfulness and worldliness of the lesson are reinforced in line 7, when the teacher makes use of direct speech ('I'm asking when I mean this word equals that one') in a kind of 'animated' footing (Goffman, 1981), as someone saying (or thinking) that; note that 'when I mean' stands as a potentially recognisable, or simulate-able, ordinary action scenario. The second case (lines 21-22) establishes what is to be done next ('I want you to form two sets of equal things, or two equal sets').

As we saw earlier, the teacher proceeds to exemplify her initial proposition. For that, she manipulates a set of cards. [HOUSE] [=] [HOUSE], she implies, means 'house equals house', which, grammatical oddness notwithstanding as the latter (e.g. absence of articles or demonstrative pronouns), has the role of translating the ordinariness of 'meaning' onto a written code. This is a question that can be addressed again in terms of how the prominent role of the referent in making order visible 'affords' grammatical alternatives. That the sentences 'a house is equal to another house', or 'this house is equal to that house' do not feature as 'less strange' utterances in the exchange between teacher and pupils can be accounted for as a case for language being designed not only to comment upon, but to map onto the formal limits of the referent (the cards), to become yet *another language*, i.e. mathematics. Ironically, the aim is precisely to suppress the production of meaning in relation to referential or metaphoric dimensions of language use (Walkerdine, 1988). In the sequence, she introduces a set of materials to the children, again cards, containing the 'equal sign' (=), and, as it is claimed in the transcript above, 'objects' (line 19) and 'numerals' (line 20), and asks them to form 'two sets of equal things, or two equal sets' (lines 21-22). Mateus, one of the pupils, promptly comes up with a candidate answer: [7] [=] [7] (line 25). The teacher's injunction following the answer consists in a 'repair' strategy (Schegloff, 1991) that hands correction over to the pupil, confirming a preference for self–repair in classroom discourse (McHoul, 1990). The coupling of the questions 'is this a set?' and 'what's lacking in order to form a set…?' (lines 26-27) implies, without saying, that Mateus' answer is to count as inappropriate, incomplete, 'lacking' at best. Arguably, the notion the teacher seems to be pursuing is that a 'set' – or any other notion for that matter, such as 'equal' – means something, that is, it is irreducible to the representation device used to convey it. The task at hand is then to show, to establish indeed, a 'meaningful' semiotic link between signifieds (things, events, vernacular meanings) and signifiers (formal inscriptions). Notice that this observation does not relate, at this point, to a general conception of a semiotic process underlying the shared use of language in the classroom. What we are suggesting is that the knowledge being put together in the example is the accountable outcome of the activities and methods of/for associating diverse elements, of forging 'visible' links; semiotic 'translation', 'modelization', and the travels from one competent world (e.g. narrative, vernacular understanding) to another (logic, mathematics) then becomes the very topic of classroom teaching and inspectability.

Again, the contested answer in lines 26-27 helps to make the point. Besides the opaque, indeterminate nature of following an instruction properly, of which the sequential mechanisms of interaction are the remedial means, the very example set at the beginning of the task ([HOUSE] [=] [HOUSE]) fails to constitute a legitimately followable paradigm, a model for a 'correct' answer. The way Mateus' intervention (line 23) arguably reflects the immediacy and availability of that model is quite interesting ('I've *already* formed'), an economy of activity that is soon questioned. Notice that Mateus' answer to the request to form two equal sets is analogous to that in the teacher's exemplary case: the repetition of a term at each extreme of the expression, separated by an equal sign indicating their equivalence:

[HOUSE] [=] [HOUSE]

[7] [=] [7]

Apart from that example, no clue had been offered on how to go about solving this apparently simple exercise. 'Materials' (lines 17-20), of course, had been made available to be used, which consisted not only of the 'numerals' that composed the pupil's answer. 'Various objects in various shapes, various colors' (lines 19-20) were presented to the pupils, as categorically distinct and sequentially prior to, 'numerals' (line 20). Retrospectively, it is easier to see that those distinctions, which operate in the request form 'I want you to form two sets of equal things' (lines 21-22), work as a 'prospective account' (Amerine and Bilmes, 1990), an account that comes to life in the teacher's contestation of Mateus' answer, as discussed above. The explanation gains further elaboration when after Mateus' failure to address the teacher's question on what was 'wrong' with his answer by saying 'one more' (line 28) – arguably orienting to the problem of 'lacking' (line 27) as a numerical one – the categorical distinctions mentioned above are used both as an account of the pupils' reasoning and as a 'corrective' device, that is, it 'indexes' the activity's projected outcome (lines 29-31: 'this is a numeral, the numeral seven (.) where are the things to mean that there are seven things here?).

The equivalence between the classes of objects separated by the equal sign cannot, however, be warranted by the existence of the sign itself, as if it had, by virtue of its presence alone, some legislative power over the meaning of 'equality'. So, although the ‘inappropriate’ character of Mateus’ (mathematically correct) answer allowed the teacher to make room for the curricular demands placed upon the meaning of ‘sets’ and ‘equivalence’, it is not the case that any two given sets of ‘things’ (as opposed to numerals) will be automatically considered ‘equal’ if they have the equal sign [=] placed between them. Thus, the ‘things’ that form sets in the task arranged by the teacher are to be interpelated not in their ‘thing-ness’, but later in terms of their ‘numerical’ equivalence, or more precisely, bi-univocal correspondence, so that the relevance of any term in the expression is accountable in terms of the others’ presence. Having set up the accountable relevance of the ‘things’ that numerals represent, the teacher can pursue on which basis is ‘equality’ to be reasoned about.

One could ask at this point whether there is a reason for the philosophical minutiae of ‘set theory’ to be present in the curricular activities of pre-school children. In the case of school mathematics, that has traditionally reflected a call for ‘understanding’, rather than ‘reproducing’ instruction. Seemingly, educational reforms in mathematics have assimilated the logicized views of modern mathematics made relevant in set theory, and given visibility in psychology and epistemology by Jean Piaget and his followers as ‘developmental’ criteria. Its instructional counterpart seems to count both on an interpretation of language as a site for reference and representation, and conversely, on an educational appropriation of such linguistic philosophy of meaning and reference into a child-friendly pedagogy, where ‘meaning’ has precedence over ‘symbolization’ or ‘representation’. The question is important insofar as it reflects the search for the kind of enlightened, non-authoritative, de-individualized ‘necessary knowledge’ (Smith, 1993) that Piaget championed.

In a sense, we are not far from concluding that the teacher is teaching semiotics theory in the sense that now and again the semiotic ‘chain’ itself is to be made visible, as part of the work of entangling ‘knowledge’ to ‘non-arbitrary’ sources. Meaning, representation, reference, shifting in and out levels are offered as the ‘telling orders’ of the actions performed (Morrison, 1981). Such semiotic ethos in the (mathematics) classroom denotes the opening, for analysis, of ‘knowledge events and what might constitute an exhibit of their understanding’ (Ibid: 245). It is, though, a ‘semiotics’ whose source of meaning is logical necessity, or identity, the could-not-be-otherwise link between activities, empirical phenomena, graphic representations and mathematical symbols. It is a discourse without a future, without metaphor (Walkerdine, 1988), based on the reiterable work of mutual reference between its terms. Its abstract features can be found as an ‘observable’, as the ‘math of the lesson’ (Macbeth, 2000) in the sequential organization of the instruction, as we saw in the examples above. Such organization is replete with ‘impressing techniques’ that allow the translation between those intermediaries to stand as a proper compliance with a formal rule.

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AN ONLINE COMMUNITY OF PRACTICE FOR PRE-SERVICE AND BEGINNING TEACHERS OF SECONDARY MATHEMATICS

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The aim of this study was to investigate how a community of practice focused on learning to teach secondary mathematics was created and sustained by pre-service and beginning teachers. Bulletin board discussions of one pre-service cohort are analysed in terms of Wenger's (1998) three defining features of a community of practice: mutual engagement, joint enterprise, and a shared repertoire. The study shows that the emergent design of the community contributed to its sustainability in allowing the pre-service teachers to define their own professional goals and values. Sustainability was also related to how the participants expanded, transformed, and maintained the community during the pre-service program and after graduation.

In reviewing current perspectives on mathematics teacher education, Lerman (2001) proposed that sociocultural theories offer useful conceptual tools for understanding teachers' learning as increasing participation in the practices of a professional community. Mathematics teachers' participation in communities of practice has been investigated as a means of supporting teacher change and innovative practice in professional development programs (e.g., Gómez, 2002). The research reported here extends this work by developing the concept of a community of practice in pre-service teacher education and its interface with beginning teaching. The aim of the study was to analyse processes through which a community is established and maintained when interaction is online as well as face to face. This paper uses pre-service teachers' bulletin board discussions to investigate how the community emerged and was sustained after they graduated and began their first year of teaching.

THEORETICAL BACKGROUND

Wenger (1998) describes three defining characteristics of communities of practice as mutual engagement of participants, negotiation of a joint enterprise, and development of a shared repertoire of resources for creating meaning. Engagement need not require homogeneity, since productive relationships arise from diversity and could involve tensions, disagreements and conflicts. Yet participants are connected by their negotiation of an enterprise linked to the larger social system in which their community is nested. Such communities have a common cultural and historical heritage, and it is through the sharing and re-construction of this repertoire of resources that individuals come to define their identities in relationship to the community. Because communities of practice evolve over time they also have mechanisms for maintenance and inclusion of new members.

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 209-216. Prague: PME.

While communities of practice are generally constituted through face to face interaction, technologies such as the Internet have opened up additional possibilities for participation. However research in teacher education highlights some of the difficulties in building online communities. When participants share few common interests or have little commitment to each other or the discussion forum, interaction consists mainly of information or empathetic exchanges or dwindles over time (Selwyn, 2000). A clear task focus and a sense of obligation to the task have also been identified as critical factors in building a teacher professional community through online discussion. This can be difficult to achieve in professional development projects that also involve face to face interaction because teachers often prefer to collaborate in person rather than in a virtual environment (Stephens & Hartmann, 2004). However other research has found that initial face to face contact is important in building virtual communities, and that providing structured tasks involving mandatory contributions does not necessarily sustain participants' interest (Hough, Smithey, & Evertson, 2004).

The difficulties reported by these studies may be associated with two underlying issues: the tension between designed and emergent communities and the question of sustainability. Instead of designing an online community in advance, Barab (2001) maintains that it is preferable to facilitate the growth of a community by adopting an emergent design so that participants build the space. The sustainability of a community of practice is related to the designed/emergent duality in that an emergent community is more likely to meet the needs of its members because they have played a part in its development and thus identify with its goals and values. The present study investigated these issues via the following research questions:

1. How did an online community of practice focusing on learning to teach emerge amongst pre-service and beginning mathematics teachers?
2. What factors might have contributed to the emergence of such a community?
3. How was the community expanded, transformed, and maintained during the pre-service course and the transition to beginning teaching?

RESEARCH CONTEXT AND METHOD

Three successive cohorts of prospective secondary mathematics teachers participated in the study from 2002-2004. This paper draws on data from the 2003 cohort and their interactions with students in the 2004 cohort. Students were enrolled in a pre-service Bachelor of Education program available to undergraduates as a four year dual degree that combined the BEd with a non-education degree, or to graduates as a single degree taken over four semesters in eighteen months. Students completed a yearlong mathematics curriculum course as a single class group during the BEd Professional Year, which corresponded to the fourth year of the Dual Degree and the first two semesters of the Graduate Entry program. The course (taught by the authors) aims to create a learning environment consistent with socioculturally oriented research in mathematics education in emphasising mathematical thinking and collaborative inquiry. During the

Professional Year all students completed fourteen weeks of practice teaching in two blocks of seven weeks. The class met twice weekly during the remaining 17 weeks of the year. Dual Degree students graduated at the end of the Professional Year while Graduate Entry students completed additional courses, not related to mathematics education, over Summer Semester (November-January), followed by a final semester comprising a ten week internship in schools (February-April) and post-internship coursework (May-June).

We established a mathematics community website via Yahoo Groups with a bulletin board, email, file sharing, and links to other websites. The advantage of such a community over Web-based course tools typically used in university programs lies in its continued accessibility to members after graduation. Our goals in establishing the website were to encourage professional discussion outside class times and to provide continuing support for students after graduation as they made the transition to full-time teaching in schools. We hoped that this virtual community might remain a source of the beginning teachers' identities if they encountered images of mathematics teaching in schools that conflicted with the professional values of the pre-service course (Skott, 2002). In the light of previous research on designed versus emergent online communities, we decided to impose minimal structure on communication. We told students the bulletin board would be an important form of communication for the course, but they were free to use it for any other purposes they chose. Also, in contrast to other BEd courses, their bulletin board contributions were not assessed or graded. Thus interest centred on how and why students chose to use the bulletin board. These questions were investigated via a group interview with the class at the end of the course. We asked about their reasons for using the bulletin board, how often they read messages, and what were advantages and disadvantages of electronic communication versus face to face interaction in class.

Although they were under no obligation to do so, students continued to post messages to the course bulletin board (UQEdMaths) after the end of the mathematics curriculum course and also after graduation. In addition, in January 2004, they decided to establish a separate Yahoo Group (uqbedmaths04) for their exclusive use while still interacting with the new cohort via the course bulletin board. All email and bulletin board messages were automatically archived on the websites. Our analysis examines messages posted to both bulletin boards and thus spans the transition from pre-service to beginning teaching. For the UQEdMaths group this includes the entire duration of the BEd Graduate Entry program (February 2003-June 2004), and for the uqbedmaths04 group the first year of its existence (January-December 2004).

A frequency count of messages was conducted to determine the distribution of messages over time and who had posted them. Messages were then categorised in a two way analysis according to the phase of the BEd program during which they were posted and the purpose for sending the message. The following program phases were identified from the perspective of the 2003 Graduate Entry students, who comprised the majority of this cohort: *Professional Year Coursework*;

Practicum; *Summer Semester*; *Internship*; *Post-internship Coursework*; and *Post-graduation*. Five categories indicating purpose resulted from a content analysis of messages: *administrative, professional, advice, information,* and *social*. Evidence that a community of practice emerged was analysed in terms of the degree of mutual engagement between participants, the manner in which they negotiated the joint enterprise of learning to teach mathematics, and the shared repertoire of resources they developed for sustaining their community. This analysis also examined how the community was expanded, transformed, and maintained over time. The group interview identified factors that may have contributed to the emergence of the community. The findings are summarised below and illustrated with sample data.

MUTUAL ENGAGEMENT: EXPANDING THE COMMUNITY

The extent of mutual engagement can be gauged by analysing how many messages were sent, when, and by whom, and by identifying whether or not members responded to each other's messages. Table 1 shows the number of messages posted to both bulletin boards. From February 2003 to June 2004, 935 messages were posted to the course bulletin board, including 207 messages sent by the authors and 534 by students in the 2003 cohort. During 2004 a further 646 messages were posted to the students' independently established bulletin board: 80 by us and 566 by the 2003 cohort. The distribution of messages initiating a new topic (about one-third) compared with those that respond to an earlier message (about two-thirds) suggests that there was genuine interaction between participants. Contributions from individual students ranged from 1 to 139 messages on the course bulletin board (19 students) and from 1 to 136 messages on the student bulletin board (14 students). Although contributions were clearly unequal, in the group interview all students insisted they checked their email regularly and read all messages, even if they did not always respond.

Table 1 also shows that online engagement of the students increased throughout the BEd program and continued after graduation. In particular, the onset of Summer Semester and post-internship coursework triggered intense discussion amongst the students. That this discussion lasted well beyond the conclusion of the mathematics curriculum course and the BEd, on both bulletin boards, implies that students found value in maintaining the sense of community engendered by their engagement.

Mutual engagement was also observed in "generational encounters" (Wenger, 1998, p. 99) between the 2003 students and newcomers entering the mathematics curriculum course in 2004. Many of these encounters were prompted by newcomers seeking advice on teaching strategies:

> Hi all. For those of you who don't know me, I am one of the 2004 batch of maths students. I was wondering if anyone could help me. I am currently tutoring a year 11 student and last week we started to cover logarithms. He didn't get it. He couldn't understand them and I will admit I wasn't too flash at explaining them. Does anyone have any strategies for this particular abstract concept or know where I could look.

Participants	Program Phase					
	Prof. Year Coursework	Practicum	Summer Semester	Internship[a]	Post-internship Coursework[b]	Post-graduation
Lecturers						
course bbd	45	36	42	24	60	--
student bbd	--	--	--	9	41	30
2003 students						
course bbd	52	87	188	101	106	--
student bbd	--	--	--	80	228	258
2004 students						
course bbd	--	--	--	33	161	--
Total	97	123	230	247	596	288

[a]Corresponds to the first period of Professional Year coursework for the 2004 cohort.
[b]Corresponds to the first practicum for the 2004 cohort.

Table 1: Frequency count of messages posted to both bulletin boards

Members of the 2003 cohort, who at that time had started full-time teaching or were completing their internship in schools, responded with strategies that had worked for them, such as checking the boy's understanding of exponents, explaining why we use logarithms, and approaching the concept via graphing inverse functions. This kind of mutual engagement between cohorts expanded the community by integrating new members and sharing practices across generations.

JOINT ENTERPRISE: TRANSFORMING THE COMMUNITY

The way in which members of the community negotiated the joint enterprise of learning to teach was investigated by examining how the content of bulletin board messages changed over time. While students were attending mathematics curriculum classes on campus, messages were most often *administrative* and referred to scheduling and assessment issues. During the practicum and internship phases the discussion turned to *professional* issues as students constructed their new identities as mathematics teachers in a school setting. Increasing connections with the wider professional community became evident here; for example, one student recounted her experience of providing graphics calculator-related professional development to other teachers in her practicum school:

> My school uses the graphics calculator in senior maths but only for text book exercises which require them to do so ... In the junior school they are unheard of. When I came up with the idea of demonstrating box and whisker plots with my grade 10 class the teachers were intrigued if not a bit wary – so I met them half way and decided to just do a class activity with the viewscreen. After much searching, we found the devices to project the screen onto the wall (still in packaging, never seen the light of day). I experimented with it for a while and before long I was taking a workshop for maths/science teachers on how to use it! I'm almost feeling like a 'real' teacher!

When the Professional Year ended students did not meet with us again for classes as a mathematics curriculum group. Instead they were scattered amongst different tutorial groups during Summer Semester and after the internship while they completed intensive courses of study on the sociology of education. During this time they struggled to reconcile their developing identities as mathematics teachers with new identities as learners in an unfamiliar discipline. This is reflected in the high proportion of messages where *advice* was sought and offered: for example, students used both bulletin boards to share their summaries of sociology course readings and give each other feedback on assignment drafts.

After graduation, *social* messages (e.g., organising social gatherings) became important for maintaining community cohesion when members, as beginning teachers, were no longer in face to face contact. *Professional* exchanges during this phase included several heated debates about ethical questions concerning treatment of socially or educationally disadvantaged students, demonstrating that a joint enterprise does not necessarily imply agreement. Expectations regarding *advice* continued after graduation and show that members had developed what Wenger (1998) described as relations of mutual accountability. For example, one graduate posted the following message after receiving no replies to his earlier request for assistance:

> Apart from exchanging ideas and information regarding the teaching profession, I believe that we in this forum should assist one another with resources. I sent an e-mail previously regarding help but unfortunately no one has helped with any resources from your schools! Ladies and gentlemen get your act together and assist!

Messages such as this suggest the graduates were defining their own goals and values regarding accountability and collegiality in ways that might transform their identities from novice teacher to emerging professional.

SHARED REPERTOIRE: MAINTAINING THE COMMUNITY

In the course of its existence a community of practice develops a shared repertoire of resources by “producing or adopting tools, artefacts, representations; recording and recalling events … telling and retelling stories; creating and breaking routines” (Wenger, 1998, p. 95). The two Yahoo websites were themselves important tools in the repertoire of resources this community used to make sense of learning to teach mathematics. Two examples illustrate how the 2003 cohort employed both bulletin boards to maintain the community and its shared history. After the internship, Graduate Entry students used the course bulletin board to organise a half day debriefing session, and they invited newcomers in the class of 2004 to attend as well as 2003 Dual Degree graduates who had started teaching:

> Hi everyone! We will have two beginning teachers [*Dual Degree graduates*] and a dozen or so half cooked teachers [*Graduate Entry students who have just completed internship*] to share our experience and hopefully many interesting stories. For all the half-cooked teachers out there please come along, I am sure that your stories would be interesting as any others because we all have different schools, different classes, different in many ways. Class of 2004 please come along because some of your questions might help us to think about our teaching approaches again.

At the debriefing session members of the 2003 group identified challenges they had experienced and sources of assistance, shared strategies for building positive relationships with students, and related anecdotes about their best and worst lessons. In the following year, interns of the 2004 cohort who had participated in this session as newcomers organised a similar debriefing, which suggests that this practice may become a routine and part of the shared history of the community.

The second example is related to the function of social gatherings as an opportunity for recalling events and telling stories as a means of expressing community membership and negotiating professional identities. After graduating and finding employment as teachers, members of the 2003 cohort used their own bulletin board to maintain social relationships by organising regular outings and dinners. However, we have observed that these events also provided occasions for quite detailed analyses and comparisons of teaching experiences in different schools similar to those undertaken in the internship debriefing session.

FACTORS CONTRIBUTING TO EMERGENCE OF THE COMMUNITY

From our group interview with the students we can identify two significant factors that contributed to the emergence of the community. First, students appreciated that participation was voluntary and not assessable. This led to more open and extended discussion compared with other university course websites where their contributions were mandatory and graded for assessment. One student explained why:

> Mandating the use of discussion lists and then basing grades upon this has led to a false sense of collegiality. [The Yahoo mathematics website] on the other hand is totally voluntary. This shifts the focus away from simply meeting criteria to pass a subject, and towards developing a sense of community.

Second, students pointed out that having face to face as well as online interaction in a small class was crucial in creating familiarity and trust so that the bulletin board became “an outlet for discussion of ideas/problems, and a relief valve for stress”. This compared unfavourably with their experience of discussion forums in other BEd courses, where up to 200 students might be posting messages on a wide range of topics that rarely were related to mathematics teaching.

CONCLUSION

We attempted to manage the tension between design and emergence in establishing communities of practice (Barab, 2001) by creating a community framework in the form of the mathematics education course website and bulletin board, and allowing our pre-service students to build the space that would meet their needs. We regard their appropriation of the course bulletin board to their own purposes and their establishment and continuing use of an alternative Yahoo Group as convincing evidence of the sustainability of this community of practice. Our analysis indicated that members of the 2003 BEd cohort increasingly took the initiative in engaging with each other and *expanding* the community through generational encounters with newcomers, defining their own academic and professional goals and values in ways that *transformed* their identities as novice teachers, and constructing a repertoire of resources for *maintaining* their community beyond graduation. According to the students, emergence of the community was associated with the voluntary and non-assessable nature of participation, and the critical importance of initial face to face interaction in creating familiarity and trust. These findings resonate with results of other studies of online communities in teacher education, which concluded that common interests, commitment to each other, and interaction that begins with personal rather than virtual contact are important in building a community that endures over time (Hough et al, 2004; Selwyn, 2000). Yet many other questions remain regarding our own role in influencing the learning trajectories of these novice teachers, and the roles of other key members of the community – those pre-service and beginning teachers who were most active in posting messages. Such an investigation may yield new insights into pre-service communities of practice that span the transition to beginning teaching of secondary mathematics.

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DEVELOPMENT OF ABSTRACT MATHEMATICAL THINKING THROUGH ARTISTIC PATTERNS

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This study examines development of abstract mathematics thinking, specifically, students' understanding of elementary geometry and symmetry groups in the context of pattern design. The findings suggest that when working with hands-on repetitious geometric art designs students develop understanding of various complex geometric concepts. Further, familiarity with mathematical concepts allows students to use abstract mathematical thinking as a tool in their artistic creations. Some results of this study were presented at PME 26.

Since ancient times until only recently, fine arts and mathematics were assumed to be basic activities of every human being. They were often interwoven and inspiring each other - Islamic mosaics, the Greek canons of harmony and beauty, and the magnificence of many architectural designs. Hence, it is only natural to encompass fine arts, especially patterns, while teaching contemporary mathematical concepts. Mathematics educators suggest that the integration of arts with mathematics may be motivating to many students. This vision is supported by the National Council of Teachers of Mathematics (2000) who calls for the introduction of extended projects, group work, discussions, and integration of mathematics across the curriculum.

In recent years, several studies demonstrated the power of the vision that integrates art and elementary mathematics, see (Willett, 1992, Loeb, 1993, Shaffer, 1997). Our earlier work (Grzegorczyk, Stilianou, 2004) examines the strategies, and approaches college students use in the art-studio setting to develop an understanding of design classification and symmetry groups.

Here, we present further results on mathematics learning when combining mathematics and art activities in an art studio-like environment by self-explorations in art (Grzegorczyk, 2000). The course contained a wide variety of accessible yet challenging problems, and served as an introduction to systematic and complex mathematical thinking, experiences and discovery used in contemporary and classic arts. Fundamental concepts in this study were geometric properties of patterns, symmetry, underlying algebraic groups, and the implications that this may have on the overall mathematics learning and attitudes of students towards mathematics.

METHOD

Participants and course activities: The study was conducted over four years with five groups of twenty-five undergraduate students, and two groups of high-school students, typically enrolled in one semester course. Students were asked to investigate and create designs with given geometric properties, and extended projects with increasing sophistication and complexity. These projects were presented to their peers for discussion, questions and comments. The classroom was equipped with

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 217-224. Prague: PME.

commercial and shareware drawing and image-manipulation programs as well as geometry and symmetry software.

Mathematics of the courses: The underling course focused on the development of the solid concept of elementary geometry, pattern generation, and various types of two and three-dimensional symmetries. Students had basic algebra skills and a vague understanding of the concept of symmetry, often disconnected from mathematics. As their discussions on works of art and the complexity of their designs progressed, they were encouraged to use precise mathematical language, simple calculations, and mathematical arguments. Students studied geometric constructions, properties of polygons (angles, rotations, reflections), and patterns and their symmetries as rigid motions of the plane. Strip patterns with seven underlying symmetry groups and wallpaper patterns, with all 17 plane symmetry groups, were analyzed or generated using various symmetries (and suitable software). The sophistication of the students and complexity of the patterns studied increased. The notion of the fundamental region for a pattern led to the study of tilings, including Escher's tessellations. Finally, the study of tilings and symmetric groups was extended to three-dimensional solids.

Data collection and analysis: For this study we collected students' sketches and designs for review and analysis. Further, students were administered pre- and post-tests that focused explicitly on their understanding of various concepts and were asked to complete a survey which focused on the students' attitudes towards mathematics.

Responses to each problem were first coded for mathematical correctness. When responses included images, then these were coded for the presence of mathematical arguments and their correctness. When responses included verbal descriptions, these, too, were coded for the extent to which mathematical ideas and concepts were used, and subsequently, for the correct use of mathematics. In this case, we differentiated between explicit, formal use of mathematics arguments and language, and general or vague references to mathematics. We also differentiated between different strategies that students employed when they approached tasks or designs that involved the use of symmetry in various forms. Finally, when coding was complete, frequencies were tallied for each code and pre- and post-interview totals were compared.

OUR RESULTS

Elementary geometry – angles and tilings. A group of 50 students were included in this part of study focused on the elementary geometry content learned by the students and subsequent attempts (or lack thereof) to utilize the knowledge to create simple designs. To code students' understanding, we used the criteria suggested by Gardner, i.e. the "ability to use ideas in appropriate contexts, to apply ideas to new situations, to explain ideas, and to extend ideas by finding new examples" (Gardner, 1993; Shaffer, 1997). Using this definition, our results indicate that all students were able to make a geometric argument in simple tilings designs. In interviews, students reported that they started to use mathematical arguments while planning their designs, both in and out of the course. By contrast, early in the of the course (pre-test), when asked to make a design which incorporates mathematical ideas and explain how these ideas are incorporated, only 5 students presented a valid mathematical argument, including 2 that used symmetry. The remaining 45 students made references to mostly measurement concepts (dividing areas in parts, measuring) and, in a few cases, attempted to construct polygons. All post-tests designs included angle calculations, symmetries and a valid (or partially valid) mathematical justification.

> Example 1: Use a compass to make a design with circles that incorporates some mathematical ideas and explain how these ideas are incorporated.

The typical picture solution on the pre-test was a random association of circles, with comments about the length of radius, or circles being round, etc. 9 designs *included co-centric circles*. All post-test designs included rosettes, 32 inscribed circles, and 10 were sophisticated. The mathematical comments included; angles calculations (41), arguments based on properties of regular polygons (23), design extension arguments (18), descriptions of fundamental regions (27), symmetries discussions (48).

> Example 2. Draw a tiling given by a *non-convex quadrilateral*.

On pre-test 0 students were able to make a design, while 42 students produced a correct picture on the post-test.

> Example 3. *Decide if it is possible to use identical regular pentagonal tiles to cover a plane. Use mathematical arguments to justify your answer.*

29 students on the pre-test said yes (and all tried to sketch the picture), 21 said no. Nobody gave a proper mathematical argument. However on the post-test 49 students gave a correct response – see example below (one produced tiling with pentagons that were not regular).

Example 4. Four weeks into the course students were assigned a free form art project, that had to include the following elements: *at least two different rosettes (as windows, flowers, whatever you like,) at least two perfect regular pentagons, an octagon and a hexagon (as windows, as gravestones, as eyeglasses, etc.), several reflections of the same object, an object scaled several times (you may draw a self-similar object), at least three different types of strip patterns with the same motif (belts, headbands, frames, decorations, etc.), at least four different types of plane patterns with the same motif (on any flat part of your artistic composition, for example on the sky, on a dress, on a wall, on floor, etc.)*

All students enjoyed the project and were able to complete it satisfactory. Many produced very sophisticated or artistic designs**.**

Recognizing and using symmetry. Our previous work was focused on the overall recognition of the existence of symmetry in designs and students' subsequent attempts (or lack thereof) to utilize symmetry principles, and we included 50 new students in the similar study. Our post-test results indicated that all students were able to make designs using mirror symmetries and rotational symmetries. In fact, all of them reported that they now regularly study and incorporate symmetry in various designs. By contrast, early in the course (pre-test), when asked to make a design, which incorporates, 90-degree rotation and no mirror symmetries only 12 students made a correct picture (other designs had at least one reflection). The remaining designs had at least one reflection. 49 post-test designs were correct. Students also developed an ability to apply the concept of symmetry to their analysis and discussions of various designs and art works.

Symmetry use	Pre-test	Post-test
Use of symmetry in doing art		
- mirror reflections	2	50
- rotations	--	50
-glides	--	43
Use of symmetry in analysing art		
- reference to mirror symmetry	15	99
- reference to rotation symmetry	--	60
- reference to glides	--	28
- reference to formal symmetry concep	--	34
- reference to lack of symmetry	--	63

Table 1: Frequency in symmetry use

While analysing highly symmetric images on the pre-test, students made references to symmetry an average of 30% of the time (total of 15 references made by 50 students). Note that all initial references to symmetry were made with respect to mirror symmetry only; no mention was made of translations, rotations or glides. However, all students mentioned symmetry on their post-test analysis, and the mean rose to 4.4 references over the same images (220 references by 50 students). These included references to mirror reflections (99 references) and rotational symmetries (60 references), glides reflections (28) as well as to the lack of symmetries (33). But perhaps, more striking is the qualitative change in students' responses to the designs they were discussing. Students used a richer, more formal, and mathematical vocabulary to describe images. In fact, 34 students made direct references to symmetry groups (i.e., dihedral, cyclic or wall paper groups).

Comments on formal understanding of symmetry. All students developed their understanding of symmetry groups, in the sense of recognition of symmetry, in designs, and in their ability to use symmetry to generate their own designs. Our coding suggested that students approached symmetry in two distinct ways: A number of students approached the mathematics of symmetry using art. Others, however, gave clear indications that, after a certain point, they used abstract mathematical objects as a gateway to art. This math-through art approach was manifested in two ways: recalling design examples when asked to discuss the mathematics (e.g., "Cyclic group C_4 is like that four-leaf rosette") or a global correspondence among types of objects in art and in mathematics (e.g., "cyclic groups are like rosettes").

The latter approach, however, that uses abstract mathematical concepts to understand art, deserves further attention. In this case, students used their understanding of a cyclic group to convince themselves that once a design is identified as equivalent to a cyclic group, there is no need to look any further for mirror symmetries – they do not exist. Students were able to use *abstract* mathematical thinking in generating their own designs and to make *short cuts* in designing repetitious patterns using certain group properties.

Combining complex ideas in 3 dimensions. We have included 150 students in part of the study. At the end of the course, after several projects that included a creation of patterns with various symmetries, tiling analysis, Escher designs (see the typical example below of tessellation designed by a student), polyhedral, their symmetries and properties, participants were assigned a complex project that was combining all ideas together. The main difficulty was coming from the fact that plane tessellations may not be easily transferable to solids. We tried to study different approaches and levels of understanding to concepts that helped students overcome the difficulty. To define the base line we administered two pre-tests, one at the beginning of the course, the second right before the project was assigned.

Pre-test 1 results (at the beginning of the course): Cube was the only polyhedral solid students could name and recognize. 5 students could do tessellation by translation. No student could explain construction of a complicated Escher design (Horsemen).

Pre-test 2 results (before the final project was assigned): All students were able to recognize and name platonic solids. All students were able to recognize Archimedean solids. All students were able to count vertices, edges and faces of a simple polyhedron (and use the Euler formula). All students were able to design a simple tessellation (using translations) and 137 students were able to design a complex tessellation (using rotations, glide translation, with no reflections – the Escher type). All students were able to explain symmetries of a complicated Escher design (Horsemen). The project required understanding of the content presented at the course, and the designing experience. Students had several days to complete the task.

> **Post-test project:** Platonic or Archimedean polyhedron has all faces that are REGULAR n-gones and exactly the same number of faces meet at each vertex.
>
> The goal of your project is to make a three-dimensional model for the solids assigned to you and decorate it with Escher-type pattern (tessellation). **We DO NOT allow the design to have MIRROR symmetries, but rotations are fine**. NOTE: The easiest

designs of CCC - Heesh type, work on every solid, ***BUT try to be more original if you can!***

1. Check if Euler's formula works for your solid. Show the calculations - number of faces, edges and vertices (corners).

2. Sketch the planar diagram of your solid on paper and show how the gluing of the faces will go. Decide on the motif of your Escher-type design. Try to come up with **a sophisticated tile type**.

3. Show the construction of a tile for your design. On your diagram show how the tiles will fit together . **Make sure that the design will fit the solid after it is put together**!

4. Make a model of your solid and decorate it with your pattern. Color it accordingly.

Results and conclusion. All students were able to build the assigned polyhedron and completed tasks 1,2,3, 4. However only 78 students (52%) come up a sophisticated original design with no errors, 48 (32%) produced an easy (CCC-type) design, 18 designs (12%) had errors (gluing did not work!), and 4 designs had many mirror symmetries. Even at the end of the course, the task turned out to be very hard, and the majority of students had problems with three-dimensional analysis of the project. While we were very satisfied with the results of the Pre-test 2, transferring flat tessellations onto solids was a new task, which required in depth understanding of the concepts of symmetries and geometry of polyhedrons and only 84% of the students had satisfactory solutions to the assignment, while only slightly above half of them achieved the suggested depth. The 16% of the incorrect designs were due mostly to the lack of understanding of solid geometry as all students were able to design plane tessellations on the Pre-test 2, and all used the descriptive Euler's formula correctly. The interviews reviled that 90 % of the students were overwhelmed by the complexity of the task, and 10% did not understand tessellation the gluing issues. 40% blame difficulty of the task on lack of three-dimensional experiences.

DISCUSSION

Our study shows that an art-studio environment and art-based instructions support development of abstract mathematics thinking. The findings suggest that when working with hands-on repetitious geometric art designs, almost all students develop understanding of various complex geometric concepts. Furthermore, familiarity with mathematical concepts allows students to use abstract mathematical thinking as a tool in their artistic creations. Willett (1992), Loeb (1993), Gura (1996) and Shaffer (1997) among others argued that an art studio can facilitate the learning of mathematics, and the mathematics of symmetry can be a meaningful organizing principle when teaching a course in this setting. Indeed, we found that our students developed their ability to detect and apply various symmetries in art designs, and to classify various patterns using familiar properties (symmetry groups). Furthermore, students regularly incorporated mathematical ideas in their own designs. This included thoughtful use of specific types of symmetries in order to achieve certain visual effects (e.g., students often explicitly mentioned that they *chose* to use a cyclic or rotational design to avoid having mirror images). Students' overall behaviour suggests that they learned to appreciate an abstract approach to patterns, and had begun to appreciate abstraction in mathematics in general.

Overall, students who participated in our study learned about the mathematical ideas of patterns and discovered a new meaning for mathematics. Shaffer (1997) in a study involving middle school students in a similar environment suggested two venues to explore as factors in students' learning: the issues of control, that is the freedom to make decisions in one's own learning (Dewey, 1938), and expression (Parker, 1984). Shaffer suggested that expressive art-based activities put students in control of their own learning. Observational evidence suggested that in our course, the freedom for students to choose the means in which to apply the concepts they find most useful in the context of art with which they feel familiar, also facilitated the learning of mathematics. Students often talked about using ideas of symmetry out of class in their own project, and the transfer of ideas of symmetry to other projects. Our results suggest a framework for thinking about the teaching of mathematics in the context of other courses. We believe patterns and their properties to be a powerful vehicle to build mathematical bridges for students that are often difficult to reach.

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RATIONAL NUMBERS AFTER ELEMENTARY SCHOOL: REALIZING MODELS FOR FRACTIONS ON THE REAL LINE

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After elementary school, students join their new learning groups with very different knowledge and approaches to fractions. Starting from the observation that this ampleness has a strong irritating potential for students, the approach is studied to model different aspects of fractions and entire numbers on the geometry of the real line and to concentrate on this as a didactic model. The changing role of this model in the learning processes of students with different levels of achievement and with different previous experiences with fractions is examined.

INTRODUCTION

This study concerns the period in mathematics classes when previous experiences and knowledge in the childhood concerning fractions have to be integrated in a systematic course. The grade in which rational numbers are systematically introduced differs from country to country. In view of studies with children (see e. g. Steencken & Maher (2003)), the reasons for these differences appear to be of traditional and organisational character. We study German fifth-graders who just have left elementary school and come into a new class-setting having various backgrounds, especially on rational numbers. It goes without saying that there is no new start on fractions for the entire class. The impact of previous learning history appears particularly complicated in the case of fractions in view of the various relations to other mathematical areas, see for instance Charles & Nason (2000).

Many school books introduce a variety of models for different purposes in the area of fractions: area models (e.g. fraction circles, paper folding, geoboards), discrete models (e.g. counters, sets), operator models, and linear models such as fraction strips and number lines, too (Millsaps & Reed (1998)). Different operations are modeled on different tools: fraction circles for the basic notion of a fraction, operator models for multiplication and division just to name a few. The literature in mathematics education concerning fractions is enormous; and many studies underline how difficult it is for students to build up internal representations of fractions and operations with them. In addition, research has provided a wealth of information on student difficulties (Behr, Harel, Post & Lesh 1992) and has given advice on ways fractions may be approached in the classroom (Davis 2003). Several studies show the difficulties students have with the various operations with fractions (Padberg 2002[3]). Besides, typical problems and mistakes of students concerning fractions are well known and also sometimes related to the model in use.

The role of imagination of fractions has received considerable attention in the literature. It is the aim of the approach investigated here to place the geometry of the real line in the center of the learning of fractions. The basic operations with rational numbers are modeled on the real line in the course over several months. It is investigated in which situations the students use the real line model and to what extent this inner-mathematical modeling enables them to solve problems of different, e. g. applied character.

DIDACTIC MODELS

Didactic models are designed as a means for the learning of a new mathematical concept. A didactic model consists of objects familiar to the learners and of well-defined operations on them. This means that the operations are in one-to-one correspondence to the formal mathematical structures. An ideal model makes it easy to introduce the mathematical syntax describing these operations.

Nesher (1989) regards didactic models especially as a means of understanding mathematical language and its properties. Cuisinair rods are a prominent example of such a model. On the other hand, the Realistic Mathematical Education approach takes every-day-contexts as the starting point of learning models (Gravemeijer & Doormann, 1999). Some recent research activities mix elements of Nesher’s and of the Realistic Mathematics Education framework, see Shternberg & Yerushalmy (2004).

Since the notions of didactic models differ considerably, we formulate the following features which we assume a didactic model to have:

- Concepts for the mind: Learning models enable students to build up internal representations of the mathematical objects in question, especially during the introduction period.
- Simplicity: Learning models are as close as possible to mathematical objects. Add-ons with an irritating potential are omitted.
- Basic Tool: Learning models are designed to help to learn mathematical contents. Their role changes from a tool for concrete actions to a basic structure which is rather present in the background.
- Flexibility: Models should leave the space for learners’ own creativity, for applications and learning material in different modes of representation.

THE LADDER MODEL FOR THE REAL LINE

For the introduction of entire numbers and of fractions, a ladder model is used. The ladder symbolizes the real line. There are rungs on each ladder, which mark the units of natural numbers. Right after its introduction, it was mentioned by some students in all of the classes that the ladder does not have an end on either side even though one only draws a part of it. Further rungs can be introduced; for example, if a dwarf wants to climb up the ladder and needs two rungs if a person of normal size needs one. The ladder obtained by doubling the rungs is called half-ladder. Another dwarf, who uses a different ladder, needs three rungs instead of one; this delivers the “one third-ladder”. The ladders thus obtained are sketched next to each other.

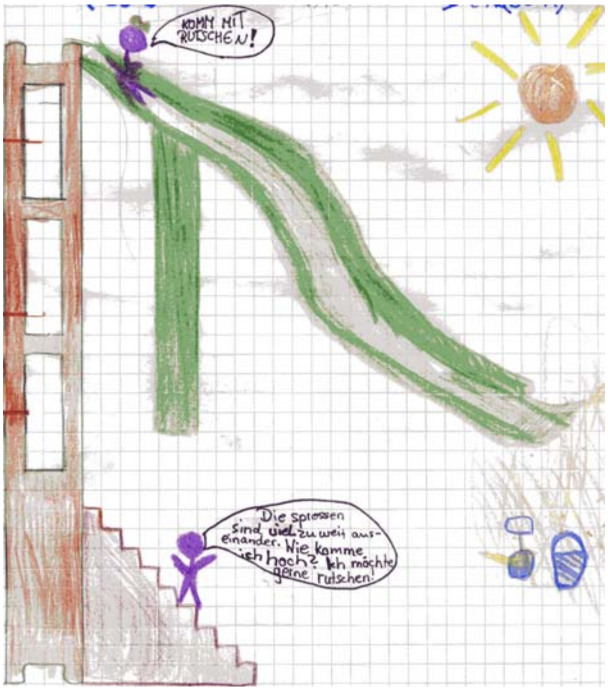

Figure 1: Story on the ladder by a student (10 years). The child on the slide says: “Come on, have a slide! The reply: “The sponges are too far apart. How can I get to the top? I’d like to have a slide!”

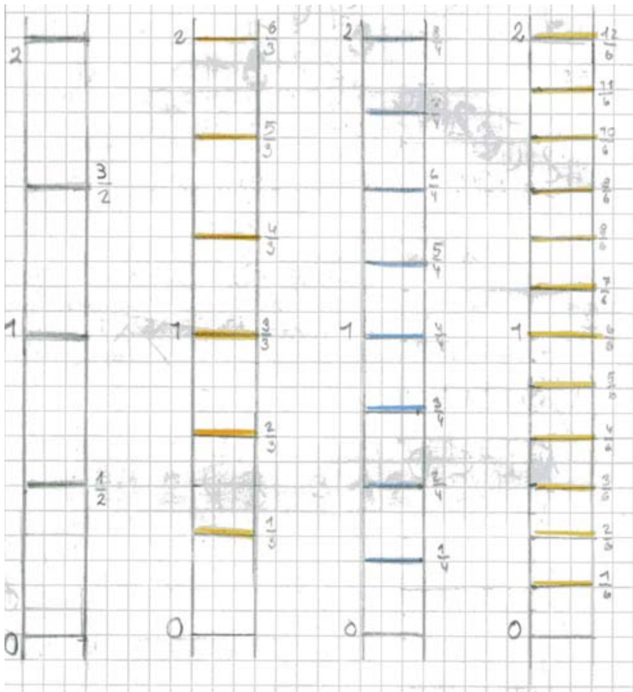

Figure 2 by a student (10 years): The one-half-ladder, the one-third-ladder, the one-fourth-ladder and the one-sixth-ladder next to each other

The ladder model can be used throughout the calculation with rational numbers:

- Positive and negative integers are sketched in different scales for high and low numbers
- Rational numbers are modeled on the ladder by partitions
- A common denominator can be found by comparing ladders with different distances between the bars.
- Multiplication of fractions can be understood by decomposing rectangles into squares of unit one and comparing their area to a normed ladder where the rungs between two neighbouring units form a square of unit one.

The courses involved in this project were based on the ladder model with the following principles:

- The ladder model is the exclusive didactic model suggested to the students as the concept for the learning of fractions. It is applied to different kinds of mathematics related to fractions, but other didactic models for fractions are not introduced as such in the mathematics courses.

- The classroom work is organized around operations of rational numbers. The main activities can be described as inner-mathematical modeling for which the ladder model is suggested to the students.
- Over a period of approximately five months, fractions and the basic operations are treated intensively. Since there is no school book of this approach, the creation of every student's own "book" is the main project for the rest of the school year. The command of rational numbers is practised continuously over two years in various contexts.
- Every student produces a card with the ladders for his or her use and is free to employ them whenever they like or to use other didactic models which they picked up somewhere.

METHODOLOGY

The approach taken here has been carried out in grade five, the first year after elementary school, and has run over the course of several years. In the last turn, it has been examined in four classes of two different grammar schools (Gymnasien) in Germany by four different teachers. As a peculiarity for the German system it should be mentioned that the – supposedly – best 30 to 40 % of German elementary school students go to grammar schools. Tests with the students involved indicate that the achievements of the students of the weaker class are at the lower end of this range - if not worse. We have no empirical research on fifth-graders who were low achievers in elementary school because curricular restrictions for other school types than grammar schools would make it difficult to carry out this project in grade 5.

The use of the model described above was the basis for the lessons in mathematics in these four classes. This study focusses on 57 students, 26 in the best and 31 in the weakest of these 4 classes. The past learning history of these students is quite diverse: they went to 37 different classes in 29 elementary schools.

An established test for previous knowledge on fractions (Padberg (2002[3])) was applied to understand what the students knew about fractions after elementary school. The test was conducted at the beginning of the mathematics course at the new school of the fifth graders, who just had left elementary school.

The study sketches a period of five months, in which rational numbers were introduced. It focusses on the following aspects of the ladder model as a didactic model:

- To what extent do the students build up internal representations of fractions and operations with them?
- How do the students with their different backgrounds use the ladder model and how does this change during the course?

- How do problem solving abilities concerning applications of fractions develop?

To investigate this, productions of all 57 students on problem solving and on applications concerning rational numbers were continuously observed and evaluated. The second author and pre-service teachers, who, both, were not involved in the planning and in the teaching of the courses, interviewed 12 students with very different previous knowledge on fractions every four weeks to observe the changing role of the ladder tool in the learning processes.

EMPIRICAL RESULTS

The test conducted in the first week is a standard test on the previous knowledge the students acquainted in elementary school (Padberg, 2003). It concerns the basic notions and operations of fractions. The two classes described here turned out to be very different with respect to their previous knowledge on fractions.

	Less than one third of the answers correct	Between one third and two thirds correct	More than two thirds of the answers correct
Class 1	7	12	7
Class 2	22	7	2

Other tests show that the distributions of high achievers and low achievers in the two classes are similar to the results in this test. However, this is no one-to-one correspondence: for instance, there are some high achievers with poor knowledge on fractions at the beginning of the course. There were 3 students (all in class 1) who came to the new school already with a good understanding of fractions and their operations.

In the first 6 weeks, entire numbers were introduced with the ladder tool. Since we focus here on fractions, we look at standardized interviews which were carried out by the second author and pre-service teachers.

We present the results of interviews with the twelve students in three categories. The first denotes basic calculations for which more complex calculations, such as expanding and reducing operations, are not necessary. The latter are listed in category 2. Applications of rational numbers are shown in the third column. The third category of applications involves applications, in particular real-life problems and measurements of time and length. We show how many students are able to answer questions in these categories without problems ("correct") and how many students tend to use the ladder tools of the course.

	Basic, uncomplicated calculations		More complex calculations		Applications	
	Correct (of 12)	tools used	Correct (of 12)	tools used	Correct (of 12)	tools used
Week 10	12	2	6	5	3	8
Week 14	12	1	9	2	7	4
Week 19	12	0	11	1	10	3

Table 1: Development in the series of interviews

In standard procedures the use of tools declines over the weeks. In the category of applications, some students used different tools than the ladder tool. Fraction circles were used. Two (of the 12 interviewed) students and about 10 of the 57 used these models at times also in calculations. They mentioned that they had seen fraction circles before the course, when brothers and sisters or parents explained them fractions.

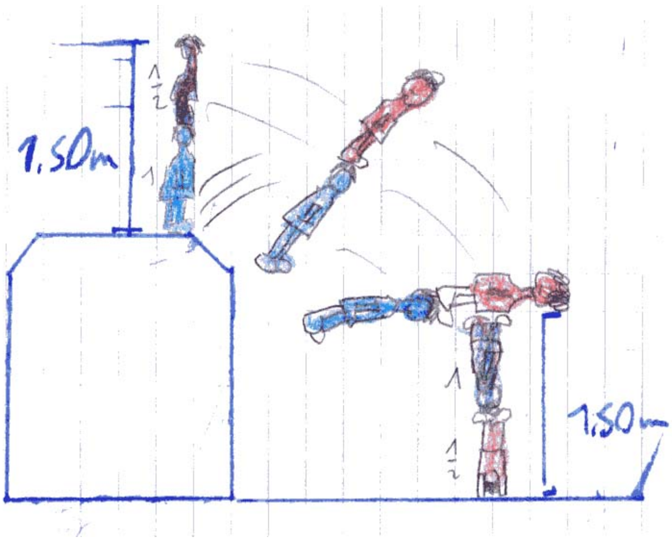

Figure 3: A Student (11 years) visualizes the additive law of commutativity

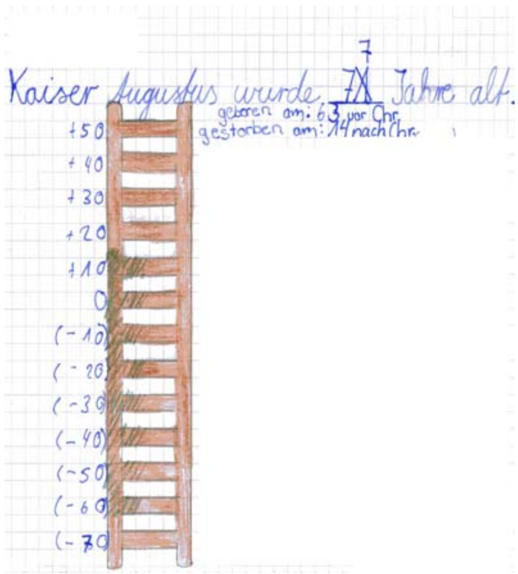

Figure 4: A student (10 years) in a history project on entire numbers: “Emperor Augustus became 77 years old.” (She calculates 14 - (-(63)) = 77 correctly, but does not consider the exact dates of birth and death.)

It should be added that many students keep using the ladder tool to approach new topics or problems. One example for this is the task to explain why the additive law of commutativity holds (see a student production in figure 3). Explorations of certain areas of mathematics were particularly successful if geometry played a role.

Written work was investigated during the whole period of the course. The self-confidence of the students in dealing with the ladder model is striking in the written material. For the class tests, the students were asked to submit a problem they regard as easy and a problem they consider more difficult and to give reasons for their rating of difficulty. Problems which were formulated by the students within the ladder setting were rated as easy. One student wrote (translation from German to English)

> "This is easy. We have done so much with ladders that I am very sure I can do it."

Most of the students built up an internal representation of the ladder tool which allowed them to perform calculations within this setting.

DISCUSSION

Table 1 is not meant to evaluate the approach investigated here. Its success depends on the long term effects of it. We also remind the reader that (very) low achievers are not likely to be in these classes due to the German school system.

The results seem to have important applications for our understanding of the students' abstraction from fraction models.

The role of the ladder tool changes over the months. At the beginning, the ladders are the basic working material for first experiences. The process of detaching from the ladders as tools for operations starts already in the first weeks. The students do not stick to the model when they feel sure about their internal representation of a fraction or of procedures involving fractions.

Once operations are linked to some internal representation, the students do not need ladders any more for those operations. But they keep referring to them if new operations are introduced or if they do not feel sure about something. The process of transforming experiences with fractions to knowledge about fractions is accompanied by a change of the role of the ladder tool. It is never absent, it becomes the back of the stage.

Interestingly, the students obtained good results in tests on the application of fractions. This might be surprising at first because the model itself is very mathematical compared to the Realistic Mathematical Education approach. Although the point of view of applications was not stressed directly in the course, the students coped well with problem questions on fractions in other contexts. This might be an indication that focussing on one model is a continuous training of basic modeling abilities. For, every operation with fractions, which might be seen somewhat more immediately in models adapted to it, has to be modeled on the ladders.

There are indications that the ladder approach can unify different previous knowledge to a common setting, especially in communicational terms. It does not mean at all that the experiences with fractions in elementary school were useless. Since most of the operations with fractions can be modeled on the ladders, the experiences seem to integrate in this approach. The fact that some students use occasionally other models is an indication that the first model one uses shapes the internal representations quite strongly. Students whose experiences with fractions were not linked to a special model seemed to have an easier start in the course.

It seems worthwhile examining the long term role of the ladder model and to study in more generality which selection of didactic models allows for a flexible use in the future school career.

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STUDENT BELIEFS ABOUT MATHEMATICS ENCODED IN PICTURES AND WORDS

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This paper presents a design for investigating mathematical beliefs. Students were asked to express their views on mathematics on a sheet of paper. Further data was collected and qualitative methods were employed to identify the beliefs encoded in these works. The data was analyzed according to established categories describing mathematical beliefs. Typical features of each category were found in the pictures. Concrete examples that support these features are provided as evidences for the represented mathematical beliefs.

INTRODUCTION

The importance of beliefs in mathematics learning is nowadays widely acknowledged (Leder, Pehkonen & Törner, 2002; Schoenfeld, 1998). As many researchers point out, the learning and success in mathematics is influenced by student beliefs about mathematics and about themselves as mathematics learners (Hannula et al, 2004; Leder & Forgasz, 2002; Schoenfeld, 1992).

Traditionally, mathematical beliefs are investigated with the aid of questionnaires or interviews. This approach is well established. However, especially for younger students, who are not yet used to this technique and who might have difficulty in reading a long questionnaire attentively, alternatives could be helpful. Bulmer & Rolka (2005) introduced pictures as a means to understand university students' views on statistics. In our study, we used a combination of pictures as well as written and oral statements for investigating student beliefs. In this paper, we focus on mathematical beliefs of fifth-graders.

MATHEMATICAL BELIEFS

Dionne (1984) suggests that mathematical beliefs are composed of three basic components called the traditional perspective, the formalist perspective and the constructivist perspective. Similarly, Ernest (1989; 1991) describes three views on mathematics called instrumentalist, Platonist, and problem-solving which correspond more or less with the notions of Dionne.

In this work, we employ the notions of Ernest (1989; 1991) and use this section to briefly recall what is understood by them. In the instrumentalist view, mathematics is seen as a useful but unrelated collection of facts, rules, formulae, skills and procedures. In the Platonist view, mathematics is characterized as a static but unified body of knowledge where interconnecting structures and truths play an important role. In the problem-solving view, mathematics is considered as a dynamic and continually expanding field in which creative and constructive processes are of central relevance.

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 233-240. Prague: PME.

METHODOLOGY

Inquiry methods and analysis

In this study, we extended the approach of Bulmer and Rolka (2005) using pictures as a means for investigating student beliefs. Additionally to the pictures, we asked the students to give an explanation of their work. The first task, scheduled for one week, was close to that in the study mentioned above:

Imagine you are an artist or a writer and you are asked to show on this sheet of paper what mathematics is for you.

This text was written on top of the A4 sheet of paper followed by a framed box for the picture. This arrangement seemed appropriate for including young students in our study.

After the submission of their work, a second task was given over a period of five days:

Explain your work by answering the following questions:

- *In which way is mathematics included in your work?*
- *Why did you choose this style for your presentation?*
- *Is there anything you would have liked to show but which you were not able to express?*

During these five days, the authors independently tried to classify the pictures according to the three mentioned views: instrumentalist, Platonist, and problem-solving. This classification was repeated later, when both pictures and texts were available to the authors.

In certain cases, a third step was carried out by the first author, who interviewed the students individually. This was considered necessary when the answers based on picture and text remained unclear.

Sample

The tasks were given to 84 students of grades 5, 9, and 11 from two schools in Germany. Among these, 61 students submitted pictures. In this paper, we focus on the sample of 28 fifth-graders, 15 girls and 13 boys. These students appear particularly interesting because they have very individual learning histories. This is due to the fact that German children leave elementary school after grade 4. In this particular case, the 28 students originate from 21 different elementary schools. The observations about the impact of learning histories and the differences between the grades 5, 9, and 11 will be published separately (Halverscheid & Rolka, in preparation).

The picturing task was integrated in the coursework of the classes. The students involved were used to work on projects over a longer period of time. The texts were meant to serve as guidelines for the students who presented their products in front of their peers.

EMPIRICAL RESULTS

In the three steps of interpretation, Ernest's categories (Ernest, 1989; 1991) proved to be suitable for the classification of the fifth-graders' views on mathematics. Among the 28 fifth-graders, 26 handed in their works; 2 students did not submit anything. From the 26 works of the fifth-graders, 24 pictures and texts could be classified according to Ernest's categories. The remaining 2 students were not very clear in their texts and, somewhat discouraged by the efforts of their peers, did not want to speak about their works either.

The method of considering pictures, texts, and – in certain cases – answers to interview questions was necessary to make a classification of the works possible. In the following table, the importance of including these three steps in the process of classifying the students' mathematical views is illustrated.

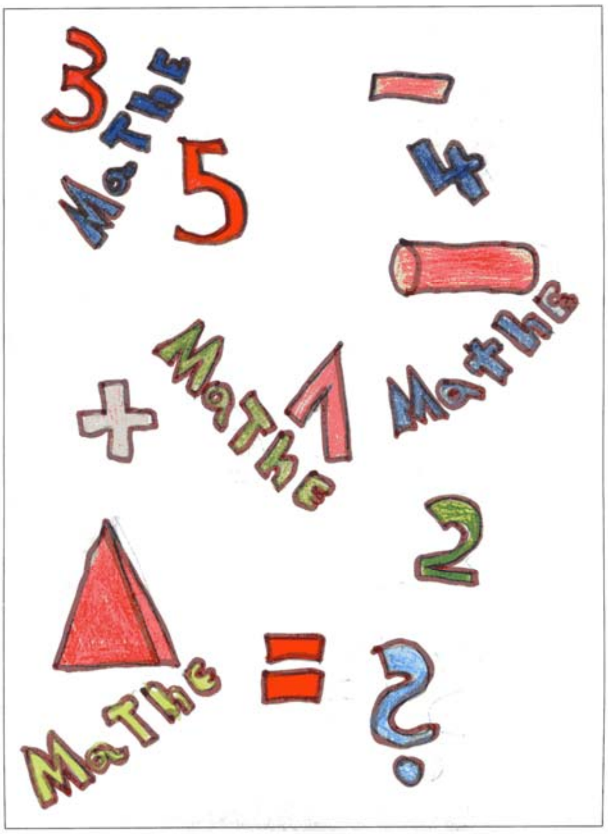
Means of interpretation	Instrumentalist view	Platonist view	Problem-solving view	Hard to classify
Pictures alone	11	0	0	15
Pictures and texts	13	1	5	7
Pictures, texts, and interviews	14	5	5	2

The following three examples serve as illustrations of the classifications according to these categories. We are interested in the students' views on mathematics and the various ways they are expressed. Therefore, the process of understanding the students' pictures, texts, and – in certain cases – comments in interviews is sketched.

Instrumentalist view

The majority of fifth-graders (14 out of 26) have an instrumentalist view on mathematics. Some students (5) stress the aspect of usefulness of mathematics in a job or in everyday life. The disconnectedness of mathematical objects is found in the pictures and is additionally supported by the texts. This proved to be the main criterion to classify these products as instrumentalist.

The picture (Figure 1) is colorful and contains different objects which the student regards as mathematical: symbols, numbers, and geometric objects. The word "Mathe", the equivalent of "math" in the German language, is written four times in different directions and at different places on the page. Next to an equality sign, there is a question mark in the lower right corner of the picture.



The girl, 10, writes: “In my picture, mathematics is found in forms, numbers and signs. I opted for this presentation because for me math has something to do with forms and numbers. Forms have something to do with geometry; that is why I associate them with math. Numbers and signs appear in my picture because math has always to do with them. […] What I would have liked to sketch: a grocery store with a cashier who cheats a client because he is not able to calculate. This should demonstrate how important it is to be able to calculate.”

Figure 1

The striking feature of the picture, also supported by the text, is the disconnectedness of the objects. It appears that no story is told, and no mathematical statement is made. The authors did not find a hint on a dynamical view of mathematics, which would, for instance, order the different symbols to a mathematical statement.

Platonist view

The Platonist view is taken by 5 fifth-graders. Most of them directly refer to historic persons, especially to Albert Einstein, although he has never been mentioned in the mathematics lessons. Discoveries of mathematics are linked in the texts with these people. It is not a mathematical process itself which is in the main focus of these works. Creativity appears as a feature of mathematics, but it is modelled on the celebrities. One picture, for instance, shows Einstein as a wizard working with numbers. The images and descriptions indicate that students with a Platonist view do not negate the importance of creativity in mathematics, but they tend to accredit authorities with it instead of engaging themselves in a creative process.

The picture’s style (Figure 2) appears simple, somewhat darkish. Only a pencil is used. One sees the 10 digits, two faces named “Archimedes” and “Al-Khwarizmi” and the formula for Pythagoras’ theorem. There is no obvious connection between the elements of the picture. No story is told and no mathematical content is constructed either. This is why the authors tended to classify this view as instrumentalist in the first step of the analysis.

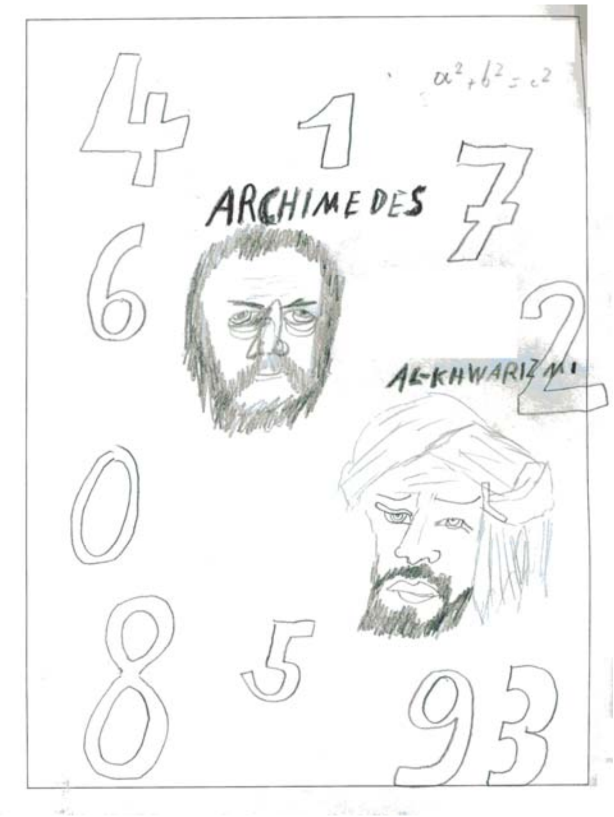


Figure 2

The boy, 11, writes: "The numbers are, of course, the most important thing because one can calculate everything with them. I added those from 0 to 9 because there is no point in writing all down. […] I included some mathematicians who deserve, in my opinion, appearing in my work. I added some geometry, namely Pythagoras' theorem: $a^2 + b^2 = c^2$. […] I chose this simple style because it is quick [to draw]. This makes more sense than a very detailed picture."

The text does not tell a story either; mathematical activities are not shown. There are indications, however, that the elements are not accumulated in a disconnected form. The student mentions that the digits from 0 to 9 represent for him numbers in general because "there is no point in writing all down". The sketched mathematicians were chosen because "they deserve appearing in [his] work". It seems that this choice of mathematicians represents a certain view on mathematics. Furthermore, it is remarkable that an 11-year-old boy refers to the formula of Pythagoras' theorem as geometry.

To be more certain about how to classify this, the student was interviewed. It became soon clear that he is very interested in science in general and has a remarkable knowledge of the history of science. He thinks that Archimedes deserves appearing in his work because many of his inventions have mathematical applications. He is fascinated by algebra and, therefore, added Al-Khwarizmi as a precursor. He wrote down Pythagoras' theorem "because it is fascinating that one can make geometry with algebra". The student does not produce mathematics in his work. In contrast to the student who designed figure 1 he looks for historical truths and links between them. This is why we classified the picture later as an example of the Platonist view.

Problem-solving view

A problem-solving view could be recognized in 5 cases. Two different approaches were taken: Three girls among the fifth-graders present numbers in an animist way. They act as individuals and produce different kinds of mathematics. Two boys, including the example illustrated below (Figure 3), chose to sketch objects which are used for mathematical activities as a sort of tools for a mathematical process. The latter case is found frequently in works by older students with a problem-solving view, whereas animist elements seem rather typical for younger students.

The picture (Figure 3) is colorful and shows pieces of fruit. Some pieces are cut in different proportions. The fact that there is a number of objects and that there are different sizes come to mind as relations to mathematics. But the exact nature of the mathematical contents remains vague as far as the picture is considered alone. This is an example for those pictures the authors were not able to classify in the first step of the analysis.

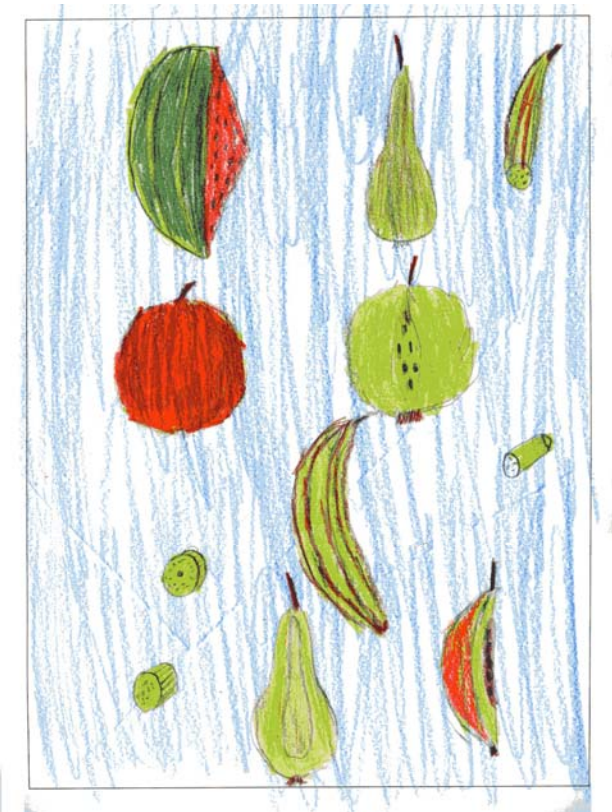


Figure 3

The boy, 10, writes: "I have thought for some time what I could draw. Then I saw a basket with fruit and decided to draw different kinds of fruit. I started with a whole apple. This is of course 1. Then I sketched a piece of banana, which stands for $\frac{1}{10}$. Then I made a quarter of a melon. I looked at it and found out that already quite a number of exercises could be made of this, e.g. $\frac{1}{10}+\frac{1}{4}$ or $1-\left(\frac{1}{10}+\frac{1}{4}\right)$. […] Then I sketched a bigger banana, half a pear, a fifth of a peach and much more."

The text stresses the proportions of the various pieces of fruit. It is rather a coincidence that the student takes fruit as an example. He just sees some while he is wondering about a reasonable topic for his work. The student regards the proportions as fractions which he uses for exercises without worrying whether the sum of a tenth of a banana and a quarter of a melon actually makes sense. At least from this point on, his considerations do not depend on the objects anymore. It is just the starting point for mathematical thoughts. It is the creative mathematical process itself which is of interest to the student. For this reason, his work and explanations contain the main features of the problem-solving category established by Ernest (1989; 1991).

CONCLUSION

Pictures consisting of several disconnected sequences, such as symbols, objects, and situations seem to parallel an instrumentalist view on mathematics. Pictures telling a story or delivering objects for mathematical activities tend to correspond to a problem-solving view. The appearance of important people in the history of science is often an indication for a Platonist view, which can also mix with an instrumentalist view or a problem-solving view. Interestingly, in the process of interpreting the pictures and texts, the Platonist view was rather difficult to detect clearly without an interview. In this regard, Ernest's category of a Platonist view might have to be seen in a more differentiated way.

Combining pictures and texts improves the empirical basis for decoding views on mathematics considerably. Furthermore, it makes better use of different students' abilities which are often neglected in everyday mathematical activities at school. Since students are normally not used to pictures and texts in mathematics, the approach presented here is appealing to those who are talented in painting and writing. Young fifth-graders, as examined here, are able to express their views on mathematics without reading and filling in questionnaires which implies certain problems regarding their age.

A word on the relationship between marks and the different views: In this study, all students falling in the problem-solving category achieve good or very good marks in their mathematics class. But there are also students with good or very good marks among those with an instrumentalist view. Students with a Platonist view seem to be often interested in the history of science; they approach mathematics and other sciences mainly by reading.

The teachers reported about several students who were normally not enthusiastic about mathematical exercises but were happy about the unusual character of this one. The realization, however, turned out to be more demanding than they previously thought. One student wrote: "Drawing a picture as homework in mathematics? Super, I thought. Then I found out that the task was not that easy after all."

Some works were surprising to the teachers involved. Since other talents than usual are necessary for this work, some students expressed worries and disappointment on their mathematical experiences very clearly – mainly in higher grades (Halverscheid & Rolka, in preparation).

The findings suggest that it is worthwhile to use this approach for investigating mathematical beliefs. If this approach is examined in more detail and with a variation of tasks in the future, it might have a potential to serve as a means of getting to know students better "beyond the purely cognitive" (Schoenfeld, 1983).

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ALGEBRA TEACHERS' WAYS OF THINKING CHARACTERIZING THE MENTAL ACT OF PROBLEM POSING

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This article deals with characterizing the mental act of problem posing carried out by middle-school algebra teachers. We report clinical task-based interviews, in which 24 teacher-participants were asked to think aloud while making up a story problem whose solution may be found by computing 4/5 divided by 2/3. The interview protocols were analysed in accordance with principles of grounded theory. Two types of ways of thinking emerged from the analysis and were validated: ***coordinating*** *and* ***utilizing reference points****. The results show that success in doing the interview task is associated with coordinated approach and utilizing a particular reference point.*

THEORETICAL BACKGROUND

This article is part of a series of reports, in progress, on the results of an NSF-funded research project, whose aim is to investigate the development of algebra teachers' knowledge of mathematics and pedagogy under a particular instructional intervention. The project is oriented within a conceptual framework called *DNR-based instruction in mathematics*, or *DNR* for short (Harel, in press a, in press b). In DNR, a mental act, such as "justifying," "modelling," "problem solving," or "problem posing," is considered with respect to observable *products* and inferred *characters* of the act; the latter is called a "way of thinking." For example, the particular solutions to a problem are products of the mental act of problem solving, whereas the problem solving approach one typically uses in solving problems is her or his way of thinking. The goal of this paper is to present ways of thinking characterizing the mental act of problem posing experienced by 24 US middle school teachers prior to the intervention.

The importance of problem posing in mathematics and mathematics education was recognized by many scholars (e.g., Freudenthal, 1973; Kilpatrick, 1987; Krutetskii, 1976; Silver, 1994). Learning opportunities associated with posing mathematical problems received keen attention of the mathematics education research community (e.g., Brown, 2001; Brown & Walter, 1993; NCTM, 2000). Also, some researchers used problem posing as a window in students' understanding of important mathematical concepts (e.g., Hart, 1981; Silver & Sai, 1996; Simon, 1993; Mestre, 2002). However, despite this recognition and attention, an activity of problem posing bears a great unrealized potential for modelling cognitive processes of individuals engaged in doing mathematics (English, 1998; 2003; Christou et al., 2005). In particular, posing problems can be a fruitful setting for modelling the knowledge base of mathematics teachers (e.g., Silver et al., 1996; Ma, 1999).

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For instance, Silver et al. (1996) introduced 71 school and preservice teachers to a set of conditions by which a billiard ball is shot from one corner of a rectangular table. They asked the subjects to write down as many questions appropriate to the situation as they could. Following this, they were encouraged to solve some of their own problems and then to generate additional questions. Silver et al. found that about 80% of the participants were able to compose some problems, though the posed problems were not always ones the subjects could solve. The results revealed complexity of the relationship between problem posing and problem solving. They also supported Kilpatrick's (1987) suggestion that many cognitive mechanisms are involved in posing problems. Silver et al. conjectured that *analogical reasoning*, *random goals generating* and *constraint manipulating* are likely to be among such mechanisms in the chosen context, and noted that much more research is needed to develop a deeper understanding of problem posing as a cognitive activity.

Ma (1999) asked in-service middle school teachers in the US and China to calculate $1\frac{3}{4} \div \frac{1}{2}$ and then to generate a story problem that matches the calculation. It appeared that, in contrast to Chinese teachers, the US teachers experienced major obstacles when making up problems involving division by a fraction: only 1 out of 23 teachers was able to compose a reasonable problem. Analysing the data—clinical interview protocols—Ma (1999) found that the teachers' deficiency in understanding the meaning of division by fractions caused their inability to generate appropriate word problems. As a result, the researcher made an important distinction between procedural versus profound understanding of mathematics. However, these studies have not addressed—at least not explicitly—the characters of the problem posing act. The study reported in this paper contributes to addressing this gap by answering the following question: What are middle school teachers' ways of thinking associated with the mental act of problem posing in the context of making up a problem whose solution is a division of fractions?

Our rationale in addressing this question is two-fold. First, as a base-line data we sought to identify important elements of the teacher-participants' knowledge base at the beginning of the DNR-based intervention. Second, the above research question is of interest by itself as it extends prior research on cognitive mechanisms of problem posing.

METHOD

Participants

Twenty four in-service mathematics teachers, 9 males (37.5%) and 15 females, were recruited for the study. The teacher participants teach first year algebra courses in urban and suburban middle and high schools in Southern California with student populations of varying socio-economic backgrounds; some of the schools were "low-achieving" and others "high-achieving". The teachers' self-reported mathematical backgrounds included 7 *mathematics majors*, 5 *mathematics minors*, and 12 whose *formal study of mathematics was less than a minor*. Twelve teachers (50%) held their academic degrees in different fields of education including 2 in mathematics education, 2 teachers did not have academic degrees, and the other 10 earned their degrees in fields other than mathematics or education. On average, the teacher participants had teaching experience of 4.75 years (SD=5.69), and taught mathematics for 3.58 years (SD=4.34); 9 teachers (37.5%) had only taught mathematics one year. This sample is not atypical since there are many US mathematics teachers with limited mathematics education or limited experience teaching mathematics.

Interview task and procedure

Each of the twenty-four teacher participants was interviewed individually for about 35 minutes. This paper is based on the first part of the interview (about 15 minutes), where the teacher participants were instructed to think aloud about the following task: "*Make up a word problem whose solution may be found by computing 4/5 divided by 2/3.*" The interviewing methodology was adapted from Erickson and Simon (1993) and Clement (2000); it was also analysed in detail in Koichu and Harel (submitted). Briefly speaking, the interviews were conducted by four members of the research team; all used the same interview guideline. The interviewers were instructed to refrain from revealing to the interviewees anything about the quality of their responses. During the interviews, when the teacher participant kept silent for more than 20-25 seconds, the interviewer prompted her or him to think aloud in a neutral manner, by saying "Keep talking" or asking "What are you thinking about?" or "What are you doing right now?"

ANALYSIS

The data analysed consisted of transcribed audiotaped interviews and notes written by the subjects during the interviews. The analysis was carried out in accordance with principles of grounded theory (e.g., Dey, 1993). We first conducted an open-ended analysis, in that we did not restrict our attention to the research question per se but attended to anything we deemed cognitively or pedagogically important; at this stage many (more than 10) categories were considered. Through a subsequent iterative process of refinement, we abstracted and classified two categories of ways of thinking, for which the data seemed to have provided rich and solid evidence.

Categories of ways of thinking

First, we characterized the mental act of problem posing by distinction between a *Coordinated Approach* (CA) and the absence of a CA, which we call an *Uncoordinated Approach* (UA). CA is indicated if at some stage of doing the interview task the teacher constructed the numbers 4/5 and 2/3 as measures of quantities, and the quantities were considered arguments of an arithmetic operation. For example, the participant can construct 4/5 and 2/3 in a form of the question "*How many 2/3 of a dollar are in 4/5 of a dollar?*" Note that by the above definition, **CA** must be assigned to the following formulation, even though it involves addition, not division: "*John has four-fifths of his birthday cake, and Elliot has two-thirds of his* *cake. How much cake do they have together?*" UA is indicated if, for example, the participant considered "*four-fifths of his birthday cake*" as a quantity measured by 4/5, then "*two persons out of three persons,*" as a quantity measured by 2/3 but did not consider these quantities as arguments of any arithmetic operation (examples will follow shortly).

Second, we characterized the mental act of problem posing by indicating *reference points*. By a *reference point* (RP) we mean a piece of knowledge the teacher participant holds as true and uses as an anchor for planning or monitoring. We identified four RPs:

"Answer" as a reference point (RP-AN): RP-AN is indicated when the teacher participant used the observation that the result of dividing 4/5 by 2/3 is equal to 6/5, or that the result is greater than 4/5. For example, one of the teacher participants argued that the idea of a discount cannot work because 6/5—the result of the operation $(4/5)\div(2/3)$—is greater than the first number, 4/5. From the context, it was clear that the teacher reflected out loud on her previous attempt to make up a problem about a discount in a clothing store. Another teacher asked himself: "*How can I obtain 6/5 of a pizza if at the beginning I share only 4/5 of a pizza?*" To us, these are indicators for RP-AN.

"Multiplicative relationship" as a reference point (RP-MR): RP-MR is indicated when the teacher attempted to make up a word problem by considering an equation of the form "$a \cdot x = b$" or considering a question like "How many 2/3 are there in 4/5 of a whole?" For example, RP-MR was indicated when one of the interviewees said, "*The word problem should be about how many 2/3s go into 4/5 of a pizza*", and, again, when another teacher said, "*My problem must lead to the equation* $2/3 \cdot x = 4/5$".

"Division by a whole number" as a reference point (RP-WN): RP-WN is indicated when the teacher participant anchored his or her word problem in a division operation in which the dividend or the divisor is a whole number. For example, RP-WN was indicated when one of the teachers said, "*First, let's try to make up a problem about division of 10 instead of 4/5*". Other examples are "*My problem should work when substituting fractions with whole numbers*" and "*It is easy to make up a problem about 4/5 divided by 3; why am I struggling with 2/3?*"

"Given task" as a reference point (RP-GT): RP-GT is indicated when the teacher participant without prompting from the interviewer talked out loud about the extent to which his or her formulation satisfied conditions of the interview task. In other words, the teacher used his or her interpretation of the interview task as a reference point. For example, RP-GT is indicated when one of the teachers noticed that her formulation is about "*multiplication by 2/3 instead of division by 2/3*"; other teachers' realizations included "*There is division by 1/3, but should be by 2/3*" and "*Gosh…what I described here is division by 3, it's not what I am supposed to do.*"

Validation of the coding scheme

We used the first 9 (in alphabetical order of the interviewee's first names) interviews to develop formulations of the categories and negotiate their understanding. For this reason, we report the levels of agreement only for the remaining 15 interviews that were coded independently by Koichu and Manaster. Five separate codes (CA or UA, RP-AN, RP-MR, RP-WN, RP-GT) were applied to each of these 15 interviews. Thus, a total of 65 judgments were required for the coding. The two authors' independent judgments were the same in 57 cases (88%). Of the 8 disagreements, 4 regarded CA, 1 RP-GT and 3 RP-WN. In 6 of these 8 disagreements, Koichu and Manaster considered each other's rationales and reached agreement by accepting one of the arguments. The two remaining disagreements were considered together by all three authors, who then reached common decisions.

RESULTS

The frequencies of the appearance of six ways of thinking in the 24 interviews are presented in Table 1.

Ways of thinking	Number of cases (Percentage)
Coordinated Approach (CA)	10 (42)
Uncoordinated Approach (UA)	14 (58)
Reference Point "Answer" (RP-AN)	17 (71)
RP "Multiplicative Relationship" (MR)	7 (29)
RP "Given Task" (GT)	18 (75)
RP "Whole Number" (WN)	5 (21)

Table 1: Frequencies of appearance of the ways of thinking

By definition, CA and UA are mutually exclusive categories, and reference points are not, that is, more than one RP can be indicated in one interview. On average, the teacher-participants used 1.96 (SD=0.95) reference points during the interview. Seventeen teachers utilized 1 or 2 RPs; 6 teachers utilized 3 or 4 RP, and only 1 teacher did not (noticeably) utilize any RP.

We now relate the ways of thinking to *success* in doing the interview task—from the analysts' point of view. We call a story problem fully *successful* if it meets the following criteria: i) each of the two given numbers (4/5 and 2/3) either appeared explicitly or implicitly in the problem; ii) at least one of the given numbers appeared in the problem as the measure of a quantity; iii) the problem included a question that has an answer based upon the information presented in the formulation; and iv) answering the question readily involved division of 4/5 by 2/3.

Only 4 teachers composed fully successful problems. They used CA and from 1 to 3 RP in the process of problem posing; all of them used RP-MR. The following resulting formulation illustrates this cluster in the data:

> **Snickers Problem**: How many mini-sized Snickers bars go into 4/5 of a full size bar if a mini-size bar is 2/3 of a full-size bar?

One teacher composed a problem that meets criteria i, ii and iv, as follows:

> **Two Groups Problems**: You and a friend are creating a smaller group of students from a larger group. You are responsible for four-fifths of the smaller group. To get that many students, you need to take two-thirds of the larger group. How many students are in the larger group?

She used CA, RP-MR and RP-GT in creating this problem.

The most difficult criterion for the teachers to meet was iv; 11 teachers formulated a problem meeting all of the other three criteria. Two examples follow:

> **Apple Problem**: Gloria picks four-fifths of 20 apples and divides them between herself and her 2 sisters. How many apples do two out of three sisters have?

> **Gas Station Problem**: I go to the gas station and fill my gas tank four-fifths full and then drive around until two-thirds of that gas has been burned. How much of the tank full of gas have I burned?

Problems from this cluster are associated both with CA (5 cases) and UA (6 cases). The teachers used from 1 to 4 reference points. In particular, all 11 teachers used RP-AN and 10 teachers used RP-GT.

The rest of the composed problems meet 2 or fewer criteria (out of 4). These (unsuccessful) problems are mostly associated with UA, RP-AN and RP-GT.

DISCUSSION AND CONCLUSIONS

Over 60% of the teachers in the current study were able to formulate some answerable mathematical question, which is consistent with Silver et al's (1996) finding that about 80% of the teachers in their study were able to do so. That study did not ask the teachers to address any specific mathematical concepts or operations, which is in significant contrast to the present study. That contrast reduced the number of teachers who produced *successful* problems to less than 20%. Indeed, the interview task to make up a story problem involving division of fractions appeared to be unsolvable for 20 out of 24 teacher participants. This finding is in a good agreement with that reported in Ma (1999). Apparently, the explanation she provided—the US school teachers lack profound understanding of mathematics—works also in our study. Moreover, we fully agree with Ma's (1999) suggestion that "in order to have a pedagogically powerful representation for a topic, a teacher should first have a comprehensive understanding of it" (p. 83).

The analysis presented above enabled us to look at the data from additional angles. We suggest that the participants' obstacles in understanding division *along* with Uncoordinated Approach to the interview task are responsible for many poor problems. Six cases indicated the following: Division by 1/3 means division into thirds, and, in turn, division by three (see, for example, the Apple Problem). Furthermore, division by 2/3 appears to be a multi-step operation of dividing a quantity into three parts and picking up two out of them (see, for example, the Gas Station Problem.) As a result, the participants confused division by 2/3 with multiplication by 2/3 or could not coordinate three multi-step operations (4/5, 2/3, $(4/5) \div (2/3)$). In other words, we suggest that Coordinated Approach to the task is a necessary but not sufficient condition of success in problem posing (see the Snickers Problem and the Two Group Problem).

We also noted that only one reference point—"Multiplicative Relationship" is clearly associated with success in doing the interview task. On the other hand, thinking of division by whole numbers sometimes misled the teacher participants. It seems that the teachers' attempts to utilize their proficiency in division of whole numbers led them away from understanding division as a multiplicative relationship and, in turn, from posing a successful problem. This finding is in line with Harel's (1995) observation that substituting given fractions by "nice numbers" in word problems is a poor teaching strategy.

The participants in our study are typical US teachers with many respects. Without any intention to make too broad generalizations, we can assume that many of the US teachers, like the participants in our study, do not see themselves as a part of a community responsible for creating problems for their students. Much should be done to change this situation and to disseminate the powerful idea of problem posing among the teachers. To achieve this goal, we need to better understand the teachers' obstacles and ways of thinking involved in mental act of problem posing. We believe that our study is a step toward this goal.

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MATHEMATICAL IMPAIRMENT AMONG EPILEPTIC CHILDREN[1]

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This study offers a set of data concerning the exploration of interrelations between neuropsychological aspects and mathematical difficulties presented by epileptic children. These children can be characterized by important neuropsychological functional disturbances in attention, memory and visual perceptual skills, which are related to mathematical impairment concerning the proper use of procedural algorithmic tools. Nevertheless, data discussed here show that epileptic children have benefited from the offer of cultural semiotic aids, which highlights the need of considering the neuropsychological foundations of mathematics activity in the context of culture.

INTRODUCTION

According to the World Health Organization (WHO), epilepsy is "(...) a chronic non-communicable disorder characterized by recurrent episodes of paroxysmal brain dysfunction due to a sudden, disorderly, and excessive neuronal discharge, (…) being one of the most prevalent neurological disorders that can be effectively prevented and treated at an affordable cost (fifty million sufferers today, 85% from developing countries - 60% to 90% of them receiving no treatment at all)" (WHO, 2006). Symptoms of epilepsy are "(...) seizures that occur at unpredictable moments, varying from frequent brief lapses of consciousness to short periods of automatic subconscious behavior or convulsions of the whole body that make the person fall over and lose consciousness completely" (WHO, International Bureau for Epilepsy, ILAE, 2006). Social stigma leads to serious social exclusion because of negative attitudes of others towards people with epilepsy: "children with epilepsy have *important problems at school*, adults may be barred from marriage, and employment is often denied, even when seizures would not render the work unsuitable or unsafe." (WHO, 2006; italics added).

As mentioned above, epileptic children's school activities are specially affected, and *mathematical learning* is in this context one of the most problematic issues for epileptic pupils (Mulas, Hernández & Morant, 2001). Nevertheless, there are very few specific research studies focusing on the neuropsychology of mathematical activities among epileptic children. As mentioned by Neumärker, psychological disorders related to mathematics are far less studied than those related to language (both oral and written) and generic memory (Newmärker, 2000). On the other hand, an important set of neuropsychological research initiatives have been focusing in

[1] Research supported by Conselho Nacional de Pesquisa, Ciência e Tecnologia (CNPq)-Brazil.

learning difficulties associated to malfunction and/or anatomical injuries in certain brain loci (see for example Miranda & Gil-Llario, 2001), without specification for delimited pathologies.

Many important symptoms due to neuropsychological malfunctions have been associated to difficulties in mathematical activity, and most of these symptoms and difficulties are very frequently related to epilepsy (see Hommet & cols., 2005; Aldenkamp & cols., 2004). This is specially the case for difficulties related to *visual-spatial abilities* (e.g., being able to distinguish 6 and 9, to properly order digits of compound numbers respecting the places of units, tens, hundreds and so on, and problems in dealing with symmetries and mental imagery to represent rotation of solids in space). Problems related to *attention* seem to make the management of superordinated and hierarchic strategies necessary to the use of mathematical algorithms specially hard, leading to difficulties in arithmetic calculations.

The authors of the present contribution adopt the theoretical assumption that mathematical knowledge emerges as an *embodied* functional product of a human brain *embedded in a socio- cultural context* (Lakoff & Núñez, 2000). Mathematical activity is seen here as a product of a *material mind* (Vygotski, 2003) subsumed in a specific social, cultural and historical context, depending interconnectedly of individual logical reasoning and cultural semiotic tools (Da Rocha Falcão, 2001). Finally, it is important to take into account the *perspective* of mathematical activity which is adopted as reference. Mathematical activity cannot be reduced to its algorithmic aspects, even though these aspects are important to school performance. Because of this approach of mathematical activity, research on mathematical impairment of epileptic children should go beyond traditional psychometric evaluations and neuropsychological correlational clinical studies. This is the theoretical and methodological effort of the present research.

METHOD

As pointed out above, this research offers data about mathematical impairment among epileptic children, trying to show *who* are these children and *what* they can and cannot do in terms of mathematical performance. Contributions from neuropsychology and psychology of mathematics education were combined in a three-stage exploratory procedure, as summarized below:

Stage A: A.1.) Survey study in the neuro-pediatric ambulatorial service of a children's hospital in Recife (Brazil), aimed to form a group of four epileptic children with idiopathic generalized epilepsy/childhood absence epilepsy (CAE)-cf. DSM-4), age levels of 9 (girl, private school), 10 (girl, private school) and 11 years (two children, a boy and a girl, both from public school), with the same pattern of drug prescription (valproic acid). A.2.) Formation of four reference groups of non-epileptic children (one group of five children for each epileptic child, with similar profile for age level, sex, type of school (public or private) and socio-economic level).

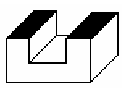
Stage B: B.1.) Diagnosis I: Application of a neuropsychological battery of evaluation tests for the epileptic group and for the four reference (non-epileptic) groups in order to establish a profile of children, adopting a set of psychometric tools largely employed and well-ranked in neuropsychological research (see Lezak, 2004). Tests and respective psychological aspects considered are summarized in table 2 below:

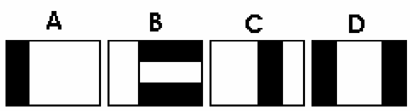
Tests	Psychological aspects
▪ Wechsler Intelligence Scale for Children-III (WISC-III)	Intelligence (total IQ, verbal IQ, executive/ manipulative IQ.
▪ Rey Auditory-Verbal Learning Test (RAVLT)	Memory
▪ Test de la Figure Complexe de Rey-Osterrieth	Visual and spatial abilities (organization and planning)
▪ Trail Making Test (parts A and B) ▪ Stroop Test ▪ Teste AC (Cambraia, 2003)	Attention and executive functions
▪ Teste de Desempenho Escolar (Stein, 1994)	General school performance

Table 1: Neuropsychological battery of tests with respective psychological focus.

B.2.) Diagnosis II: Evaluation of mathematical school performance for epileptic children and reference groups using Evaluation Instrument DII. The main goal of using this instrument was to establish a trustworthy school mathematics profile of all children. This instrument consisted of a modified version of a set of 20 questions conceived for evaluating mathematical school performance at the end of the first level of fundamental teaching for Brazilian children (1st to 4th school level, 6-8 to 9-11 years of age). It was built in order to cover five sets of mathematical school domains, as described and exemplified below:

- Algorithmic abilities and comprehension of numerical decimal system : "Write a number formed by 2 hundreds, 7 tens and 5 units", "Do the following operation: **847 + 5 + 98**"
- Additive structures: "A friend and I like very much stickers. Yesterday my friend came to my home to visit me, and I gave him four stickers and he gave me six; at the end I had fifteen stickers. How many stickers did I have before my friend's visit?"
- Multiplicative structures: "We are going to have a birthday party, and we want to give two balloons to each invited child. How many children will we be able to invite if we have eighteen balloons?"
- Mental imagery of geometrical properties of solids: look at the object below and choose among the options (A to D) which would be its best representation when it is seen from above".

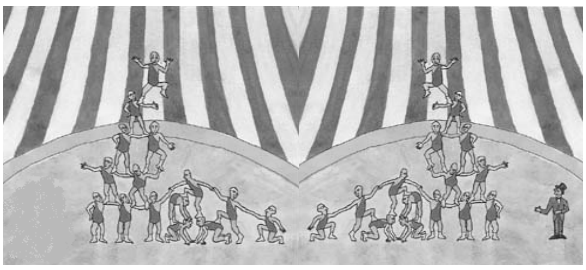
- Object: 

Options: 

- Comprehension of cultural measures (e.g., time): "I wake up at 6:30 a.m., take a shower and go to school; my classes begin at 7 o'clock a.m.. I leave school at 12:30 p.m. and come back home to have lunch. In the afternoon I do my homework, and I go to bed at 8 o'clock p.m. Complete the clocks below, drawing the pointers of hours and minutes accordingly to the moments of the day mentioned:"

Stage C: Diagnosis III: Evaluation of mathematical performance of epileptic and four non-epileptic children, (each non-epileptic children taken at random from each of the reference groups), using another evaluation instrument (Evaluation Instrument DIII), conceived in order to highlight situations where procedural-algorithmic and conceptual aspects of mathematical activity could be distinguished. This Stage, therefore, was crucial for the purposes of the present study, since we presumed that mathematical impairment of epileptic children had an important component represented by difficulties in the spatial representational execution of algorithms. In other words, mathematical impairment of epileptic children would be *procedural* rather than *conceptual*. Evaluation Instrument 2 consisted of a set of questions aimed to explore aspects connected to analytical visual spatial reasoning, covering the following aspects:

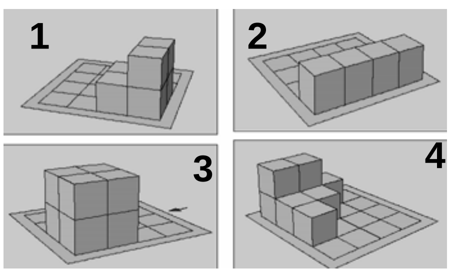
- Ability to identify, analyze and complete mirror-like complex images. This ability is considered here as a psychological precursor for the geometrical concept of symmetry.



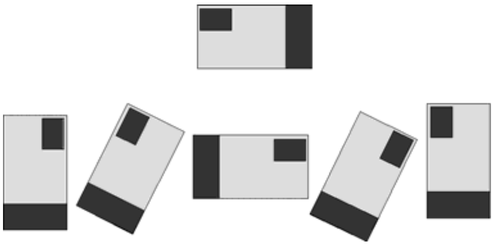
"Is there anything lacking in the big picture? Where must the small picture of the man be placed in the bigger picture?"

[Illustration adapted from the game "Mirror", Freudenthal Institute, 2005].

- Ability to manage visual imagery of solids. In this task subjects were firstly asked to establish the quantity of elements composing each of the four sets based only in visual imagery and visual spatial reasoning (item 1 below). Afterwards, the same subjects were asked to build similar sets of elements using brick layers (Lego™ bricks) (item 2). 1."How many blocks are there in each set?" 2. "Build similar sets using Lego bricks". [Illustrations from the game "Build Free", Freudenthal Institute, 2005]



- Spatial orientation 1: ability to manage imagery of solids rotation and translation in space:



"Look at this figure on the top; you have five other figures downwards, in different positions.

To which of the figures downwards correspond the figure in the top when it is rotated"?

- Spatial orientation 2: ability to adequately represent and operate algorithms of addition with the aid of colors to distinguish units, tens and hundreds:

"Operate the following addition: **847 + 5 + 98**"

Hundreds written in red.

Units written in green.

Tens written in blue.

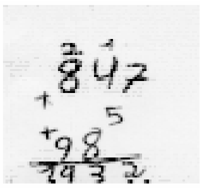


Figure 1

Epileptic children with particular difficulty in organizing properly the algorithmic procedure to operate addition (see figure 1 on the left) were expected to get around this difficulty with the representational aid of using colors associated to the place-value system of numeric decimal system.

RESULTS

Data from stage B mentioned above were categorized and encoded for treatment by descriptive multidimensional analysis, combining hierarchical cluster analysis (figure 2) and factor analysis (figure 3). Both analyses show a clear separation of epileptic children (subjects 13, 1, 7 and 19) from the rest of the group. Hierarchical cluster analysis (figure 2) produced a first partition (A) opposing the epileptic child 19 to the rest of the group; the second partition (B) opposes epileptic children 13 and 1 to groups c1 and c2. The first (and more important) factor from factor analysis opposes epileptic children (1, 13, 7 and 19, having projections over the left side of the factor) to the others (right side of the factor); the second factor opposes subjects 1, 13, 7 (downside) to subject 19 (upside), which will help us to explain the isolation of subject 19. Data from a complementary factor analysis based on modalities of variables from stage B, crossed with clinical analysis of data from stage C allow us to go further in the interpretation of partitions and separations in the factorial plan: the most important contributions for the left side of the first factor (the "epileptic" side) were, in decreasing order of importance concerning their respective contributions: 1. Difficulties in operating algorithms for addition (being able to put hundreds under hundreds, tens under tens and units under units (see fragment of protocol reproduced in figure 1 above); on the other hand, it is important to mention that these epileptic children have benefited of operating with numbers displaying hundreds, tens and units written in different colors, according to data from Stage C; with this kind of representational-chromatic aid, epileptic children could organize (and operate) properly the algorithm of addition, as expected by us.

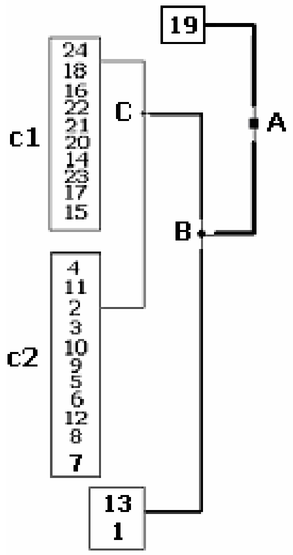


Figure 2: hierarchical cluster representation

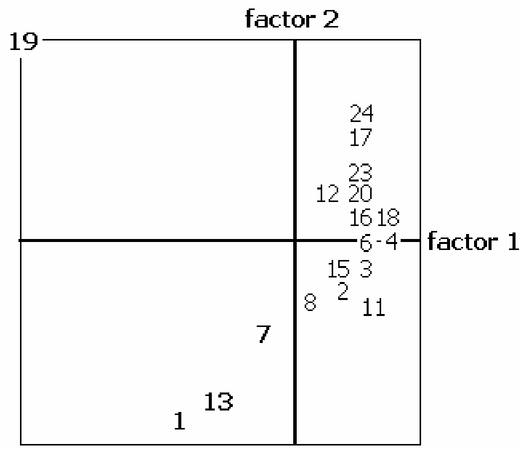


Figure 3: factorial plan (factor analysis from nominal data), factors 1 by 2.

Even more important: these children could properly justify the role and interest of colors use, pointing out that different colors were representing different place-values (hundreds, tens and units). This clinical aspect allows us to discard the interpretation of epileptic children's improvement only as a result of strictly and simple perceptual combinations: blues under blues, greens under greens and so on. In fact, the semiotic aid represented by color-use allowed these children to show a mathematical conceptual knowledge that they could not mobilize under usual conditions. 2. Difficulties in distinguishing numbers like 6 and 9. 3. Normal memory-span at RAVL-Test, with difficulties in progressing along repeated essays. 4. Percentiles 25 (under the reference mean for the age group) in Rey-Osterrieth Complex Figure for copy and also for memory reproduction. 5. Low performance (inferior to the expected mean for the age group) in concentrated-attention tests (AC Test). The only relevant contribution to the right-side of the factor 1 (the "non-epileptic" side) was a normal (mean) performance in concentrated-attention tasks (AC Test). This difference in contributions between these two sides is normal in this case, since the non-epileptic group is more heterogeneous and numerous than the epileptic group. Factor 1, then, operates a partition between epileptic and non-epileptic children in terms of visual spatial organization (data from Rey-Osterrieth Complex Figure for copy and memory), and processes of attention. These results can be completed by clear differences observed in two of the tasks from Stage C: i) epileptic children consistently failed in the first item of the tasks aimed to evaluate ability to manage visual imagery of solids (to say the number of blocks in the sets), while all non-epileptic children have produced a right answer for this task. Nevertheless, it is important to mention that all epileptic children have corrected themselves when they were answering part.2 of the task (to build similar pieces as showed in the illustration, using Lego brick layers); ii) epileptic children have once again consistently failed in the task Spatial orientation 1, related to manage imagery of solids rotation and translation in space (no right answers at all), while non-epileptic children have consistently produced right answers for this task.

The present analysis must finally take into account information given by the second factor in importance, factor 2. Only epileptic children have relevant contributions (i.e., equal or superior to the mean of contributions) to this factor, which explains the opposition of epileptic subjects 19 (superior side) and 1, 13 and 7 (inferior side). Shortly, subject 19 presents a deeper degree of impairment of cognitive processes related to verbal memory (difficulties in progressing across repeated essays in Rey Auditory-Verbal Learning Test) and visual perceptual skills (as shown by the mixing-up of numbers like 6 and 9).

FINAL REMARKS

Epileptic children have clearly shown difficulties in mathematical activity, in processes related to attention, memory and visual perceptual skills, the latter being the most salient and unifying aspect of the group. Such data are in line with other research initiatives (Schubert, 2005). Visual perception refers to the process of organization and interpretation of visual information, covering abilities such as visual discrimination, visual memory and visual spatial organization. As shown by data presented here, these abilities are directly related to *procedural* aspects of mathematical activity, like digit discrimination, spatial organization of written numbers and imagery and resolution of geometrical problems. Nevertheless, it is important to highlight that this peculiar kind of impairment of mathematical activity shown by epileptic children does not necessarily imply in conceptual mathematical insufficiency, since even the most compromised epileptic subject (19) has benefited from the offer of representational-chromatic aid (hundreds, tens and units represented by different colors); all epileptic children could explain what the different colors were representing (red for hundreds, blue for tens and green for units), showing that they were aware of the logic of the place-value in decimal system. On the other hand, data from Evaluation Instrument DII have not allowed the establishment of significant mathematical conceptual gap between epileptic and non-epileptic children. The use of the chromatic aid mentioned above, and also the possibility of manipulating brick layers to build equivalent sets of solids represented bi-dimensionally in a written illustration, allowed the bypass of a specific difficulty related to visual spatial organization. These clinical data call attention to two important aspects: first, mathematics activity covers both procedural and conceptual aspects; it is particularly important to keep this in mind and look attentively to the difficulties shown by mathematics learners. Accordingly, diagnosis of mathematical impairment should go beyond isolated scores of psychometric and mathematical evaluation tools. Second, as very well expressed by Vygotsky, semiotic tools from culture can change negative aspects of deficiency into positive aspects of compensation (Vygotsky, 1993).

Keeping this in mind is important to understand and help not only difficulties of epileptic children, but also of children in general.

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PATTERNS OF PARTICIPATION IN NETWORKED CLASSROOMS

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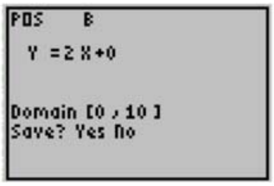
We study the combination of visualization software in the form of SimCalc MathWorlds with wireless Networks on student participation in Algebra classrooms. Such technologies allow students to create mathematical objects or motions on hand-held devices that can be aggregated within parallel software on a teacher's desktop computer then publicly displayed and analyzed. We use work from Linguistic Anthropology to analyze the rich participation frameworks that are evident in such situations, focusing on the shifting roles of speech and physical actions (e.g., gesture and deixis), as students make mathematical meaning at a social level.

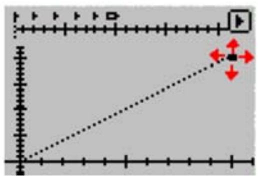
FOCUS OF STUDY: VISUALIZATION AND PARTICIPATION

This paper focuses on the present findings of our National Science Foundation funded project investigating the impact on students' participation and engagement in high school algebra classrooms that use a combination of rich visualization software and wireless networks. We use theoretical perspectives from Linguistic Anthropology (Duranti, 1997) to explain new forms of participation frameworks that are evident in our classrooms.

Recent work (Hegedus & Kaput, 2004) developed a series of activity structures with such a combination of technologies to create learning environments that utilized the natural, physical and social set-up of the classroom. Students can create mathematical objects on hand-held devices (such as graphing calculators) and send their work to a teacher computer, which is projected on her whiteboard. Due to advances in wireless communication and interactivity between desktop PCs and hand-held devices, the flow of data around a classroom can be very fast allowing large iterations of activities to be executed during one class. But it is not just an advance in connectivity but in the development and application of curriculum that maximizes such an innovation. Our prior work created activities that allowed students to make functions in SimCalc MathWorlds on the TI-83/84+ graphing calculator that could then be aggregated by a teacher into MathWorlds running in parallel on a PC, using TI's Navigator wireless network. Such an action by the teacher, though, was not done in an arbitrary fashion (i.e., collect all work) but in a mathematically meaningful way. For example, each student is in a numbered group (say 1 through 5), and has to create a position function that can animate an actor (in the **World**) at a constant speed equal to their group number for 10 seconds. So group 2 can use MathWorlds to create a function algebraically (see figure 1), i.e., $y=2x$ on the domain [0,10], or graphically by building a linear function, and dragging a hot spot attached to a line segment with left endpoint at the origin out to (10, 20). When all functions are submitted by the students, or aggregated by the teacher, then a family of functions, $y=\mathrm{m}x$ (m=1 to 5) will be displayed. This simple activity can be extended into an activity structure that uses group count-off indices in general to distribute mathematical variations across a whole class of students. This latest innovation expects more active participation by students since every student is required to contribute, but not only that, engagement is potentially affected even if a student does not contribute; the aggregate (displayed by the teacher computer) exposes this and, potentially, errors that some students might have made upon analysis of the collection. This is a main working hypothesis and our study has investigated the reality of such a claim.



Algebraically editing motions



Graphically editing motions

Figure 1a: MathWorlds for the TI 83+/84+

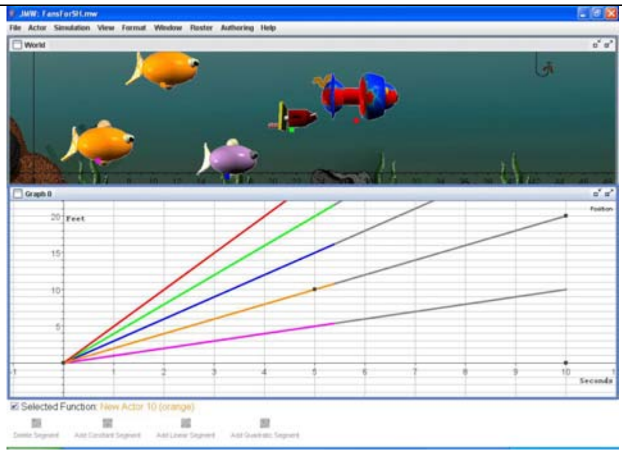


Figure 1b: Aggregation in MathWorlds for the PC

Our work has focused on using Linguistic Anthropology to understand the framework of participation in our classrooms and how it changes across activity structures and use of networked classrooms. We present a case study of our work, which analyses one intense classroom episode and the role of such technology. In building on Goffman's work (1981), we are particularly interested in the intersection of both (what we call) the *discourse* and *physical action* spaces, i.e., the role of language, natural, technical and metaphorical, as well as gesture, deixis (e.g., pointing), and posture. We regard these two spaces as intimately linked and so our analysis investigates each of these features to make sense of the impact of the technology on participation and engagement.

METHODOLOGY

Our main study has conducted several common teaching experiments in grade 9 Algebra classrooms across three medium-to-low achieving districts with teachers of varying experience. The teaching experiment consisted of the implementation of a 3-week unit that replaced a chapter and half of material in the text used by the participating schools (Bellman, Bragg, Chapin, et al., 1998). Our participant teachers collaborated with us prior to the interventions to agree on a set of curriculum materials that we had produced with the MathWorlds software that focused on linear functions (y=mx+b form), slope as rate and variation. These materials had been developed over several years and field-tested in a variety of high school and college freshmen classes.

Each class was recorded with two digital cameras, one focused on the teacher and the whiteboard space where MathWorlds was projected and the other positioned at the front of the class focused on the students using a wide-angled lens to pan out and observe whole class dynamics as well as small group interactions. Both cameras were used as roaming cameras when the class was involved in small group work. The camera placement and focus is largely guided by our research questions and inquiry on the types of participation and engagement exhibited in class both from a linguistic and physical perspective. Our rich inter-related datasets allow us to examine the impact of the technologies from a teaching and learning perspective. This paper focuses on analyses of some of our classroom video data.

APPLICATION OF LINGUISTIC ANTHROPOLOGY

We will first outline briefly the theoretical perspectives of Linguistic Anthropology and then use them to unpack the impact of the technology on participation, engagement and learning using vignettes from our classroom intervention described above. Linguistic anthropology combines the study of language and culture as one of the main sub-fields of anthropology. Linguistic anthropology is "*not just interested in language use but language as a set of symbolic resources that enter the constitution of social fabric and individual representations of the world*" (Duranti, 1997, p.2). Researchers in the field see the subjects of their work, speakers, as **social actors** that are members of complex, interacting communities. Our analysis has profited from this theoretical perspective as to study the interactions and learning cycles within the SimCalc Networked classroom but we cannot only focus on the use of language but the interactions and physical expressions that occur between students based upon the publication and representation of their work in a social workspace. The computer software, projected onto a whiteboard display, becomes an "active participant" as much as any human in the classroom, as a harvester, presenter and facilitator of students' mathematical work.

Analysis 1. Production Formats

A key contributor to the field of linguistic anthropology who primarily focused on participation structure is Goffman. We focus on one aspect of his work to analyze the new forms of participation evident in our connected classrooms. Goffman (1981) argues that a person can identify themselves in three ways in a discussion, (i.e., the pronoun "I" can refer to three distinct roles) namely, **animator** (person who gives voice to a message that is being conveyed), **author**, one who is responsible for the sentiments or words being expressed, and **principal**, person whose beliefs are being expressed. One person could have all three roles, but they can often be separate, e.g., a press release from the Whitehouse where the President (as Principal) might have a speech written for him (an Author) that is delivered by a spokesperson (an Animator). These three roles constituted what he called a **production format** of an utterance. In addition to roles of speakers, he determined two terms for "hearers"—**ratified participants** (those entitled and expected to be part of the communicative event) and **non-ratified participants**. This leads us to understanding not only what speakers know and want but also how speakers design their speech in the on-going evaluation of the recipient(s). This is described by the notion of **recipient design** (Duranti, 1997, p.299). The design of speakers and hearers is called the ***Production Format***. We are especially interested how within this format students choose to participate whether they are ratified or not and what role the technologies play in this process.

This structure has helped us unpack the communicative complexities that appear to evolve in a networked classroom. What is fundamentally new for us is the role of the *aggregation space* where students' contributions are displayed in a public display space. We analyze an episode from one of our classrooms. The students are working on an activity we call "arrows":

You and your partner will start at different positions. You are positioned G (your Group-number) away from 3 feet. The person with the odd count-off # will start to the right of 3 feet. The person with the even count of # will start to the left of 3 feet. You and your partner must meet at 3 feet at the same time. You and your partner will determine the amount of time you will travel for. The group CANNOT travel for the same amount of time, only you and your partner can. You must create a linear expression for your motions.

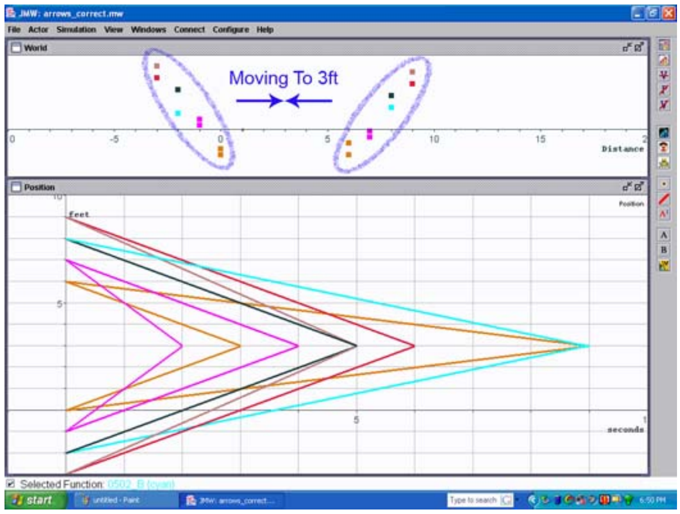


Figure 2: Arrows following a correct aggregation in MathWorlds

Figure 3: Classroom Setup

After their work has been collected into MathWorlds on the computer, they begin by looking at the view of the **World** where the animations of all their graphs occur (see Figure 2 for a correct set of contributions). The first error in the world is the starting position of two students, Jess and Alyssa. They should have started at -2 and 8 feet, respectively (see Cyan colored dots in figure above). Instead, they both began at zero. When the motion is run, two students do not stop at three. They also do not travel for the same amount of time. There is some debate among the students as to how to correct the motion of these actors. Robert (R) and Kirsten (K) believe that the actors have a domain that is incorrect (see figure 3). Kirsten then suggests that maybe one of the students didn't change their slope to be negative. Alyssa (off camera) recognizes that the actors with the incorrect motions are Jessica and herself but remains quiet thereafter. Alyssa thinks that she did not make the slope negative, but on running the motion again the(ir) two actors are moving in the correct direction.

Nick (N) then begins to argue, rather forcefully, that the domains are wrong. Kirsten believes that the domains do not matter, and that you can always end at three. Nick, frustrated, says that "you'll keep going if the domain isn't changed". The teacher asks him to listen to Kirsten, who explains that if you go for a longer amount of time, you will not pass three, you will (just have to) go slower. Nick believes this will only work if you have the right slope. Luke goes to the board and draws the action of two actors as if they are being "traced." He shows two actors traveling at the same rate, and the one that travels for a longer amount of time goes farther. Kirsten then goes to the board to show that an actor can travel the same distance in more time if they go slower. Nick is emphatic, and says then you would have to change the slope.

In this example the students who created the two wrong motions (the **Principle**) are not voicing their beliefs on the motions they created, yet two other students (and later more) are **Animators** of their constructions, during an analysis of the work of the whole class. One might think of the students with the wrong answer as a Principal, yet the representation of their work has been projected from their own local workspace on the calculator to a parallel, yet different, object in the aggregated environment. And the Computer (which projects their work) can be thought of as an **author** of their work through creating/or re-creating or publicizing their work. We have often seen that students do not always identify with "their" object after it has been collected and displayed publicly. So whilst the mathematical function has been originally authored by, and represents the beliefs of, the student, it might be perceived differently by the student when in public display. The public display of all students' contributions has fuelled a group analysis of the overall system of motions. Note the teacher has not chosen to show the graphs of their motions yet and says very little (just asks Nick to listen to Kirsten). MathWorlds on the computer can be thought of as a ratified participant in this communicative event, and although, non-verbal, is a voice box for the class analysis. The computer software is also an **animator** for the set of beliefs for the whole class. This has only been made possible through the integral role of classroom connectivity. Having a representation of their linear expressions they submitted that can be executed (i.e., press play to run a simulation of all their motions at once), the teacher has created a further role for the computer environment that tests the conjectures of the students in debate. In fact neither Nick, nor Kristen are wrong but they are approaching it from different perspectives. If you fix the domain (which is up to the individual members of the group) then you would have to change the slope of the graphs; if you fix the slope then you would need to change the domain. We also believe that although Jessica and Alyssa are ratified participants in the classroom, their work has led to a more general focus for analysis by their classmates, and they choose not to participate. We believe the ratification process has occurred not just through some students choosing to begin a communicative event, but primarily with the computer (following aggregation and execution of the representation) to highlight (and potentially ratify) whose object is to be discussed. The computer does not point (physically) to two members of a class. In fact the discussion is around the two objects and other contributions and NOT at the contributors. This has been done non-verbally, which we believe is a new result of classroom connectivity that can help students reason and generalize mathematical variation as well as analyze strategies and misconceptions.

ANALYSIS 2. PARTICIPATION IN TIME AND SPACE

Studies of language use do not always refer to the material world or the built environment through which meaning is mediated and made sense of. A major exception is the study of *Deixis*, which examines the properties of linguistic expressions (indexes) that cannot be interpreted without reference to a nonlinguistic context of their use (Duranti, 1997). Deixis extends to the use of gestures, movements, posture and gaze as well as pointing acts used in collaboration with speech. We continue to analyze our classroom episode focusing on how participation is effected by the role of deixis and physical action, and how the public workspace has become a motivator for debate and analysis of other students' thinking (Radford Demers, Guzman & Cerulli, 2004). We begin with the teacher asking A to speak:

T Go ahead Amanda.

A  If you do it again *{A is standing at her desk pointing to board}* and you watch the bottom two people on that, the bottom two dots. *{She is referencing two actors that have a correct motion, but go much slower than the remainder of the class.}*

T Do you want me to go back to the beginning?

A Yeah.

N Yeah, but the longer you go…

R It doesn't matter.

The teacher (T) has ratified Amanda (A) as the primary speaker. Amanda has decided to focus on two other motions that are correct but explain how you can have a longer duration but, depending on where you start, the speed will be different. She needs to stand up and point (note the two fingers which actually move up in down in reality) to focus the attention of her analysis.

N

{N is standing at front of class, facing class, next to teacher, T}

If you go longer then you gotta make your slope …

{holds two hands apart at waist level, brings them together} shorter

A It doesn't…

R No, no, 'cause if…

Nick (N) is frustrated and although what he says is correct he is not interpreting what Amanda (A) and others have said previously. He decides to not only stand up but also face the class. The motion of his arms through space describes what he is saying and he uses his posture to try to convince the class. He has interrupted Amanda, the primary speaker. He has sat on his desk perpendicular to the class up till this point, showing some resilience towards his classmates making sense of the aggregation.

N How're you gonna tell me? *{open arms, is still facing class, trying to lead the discussion}*

R … that part of your graph has the same slope, so that means they both have the same slope, you can't change it. *{R points both index fingers towards each other, then moves both hands together to cross fingers}*

{A gets out of seat, heads to front of class, then returns back to seat but remains standing}

A Plus, the bottom two have this…

T Robert, what you're saying is, they both have the same slope?

R  Well you, those two, kept going they were partners. *{A is standing, but partially turned around to face R. R repeats action from hands described above}* They both had to have the same slope, and they have to keep that slope so they can meet at three.

A Well they didn't because…

R In their amount of time, then they put too much time, and they went past it. *{again repeats motion with hands, A sits down in seat and turns towards K}*

Robert (R) has begun to support Amanda in making sense of the situation. His use of gesture is an important indicator of this process. The motions of index fingers mimics the motions of the objects in the aggregated display that is under discussion. Amanda who was the primary speaker, looks behind and appreciates Robert's analysis (prompted first by his verbal description but followed by his gestural actions—see when she partially turns her head) and sits down. This interaction leads to resolving the two motions by changing the domains of the functions helping K understand.

REFLECTIONS

This paper has focused on the impact on classroom participation of networked classrooms, which allow the aggregation and public display of students' mathematical work on hand-held devices in a parallel software environment on a teacher's computer. We have used the theoretical perspective of linguistic anthropology to deconstruct the categories of speaker and hearer to analyze the complexity of participation that occurs when students generalize their own and their classmates' contributions to this public workspace, recognizing speech acts as an activity of socio-historical depth. In addition, emphasis on participation reframes speech not only in terms of oral but spatial expressions. This analysis can give us tools to understanding particular points in a classroom discussion when the dominant discourse is challenged in subtle but effective ways. The networked classroom appears to propitiate a rich set of communication events where analysis of mathematical variation is brought to a social plane where students can understand the core mathematical ideas in focus from a collaborative perspective. Our future work will look at significant shifts in content knowledge of particular students from our quantitative datasets with respect to the various forms of their participation highlighted in our present analysis with the aim of understanding whether such *participatory learning* can impact engagement, and realization and understanding of abstract mathematical concepts.

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TEACHER TRAINING STUDENTS' PROBABILISTIC REASONING ABOUT COMPOUND STOCHASTIC EVENTS IN AN ICT ENVIRONMENT

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Sixteen teacher training students participated in a teaching experiment involving probabilistic tasks and use of ICT simulation tools. Using the theoretical model offered by Noss and Pratt (2002) the focus was on students' naïve mathematical knowledge and how the ICT-environment structured the students' interpretation and focus in the situation. This paper reports from the preliminary analysis where two dominant naïve strategies were identified. In addition we offer some insight into the role the ICT environment plays for the students in their meaning-making process in this particular setup.

BACKGROUND

The current Norwegian national curriculum (L97) has a high focus on problem solving activities as a natural part of the teaching process. Still, recent international studies, like PISA (Kjærnsli et al. 2004) and TIMSS (Kjærnsli et al. 2004), show that the problem solving competence of Norwegian students is quite poor compared to that of students in other countries. Another didactical dimension seen as important by the curriculum reformers in Norway is the role of ICT in the learning process. To answer the call for research (in Norway) into these two dimensions the present study was undertaken, with a focus on students' probabilistic reasoning when working within an experimental setting with access to ICT-tools. Investigating students' probabilistic reasoning within an ICT-environment is a recent endeavour (Pratt, 1999; Stohl, 2000; Paparistodemou; 2004). However, these studies indicate that such an environment can in fact enhance students' learning. The focus in the current study is on how students reason when they deal with compound stochastic events - and how their reasoning is being structured by the setting.

PROBABILISTIC TASKS INVOLVING USE OF MULTIPLICATION

When dealing with problems like "throwing of two dice", the normative way of modelling this situation is to see it as composed of two subevents which are each much simpler to deal with. By understanding the multiplicative structure involved in the situation, one can calculate the probability getting "two sixes" this way $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$.

For many students this modelling is far from natural (in the first stage of the learning process). What can make them see and understand the multiplicative structure involved? The pedagogic setting chosen in the present study offers explorative possibilities, and by using ICT simulation tools the students could collect small and large samples of data. In other words *experience*, i.e. collecting data, is seen as an important part of the participants' meaning-making process. Discussions with other students and with the teacher, and pencil and paper activities are other important resources in the setting.

THEORETICAL CONSIDERATIONS

The current research builds on a theoretical model offered by Pratt and Noss (2002). The main purpose of the model is to describe how probabilistic knowledge evolves and what part the structuring resources play in this process. The model consists of five elements:

1. A description of the nature of naïve mathematical knowledge
2. A setting that encourages students to make conjectures and provide tools with which they can express ideas and test conjectures.
3. A detailed elaboration of the nature of new knowledge and its relation during evolution to prior knowledge.
4. An understanding of the relation between new knowledge and the setting in which that knowledge is constructed
5. The degree of dependency of context

In the present study we will limit ourselves to looking a bit closer at items 1 and 4, focusing on *naïve strategies* used by the students (item 1) and how the setting affects their reasoning during the meaning-making process (item 4). Before we formulate precise research questions we give a brief description of the software involved.

THE ICT-ENVIRONMENT

The software *Flexitree* simulates one or more marbles rolling downwards on a board with a system of channels (several different boards are available, see Figure 1). Several of the systems are ICT-versions of the devices used by Fischbein et al. (1975) in an earlier study. Where the marbles encounter a crossroad and the continuation is decided by a random mechanism. In the initial setup the probability for going left (and right) is 50 %, but the student can use the *magnet-device* to alter this probability. Included in the ICT-environment are a frequency-table, a diagram and a simulation-tool (where up to 100 000 marbles can be simulated to be released) (see Figure 2). The ICT-environment offers 9 different setups (see figure 1).

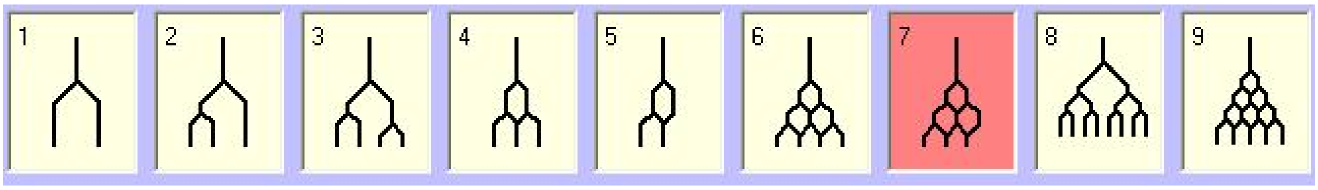

Figure 1. A picture of the nine setups in Flexitree

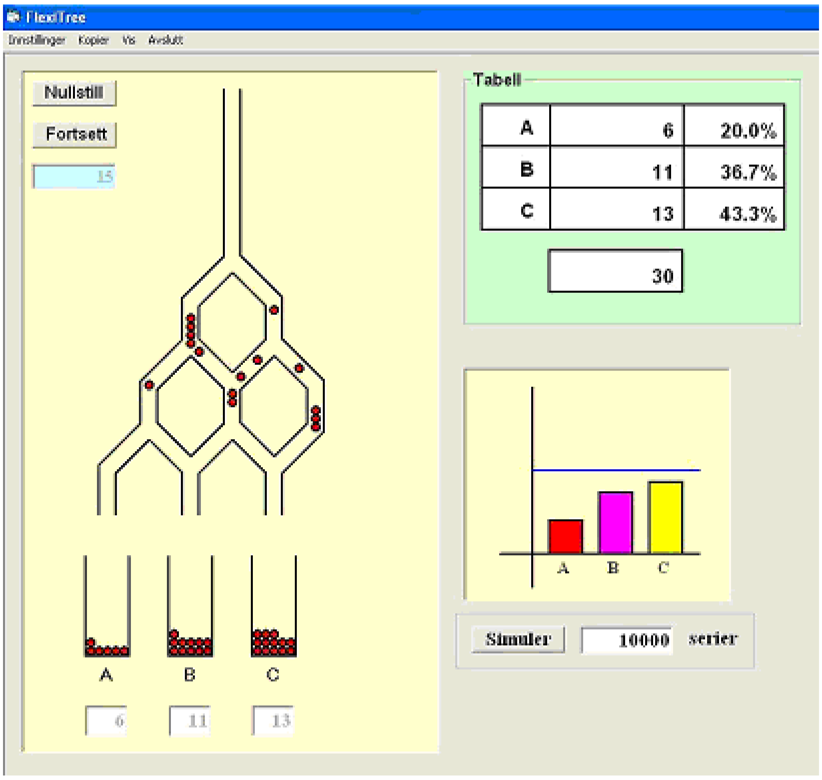

Figure 2. A picture of Flexitree

RESEARCH SETTING AND RESEARCH QUESTIONS

Students were working in pairs with access to the software Flexitree. A protocol was designed that involved the following stages:

Stage 1) The students were given a written test individually that involve several of the situations in the ICT-environment (se figure 2).

Stage 2) Pairs of students discussed the tasks, still without access to the ICT-environment, making conjectures and arguments for their view on how to solve the tasks.

Stage 3) In their further exploration of the tasks the students now have access to Flexitree. In this stage the magnet-device was not introduced to the students so that the probability for going left (and right) was 50 % in each crossroad.

Stage 4) In this stage the magnet-device was introduced to the students. Some new tasks were given to the students in connection with this ICT-tool.

In the current paper we limit ourselves to looking a bit closer at the students' meaning-making process in stage 2, 3 and 4. The following research questions were formulated:

- What naïve strategies do the students use to solve the probabilistic tasks?
- What role does the ICT-environment play in students' meaning-making process?

METHOD

To respond to our research questions we needed in-dept data, which ruled out standard written test and structural interview (Ginsburg, 1981). To be able to answer our research questions we wanted to shape a didactical situation where several different factors could be of importance in the students learning process:

- Discussions with fellow students
- Input from the teacher
- Influence from the ICT-environment.

On this basis we therefore see this research setting as a clinical interview situation (Pratt, 2000). To respond to this, pairs of students were working together (except in stage 1) with and without ICT-tools. Both video- and audio-recording were used. The role of the teacher (researcher) was as a participant observer, attempting not to interrupt the students unless they seemed to be stuck or to get them to express (more clearly) their reasoning in connection with the tasks. In our analysis of the situation we see the role of the ICT-environment as twofold (Noss and Hoyles, 1996):

- For the researcher: As a window on students' reasoning.
- For the students: As a window on abstract mathematical knowledge.

RESULTS

From the data we identify two dominant naïve strategies which students apply when attempting to solve the tasks given to them. We refer to these strategies as the *path counting strategy* and the *division strategy.* The idea of the path counting strategy is to count the number of possible ways leading to a box. Assuming that all paths are equally likely, the probability for a marble to end in a given box will then equal the fraction:

$$\frac{\text{number of paths leading to the box}}{\text{total number of paths in the setup}}$$

To open up a discussion of how the students' thinking evolves in the pedagogic setting, we consider a short negotiation between Per and Pernille. After entering stage 2 they quickly agree on their interpretation (and answers) regarding setup 1 to 3, but when it came to setup 4 Per initially suggested that the three boxes had an equal chance.

Pernille: I have another suggestion. There are two paths leading to B.

Per: Oh, yes.

Pernille: But there‘s only one path leading to A and C.

Per: Mm, that's true

Pernille: They are more likely to end up in box B

Per: Yes. I did not think of that.

So in the beginning of the discussion it seemed as if Per didn’t have any clear strategy (or it appears that he have the equiprobabilistic misconception), but after a short negotiation with Pernille he adopts the path counting strategy. Later in the exploration of other (more complex) setups (e.g. setup 9), both students seem quite comfortable in using this strategy. This strategy works well when both directions have equal chances in the junctions (50-50), and the trees are symmetrical. In system 5 and 7, however, this is not the situation. The lower branch on one side of the tree is truncated, leading the marbles into the neighbouring box. In setup 7 (figure 2), the path counting strategy gives one possible path leading to box A, and three paths to each of the boxes B and C. In the beginning of setup 7 the two students agreed in the probability distribution {1/7, 3/7, 3/7} for the three boxes {A, B, C}. (In other word, they use the path counting strategy). In stage 3 the ICT-tool was used to verify the results from stage 2. A cognitive conflict arose when the results from the simulation did not correspond with their presumptions. We enter at the beginning of setup 7 (stage 3):

Per: What have you got on setup 7?

Pernille: Three seventh

Per: How much is that in percentage?

Pernille: (Calculating) 14.28 and 42.85

Per: OK (simulates 100 00 marbles, 7 sec pause, simulates 100 000 more, 8 sec pause) 42.75? I think we are wrong. I have simulated 200 000.

Pernille: Why does C have a higher chance than B? I would say that B and C were equal.

Per: Yes, because there are three (paths?) on each.

(The students count paths again and discuss several possibilities)

The students showed a high degree of confidence in the outcome from the simulations, and believed that they had made a mistake. As a consequence of the simulation result, the students now try to find new explanations:

Per: Could it have anything to do with this? (Pointing at the “missing branch”)? There is just one possible way here (pointing at missing branch at the right-most path leading to box C), while there is two possibilities here (pointing to the crossroad to A and B)?

Pernille: Yes, certainly, the chances for B are reduced because they can go here (pointing at the path to box A)

Per:	Here you have fifty-fifty, and there you have fifty-fifty, but here you have hundred on C. Then it becomes …12.5, and 37.5 and 50

In the last part the students seem to leave the path counting strategy and rather opt for the division strategy, where the probability of ending up in a box is based on repeated use of division.

Another situation that shows how the ICT-environment influences the students' thinking is when two students, Harald and Ingri, are working with setup 2 with magnet device (shown in Figure 3). The grey squares symbolize "magnets", which can change the probabilities for the marbles to go right or left in a crossroad. The magnets can be tuned to give the wanted probabilities in percents.

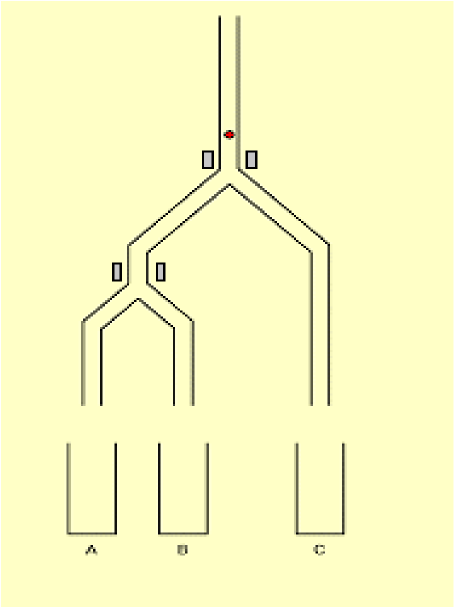

Figure 3

The task was to figure out at which percentage the magnets should be tuned, to give equal probabilities for ending up in all three boxes. Ingri and Harald agree that the probability for ending up in box A and B should be 25%, and the each. The discussion continues:

Harald:	This one (he points at the upper magnet) should be set to give 25% chance to the right direction
Ingri:	Yes
Harald:	We don't have to adjust this one. (pointing at the magnet down to the left) It is set to 50% initially.
Ingri:	Yes, we set 25% here, and we have 50% here.
Harald:	This was simple. This should be right.
	(the simulation of 10 000 marbles gives the distribution 37.5%, 37.5% and 25% for box A, B and C, respectively)

After some reflection Harald suddenly utter: "but we can't have 25%, we must have 33%", so the result from the simulation had an important impact on the students' interpretation of the situation, and made the student to reconsider their solution.

DISCUSSION

The aim of this study has been to investigate students' probabilistic reasoning, and how it is influenced by the introduction of an ICT-environment.

We found that the students base their calculations on some naïve strategies. The strategies used are dependant on the task. In stage 1-3, all the marbles have 50% chance to go either way in the junctions. Here the path counting strategy is preferred by a majority of the students. The other dominating strategy, the division strategy, involves division of probability where the roads split, and addition where the roads meet in a junction. We do not find any strategy based on the product law, as we could expect, used by the students in this study. An explanation could be that the participating students had a rather weak background in mathematics. None of the students had studied mathematics after the first course at college. Another possible explanation could be that the students had a quite physically interpretation of the situation, thinking of how the marbles distributed down through the system instead of calculating probabilities by use of a mathematical law.

The other result from this study is how the ICT environment influenced on the students' thinking. We have seen two examples where the ICT simulation tool had a crucial importance in the students attempt to make meaning of the stochastic situations.

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LEARNING TO PROVE WITH HEURISTIC WORKED-OUT EXAMPLES[1]

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In this paper, we discuss heuristic worked-out examples as a tool for learning argumentation and proof. This learning environment is based on traditional worked-out examples that turned out to be efficient for the learning of algorithmic problem solving. The basic idea of heuristic worked-out examples is to make explicit different phases in the process of performing a proof. The results of an intervention study with 243 students from grade 8 are presented. They suggest that this learning environment is more effective than the ordinary instruction on mathematical proof.

PROOF COMPETENCIES IN THE MATHEMATICS CLASSROOM

It is an important aspect of mathematics education to foster students' abilities of reasoning correctly and arguing coherently (NCTM, 2000). However, many students have difficulties to learn mathematical reasoning, argumentation, and proof. International studies on mathematics achievement and specific studies on students' proof competencies revealed that proving is a demanding mathematical activity. Some studies provide evidence that there is a significant influence of the specific classroom. Accordingly, some teachers may be regarded as more successful in teaching mathematical proof than others. The findings suggest that it might be possible to define learning environments for the mathematics classroom that are apt to enhance students' proving competencies.

The ability to argue in a mathematically correct way and to generate a proof asks for certain prerequisites, including the knowledge of mathematical concepts and heuristic strategies, their application in a problem situation, the use of metacognitive control strategies, as well as an adequate understanding of the nature of proof in mathematics (Schoenfeld, 1983). Several empirical studies from different countries and cultures indicate that many students lack one or more of these facets of proof competence (Healy & Hoyles, 1998; Hoyles & Healy, 1999; Lin, 2000; Reiss, Hellmich, & Reiss, 2002). Our own research in Germany reveals important differences between students with respect to their level of competence. In a study with 8th grade students, we focused on their proving competence. They were asked to solve items demanding basic knowledge (level I), simple argumentation (level II), and more complex argumentation (level III). The results gave evidence that low-achieving students were hardly able to deal successfully with items on level III whereas high-achieving

[1] This research was funded by the "Deutsche Forschungsgemeinschaft" (German research council) within the priority program „Bildungsqualität von Schule" (RE 1247/4).

students performed well on level I and level II items and satisfactorily on level III tasks (e.g. Reiss, Hellmich, & Reiss, 2002). Furthermore, the results of our study indicate significant differences in achievement between the classrooms. A multilevel analysis revealed that the mean pre-test scores on the classroom level had greater influence on the individual post-test achievement than the individual pre-test scores (Heinze, Reiss, & Rudolph, 2005).

Some studies suggest that students approach a proof task by exploration. They search for empirical evidence and using case-based reasoning, which might encompass adequate ideas for a proof. However, most students have difficulties bridging the gap between empirical reasoning and a valid mathematical deductive reasoning (e.g. Hoyles & Healy, 1999). They probably lack strategies for identifying mathematical arguments that support their empirical ideas and for generating mathematical evidence. This is probably due to an inadequate understanding of the nature of mathematical proof. According to Harel and Sowder (1998) even university students may lack an adequate scheme for mathematical proofs.

LEARNING FROM WORKED-OUT EXAMPLES

In recent years, worked-out examples have received increasing attention from cognitive psychologists as well as from educational psychologists. Many studies give evidence that learning from worked-out examples is superior to the "ordinary" instruction, particularly in well-structured domains such as mathematics (for an overview see Sweller, van Merriënboer, & Paas, 1998). A worked-out example consists of a problem, its solution steps and a solution. Accordingly, learning from worked-out examples means to understand how a solution is generated and why it will work. In this respect they differ from examples or problems, which are usually presented in the mathematics classroom and which are to be solved by the students.

In the regular German mathematics classroom, examples generally initiate a problem-based learning: Students solve the problem and should simultaneously understand the process and the scheme of the solution. Since the problem solving process requires a large amount of the working memory capacity, there are few resources left for the process of understanding and the acquisition of generalizable problem solving schemata. Worked-out examples are already solved problems and might therefore foster the adequate use of a student's cognitive resources. Moreover, there might be positive learning effects of worked-out examples because learners prefer to rely on examples rather than on text information.

However, using worked-out examples in the mathematics classroom cannot guarantee effective learning. For example, Renkl (2002) stressed that the quality of self-explanations significantly contributed to the learning success. Based on empirical results he focused on fostering self-explanation activities during the learning with worked-out examples by instructional explanations.

LEARNING TO PROVE BY HEURISTIC WORKED-OUT EXAMPLES

The idea of heuristic worked-out examples

Through their own work, mathematicians know the difference between the proving process and the proof as an outcome of this process. The process of proving may include various approaches that will (or will not) lead to a correct proof. However, the final proof will hardly reflect the process of generation. In order to differentiate between process and outcome of proving, Boero (1999) describes an expert model of this process. This model distinguishes different phases and gives insight into the combination of explorative empirical-inductive and hypothetical-deductive steps during the generation of a proof. The first phase is (1) the production of a conjecture. This includes the exploration of a problem leading to a conjecture as well as the identification of arguments to support its evidence. (2) The precise formulation of the statement defines the second phase. It aims at providing a precisely formulated conjecture as a basis for all further activities. The third phase combines (3) the exploration of the conjecture, the identification of mathematical arguments for its validation, and the generation of a rough proof idea. The first three phases are part of the "private work" since exploration might lead to errors or at least to preliminary formulations within the proof. Only the following last three phases are subject to public communication. They include (4) the selection and combination of coherent arguments in a deductive chain, (5) the organization of these arguments according to mathematical standards, and sometimes (6) the proposal of a formal proof.

This expert model indicates that a mathematical proof as solution of a proof task gives only an incomplete representation of activities performed during the proving process. Consequently, a worked-out example consisting of a problem formulation and its (perfect) solution will not reflect the solution process but simply display the product. For this problem Reiss and Renkl (2002) introduced the idea of "heuristic worked-out examples", which combine Schoenfeld's results of on the teaching of heuristics for problem-solving, and the concept of worked-out examples. Schoenfeld (1983) investigated experts' thinking processes during problem solving and found out that they used various heuristic methods. Moreover, experts were able to manage these heuristics properly. Schoenfeld (1983) taught students some of the heuristics and showed them how they ought to be applied in different kinds of mathematical problems. This approach, namely making heuristics explicit, was used by Reiss and Renkl (2002) in order to design heuristic worked-out examples that did not provide simply the final solution steps, but heuristic strategies that guided the problem solving process and lead to the final solution.

A learning environment based on heuristic worked-out examples

Heuristic worked-out examples for mathematical proofs are based on Boero's model of the proving process. Each worked-out example consists of a collection of work sheets, which offer a mathematical problem and a heuristic solution process to the students. In their work, the students alternate between phases of guided exploration and more reproductive phases. One important aspect is that students are asked to identify arguments leading to a solution and to combine these arguments in order to get a coherent proof. Moreover, this proof is presented in detail in the worked-out example (Reiss & Renkl, 2002).

For the implementation of heuristic worked-out examples into the mathematics classroom, we developed a teaching unit on proof in elementary geometry (grade 7/8). The main aspects of the development may be described as follows:

- Each heuristic worked-out example is structured according to Boero’s model of the proving process. Students are supposed to explore a problem situation, to formulate a conjecture, to explore the conjecture, to identify appropriate mathematical arguments, and to generate a rough idea of the proof. Moreover, they will select coherent arguments, combine them in a deductive chain, and organize these arguments into a proof.
- Heuristic worked-out examples are embedded into different stories. In each example two or three (hypothetical) students encounter a problem situation they wish to solve. Hence, the learner can follow the proving activities of the protagonists, which are accompanied and structured by explicit explanations from a meta-perspective.
- Every worked-out example provides important geometry knowledge, which might be useful in the specific context. Thus, the students may concentrate on the proving process rather than on the recapitulation of facts.
- Students are encouraged to perform self-explanation activities by working with integrated exercises and short texts with blanks. The students are asked to make drawings, to measure angles and sides of geometrical figures, to give their own conjectures, to complete statements, and to look back at the end of the proving process. Moreover, there are explanations that should help the students to understand the structure of the heuristic process.

Summarizing these aspects, heuristic worked-out examples combine characteristics of traditional worked-out examples with aspects of the heuristic proving process. Heuristic worked-out examples provide scaffolding and might on the other hand encourage students to perform their own mathematical activities.

RESEARCH QUESTIONS

In this study we investigated to what extent learning mathematical proof can be fostered by the implementation of heuristic worked-out examples in the mathematical classroom. The following research questions were addressed:

1. Do heuristic worked-out examples influence students’ proof competencies? In particular, are there positive effects on students’ argumentation competencies?
2. Are there significant differences between students learning by heuristic worked-out examples and students participating in regular mathematics instruction?
3. Will low-achieving and high-achieving students have the same success when learning by heuristic worked-out examples?

SAMPLE, METHOD, AND INSTRUMENTS

The sample comprised 243 grade-8 students from Germany. They were assigned to an experimental group (150 students in six classrooms) and a control group (93 students in four classrooms) depending on their results in a pre-test on reasoning and proof and a questionnaire on interest and motivation with respect to mathematics (both administered at the end of grade 7).

All students took part in a regular teaching unit on geometrical reasoning and proof at the beginning of grade 8. At the end of this unit, the experimental group worked with heuristic worked-out examples whereas the students of the control group received regular instruction on reasoning and proof. Students of the experimental group were asked to work individually with three heuristic worked-out examples for one hour each. During this work they were encouraged to discuss their problems with other students. After finishing their work, the teacher discussed the proof and the proving process presented in the example with the students. Working on each heuristic worked-out example took about 75 minutes plus the time for homework. Shortly after, all students took part in a post-test on reasoning and proof in geometry.

All tests were used in former studies (e.g. Reiss, Hellmich, & Reiss, 2002). In particular, the mathematics pre-test and post-test could be scaled unidimensionally in one latent dimension by the Rasch model, if the items were rated in a dichotomous way as correct or incorrect.

RESULTS

Pre-test results

The pre-test results support former findings (Reiss, Hellmich & Reiss, 2002). The pre-test on reasoning and proof in geometry had an overall mean of M = 60.9 % (SD = 15.8) of the test points. The individual test scores of the sample can be fitted to a normal distribution (Kolmogorov-Smirnov-Z = 0.944, p = 0.335). The test scores differ slightly, but not significantly between experimental and control group (experimental: M = 62.3 %, SD = 15.5, control: M = 58.5%, SD = 16.2, t = 1.86, df = 241, p = 0.64). A more detailed analysis of the pre-test results according to the levels of argumentative competencies (cf. section 1) is given in table 1.

Percentages of test points	Pre-test	Level of competency I	Level of competency II	Level of competency III
Experimental group (N = 150)	62.3	78.3	68.9	33.4
Control group (N = 93)	58.5	70.4	71.1	31.0
Total (N = 243)	60.9	75.3	69.8	32.5

Table 1: Results of the pre-test.

The t-test shows a significant difference between experimental and control group for basic competencies in geometry ($t = 3.46$, $df = 241$, $p < 0.001$). The effect was due to a specific pre-test item unfamiliar to the students of two control group classrooms. However, this difference between control and experimental group may be neglected, as the mathematical content of this item was important neither during treatment nor in the post-test. The results of items asking for argumentative competencies (levels II and III) reveal no significant difference between the groups ($t = -0.52$, $df = 241$, $p = 0.60$ and $t = 0.81$, $df = 241$, $p = 0.42$).

Post-test results

The mean score of the post-test on reasoning and proof was lower than the pre-test mean score ($M = 51.0\%$, $SD = 17.9$). This result was not surprising since the post-test encompassed more items asking for mathematical reasoning. It was closely related to the grade 8 teaching unit. The individual scores of the sample could be fitted a normal distribution (Kolmogorov-Smirnov-Z = 0.909, $p = 0.380$). Comparing the mean post-test scores of the experimental and the control group we found a significant difference between experimental and control group. The experimental group achieved much better results in the post-test than the control group (experimental: $M = 54.2\%$, $SD = 17.1$, control: $M = 45.9\%$, $SD = 18.0$, $t = 3.59$, $df = 241$, $p < 0.001$). The effect size $d = 0.47$ indicates a medium effect. Analyzing the post-test according to the different levels of argumentative competencies shows the results presented in table 2.

Percentages of test points	Post-test	Level of competency I	Level of competency II	Level of competency III
Experimental group (N = 150)	54.2	71.9	61.8	30.8
Control group (N = 93)	45.9	68.1	54.1	17.6
Total (N = 243)	51.0	70.4	58.8	25.7

Table 2: Results of the post-test.

The differences between experimental and control group were significant for the scores of competency level II ($t = 2.38$, $df = 241$, $p = 0.018$, $d = 0.31$) and competency level III ($t = 4.38$, $df = 241$, $p < 0.001$, $d = 0.59$). There are no significant differences for the scores of competency level I ($t = 1.22$, $df = 241$, $p = 0.223$).

In summary, the analysis of the post-test results indicates that the experimental group achieved significant better results than the control group. This is particularly true for the argumentative items on level II and III, i.e. for items related to the competencies in reasoning and proof, whereas there is no significant difference with respect to items requiring basic competencies in geometry.

Further results at a glance

The ten classrooms differed in their pre-test as well as in their post-test results. However, the scores of the different classrooms for the post-test indicate a tendency that all experimental classrooms took advantage from working in the learning environment. This becomes more apparent for items of competency level III. However, this result should not be overestimated. Controlling classroom effects on a statistically reliable basis would have required a larger number of classrooms.

In order to identify possible differences with respect to the learning gains of different achievement groups, the sample was divided into three groups, namely a lower, an average, and an upper achievement group according to the pre-test results. The post-test results show that low-achieving and average-achieving students benefit most from a learning environment based on heuristic worked-out examples (the effect size d is between 0.48 and 0.74). We were able to identify a specific learning gain for argumentative competencies. For high-achieving students we found no significant difference between experimental and control group, though there is a tendency in favor of the experimental group for competency level III items.

DISCUSSION

In this study we investigated to what extent learning to prove can be fostered by the implementation of heuristic worked-out examples in the mathematical classroom. Based on the positive learning effects of “traditional” worked-out examples we expected better post-test results for the experimental group than for the control group. As described in section 6.2 the students of the experimental classrooms obtained significant better results. A detailed analysis of the data revealed that this positive effect was due to a higher achievement of the experimental group for items of competency level II and III. Accordingly, these students were able to increase their performance level for items that required mathematical argumentation. With respect to different achievement groups we identified a major achievement gain for low-achieving and average-achieving students. The results indicate that heuristic worked-out examples might improve students’ achievement on reasoning and proof in the mathematics classroom. Moreover, they suggest that low-achieving and average-achieving students may take particular advantage of this learning environment. On the one hand, heuristic worked-out examples provide a scaffold for learning but on the other hand, they enhance self-explanation activities and foster students’ self-determined learning. It is probably this mixture that is appropriate for initiating robust learning processes. The fact that high-achieving students could not benefit in a similar way from the learning environment might have an explanation in the topics introduced during instruction. The students were assigned to the achievement groups according to their pre-test results. However, distinguished pre-test results are linked to an appropriate understanding of mathematical argumentation and proof. We assume that heuristic worked-out examples emphasize aspects of the proving process those students are already familiar with to some extent. Possibly, the structured learning environment did not activate high-achieving students appropriately and they did not work with the material as motivated and concentrated as other students. It remains to be seen whether more challenging and/or more difficult problems might have a positive effect on high-achieving students. Moreover, it should be investigated whether and to what extent heuristic worked-out examples could be complemented by forms of instruction that provide even more openness in problem solving.

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TEACHER ACTIONS: ENHANCING THE LEARNING OF MENTAL COMPUTATION IN YEAR 2

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This paper examines teacher actions during a teaching experiment aimed at enhancing Year 2 students' mental computational strategies. Specific teaching instruction was conducted by the classroom teacher while the first author acted as participant observer. The teacher was provided with a theoretical background for mental computation and support materials for the development of the instructional program. The lessons were designed to enable students to access a range of representations to build mental models in order to calculate efficiently. The results indicate that these elementary students are not only capable of participating in class discussions on computation methods; they are also able to develop a range of computational methods of increasing sophistication.

INTRODUCTION

In the past, the focus in developing computation was on learning (often by rote) and practising standard written computational methods (Anghileri, 2005). However, the focus today is on developing "mathematical thinking and communication to prepare them for the world of tomorrow" (Anghileri, 2005, p. 2). This has resulted in a change in what we value in mathematics. This change is reflected in many syllabus documents (e.g., NCTM, 1989; QSA, 2004). This change is new for many teachers; it requires a shift in their beliefs and attitudes about content and pedagogy in maths. It is making this change that needs to be supported.

Strategy		Example of addition strategies for 28+35
Counting		28, 29, 30, ... (count on by 1)
Separation	*Right to left (u-1010)*	8+5=13, 20+30=50, 63
	Left to right (1010)	20+30=50, 8+5=13, 63
	Cumulative sum or difference	20+30=50, 50+8=58, 58+5=63
Aggregation	*Right to left (u-N10)*	28+5=33, 33+30=63
	Left to right (N10)	28+30=58, 58+5=63
Wholistic	*Compensation (N10C)*	30+35=65, 65-2=63
	Levelling	30+33=63
Mental image of pen and paper algorithm		Child reports using the written method taught in class, placing numbers under each other, and carrying out the operation, right to left.

Table 1: Mental computation strategies for addition and subtraction (based on Beishuizen, 1993; Cooper et al., 1996)

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Some research suggests that higher achieving students will naturally employ a range of mental computation strategies (Askew et al., 1997; Heirdsfield & Cooper, 2004b; Steinberg, 1985); while below average students will rely on counting procedures (Askew et al, 1997; Heirdsfield & Cooper, 2004a). See Table 1 for an explanation of mental computation strategies. Therefore, in a classroom situation, it is important to cater for all students by providing rich learning environments to enable the development of efficient computational strategies (Fuson, 1992).

THEORETICAL FRAMEWORK

The theoretical perspective adopted utilises the role of mental models in assisting students to construct their understanding of specific mathematical concepts. In this study the selection of appropriate mental models has been essential for the construction of various and sophisticated mental computation strategies. The literature argues that the model/representation chosen must (a) represent the relations and principles of the domain, (b) engage various modalities (e.g., kinaesthetic and visual), and (c) be unambiguous (English, 1997). Teacher actions that support the appropriate use of these models are critical to the process of student construction of understanding. It is argued that (a) the use of concrete materials must directly relate to the mathematical concept being studied, (b) recognise student potential as well as pre-existing constructions, and (c) engage students in active participation (Davis & Maher, 1997).

THE STUDY

This research adopted a case study design in which a teaching experiment was conducted (Lesh & Kelly, 2000). The researcher sought to produce an environment that was supportive and collegial where the teacher and researcher collaboratively planned the instructional program. Mental computation did not feature in the old mathematics syllabus; while the new syllabus (QSA, 2004), which was in draft form at the time of the study, requires a significant shift in beliefs and attitudes about teaching content and pedagogy. Therefore, the students had no previous exposure to mental computation. The instruction was also new to the teacher as the old syllabus was limited to traditional algorithms for addition and subtraction, and the draft syllabus did not indicate what mental computation strategies, beyond number fact strategies, might be appropriate for young children.

Participants

Twenty-one Year 2 children (average age 7 years 6 months) and their teacher participated in this study. They attended a school serving a predominantly middle class community in an outer suburb of Brisbane, Australia.

Data Collection and Analysis

The data collection methods used included *observation* where all lessons were video taped and field notes taken by the first author, students were individually interviewed prior to the instruction and they were again individually interviewed on the completion of the series of lessons. Individual interviews were necessary so as to accurately ascertain the mental computation strategy used by the student. The interview involved stimulus pictures and numerals being presented on card to the child, while the interviewer verbalised the word problem. The students did not use pen and paper. This paper reports on the strategies used for 3 addition and 3 subtractions questions. These questions were a direct reflection of the teaching that occurred during this teaching experiment.

Procedure

An initial consultation with the teacher focused on the aims of the project and the anticipated format. At this meeting background reading was provided to familiarise the teacher with the philosophy and theoretical background of mental computation. Pre-instruction student interviews were then conducted with student base knowledge subsequently outlined to the teacher. This knowledge guided the development of the instructional program. The program of one, half hour lesson per week for eight weeks was then implemented. After each episode, the teacher and researcher reflected on the outcomes to inform the subsequent episode. The implementation phase was followed by post-instruction interviews.

RESULTS AND DISCUSSION

Student Results

The results from the pre and post interviews indicate growing sophistication of the children’s mental computation strategies. In the next section, how teacher actions have supported this change is discussed. Table 2 details the change in strategies from pre to post interviews for six items: 20+30, 26+9, 36+99, 30-10, 46-20, 30-19.

Strategy	20 + 30		26 + 9		36 + 99		30 – 10		46 – 20		30 – 19	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Inappropriate /no strategy	9	1	10	3	11	7	11	4	13	8	17	9
Counting	5	1	10	7	7	2	2	1	4		2	3
Separation												
Right to Left			1	1								1
Left to Right	7	19		2	1		8	16	4	13	2	5
Aggregation												
Right to Left												
Left to Right				1								
Wholistic												
Compensation				6	3	11						3
Levelling				1		1						

Table 2: Frequency of Mental Computation Strategies used for Addition and Subtraction (n=21)

Teacher Actions

Over the course of the eight-week intervention a range of models, namely, hundred board, bundling sticks, number ladder, and number line (with intervals of 10 marked and labelled, and internals only marked), were used to support student learning. The models supported the pedagogy adopted by the teacher where she used three main approaches: (a) direct questioning where specific students were required to explain their computation strategy, (b) general questions where several students were encouraged to make a contribution to the discussion on solution methods, and (c) a technique that combined the questioning and modelling, allowing her to share her own thinking process and model that process with the representation used.

In the first lesson the hundred board was utilised to identify the position of a particular number. It was then used to identify the pattern in numbers and how counting in tens starting at any position can be achieved. Each student had their own hundred board and a counter to mark the starting position. When locating a number on the hundred board the teacher looked for different methods to enforce the notion that there is no one correct way.

> Teacher: Everyone put their counter on twenty-six. Well done. How did you find twenty-six Lachlan?
>
> Lachlan: I put my finger down the line.
>
> Teacher: So what were you looking for?
>
> Lachlan: The twenties and then I looked along six.
>
> Teacher: Did anyone find twenty-six a different way?
>
> Helen: Yes, I looked for six and went down to twenty-six.

When the teacher directed her students' attention to the patterns in the hundred board she again looked for multiple methods.

> Teacher: We counted on ten from nine and we got to nineteen. Let's count on ten more. Where will that take us? Look for the pattern. Let's start at nine.
>
> Whole Class: 19, 29 … 99.
>
> Teacher: What is happening with this pattern?
>
> Mark: They are all in the same row. (Student means column)
>
> Mary: They all end in nine.
>
> Jane: They are all counting in tens.
>
> Teacher: Yes, all good answers. Well done.

The second lesson employed the same questioning techniques of lesson one as the students continued to familiarise themselves with the hundred board and the inherent patterns therein.

Teacher: Put your marker on the number ten more than twenty-four. Mark, how did you find ten more than twenty-four?

Mark: I just went straight under twenty-four.

Teacher: Why did you go under twenty-four?

Mark: Because that is the same as counting on ten.

When students used language inappropriately, the teacher engaged them in a discussion, such as the following.

Teacher: Jean, how many have I added if I go from six to thirty-six?

Jean: Three

Teacher: Jean thinks she has added on three. Who agrees? (No hands go up) What have I added on then if I haven't added on three?

Tom: Three lots of ten.

Teacher: How else can we say that?

Mary: Thirty.

Teacher: Who agrees it is thirty? (Hands go up)

The teacher went on to discuss the difference between three and three leaps of ten.

In the second lesson the students were also introduced to adding and subtracting 9. In this instance the teacher demonstrated using the large hundred board and demonstrated her own thinking.

Teacher: I am on ten but I only want to jump forward nine spaces. Who can think of a really fast way to do that?

Bret: Go diagonally. (See below for further discussion)

Teacher: Does anyone have another way?

Sue: I counted in three's.

Teacher: You were very clever to do that Sue. Now I am going to show you my way. I could add on ten and that will get me to twenty but I only want to add on nine so I just go to the number before twenty and that is nineteen.

The students quickly adopted this method as one child, Nick, demonstrated.

Teacher: This time I want you to add on nineteen to seventeen.

Nick: Thirty-six.

Teacher: What did you do Nick?

Nick: I went down and then down and then back one.

Teacher: What does down and down mean?

Nick: Adding on two tens – which is twenty. Then you go back one.

Teacher: Bret, can you go diagonally when you add on nineteen?

Bret: No.

The teacher had picked up on Bret's earlier contribution of moving diagonally on the hundreds board. However greater discussion on why this is not always a successful method was not pursued and would have been beneficial.

The number ladder was introduced in the third lesson to reinforce counting forward and backward in 10s; and as an intermediate step to using a number line. A number ladder was drawn on the board and students engaged in several counting games where they rolled a die which resulted in either 10, 20 or 30 indicating the counting on or back by that amount.

The second model used in this lesson and the next was bundling sticks. The students were to make a number using bundles of 10 for example, 60. Discussion ensued about the different combination of 10 that could be used. Some students held 5 lots of ten in one hand and one in another, some held 3 lots of ten in one hand and 3 in another, and so forth. In the following lesson the students used bundling sticks to count on or count back in 10s by physically moving the bundles of ten across the desk. However, many of the children saw little connection between the bundling sticks and counting in 10s. Bundling sticks were abandoned after this lesson.

In lesson four the number line with graduations of ten was introduced to develop further counting in tens. To familiarise the students with the number line, the teacher engaged them in a discussion on what numbers could be on the number line if counting in tens.

Teacher: What numbers would I find on my number line?

Jim: Ten.

Bret: Thirty.

Mary: Twenty.

Teacher: So if I were counting in tens what number would be here? (Pointing at a position on the number line drawn on the board.)

Luke: Fifty.

Teacher: Yes.

The students then used their own number lines and drew in leaps of 10, 20 or 30 forwards or backwards on the number line. The use of the number line was revised in lessons 5 and 6 to include the graduated number line so that all whole numbers between the tens could be identified. During these lessons the teacher again orientated her students to the graduated number line by asking direct questions.

Teacher: Between which two, tens would I find thirty-four?

Helen: Thirty and forty.

Teacher: OK, how do we then find thirty-four, Lily?

Lily: Go to thirty and just count on four.

Teacher: How would you find sixty-eight?

Lily: I would go between sixty and seventy and count on eight.

Teacher: Look at the number board. Is sixty-eight closer to sixty or seventy?

Lily: Seventy.

Teacher: So how do we get to sixty-eight?

Tom: Count back two.

Teacher: Each time I want you to think about which ten the number is closer to. For example; is it closer to seventy or eighty if the number is seventy-three?

Bob: Seventy.

The technical aspects of using the number line to locate any number were concentrated on during the final two lessons of the series. The location of a number on the number line resulted in a discussion on who agrees with the result and why. This explanation was an essential part of the discussion. Students had become willing to contribute realising that there were many ways of describing how an answer was achieved, for instance, by talking about the hundred board or taking leaps on a number line. Similar discussions to earlier lessons continued. Students again counted on and back in 10s, 20s and 30s starting from any number identified by the teacher.

CONCLUSIONS

From this study, it was evident that young children are able to engage in mathematical discussions where different solution methods are acceptable and the community of learners co-constructs understanding of mental computation. This is a critical finding as an ability to engage in such discussions lays the foundation for these children to think and work mathematically.

In this study, the children's mental computation strategies developed from less sophisticated to more sophisticated methods as indicated in Table 2. The literature (e.g., Askew et al., 1997; Murphy, 2004) argues that assisting the average to below average children to make connections with the mathematics they know will lead them to a greater depth of understanding resulting in a wider repertoire of solution methods. This study supports this finding. We agree with Goos et al. (1999) that by giving students access to the discussions of the above average students allows them to participate in a community of practice where they have 'reflective inner dialogue' (p. 59) where the results of this dialogue can be seen in the change in strategy used.

Students were introduced to a variety of models to assist them in developing efficient mental computation strategies. The ability to map between these models is of significant pedagogical consequence and a very effective teacher action as the use of one representation might support other representations aiding the construction of understanding.

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EARLY CONCEPTUAL THINKING

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A pupil's mathematical development is aimed at a procedural rather than a conceptual style of thinking. Both types are characterised and we illustrate the consequences which neglecting conceptual thinking can bring. We describe a fairy tale context, which enables us to investigate conceptual thinking, its diagnosis and development of pupils of Grade one. Action and clinical research was carried out and some mental phenomena describing the thinking processes of pupils in the given context were found.

INTRODUCTION

Many researchers claim that often pupils' mathematical knowledge is only mechanical. Beginning in the elementary school, calculative skills for the basic four numerical operations, that is a procedural understanding of mathematics, are emphasised. The following two examples illustrate the lack of conceptual thinking even of students interested in mathematics and the low level of their ability to see a set of mutually connected objects in a mathematical problem.

Illustration 1. In the entrance examination to the Faculty of Mathematics and Physics, forty-three applicants solved the equation: $(x + \sqrt{2})^2 + 2 = 3(x + \sqrt{2})$ (*)

All the solvers squared the expression in brackets on the left and multiplied the brackets on the right and then solved it in a lengthy way and with mistakes. None of them used substitution to provide a quick solution (Hoch, Dreyfus, 2005).

Illustration 2. More than 50 students from the same faculty, future secondary mathematics teachers, solved the following problem in a test: Find $|(3 + 2i)/(3-2i)|$. Only one student wrote the answer immediately and he explained it using a fraction $\sqrt{13}/\sqrt{13} = 1$. The others used a standard procedure: to multiply the numerator and denominator by $3 + 2i$. They got the expression $|(5 + 12i)/13|$ and finally the result 1.

Other researchers have reported similar experiences at university level. For instance, Schonfeld (1985) points to a number of procedures used in integral calculus and adds: "For obvious reasons, this particular strategy, trying a series of techniques in a particular order, can result in remarkably inefficient problem-solving performance."

This article characterises the procedural and conceptual approach of a pupil to a problem. We will emphasise the disproportional development of these complementary meta-strategies and show the possibilities of developing the conceptual meta-strategy in a context which can be applied as early as the first grade of the primary school.

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 289-296. Prague: PME.

THEORETICAL FRAMEWORK

First, we will characterise the solving procedures at a procedural meta-strategy level as follows:

1. A solver places the problem into a certain area (a certain topic);
2. He/she activates those procedures in his/her mind, which concern the topic in question;
3. In the problem, which can contain several indications to take action (in equation (*) they are squaring the binomial on the left and multiplying the brackets on the right), he/she decides on the order in which they will be carried out;
4. After the first step, he/she repeats point 3 until the problem is solved or until he/she loses his/her way;
5. Thus, the solver becomes more skillful in problems of the given type.

The characteristics of the solving procedure at a conceptual meta-strategy level is as follows:

1. A solver creates an image in his/her mind about the problem as a whole;
2. He/she analyses it to find its inner structure;
3. He/she looks for the key element or relation in the situation; this concerns an insight into the relationship between given and unknown elements;
4. As soon as the key element or key relation is found, he/she constructs a solving strategy;
5. The above process leads the solver towards a higher level of understanding the situation in question.

In illustration 1, a solver with a procedural style understands the square of the binomial as a challenge for action – squaring. A solver with a conceptual style considers the equation as a whole and notices that the repeated expression ($x + \sqrt{2}$) is the key element of the problem and that it should be taken as an elementary object Deal with the compound term as a single entity and through an appropriate substitution recognises a familiar structure in a more complex form.). Then, he/she substitutes y for the expression ($x + \sqrt{2}$) and this substitution changes equation (*) into the simple equation $y^2 + 2 = 3y$.

In illustration 2, a solver with a procedural style understands the complex number in the denominator of the fraction as a challenge to multiply the denominator and numerator by a conjugate number. A solver with a conceptual style considers the fraction as a whole and notices the relationship between the numerator and denominator (as these are conjugate numbers, they have the same absolute value). This is a key realisation which provides then immediate strategy $|(3 + 2i)|/|(3\text{-}2i)| = 1$.

Nearly all standard tasks which primary and secondary students (and often university students, too) meet are oriented towards a procedural meta-strategy. Even though it is possible to solve them using the conceptual meta-strategy in a quicker and more elegant way, the standard strategy leads to success, too. Often teachers suppress the efforts of some pupils to use conceptual solutions. They claim that any "non-standard” procedure is confusing for the less able students and therefore it is not appropriate to introduce it in mathematics lessons.

Exceptions to the above are word problems which must be first considered as a whole or combinatorial problems or geometric constructions. These problems are traditionally considered as the most difficult.

DESCRIPTION OF THE CONTEXT

Father Woodland is a fairy tale figure who looks after different animals and organises tug-of-war games. The weakest animal is a mouse (M). Two mice are as strong as one cat (C). A cat and a mouse are as strong as a goose (G). A goose and a mouse are as strong as a dog (D). Other animals are introduced in a similar way, too, however, they will not be considered here. Each animal is represented by both a picture and an icon (see fig. 1 – the picture was drawn by D. Raunerova).

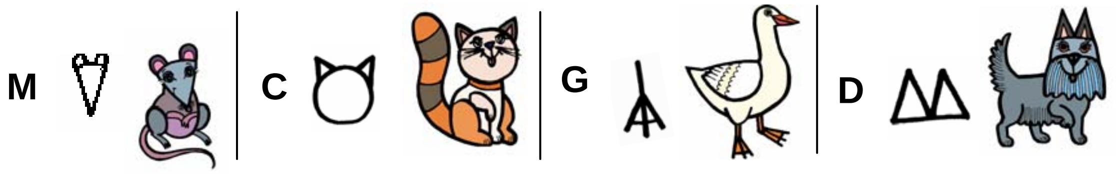

Fig. 1

Tug-of-war games take place on a playground which consists of two circles, one red and one blue. A group of animals go to each circle and they start pulling at a rope which lies between the circles. The task is to (a) decide which group is the stronger and (b) add some animals to the weaker group so that the two groups are equally strong. A situation in which there are two cats and one goose in the red circle and a dog and a mouse in the blue circle will be symbolised here as: {CCG} ~ {DM}. Pupils got these problems in an iconic way (see fig. 2).

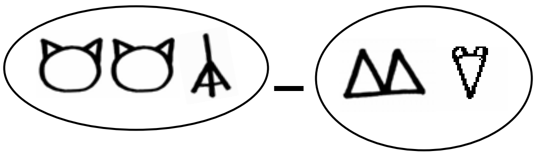

Fig. 2

In our experiments and teaching experiments at grades 3 to 5, we also used more difficult problems. For instance, the tug-of-war game can take place at the time of a carnival when some of the animals wear masks. These are marked by X and Y. In these problems, both red and blue groups are equally strong. For example, there is a cat and two animals marked X in the red group and two dogs in the blue group. The situation is symbolically depicted as {CXX} = {DD}. We can formulate demanding problems such as a system of equations with two unknowns: {CXX} = {DD}, {XY} = {D} or a system {YXX} = {DM} and {XYY} = {DG}. The problems were solved in a manipulative way using counters, in some cases the pupils drew the situation.

In this context, it is possible to formulate very demanding problems such as Diophantine equations and to build a preconcept of the lowest common multiple and greatest common devisor. However these and other types of problems will not be considered in this contribution.

Note: Let us add that already in 1942, Polish mathematician Karol Borsuk created a desk game (focused on stochastic thinking) in which he used animals as the bearers of quantities (Hoffmann, 2002).

METHODOLOGY

Our first experiments in which numbers were represented by semantic objects were realised in 1972. We began the systematic research of the conceptual thinking of a child/student in the 1990s, namely with the concept of infinity (Jirotková, 1997) and the concept of triad (Kratochvílová, 2002). The research in the context of Father Woodland started in September 2005. Eight semi-structured interviews were carried out with three Grade 1 pupils, three Grade 2 pupils and two Grade 3 pupils. The experiments were video-recorded and the pupils' graphic records were collected. At the same time, three cooperating teachers implemented the context in Grade 1 class.

Gray & Tall's theory of procept was used for the analysis of experiments (Gray & Tall, 1994), especially the role of signs in the building of procepts. We also implemented APOS theory (Dubinsky et al., 2005). However, the main tool for our analysis of the concept creation process was the theory of generic models (Hejný, 1988; Hejný, Kratochvílová, 2005).

EXPERIMENTS AND ANALYSES

Because of the page limit of this contribution, we will only include fragments of our research. We will focus on two areas: objects and relationships.

Objects and their three representations

Each object (e.g. M) is given to a child in three different languages: verbal ("mouse") and two visual ones (iconic and pictorial). The child works in all three languages and moreover, he/she can use the available signs or can create others her/himself. If he/she creates the signs, it can be an exact copy or a modification.

We noted three interesting phenomena in the given three languages:

The first was that our assumption that the iconic language will be incomprehensible to some children was not correct. All children (from experiments and from experimental classes) accepted the icons spontaneously.

The second phenomenon concerns motivation. Children interested in art tried to draw the icons or pictures. When asked to record in a graphic way {D} = {GM}, Tyna (age 7:4) chose pictorial language (fig. 3). This picture is very good for a seven year old. The picture of a dog is markedly different from the artist's picture of dog and the picture of mouse also lacks a noticeably triangular face. These differences points to the child's creativity and the motivational power of the context. On the other hand, children who are motivated cognitively considered drawing icons or

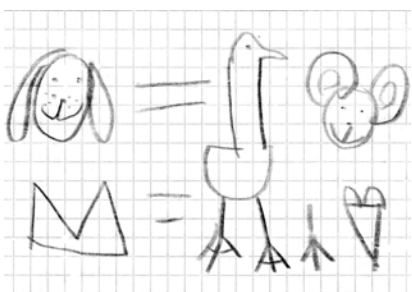

Fig. 3

pictures to be unnecessary. They felt that they kept them from solving problems which they found attractive. If asked to depict a situation in an iconic way, they tried to economize (see Victor's icon in fig. 4). From the point of view of mathematics, the second type of child seems to be more advanced. However, we feel such assessment to be one-sided if we regard the child's personality as a whole. Children interested in art devote more energy to drawing pictures than to a cognitive activity but on the other hand, this fact draws them to this, strictly speaking, cognitive context.

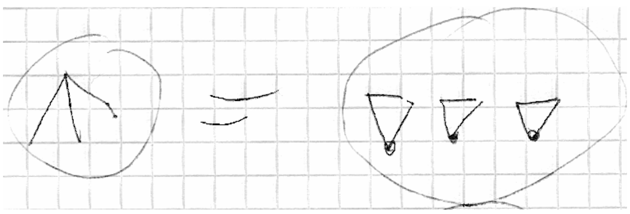

Fig. 4

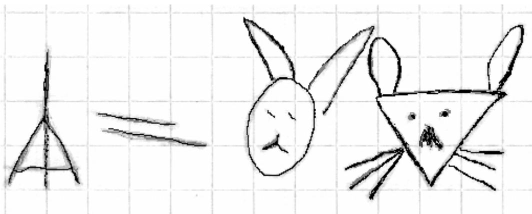

Fig. 5

The third phenomenon concerns the mingling of the two visual languages. Aysa (age 8) in fig. 5 drew the equality {G} = {CM}. There is an icon on the left and pictures rather than icons on the right. It is probable that the icon of the goose is not complemented by its picture because she was not able to draw it. But the important point is that the heterogeneity of the languages is not a cognitive obstacle for Aysa. An analogical phenomenon appears in mathematical thinking in many areas. For example, if a Grade 6 pupil writes 0,25 + 0,25 = ½, on the one hand we can see that he/she understands the relation well as the record contains both decimal numbers and a fraction. On the other hand, the heterogeneity of the languages gives a certain drawback to the record. If the same thing was recorded as 0,25 + 0,25 = 0,5 = ½, then it would be clear that the first equality points to the arithmetic thinking inside the context of decimal numbers and the second equality interprets the decimal number in the language of fractions. This approach can be seen in fig. 3, where in the upper line, the equality is given in the language of pictures and in the bottom line (to which the legs of the goose from the upper line stretch), the equality is given in icons. Again, we do not dare to say that any of these ways is better than the other. The polarity of heterogeneous/homogeneous languages will be studied further.

Relationships

Some children are not able to accept the association between an animal and quantity. They like the pictures of animals and the icons and the correspondence picture ↔ icon, but they understand the fact that a cat is as strong as two mice only for the given cat and the given two mice we are speaking about. If we change the mouse for another one, the situation can change for them.

Some children are able to accept the association but they are unable to use the causal manipulation (e.g. if the equality {G} ~ {CM} is violated by adding M on the right hand side, they do not know that the equality can be reestablished by adding M on the left hand side). Another example of such thinking is given below.

Fragment 1. Jan (Grade 1, 7 years) built the equality {C} = {MM}. The experimenter moved the icons to the upper part of the playground and asked Jan to create the same equality below the first. Jan made it. Next, the experimenter "added" cats on the left, he put them together. She asked Jan what would happen to the mice. Jan put the mice together for a while and then took one away. Thus the "equality" {CC} = {MMM} was arrived at.

Ex: Is it correct now?
Jan: Yes.
Ex.: And if we separate the cats again, what will happen to the mice? (He separeted the cats.)

Without speaking, Jan took the removed icon of mouse he had removed and created the original situation.

If the child accepted the conventional assigning of quantity to individual animals at the beginning of the interview, in several cases there was a change in his/her image: the convention was replaced by school experience (fragment 2) or life experience (fragment 3).

Fragment 2. Ayse (age 8). She was a little tired after 15 minutes. She had been solving the situation {C} ~ {MMM}.

Ex: Who was stronger now?
Ay: Those three mice.
.....
Ex: So what would you do so that they are equally strong?
Ay: Well. (She adds 2C to the left.) Now they are equally strong. Because there are three there and here too.

Comment. Ayse stopped considering the strength of the animals. She can only see number 3 behind groups CCC and MMM. It is possible that the change of the context was caused by the word "three" which was said. The cause may also be the formulation of the experimenter's question which reminded her of school. In textbooks the objects are nearly always counted in pieces. (In our experiments with coins, a Grade 2 pupil said that for five 1-CZK coins one could buy more than with two 5-CZK coins (the coins were on the desk).)

Fragment 3. When the experimenter challenged the school context for Ayse, she changed it into the school problem.

Ay: Two mice... three mice ... Well, two mice are ... (she took them again) ... not strong enough. If we add one big fat mouse...

Comment. The initial inequality was not understood by Ayse as the task "to fill" but as a challenge "to think". The conventional strength of animals receded and the life experience prevailed which said that not all mice were equally strong. In the language of APOS, there was no encapsulation of the concept of "mouse – bearer of conventional quantity". In the language of procept, we can say that the procept of the concept in question was not created yet. In the language of generic models, it can be said that the child still understands the objects as isolated models, the generic model of the concept of "mouse" was not yet created. The same phenomenon was recorded by the cooperating teachers. They felt that a large number of pupils changed their image about the quantity assigned to the animals.

A small number of pupils did not change their image at all. They immediately understood the idea of the game (fragment 4) and structured the set of animals (fragment 5). Some even assigned numbers to the structure of animals, that it they saw the isomorphism: M ↔ 1, C ↔ 2, G ↔ 3 etc. (fragment 6).

Fragment 4. Victor (age 8) was a very bright boy. The equalities {C} = {MM} and {G} = {CM} had been introduced to him.

Ex: ... you know that the goose is as strong as ...
Vi: ... as three mice (he is drawing fig. 4)
Ex: or the goose is as strong as ...
Vi: ... a cat and a mouse. (he is drawing)
Ex: and the dog? ...
Vi: We do not know it yet how strong it is.

Comment: Victor immediately understood that by axiom, each animal is assigned some strength. A concisely drawn picture showed the clear orientation of the child towards the cognitive area.

Fragment 5. As soon as Victor learnt the strength of the dog, he placed it into the structure.

Vi: D is as strong as G and M. It is so far the strongest. He was drawing an iconic record of the relationship.

Fragment 6. Victor solved an easy task and then he was given the task {GM} ~ {CMM}.

Vi: A cat has two mice, that means that there are four mice, here as well, so they are equally strong, ... no they aren't, ... they are, they are.
Ex: Why?
Vi: Because there is one mouse and you can divide the goose into three mice and you will get four mice. We will divide the cat into two mice and two mice and two mice are four mice, so it is the same. So they are equal.

Comment. Victor immediately transfers the situation from the language of animals to the language of numbers and solves it there. He is prepared to solve more demanding problems.

DISCUSSIONS

The above context of Father Woodland is a suitable way to investigate the conceptual thinking in pupils not only at the elementary level. It will be interesting to follow the development of the ability to think conceptually in these observed pupils.

The Father Woodland context is also a diagnostic tool enabling us to characterise both cognitive and meta-cognitive styles of pupils. This is confirmed by the cooperating teachers who claim that the tasks on animals revealed marked differences in the quality of thinking of individual pupils. In cooperation with them we have been elaborating educational techniques which can help children who have problems in this area. It mainly concerns the building of understanding of the difference between a quantity (expressed in units) and a number (expressed in pieces). The techniques make use of the context of coins, lengths, volumes and weighs and utilise drama and manipulation.

More demanding tasks are used for older pupils, mainly Diophantine equations formulated in the language of the Father Woodland context.

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DIVERSITY IN THE CONSTRUCTION OF A GROUP'S SHARED KNOWLEDGE

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We describe and analyze episodes taken from a long-term research project, whose main goal is to investigate the constructing and consolidating of knowledge in elementary probability. Specifically, we follow the constructing and consolidating of "shared knowledge" by a group of three students in one of the project classrooms. The RBC model is used as the main methodological tool. We found that the group constructed shared common basis of knowledge, which enable them to continue the constructing of a new knowledge. We also found that this knowledge flows from one student to the other, where many times each partner has her own way of constructing knowledge.

INTRODUCTION

The relationships between individual students' knowledge and what might be called the "shared knowledge" of the ensemble is a fascinating issue, both from cognitive and socio-cultural points of views. We consider ensemble in the sense of Granot (1998) as "the smallest group of individuals who directly interact with one another during developmental processes related to a specific context" (p. 42).

However, the researcher who plans to observe and analyze in detail processes of *constructing* knowledge in an ensemble, in a given context, and along a time segment, in which some learning occurs, will face great difficulties: The observation and documentation processes are complicated, data are usually heavy and there is no systematic clear-cut methodology for analyzing them. The individuals' diversity of constructing knowledge within the ensemble is an additional crucial aspect, which makes it hard to define the shared knowledge of the ensemble. All these difficulties grow as the number of the ensemble's participants becomes larger.

Many researchers are aware of the difficulty individual diversity presents for defining shared knowledge. For example, Cobb and his colleagues analyzed the collective learning of a classroom community in terms of the evolution of classroom mathematical practices (Cobb, Stephan, McClain, & Gravemeijer, 2001). For this purpose, they felt the need to coordinate "a social perspective on communal practices with a psychological perspective on individual students' diverse ways of reasoning as they participate in those practices" (p.113). They discussed the notion of *taken as shared activities* of the students in the same classroom, where taken as shared learning is such an activity. The following is their explanation for using the above term:

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 297-304. Prague: PME.

> "We speak of normative activities being taken as shared rather than shared, to leave room for the diversity in individual students' ways of participating in these activities. The assertion that a particular activity is taken as shared makes no deterministic claims about the reasoning of the participating students, least of all that their reasoning is identical." (p. 119).

The main goal of the present paper is to follow and investigate processes of *constructing and consolidating of "shared knowledge"* within an ensemble of students learning together. The data form part of the corpus of data of a long-term research project, whose goal is to investigate the constructing and consolidating of knowledge in elementary probability. Because of the detail needed in order to understand and interpret these processes, we chose to focus on a group of three students in a classroom. Since we are aware to the potential diversity of constructing knowledge processes within the group of three, we will relate to the individuals' processes of constructing knowledge concerning the learned issue, and to the interactions between individuals, and the flow of knowledge from one student to the other. Than we will emphasize the "group's shared knowledge", which is the group's common basis of knowledge within these processes. This common basis allows the three students to continue to work together during further learning activity, in which the consolidation of the shared knowledge might be evidenced. Thus the research focuses on the constructing processes as well as on the constructs at a given point of time, and also on their consolidation, whereby personal diversity and the unique flavor of each individual is observed and analyzed.

We grouped the relevant data in narratives, taken from the activities of various groups from different schools, but all on tasks belonging to the same sequence designed for learning elementary probability. The flow, in which the "shared knowledge" is constructed out of the individuals' knowledge, shows many variations. Some of these are exemplified in one narrative, which we present here.

THE RBC MODEL

The RBC model will be used as the main methodological tool for describing and analyzing the constructing of shared knowledge and its consolidation (Hershkowitz, Schwarz, & Dreyfus (HSD), 2001; Dreyfus, Hershkowitz, & Schwarz; (DHS), 2001). The RBC model is a theoretical and practical model for the cognitive analysis of abstraction in mathematics learning. This model suggests constructing as the central process of mathematical abstraction. Processes of new knowledge construction are expressed in the model through three observable and identifiable epistemic actions, *Recognizing, Building-with, and Constructing* (whence RBC). Constructing of new knowledge is largely based on *vertical re-organizing of existing knowledge constructs* in order to create a new knowledge construct. Recognizing takes place when the learner recognizes that a specific knowledge construct is relevant to the problem s/he is dealing with. Building-with, is an action comprising the combination of recognized knowledge elements, in order to achieve a localized goal, such as the actualization of a strategy or a justification or the solution of a problem. The actions of recognizing and building-with are often nested within the action of constructing.

Moreover, constructing actions are at times nested within more complex constructing action. Therefore the model is called the "nested epistemic actions model of abstraction in context", or simply the "RBC-model". A more detailed discussion of the RBC model may be found in the 2 papers above, in which two case studies of students in laboratory settings were analyzed and led the researchers to initiate the elaboration of the model. After starting with an interview with a single student in the first paper (HSD, 2001), the researchers turned to the observation of dyads working in collaboration in the second (DHS, 2001). In this second case study, the construction of knowledge of the dyad and the construction of a new construct of knowledge of each individual in the dyad were investigated by analyzing interactions between the two students. Interaction was investigated in detail as a main contextual factor determining the process of abstraction. From this point of view, the present article is a continuation of the DHS paper. Since then, The RBC model has been validated and its usefulness for describing processes of abstraction of other contents, and in a variety of contexts has been established by a considerable number of research studies by our group as well as by others (e.g., Bikner-Ahsbahs, 2004; Dreyfus & Kidron, in press; Williams, 2006).

Later studies investigated the consolidation of the new knowledge constructs. Consolidation is expected to occur in learning activities that follow the one in which the new knowledge construct first emerged. Evidence for consolidation might be found in the epistemic actions in these following learning activities. And indeed research showed that the RBC model can be extended to processes of abstraction and its consolidation on a medium term time-scale (Dreyfus & Tsamir, 2004; Tabach, Hershkowitz & Schwarz, in press; Monaghan & Ozmantar, in press).

THE RESEARCH PROJECT IN THE CONTEXT OF PROBABILITY

In our current research project, the focus is on students' learning during sequences of activities with a high potential for constructing and consolidating. The "RBC model" was expanded to the "RBC+C" model, where the second C stands for Consolidation.

It was decided to focus in this project on the basic concepts of probability, for several reasons:

- Probability is part of the 8th grade curriculum; the research thus inserts itself naturally in the activity of the school year and contributes to the learning in the experimental classes.
- The topic of probability has relatively little interaction with other topics.
- Intuition plays meaningful roles in probabilistic thinking, because probability offers many exciting connections to daily life (e.g., Falk, Falk and Levin, 1980; Konold, 1989). Moreover intuition might lead to wrong conclusions and hence to surprises and conflicts (Kahneman and Tversky, 1972). Students' initial knowledge is thus undifferentiated, as described by Davydov (1972/1990) and may become articulated and abstracted in adequate activities.

- The hierarchal structure of probability makes it possible to design a sequence of tasks that offers opportunities for constructing of a set of concepts and processes and their consolidation.

A unit consisting of a carefully designed sequence of activities for learning elementary probability has been developed and used with pairs of students as well as in classrooms. The unit includes:

- (i) A written pretest to be answered individually;
- (ii) Five activities (about ten lessons), constructed as sequences of problem situations for group investigations, for whole class discussions, and for (mostly individual) homework assignments;
- (iii) Three post-tests: a written post-test, an interview, and a game/interview, all to be carried out individually.

THE STUDY

Five different teachers taught the unit in eighth grade classrooms, in four different schools. The regular classroom work included group work, whole class discussions led by the teacher, teacher demonstrations, homework discussion, and tests. In each class, a focus group of two or three students was chosen by the researchers and the teacher. The choice criteria were average ability and good verbalization.

In each lesson one or two researchers were present, and documented the lesson by means of two video cameras. One camera focused on the focus group along all the unit's lessons, and the second camera focused on the teacher and the activity of the class as a whole. The researchers also took field notes and collected students' written work.

Having access to a group's work over all ten lessons of the unit allowed us to focus on consolidating processes in addition to constructing of knowledge processes. For the purpose of analysis, narratives concerning the construction and consolidation of knowledge were chosen for groups working in different classrooms but on the same task sequence. Particular attention was paid to the social interactions (group, student-teacher) and frameworks (whole class, small groups, individuals) within which the epistemic actions occurred. Because of space limitations, we will here present and analyze only one narrative from one group.

CONSTRUCTING 2D SAMPLE SPACE – PRINCIPLES AND TASKS

The unit deals with the overall construct of Sample Space, and is organized in three hierarchical stages:

- I. Sample Space in one dimension (1d SS). A simple event in such a sample space is, for example, to obtain 3 when throwing a die.
- II. Sample Space in two dimensions (2d SS), for cases where the possible simple events in each dimension are equi-probable; in such cases, the 2d simple events (expressed as pairs) can be counted and organized in a table,

and the probabilities of complex events can be counted or calculated from the table. A simple event in such a sample space is, for example, to obtain 3 and tails, when throwing a die and a coin.

- III. Sample Space in two dimensions, for cases where there might be a few possible simple events in each dimension, which are not necessarily equi-probable, but whose probabilities are explicitly given; in such cases, the 2d simple events can be organized, in an area diagram, from which the probabilities of more complex events can be calculated.

The data in this paper will be taken from students' activity in stage II of the learning unit and mainly concern one epistemic principle of 2d SS, which we call principle E1: *A simple event in 2d SS consists of a pair of simple events, one in each dimension*. Example: The possible outcomes on each of two dice create pairs of numbers as simple events. In 2d SS, constructing E1 is a necessary condition for constructing other principles, for example principles concerned with the collection of all possible simple events (E2) and with the relevant events for a particular problem situation (E3).

The following probability tasks from Activity 3 (Q1 & Q2) were used in this study:

Activity 3, Q1:

1a Yossi and Ruthie throw two white dice. They decide that Ruthie wins if the numbers of points on the two dice are equal, and Yossi wins if the numbers are different.
Do you think that the game is fair? Explain!

1b The rule of the game is changed. Yossi wins if the dice show consecutive numbers.
Do you think the game is fair now?

Activity 3, Q2:

2 We again throw 2 regular dice. This time we observe the difference between the bigger number of dots and the smaller number of dots on the two dice. (If the numbers on the two dice are equal, the difference is 0.)

Make a hypothesis whether all differences have equal probability. Explain!

It is important to note that activity 3 is the first one in the unit, which deals with 2d sample space, and hence Q1 and Q2 were the first time the students in the study dealt with 2d sample space. (Activities 1 and 2 deal with 1d sample space.) Even more importantly, our epistemic analysis showed that there is no way to deal with these questions without constructing E1.

CONSTRUCTING E1 TOGETHER

In this narrative, the discourse among three girls, Yael, Rachel and Noam, shows how shared knowledge concerning E1 is constructed. The three girls start by discussing Q1a. Yael counts pairs of numbers and reaches 27 pairs for Yossi and Ruthie together.

21 Noam What are you doing?

22	Yael	I did: What are the chances… (1,1) (2,2) (3,3) (4,4) (5,5) (6,6) and (1,1) (1,2) (1,3) (1,4) and (2,1) (2,2).

It seems that Yael is already busy in counting all the relevant pairs for Ruthie's and Yossi's chances to win (E3), and wants to know what are all possible events (E2). We don't know when and how Yael has constructed the meaning of a simple event in 2d Sample Space (E1). In any case, she seems to recognize E1 right from the beginning of her work on Q1, while trying to build-with it further constructs.

Rachel and Noam haven't yet constructed the meaning of E1, of simple events as pairs of numbers; thus they don't understand what, how, and why Yael is counting. Yael does not realize that her friends are not aware that events in 2d SS are represented by pairs (23-27):

23	Rachel	I don't understand what you are doing?
24	Yael	It is because I have to know what is our whole, like: What are all the possible outcomes that might be, and all these outcomes are either (1,1) (2,2) (3,3) (4,4) (5,5) (6,6) or (1,2) (1,3) (1,4) (1,5) (1,6) and then 2 [meaning that now she has to count the combinations of 2 with the other numbers].
25	Noam	Yael, I don't understand anything of what you are doing.
26	Rachel	Nor am I.
27	Yael	Listen, there are some possibilities that 1 will appear: (1,2) (1,3) (1,4) (1,5) (1,6) and we finished with 1, now 2: (2,3) (2,4) (2,5) (2,6).
28	Rachel	O.K., O.K. we understood that, but why are you adding? I don't understand.

Although it is not evident from Rachel's utterances, that she has already constructed E1, Rachel may already share with Yael that one has to count pairs, because she only asks about Yael's conclusion concerning the number of events in the sample space, and not about the nature of simple events in 2d SS. More convincing evidence for Rachel's construction of E1 follows (Rachel 64).

Along the above discussion, Noam expresses objection without any evidence for understanding. She is still in the process of constructing E1, and manages to complete it by confronting her friends with her misunderstanding:

58	Noam	Look you don't… you did as if one side of the die is 3 and the second side is 4 and you did 3 plus 4 and it is as if…
59	Yael	I didn't do 3 plus 4. I will tell you exactly what I did…
60	Noam	No, one second, second. That's what I understood of what you did.
61	Yael	I will explain…

But Noam now wants to explain herself:

62	Noam	One minute! No! You have to do 3 and 4 it is one possibility, and 4 and 5 is a second possibility, so it is two [possibilities].
64	Rachel	That is what she did; (3,4) is one possibility and (5,4) is one possibility.

Noam now appears to have constructed E1 (62), but she still does not represent it as pairs. Rachel (64) provides additional evidence for her own constructing of E1. (It is the first time for Rachel to speak in terms of pairs.).

Additional evidence for the fact that Noam has constructed E1 is that later, while the three girls' work on Q2, Noam explains what is the meaning of the "differences of outcomes" on the two dice:

109	Noam	… as if we look at the difference of one die and the other die; (2,2) then the difference is 0.
110	Rachel	And if we have 1 and 5, then the difference is 4.

We may see here, that after a while, in a later activity (Q2), Noam as well as Rachel recognize the pair nature and its formal representation for simple events of the 2d SS (E1) and use it for building-with it the explanation for the meaning of the differences (Noam 109 and Rachel 110). Thus both of them gave evidence for consolidating E1.

CONCLUDING REMARKS

Constructing and consolidating: Each girl's individual knowledge of E1 was constructed and seems to be consolidated. Yael showed from the very beginning of the group common work on Q1a, that she has E1 construct. But, the questions of Noam and Rachel, the explanations of Yael, and the self-explanations of Noam and Rachel, in the course of constructing this knowledge, have a crucial role. Examples:

1. The insistent questions of Noam and Rachel, led Yael to repeat her counting pairs. While organizing the counting, Yael is producing evidence for her consolidating of the E1 principle (21-27), while counting all possible events and relevant simple events (E2 & E3).
2. Noam puts the blame of her mistakes on Yael (58), and then accepts Yael's refutation (60). This leads her to explain in her own words that she has to relate to pairs (62).
3. Noam and Rachel (109, 110) provided evidence for their consolidation of E1 when using E1 for explaining the meaning of the differences.

In the analyses of the above narrative we exemplified some of the effects the three girls had on each other's constructing and consolidating of knowledge.

“Shared knowledge”: As we exemplified at the data above, the three girls now constructed and consolidated E1 and used it in a new task (Q2). Thus E1 is the “shared knowledge” which enable them to continue working together.

We also showed how this knowledge flowed from one girl to the other, where many times each partner has her own way of constructing knowledge, which evolves from a different need, at a different point of time, and also the construct of each individual might varied from one individual to the second, (for example, Noam’s informal representation of pairs).

In short, at this point in time principle E1 appears to be a shared common basis of knowledge for the group, and the group may continue the constructing of a new knowledge and/or may show the consolidation of this knowledge in follow up situations.

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STRUCTURE SENSE VERSUS MANIPULATION SKILLS: AN UNEXPECTED RESULT

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This paper presents a refined definition of structure sense and some of the results of a questionnaire given to 165 advanced level mathematics high school students with the aim of measuring their structure sense. It was expected that these students would display a high level of manipulation skills and a lower level of structure sense. The results did not correspond to this expectation.

Students who have previously displayed proficiency at using algebraic techniques often have difficulty in applying these techniques in unfamiliar contexts. It must be emphasized that the students under discussion are not those who are generally recognised as poor learners or even just weak at mathematics. On the contrary, they are the higher ability students who are learning mathematics at an intermediate or advanced level in Israeli high schools. Yet despite their previous excellent grades in mathematics many of these students prove to be poor at applying their algebraic knowledge. For example we have observed that some students display a difficulty in taking the first algebraic steps necessary for the solution of the equation $3\cos^2 x - 2\cos x = 1$, despite the fact that they can easily solve $3x^2 - 2x = 1$. These equations possess the same quadratic structure, but this is not apparent to all students. We attribute difficulties of this kind to a lack of structure sense.

Linchevski and Livneh (1999) first used the term structure sense when describing students' difficulties with using knowledge of arithmetic structures at the early stages of learning algebra. Hoch (2003) suggested that structure sense is a collection of abilities, separate from manipulative ability, which enables students to make better use of previously learned algebraic techniques. This was illustrated by presenting students' various attempts to prove: $\frac{k(k+1)(k+2)}{3} + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$. Hoch & Dreyfus (2004) described algebraic structure as it applies to high school algebra and gave a tentative definition of structure sense. The presence of brackets was found to help students see structure. Hoch & Dreyfus (2005) developed the definition of structure sense further and examined how it could be used to explain students' problems with factoring. Pierce and Stacey (2001, 2002) called a similar collection of abilities algebraic expectation - recognition of conventions and basic properties; identification of structure and key features; ability to link representations.

We refer to ability to apply procedural knowledge as manipulation skills. The manipulation skills under discussion here are techniques for factoring expressions and for solving equations. These techniques are taught traditionally with most of the

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 305-312. Prague: PME.

emphasis on the procedures ("how to") and little if any emphasis on the concepts ("what" and "why" and "when"). This is an example of encouraging instrumental as opposed to relational thinking. Skemp (1976) described instrumental understanding as mathematical usage of rules when solving problems without necessarily knowing why the rule is valid (rules without reason) and relational understanding as the ability of deriving rules, interpreting and possibly proving, to see them as rules in a net of concepts (knowing both what to do and why). Students seem to be satisfied with understanding instrumentally, whereas many teachers want them to understand relationally. It is difficult to assess whether a person understands relationally or instrumentally. Barnard & Tall (2001) called the process of learning to carry out a solution procedure by rote "procedural compression". Bannerjee & Subramaniam (2005) showed preliminary evidence in favour of a structure-oriented approach strengthening both procedural knowledge and structural understanding of arithmetic expressions. We now present our refined operational definitions of manipulation skills and structure sense, and then the results of an empirical study to measure structure sense.

DEFINITIONS

A student is said to display manipulation skills (MS) if s/he can:

- Solve an equation or factor an expression when given explicit instructions.
- Substitute correctly in a given formula.

A student is said to display structure sense (SS) for high school algebra if s/he can:

- Recognise a familiar structure in its simplest form. (SS1)
- Deal with a compound term as a single entity and through an appropriate substitution recognise a familiar structure in a more complex form:
    - where the compound term contains a product or power but no sum. (SS2a)
    - where the compound term contains a sum and possibly also a product or power. (SS2b)
- Choose appropriate manipulations to make best use of a structure:
    - where the structure is in its simplest form. (SS3)
    - where the compound term contains a product or power but no sum. (SS3a)
    - where the compound term contains a sum and possibly also a product or power.

In Table 1 we present examples to illustrate the different parts of these definitions. These examples, based on an algebraic structure - the difference of two squares - that students meet in high school, appear in a questionnaire designed to measure structure sense, to be described later.

	Ability	Example	Comments
MS	Factor expression, explicit instructions	Factor $36-y^2$, given the formula $a^2-b^2=(a-b)(a+b)$	No recognition of structure
SS1	Recognise structure in simplest form	Factor $81-x^2$	No formula given
SS2a	Deal with compound term (product or power) as single entity, recognise structure in complex form	Factor x^4-y^4	Deal with x^2 and y^2 as single entities
SS2b	Deal with compound term (sum) as single entity, recognise structure in complex form	Factor $(x-3)^4-(x+3)^4$	Deal with $(x-3)^2$ and $(x+3)^2$ as single entities
SS3	Choose appropriate manipulations to make best use of structure in simplest form	Calculate 1001^2-999^2 without using a calculator (Not in questionnaire)	Recognise the structure and the advantage of factoring
SS3a	Choose appropriate manipulations to make best use of structure(compound term contains product or power)	Factor $24x^6y^4-150z^8$	Extract common factor 6 and deal with $2x^3y^2$ and $5z^4$ as single entities
SS3b	Choose appropriate manipulations to make best use of structure(compound term contains sum)	Prove that $(x+y)^4=(x-y)^4+8xy(x^2+y^2)$	Subtract $(x-y)^4$ from both sides of equation; deal with $(x+y)^2$; $(x-y)^2$ as single entities

Table 1: Examples to illustrate the definitions

METHODOLOGY

A group of 176 students completed a questionnaire containing twelve items. These students were from seven 10th grade classes, learning mathematics in the advanced stream. The questionnaire was administered during the second half of the school year. The first four items of the questionnaire were considered to be qualifying items. Only students who answered at least three out of the four qualifying items were included in the final sample. Eleven students were thus disqualified. The answers of the remaining 165 students to the remaining eight items were examined and used to construct a score for manipulation skills (MS) and a score for structure sense (SS).

QUESTIONNAIRE

We designed a questionnaire to measure structure sense and manipulation skills among students studying mathematics at intermediate to advanced level from the second half of tenth grade onwards. In fact a total of twenty-four different items were divided into two questionnaires, each containing twelve items. Four algebraic structures were examined - $a^2 - b^2$, $ab + ac + ad$, $a^2 - 2ab + b^2$ and $ax^2 + bx + c = 0$ - each at six different "levels" - i.e. requiring MS, SS1, SS2a, SS2b, SS3a or SS3b.

The first four items in each questionnaire – the qualifying items –were one from each structure, two requiring only manipulation skills, two requiring SS1. The remaining sixteen items - eight in each questionnaire, divided up in as balanced a manner as possible, - were four items from each structure, and within each structure were one item requiring SS2a, one requiring SS2b, one requiring SS3a and one requiring SS3b.

For example – this item: "Solve for x: $(2+3x)^4 - 12 = 4 + 12x + 9x^2$" possesses the structure $ax^2 + bx + c = 0$ and requires structure sense SS3b. One way to solve it is to factor the right hand side to get $(2+3x)^4 - 12 = (2+3x)^2$ and then recognise the structure – a quadratic equation in $(2+3x)^2$. An alternative approach is to multiply out $(2+3x)^2$ on the left hand side to get $(4+12x+9x^2)^2 - 12 = 4 + 12x + 9x^2$ and then recognise the structure – a quadratic equation in $4 + 12x + 9x^2$. A student who uses either of the above methods is considered to be displaying structure sense SS3b. A student who solves this equation correctly, or with only one minor error, is considered also to be displaying manipulation skills. A student who multiplies out to get $84x + 207x^2 + 216x^3 + 81x^4 = 0$ is considered to be displaying manipulation skills but not structure sense.

RESULTS

Each answer was coded according to which structure sense was used, and how accurate the calculations were. Let us look for example at students' answers to one of the items in the questionnaire.

Factor: $(2x+3)^2 - 12(2x+3) + 36$.

This question was given to 88 students. Eight of them left it blank. Thirty-five students opened brackets and did not factor. The four among them who did so correctly (or with one minor error) were coded as displaying manipulation skills but no structure sense. The other 31 students made computational mistakes, and thus were coded as displaying neither structure sense nor manipulation skills.

Twenty-two students opened brackets and then factored. [$4x^2 - 12x + 9 = (2x-3)^2$] Here there is recognition of the structure $a^2 - 2ab + b^2$, dealing with the compound term $2x$ as a single variable. The 20 students, among the 22, who factored correctly, were coded as displaying manipulation skills in addition to SS2a; the remaining two were coded as SS2a without MS.

Twenty-three students factored directly. [$(2x+3-6)^2 = (2x-3)^2$] In this case there is recognition of the structure $a^2 - 2ab + b^2$, dealing with the compound term $2x+3$ as a single variable. The 15 students, among the 23, who factored correctly, were coded as displaying manipulation skills in addition to SS2b; the remaining eight were coded as SS2b without MS.

Each student was given a structure sense score corresponding to the "highest level" of structure sense he displayed. (Although the types of structure sense are not considered to be hierarchical, the questionnaire was designed in such a way that it would be extremely unlikely, for example, for a student to display SS3b but not SS2b. In fact, the results showed that no student did so.) Each student was given a manipulation skills score according to the number of questions he manipulated correctly (with or without structure sense) or with only one minor error.

Table 2 summarises the categorisation of the 165 students into nine groups according to high, medium or low manipulation skills (MS) and high, medium or low structure sense (SS).

The high structure sense group contains the students who solved at least one question using SS3a or SS3b. The medium structure sense group contains the students who solved at least one question using SS2a or SS2b, but did not solve any questions using SS3a or SS3b. The low structure sense group contains the students who did not solve any questions using SS2a, SS2b, SS3a or SS3b.

The high manipulation skills group contains students who solved at least 5 out of 8 questions with no more than a minor error (with or without using structure sense). The medium manipulation skills group contains students who solved 3 or 4 out of 8 questions with no more than a minor error. The low manipulation skills group contains students who solved less than 3 out of 8 questions with no more than a minor error.

	HIGH SS	**MEDIUM SS**	**LOW SS**	*Total*
HIGH MS	**5.4**	**6.1**	**0**	**11.5**
MEDIUM MS	**11.5**	**19.4**	**0**	**30.9**
LOW MS	**17**	**27.9**	**12.7**	**57.6**
Total	**33.9**	**53.4**	**12.7**	**100**

Table 2: Percentage of students in each MS / SS category. (N=165) MS = Manipulation skills. SS = Structure sense.

DISCUSSION OF RESULTS

Referring to Table 2 we see that approximately a third of the students have high structure sense, half have medium structure sense and a sixth have low structure sense. This is more or less what we might have expected from high achieving students. However the picture we get when we look at manipulation skills is somewhat different. Only a sixth of the students have high manipulation skills, and this in spite of the fact that one needed major mistakes in at least 4 out of the 8 items in order to be classified as less than high. A third have medium manipulation skills and half have low manipulation skills. The distribution of manipulation skills does not seem to vary between the different levels of structure sense except in the lowest group where low manipulation skills seems to go hand in hand with low structure sense.

These results surprised us. The majority of these high achieving students did not manage to do more than half of the exercises accurately (and remember, we did not demand complete accuracy – minor errors were allowed!). This is just about the opposite of what we had expected. Our experience had taught us that these students reached the advanced stream by having earlier proved their mastery of sets of rules for transforming algebraic expressions and for solving equations. This mastery would have been displayed in the type of written exams which include a series of exercises similar to those recently solved in class, to be solved by working mechanically, according to the rules. These exams test the how and not on the why, and thus may encourage instrumental as opposed to relational thinking. We expected these students to maintain a high level of instrumental proficiency (manipulation skills), and that any difficulties they displayed would be on a relational level (structure sense). Yet clearly the students are not performing at a high instrumental level.

Referring to the example presented above, we see that half of the students used structure sense. Approximately 80% of them did so accurately. However this leaves the rather daunting result that only 35 out of 88 advanced stream tenth grade students succeeded in factoring the apparently simple expression $(2x+3)^2-12(2x+3)+36$. How can this be explained?

Barnard & Tall suggest that *“Great success in calculation may be developed with a huge range of connected ideas, some meaningful, some rote-learnt”* (pp. 94-95). We might suspect a lack of connection between our students’ rote-learnt ideas and their meaningful ideas – their structure sense. Our assumption was that students who are adept at using certain algebraic techniques (high MS) experience difficulties with applying these techniques in unfamiliar contexts due to lack of structure sense. Yet now we see that our assumption of high manipulation skills is unfounded. So perhaps these students, who have succeeded so far in mathematics despite their low manipulation skills, have done so thanks to their structure sense. How could we test this? We could administer the questionnaire to intermediate stream students and compare the results. If we found similar levels of manipulation skills and lower levels of structure sense, then we could say that structure sense contributes to success in school mathematics. But if we find even lower levels of manipulation skills, then we have to search for an answer to the question – why do our students have such poor manipulation skills?

Would improving structure sense improve also manipulation skills? We looked at ten items, five from each questionnaire, which according to the student response were the least difficult. Each item was attempted by a different number of students (we discarded blank responses). Of the 382 solutions using structure sense, 22% contained more than one error while of the 283 solutions not using structure sense, 94% contained more than one error. This may be because using structure sense leads to shorter, more efficient solutions, and thus leaves less room for calculation errors.

CONCLUDING REMARKS

Our findings show that the majority of the high achieving mathematics students that we tested do not use a high level of structure sense when solving exercises requiring the use of algebraic techniques. Students who do use structure sense to solve an exercise make fewer mistakes than those who do not. We were surprised to find that most of these high achieving mathematics students display very poor manipulation skills. We are working on a method to develop students' structure sense. We suggest that this might improve their overall performance by reducing the amount of calculations they will need to perform.

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YOUNG CHILDREN DEVELOPING PLACE VALUE UNDERSTANDINGS

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In developing an understanding of place value the skills of reading writing, ordering and interpreting numbers are all important to demonstrate the multi-unit aspect of seeing a bundle of ten as a unit. In a study which interviewed children at the start of their grade 1 year the results show a fifth of the children have all four of those skills for two-digit numbers. The study explores the dependence of these four aspects of place value on each other and finds that there is no clear progression between them, although the children generally found it more difficult to model numbers using bundles of ten than to read, write or order two-digit numbers.

BACKGROUND TO THE STUDY

The fundamental idea of the decimal place value system is treating a group of ten as a single unit. This is used not only in naming and recording but in calculation techniques. Children's learning of number is initially unitary not recognising the importance of this group of ten even though they may know and work with numbers past twenty but they need to develop a multi-unit conceptual structure (Fuson, 1990, 1992; Jones, Thornton, Putt, Hill, Mogill, Rich & van Zoest, 1996). Many authors have proposed frameworks for understanding numeration (Bednarz & Janvier 1982; Boulton-Lewis, 1996; Jones et al., 1996; Payne & Huinker, 1993). These frameworks generally include the aspects of reading a number written in symbolic form, writing a number in symbolic form and interpreting a number in terms of it consisting of tens and ones (and higher groupings) often through demonstration with concrete materials. Some also refer to ordering sets of numbers.

Reading and writing numbers for English speakers can be difficult as the structure of language is not consistent. The teen numbers particularly are not said in the way in which they are written. The language structure is much clearer once numbers past twenty are explored. The language difficulties of the teen numbers are not present for some languages while for others there are further aspects of language which may cause confusion. For example in French the seventies, eighties and nineties do not follow the same pattern as the thirties to sixties. The written symbolic representation is consistent.

The other difficulty with the translation between symbols and the spoken language is the zero place holder. Although number systems have existed for at least five thousand years, and over two thousand years ago people such as the Egyptians and

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 313-320. Prague: PME.

the Greeks were working with systems that had a base of ten and a way of representing very large numbers with symbols, the zero as a place holder was not accepted in Europe until about 1200, less than one thousand years ago (Eves, 1982). It is not surprising that the use of zero as a place holder causes difficulty for some children.

One skill of numeration is the ability to order numbers. Early ordering can be based on a child checking the counting sequence for the number said later. Children are unlikely to use this approach if the numbers are not close together in the sequence. Ordering numbers presented in symbolic form could be done before children have completely understood the verbal form of the numbers. Unless the sets of numbers though are carefully chosen a child can order numbers correctly for reasons not reflecting place value understanding. For example a child may claim 79 is larger than 32 because 9 is the largest number present. This child would have difficulty with 29 and 53.

Interpretation of the numbers, particularly through representing them with concrete materials based on multi-unit structures such as bundles of ten or MAB, thus demonstrating the partitioning of a number such as 36 into three tens and six units is another aspect of place value understanding.

These four aspects of reading, writing, ordering and interpreting numbers formed the basis of the framework used in the Early Numeracy Research Project (ENRP).

THE RESEARCH PROJECT

The research reported here is part of a large project, the ENRP, which focussed on numeracy in the early years of schooling. The project had many aspects including an extensive professional development program and a study of effective teaching (Bobis, Clarke, Clarke, Gould, Thomas, Wright, Young-Loveridge, & Gould, 2005; Clarke, Sullivan, & McDonough, 2002; Horne, & Rowley, 2001; McDonough, & Clarke, 2002).

As part of the project a framework of Growth Points, based on research literature, was developed in nine domains: Counting; Place Value; Addition and Subtraction Strategies; Multiplication and Division Strategies; Time; Length; Mass; Properties of Shape and Visualisation. In each domain the framework hypothesised a learning trajectory along which most students could be expected to proceed. The Growth Points represented large indicators in any domain. The development of this framework has been described elsewhere (Clarke, Sullivan, Cheeseman, & Clarke, 2000; Clarke, Cheeseman, Gervasoni, Gronn, Horne, McDonough, Montgomery, Roche, Sullivan, Clarke, & Rowley, 2002)

The particular domain of interest here is the domain of Place Value and these Growth points are shown in Figure 1. While the titles of the growth points as they were set focus on the number of digits they could more effectively be interpreted in terms of unitary, ten-structured, multi-unit and extended multi-unit concepts.

0. Not apparent
   Not yet able to read, write, interpret and order single digit numbers.
1. Reading, writing, interpreting, and ordering single digit numbers
   Can read, write, interpret and order single digit numbers.
2. Reading, writing, interpreting, and ordering two-digit numbers
   Can read, write, interpret and order two-digit numbers.
3. Reading, writing, interpreting, and ordering three-digit numbers
   Can read, write, interpret and order three-digit numbers.
4. Reading, writing, interpreting, and ordering numbers beyond 1000
   Can read, write, interpret and order numbers beyond 1000.
5. Extending and applying place value knowledge
   Can extend and apply knowledge of place value in solving problems.

Figure 1: Place Value Growth Points.

Based on this framework children in grades 0–2 were assessed in their numeracy understanding using a structured one-on-one interview. The interview has been described as a "choose your own adventure" (Clarke,) as in each domain the interview finished when a child showed a lack of success. For the project each child was assigned a Growth Point in each domain. Figure 2 shows the spread of children across the growth points for all 5569 students who were interviewed at the start of their grade 1 year.

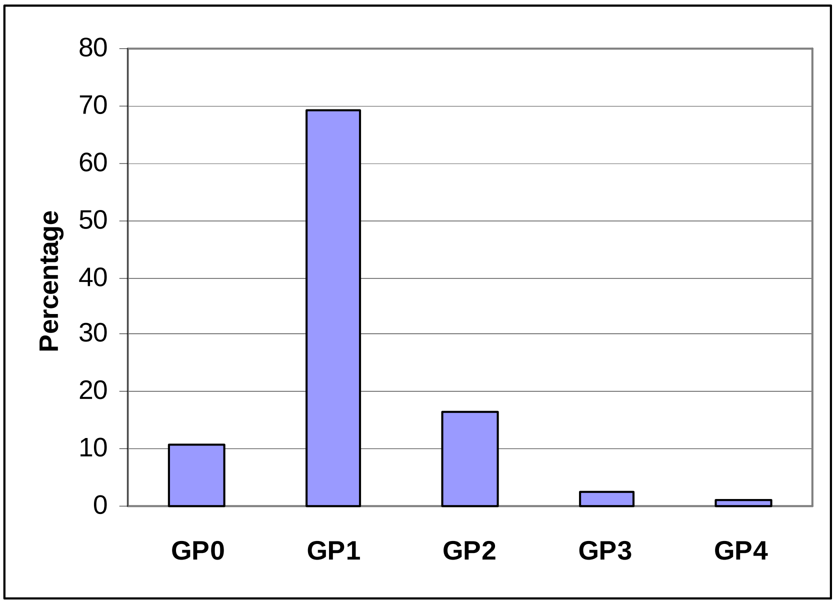

Figure 2: Children achieving Growth Points in the Place Value domain at the start of Grade 1.

From this it is clear that nearly 70% of students at this level have mastered one-digit numbers but have some difficulty with at least one of reading, writing, ordering or interpreting two digit numbers. This raises the question of where the difficulties lie.

There was no data base made of all responses to the questions for the over 13000 students who were interviewed so a random sample of the interview records of 200 students who were just beginning grade 1 (mostly age 6) was drawn from the overall collection of interview scripts.

The skills the particular questions of interest required children to demonstrate were

1. Reading numbers (3, 8, 36, 83, 18, 147, 407, 1847)

2.a Writing numbers on a calculator (7, 47, 60, 15, 724, 105, 2469, 6023)

2.b Reading numbers from a calculator where the digits change place value

3. Ordering a set of numbers

4. Interpreting a 2-digit number using bundling or a group of 10 as an object

5. Interpreting and reading missing numbers from a two-digit chart

6. Interpreting and reading missing numbers from a three-digit chart

7. Interpreting task involving 10 more across a hundreds boundary

8. Interpreting task involving 100 less across a thousands boundary

9. Reading and ordering numbers greater than 10 000

10. Approximately placing numbers of varying sizes on open number lines with a variety of endpoints marked

This section of the interview terminated when children were unsuccessful at tasks involving a particular number of digits. For example if a child was successful in all of the two digit tasks in questions 1-5, read the three digit numbers but made errors in writing the three digit numbers the child was not asked any questions past question 6.

This framework of Growth Points, however, reflects only major indicators of children's understanding, including in each of the first four growth points all of the four aspects of reading, writing, ordering and interpreting numbers. The question arises in what order children develop these understandings. For example do children at grade 1 level generally read numbers before they write them? Or do they order numbers before either reading or writing them. Since the level was grade 1 the analysis focussed specifically on two-digit numbers, reflecting a tens based but not a complete multi- unit structure.

RESULTS AND DISCUSSION

Reading numbers.

Question 1, which required children to read numbers from cards, showed only 1 of the 200 students was unable to read the single digit number but a further 43 (22% overall) were unable to read all three of the two-digit numbers. Generally the unsuccessful students either could read none of the two digit numbers or made errors on the teen number (9%). A further 48% were unable to read both three digit numbers with 28% having difficulty just with the 105. There were 16% of students who were able to read all numbers presented.

Question 2b was a second reading task where there were no zeros or ones used in the numbers but the digits change place value. Each child chose the digits. All successfully read the single digit they had chosen but 27 (14%) were unable to read the two-digit number. A further 47% were unable to read the three-digit number and only 10% could read four digits successfully.

This shows the wide spread of children at the start of Grade 1 with some able to read four-digit numbers and others unable to read two-digit numbers. It also shows the expected difficulties of the teen numbers and the zero which were signalled in the background discussion.

The seeming greater difficulty with reading two-digit numbers from the cards can be explained by the inclusion of a teen number on the reading cards which did not arise in the calculator task. Exclusion of the teen number gave 14% in both questions having difficulty reading other two-digit numbers. For those who could read four digit numbers from the cards the difficulty with the calculator for some seemed to be the confusion they had with the digits changing places so three in one number became thirty in the next and three hundred in the following one.

Writing numbers

Question 2a, which required children to write numbers on a calculator following a verbal prompt showed all students could successfully write single digit numbers. However 38% had some difficulty with two-digit numbers and a further 45% had difficulties with three-digit numbers. Only 5% were successful at writing all including the four digit numbers. Again the teen numbers and the numbers including the digit zero proved the most difficult.

Writing a number on a calculator is different to writing it with pen and paper. The children had all used calculators in their first year at school (Grade 0) and were familiar with them. The decision to use the calculator was to avoid difficulties with fine motor control while still determining whether the child could represent numbers symbolically. In order to enter the number on the calculator the child has to be able to recognise the digit on the keypad rather than recall how to form the digit.

Comparison of reading and writing

There were more children who had difficulty writing than reading two-digit numbers but more who could write all the numbers than read them all. The differences though are really very small.

Of the students who could write two-digit numbers, 13% could not read two-digit numbers and 22% of them could successfully read numbers of three-digits or more. On the other hand 33% of the students who could read two-digit numbers had difficulty writing them on a calculator and 10% could successfully write three-digit or larger numbers. This suggests reading and writing are very similar with most children tending to master two-digit reading and writing at the same time though some children master one before the other.

Ordering

Of those who could write two-digit numbers 13% could not order them while 39% could order at least three-digit numbers. Of those who could read two-digit numbers 15% could not order them while 37% could order at least three digit numbers. This indicates that some children found ordering a little easier than writing or reading, but others did not. It seems that the order in which children learn to read, write and order two-digit numbers varies and this confirms the decision to include them together as part of the one Growth Point.

Interpreting

The two tasks for interpretation of two digit numbers involved modelling a written number using bundles of ten and single sticks and finding and describing how to find a missing number on a hundreds chart with a group of missing numbers. The text for the first interpreting task is shown in Figure 3. The italics are the words which tell the interviewer what to do while the other text tells the interviewer what to say.

Bundling Tasks
Ask the child to unpack the icy pole sticks.
Here are some icy pole sticks in bundles of ten. *(Offer the chance to check a bundle if it seems appropriate).*
Here are some more loose ones.
Show white card for 36.
a) Get me this many (icy pole) sticks.
(If child starts to count all in ones, interrupt and ask them if they can do it a quicker way with the bundles. If they can't, ☹ → **C***)*.*
b) Tell me how you worked that out. ☹ → C*

* This instruction is to go to the next section if the student gives an incorrect response.

Figure 3: First two-digit interpretation question

Of the students who could read, write and/or order two-digit numbers 64%, 45% and 65% respectively could not interpret two-digit numbers and 16%, 26%, and 14% respectively could interpret three-digit numbers as well. This shows that generally students found interpretation more difficult than the other skills and it was often the stumbling block to students achieving Growth Point 2. This is not surprising as the interpretation tasks really require seeing the importance of the ten bundle.

What perhaps is surprising is that there is no clear order emerging but rather the four areas of reading, writing, ordering and interpreting, develop for different children in slightly different orders. This supports the joining of these four as a single Growth Point and encourages the recognition of the interplay between these four skills.

CONCLUDING COMMENTS

While all four place value skills of reading, writing, ordering and interpreting are closely linked many students read and order two-digit numbers slightly ahead of writing them. Of all the students in the sample 75% could read two-digit numbers (or greater), 75% could order two-digit numbers or greater, while 60% could write them on a calculator and 37% could interpret them. Interpreting two-digit numbers requires a greater connection to the concept of the tens structure of the place value system. The linking of these four skills into the one growth point as part of a learning trajectory is supported by the data.

The difficulty with zero reported by Bednarz and Janvier (1982) is clear from the data as is the understandable difficulty English speaking children have with the teen numbers.

There has been discussion in the literature about the choice of materials used in class and the place of concrete models to help understanding of the ten-structure of the system (Baroody, 1990; Boulton-Lewis, 1996; Fuson, 1990). This raises the question of how early teachers should structure experiences with the base ten nature of our number system. These children were mostly aged 6 though some were still 5. They had had one year at school and had used calculators and a variety of materials to represent numbers. Some of the children could read, write, order and interpret three digit and four digit numbers. Does a curriculum which sets mathematical concepts at grade levels rather than looking at development more generally mean that these children have to mark time? One aspect to investigate further is whether the children who at the end of grade 1 are still having difficulties with two digit place value remain in the lower part of the class throughout primary school.

Acknowledgements This research was conducted while the researchers were at the Australian Catholic University. The original data on which this research is based was funded by the Victorian Department of Employment, Education and Training, the Catholic Education Office (Melbourne), and the Association of Independent Schools Victoria.

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NEW INSIGHTS INTO LEARNING PROCESSES FROM SOME NEUROSCIENCE ISSUES

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We recall a Houdé's experiment in which people performing logical tasks like Wason test are observed by means of functional brain imaging techniques, and some correlated neurobiology theoretical constructs. Then we discuss possible outcomes of these results on models of cognitive dynamics and, more generally, on Mathematical Education. Finally, we show how the above theoretical issues, together with some new experimental data, support our resonance model of cognitive dynamics.

INTRODUCTION

In (Guidoni, Iannece & Tortora, 2005) a model of cognitive dynamics is proposed, whose distinctive feature is the core relevance of a *basic resonance dynamics* assumed to work at the root of all the modulations (from perception to abstract thinking) and interferences characterizing the knowledge of an individual. Resonance implies in particular that a continuous shifting from one cognitive dimension to another in a mutual progressive enhancement is, by itself, a specific feature and a specific goal of the learning process.

In this paper we want to show how some results by Houdé on logical thinking obtained by means of *brain imaging techniques* (BIT in the sequel) enrich and support our model and suggest new implications for the learning process. In order to do this, we will compare two different interpretations of the results of the classical Wason test[1], which lead to two completely different hypotheses on what the "natural" cognitive behaviours are, with all their consequences for Mathematics Education (ME). The first interpretation is based on the theory of Evolutionary Psychology (EP), the second one on assumptions in experimental neurobiology, namely on Damasio and Houdé's hypothesis of a strong connections existing between knowledge and emotions[2], and on Changeux epigenetic theory.

In the first section we summarize our basic theoretical framework, while in the second one, after comparing two interpretations of Wason test, we show why some constructs from EP are not adequate enough for ME. In the third section, in order to test the potential impact and to illustrate the possible outcomes for ME of our

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[1] For other results about Wason test in Mathematics Education research, see also (Inglis & Simpson, 2004)

[2]According to them, in our brain there is a specific region where the systems involved in emotions/feelings, in the attention and in the operative memory interact so deeply that they can be considered as the common source of the energy for outward (motory) as well as for inward actions (reasoning, thought).

assumptions, we present two experimental data: the first one is a modification of Houdé's experiment, where the intervention of the experimenter is substituted by a cooperative learning environment; the second one is a brief excerpt from an activity with young children. Finally, in the last section we discuss how experimental evidences allow us to integrate neurobiological data in our model of cognitive dynamics.

SOME LINES OF A THEORETICAL BACKGROUND

For a general theoretical framework concerning ME research, we refer to (Tall, 2004), where the author draws a comprehensive picture of the main psychological theories of cognitive growth in mathematics, from Piaget onward. In particular Tall notices in this field an emerging strand, given by researches into brain activity, where a lot of experimental data have been collected, from those concerning innate numerical competencies (e. g., Wynn 1992-1996, quoted in Devlin, 2000)), to those based on BIT applied to subjects engaged in elementary arithmetic tasks (Dehaene, 1997), (Butterworth, 1999).

BIT are nowadays largely employed in several different contexts, providing experimental evidence about which brain areas are involved in specific tasks (see (Changeux, 2002) for references). Of course, it is still too early for a global model of brain functioning, that could connect "local" data provided by neurophysiology experiments with psychology global models describing and interpreting observable behaviours. Nevertheless it should be an unvaluable task to try to integrate all the now available brain-founded researches supported by BIT in any model of natural cognitive dynamics and of learning processes. Along this path many "solid" beliefs might perhaps be put in doubt.

For instance, the discovery that a sense of numerosity is innate has overthrown Piaget's theory about children's acquirement of numbers. To show this, in (Devlin, 2000) Piaget's famous experiments on rearrangements of objects in a line are recalled: doubts are advanced about his conclusion that five years old children cannot master the principle of quantity invariance and hence the idea of number; and the criticism of (Mehler & Bever, 1967) about the modalities of these experiments, and their reliability, is revisited. Our intention is to show that, analogously, a recent study concerning logical thinking (Houdé *et al.*, 2000), where behaviours within Wason-like tests are observed, puts into discussion some issues from EP, as described in (Cosmides & Tooby, 1997). This seems to us quite interesting, since in (Leron, 2004) such results are utilized to infer that "formal mathematics"[3] clashes with human nature. In next sections we will discuss two interpretations of Wason test, and their outcomes for ME; for this purpose we recall here some theoretical constructs from EP.

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[3] The notion is from (Leron, 2004), even if we prefer the classification in (Tall, 2004), with his notion of "formal world".

In (Cosmides & Tooby, 1997) the authors, starting from the observation of many cognitive behaviours, propose an evolutive theory of the brain and consequently a definition of human nature. They hypothesize that, for the sake of survival, some neuronal circuits have been developing on a biological temporal scale, specializing for specific purposes: a sort of stable modularity of the brain. For example, the following "definitions" of human nature and of common sense are taken from (Leron, 2004): "*Human nature* [can be seen] *as a collection of universal, reliably-developing cognitive and behavioral abilities – such as walking on two feet, face recognition or the use of language – that are spontaneously acquired and effortlessly used by all people under normal development … Common sense is the cognitive part of human nature – the collection of abilities people are spontaneously and naturally "good at"*". (Leron, 2004, p. 217-218)

The previous definitions allow Leron to attribute the difficulties that students encounter in dealing with formal mathematics to some strength (not weakness) features of our cognitive structures[4]. In this way EP happens to be a good theoretical framework for interpreting "natural" difficulties. But for the learning process, it is not enough to consider what human beings effortlessly grasp from their environment: what is more important is what they can learn in assisted contexts. According to Vygotskij, what an individual can learn by himself is different from what he can learn under the guidance of an expert adult or in a cooperative context. From (Guidoni, 1985) we draw a useful distinction between spontaneous and natural. The culture itself, even including formal mathematics, is in a sense "natural", like any human production, but it is not "spontaneous"; nor is spontaneous that an individual could go again by himself over the whole historical process of knowledge construction.

Therefore we believe that other models of mind, originated from the previously quoted neurobiological studies on the brain, can better help to understand the learning process. We refer in particular to (Changeux, 2002) (but see also (Damasio, 1994)). According to Changeux' *epigenetic* theory, in human beings the genes substantially determine some circuits and systems of neurons in the ancient part of the brain. These "innate" structures not only guarantee the survival of the individual, but also, after birth, influence and constrain the development of structures in the evolved part of the brain, which will be shaped all along life in the interaction between the individual and his social and cultural environment. In other words, a high level of plasticity characterizes the complex neuronal structure of the brain, which is continuously remodulated as a consequence of learning processes. In (Changeux, 2002) many experimental data are reported to support these theoretical assumptions: for instance he shows how the acquisition of writing abilities modifies synaptic links in the brain (*ibid*., Ch. 6, Section 7).

[4] Notice that Devlin already made a similar remark, concerning the difficulties that most people face in memorizing the multiplication tables: "*Such a widespread problem with multiplication is surely due to a peculiarity of human brain that would deserve deep investigations, instead than be object of surface criticisms*". (Devlin, 2000, page 81 of the Italian edition, our translation)

TWO DIFFERENT INTERPRETATIONS OF WASON'S TEST

Cosmides and Tooby started from a large literature that shows that people perform very poorly in detecting logical violations of if-then rules in Wason selection tasks (Wason, 1983), even when these rules deal with familiar content drawn from everyday life. On this basis, they built a new sperimentation that shows how people who ordinarily cannot detect violations of if-then rules can do so easily and accurately when that violation corresponds to cheating in a situation of social exchange (see (Cosmides & Tooby, 1992), where several other references on Wason test can be found). From this they infer, according to the EP point of view, that the evolved architecture of the human mind would include inference procedures that are specialized for detecting cheaters. But Cosmides & Tooby are mainly interested in human nature, in the sense above quoted, that is in everything a human being can do without effort, activating predetermined and stable brain circuits. So we believe that the interpretation in (Leron, 2004) of Wason test suffers from this limitation, in concluding for an inherent difficulty of people in dealing with abstract conditional statements.

A slightly modified form of Wason test is proposed in (Houdé *et al.*, 2000), a study in functional neurology inspired to Changeux' above quoted theoretical frame and moreover to the assumptions reported in (Damasio, 1994), according to which rationality and emotions are always interwoven in any human cognitive behaviour. In Houdé's version, the logical implication to be falsified is expressed in terms of correspondence between colours and geometric forms. The failure percentage in people performances is similar to that observed by Cosmides & Tooby and by others, but Houdé gives a completely different interpretation. He asserts that perception-based modalities of reasoning are automatically activated by geometric forms and colours[5], and that only the *inhibition* of these modalities can allow logical reasoning to be put in action, and the test to be correctly solved.

Indeed, the monitoring of the subjects engaged in the task, by means of BIT, reveals that the active brain areas are respectively: a zone in the back part of the brain, specialized in the processing of perceptive informations, when the answer is incorrect; and a frontal zone involved in logical reasoning but also committed with emotions, in case of successful answer (for further details see (Changeux, 2002), Ch. 3, Section 9).

Therefore so many failures in the task could be ascribed to a sort of inability in inhibiting perceptive reasoning modalities. Going on, the author raises the question how to favour this inhibition. To this purpose, a second step in the experiment is designed, where the sample is divided into two groups, submitted to two different "learning scenarios". In the first one, called "cold", the experimenter gives detailed

[5]A similar remark applies also to cards with letters and numbers. Indeed, the analysis of incorrect answers, together with some interviews, show that most people address their attention to perceived things, pointed by the words (vowels, even numbers) occurring in the question, so limiting their control to these ones.

information about the logical aspects of the test. In the second, “warm”, scenario, the experimenter gives warnings against possible traps hidden in the task, recommending to avoid reckless answers.

Submitted again to the test, the first group, despite having substantially understood the logical structure of the test, fails again with minor improvements, while the second group gives correct answers with a percentage as high as 90%. The monitoring in this second phase shows that the same brain areas as before are active for the group involved in the cold scenario, while prefrontal areas of the brain are active for the group involved in the warm one.

TWO KINDS OF EXPERIMENTAL DATA

We start from the results of Houdé on Wason-like tests, and particularly from his emphasis on the necessity of inhibit a “perception” approach to allow logical reasoning. Our hypothesis is that the capacity of activating this inhibition can be considered, in Vygotskij’s words, a higher mental function, and therefore it can evolve along a learning process, hopefully in a stable enough manner. Following again Vygotskij, we claim that any accomplishment can be better achieved in an interindividual work, before becoming an intraindividual resource. Therefore we have conceived and realized a simple experiment on Wason test, in which a cooperative way of working substitutes the experimenter’s intervention in the “warm learning scenario”.

The population of our test has not been large, the organization and the modalities have been quite informal and, moreover, we do not dispose of the technical apparatus of neuroscientists, but nevertheless we believe that some significant indications can be inferred from it. Our sample consists of about 120 subjects of different ages and instruction levels, even if many of them (more than one half) are students of a Mathematics Education course for prospective teachers, which means graduated in Mathematics or in Physics, and some in Engineering or Economy. Almost casually, some of the volunteers were submitted to the classical version of Wason test (cards with letters and numbers), without further information or warnings: in coherence with all reported results, the majority of them didn't respond correctly, almost independently from their age or instruction level. The other volunteers were divided into groups of two or three individuals and submitted to the same test with the task of discussing until coming to a shared solution. In this case, results are strikingly different, inasmuch about 80% of the groups give a correct answer.

We conclude reporting a brief excerpt from an activity with 2^{nd} degree children[6], in order to show that the same dynamics are at work: collaborating among peers allows to recognize that perceptive thinking is not always effective, as a first step toward a

[6]The activity is part of a long term didactical strategy (see (Guidoni, Mellone & Pezzia, 2005) for details), which starts as early as at a pre-scholar stage, where we systematically propose both qualitative modelization processes from everyday life phenomena and “problems for thinking”.

conscious inhibition of it. Of course, because children are very young, the task is different: this time a problem is presented where three different conclusions are already drawn. The task consists in discussing and comparing these three ways of reasoning, instead than producing autonomous ones. The problem is the following:

Three friends, during a train trip in Ireland, pass near a beautiful lawn and see a black side of a sheep browsing. The first guy says: "Look, in Ireland sheep are black!". The second one: "No, you can't say that all sheep are black, the only thing you can say is that in Ireland there is at least a black sheep". The third one intervenes saying: "The only thing we can say is that in Ireland there is at least a sheep with a black side". Who of the three is right?

At the beginning, the children are almost equally divided as supporters of the first two friends, both referring to perceptive register. Then the negotiation starts:

> Tommy: "I agree with the first guy, since there aren't countries with only one sheep".
> Luca: "But the second friend doesn't say this! He says that there is **at least** a black sheep, not that there aren't other sheep!"
> Tommy: "It is sure that there are other sheep, so the first friend is right!"
> Luca: "We haven't being asked to say which assertion is more likely, but which is true, on the basis of what they see. It is likely, but not sure, that there are other sheep, and moreover we don't know their colours!"
> … Stefano: "But, … I think there is a trap! The sheep could have been bought in Scotland!"
> Dea: "Right! Or she could have been dyed, or she **looks** black in the shadow of the train!".

As it can be seen, logical reasoning begins to appear. In the sequel of discussion some interest arises toward the third assertion, while the first one is abandoned, and the second one gains new supporters. In any case, perception-based and logical thinking are continuously intertwined. Of course, a long path is necessary before the class become able to consciously shift from one to another, and a longer one before every child can do it individually. But every long trip requires a first step: we recognize it in the following intervention:

> Giulia: "Well, I don't really think there is a sheep half black and half of a different colour. Surely, the sheep is all black. O.k., o.k., we are joking, we are playing at being true mathematicians!"

DISCUSSION AND CONCLUSIVE REMARKS

"*... informal mathematics is an extension of common sense, and is in fact being processed by the same mechanisms that make up our everyday cognition ... [while] ... the thinking involved in formal mathematics is not an extension of common sense: it either can't find suitable abilities to co-opt, or it can even clash head-on with what for all people "comes naturally"*". This conclusion is in (Leron, 2004), pages 219-220.

Taking into account our theoretical framework, we are inclined to a different conclusion. It is surely true that the brain circuits, specialized in processing sensorymotor sygnals, spontaneously come into play, since such a mechanism (called *somatic marker* in (Damasio, 1994)) allows quick decisions and ultimately guarantees our survival; but it isn't true that there aren't "abilities to be coopted", when a cognitive task requires logical thinking, as Houdé's experiment clearly shows. Therefore, the problem is how to induce an autonomous capacity of inhibiting the circuits that immediately come into play, or, better, of putting into resonance the two "ways of working" of the brain, in a mutual progressive enhancement.

On the other hand, the problem is not confined to logical thinking nor to mathematics. Indeed, recent philosophical as well as scientific studies on perception (see, among all, (Bozzi, 1990)) show without any doubt that Piaget was wrong in thinking that the culturally evolved representations substitute the naïve ones. Quoting again from Houdé: "*…le cerveau de l'Homme…est une sorte de jungle où le competence du bébé, de l'enfant et de l'adulte, sont à tout moment susceptibles de se télescoper, d'entrer en competition, en même temps qu'elles se construisent…*" (Houdé, 2000). This is why we underline the importance, in cognitive growth, of putting these dimensions in resonance.

Our experiment on Wason test suggests that perhaps the "inhibition" capacity can be achieved in an interindividual work before becoming an intraindividual resource. However, we believe that this hypothesis deserves further investigation: in particolar, it should be interesting to monitor, by means of BIT, a cooperative context, in order to throw more light on the question.

A final remark concerns a didactical hypothesis, underlying the second case reported in the previous section. We believe that a very early and systematic experience with real modelization processes would induce a relatively stable capacity of conscious inhibition of brain areas, unsuitable for accomplishing logical tasks. This follows from the fact that any modelization process, performed in a Vygotskijan situation, entails comparison between cognitive games, and therefore favour the acquisition of metacognitive competencies, those allowing to make judgements and decisions, namely to choose among various ways of reasoning the most suitable one for a given goal.

We have not yet enough evidence for our hypothesis, and so the observations of children reported above are mainly presented as a possible context of investigation of our idea. In this sense, they seem to us very promising.

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FOSTERING CONCEPTUAL MATHEMATICAL THINKING IN THE EARLY YEARS: A CASE STUDY

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In this paper[1] *we investigate how teachers can foster conceptual mathematical thinking in 5-6 year-olds. We define conceptual mathematical thinking as instances where pupils show some of the mathematical skills associated with successful mathematical thinking, as first described by Krutetskii (1976). We investigate the impact of some well-defined sociomathematical norms observed in the classroom that help foster such thinking at this very early age. Here we present one of 5 case studies and we conclude that it is in the classrooms of teachers who view mathematics topics as connected to each other and who encourage their pupils to negotiate mathematical meanings with their peers that we observed pupils engaged consistently in conceptual mathematical thinking.*

INTRODUCTION AND THEORETICAL BACKGROUND

Teaching mathematics in the early years in the UK revolves around a rather narrow definition of numeracy (NNS Framework, 1999), which includes "understanding of the number system, a repertoire of computational skills and an inclination and ability to solve number problems in a variety of contexts" (ibid. p 4). In this paper we investigate if it is possible to foster successful mathematical thinking skills, which are not necessarily connected to computation, at the very beginning of primary education. We define *conceptual mathematical thinking* as instances of pupils' actions while tackling tasks set by the teacher that incorporate some of the following features:

- choose appropriate and effective strategies for problem solving
- adapt pre-existing strategies to the current problem
- generalise rapidly and broadly
- be flexible with mental processes
- grasp formal structures

These abilities are among those first described by Krutetskii (1976) in his work with mathematically able schoolchildren, and have since been adopted (Bishop, 1976, Watson, 2001) as some of the indicators of successful mathematical thinking for learners of mathematics throughout the primary school and in more advanced

[1] We would like to thank the Economic and Social Research Council in the UK for their financial support of the study from which this paper draws. (RES-000-22-0851).

mathematical studies. Moreover, we approached life in the classrooms from the point of view of social interaction (Cobb & Yackel, 1996). In the naturalistic set of the primary classroom we asked which sociomathematical norms, if any, are conducive to pupils consistently using conceptual mathematical thinking. By sociomathematical norms we mean

> ….normative aspects of classroom interaction that are specific to mathematics. (Cobb, 1998, p 2)

These sociomathematical norms are negotiated between the pupils and the teacher in the classroom throughout the academic year in a non-overt way, and originate in the teachers' beliefs on what it means to learn mathematics, how pupils learn mathematics and how teachers teach in order for the pupils to learn mathematics.

METHODOLOGY

The study was structured around (5) multiple case studies (Stake, 2005), of Year 1 classrooms (i.e. 5-6 year-olds). Each case study was built within a 3-week cycle of observations of mathematics lessons and interviews with the teachers and pupils. During an initial interview with the teacher, we asked him/her to nominate 6 target pupils, a boy and a girl in the high, medium and low achieving sets (all the schools we visited grouped their children in achievement sets) and explain why they had decided to nominate the pupils. These pupils were the focus of the observation to follow. We also asked the teacher to show us his/her lesson plans for the following three weeks so that we would have an idea of the material that was about to be covered. Prior to the period of observation we held diagnostic interviews with each of the target children individually. We used these interviews to understand the degree of familiarity that each pupil had with the mathematics he/she was going to encounter during the period of observation. Following these introductory interviews we observed each class for 3 weeks every day during the hour dedicated to numeracy.

The focus of our observations were the teachers, during the first and the last part of the lesson, and one or two of the target pupils during the main teaching activity[2]. Once a week we video recorded the lesson and on the same day we held stimulated recall interviews (SR) where the teachers were asked to comment on a part of the videoed lesson. The section of video was chosen by the two authors and usually focused on a whole class teaching part of the lesson. The procedure of the SR was carefully designed in order to minimise some of the problems associated with this method of data collection (Lyle, 2003). ***The interview data*** (with the teachers) were coded (using the software TamsAnalyzer, http://tamsys.sourceforge.net//osxtams/) to create insight into the teachers' beliefs in terms of how to teach to become numerate, what it means to be numerate and how children learn to be numerate. We use the term "numerate" as this is the main

[2] Mathematics lessons in the UK follow a 3-part layout: the first and the last part involve whole class teaching and in the middle part pupils tend to work individually or in groups on activities set by the teachers.

emphasis of teaching mathematics in the English primary classroom. ***The observation data*** were analysed in two different ways, according to the part of the lesson in question. From the observation of the whole-class teaching we were able to glimpse how teachers put into practice the beliefs expressed during the interviews. In the observation of the target pupils during the main teaching activity, we were able to group incidents where we thought we could detect pupils engaged in conceptual mathematical thinking. These incidents were then grouped further according to the kind of mathematical skills we thought we observed following the definition of *conceptual mathematical thinking* adopted. For each lesson we also produced a table of the tasks given to the children. The other data – diagnostic interviews with the pupils, lesson plans etc. - were of help to us in order to create a rounded understanding of the classroom, the pupils and the teachers we were observing. For each of the teachers in our study we wrote ***a teacher profile***. These narratives were composed using and expanding on the data analysis technique 'codeweaving' (Saldana, 2005). In the profiles we have "woven" into the narrative codes from teacher interviews and instances from the observations which relate to our research question, inserting in the text verbatim quotes of interviews and extracts from field notes in order to preserve the original language of the participants and the fieldworkers (Saldana, ibid.). The teacher's profiles aim at describing the teachers and the life in their classrooms during the period of observation. In what follows we present a much- shortened version of the profile we wrote for one of the teachers in the study[3].

KATE

Kate teaches a Y1 class in an independent school in the East of England and qualified 5 years ago. Central to her beliefs about teaching and learning is the need to make connections both between different parts of mathematics and between different subjects in the school curriculum. She explains this many times during the interviews:

> K: I think … I think that that is a very important role of the teacher … by the time the child is 5 they have put down most of their brain… their brain is fully formed and our role is just to build the bridges, isn't it? Which is why the most effective sort of teaching is cross-curriculum teaching, it is linking in things.

Kate's classroom is a very lively environment and the pupils seem to be engaged in their mathematical tasks in a meaningful way. During the observations of her lessons the constant making connections become clear. As early as the second week in Y1, while introducing the ordering of the non-negative integer numbers on the number line and introducing the language of 'before, after and in between', we observed:

[3] We adopt the following transcript conventions: […] repetitive or irrelevant words omitted; (*italics*) explanatory note added by the authors; … a short pause in the sentence. K is the teacher, AC is the second author of this paper and other capital letters are children in Kate's class. In order to preserve anonymity the names of the teacher and the pupils have been changed.

Now Kate talks about 'before'. She asks before 5? Chris? He answers 4. She draws

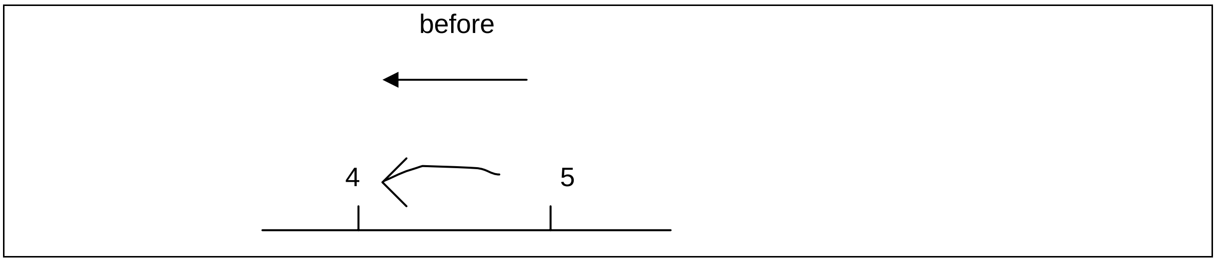

Fig. 1: Kate's drawing on the board

and asks the children what they see. Odette looks at 5, counts one 'bunny jump' back. Kate asks if we started from 0, then she asks if there is a better way to do this. Samuel? He replies he is thinking about it. She asks Brendan. He replies look at the first number (5) and then take away.

In this very short snapshot we can see one of Kate's constant characteristics of teaching: connecting different parts of the curriculum. Sometimes these connections originate from Kate, and sometimes, as in the extract above, originate from the children. The occasion for making a connection is never lost. We came across many instances of conceptual mathematical thinking in Kate's class. A few days after the incident reported above we observed:

Kate now writes on the board:

5-2=... 4-2=… 3-2=… 2-2=…

Then she goes around the class. Brendan is still playing and Kate asks him what is

3-2=. Brendan replies 1. Kate asks him why and he says that it is because 2+1=3.

Here we can see how Brendan has adopted an existing strategy to solve the current problem. In the previous week the class had seen simple addition facts, and on the observation day (a Monday) the teacher was starting to introduce subtraction. Here Brendan has been able to use what he knows about addition to solve a simple subtraction problem. We observed Brendan using this strategy in more than one instance during the 3-week observation period, leading us to believe that it is indeed a strategy and not just an incidental observation by the child. It is also important to observe here that Kate reacts positively to Brendan's comment, and later on in a post-task interview she explains why she thinks this is a very good strategy and says that she herself uses it to solve easy subtraction.

Discussion

During the time we spent in Kate's classroom we observed many instances of conceptual mathematical thinking with her pupils appearing to solve mathematical tasks in an non-procedural way much of the time. One of the main characteristics of Kate's teaching, which become established as a very clear sociomathematical norm in her classroom, was that she gave very sophisticated responses to pupils' strategies, even when those were very different from the strategy she was using herself. Thus she valued pupils' contribution to the lesson and established a view of mathematics learning that is negotiated between all the participants to the mathematics lesson.

This in turn encouraged pupils to try and use their own strategies in a variety of problems and helped them make generalisations, as in the case of simple addition fact reported in Kate's profile. Moreover, the constant making of connections between different parts of mathematics and different subjects in the curriculum allowed the children not to feel that mathematics topics are isolated from each other, and facilitated generalisation and transfer of known facts to different contexts, when this is possible. An example of this occurs when Brendan suggests that taking away is related to adding and explains why. Kate's response is to investigate this strategy further, to link subtraction and addition, and to make it available through discussion to the other pupils in her class. This does not imply that Kate does not teach procedures or does not recognise that there are strategies more effective than others in solving problems. Rather it is in the way she responds to strategies other than her own, that we see how she actively supports independent mathematical thinking in her pupils.

A critical incident in Kate's classroom: Kate is using a child's error as a prompt to clarify a basic fact about subtraction, namely that when you subtract "the largest number comes first" (Let's remark here that such basic fact holds true in the case, as in a Y1 class, that the only numbers that exist are the non-negative integer numbers). She has drawn on the white board a number line from 0 to 10 and the sum 3-4=. She holds up 3 teddy bears and asks the class how could she take 4 teddy bears away from the three she is holding. She then asks the children what could 3-4 be and receives a variety of answers from the pupils, from 0 to 2 until she asks Samuel:

K: Samuel?

S: Minus one.

K (*very quietly and laughing*): Minus one. Mark?

M: Zero or nothing.

After the children finished suggesting answers Kate told the class:

K: Now Samuel's answer – I've got to say he deserves a sticker because when you've long finished with me you will learn that zero is not actually the smallest number. All right Samuel, and you are quite right. […] The number line extends to infinity this way *(extends number line on board to left of zero*) as well as this *(pointing to number line to right of zero*) way and there are minus numbers that go continuing along this (*left*) side of zero (*writes '-1', '-2'*) and although you can't take four teddies from three teddies when you are ready to get rid of all your cubes, all your teddies you'll find that you can create a completely new family of numbers which have a minus in them and that was a very, very clever answer.

After the mathematics lesson finished, Kate called Samuel and asked him to explain to the other pupils how he knew that 3 minus 4 equals -1. This is what Samuel said:

K: That was a very clever answer. How did you get it?

S: From the underground car park.

K: Oh, from an underground car park. We can tell the children that. Wow! Samuel is going to tell you something amazing. Listen to 'Brains of Britain' here. How did you know that three take away four is minus one?

S: Because I went to the underground car park at the library and there was minus one, minus two, minus three (*the numbers on the buttons in the lift)*.

In the stimulated recall interview with Kate at the end of the school day (just by chance it happened that on that day we were videoing the lesson), we asked her why she had chosen to show the children in her class the sum 3-4= and what kind of answers she was expecting. She replied that she wanted to reinforce one of the basic subtraction facts, namely that ‘in subtraction the largest number goes first’ and that if this is not the case, then it is not possible to do the subtraction in the context of the non-negative integer numbers. She was using the teddy bears simply to illustrate this. When we asked her what she thought when she heard Samuel’s reply:

K: That is why I probably paused a bit and in my mind I thought how do you deal with this Kate? I couldn’t be exuberant and show it to the class straight away. I had to think about what was I going to do because I felt that this was something, and I believed that this was way ahead of anybody else in the whole class so I felt it was important that I made Samuel feel good…And in terms of lesson objectives, to open doors to other children. So I painted a picture of the car park and going below so that when they go shopping with their parents they might suddenly connect… see that mum, that says minus and then a conversation will ensue…

AC: Right. So it is giving them options if you like.

K: Yes. I call it opening doors I suppose.

In Kate’s reaction to Samuel’s reply we can again see the emergence of a very specific sociomathematical norm in the classroom: negotiating mathematical meaning with all the pupils in her class and “open doors” so that each one of them might reflect upon and incorporate Samuel’s answer in the growth of their mathematical understanding.

Discussion

The critical incident we described above exemplifies Kate’s reaction to a potentially challenging episode. Here Samuel has correctly answered a question that, in Kate’s planning, was supposed to lead the children to think had no answer, hence challenging the lesson plan and learning objectives that Kate meant for that lesson. If we break down the actions in the classroom we can see that:

- *Kate is implementing a strategy that is correct in the number context considered* (i.e. non-negative integer numbers): we have observed above that Kate teaches her pupils strategies and follows the learning objective suggested by the National Numeracy Strategy;

- *She analyses why the case 3-4= does not work* (i.e. on this occasion as -1 is outside the number context considered). Kate often uses (anonymised) children’s mistakes to illustrate mathematical facts.

- *She accepts the answer that generalises the number context taken into consideration and shows that the suggested strategy indeed works when the number context is* *changed* (Samuel's shift to the negative integer numbers): Kate is open to any suggestion coming from her pupils, even when the suggestion changes the aims and objectives of the lesson. This sociomathematical norm of the classroom, which we have observed being used previously in other situations, it is used here in a very effective way to investigate and negotiate a new mathematics concept.

- *She reinforces that such answer is indeed correct:* this is a very important step as Kate fully acknowledges an answer that jeopardises the strategy she herself was proposing.

- *She investigates how such answer was reached:* hence, no mathematical fact is just dismissed as true or false. Moreover facts which are unexpected and more advanced than expected are justified and discussed between the pupils and the teacher.

This sequence of actions on the teacher's part establishes some very defined sociomathematical norms. Kate reinforces the truth of Samuel's answer, accepts his generalisation, discusses it with the rest of the class, hence making it available to the other pupils, and generally supporting the belief that growth of mathematical understanding is reached through interpersonal negotiation of meaning.

CONCLUSIONS

The aim of this paper was to examine if it is possible to foster mathematical skills in very young pupils which not only focus on becoming numerate, but which will be useful throughout the pupils' careers as learners of mathematics. Viewing classroom life from the point of view of social interaction, we tried to understand what sociomathematical norms are conducive to pupils using conceptual mathematical thinking in a consistent way. We believe that that the above critical incident in Kate's classroom is not a coincidence. On examining the data from the 4 other classrooms in this study we noted that the pupils tended towards a much more procedural approach to the tasks when

- the classroom sociomathematical norms were very different from those in Kate's classroom, or
- mathematics learning was not viewed as something that is constantly negotiated between teacher and pupils, or between the pupils themselves, or
- mathematics itself was viewed by the teacher as consisting of a discrete set of facts isolated from each others.

In such cases we observed the pupils constantly engaged in reproducing the strategy modelled by the teacher rather than engaging in devising a problem-solving strategy by themselves, and in no other class we witnessed critical incidents comparable to the one involving Samuel and the negative numbers. To conclude, we believe that it is crucial that, from the earliest stages of schooling, children's ability to use conceptual mathematical thinking should be fostered and that it is possible to isolate factors related to the social interaction in the classroom between the teacher and the pupils, and the pupils themselves, that make fostering such skills possible.

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THE ROLE OF MATHEMATICAL CONTEXT IN EVALUATING CONDITIONAL STATEMENTS

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Adrian Simpson — University of Durham

Recently there has been increasing interest in the mathematics education research community about the role of logic in the teaching, learning and production of mathematics. In this paper we investigate how conditional statements are evaluated by successful mathematics students, and argue that the role of context is vital to determine the manner in which this evaluation proceeds. We use two versions of the so-called Labyrinth Task, one in it's original context and one in an overtly mathematical context. We report results that indicates that the manner in which conditional statements are evaluated on these tasks differs depending on the context. These results are supplemented by data from a qualitative task-based interview study.

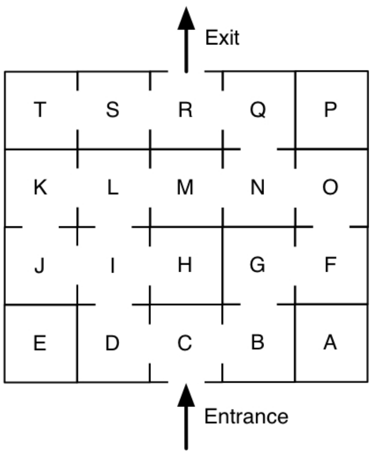
Logical implication is seen as being one of the most important structures in mathematics, and researchers have argued that coming to terms with it is vital for developing an understanding of proof (Durand-Guerrier, 2003; Weber & Alcock, 2005). Our goal in this paper is to describe a psychological framework that explains the processes involved in evaluating conditional statements in everyday language, and to explain how these processes differ in mathematical contexts. To do this we first describe a task used by several researchers and teachers to investigate the role of logic in mathematical reasoning.

THE LABYRINTH TASK

Durand-Guerrier (2003) introduced the so-called Labyrinth Task into the mathematics education literature. In this task participants are presented with a maze, and told that a person *X* managed to pass through it without using the same door twice.

They are then asked to categorise a series of statements as being either *true*, *false,* or that *there is not enough information to tell* (can't tell):

1. *X* crossed *P*.
2. *X* crossed *N*.
3. *X* crossed *M*.
4. If *X* crossed *O*, then *X* crossed *F*.
5. If *X* crossed *K*, then *X* crossed *L*.
6. If *X* crossed *L*, then *X* crossed *K*.



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The answers to the first 5 statements appear to be relatively straightforward, but for statement 6 the answer becomes less clear. Durand-Guerrier (2003) argued that the correct answer is "can't tell" because it is impossible to know whether *X* passed through *K* or *I* before s/he passed through *L*. When administered to 15-16 year olds apparently this was the answer given by 60% of students, "especially those deemed good at mathematics" (p.9). However, the students' teachers apparently disagreed:

> Surprisingly, some teachers considered this answer to be wrong! (p.8)

The teachers believed that the correct answer was "false", since according to Durand-Guerrier's analysis, they had interpreted the statement as "for all *X*, if *X* crossed *L*, then *X* crossed *K*". Suggesting that "can't tell" is the "natural" answer for students, Durand-Guerrier worried that the teacher's interpretation of the statement causes a didactical obstacle for students:

> It is necessary to overcome the opinion that every implication met in the classroom is a relation between propositions which is either true or false and that carries necessity. Indeed, *implication between propositions carries no necessity, but is a set of possible cases for truth values.* (Our emphasis, p.29).

The idea that implication is a set of possible truth values may be *logically* correct, but there is a multitude of research that suggests that it is not *psychologically* correct. In the next section we briefly discuss some of this work: Evans & Over's (2004) theory of conditionals based on the so-called Ramsey Test.

THE RAMSEY TEST – CONDITIONALS IN EVERYDAY LANGUAGE

According to Ramsey (1931), when people judge the truth/falsity of a conditional in natural language they are "hypothetically adding *P* to their stock of knowledge and arguing on that basis about *Q*", they are in effect "fixing their degrees of belief in *Q* given *P*" (p.247). This idea – that to judge $\mathbf{P}(P\Rightarrow Q)$ a person judges $\mathbf{P}(Q|P)$ rather than $\mathbf{P}(Q \text{ or not-}P)$ – has become known as the Ramsey Test. (Here $\mathbf{P}(X)$ indicates the level of belief that a person has in event *X*. This is clearly related to, but not necessarily identical to, the probability of event *X*.)

The notion of the Ramsey Test is a non-trivial model of the manner in which people judge conditional statements. Such a model is at odds with both formal logic and other psychological accounts of conditional statements, specifically Johnson-Laird & Byrne's (2002) influential mental models framework (for a full discussion of the difference between these theories see Evans, Over & Handley, 2005).

To illustrate how the Ramsey Test operates, consider the statement "if you're in Birmingham, then you have a good choice of Indian takeaways". This statement is judged by hypothetically supposing that you are in Birmingham, and then considering the availability of Indian food, given this supposition and your existing knowledge and beliefs. Note that this process is both psychologically and logically different to the truth evaluation of formal material conditionals. A material conditional $P\Rightarrow Q$ is true whenever *P* or not-*Q* is true. Thus if you are not in Birmingham and the statement is evaluated as a material conditional, then it is automatically true. But evaluated as a suppositional conditional using a Ramsey Test it may be true or false depending on the individual involved's beliefs. There is an increasing body of evidence that supports the notion that the Ramsey Test is an accurate model for how humans judge conditional statements (e.g. Evans & Over, 2004; Hadjichristidis et al., 2001).

How then, does the Ramsey Test apply to the Labyrinth Task? The participant hypothetically adds "X crossed L" to their stock of beliefs and then evaluates their degree of belief in "X crossed K". Given the layout of the maze it is clear that

$$\mathbf{P}(X \text{ crossed } K \mid X \text{ crossed } L) = 0.5$$

so Evans & Over's (2004) theory of suppositional conditionals would predict that most people might categorise "if "X crossed L, then X crossed K" as "can't tell", as they neither have strong belief nor disbelief in the statement.

However, as we have seen Durand-Guerrier (2003) reports that the mathematics teachers who administered the task believed that the correct answer was "false". It seems clear that they were evaluating the statement in a somewhat differently to the manner which Evans & Over's (2004) theory predicts. The purpose of this paper is to investigate the role that mathematical context plays in the evaluation of conditional statements with the Ramsey Test.

METHOD

We were interested in discovering exactly how successful mathematicians evaluate the Labyrinth Task, and whether mathematical context plays a part in this. To this end we administered two versions of the task. The first version was identical to the original task reported above, and the second was phrased in an overtly mathematical context:

> Your friend X is interested in a sequence of real numbers, (a_n). X writes down sentences about the sequence. For each of the sentences you must decide whether it is *true, false,* or whether there is *not enough information* to tell.
>
> Place each of the following statements into the categories: *true, false* or *can't tell.*
>
> 1. $a_k=4$ for some $k \in \mathbf{N}$.
> 2. $a_{46} \in \mathbf{R}$.
> 3. $a_n \to ¾$.
> 4. If $a_{n+1} > a_n + 1$ (for all n), then $a_n \to \infty$.
> 5. If $\sum a_n$ converges, then $a_n \to 0$.
> 6. If $a_n \to 0$, then $\sum a_n$ converges.

Parts 4,5 and 6 of this task were designed to be isomorphic to the maze task, but set in a mathematical context. Thus presumably Durand-Guerrier (2003) would argue that the correct answer to part 6 is "can't tell" as there are some sequences (a_n) for which this statement is true (from a formal logic standpoint), but some sequences for which it is false. For example the sequence $a_n = n^{-2}$ both tends to zero and its associated series converges. Thus, since both P and Q are true, "if P, then Q" formally is also true. However the sequence $a_n = n^{-1}$ tends to zero but its associated series does not converge, thus "if P, then Q" is false. So (it could be argued) there is not enough information to tell whether statement 6 is true or false.

The participants were 433 first year mathematics undergraduates from the first author's institution. All the students in our sample had been highly successful school level mathematicians, and had typically achieved A Level grades of AAB or higher. The cohort was randomly split into two equal groups and each group were given either the original or the mathematical version of the task. The task was administered as part of a biweekly test that formed a minor part of the assessment[i] of a first year Foundations of Mathematics course (which included sections on logic and implication). All the students were simultaneously taking a course in Analysis and so should have been familiar with the terms used in the mathematical version of the question.

RESULTS

In this section we report the results of part 6 of each version of the labyrinth task. These figures are shown in Table 1. It can clearly be seen that the range of responses was different between the versions. In the original version participants were fairly evenly split between the 'false' (44%) and 'can't tell' (54%) responses, with almost no one selecting 'true' (2%). However, in the mathematical version a large minority of participants selected 'true' (30%), and few selected 'can't tell' (14%).

Answer	**Original**	**Maths**	**Total**
T	2	30	16
F	44	56	50
C	54	14	34
n	216	214	430

Table 1: The breakdown of responses, as percentages, to statement 6.
T – true, F – false, C – can't tell.

The difference between the responses (by test version) reported in Table 1 is highly significant, with a large effect size, $\chi^2=107$, df=2, $p<0.001$, $\phi=0.498$.

However, we were concerned that some of the differences between the test versions could be attributed to poor subject knowledge in the mathematics version. For example, it is hard to see how any structural property of conditional statements could lead participants to judge part 6 of the mathematical version to be true. To try to mitigate this distorting effect we removed all participants from our analysis who answered part 5 incorrectly. That is to say that we were only interested in the responses (to part 6) of participants who were sufficiently aware of the properties of sequences and series to answer part 5 of the mathematical version correctly (although, for consistency, we also removed the 7 participants who incorrectly answered part 5 of the original version). After removing these participants from the analysis, the percentage of 'true' answers to the mathematical version was reduced from 30% to 9%. These data are shown in Table 2.

Answer	Original	Maths	Total
T	2	9	5
F	44	73	56
C	54	18	39
n	209	146	355

Table 2: The breakdown of responses, as percentages, of those participants who answered part 5 correctly, to statement 6.

The figures in Table 2 are clear. The mathematical version of the task elicited many more "false" responses than did the original version. This difference is highly significant, with a moderate effect size, $\chi^2=48.9$, df=2, $p<0.001$, $\phi=0.371$.

DISCUSSION

The results from this study are interesting for several reasons. Recall that Durand-Guerrier (2003) reported that approximately 60% of 15-16 year old students responded with "can't tell" to statement 6 of the original labyrinth task, and noted that this response tended to be given by students of high mathematical abilities. Our results cast doubt upon this interpretation. Our sample of extremely able 18-19 year old students were fairly evenly split on this item. If there was some correlation between mathematical ability and answering "can't tell", we would expect a substantially higher percentage of our participants to have answered in this way.

It is also clear that the context in which the question is set has a significant influence upon responses. Conditional sentences in mathematical contexts appear, across the sample, to be treated differently to conditional sentences in non-mathematical contexts. Paradoxically, the mathematical context appears to bias highly able students towards what Durand-Guerrier (2003) believed was the mathematically incorrect answer.

THE RAMSEY TEST – CONDITIONALS IN MATHEMATICS

How then can we account for these results? It seems that a mathematical context fundamentally alters the manner in which conditional statements are evaluated. But how? To see how mathematicians evaluated statement 6 in the labyrinth task we conducted 11 task-based interviews with a range of university level mathematics students. Participants were asked to solve the original task whilst speaking out loud.

In the following extract we report how one student, Rachel, responded to the original version of the task. Rachel is a postgraduate student, and had been a teaching assistant on the Foundations of Mathematics module in which formal logic is taught to first year undergraduates.

Rachel: This one [statement 6] is wrong.

Interviewer: Why?

Rachel: Well the statement is saying that if he crossed *L* then he definitely crossed *K*, which is not true. Because you could have gone *I-L-M* and then leave the maze and then you wouldn't have gone through *K*, I mean it would have been a possibility to go through both, but it's not a necessity, which makes the statement wrong.

Here Rachel is clearly not using the Ramsey Test to evaluate statement 6. Instead she interprets the conditional statement as demanding that *X necessarily* has to have gone through *K* if s/he went through *L*. The interviewer asks what would happen if more knowledge about the route became available:

Interviewer: OK. How would you react if I told you what the route was? [Describes a route that does go through *L* and *K*]. How would that affect [statement 6]? So if the person did go through *K* and *L*?

Rachel: Well it's still wrong. Because this is just a conditional thing saying that if this happens then something else happens and this, you know, this has got to be true for all routes that cross *L* not just the particular one chosen. You know, as I said, you can go through *K* and *L* and still leave the maze without going through any door twice. So it's a possibility, so it's not wrong in the sense that it can never ever happen, but it's an implication that you can't make.

So Rachel clearly believed, contrary to Durand-Guerrier's logical analysis, that implication *does* carry necessity, for her it is *not* merely a set of truth values. When the interviewer points out the truth table for "$P \Rightarrow Q$" and argues that an analysis along these lines suggests that the sentence could be either true or false depending on the particular route, Rachel remains unconvinced:

Rachel: I don't believe your argument, I'm sorry.

Interviewer: So where's the flaw in my argument?

Rachel: Umm, I don't know… umm, I don't know, that's the problem I have at the moment […]

Interviewer: So if you were teaching some first years and [the labyrinth task] was a question on their exam, what answer would you hope that they'd give to number 6?

Rachel: That's a tricky question. If they'd just done logic and they'd drawn a truth table and said, you know, you've got both in the last column so you can't tell which it is, I suspect I would feel obliged to give them full marks.

Whilst Rachel, a successful mathematician, clearly understands the argument based on truth values, she remains unconvinced by it. Nevertheless she grudgingly accepts it may be 'correct' in some unnatural formal sense.

Although the transcript indicates that Rachel has all the information required to perform a Ramsey Test successfully and deduce that $\mathbf{P}(Q|P)$=0.5, she resists. Instead Rachel seems to be demanding that, for a conditional statement to be true in mathematics, $\mathbf{P}(Q|P)$ must be equal to 1. That is to say that for the statement "$P \Rightarrow Q$" to be evaluated as "true", once P has been added hypothetically to her stock of knowledge, she is demanding to be able to conclude Q with absolute certainty. Thus the Ramsey Test appears to operate differently in mathematical contexts for mathematicians than in general day-to-day life.

This idea of a modified Ramsey Test has strong connections with Weber & Alcock's (2005) notion of a *warranted conditional*. Drawing on Toulmin's (1958) work on informal logic and argumentation, they define a conditional "$P \Rightarrow Q$" to be *warranted* if the consequent Q necessarily follows, by some valid mathematical procedure, from the antecedent P. They suggest that a conditional statement is invalid in mathematics unless it is warranted. Recall our example from the mathematical labyrinth task:

If $a_n \to 0$, then $\sum a_n$ converges. (*)

In Weber & Alcock's terms, this statement is unwarranted as the consequent does not necessarily follow from the antecedent. However the statement is true for certain sequences (a_n). In the language of Ramsey (1931) and Evans & Over (2004). Weber & Alcock are saying that when evaluating this statement a person hypothetically adds the belief that (a_n) tends to zero to their stock of knowledge, and evaluates their degree of belief in the series converging. If their degree of belief is not 100%, or thereabouts, the conditional is rejected as unwarranted and false.

Evaluating the Ramsey Test in mathematical contexts may be a non-trivial matter, and in some circumstances it may rely more upon general knowledge of the subject matter than it does on the actual argument contained in the proof. Indeed it has even been argued that acceptable mathematical proofs routinely contain 'gaps' that break the chain of implications justified by 'valid' mathematical warrants (Fallis, 2003).

WHICH RAMSEY TEST? THE ROLE OF CONTEXT

Our results clearly indicate that the majority of first year undergraduates evaluated statement (*) as being false, suggesting that they conducted a modified version of the Ramsey Test. However, for the original version of the task roughly half the sample answered "false" and half answered "can't tell". So if our analysis is correct than there was no clear agreement whether to use the modified version of the Ramsey Test or the standard version. We argue that this is because the context was less clearly mathematical in this version. The labyrinth task is not overtly a mathematical question, despite appearing in a mathematics test. However, the mathematical labyrinth task is visibly mathematical: it refers to subject matter from real analysis.

We believe[ii] that mathematicians judge everyday conditionals – such as "if you're in Birmingham, then you have a good choice of Indian takeaways" – in the same manner as the rest of the population. Namely, according to Evans and Overs's (2004) theory, they conduct a standard Ramsey Test to fix their degree of belief in Q given

P. But when in the mathematics classroom, the lecture theatre or the office, they seem to behave differently: they use a modified version of the Ramsey Test, which demands that $\mathbf{P}(Q|P)$ is equal to 1.

We, therefore, believe that Durand-Guerrier (2003) should not have been surprised that the teachers she spoke to considered “can’t tell” to be the incorrect answer to part 6 of the labyrinth task. The teachers were evaluating what they believed to be a mathematical statement in a manner appropriate for a mathematical context.

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[i] The assessment was set up in such a way so that the students’ overall mark would be improved if they performed well on the experimental question, but that if they scored below their average mark for the rest of the test, the experimental section would be ignored. Thus all the participants had an incentive to take the experimental questions seriously, but would not be disadvantaged by a poor performance on this section.

[ii] Note, however, that we have no empirical evidence to back this belief up. More work is needed on individual differences in contextual reasoning behaviour.

KNOWLEDGE FOR TEACHING FRACTION ARITHMETIC: PARTITIONING DRAWN REPRESENTATIONS

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I report on knowledge that two U.S. 6th-grade teachers deployed when using linear and area representations to teach fraction arithmetic for the first time. Both teachers were using the Bits and Pieces II unit from Connected Mathematics. Data came from videotaped lessons and interviews. Neither teacher appeared to see solving problems with drawn representations as a source of experience from which students could deduce written symbolic procedures. The results suggest that the role of representations in mathematical knowledge for teaching needs still closer examination as we try to support sense making in classrooms.

CONTEXT AND OBJECTIVES

Research on teacher knowledge has expanded from studies of teachers' subject-matter knowledge of various content areas to the organization of teachers' knowledge for teaching particular content to students (e.g., Ball, 1991; Ball, Lubienski, & Mewborn, 2001; Borko & Putnam, 1996; Ma, 1999; Shulman, 1986). As part of this development, current discussions of teacher knowledge are often framed in terms of subject matter, pedagogical, and pedagogical content knowledge (e.g., Borko & Putnam, 1996). When introducing the notion of pedagogical content knowledge, Shulman (1986) emphasized knowledge of students' thinking about particular topics, typical difficulties that students have, and representations that make mathematical ideas accessible to students. Building on the notion of pedagogical content knowledge, Ball et al. (2001) emphasized the importance for the field of examining how teachers use mathematical knowledge in the course of their work and argued that, with respect to methods for research, this implies starting with practice and working back to infer mathematical knowledge that supports both routine and non-routine aspects of practice.

Central questions about the mathematical knowledge teachers use in practice include: What mathematical knowledge do teachers need? Where and how do teachers use such knowledge? How can teachers develop such knowledge? For two reasons, I approach these questions with a focus on knowledge used when teaching with drawn representations. First, reform-oriented curricula in the United States often place new demands on teachers and students to interpret, reason with, and make connections among representations. Second, when research on teacher knowledge has attended to representations (e.g., Ball, 1993; Ball & Bass, 2000; Eisenhart, Borko, Underhill, Bright, Jones, & Agard, 1993; Shulman, 1986), most often it has not focused on fine-grained features of representations that recent research has demonstrated can be central to students' sense making (e.g., Izsák, 2005; Steffe, 2004).

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Data for the present report come from case-studies of two U.S. sixth-grade teachers who were teaching fraction arithmetic using reform-oriented materials for the first time. The materials were from the Connected Mathematics Project (CMP; Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002). These materials use linear and area representations to help students develop understandings of fractions and fraction arithmetic. The teachers focused on various teaching opportunities in the CMP materials but neither focused on opportunities to develop numeric algorithms from experiences reasoning with drawn representations. The two case-studies provide related, but different, insights into knowledge that could support more fully teaching fraction arithmetic with drawn representations.

BACKGROUND

In addition to theoretical perspectives on teacher knowledge cited above, several results from research on students' construction of fraction understandings helped to explain the (possibly tacit) knowledge that the two teachers used when enacting the CMP activities. Researchers have examined the roles that students' whole-number knowledge can play when constructing understandings of fractions. Streefland (1991) used the term N-distractors to refer to a range of phenomena in which students use their whole-number knowledge to misinterpret fractions. Particularly relevant to the present case-studies are reports of elementary students who interpreted numerators and denominators as *pairs of whole numbers*; the denominator denoted the cardinality of the set and the numerator the cardinality of a subset. Students using this interpretation have made errors when failing to maintain the correct unit—for instance, when misinterpreting two units, each divided into four pieces, as one unit divided into eight pieces (e.g., Ball, 1993, p. 165-166).

Other researchers have examined how students' whole-number knowledge can support construction of fraction knowledge. For this report, I focus on some results emphasized by Olive and Steffe (Olive & Steffe, 2001; Steffe 2001, 2003, 2004), whose research has examined how elementary students constructed knowledge of fractions as they reasoned about lengths and areas. *Iterating* (Olive & Steffe, 2001; Steffe, 2001) involves taking a length segment and joining copies end to end to make longer segments. For instance, students might iterate to test an estimate for the length of one fourth of a unit segment. *Recursive partitioning* (Steffe, 2003, 2004) is defined to be taking a partition of a partition in the service of a non-partitioning goal. For instance, to understand the result of taking 1/3 of 1/4, students might begin by partitioning a unit into four pieces and then partitioning the first of those pieces into three further pieces. Determining the size of the resulting piece is a non-partitioning goal, and students could accomplish this in more than one way. Students might simply iterate the resulting piece and count to see that 12 copies fit in the original unit. This solution requires decomposing an initial unit into a unit of units (one unit containing 12 twelfths). Alternatively, students might recursively partition by subdividing each of the remaining fourths into three pieces. In contrast to the first solution, recursive partitioning involves decomposing an initial unit into a unit of units of units structure (one unit containing 4 fourths, each of which contains 3 twelfths). The first solution is based on *2 levels of units*, the second on *3 levels of units*.

METHODS AND DATA

Data for the present report come from a larger study of teaching and learning mathematics that is being conducted by a team of researchers in a rural middle school in the Southeastern United States. The school first adopted the CMP materials in the 2001-2002 school year and has a racially and economically diverse student body. Data for the first teacher in the present report were collected in Spring 2003, and for the second teacher in Spring 2004. Both teachers began the transition to reform-oriented materials with limited professional support. The district hired a consultant to help each grade level select and prioritize units for the first year. The research project provided further support starting in Spring 2003.

Members of the research team videotaped each teacher's lessons during the same class period every day for 4 to 5 weeks. Each afternoon we analyzed that morning's lesson for the mathematical ideas, problem-solving strategies, and representations that, from our perspective, seemed central. Oftentimes, excerpts during which the teacher and students had difficulty understanding one another as they discussed drawn representations provided broader access to the range of knowledge teachers engaged than those places where lessons progressed smoothly. We replayed such excerpts in student and teacher interviews.

I conducted weekly, semistructured interviews with three to four pairs of students selected from the same classrooms to represent a cross-section of achievement. I had the students work on tasks that posed similar questions and used similar external representations to those in the lesson excerpts. I probed students' thinking for the mathematical understandings that they brought to bear on the problems, including understandings of linear and area representations. I then had the students watch the video clips and asked them to comment on what they thought their teacher wanted them to learn. As the interviews progressed, I moved back and forth between tasks and clips to access ways that students used their current understandings of the content to make sense of the lessons.

I then worked with other members of the research team not listed as authors to plan weekly teacher interviews that used the same lesson excerpts and related student interview excerpts as prompts. Researchers asked the teachers to summarize their approaches to preparing and conducting the lessons, to examine student work from the lessons and interviews, to comment on what students understood and where they struggled, and to discuss how they might address students' observed difficulties in future instruction. These interviews provided further access to understandings of the mathematics (including representations) and students that teachers used during the observed lessons.

Once the data were collected, I conducted further, more detailed analyses of the videos using a version of the constant comparative method described by Cobb and

Whitenack (1996) for conducting longitudinal analyses of classroom videorecordings. These analyses used talk, gestures, and inscriptions as evidence for teachers' and students' understandings of the content and the lessons. I treated knowledge teachers evidenced in interviews as confirming evidence in cases where it appeared consistent with knowledge teachers evidenced in lessons. In cases where knowledge evidenced in interviews appeared inconsistent with knowledge evidenced in lessons, I reexamined both sets of data and tried to refine my interpretations to achieve a consistent account of what teachers said and did in both contexts. Finally, I examined the teacher's edition to determine which mathematical ideas the CMP materials emphasized and how the materials presented the role of external representations in the activities.

ANALYSIS AND RESULTS

Bits and Pieces II develops fraction arithmetic through problems in which fractions are embedded. Many of the problem situations can be modeled using length or area representations. Teachers are to help students develop their own strategies and to inject further ideas for students to consider. In my reading, the CMP materials intend for students ultimately to construct written symbolic methods based on their experiences reasoning with drawn representations. The main result I emphasize is that, for different reasons, neither teacher attended to opportunities for *deducing* the written symbolic methods from the drawn methods. Instead, they understood drawn and written symbolic methods to be *parallel* methods that led to the same answer.

Example 1: A Case of Latent Resources for Partitioning

Ms. Reese used two methods for partitioning unit intervals when teaching students how to add and subtract fractions on number lines. One method was related to iterating, the other to recursive partitioning. When using the first method, she simply added tick marks from left to right. For instance, to partition the interval from 0 to 1 into eighths, she marked 1/8, 2/8, …, 7/8 in order and moved the location of the 1, if necessary, to create intervals of equal length. When using the second, she created the partition in stages by marking half, then 1/4 and 3/4, and finally the remaining eighths. She used the first method most often but used the second in special cases where she wanted accurate drawings. She identified a common denominator before partitioning and did not connect taking partitions of partitions with finding common denominators.

Analysis of the lesson and interview data revealed a set of understandings that directed her attention toward the first partitioning method more often than the second. First, she wanted all students to find and understand at least one reliable method for adding and subtracting fractions and so valued both number line and written symbolic methods because having more than one method provided choices for students. Second, from Ms. Reese's perspective, showing each fraction as a length and the sum as the final location on the number line was much more important than the process by which the number line representations were produced, and she thought partitioning unit intervals from left to right (the first method) was more accessible to her students than partitioning in stages (the second method). Finally, Ms. Reese thought her students were ready to replace drawn with more advanced symbolic methods. Thus, she did not want to spend much class time discussing different partitioning methods.

Interviews with some of Ms. Reese's students revealed that the first partitioning method reinforced, unintentionally, some difficulties maintaining a fixed whole. One pair of students even drew a number line that had "1" and 12 twelfths in two separate places. They explained that the "1" was an estimated location and 12 twelfths was an exact location. The students did not erase the "1" and became confused when trying to reason with it and 12 twelfths in separate locations. Ms. Reese watched this excerpt and commented that she had "no idea" why her students would reason like this. My conjecture is that they inferred this from her left-to-right partitioning method. Moreover, when I told the students to think of the "1" as an exact location, they began reasoning about partitions of partitions. Thus, both Ms. Reese and her students evidenced knowledge central for using number lines to develop fraction addition with unlike denominators, but that remained largely latent during the lessons.

Example 2: A Case of Inflexible Units

In the case of Mrs. Archer we collected data on the preceding *Bits and Pieces I* unit as well as on *Bits and Pieces II*. *Bits and Pieces I* introduces various fraction interpretations and representations, and during this unit Mrs. Archer and her students began to solve problems that involved products of fractions and whole numbers. The problems used thermometers to show progress toward fund-raising goals. The most common method was to determine the value of one equal part of the whole number (interpreting fractions as pairs of whole numbers) and to use repeated addition or multiplication to determine the final answer. For instance, to determine 3/4 of 640 the class found that 1/4 of 640 was 160 and calculated 160 + 160 + 160 = 480 and 3 x 160 = 480. This solution was also discussed in the CMP materials, and lesson and interview data suggested that many students understood it. Mrs. Archer and her students apparently attended explicitly to only 2 levels of units (640 is 4 groups of 160).

Mrs. Archer had more difficulty introducing products of proper fractions, problems in which 3 levels of units become central. As suggested in the CMP materials, she introduced the product of two proper fractions using the rectangular area model (interpreted as a pan of brownies) to help students solve 1/2 x 2/3. As shown in the teacher's edition, Mrs. Archer first partitioned the unit square vertically into thirds and then horizontally into halves. In so doing, she showed 1/2 of the whole pan, not just of the 2/3. She told students the answer could be found where the two fractions overlapped and that this demonstrated "why" 1/2 x 2/3 = 2/6. She confirmed this by multiplying numerators and by multiplying denominators, a computation she had already discussed a few times. Several students thought that the diagram showed the answer to be 5/6 because 5 of 6 pieces were shaded. In response, Mrs. Archer simply repeated her instructions to look at the overlap.

Mrs. Archer had students then work on further CMP problems that introduced “part of the part” of a brownie pan. As Mrs. Archer helped students who were confused by the “part of the part” language, she gradually changed the way she drew brownie pan representations. In so doing, Mrs. Archer apparently began attending (at least implicitly) to 3 levels of units, but she did not do so flexibly. As one of several examples, Mrs. Archer first rejected several solutions to 1/2 x 2/3 in which students partitioned the whole pan into thirds and then, without any further partitioning, pointed to one of two shaded thirds as their answer.

In interviews, students drew a variety of brownie pan solutions that did show part of a part and that suggested initial attention to 3 levels of units, but that did not suggest the computation algorithm. For instance, some students created vertical partitions for both factors, even for problems like 2/5 of 3/4. In such cases, students could not relate the part of the part back to the whole to determine the product. When reviewing students’ brownie pan representations for fraction multiplication, Mrs. Archer asserted that the purpose of drawn representations was to show "why," by which she meat that the drawn representations showed part of a part of a whole and gave the same answer as the numerical procedure. She did not appear to see that combining vertical partitions for one factor with horizontal partitions for the second led to opportunities for constructing algorithms. More generally, Mrs. Archer said that drawn representations helped students connect fractions to “real life” situations and stated that methods based on drawn representations were an alternative to computation algorithms. Thus, like Ms. Reese, Mrs. Archer maintained drawn and symbolic methods as parallel.

CONCLUSION

The cases of Ms. Reese and Mrs. Archer are both instances in which analyzing teachers’ and students’ use of representations in fine detail can provide insight into knowledge necessary for supporting effective practice with reform-oriented materials. Central to both examples were pedagogical affordances of different methods for creating partitions. Ms. Reese was unaware that partitioning from left to right, a method that she thought would be more accessible to students, in fact seemed to compound problems for students who did not yet maintain a fixed unit. Methods based on partitions of partitions, which she thought were unnecessarily involved, seemed to address difficulties these same students were having. Mrs. Archer, on the other hand, was not flexible enough in her perspective on area models of fraction multiplication to build on the variety of ways that her students were beginning to construct 3 levels of units. In fact, Mrs. Archer seemed to be attending to a third level of unit more explicitly as her lessons unfolded and students asked about the "part of a part" language in the CMP tasks. From my perspective, that neither teacher attended explicitly to taking partitions of partitions and the different 3-level structures that can result played a central role in limiting students' opportunities to use experiences with drawn representations for constructing computation algorithms. Finally, the data on Ms. Reese demonstrate that understanding mathematical knowledge for teaching is not simply a matter of cataloguing the contents of teacher knowledge, but also understanding the contexts in which teachers use that knowledge.

The research reported in this article was supported by the National Science Foundation under Grant No. REC-0231879. The opinions expressed in this paper are those of the author and do not necessarily reflect the views of NSF.

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INQUIRY COMMUNITY IN AN ACTIVITY THEORY FRAME

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A developmental research project, Learning Communities in Mathematics (LCM) bases its activity on the theoretical concepts of inquiry and community. It seeks to create knowledge and improve practice in the learning and teaching of mathematics through developing inquiry communities between teachers in schools and didacticians in a university setting. Analysis of data requires a recognition of the complexity of socially embedded factors, and we draw on activity theory to address complexity and deal with issues and tensions related to learning within the project. This theoretical paper presents our early thinking in analysing inquiry community within an activity theory frame.

We present a theoretical paper related to a research project, LCM[1] (Learning Communities in Mathematics) introduced in previous papers. The theoretical basis of our project is *community of inquiry* which is addressed in Jaworski (2004, 2005 and in press). In Goodchild & Jaworski (2005), we introduced an activity theory frame for analysing data within the project; here we provide a more detailed theorisation.

INQUIRY COMMUNITIES AND DEVELOPMENTAL RESEARCH

Teachers and didacticians each bring specialised knowledge to developing teaching, and hence learning, of mathematics. *Together* we can use, and *explore* the use of this knowledge in order to improve the mathematical learning experiences of students in classrooms and to know more about the creation of good opportunities for learning. The words “together” and “explore” adumbrate the concept of *inquiry community* (Wells, 1999). Fundamentally, inquiry and exploration are about *questioning*: asking and seeking to answer questions. Together, we ask and seek to answer questions to enable us to know more about mathematics teaching and learning. Moreover, the asking of questions is a developmental tool in drawing students, teachers and didacticians into a deeper awareness of their own actions, motives and goals (Jaworski, 1994; Mason, 2001).

Thus we engage in *developmental research*: research which both studies the developmental process and, simultaneously, promotes development through engagement and questioning. We recognise engagement in *inquiry activity* within a well defined *community* as a significant means of *coming to know*. Thus, a major aim of our critical questioning approach is to take us, as a community, deeper into knowing mathematics teaching and learning. Not only are research questions defined and explored (through suitable data collection and analysis), but the whole research process is subject to question and exploration. We look critically at our research activity while engaging in and with it (Chaiklin, 1993).

Teachers in the project belong to particular school communities within the school system, functioning within educational norms in society, and a political framework. Everyday factors such as curriculum, school timetables, responsibilities of teachers, time and energy afford and constrain what is possible (Engeström, 1999). We would not describe such communities as communities of inquiry, although they might be termed *communities of practice* (Wenger, 1998)[2]. They are groups of people dedicated to specific activity with established ways of thinking and doing which may be questioned but are most often taken for granted in the everyday momentum. Didacticians belong to a university community in which research activity is a norm involving familiarity with inquiry at formal levels. As teacher educators, didacticians are expected to question and theorise teaching in schools. However, questioning or theorising their own activity (of teaching and research) may not be a community norm. Thus, a desire to generate communities of inquiry within the project requires a serious addressing of the activity and goals of these various communities and a searching for ways of generating the kinds of thinking and coming to know that we expect from inquiry activity (Cochran Smith & Lytle, 1999; Wells, 1999)

The LCM project was designed by didacticians who sought funding and had initial responsibility for the project. Schools and teachers were recruited after funding had been secured (Jaworski, 2005). Initial activity was motivated by the need to establish a project community and to start to understand jointly what *inquiry* might mean within the project. Didacticians have designed activity to create opportunity to work with teachers, to ask questions and to see common purposes in using inquiry approaches that bring both groups closer in thinking about and improving mathematics teaching and learning. We design workshops, and tasks for workshops, through which parallel design activity can start to take place in schools. This design process is generative and transformative (Kelly, 2003). We use tasks necessitating inquiry to generate inquiry activity through which a joint community, with common goals can emerge.

Workshops have encouraged all of us to do mathematics together, to inquire in tackling mathematical problems, to raise questions about learning and teaching and to start to think and plan for the classroom. In schools, teacher teams, with didactician support, follow up experiences from workshops to explore possibilities for inquiry activity in classrooms, engaging themselves in inquiry through their design of tasks for students. These words express, simply, aspects of the project design and of its implementation but they underestimate the complexity of the process and the problematic nature of interpreting project goals into the realities of engagement in the project (Goodchild & Jaworski, 2005; Jaworski, 2005). We see tackling issues and tensions as forming the essence of our learning: at a practical level, for the project to make progress; and, at a theoretical level, to conceptualise their role in our learning development, both theoretical and practical. It is here that we are exploring the use of activity theory as an analytical framework and toolbox.

ACTIVITY THEORY (AT)

Key concepts and terms

Activity theory develops from the work of Vygotsky, particularly his arguments that cognition arises through the internalisation of external operations that occur in sociocultural contexts (Vygotsky, 1978). In identifying an intermediate link in the stimulus-response process, Vygotsky proposed the notion of a "complex mediated act" which "permits humans … *to control their behaviour from the outside*. The use of signs leads humans to a specific structure of behaviour that breaks away from biological development and creates new forms of a culturally-based psychological process" (1978, p. 40, italics in original). Through consideration of sociocultural artefacts that mediate between stimulus and response, the idea of a complex mediated act has been developed further. For example, following "the tradition of the theory of activity proposed by A. N. Leont'ev", Wertsch refers to "goal-directed action" and writes, "human action typically employs 'mediational means' such as tools and language". He goes on to emphasise that "the relationship between action and mediational means is so fundamental that it is more appropriate, when referring to the agent involved, to speak of 'individual(s)-acting-with-mediational-means' than to speak simply of 'individual(s)'" (1991, p. 12). A. N. Leont'ev makes the following point, "in a society, humans do not simply find external conditions to which they must adapt their activity. Rather these social conditions bear with them the motives and goals of their activity, its means and modes. In a word, society produces the activity of the individuals it forms" (1979, pp. 47-48). So, according to Wertsch (p. 27), rather than "the idea that mental functioning in the individual derives from participation in social life", "the specific structures and processes of intramental processing can be traced to their genetic precursors on the intermental plane".

The key idea for us here is that human activity is motivated within the sociocultural and historical processes of human life and comprises (mediated) goal-directed action. According to Leont'ev, "Activity is the non-additive, molar unit of life … it is not a reaction, or aggregate of reactions, but a system with its own structure, its own internal transformations, and its own development" (p. 46). He proposed a three tiered explanation of activity. First, human *activity* is always energised by a *motive*. Second, the basic components of human activity are the *actions* that translate activity motive into reality, where each action is subordinated to a conscious *goal*. Activity can be seen as comprising actions relating to associated goals. Thirdly, *operations* are the means by which an action is carried out, and are associated with the *conditions* under which actions take place. Leont'ev's three tiers or levels can be summarised as: activity ←→ motive; actions ←→ goals; operations ←→ conditions.

Leont'ev writes emphatically about the movement of the elements between the 'levels' within an activity system: activity can become actions and actions develop into activity, goals become motives and vice-versa, similarly with operations-conditions. The crucial differences seem to be: first, goals are conscious, if the motive of activity becomes conscious it becomes a motive-goal; second, motive is about an energizing force for the activity and the actions, it is not something that is attained but rather drives the activity forward; on the other hand goals are results that can be achieved. Leont'ev writes "The basic 'components' of various human activities are the actions that translate them into reality, We call a process an action when it is subordinated to the idea of achieving a result, i.e. a process that is subordinated to a conscious goal" (pp. 59- 60).

Exemplifying AT terms and concepts in the LCM Project

Here we exemplify briefly the concepts and terms above with reference to examples from the LCM project. These examples (rows in the table below) are deeply related to each other and so could be considered elements of one complex activity system. We separate them artificially to show elements of the three levels (the columns). Exemplification is an oversimplification, but serves the purpose of clarifying how we see these terms and concepts fitting our project and serving as a basis for analysis. In each case actions and operations are only examples of many possibilities.

Activity (System) & Motive	Actions & Goals	Operations & Conditions
Developmental research, whose motive is to study developmental processes and, simultaneously, promote development in the learning and teaching of mathematics.	Asking researchable questions, collecting and analysing data leading to findings or outcomes related to new knowledge and/or practice.	Making methodological decisions related to principled and effective ways of collecting and analysing data to address research questions.
A school, whose motive is to educate pupils.	Organising teaching groups and designing lessons to promote learning according to the declared curriculum.	Choosing topics and planning classroom tasks according to the school's approaches to addressing the curriculum. (Planlegg et opplegg[3])
The LCM project as community of inquiry, with motive to provide the environment and modes of action for teaching development to be realised.	Creating opportunities for working together and engaging in inquiry to achieve a working community with practical knowledge of inquiry processes.	Teachers and didacticians working in groups in workshops on mathematical problems to exemplify inquiry processes and develop common understandings.

Mediation in goal directed action

Within an activity system, goal-directed actions are mediated by tools and signs as represented by the basic mediational triangle deriving from Vygotsky and developed further by Leont'ev (Figure 1) (e.g., Engeström, 1999; Vygotsky, 1978). Here we see the human subject or group seeking to achieve a goal or object, mediated by some tool or sign where the nature of the tool or sign is deeply embedded in the activity. In recognition of this deep embeddedness, Engeström expanded the basic triangle to the "complex model of an activity system" (1999, p. 31) to recognise mediation by or through *community*, *rules* of activity and *division of labour* within the activity system. Each of the connections within the expanded triangle indicates possible mediational means within the system. The double arrows throughout indicate dialectical dependencies between the elements of a system.

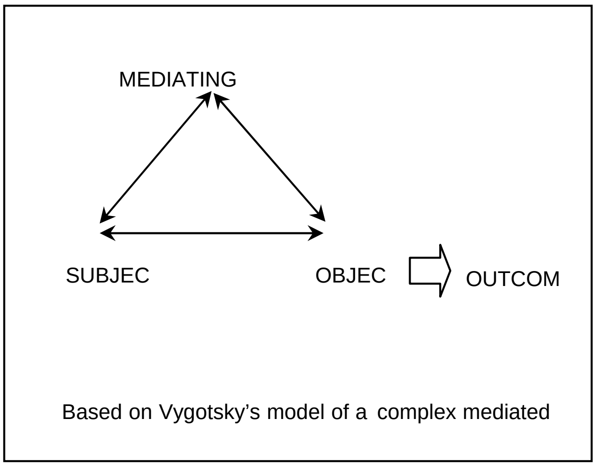

Figure 1

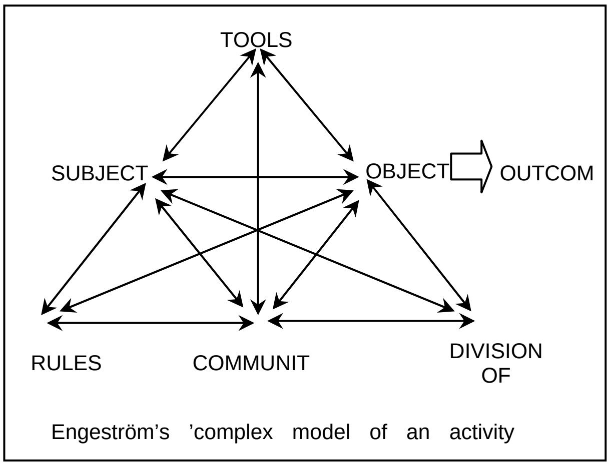

Figure 2

LCM AS AN ACTIVITY SYSTEM

Within the LCM project we engage in research that seeks to promote development in teaching and learning mathematics and to study that development. We believe this can be achieved through the development of an inquiry community comprising teachers and didacticians. This is the energizing force of the project, it is the motive for the activity of the project. The motive provides a rationale for the activity, and an incentive for the actions that comprise the activity. The actions are 'energized' by the motive but they are directed towards achieving some conscious goals, achievable results that will arise from the actions. For example, we want to achieve a sense of community, so we organise workshops and within those workshops opportunities for teachers and didacticians to meet together, work together and discuss together. Individual didacticians also spend time on their own seeking out mathematical activities for the workshops; in this action the goal is to find tasks that show potential to be of use. The result of this time spent is an 'oppgave'[3]; neither the action nor the 'oppgave' is the central purpose of the project, but the motivation for the action of finding the tasks is the same 'energizing force' (motive) of the project.

In the project activity system a number of mediational means are available to support or enable the actions that comprise the activity. For Vygotsky the main tool was language and the project seeks to develop a language of inquiry within the community. It has been emphasised that the asking of questions is fundamental to inquiry. In this respect the questions are an important tool or artefact within the envisaged activity system. The workshops, tasks, research literature, meetings of teachers in school or didacticians at the college also have a role as 'mediating artefacts'. The rules of the activity system, now, include rules underpinning rigorous research and rules governing teachers' and students' work in school, such as following the national curriculum. However, through the project we anticipate that our understanding of the developmental research paradigm will grow and as teachers engage to a greater degree in teaching characterised by inquiry processes it is possible that their interpretation of the curriculum 'rules' may change. At the outset of the project, there were a number of separate communities, the community of didacticians and a number of school communities, each pursuing their own activity largely independent of the others. In coming together within the project it was recognised by each community that we can learn together and develop our practice. The final item in Engeström's model is 'division of labour'. Inevitably in the envisaged activity system didacticians, teachers and students will have distinct roles that engage them in different tasks and actions. As teachers increasingly recognise and value their own research potential and didacticians participate in school and classroom, we anticipate that the division or labour will change. Thus we see, and expect to see, developments in the activity system as we engage in it.

THE TRANSFORMATIVE NATURE OF THE ACTIVITY SYSTEM

The LCM project emerges from a vision of an activity system whose motive is to engage, collaboratively, didacticians, teachers and students in developing and researching the teaching and learning of mathematics through processes of inquiry. At the time the project was proposed, this activity system did not exist, nor did it exist when the proposal began to be implemented. Now, halfway through the initial funding period, an activity system exists but we question the extent to which it fulfils what was envisioned. The vision is of a coherent community of co-learners taking roles relevant to the nature of their participation with responsibility as partners within the project. For example teachers and didacticians might be both insider and outsider researchers: insiders, as they seek to explore and develop their own practice and outsiders as they explore characteristics in their students' learning and understanding of mathematics, or in the activity of their co-participants (Jaworski, 2005).

The words above point not only to possible divergence between original goals and current activity, but to the transformative nature of the process in which we engage and the problematic nature of what we experience. In promoting development of inquiry communities we are motivated by theoretically warranted visions of transformation in mathematics teaching and learning. In the reality of project implementation, we recognize people, relationships, existing systems, ways of being and thinking and obstacles to change. Every event emerging in research embodies a complex story (e.g., Goodchild & Jaworski, 2005). We learn about the development of inquiry communities as teachers and didacticians act together and embrace notions of inquiry. We have talked about using inquiry as a tool leading to development of inquiry as a way of being (Jaworski, in press). We see 'inquiry as a tool' in many circumstances within the project. For example, tasks designed for workshops promote joint asking of questions and associated exploration. Tasks, questions and exploration are tools mediating activity. Use of these tools involves goal directed actions comprising condition-related operations. However, 'inquiry as a way of being' is a motive-goal (i.e., where the motive becomes conscious) of activity, rather than an outcome from it. Achieving an inquiry stance is ongoing and problematic.

Our developmental process is a struggle with developing thinking related to intransigencies in everyday activity. While workshop activity might prove inspirational in illuminating concepts of inquiry, and promote associated actions in school activity, the actions and emergent thinking contend with the demands of school activity, and established ways of thinking about it. Reports at a workshop of actions deriving from the previous workshop, reveal activity and thinking that both indicates elements of progress, and reveals limitations and constraints in vision and practice. As the project progresses, our analysis of data both charts the nature of development and reveals the problematic nature of that development. We are submerged in the complexity of relationships, interactions, demands from established communities with their deeply embedded ways of thinking, and our ongoing struggles with changing thinking and practice according to theoretical principles. We both recognize our situation as a complexity of activity systems, and draw on analytical frameworks in AT to navigate the complexity.

The AT triangle offers a unit of analysis for all levels of the project. Its value lies in the possibility of exploring the mediating elements and the dialectical relationships between elements. As the project is intended as developmental research the concern is to engineer, monitor and research changes within the activity system. Engeström suggests that contradictions and tensions take a central role as sources of change and development and thus the model can be used as both development and research tool in that it draws attention to those points where contradictions or tensions exists, whether these be within the elements or the dialectical relationships between the elements and thus prompts "a search for solutions … (that) reaches its peak when a new model for the activity is designed and implemented" Engeström (1999, p. 34). Engeström refers to this process as "the expansive cycle" (ibid. p. 33).

It is easy to be discouraged when plans do not result in envisioned outcomes. Particularly in the activity in schools, there have been many obstacles to realisation of teams of teachers working in inquiry modes. The developmental nature of the project is that we work with the perceived obstacles and through this work relationships develop and forms of activity emerge that could not have been predicted in the original design. We work with what we have, and rethink according to theoretical principles and emerging reality. Periodically, in the cycles of activity and thinking, a recognisably new way of acting and thinking emerges, and becomes new activity. We see this as an expansive cycle. Its importance for the project is twofold. Firstly it is manifested in a build up of tension with transformative power in that it promotes a new wave of activity with clear actions and goals. Secondly, and possibly most powerfully, it creates new learning in which we gain new insights to theoretical realisation in social and historical complexity.

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Notes

[1] The LCM Project is supported within the KUL Programme (Knowledge, Education and Learning) of the Norwegian Research Council. Project number 157949/S20.

[2] We have demonstrated a fundamental difference between community of practice and community of inquiry (e.g., Jaworski, in press). Space does not allow articulation here.

[3] Key Norwegian concepts and terms have entered into our vocabulary and are difficult to replace simply in English. *Planlegg et opplegg* refers to teachers' planning of tasks for the classroom, and their resulting lesson plans. Tasks and all their related feature are called *oppgaver*.

GENDER DIFFERENCES IN PATTERNS OF STRATEGY USE AMONGST SECONDARY SCHOOL MATHEMATICS STUDENTS

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Recent research of gender differences in mathematics education has indicated that boys and girls can often be differentiated in terms of the strategies used in response to mathematical problems. Using an experimental method based on Mevarech and Stern (1997), the present study aims to investigate patterns of strategy use in response to rate of change problems across two trials. Results indicate that one group, consisting mostly of girls, tends to use the same strategies in both trials, while a second group, mostly boys, tends to change their strategy between trials. This extends findings of research involving differences in strategy in static problem situations and may help to explain continuing gender differences in both achievement and pursuance of school mathematics.

The literature on gender differences in mathematical cognition has been steadily moving from the general to the more specific over the past two decades. Hyde, Fennema, & Lamon (1990) found that a small difference favouring males emerged in high school and college. They also found that the difference was greater among higher achieving students. Hedges and Nowell (1995) found a similar pattern of achievement, but also investigated the ratio of boys to girls among the top 10% of scores in standard mathematics tests. They found that among these top-scoring students, there were more boys than girls. Interestingly, Hedges and Nowell also note that while the gender difference in achievement in mathematics has been decreasing over the last several decades, the difference in the numbers of boys and girls amongst the highest achievers has remained constant.

Research investigating the reason for any gender differences in achievement has been more recent. Some studies have linked differences in achievement to differences in attitudes towards, or anxiety related to, mathematics (e.g. Ashcraft & Kirk, 2001; Nosek, Banaji, & Greenwald, 2002; Skaalvik & Skaalvik, 2004; Vermeer, Boekaerts, & Seegers, 2000). Others have associated differences in achievement with differences in ability in ‘math-fact retrieval’ (Royer, Tronsky, Chan, Jackson, & Marchant, 1999). There is also a growing literature concerned with gender differences in strategy use. For example, Fennema, Carpenter, Jacobs, Franke, and Levi (1998) showed that although there is no difference between boys and girls in proficiency and ability, there is a difference in the kinds of strategies that are used (for addition and subtraction problems). Girls tend to use traditional/taught strategies while boys tend to use invented strategies. Carr has shown that boys in their first year of school tend to use retrieval rather than algorithmic strategies when solving arithmetic problems due to their emphasis on the social impact of strategy choice (Carr & Jessup, 1997). Gallagher and de Lisi (1994) showed that boys were more likely to match strategy use with problem demands in questions on the SAT-Math paper than were girls. In a later study, the results of Gallagher and de Lisi (1994) were replicated (Gallagher et al., 2000) and the authors concluded that, ‘strategy flexibility is a source of gender differences in mathematical ability’.

The research discussed above shows that strategy use varies according to gender in response to specific problems. The present study aims to investigate gender differences in patterns of strategy use by comparing strategy decisions across problem solving instances. Siegler (1987) has shown that it is important not to rely on a single snapshot of data when investigating children’s strategy decisions. The way that a child solves a problem in one instance does not necessarily tell the researcher very much about how that problem has been attempted in the past or how it might be attempted in the future. Averaging data across either children or trials can be misleading. It would seem that further investigation of gender differences in strategy use might be valuable.

It seems that gender differences should be expected in the ways that children apply strategies across problem situations – the way that children transfer knowledge across contexts. An experimental method used in Mevarech and Stern (1997) provides a means of investigating such differences. One of the experiments reported in Mevarech and Stern (1997) involves children working on a set of problems related to rates of change of lines on a graph. There are two, isomorphic, sets of problems. One presents graphs that represent realistic situations; another presents graphs in a more abstract form. Half of the children in the experiment were presented with the realistic problem set followed by the abstract set one week later. The other half of the children received the sets in the reverse order. Mevarech and Stern were using this method to investigate differences in transfer effects depending on context. However, the method seems equally appropriate for the purposes of the present study.

The problems used in Mevarech and Stern (1997) lend themselves well to an investigation of patterns of strategy use. There are many possible strategies available to children, leading to both correct and incorrect answers. Also there is a clear distinction between reading off points from a graph and making judgements about rates of change, meaning that a wide range of children will access to these problems without already having been taught strategies for finding solutions.

The present study aims to use the experimental method of Mevarech and Stern (1997) to investigate differences between boys and girls in terms of patterns of strategy use across two sessions.

METHOD

Participants

Participants were 13-14 year old children from two complete classes one each from two schools in Nottinghamshire. Both schools’ Mathematics departments set children in terms of ability and both classes sampled were set 3 of 5, with set 1 being the most able.

The children were initially asked to complete Raven's Standard Progressive Matrices (Raven, 1976). This test is widely used to assess the non-verbal intelligence of children aged 8-16. There are 60 items in total, divided into five groups of 12. Each consists of a pattern in which there is a missing part. Children are required to select the correct part from a number of alternatives.

Lists for both boys and girls were drawn up in order of Raven's score. Alternate children were then placed in group 1 and group 2. The result of this process was four groups of children labeled 'male group 1', 'male group 2', 'female group 1', and 'female group 2'. The four groups had approximately equal average Raven's scores.

Tasks

Three isomorphic sets of problems were used, adapted from the study of Mevarech and Stern. Each task took the form of three printed A4 sheets stapled together. The top half of each sheet showed a graph – all of the questions in the set referred to the same graph. The instructions advised children that could do any workings out on the graphs if they thought it might help, and also that they should pay careful attention to their explanations when asked for.

On the second page there were 6 questions (1.a-c and 2.a-c) that asked children to read values from the graph given a value on one axis. The part c questions asked children to work out how much the value on the y-axis increased as the value of the x-axis increased. On the third page there were 3 questions (3. 4. and 5.) that were taken directly from Mevarech and Stern (1997), with only the wording changed in order to improve children's understanding of the questions, based on previous pilot work. These asked children about the rates of change of the two lines on the graph and also asked children to explain how they decided on their answer. For example, one question asked children, "after 1984, did the income of company A increase faster, slower or at the same rate as the income of company B?".

The only difference between the three sets of problems was the context. The sparse context problems involved a graph with axes labelled 'x' and 'y', and lines labelled 'line A' and 'line B'. There were two sets of realistic context problems; one involved a graph with axes labelled 'income' and 'year' and lines labelled 'company A' and 'company B', while the other involved a graph with axes labelled 'amount of water' and 'time' with lines labelled 'tank A' and 'tank B'.

Procedure

Children were administered the two tests, one week apart. Children in group 1 were initially given the sparse task, with the realistic task a week later. Children in group 2 were given the tasks in the reverse order.

Tasks were administered individually, taking approximately 15 minutes to complete in total. Each participant was asked to complete a set of problems. Once the set of problems was completed, the experimenter checked the explanations given by the children to ensure that enough detail had been given in order to determine the strategy employed. Strategies used to judge relative rates of change included using:

- Relative gradient/steepness of lines
- Relative height of lines
- Fact that lines meet at a point
- Fact that the points on the lines 'line up'
- Fact that two lines start apart and one 'catches up'
- Calculation of gradient
- Fact that the 'step' between points was greater for one line than the other

The experimenter had the opportunity to ask each child for more information at the end of each session in order to elicit missing answers and to obtain further information regarding strategy. Questions were asked with the intention that children should not be lead towards one explanation or another and were as neutral as possible, such as "could you tell me a bit more about how you did this one". The experimenter's responses were also as neutral as possible, intended to give no indication as to the correctness or otherwise of answers or of explanations.

RESULTS

Initially, confirmation was made that the groups were matched appropriately and that there was no difference in baseline ability (measured using Raven's Matrices) between either groups 1 and 2 ($t=0.4$, $p>.05$) or between boys and girls ($t=.\ 0.587$, $p>.05$).

There was no difference in average score on either trial between girls and boys (First trial $t=1.019$, $p>.05$, Second trial $t=1.899$, $p>.05$). Nor was there a difference in improvement between trials between girls and boys ($t=0.669$, $p>.05$)

Boys' improvement between trials is related to problem order ($t=2.797$, $p<0.01$), with greater improvement experienced by those completing the abstract task followed by the realistic task. This improvement seems to be due to a reduced score for the abstract task set in the first week when compared with the figures shown for the other three groups. Girls' improvement is not related to problem order ($t=1.16$, $p>0.05$).

To assess the consistency or flexibility of strategy use of boys and girls, correlations are taken between children's use of successful (i.e. use of steepness or gradient) strategy in the first and in the second trial (see table 1). Fisher's transformation shows the difference between boys' and girls' correlations is significant ($z=1.32$, $p<0.05$), with girls showing greater consistency in use of strategy than boys.

	Pearson Correlation Coefficient	Significance (2-tailed)
Girls	0.504	0.012
Boys	0.180	0.308

Table 1: Correlations between use of steepness in first and second trials for boys and girls

Table 2 shows correlations between the use of relative heights (the most commonly used inappropriate strategy) to make judgments about rates of change. Fisher's z transformation in this case again shows the difference between boys' and girls' correlations to be different ($z=2.82$, $p<0.05$), with girls showing greater consistency in use of strategy than boys.

	Pearson Correlation Coefficient	Significance (2-tailed)
Girls	0.694	<0.001
Boys	0.059	0.742

Table 2: Correlations between use of relative heights in first and second trials for boys and girls

As many boys as girls used each of the two strategies analysed above. In summary, there were no differences between girls and boys in terms of average score or types of strategies used. However, there were sizable differences in terms of the patterns of strategy used demonstrated by boys and girls when comparisons were made across trials.

DISCUSSION

The most striking finding to be discussed here is the consistency of girls and the flexibility of boys with regard to patterns of strategy use between trials. The data show a distinct difference between boys and girls in terms of the likelihood of using a particular strategy on the second trial given its use on the first trial.

Other researchers have claimed that strategy flexibility is at least one cause of gender differences in mathematics test scores. For example, as discussed above, Gallagher and de Lisi (1994) and Gallagher et al. (2000) both show that boys are more likely than are girls to match a strategy with a problem in an appropriate way. Additional studies have also shown that boys and girls differ in terms of their choice of strategy in response to particular problems (Carr & Davis, 2001; Carr & Jessup, 1997; Carr, Jessup, & Fuller, 1999; Davis & Carr, 2001; Fennema et al., 1998). The present study has shown that boys and girls differ not only in terms of strategy choice is response to individual problems, but also in terms of patterns of strategy use across multiple problem situations. While further research would be needed in order to generalize this finding to children of other ages and abilities and to other topic areas in mathematics, these differences do fit into a pattern of existing research. In addition to research on differences in strategy use in mathematics, Ridley and Novak (1983) have suggested that strategy use may be the source of gender differences in achievement in science, with girls tending to rely on rote-learning more often than do boys. Concept flexibility is also thought to be a key factor of dyslexia. Dyslexic children often able to both read words that they cannot spell and spell words that they cannot read, thereby demonstrating a lack of flexibility (Bryant & Bradley, 1985).

These differences in strategy use across problem situations may help to explain the persistence of the difference in the numbers of boys and girls amongst the highest achievers in mathematics relative to differences amongst the general population. Intuition suggests that in order to gain a place amongst the highest achievers, creativity or invention in problem solving is required over and above rote learning of problem solving processes. Therefore consistency in using recognized strategies may be detrimental to gaining a place amongst the highest achievers. A similar argument could be used to explain why boys are over-represented amongst the lower achievers in mathematics, as in this case an ability to consistently apply learned (successful) strategies will be favoured over flexibility.

The demands of the mathematics curriculum are such that there is often little time to spend on the development of mathematical thinking alongside the knowledge required to pass exams. In their attempts to make sense of mathematics, especially in the co-ordination and integration of concepts, children may be differently motivated and differently skilled. If this kind of understanding is left to develop without specific teaching, then a sizable proportion of children may be disadvantaged.

It is likely that a predilection for consistency in applications of strategies to problems would have negative consequences for a student's long-term achievement. Consistent application of rote-learned strategies to problems might well be enough to ensure high achievement up to the age of 16 (GCSE examinations in UK) but would probably not be sufficient beyond that point. Data for A-level (for students aged 16-19) examination results for recent years show that many more boys complete the course than do girls ("A-level results by subject 2004," 2004). Any possible link between consistency of strategy use and participation in post-compulsory mathematics education surely deserves further study.

In the classroom, the findings presented here might be considered when planning the introduction of a new topic to a mathematics class. It seems that children's first encounter with a topic will have a considerable effect on their understanding, also that different children need different considerations. For example, one group of students, mostly boys, might benefit from a consideration of the contexts used in introducing new topics. Findings from the present study indicate that boys show greater improvement when moving from a more abstract to a more realistic context. This may be due to the constraining effects of the more abstract context that forces children to attempt a solution used logical and mathematical methods; methods that can be readily applied to similar problem situations. Where boys are initially presented with problems set in realistic context, they may attempt a solution using more practical, commonsense methods, which do not transfer so readily to other problem situations. Another group of students, mostly girls, might on the other hand benefit from increased reflection and consideration of multiple methods in order to avoid reliance on familiar strategies.

It is useful to know that there are two types of problem solving behaviour in a classroom. It may be, on the other hand, detrimental to say that there is a gender difference in problem solving behaviour without an understanding of why such a difference might exist. This is due partly to the fact that not all boys and girls fit their respective patterns, also that the differences in behaviour may at last in part be caused by teacher and peer expectations. It is almost certainly not appropriate to direct teaching in different ways to girls and boys within a class. Further work in this area is likely to include the investigation of those factors influencing children's strategy decisions in order to understand why it is that patterns of strategy use differ.

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POSITIONING OF A SUBJECT BASED AND INVESTIGATIVE DIALOGUE IN PRACTICE TEACHING

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This paper refers to a project where the preconditions for a subject-based and reflective approach in the context of practice teaching in teacher education are investigated. Second-year student teachers and their tutors were invited to a collaborative investigation. This paper focuses on inquiring the preconditions for including a collaborative and subject-based discussion within the conversation in practice teaching. An educative and an evaluative approach are identified and the contradiction between them discussed. The qualities of the conversation are described in context of the students' awareness of the kind of conversation they participate in, and the influence it has on their learning.

BACKGROUND

The Norwegian teacher training curriculum focuses on practice teaching as an important part of the teacher education. A stronger bond between practice teaching and other parts of the study programme is emphasised. More than in the past, the curriculum now[1] focuses on the mathematics educators' responsibilities concerning practice teaching.[2] Practice teaching is to be seen as an arena for learning subject based knowledge as well as teaching.

Practice teaching has a strong tradition and can be seen as a subculture within teacher education. The corps of tutors (special trained school teachers) has a high degree of autonomy in respect of tutoring the students. This tradition has well-articulated and clearly defined position. The tutors and the students observe each other when they are teaching. They discuss the class sessions both beforehand and afterwards. The tutors are responsible for establishing the conversation, and the post-teaching conversations are seen as important parts of the didactical[3] conversations that take place regularly.

By taking part in the didactical discussions in practice, teacher educators are able to study and participating in the interaction with students[4] and their tutors. The study this paper refers to, intended to establish a stronger emphasis on collaborations in the

[1] http://www.dep.no/filarkiv/235560/Rammeplan_laerer_eng.pdf

[2] As well as those in other subjects

[3] The term didactical refers to didactics as it often is used in a Norwegian tradition. In this paper, the notion more explicitly refers to an area constituted by "how (mathematical) knowledge is developed, used and communicated". Didactics implies (theoretical) considerations relevant for educational practises (inside and outside schools) and deals with conditions for learning, using and communicating knowledge.

[4] *Student* refers to teacher student and *pupil* to children in primary school.

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 369-376. Prague: PME.

field of practice teaching. As mathematics educators (M)[5] we chose to focus on the post-teaching conversation as a forum for conversations between students (S), tutors (T) and mathematics educators (M). We wanted to explore the didactical potentials of these conversations.

Knowledge derived from the LCM-project (Learning Communities in Mathematics) at Agder University College is relevant for our work. The Agder-project introduces mathematical tasks that teachers and researchers explore together. Their common understanding provides basis for the implementation of projects in classrooms and for further research in context of investigative cooperation (Jaworski, 2004; in press). Our ongoing investigative approach has links to the LCM-project, although our initial focus is on the students' teaching practice.

PRELIMINARY FINDINGS AND RESEARCH FOCUS

Two groups of students and their tutors were invited to join in a collaborative inquiry that aimed to investigate the post-teaching conversation and its potentials.[6] Each group and the Ms observed teaching-learning situations where students were actively engaged with the pupils. These observations served as a common reference point for post-teaching conversations. As Ms, we brought a meta-perspective into these conversations. In order to gain insight into the nature of such didactical conversations and to explore how suitable preconditions for such conversations can be established, the position of mathematics[7] in practice teaching were investigated and also the role of the teachers from the university college, especially the mathematics educators. The students and their tutors were invited to join in a dialogue which would foster our joint understanding of the qualities and the conditions for interaction. The discussion of quality was linked to what was to be seen as fruitful for the students learning. Already in the preliminary conversations effort was made to create an awareness of the kind of communication we were participating in.

We identified the most common approach to be evaluative. All the participants expressed a familiarity to the post-teaching conversations as "evaluative conversations" where the focus was on the learning/teaching session interpreted in terms of what had or had not worked well, what could or could not have been done, and why specific choices had been made. This kind of conversation represented a "what-happened" or retrospective perspective. A conversation in this evaluative mode might, as was explicitly stated on several occasions, enlighten students regarding the consequences of their later teaching and learning sessions. However, this approach relies to a dominant subordinate relationship and is to a high degree directed towards the past - an evaluation of what has been done.

At an early stage it was made clear that, as M, we intended to challenge the discourse developed within the post-teaching conversation by having beyond evaluation to

[5] We refer to the authors of the papers as mathematics educators and researchers.
[6] Five students, three tutors and two mathematics educators (researchers) participated in the project.
[7] Mathematics and mathematics education.

initiate or stimulate an educative[8] discussion about the mathematical content, teaching and learning. We emphasised that discussions should be based on reflections about what happened in the learning sessions without the common "what-I-did-well-or-wrong-focus". We discussed what a *subject based discussion* could imply and applied the term *ongoing investigative dialogue in the context of mathematics* to the kind of didactical communication we sought to develop. We recognized the importance of such communication even before we were able to explain what *a subject based and ongoing investigative dialogue*[9] should be. Accordingly this became an interesting issue for all the participants to elaborate on.

Our efforts to foster an alternative kind of discussion implied a challenge to the established discourse. We interpreted this as indicating that our interference made the characteristics of the established discourse more visible to all the participants.

The established discourse and the discursive possibilities were studied in the context of the students learning. We offered a common focus: *How to describe an ongoing investigative dialogue? And how do we envisage the didactical conditions for including a subject based and ongoing investigative dialogue within the didactical conversation in practice teaching?*

METHODOLOGICAL APPROACH

The project is initiated and implemented by the mathematics educators (M1 and M2). It is based on data collected during observations and post-teaching conversations in the field of practice. One of the groups of students were in first grade and the other in second. The project was organized in these phases:

1. Observation of sessions in which S,M and T observed students' teaching. These observations generated references for 2. During the session notes were taken, some of which were detailed and well-developed text.
2. SMT conversations based on these observations. The conversations were intended to addressing the issues mentioned above: How an investigative dialogue can develop and can be understood as part of the didactical conversations. In addition, it was an issue to investigate how the role of the mathematics educator can be viewed. Notes from phase 1 provided the basis and notes from phase 2 were written up.
3. Individual interviews with the students and the tutors based on the theme in 2 was recorded.

Emphasise was made on the development of subject-based approaches as part of the post-teaching conversation, and the development of a conversation about the conversation itself (2). In order to gain insight into the positioning of subject-based and investigative perspectives within the didactical communication in practice, we tried to frame the interviews as investigative dialogues (3). By implication therefore it became important not to "pose questions for them to answer" - or at best to

[8] We use *educative* as the English word that we find closest to the Norwegian *dannende (danning)*
[9] It is difficult to find a sufficient translation for the Norwegian: *den faglig fortsettende samtale*. We have chosen *ongoing investigative dialogue about mathematics*. The meaning is elaborated further on page 7.

minimize this. We invited the participants to join us developing and sharing insights; to expound our perspectives; to turn and twist issues and search for possibilities. We build upon a dialogical approach developed by Alrø & Skovsmose (2002) where they underline dialogue as a conversation of inquiry:

> "Entering an inquiry means to take control of the activity in terms of ownership. The inquiry participants own their activity and they are responsible for the way it develops and what they can learn form it. The elements of shared ownership distinguish a dialogue as an inquiry from many other forms of inquiry where, for instance, an authority sets the agenda for the investigation and the conversation." (p.119)

This chosen methodological approach had implications for the analysis. The post-teaching-conversations and the interviews were analytical processes in the sense that we discussed how we understand and develop relevant concepts. All the participants made serious contributions to the analytical processes.

EXCERPT FROM POST-TEACHING CONVERSATION

"The back table did not follow"

T1[10] started the post-teaching conversation by asking S1 to comment on the lesson he had been in charge of. The other students were asked to add their comments, as were the mathematics educators.

Phase 1. S1 had been working with 28 six-years-olds on Lurvelegg[11] and number-concepts. The children had made models, they had drawn pictures and they had song and danced to illustrate the story about Lurvelegg. We had enjoyed ourselves as we observed how S1 managed to handle various aspects of number concept in a flexible and creative classroom dialogue. The children helped to chose the symbols to be used (drawings); they negotiated numbers, letters and positions and they discussed and made changes on the blackboard in interaction with S1.

Phase 2. S1 started the post-teaching conversation by stating: "It went pretty well as planned." "Most of the pupils followed what I said and grasped it, I think." He seemed satisfied. "We noticed that the back table did not follow," T2 commented.[12] The conversation turned to issues about behaviour, norms and limits. After a while, M2 shifted the focus on how communication can foster linguistic- mathematical creativity and influence the development of number concepts. She pointed out that the teaching by S1 served as an excellent example to use as background for an elaboration. The group members joined in, but S1 seemed to resist entering the discussion. Several times he responded with utterances like "I muddled the order..." "I should have done…" "No, I did not…" "I tried to…". M2 tried to reassure him that she did not think this was the case. It was obvious however, that S1 did not

[10] There are two tutors in this group, T1 and T2.

[11] *Lurvelegg* is a cultural fantasy-figure. He has one eye, two noses, three ears and four legs….

[12] The pupils were organized in groups. *The back table* refers to one of the groups seated back in the classroom.

change his focus to join the discussion. He returned to evaluate what he might not have done well enough.

M1 intervened, emphasising that S1 had taught a very good lesson. Her description was detailed and had a strong evaluative component. She described his muddling with numbers, signs and order as a brilliant didactical example. She gave grounds for why the lesson had been very good: "You invited the children to play and argue, and they joined in. You handled a variety of number-concept features brilliantly," she said. M1 stressed that we were interested in going beyond a discussion of what was good and was not; that we wanted to establish a basis for a continuing enquiry independent of wether he had succeeded or not. This appeared to help; S1 seemed more relaxed and satisfied. Nevertheless he soon returned to comments like: "Yes, I should have…."

Phase 3. During the interview later on we (Ms) told S1 how we interpreted the way he interacted as resistance and he commented: "Yes, that is how I handled it, because that is what we are used to do. In practice teaching last year we always focused on what we did well enough and what we did badly. The only interest we were supposed to have was about what we did badly. The post-teaching conversations focused on *what should have been done differently*. That was the point. As if…well, the evaluation was introduced by: *yes…but...you should have done so and so*. The discussions we have had this year has been fantastic in comparison." He was referring positively to the sessions led by his tutors this year.[13]

S1 went on to say that he had felt relaxed initially since he felt that the children had been active and well focused. The comment about *the back table* had changed his attitude. T2's comment had been impossible to erase from his mind and he could not concentrate on the discussion. Even at the time of the interview the back table was what he remembered most clearly from the lesson and from the post-teaching conversation. He referred to it as something that disturbed him: *they did not follow, and he had not recognised this*. He claimed that even when we discussed the function of such utterances and tried to see how it might restrain or stimulate the didactical conversations, the bad feeling in his stomach constantly repeated: *The back table did not follow*.

CONFLICTING APPROACHES

Drawing upon Foucault, we view the discourse experienced in the post-teaching conversations as being generated institutionally (Mellin-Olsen, 1991; Popkewitz & Brennan, 1998). The participants knew which questions were relevant, and which were not. They knew which way questions should be posed and responded to. When we visit different groups, we find that although the communication pattern differs from group to group, there are marked similarities. The post-teaching conversations have established a discursive identity. When analysing the data, the power of this discourse became evident.

[13] The students have two periods of practice teaching each year. The present period is their fourth.

When present, the Ms most often are regarded as visitors. We are entering, are confronting or confronted by a discourse that is already established when we join in the post-teaching conversation. The discussion has started earlier.[14]

S1 initiated the discussion by commenting that "it developed all right" and that "most of the pupils followed what I said and grasped it, I think." This comment can be viewed in connection with a comment he makes later when he "defends" a fellow-student (S2): "Of course the focus of S2 is on wether the pupils really did what he intended them to do. What he had planned that they do. That was his focus!"

S1's focus was questioned by T2 when he makes a comment about the back table that did not follow. This led to a change in his attitude. His body-language, his silence, his tone of voice when he was talking and the brevity of his utterances all indicated to us that he felt confused and insecure. This was highlighted by the evaluative comments: "I muddled the order..." "I should have done…" "No, I did not…" "I tried to…".

We tried to challenge him by leaving the evaluative mode. However, even when M1 praised S1's teaching and then explicitly explained that we intended to have a conversation concerning children's learning of number concept without discussing what was done well or badly, he did not fully participate. S1 never regained the relaxed and well confident attitude that he had had in the beginning.

Later in the interview S1 commented on what had happened by referring to what they "were used to". He referred to experiences from the first year of his study when he states that "the only interest we should have (…) was what we did badly (…) the evaluation afterwards was introduced by: *Yes…but...you should have done so and so*." S1 described the changes in approach from previous year. By so doing, he also strengthens that the foundation was laid early in his studies. Later in the interview, he commented on the discussion concerning a fellow-student: "That is how it is - we easily retreat to the perspective of evaluation (…) it is difficult to leave ones focus and look for other aspects (…) The focus is on *getting the pupils to do what we have planned* ".

S1's comments strengthen our view that the dominant approach is evaluative. He moves actually the focus still further. Rather than evaluating what was good, or could have been better in a broader sense, he simply questioned weather he had succeeded in getting things done as planned.

S1 is describing characteristics of the discourse of practice teaching by describing what is considered relevant to talk about (and in which ways) and what is not. He describes a discourse imprinted by evaluative connotations. He also describes how the discourse is implied as a frame for interpretation. The comment about the back

[14] The fact that we enter a discussion that we interpret (also by the participants telling us) that started earlier, makes an analytic perspective based on M.M. Bakhtin relevant. In a perspective of continuity we do not see a discussion as started and ended. A discussion is to be seen as a complexity where a variety of discussions are brought to the fore (Johnsen Høines, 2002; 2004). These perspectives are important in the project, but are not developed further in this paper.

table is interpreted by the student in the light of how such conversations have been practiced.

Analyses of the transcripts[15] from the post-teaching conversations and interviews highlights the aspects referred to above. It is possible to identify strong evaluative tendencies characteristic of the discourse as it is articulated by all the participants. By trying to foster a didactical discussion without a strong evaluative focus, we challenged an established discourse. The established discourse became more clearly visible when viewed in the light of an approach that the participants described as unfamiliar. The fact that an alternative discussion was emphasised and the questioning of what kind of discussion we were trying to develop, influenced the way the established discourse was investigated.

We found the students inquisitive and cooperative when it came to analysing the interactions. The paragraph in which we discuss "the back table did not follow" sequence shows traces of how a collaborative inquiry has developed. We have identified two different approaches:

An evaluative approach that aims to focus on a discussion of the learning/teaching session in terms of what was considered to work well and what did not, what could have been done (and was not), why choices had been made. It represents a *what-happened* perspective, a retrospective perspective. A conversation within this mode might discuss the consequences one sees for later teaching and learning sessions. However, the perspective is to a large degree directed towards the past through the evaluation of what has been done.

A ongoing investigative dialogue implies an educative approach that aims to explore how the situation might generate discussions for further development; a future-oriented perspective. It is generated in the practice teaching situation, released from the evaluative aspects and developed as a subject-based interest; as a foundation for subject-based reflectiveness. This perspective implies a continuing, investigative and dialogical approach. (Alrø & Skovsmose, 2002, Johnsen Høines, 2002)

These two approaches are very different and may appear to be conflicting approaches. They do not occur as equal. The evaluative discussion has a strong tradition in the didactical conversations in practice teaching and in the post-teaching conversations. It is constructed in exchanges between the tutors and the students[16]; it is embedded in the discourse shared with other tutors. The dominant position of the evaluative discussion can be attributed due to the discourse that has developed in context of practice teaching (Mellin-Olsen, 1991). Findings in our project indicate that a strong evaluative approach might restrain the development of an approach described here as continuing, investigative, dialogical and subject based.

The two approaches can be represented by individuals in the sense that each participant brings their personal approach into interaction with others. It can also be identified as intrapersonal, in the sense that each person moves between different approaches in the process of gaining understanding. Our analyses reveal that this is

[15] The limitation of the paper restricts us from presenting broader analyses here.
[16] and to a less degree by the teachers from the university college

the case for all of us; both perspectives were evident in the argumentation of each participant. We move between these partly conflicting perspectives, intrapersonally and interpersonally (Johnsen Høines, 2002: 77; 2004: 65). This can be described as identifying two different dialogues as part of a dialogue.

A FOCUS ON DIALOGUE AND LEARNING

By discussing the nature of the conversation, the project also focused on the quality of students learning. The collaborative inquiry that was developed between the students, their tutors and mathematics educators identified two different approaches - or two different kinds of dialogues - as part of the conversation. The two kinds of dialogue show to have different qualities, they serve different purposes and, as also elaborated by Alrø & Skovsmose (2002), they influence the learning in different ways. The importance of the ability and readiness to move between different approaches was realised. In this project we have focused on the learning in teacher practice. It became evident that awareness about the nature of the conversation effects the learning itself.

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EVERYDAY COMPUTER-BASED MATHS TEACHING: THE PREDOMINANCE OF PRACTICAL ACTIVITIES

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I address everyday interaction in computer-based mathematics classrooms from an interpretative sociologist´s point of view. Findings indicate that mathematics teaching in on-computer environments (mainly using CAS) is a „practically dominated" complex of activities. Doing at the computer is the sense-making reference point, though mathematics itself can be in the foreground. The empractical talk at the computer is indeed computer-related talk and the interactivity of the programs seem to cut short mathematics even more.

BACKGROUND: RESEARCH ON COMPUTER-BASED MATHEMATICS TEACHING AND ON EVERYDAY TEACHING IN OFF-COMPUTER CLASSROOMS

Particularly French research has tackled the issue of learning mathematics supported by computers and calculators, basing upon an elaborate theoretical concept of tool and appropriation (Artigue, 2002). Research has shed light on the process of instrumentation (Guin, Ruthven & Trouche, 2005; Kendal & Stacey, 2001; Laborde, 2001; Lagrange, 1999; Ruthven, 2005): on types of student behaviour and its development over time, on teachers´ approaches to technological systems and their styles of use in classrooms, on didactical arrangements, their specifics, and their general conditions of implementation. Altogether, findings indicate that instrumentation is all but a simple matter. While at its beginning research was more or less individual-oriented, nowadays the social organization of learning processes and institutional conditions have moved in the focus as well. However, the very course of interaction has not been paid much attention yet. How teachers act (verbally), how students do, how they together establish everyday teaching and learning at and in the face of computers are still open questions to a large extent, and suggested for future research (Drijvers & Gravemejer, 2005).

As for off-computer mathematics teaching the „social turn" (Lerman 2000) has increased researchers´ interest in everyday (verbal) processes. A distinct branch in this research is the one drawing on interpretative sociology; that is, researchers conceptualize mathematics teaching as a negotiation process and make this a fruitful means for the analysis of the negotiated mathematical objects as well as the negotiation practices (Bikner-Ahsbahs, 2001; Cobb & Bauersfeld, 1995; Jungwirth, 1996; Krummheuer & Brandt, 2001; Mehan, 1979; Voigt 1994). Interaction is an emergent process: It has its own dynamic and can lead even to results that are diametrically opposed to initial aims of participants. Emphasis on the processuality of teaching, however, does not underestimate the constitution of order and its regulating

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 377-384. Prague: PME.

potential. On the contrary, empirical analyses reveal to what an extent interactional patterns are (re)produced day by day. This may hamper innovation, but on the other hand a smooth-running everyday teaching is achieved.

HOW THEY MAKE THEM HAPPEN: A STUDY ON EVERYDAY COMPUTER-BASED CLASSROOMS

In my research being subject of this paper I have examined interactive processes in ordinary computer-based mathematics classrooms from the point of view of interpretative sociology. It is part of the current project of H. Jungwirth & H. Stadler (supported by the Austrian Ministery of Education, Science, and Culture) comparing on-computer mathematics and physics teaching with respect to the interactive constitution of relations to subject matters. As these depend on the interactional conditions in general, attention is payed to (verbal) methods of teachers and students and resulting patterns of negotiation and participation.

Theoretical approach and research questions

A sensitizing concept for my research is the separation of activity complexes into verbally and practically dominated ones (Fiehler, 1980). In the first, relations between the activities of the participants are managed by utterances; mere doing (if it takes place) parallels conversation without determining it. In the latter carrying out the practical activities is essential; talking is related to, or determined by them. Genuine conversation is not impossible, but may take place only if doing permits such talking. Practically dominated activity complexes thus are particular interaction systems: Firstly, they provide two modes for participants to build and keep up mutual relations, and secondly, talking has its specific functions. Taking up the term of Bühler (1982) it is „empractical“ talk. It is less connex, less coherent, but more linked to the context than conversational talking (Brünner, 2005; Fiehler, 1980). My particular approach implies the following research questions: Is it possible to classify computer-based mathematics classrooms clearly in these terms? In which modes do teacher and students organize their use of the computer? What is simply done, what is verbally expressed? What are the reference points of (empractical) talking, and what are its characteristics on a linguistic level? So what kind of mathematics comes out of negotiation? In terms of the French didactics my study approaches the theoretical discourse in classrooms.

Method

It is a study in the tradition of the (German speaking) interpretative research within mathematics education (Beck & Maier 1994; Jungwirth, 2003; Maier, in press); that is, it is based on Grounded Theory as overall methodology but oriented by hermeneutics in interpretation matters in detail. Data base are transcripts of video records of 21 mathematics lessons in several classes in Austrian secondary schools. Teachers did their common teaching without any instructions from my side, or any didactical collaboration with them. Yet they have dealt with computer integration more than the average colleague. The software reflects the situation in Austria:

mainly CAS (including calculators), and in a few lessons Excel and Cabri. Beginners and rather skilled classes feature both in my study. As for mathematics, classical contents for CAS, like derivatives, or sequences dominated, additionally there were tasks from trigonometry, financial applications, and plane geometry.

FINDINGS: THE PREDOMINANCE OF PRACTICAL ACTIVITIES

As I report work in progress I give an outline of the present findings and do not address consequences for teaching, or teacher education. Analyses so far indicate that computer-based mathematics classrooms are activity complexes in which doing dominates. The work at the computer is the very reference point, although there are phases within the teaching process in which mathematics itself is in the foreground. The empractical talk is „computer-related“ talk (this term I have chosen to stress in a neutral way the relevance of the visibility and physical presence of the artifact).

Regarding speech itself the dominance of activity-related utterances is striking. Directive speech acts like more or less specified commands, or instructions concerning input activities, and representations in form of (self)monitoring of such activities are typical (for example, S: Well now draw it; T: Go back to the y editor; T: Now I zoom in that). Deictic expressions („here“) and just pointing at the referred object are used by students and by teachers. There is the screen as a visible reference point, after all, and so it would do. In this mode students solve their tasks. For example, Karen and Paula (10th graders, using Derive) try to prove their conjecture about the monotony of the given sequence. They look back and forth on their commands. The verbalization of the screen entries is not a self-monitoring activity only. By that they establish a shared solution. It does not matter that their utterances partly remain fragmentary as they are integrated in their doing at the computer.

01 Pa:	Do away this (points to the screen) this is wrong, too (pause 3 sec) and
02	what will be if we, now we have to use that solve once again oh yes it is
03	already there. Now (pause 3 sec)
04 Ka:	(a?)
05 Pa:	Well solve (points to solve in the menu bar)
06 Ka:	Right. This is ehm by solve we have by solve by solve we did it before yes
07 Pa:	Solve expression n

Teachers, too act in that mode when they handle the computer and talk simultaneously. Management and control of practical activity at the computer is of prime importance. This can reach even such a state that handling matters can interrupt the discussion of other, already well-established topics. For example, input problems and their solution can stop mathematical interpretations of the output. In other words: The development of a lesson is strongly influenced by the actual computer-related requirements. By the following examples I want to demonstrate how much practical activities prevail in teacher-student-interaction. The first one is taken from the proving of another monotony conjecture in another 10th form (using Voyage 200). The not-negativity condition for n should come up for discussion in the classroom. Yet before, students should try to solve the monotony inequation. Jana´s remark aims at a basical problem. Its discussion could contribute to the answer to the „postscript“ question of the teacher. But he does not involve in Jana´s problem, and she gives in. His asking for reasons turns out to be a rhetoric question; the solution at the computer only is on the agenda.

01 T: All enter the term and try to solve for n by solve. In that the calculator will

02 have a certain problem. The question is why is there such a problem. yes´

03 S1: I cannot find the < could you please?

04 T: Left below (Jana rises her hand)

05 Ja: The calculator cannot refer to n generally, can it?

06 T: What can it not do?

07 Ja: Refer to n

08 T: To n. n is an ordinary variable has no consequence ...

In the second example a mathematical question is turned into an input problem. 12th graders deal with a financial application (redemption problem) by the use of Excel. Already before the scene Susan addressed her lack of mathematical knowledge, and the neighbour student began to tell her, but soon the answer turned into an instruction to complete the Excel table. The teacher remains on this level. Maybe it is beyond his horizon that she could have difficulties with such a simple mathematical matter but, on the other side, he mentions an equally simple command.

01 Su: Instalment minus? (types)

02 Te: Minus the interests. (the teacher comes to their desk)

03 Su: Minus the interests (maintains the pitch, hesitates)

04 T: Yes of course. Copy, into the last column before one you copy the

05 instalment ...

Results above reveal that verbal interaction is oriented to and embedded in on-computer activities. Regarding quantity they indicate rather a large amount of verbal activity. There are, however, as well phases in teaching in which utterances are so rare, or so isolated that it is not real talking that takes place; talking breaks up (Jungwirth 2005). Similar phenomena are question-response-episodes in both modes, the verbal and the practical, or in the latter only. For instance, instead of asking another one a student can look at the screen of the other and read off the information s/he needs. Another typical case is that a verbal question is settled by an on-computer activity (typing, or mouse-click) of the addressee paralleled by fragmentary utterances at most. The asking student can watch and follow the solution on the screen. Whether the latter happens is open to doubt but at least s/he is then ready for the next step of task solving; and that is crucial. Not among students only help is managed in that mode, teachers´ helps are as well. In the example (10th graders using Derive) the graph of the sequence x(n) should be drawn. Sophia and her neighbour did not get the expected result (because of their input x.(n) instead of x(n)).The teacher corrects the input, makes the graph again available at the screen, and tells them what was wrong, but does not take the event as an occasion for a more fundamental discussion of the failure, and the students do not insist.

01 T:	Did you twice that sign, press our button?
02 So:	Yes. (the teacher takes the mouse, scrolls, silence 8 sec)
03 T:	(Well?) (silence 4 sec) What is wrong with the input. On the right side
04	this times n this does not belong there (silence 8 sec, corrects the input
05	and the graph) okay? Don´t calculate times n but only the function ...

Finally, I turn to the subjects which are negotiated in the computer-based classrooms in my study. Within preparatory talk about inputs, or within reflections when the actual input acitivity has been finished, software-related aspects are considered. These can be structural elements that reach beyond the given task (T: Functions in Derive one does not enter by y = or f(x) = but directly only). Besides, and these are the dominating events, teachers and students raise command, or declaration questions, or describe what is on the screen (S: Please what did you do for getting the result below - T: I´ve clicked on the approximation; T: We see here the line 2.6)

Mathematics (in the sense of an off-computer body of knowledge for its own) is included, too. Teacher and students deal with mathematical issues at some length within preceding, general preparation of computer activity only. Dealing with mathematics, however, that is integrated in on-computer activities (in preparations of the actual step, or within talking in the face of the actual output) tends towards mentioning things more or less briefly, or even may look like a token activity. The following episode gives an example for this. It happened in an 11th form within calculation of extreme values using Mathematica (which of all rectangles with given circumference has the maximum area?). The class made a list of length, width, area of all (integer-sized) rectangles with circumference 20. Further questions could be discussed in this context (for instance, what about decimal dimensions). Yet another step at the computer is prior, and the following dealing with such dimensions is wrapped in operations at the computer again.

01 T:	That´s okay, we can see if x is zero, the width
02 S1:	10
03 T:	The area
04 S2:	(Zero?)
05 T:	Okay. But now I want to have a heading for the column x y z ehm x y the
06	area (to the sherpa-student) this can be done in <u>that</u> way ... Couldn´t there
07	be a value in <u>between</u>, for instance, 5.1 ... Well, for which dimensions
08	shall we get the biggest. We zoom into the table now ...

CONCLUSION: (RE)ESTABLISHING PRAGMATIC TASK SOLVING

It is a matter of fact that in off-computer mathematics teaching there are not always discoveries, deeper implications, or elaborate proofs on the agenda. In particular, interpretative research has revealed that even if teachers aim at profound understanding, explications and argumentations may remain undeveloped in interaction. Students, and teachers as well, just address what has been „done“ and what has „resulted“ from operations. In on-computer mathematics teaching, however, the portion of „settling the affairs“ seems to increase. A pragmatic dealing with tasks takes place. Students and teachers as well practise task solving in this way. From my theoretical stance their acting is not a personal failure as already mentioned. (Indeed, teachers´ materials in my study give evidence of their aspirations to high-level cognitive development.) This increase springs from two processes, or in other words: It is two processes that establish the dominance of doing in the practical activity complex.

Firstly, there are necessarily activities at the computer which take time (even experts are busy with them), and, within those, teacher and students tend to refer verbally to what is going on. On the assumption that utterances in the course of practical activities have to meet the communicative requirements of these, it is just appropriate that participants address what has been already done, is up now, will be done next. Secondly, computer-related activities prevail because common (at least in the German speaking mathematics teaching) decomposition of task solving into small pieces of subsequent actions is reproduced. Tasks are, in the extreme case, run through input after input, output after output. So per step there is, under pragmatic perspective, at most a piece of mathematics that can be talked about. The „interactivity“ of the software (no such from the interpretative stance) provides an appropriate material basis for the reproduction of that habit of decomposition.

Sticking human activities to material processes is a general phenomenon to which sociological research on technology and society has paid attention for rather a long time (Joerges, 1988; Latour, 1991). My findings can be interpreted within that framework. Drawing on Latour´s analysis that technology re-establishes social occurences a further fixation of a well-known mathematics-related pattern takes place. The humans-and-computer-network (Borba & Villareal, 2005) is just not very innovative, so to speak. So nothing extraordinary happens in on-computer mathematics classrooms. However, with respect to the mathematical (in the sense above) process itself, teaching appears curiously fragmented and reduced. My study so far indicates in accordance with many others that instrumentation is a protracted process. It seems that it is fundamentally arduous, because the empractical talk is not a very appropriate means for mathematically substantial discussions. The discourse in classroom - referring to the French didactics again - seems to be rather weak in the beginning of instrumentation already.

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FACILITATORS FOR CHANGE OF ELEMENTARY TEACHER STUDENT'S VIEW OF MATHEMATICS

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In this article we consider how elementary teacher students' views of mathematics changed during mathematics education courses. We focus on four students. At the beginning of mathematics education course, two of them had mainly positive views of mathematics with a task-orientation and the other two had negative views of mathematics with an ego-defensive sosio-emotional orientation. The biggest changes were observed on views of teaching and learning mathematics. Moreover, ego-defensive orientation changed towards social-dependence orientation. The most central facilitators of change seemed to be handling of and reflection on the experiences of learning and teaching mathematics, exploring with concrete materials, and collaboration with a pair or working as a tutor of mathematics.

INTRODUCTION

In this paper we present results of a research project on elementary teacher students' views of mathematics. In our earlier studies we explored the structure of 269 students' view of mathematics at the beginning of teacher education: 43 % of students had positive, 35 % neutral and 22 % negative view of mathematics (Hannula, Kaasila, Laine & Pehkonen 2005). It seems that students who had positive view of mathematics, were in general task-oriented (Kaasila, Hannula, Laine & Pehkonen 2005a), and many of those who had negative view of mathematics had an ego-defensive orientation (Kaasila, Hannula, Laine & Pehkonen 2005b). A negative view can seriously interfere students' becoming good mathematics teachers. On the other hand, student teachers who have experienced only success in school mathematics may find it hard to understand pupils for whom learning is not so easy (Kaasila 2000). All these things imply great challenges for teacher education.

In the focus of this article are four elementary teacher students: two of them had mainly positive and two negative view of mathematics at the beginning of mathematics education course. We analyse how their views of mathematics changed during this course, and seek the main facilitators for change.

Mathematical identity and the view of mathematics

People often develop their sense of identity by seeing themselves as protagonists in different stories. What creates the identity of the character is the identity of the story and not the other way around. (Ricoeur, 1992) Sfard & Prusak (2005) define identities as collections of those narratives that are reifying, enforsable and significant. Like they, we see that different identities may emerge in different situations.

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The view of mathematics is an important part of a person's mathematical identity, and consists of one's knowledge, beliefs, conceptions, attitudes and emotions. In the view of mathematics we distinguish three components: 1) The view of oneself as a learner and teacher of mathematics, 2) the view of mathematics and its teaching and learning (Pehkonen & Pietilä 2003) and 3) the view of the social context of learning and teaching mathematics. (Op 't Eynde at al. 2002) Self-confidence that pertains to the component 1 has a central role in the formation of view of mathematics.

Socio-emotional orientations

The model of learning orientations describes how motivational and emotional dispositions develop interactively in learning situations. Lehtinen with colleagues classified three categories of socio-emotional orientations: 1) Task orientation is dominated by an intrinsically motivated tendency to explore and master the challenging aspects of the environment. The student's initial cognitive appraisal of task cues consists of recognising the task as intelligible. Curiosity and interest arise. 2) In social-dependence orientation student adaptation is dominated by social motives (e.g. seeking help from the authority), and she/he is not very willing to make independent efforts: she/he easily becomes helpless. Positive emotions are connected with expected satisfaction of the teacher. 3) Ego-defensively oriented student adaptation is dominated by self-defensive and self-protective motives. The student will be sensitised to task difficulty cues anticipating a negative response from the teacher. She/he does not concentrate intensively on the task, and may try to find some compensatory tactics in order not to "lose face" (e.g. Hannula 2005). The student's expectations of success are low. (Lehtinen et al. 1995)

The phases of teacher change

We have constructed a model that includes the phases of teacher change by combining some central elements of Smith, Williams & Smith's (2005) and Senger's (1999) models: 1) Problematizing current beliefs and practices: the students think their views of mathematics are not the best possible in order to teach pupils effectively; 2) Being aware of a new way: students create new personal visions of what mathematics learning and teaching should look like; 3) Exploring and testing alternative beliefs and practices during the mathematics education course or in practice of teaching or verbalizing new beliefs; 4) Reflective analyses of benefits: Students become more convinced of new beliefs they adopt; 5) Views of mathematics and teaching practices change.

METHOD

Behind this paper, there is a research project "Elementary teachers' mathematics" (project #8201695), financed by the Academy of Finland (see e.g. Hannula et. al 2005): the project draws on data collected on 269 trainee teachers at three Finnish universities (Helsinki, Turku, Lapland). The students at the University of Helsinki consist of 'normal' students (HU1) and students who are studying while working in the schools (HU2). In contrast to the other universities, the mathematics education course at the University of Lapland (LU) is given in the second year. Two questionnaires measured students' mathematics-related experiences, their views of mathematics and their mathematical competencies in autumn 2003.

We chose 21 students and carried out interviews with them in autumn 2003. Six of them had a positive view of mathematics. Their self-confidence registered within the top 15 % and their mathematics achievement in the test was within the top 30 %. Eight of the students had a negative view of mathematics. Their self-confidence registered with the weakest 15 % and in the test the weakest 30 %. The remaining seven students had a neutral view of mathematics. Here we will answer the following research questions: How do elementary teacher students' views of mathematics change during mathematics education course? What facilítators promote the change?

In this report our focus is on four female students: at the beginning of mathematics method course Kati (LU) and Sini (HU1) had had positive and Erja (HU1) and Aila (HU2) negative views of mathematics. In upper secondary school Erja and Aila had selected general and the others advanced mathematics courses. Aila had over 3 years experience of working as an elementary school and as a kindergarten teacher, while the others had very little teaching experience. These four students were selected as representative of a wider spectrum of changes manifested among students either with positive views and task orientation or with negative views and ego-defensive orientation at the beginning of the course.

In the first interview, students reported their mathematical autobiographies that revealed how students had constructed their mathematical identities. Autobiographies included students' personal experiences in learning and teaching mathematics and ways they handled them. In the second interview, in spring 2004, students told which parts of their views of mathematics possibly had changed during mathematics education course. The post-test in mathematics consisted of four tasks measuring understanding of rational numbers and division.

In the narrative analysis, we attempted to recognize the parts of the data that appeared to be central in changing the student's view of mathematics: we constructed the narrative of change. The plot serves to recognize the contribution certain events make to the development and outcome of the story (Polkinghorne 1995). We also paid attention to the language, including the method of narration and vocabulary before and after the turning point. Finally, we compared systematically students' narratives.

RESULTS

Memories from school and socio-emotional orientations

In our earlier studies we have desribed the relationship between Kati's, Sini's, Aila's and Erja's school memories, socio-emotional orientations and views of mathematics at the beginning of mathematics education course. Task orientation was the most central explainer of Kati's and Sini's positive views of mathematics. For example, Kati enjoyed at school her insights as well as the fact that she went '*behind the formulas'*. (Kaasila et al. 2005a) Erja and Aila had an ego-defensive orientation: Aila told she had no need to learn mathematics:*"I lack the capability to learn math. I don't need it (math). I will success in my life without it."* Pleading opposite values seems to be a rhetorical device used in ego defensive talk. In first interview, Erja did not tell negative experiences from her years at school. In the first test of mathematics she had answered only a few questions. When we asked from this, her safeguard gave up: *"I thought how I dare to give the test paper back."* (Kaasila et al. 2005b)

The narratives of change during mathematics education course

Kati's case: Kati's positive view on herself as a learner of mathematics remained unchanged through the course. The negative experiences of teaching other subjects during her first period of teaching practice (before mathematics education course) seemed to have influenced negatively Kati's view on herself as a teacher of mathematics. After mathematics education course her view as a teacher of mathematics changed towards more positive. One of the main facilitators was Kati's role as a tutor for other students: *"Acting as a tutor of mathematics was a very important experience for me, although it was sometimes quite hard work. I taught ten students before they were going to the test. It was very nice to see when they got AHA! -experiences. I was very happy about their success."*

Kati's success in both tests of mathematics was very good, and she had 'a strong proficiency in mathematics'. Both before and after the course, Kati had a task orientation and mathematics was one of the most pleasant subjects for her. Kati's view of teaching mathematics changed clearly during the course: *"I got a broader view of mathematics, and its many relationships to everyday situations. The use of manipulative models increased my understanding very much."* From the teacher students' mathematical autobiographies (Kaasila's 2000) that were read during the course, Kati identified with Sirpa: *"We both enjoy the challenges of mathematics"*.

Sini's case: Sini's and Kati's cases had much in common. Sini's view on herself as a learner *'did not change at all, it was positive already before'*. At the beginning of the course, Sini's view on herself as a teacher of mathematics was good. During the course she had worked 8 weeks as a substitute teacher, and got positive experiences from it: *"I want my pupils to understand math's meaning and beauty."* The course was '*very productive, but also very fraught' for her*. During the course, Sini had taught mathematical contents to her pair: *"At the beginning my pair did not have a positive view of mathematics, but she succeeced well in the final test, and was satisfied with my teaching."*

Sini's view of teaching mathematics changed noticeably: *"It is more like fun than earlier"*. After the course she emphasized the role of manipulative models. Both before and after the course, Sini had a task orientation, and mathematics was one of the most pleasant subjects to teach for her. Her success in both tests was good.

Aila's case: Aila's view on herself as a learner of mathematics changed noticeably: Before the course she said: *"I have blamed myself for my learning difficulties"*. After the course **s**he told:*"The reason of my learning difficulties is outside me: I was against my teachers and teaching I got."* In the first interview, Aila had told about a negative teaching experience: She had tried to teach one boy, but could not help him. After the course her view as a teacher of mathematics had changed *'somewhat towards the positive direction: now I am more eager than before to teach math'.*

Aila's ego-defensive orientation had changed into a sosio-emotional orientation: After the course she no longer denied the value of mathematics for her. When we interviewed her about the tasks in the final test, her social-depedence orientation was revealed by her tendency to seek hints from the interviewer: *"I don't dare to say anything, because I don't know what to say"*. Aila crystallized the meaning the course had for her: *"In my head the whole picture of math has become clearer, the main concepts and their relationships…If my teachers at school had used concrete materials, I would have understood."* Mathematics had changed from an unpleasant to a neutral subject to teach. Aila's proficiency in mathematics had changed during the course, and her success in the final test was average.

Erja's case: Also Erja's ego defensive orientation changed during the cource into a sosio-emotional orientation: *"I really don't feel ashamed that math is difficult for me. I have very openly told (to other students), that I don't understand some contents."* After the course she had many signs of social-depedence emotion: *"I need someone who is teaching me contents… With my father we systematically studied the contents of the whole course.... My goal is to pass the exam."* Erja's view of herself as a learner of mathematics changed a little: *"I noticed that I am not so stupid… I will understand contents if I work hard."* However, she told: *"I am not talented in mathematics. I have problems with basic contents."* The reason for a more positive view was the other student who taught Erja: *"My pair was very supportive."*

At the beginning of the course, Erja had had a negativive view of herself as a teacher of mathematics: *"Because I have not liked math myself, it is difficult for me as a teacher to say that math is wonderful."* Also this view had changed a little bit: *"Yet, I feel it difficult to teach math, but I know I can teach it…"* View of teaching mathematics had changed somewhat: *"To teach math is much more than I thaught earlier."* She found the use of manipulative models sometimes *'very difficult'*, and 'mathematics was still '*the most insecure subject to teach'* When we asked about the tasks of the final test, Erja's talk changed clearly: She gave many explanations why her success had been poor. We interpret this as a sign of an ego-defensive orientation.

Summary of changes (Table 1): The biggest changes occurred in Aila's views: her view of herself as a learner of mathematics had changed noticeably and her attitude towards teaching mathematics had changed from unpleasant to neutral. Aila also had increased her mathematical proficiency, and both Aila and Erja had changed their socio-emotional orientation from ego-defensive towards socially-dependence orientation. Everyone's views of teaching mathematics had changed towards a broader perspective. Kati's view on herself as a teacher of mathematics had changed most radically.

	Phase	Kati	Sini	Aila	Erja
Memories of school mathematics	S	+++	++-	---	---
Course selection and success in Matriculation Examinations' mathematics test	S	Advanced Good	Advanced Good	General Poor	General Not participicated
Socio-emotional orientation	1	Task	Task	Ego-Def.	Ego-Def.
Success in mathematics test	1	Good	Good	Poor	Poor
Attitude towards teaching mathematics	1	Pleasant	Pleasant	Unpleasan t	Unpleasant
View of self as a mathematics learner	1	+++	++-	- - -	- - -
View of self as a mathematics teacher	1	-- +	++-	- - +	- - -
Success in mathematics test	2	Good	Good	Average	Poor
Attitude towards teaching mathematics	2	Pleasant	Pleasant	Neutral	Unpleasant
View of self as a mathematics learner	2	+++	+++	++-	- - +
View of self as a mathematics teacher	2	+++	+++	++-	-- +
Socio-emotional orientation	2	Task	Task	Social-D.	Social-D.
Change in view of mathematics teaching	2	Big	Big	Big	Medium

Table 1. The changes on teacher students' views of mathematics.

Central facilitators of change: 1) handling of and reflection on the experiences of learning and teaching mathematics (all), 2) exploring with concrete materials (all) and 3) working with a pair (Sini, Aila, Erja) or as a tutor of mathematics (Kati).

1) We have used many ideas to help teacher trainees' handle experiences with mathematics: a) Students shared their experiences by telling stories about school time memories (all); b) Students draw schematic pictures of their views of mathematics at the beginning and end of the course (Aila); c) Bibliotherapy, which means the use of reading to produce affective change and to promote personality development (Lenkowsky 1987). Students read the six mathematical biographies included in Kaasila's (2000) dissertation, selected the one that most closely resembled their own background, and studied this case in detail (Kati).

2) The mathematics education courses gave students the opportunity to explore different contents by themselves using manipulative models (all). Students taught mathematics in teaching practice, immediately after the course (Kati).

3) Students worked in pairs, in order to be able to ponder learned contents together (Aila, Sini, Erja). Students had an opportunity to take part in remedial sessions (Aila, Erja). Tutors of mathematics helped trainees whose mathematical proficiency was lower than average (Kati).

DISCUSSION

It seems that mathematics education course can influence teacher trainees' views of teaching and learning mathematics and views of themselves as teachers of mathematics (see also Kaasila 2000; Pietilä 2002). On the other hand, it is not easy to influence students' views of themselves as learners of mathematics. The view of mathematics consists of a hard core including the student's most fundamental views (cf. Green 1971). This study supports the idea that students' views of themselves as learners of mathematics belong to a hard core of the view of mathematics. Yet, Aila's case manifests that also this kind of change is possible: Earlier Aila had thought that it is her fault that she is not good in mathematics. It is a question of an uncontrollable cause that is mostly internal (Weiner 1986). After the course she defined her mathematical past in a new way: the reason for difficulties was in the way she had been taught. Considering the significance of the results it is important to take into account the rhetorics of self-development, which manifested in all students' talk: sometimes it can obscure the views students really have (see Kaasila et al. 2005b).

The model of teacher change, in which we applied and combined some central elements of Smith's *et al* (2005) and Senger's (1999) models, seems to desribe well the phases through which teacher trainees' views of mathematics develop.

On the grounds of this study we give some careful guidelines for the contents of mathematics education course: 1) Handling recollections has been an effective facilitator in this and our earlier studies (Kaasila 2000; Pietilä 2002): If the student reflects occasions in their mathematical autobiography and gains an insight that the interpretations of events can be changed, it can free them to search new perspectives of their mathematical past and future. 2) Exploring with concrete material helped students to understand learned topics better, and they consider it important that they can experiment things that are similar to what they will teach in the future to their own pupils (Pietilä 2002). 3) Teaching a pair or acting as a tutor of mathematics seems to influence at least as much the teacher's (tutor's) view on oneself as a teacher of mathematics as the pair's view of oneself as a learner of mathematics.

The change from ego-defensive towards social-dependence orientation is an important step. Later it could be interesting to study how mathematics education course could change anxious students towards task orientation: to enjoy autonomously AHA!-experiences and see the beauty of mathematics.

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THE MATHEMATICAL BELIEFS OF TEACHERS ABOUT APPLICATIONS AND MODELLING – RESULTS OF AN EMPIRICAL STUDY

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The paper presents the results of an empirical study on the meaning and the role of applications and modelling in mathematical beliefs of teachers. Within the framework of the evaluation of an innovation project, 41 teachers were asked in written form about their beliefs concerning mathematics as a discipline and concerning teaching and learning of mathematics. Based on classifications of mathematical beliefs in static and dynamic conceptions, 8 teachers were interviewed concerning the meaning of applications and modelling. It became clear that although teachers were convinced to considering applications and modelling for daily school practice they still argued for mathematics and mathematics teaching in which applications and modelling only played a minor role.

INTRODUCTION

Since the last decades the didactic discussion has reached the consensus that applications and modelling must be given more meaning in mathematics teaching. They are considered important for achieving the central goals of mathematics, such as to enable students to understand and to master situations in everyday life and to master problems they experience in the world they live in. However, international comparative studies on mathematics teaching carried out during the last years, especially the PISA Study, have demonstrated that worldwide young people have significant problems with application and modelling tasks and show low performances in these kinds of problems. Various empirical studies point out that, among other reasons, one reason for these low performances is the small relevance applications and modelling play in daily school practice despite the didactical debate, which is supported by the political debate (see e.g. Blum et al. 2002). The assumption that beliefs of teachers about mathematics and mathematics teaching play an important role for lessons, leads to the question of what are the beliefs of mathematics teachers about applications and modelling and to what extent do teachers practice teaching of contextual examples as called for worldwide within the framework of innovation projects.

In this contribution results will be presented which demonstrate that the beliefs of teachers about mathematics and mathematics teaching are coined by formalistic and schematic imaginations and that applications and modelling only play a minor role in it. Besides that it came out that formalistic or schematic imaginations are obstacles for applications and modelling in teaching.

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FRAMEWORK AND DESIGN OF THE EMPIRICAL STUDY

The study was conducted in connection with the evaluation of a pilot programme of the German government and the federal states which is aimed at increasing the efficiency of mathematical and scientific teaching (so-called SINUS-programme). This innovative programme, carried out during the period 1998-2003, aimed at fundamental changes in mathematics teaching in two directions, namely (1) a change in tasks as practised in lessons, and (2) a change of the dominant learning and teaching structures meaning a stronger integration of applications and modelling examples. Over the entire period teachers were also offered further professional development opportunities via internal and external initiatives. Furthermore, the participating teachers were asked to try out already existing material and to develop new material through teamwork. Teachers were given access to a great amount of material - which had been developed all over Germany within the framework of this innovation programme - by a special server. The material consisted of teaching units and real world examples or materials allowing individualised work of the students. Proposals for opening already existing closed tasks from textbooks or for variation within a group of tasks were developed.

The study, for which the results will be described below, is restricted to the evaluation of this programme at the six participating schools in Hamburg, the second largest city in Germany. Due to organisational considerations, the evaluation is limited to a period of only one and a half year which suggests that large scale changes could not be expected.

The evaluation study started when the students of the 6 participating schools attended year 7 and 8 and ended when they were in year 8 and 9. The study is divided into different components: One component was more quantitative based in which the development of mathematical literacy as well as students' beliefs within a greater sample of students was examined (for results of the study see for example Kaiser, Willander 2005). In the second qualitatively oriented component the mathematical belief systems of the involved teachers were examined.

The theoretical approach of this study refers to the discussion about beliefs as mental constructs that represent the codification of people's experiences and understandings (Schoenfeld, 1998). The teachers' beliefs can be distinguished amongst others by the fact whether they refer to the nature of mathematics as discipline or to mathematics teaching and learning (see Thompson 1992). The study starts from the classification system of mathematical beliefs developed by Grigutsch (1996) and elaborated by Grigutsch, Raatz, Törner (1998) concerning teachers. Grigutsch categorizes students' beliefs mainly by four aspects of mathematical belief systems which refer to the nature of mathematics as a discipline: Mathematics can be understood as a science which mainly consists of problem solving processes (aspect of process), as a science which is relevant for society and life (aspect of application), as an exact, formal and logical science (aspect of formalism) or as a collection of rules and formulae (aspect of scheme).

Methodologically, the study was qualitatively oriented and applied methods from qualitative social science. Furthermore, the applied empirical methods concerning the choice of sample, data analysis and data interpretation are based on the theoretical considerations of Grounded theory (see Strauss, Corbin 1998). Grounded theory was suitable for this study because it is a scientific-theoretical style of research and simultaneously provides various single techniques by which a theory can be developed in a systematic way grounded within the data received from interviews, field observations etc.. Especially for this research object, it seems to be perfectly adequate, because until now, sufficiently established theoretical knowledge was not yet available or only available by a few studies (see Wilson, Cooney 2002; Lloyd 2002). Grounded theory is based on comparative analyses meaning permanent construction of comparison which in the presented study was done repeatedly during the various phases. Furthermore, the study refers to theoretical coding, meaning that data related in-vivo-codes are developed and transformed into theoretical codes.

In this study, all teachers involved in mathematics teaching of year 7 and 8 students of the six participating schools have been asked about their beliefs concerning mathematics as a discipline and the teaching and learning of mathematics at the beginning of the project and after one and a half year. This was done in written form via open and closed items. The following aspects were covered in the questions: beliefs about mathematics as a discipline, about the nature of mathematics teaching and the underlying goals of mathematics teaching, about the teaching and learning of mathematics. Altogether 41 teachers participated at the beginning and 29 at the second questioning. With 16 teachers who were chosen for certain theoretical criteria, partly standardised interviews were carried out deepening the already mentioned aspects of the mathematical belief systems of the teachers. 8 teachers were interviewed at the beginning and 8 at the end of the study, 8 of these interviews were analysed in detail using methods from Grounded theory. For this, the additionally carried out interviews with teachers became the basis for further validation of the results (for details see Ross, 2002 and Kornella, 2003). In general, triangulation of data sources was accomplished for validity and reliability purposes.

RESULTS OF THE WRITTEN QUESTIONNAIRE AT THE BEGINNING AND THE END OF THE STUDY

The written questionnaire at the beginning of the study shows a clear dominance of static beliefs about the nature of mathematics, meaning for teachers mathematics meant exact mathematical thinking and exact ways of working as it is described in the formalism-oriented approach. Likewise, beliefs about the nature of mathematics teaching can be classified as static with both, formalistic and schematic perceptions dominating. Beliefs concerning the objectives of mathematics teaching are schematic, dominated by the teaching of rules and formulae. While beliefs concerning the teaching of mathematics are predominated by static aspects, with beliefs concerning the learning of mathematics the dynamic aspects prevail. Taken together, it becomes obvious that for the whole group of examined teachers applications and modelling play only a minor role in their beliefs about mathematics and mathematics teaching.

The second written questionnaire conducted after one and a half years showed that the mathematical beliefs of the examined teachers changed only rudimentarily. However, there were clear indicators that beliefs concerning the nature of mathematics and mathematics teaching shifted towards a more application related orientation for which applications and modelling are important. Furthermore, beliefs concerning the ways of teaching and learning tended to a more dynamic viewpoint as consequence of the modified role of the teachers. This makes clear that changes have not gone further than to the level of „surface“ beliefs, meaning that changes are not deeply rooted within the belief system. The processes of change did not last long enough to effect changes in the „deep“ beliefs, beliefs that function as central anchor points (see Pehkonen 1994). This is not astonishing at all because the study's duration of one and a half year was quite short.

In the following section, these mathematical belief systems are investigated more intensively as well as the role applications and modelling play. These aspects were analysed more in detail by means of intensive interviews.

RESULTS OF THE IN-DEPTH-INTERVIEWS

First, the results of the 4 interviews from the first round will be described, followed by results from the second interview round with 4 interviews too. For detailed analyses, 16 teachers interviewed at the beginning then were reduced to the number of 8 teachers who were chosen based on the results from the first written questionnaire. The leading criterion was that the teachers could be classified exactly according to one of the four streams of mathematical beliefs as described by Grigutsch (1996) and Grigutsch, Raatz, Törner (1998). In the first round, we chose one teacher for each category of beliefs - formalistic, schematic, process-oriented and application oriented beliefs, likewise for the second round. The most determining aspect for the choice of teachers was that beliefs concerning mathematics as a discipline as well as concerning the teaching and learning of mathematics corresponded strongly with one of the categories. This must be regarded as a fundamentally different approach compared to what is described in the existing literature in quantitatively oriented studies, which have asked about the distribution of the single aspects of beliefs within a whole sample.

Results of the first interview round

In the following, while referring to exemplarily statements, the positions of teachers classified as formalistic, schematic or process-oriented shall be put into concrete terms.

First about the as formalistic classified teachers: This teacher describes his view about mathematics as follows: Mathematics is at first a "formal language", in contrast to colloquial language "not redundant", "precise" and „logical/consistent“. According to this teacher's opinion there is only a weak relation between mathematics and everyday teaching: 'For me mathematics is … not always, sometimes yes, …. has also a relation to life.'

To some degree the position of the as schematic classified teacher shows similarities to a formalistic view of mathematics: For him mathematics is reduced to the accumulation of rules and formulae. Mathematics is ‘the logical sequence of formulae’. Non-mathematical applications do not form a constitutive part of mathematics. In mathematics lessons students learn ‘the basic conditions of mathematics’, ‘and everything else comes from the other subjects, there one continues to calculate.’

For the teacher with process oriented beliefs towards mathematics, mathematics is understood as an intellectual exposition of problems. This goes along with the fact that there is only seen a weak relation between mathematical subject knowledge and the real world. In the interview the teacher explained that mathematics might even be replaced by playing chess, because mathematics is aimed at developing thinking abilities.

The teacher, for which the aspect of application plays a central role, made clear in the interviews that the aspect of application had a fundamental meaning for her: ‘What shall I do with mathematics, if I cannot apply it somehow for my life?’ However, not its profits, but the training of ‘critical questioning’ is as important as the training of thinking abilities for her.

Results from the second interview round

As mentioned earlier, it became obvious in the second round of questionnaires that only slight changes within the mathematical beliefs of the teachers had taken place towards a greater relevance of application and modelling examples. For this reason, the second round of interviews focussed on questions on how the involved teachers handle applications and modelling, whether they regard them as useful for their own lessons and whether they would possibly include them into their lessons. Like in the first round, here again 4 teachers were interviewed (as already noted above, originally 8 teachers were asked) who again represented four categories of mathematical beliefs. The selection was done based on the written statements.

The results of the in-depth-interviews are as follows: Teachers with mathematical beliefs, in which the aspect of application only plays a minor role, interpreted application oriented beliefs about the nature of mathematics or the nature of mathematics teaching in a way by which they became appropriate for their own mathematical beliefs. In detail:

Teachers with a process oriented understanding of mathematics and mathematics teaching stress the many chances which exist for developing solutions and reduce applications and modelling on this aspect.

In contrast to that, teachers with schematic mathematical beliefs restrict applications and modelling to examples that enable easy mathematisations or lead directly to a formula.

For teachers with formalistic beliefs, the context nearly does not play any role.

The modifications of the nature of applications and modelling and of their functions became clear especially in connection with one modelling problem for which the needed data had to be estimated from a realistic context and non-mathematical knowledge had to be activated. The problem reads as follows (Herget, Jahnke, Kroll 2001, 20):

> The problem shows a photo of a child and a monument in Bonn, capital of Western Germany before unification. The monument shows the head of Konrad Adenauer (1876-1967), first chancellor of Western Germany in 1949 – 1963. It is asked how high a monument must be in order to show the entire body of the chancellor based on the same scale. A possible approach is to estimate the age of the child and then to determine the height of the child in correlation to its age. Then, by measuring the child, one gets the scale of the photo on the problem sheet. Then, considering the proportion of head and body of an adult, it is possible to calculate the monuments height.
>
> For comparison reasons, at the end of the problem the height of the American Statue of Liberty (46 m) was noted beside its photo.

This modelling problem was handed to all teachers and then they were asked whether they would deal with it in their lessons, and if yes how they would do.

At first it became obvious that all teachers thought it desirable to discuss contextual and modelling problems at school. Additionally, in their opinion they were really practicing contextual and modelling problems. Nevertheless a deeper look at their understanding of applications and modelling tasks makes clear that this picture has to be differentiated.

A scheme oriented teacher rejects the problem at the beginning because the problem is presented with too much text. He said that from his perspective, ‘at the first moment he did not know what to do with it because for him the item refers to the subject history: ‘lived then and then...(laughing) historical theme’. He refuses estimation because according to him it does not provide ‘anything tangible’. In his opinion, this task ‘in connection with mathematics does not contain anything to hold on for children’. ‘They don’t know what to do with the 46 meters written below’.

He transformed the problem into a closed calculation task containing the needed information. Because of the difficulties students would face according to his assumption, he split the task into small steps for which he gave detailed working instructions.

The formalistic oriented teacher views the task less negatively but classifies it as quite advanced. He is afraid that the students might get confused and therefore he wants to set exact mathematical main points, for instance measuring, proportions, scale. In his modification of the task he split the task into several sub-tasks too, including exact questions, which referred clearly to a unique mathematical concept to be used. Like the scheme oriented teacher he also wanted to give the needed data.

The two teachers with process and application oriented beliefs show significantly different reactions, although they were both quite enthusiastic about the learning potential of the problem:

The application oriented teacher even deletes the question ‘in order to let the students develop the questions by themselves’ which might go into different directions. As an example she developed the following questions: ‘What is the head’s weight?’ ‘Why does he have such big ears?’

The process oriented teacher regards the problem as a good idea which ‘one might easily develop further’. She asks for instance: ‘How many people one could put into such a head?’. But if needed, she would consult the biology textbook in order to avoid confrontation of girls and boys.

In summary, the study makes clear that process oriented beliefs about the nature of mathematics and mathematics teaching do not create obstacles for application and modelling in mathematics teaching, even if applications and modelling problems in mathematics teaching are often shortened. With process oriented beliefs the real world context does not play the same role as with application oriented beliefs. In contrast to that, scheme and formalism oriented beliefs build high obstacles for application and modelling problems in mathematics teaching, because the nature of contextual and applied problems are not compatible with those beliefs.

Altogether, this study demonstrates that the evaluated project of innovation has effected slight changes on the level of surface beliefs. Thus the interviewed teachers often stressed the importance of applications and modelling for mathematics teaching, but the examples should not be too complex nor demand too much non-mathematical knowledge. Such examples were modified and changed by them which indicate that on the level of deep beliefs there were no significant changes.

Finally, it can be stated from the study that teachers and their beliefs concerning mathematics must be regarded as essential reasons for the low realisation of applications and modelling in mathematics teaching. Furthermore we can conclude that fast changes by short-termed projects might not lead to a change of the relevance of applications and modelling in school reality. In order to promote real world and modelling examples within mainstream mathematics education, the integration of applications and modelling into teacher education at universities and in-service-training for teachers seem to be necessary.

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AN ANALYSIS OF SOLVING GROUPS OF PROBLEMS (TOWARD THE STUDY OF PROBLEM SOLVING INSTRUCTION)

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This article examines how students solve groups of problems, finding (or not finding) connections between them. The research discussed here deals with solving problems that involve absolute value. Using the results of this research as a starting point, the author poses questions for further investigation of the way students think while working on groups of variously interrelated problems. It is argued that such an investigation can improve our understanding of the necessary conditions for effective problem solving instruction.

INTRODUCTION

Analyzing the vast literature connected with problem solving, Lester (1994) calls attention to the fact that problem solving instruction requires further study. Indeed, even twelve years later, rather little is known about this topic. Vygotsky (1982) distinguished between two sides of the teacher's work: as the organizer of the instructional environment and as an actual component of this environment (Vygotsky compared the teacher to a tram conductor and a rickshaw driver—in the former case, the worker mainly controls the vehicle; in the latter case, he is also one of its parts). The organization of the environment, which is important in any lesson, is particularly significant during problem solving. It seems, however, that we still know very little about how the environment ought to be organized, and in particular, about how groups of problems should be selected and what influence solving such groups of problems can have on students. Therefore, it seems important to study not so much how students work on individual problems, but how they work on groups of problems.

The present article reports on research into the difficulties experienced by students while working on a complicated mathematical concept, command of which presupposes the ability to establish connections between different areas of mathematics by using various forms of representation (specifically, the research focused on the way in which students study the concept of absolute value). Using this report as a starting point, we attempt to outline a program for further research. The author is currently working on certain aspects of this program, but it deserves the attention of other investigators as well.

SOME THEORETICAL CONSIDERATIONS

The very concept of a problem is usually defined as a situation in which a goal is to be attained and a direct route to the goal is blocked (Kilpatrick, 1985). It is clear, however, that in actual practice there is a great deal of room for interpretation—for example, to what extent should the teacher be permitted to hint at roundabout ways toward the solution, so that the assignment still remains a problem and not becomes a routine exercise? It is not likely that a general and precise answer can be given to such a question.

The Russian researcher Kalmykova (1981) has noted that

> "No matter how familiar a problem is to a student, its concrete contents differs from any problem that the student has already solved: a person must re-code (translate) it into the language of scientific terms... that is, the solution requires specifically intellectual, and not merely mnemonic, activity; it does not turn into an act of memory" (p. 20).

Consequently, according to Kalmykova, *productive* and *reproductive thinking* should not be juxtaposed as opposites, but ought rather to be seen as lying on a kind of continuous spectrum, with the most striking examples of creative thinking at the top; reproductive thinking — a good part of which consists of new intellectual activity — somewhere in the middle; and finally, the most extreme forms of reproductive thinking, which can in effect no longer be considered thinking at all, at the very bottom.

Problems that require students to establish connections—including those that involve performing multiple re-codings—appear to us in general to be aimed specifically at relatively high levels of thinking. Dreyfus (1991) notes that making links between parallel representations, and integrating the representations and flexibly switching between them, represents the highest stages of learning processes that presuppose a high level of abstraction (as in grasping the fact, for example, that the concept of the function can be expressed in different ways). Such problems can be solved through reproductive thinking (in those cases when they are not being solved for the first time), but—it is worth repeating—reproductive thinking of a high level. Problems that involve constructing a relationship are considered important for developing understanding (Carpenter and Lehrer, 1999). Therefore, the scientific literature (Coxford, 1995) pays special attention to "connectors," concepts and objects that make it possible to link different themes. Finding connections is an important part of problem solving instruction. We were interested in the extent to which students see existing connections when they solve typical problems (i.e. problems that they are generally familiar with) that involve the concept of absolute value; that is, we were interested determining the level of reproductive thinking that they display in working on such assignments.

ABOUT PROBLEMS THAT INVOLVE ABSOLUTE VALUE

The concept of absolute value occupies a rather important place in the school curriculum and therefore the scientific literature has devoted a considerable amount of attention to how this concept is studied and how it can be studied (Wallace, 1988; Yassin, 1991; Parish, 1992; Horak, 1994). It may be argued, however, that as a rule researchers have been interested in developing practical recommendations for teachers aimed at improving the teaching of how to apply the concept of absolute value—a concept that is useful and even necessary in studying such subjects as

Calculus. Such an approach is entirely justified, since it has long been noted (Bratina, 1983) that difficulties in working with this concept turn out to be the cause of difficulties in working with other fundamental concepts. The present study has a different focus. What is important for us is the fact that the concept of absolute value is examined in school through the use of a variety of different representations: it is defined both as distance and in word formulas (with reference to various specific examples); it is illustrated by means of graphs and it is actively employed in the course of various algebraic manipulations. Thus, although it is true, as Kalmykova argues, that re-coding takes place during the independent solving of virtually any problem (for example, of the equation $x+1=3$ after going through the solution to the equation $x+1=2$); nonetheless, in solving problems that involve absolute value, re-coding takes place at a significantly higher level. In solving the equation $|x-1|=1$, the student can either make use of algebraic manipulations, or think about the distance from the point 1 on the number line, or to imagine the intersection of the horizontal straight line $y=1$ with the graph of the function $y=|x-1|$. All of these respective algorithms are usually studied in school. The present study was aimed at finding out the extent to which students are capable of seeing the connections between them—and thus demonstrating a genuinely high level of reproductive thinking in using the algorithms that they had been taught. Therefore, we were interested not so much in the way in which students solved one or another isolated problem, but in the way in which they solved groups of interconnected problems.

EXAMPLES OF STUDENTS' DIFFICULTIES IN SOLVING SPECIFIC PROBLEMS INVOLVING ABSOLUTE VALUE

In solving problems that involve absolute value, students experience both specific difficulties associated with specific problems and difficulties of a general character. Thus, for example, the generic extension principle (Tall, 1991), which states that patterns observed in one context can be transferred to other contexts, finds expression in the solving of problems that involve absolute value—as when equations involving absolute value are solved as if they were linear equations, despite the absolute value notation. Chiarugi et al. (1990) undertook a systematic study of students' difficulties in solving problems that involve absolute value. They observed confusion in applying the concepts of domain and range, as when, for example, students believe that the expression $|x-1|$ takes on two values for a given value of x. It has likewise been noted (Sink, 1979) that students sometimes see a contradiction between the fact that absolute value is always nonnegative and the fact that the minus sign appears in the course of solving the problem. Students demonstrate a higher level of problem solving when they make use of geometric representations than in those cases when they must employ only algebraic manipulation. Another important result is the finding of Chiarugi et al. (1990) concerning the extreme difficulty that students have in solving problems in which any of the aforementioned difficulties are added on top of others that the problem might possess (for example, when it involves using variables—parameters rather than numbers). The *combination* of difficulties makes the problem considerably more difficult.

METHODOLOGY OF THE EMPIRICAL STUDY

Our study had two stages. During the first stage, a large group (117 students) of eighth-graders and twelfth-graders were given a written assignment. The study was to a certain extent structured on the model of the work of Chiarugi et al. (1990) and in general (with minor exceptions) it confirmed their results. The present article will not address the details of the study (Karp, Marcantonio, in preparation). Next, a series of interviews was conducted (14 with eighth-graders and 10 with twelfth-graders), in the course of which the students were given relatively standard problems (e.g. students were asked to discuss how many solutions the equation $|x+1|=3$ had and to solve an equation of this type; they were given several graphs and asked to identify the one that represented the function $y=|x+1|$; and so on); however, the aim of the interviews was to determine whether the students saw any connections between these problems, with the interviewer himself sometimes explicitly pushing them to discover such connections. It should be noted that, by contrast with the study conducted by Chiarugi et al., the participants of this experiment were not initially given a definition of absolute value. On the contrary, the students were asked to formulate a definition themselves, and only after it was determined that they were incapable of doing so was the definition given to them by the interviewer. All of the interviews were audiotaped. All of the participants of the experiment attended a public school located in the New York area with a primarily Caucasian population (97%). Over 75% of this school's graduates go on to attend a 4-year college.

ON THE RESULTS OF THE EMPIRICAL STUDY

In the overwhelming majority of cases, the students find no connections between the problems they are given, solving them as completely independent assignments and not taking away any lessons from experience acquired during the course of the interview. Thus, for example, the answer to the question "How many solutions does the equation $|x+1|=3$ have?" was often obtained by trial and error.

Interviewer:. How many solutions do you think you would get if you could solve that?

Bob: [Thinks for few seconds]. Two.

Interviewer: Why do you think two?

Bob. [Thinks again for a second]. Wait, now that I think about it, it's not two, because I was thinking it would be -2 and 2. But it was 2, it would work, but -2 wouldn't because negative one would become one.

Interviewer: OK, so you just think one solution then? Because 2 is the only one that would work in there.

Bob: No.

Interviewer: OK.

Bob: Two, ‘cause if actually…, two, because -2 and 1.

Interviewer: Okay, how many answers do we have?

Bob: Two.

Interviewer: Two, you think two and what are the answers? You think 2 going to work?.

Bob: 2…, 2 and -3.

Interviewer: All right, so -3. If I put -3, does that work?

Bob: Wait…no, because…that would only take up to positive 2. So maybe -4.

The student’s behavior in this instance is rather typical, including the initial confusion between the range and the domain of the expression, which first gave rise to the idea that the equation has two solutions. The obtained result, however, in no way influences the student’s further behavior, when he must solve an analogous equation.

Interviewer: See if you can solve number 6 [equation $|2x+3|=7$].

Bob: All right. [Takes about 30 seconds to come up with a single solution of x=2, simply dropping the absolute value notation in solving the problem.]

Interviewer: Oh, did you get it? So you got x equals 2? Ok.. That the only answer?

Bob: What?

Interviewer: That the only answer?

Bob: Um.

Interviewer: That’s okay if that’s it. Is that it?

Bob: Yeah.

In this way, the fact that the given equation has one solution, and not two, like the previous one, elicits no surprise from the student. Naturally, not all of the students interviewed responded in precisely this way. For example, many better mathematically prepared twelfth-graders, who came from a Calculus class, simply solved both of the given equations in the same way (and solved them correctly), so that no contradiction arose (and no shift from one approach to another took place).

A similar situation arose when students had to select a graph that corresponded to the function $y=|x+1|$. Very often, the graph of the function $y=x+1$ was picked as the answer—because, as the students explained, the graph of the function $y=|x+1|$ has a y-intercept equal to 1 and its slope is also equal to 1. Here, too, the fact that the value 3 had to be assumed twice—as the students had just determined—was in no way taken into account.

The list of similar examples can be continued. In particular, it should be noted that in their reasoning, the students practically never made use of the definition of absolute value.

CONCLUSIONS OF THE EMPIRICAL STUDY

It is fair to say that in most of the cases studied, the students had assimilated only individual algorithms (and again, with varying degrees of depth and firmness). They had formed no unified semantic field (Lins, 2001)—no interconnected system of knowledge growing out of a knowledge kernel. The students did not need a definition—instead, they made use of concrete algorithms which told them what they had to do. Moreover, these algorithms were also completely cut off from one another—there was one algorithm for making calculations, another for solving equations, a third for constructing graphs, and so on.

It is clear that under such circumstances, the students must rely mainly on memorization in solving problems (a fact that the students themselves did not conceal), and that any talk of reproductive thinking at a high level would be out of place. The problems examined above may have been more difficult than certain others, but this was only due to a greater number of steps in the algorithm used in solving them. There was no evidence of any complex re-coding (not even one based on a given model) that could be considered a sign of a high level of reproductive thinking.

SOME QUESTIONS FOR FURTHER RESEARCH

Analysis of the programs and materials used in the instruction of the participants of the interviews warrants the conclusion that, in fact, they were almost never taught to look for connections between different representations and in this way to achieve a certain degree of control over the actions they performed. According to Lins (2001), a key role in the formation of a semantic field is played by "justification." Comprehension is constructed precisely through the justified expansion of some base of knowledge. The educational materials also contained virtually no instances in which the algorithms applied were justified. The conclusion of our research may be said to consist in the following finding: on their own, the students established no connections between individual algorithms learned in isolation from one another (leaving aside certain exceptional cases), even in those instances when the algorithms were actually learned—which was also something that by no means always happened.

It is natural to ask to what extent the work of the teacher in posing problems that require establishing connections between various representations affects the thinking of the class. What happens when students not only systematically solve problems that involve the application of algorithms—connected with algebraic manipulations or with geometric and graphic representations—but also problems that require them to establish such connections? How successfully does the class deal with such problems in the future, and to what extent is the skill of making comparisons that is acquired by the students transferred by them to other problems that were not discussed in class? In particular, to what extent does their control and monitoring of their own solution (even by simply comparing it with other solutions of the same problem) become *reproducible* when attention is paid to it in the classroom? Further research is necessary in order to answer such questions.

EXPANDING THE CONTEXT OF THE STUDY

The possibilities for connections among the problems in a group are not exhausted by the application of different representations. A block of problems is a rather complicated object and its structure and morphology merits study (Karp, 2002; Watson & Mason, 2005). For example, blocks of problems can be organized in such a way as to induce to students to make generalizations. In such instances, the sequence of problems begins with problems devoted to quite concrete and specific cases and moves on to increasingly general ones. There exist wonderful examples of problem books that are structured in precisely this way. It may be argued, however, that we have precious little information concerning the effectiveness of such an approach to instruction. That is, how successful are students at making generalizations when dealing with problems that are close to the ones which they initially studied? Likewise, we have little information about the extent to which the skill of thinking in terms of generalizations is carried over into other classes of problems.

CONCLUSION

This article contains more questions than answers. Answering them is no simple task, if only because it is not easy to separate the impact of problem blocks from other parameters that influence the students, first and foremost the various ways in which they are influenced by the teacher. Nevertheless, it is important to search for these answers, not only in order to achieve a better understanding of the way students think—and a better grasp of what and how their thinking is influenced by—but also in order to put teaching, and the writing of teaching materials, on a firmer scientific footing. In the absence of such research, calls for teaching students the craft of finding connections might remain nothing more than calls.

Acknowledgements. The author expresses his gratitude to his graduate student Nicholas Marcantonio. The results of a study conducted jointly with him have been used in this article.

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ADVANCING LOW ACHIEVERS IN SECONDARY SCHOOLS BY USING NON-PROFESSIONAL MATHEMATICS TUTORING

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This paper describes a unique project, carried out in a secondary school where the student population includes many low achievers in mathematics. Non-professional volunteers, aged eighteen to nineteen, served as mathematics tutors throughout the school year, teaching small groups of students in an intensive program. In spite of apparent limitations of using this type of tutoring, the effect it had in most cases on students' attainments was substantial. The work model of the tutoring is outlined and some results concerning the students' gains in achievements are given. It is suggested that affective factors, as well as the instruction given to tutors by a specialized counselor, have played a major role in maintaining successful tutoring.

INTRODUCTION

In mathematics, as well as in other subjects, tutoring[1] in small groups is perceived as an efficient way to advance at-risk students and low achievers at all grade levels (Balfanz et al., 2002; Brophy, 1996; Slavin, 2002). Due to the high costs of professional tutoring, there is a growing interest in paraprofessional tutoring, reflected in recent literature. Leblanc et al. (1995) describe a tutorial program in which tutors are preservice teachers. Other studies refer to college students who are trained to serve as tutors (Nesselrodt & Alger, 2005; Shulman & Armitage, 2005; Fitzgerald, 2001). In these cases, tutors were carefully selected to match the programs' goals. However, within the research literature on mathematics tutoring, there is almost no documentation of volunteer tutors who cannot be considered as paraprofessionals, but rather as non-professionals, having no academic backgrounds. One possible explanation for this is the plausible consideration that non-professionals can hardly be regarded an adequate source for mathematics teaching: Mathematics (and secondary school mathematics in particular) is far too complicated a subject to be taught by people whose mathematical preparation is limited to their own high school experiences. To illustrate this argument, let us consider an example. Roy, a 19-year-old volunteer, was asked how he would solve the following problem: *The side of a given isosceles triangle is 2 cm longer than its base. The triangle's perimeter is 19 cm. Find the lengths of the triangle's sides*. Roy solved the problem by a "guess and test" strategy: He started with a 5 cm guess for the side, obtaining a base length of 3 cm and thus a perimeter of 13 cm. After two modifications he found the correct answer. When asked if he could produce this answer using an equation, he seemed perplexed and said he was never good with equations. Roy's solution can be regarded as an informal mathematical product (Karsenty et al., in preparation), which may be

[1] The term 'tutoring', as shall be used in this paper, refers to adult-to-child tutoring (Slavin, 2002), as opposed to peer tutoring.

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acknowledged as a satisfactory answer if given by an 8th grade student. However, it seems only natural to expect a tutor for secondary school math to be able to produce a formal argument in order to solve this task. Roy's apparent difficulty to do so demonstrates the problematic nature of using non-professionals as math tutors. The situation appears to be even more problematic if we consider some affective factors as well. For instance, of the nine tutors involved in the project to be reported in this paper, seven have agreed with the following statement given in a questionnaire: "*When I graduated from high school, I was glad I didn't have to study math any more*". Therefore, the idea of using non-professionals as mathematics tutors for secondary school students does indeed raise many doubts. However, my argument in this report is that this idea is nevertheless worth considering, as it turns out that non-professional tutoring may enhance students' learning, despite its apparent limitations.

BACKGROUND AND RESEARCH QUESTIONS

The study described in this paper was conducted as part of a wider project, named the SHLAV[2] project, which started off on 2004 and is still in its pilot phase. The project's goal is to advance low achievers, from educationally disadvantaged environments in Israel, who are at risk of failing the Matriculation Exam in mathematics[3]. This project continues previous projects conducted at the Weizmann Institute of Science in the past fifteen years, which focused on low achievers in mathematics. These included: (a) Design of special learning and teaching materials for low-track students in secondary schools, building largely on students' common sense and informal reasoning (see Arcavi, Hadas and Dreyfus, 1994); (b) Research investigating learning and thinking characteristics of low achievers in mathematics, based on classroom observations and interviews with students (Karsenty & Arcavi, 2003); and (c) In-service teacher courses for mathematics teachers of low-track students. The novelty of the SHLAV project is in creating a unique framework where the inputs of the previous work could be exploited in order to address the needs of low achievers from *inside* the school system, through direct contact with students and teachers. The project's team has designed a model for a professional role, carried out in schools, defined as a counsellor focusing on difficulties and low achievement in mathematics. During the school year of 2004-5, the author of this paper has been practicing this experimental position within a secondary school that served as a pilot case (herein named ML school), located in an Israeli city known to have a low percentage of students who pass the Matriculation Exam in mathematics. The ML school’s student population consists of a large portion of new immigrants and youth from low socio-economic backgrounds. Main activities carried out as part of the counselling (which was preformed during two full days a week) included identifying students in need of

[2] This Hebrew name is acronym for "Improving Mathematics Learning". The SHLAV project is a joint initiative of the Davidson Institute for Science Education and the Science Teaching Department of the Weizmann Institute of Science, supported by the Clore Israel Foundation.

[3] The Israeli Matriculation Exam in mathematics is a final exam taken at the end of high school, and is compulsory for receiving a Matriculation Certificate. This certificate is a prerequisite for applying to any higher education institute, and is also a necessary requirement for many jobs.

intensive support in mathematics, diagnosing students' difficulties, instructing teachers, and conducting teaching sessions in small groups with some of the students.

After two months in the ML school, it became clear that there were a considerable number of low achievers in mathematics whose problems could hardly be addressed within the limited resources offered by the school. Through sessions with individual students, observations, and discussions with teachers and counsellors , we identified many students whose failure in mathematics could not be attributed to lack of cognitive capabilities, but rather to affective, social or behavioural problems (e.g., tendency to be easily distracted, constant need of attention, lack of motivation, low self-esteem, unsupportive attitude of parents, etc.), that have gradually led to serious deficiencies in mathematical knowledge of earlier years. As expected in light of previous research (Karsenty & Arcavi, 2003; Chazan, 2000), quite a few of these students were able to demonstrate sound mathematical reasoning when placed in more intimate and supportive learning environments, such as tutoring in small groups. Yet, the school's mathematics teaching staff, however dedicated, could not provide more than few opportunities for such circumstances.

It is not surprising, therefore, that when the school principal received an offer to take on five non-professional volunteers for mathematics tutoring, she could hardly decline it. Volunteers were 18-year-old high school graduates from different places in Israel, who participated in a year-long pre-army service program (operated by a social organization), that took place in the neighborhood where the school was located. The SHLAV counsellor assumed the responsibility of organizing and supervising the work of these young people. The project team looked upon this situation as an opportunity for conducting an action research, investigating the effectiveness of non-professional tutoring. Of the research questions that immerged from this situation, I will concentrate here on two, as follows:

1. Can non-professional tutoring improve students' achievements in mathematics, and to what degree?
2. After experiencing mathematics tutoring, what factors are considered by non-professional volunteers as most contributing to their success as tutors?

THE WORK MODEL OF THE VOLUNTEERING TUTORS

This section outlines how the tutoring process was designed and implemented. It relates to the work of the five volunteers, mentioned above, who worked in the ML school throughout the 2004-5 school year, and to the work of four new volunteers, aged 19, who worked during the first term of the 2005-6 school year.

As volunteers were sent by a certain organization, their work was monitored by the local representative of this organization (the abbreviation LR will be used in the following in reference to this role). The LR introduced a set of nonnegotiable conditions, of which the two most significant were: (a) Each volunteer will teach mathematics to one group of two to four students, for 8-10 hours per week. Sessions were to be conducted during school time and use all the time slots intended for math classes in the students' schedule, plus few slots "borrowed" from other subjects to complete the requested amount of hours; (b) Volunteers will work with the same students for about two months, towards a specific predetermined goal. After this period, students will return to their regular math classes, while volunteers take on a new group and a new goal. It should be noted that this model of work was not considered by the ML school staff as optimal, due to several shortcomings which will not be specified here. Instead, the staff preferred models such as the one described by Fitzgerald (2001), in which tutoring hours are spread over longer periods of time, with a smaller number of weekly hours. However, staff members aligned with the LR's terms and were fully cooperative throughout the school year. Within the given terms, the SHLAV counsellor had a free hand in planning the tutors' work. The first decision made was that all volunteers will work with students from the same grade level towards the same goal during each of the two-month periods. The rationale underlying this decision was that tutors would thus be able to create a "tutoring team", share ideas, cooperate in preparing their sessions and support one another if problems arise. Then, for each tutoring period, the following steps were taken:

1) *Deciding on the grade level and the mathematical goal of the tutoring period*. Decisions took into account factors such as the number of unsuccessful students in a grade level and the centrality of certain mathematical contents in upcoming years. Thus, for instance, the first tutoring period was dedicated to linear functions (studied in ninth grade), since failing to understand this subject may cause continual failure in linear programming, analytic geometry and derivatives as slopes of tangent lines.

2) *Selecting students for tutoring*. Criteria for students' selection were: (a) The student had attained low grades in mathematics in previous years; (b) The student was not diagnosed as having cognitive abilities below the age norm (i.e., students with distinct learning disabilities were not selected for tutoring - except for one case, referred to later on - since teaching these students seemed an excessive challenge for young unprofessional tutors); (c) The student did not have a record of physical violence (however, students with discipline problems were selected); and (d) The student and his parents/guardians have agreed to participate in the tutoring program.

3) *Matching students and tutors*. In accordance with Tingley (2001), assigning students to a certain tutor took into account, as much as possible, the tutor's preferences as expressed in a preliminary discussion with the SHLAV counsellor. Tutors wished to create the best fit between their "teaching personalities" as they perceived them, and their assigned students. For instance, some tutors preferred working with students who were considered slow and cooperative learners. Others specifically requested to work with students who were known to be "troublemakers", but had less learning difficulties (of course, these are merely illustrative, simplified profiles of students. There were many nuances to consider within the spectrum of students' learning profiles). Sometimes there were gender preferences as well.

4) *Launching*. Great emphasis has been put on creating a positive atmosphere about the tutoring process. Students were called to the principal's office for an opening session. The principal congratulated them for being selected for the tutoring project, and introduced it as a beneficial opportunity. Tutors then conducted acquaintance

conversations with each of their assigned students. Finally, tutors visited students' homes and introduced the project to parents. The purpose of all these activities was to establish a personal commitment of students towards active participation.

5) *Instructing tutors*. Parallel to the preparations described above, tutors received a full-day instruction from the SHLAV counsellor. Instruction related to mathematical and pedagogical aspects of the specific teaching tasks, as well as affective components to be considered when teaching low achieving students. Tutors were given learning materials, and were advised on how to use them as a basis for composing individual work assignments. Then, on their own time, tutors practiced the material both individually and through group work, and prepared worksheets.

Once these stages were completed, the tutoring period began. As said, 8-10 sessions were conducted per week, and at the end of each week all students took a test. During the entire period, tutors met regularly with the SHLAV counsellor, to report on students' work, discuss problems and receive further instruction. The LR was present at school during most sessions, to supervise the process and attend frequent problems, such as disruptive behaviour or negative interactions between students. In two cases students were removed from the project. At the end of the period, students took a test, composed especially by the SHLAV counsellor to assess the degree to which the pre-specified goal was achieved. On the two last days of each period, tutors conducted full-day learning sessions, held outside the school[4]. These events were looked upon as "final marathons", performed in a friendly atmosphere, with food and candies supplied to students. The official end of each period was noted by a social evening gathering, to which students and parents were invited. The principal summarized the learning process and parents were informed about their children's progress[5].

DATA COLLECTION

The current report refers to all three learning periods completed during the 2004-5 school year, and to the period completed in the first term of the 2005-6 school year. Collected data included the following:

1. Students' grades in mathematics before entering the tutoring program, as appeared in their most recent grading records.
2. Students' grades in the weekly tests and in the final test of the learning period.
3. Details on volunteers' schooling backgrounds.
4. Field notes of staff meetings and of instructional sessions held with volunteers.

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[4] Sessions took place either at the Davidson Institute or at a local youth center, in order to enhance students' feeling that this is a special day, which requires special effort.

[5] This situation involved some delicate considerations, as in each learning period there were few students who did not advance as much as they have anticipated (see results). Some decisions had to be made, therefore, in regard to communicating their attainments during this final event. Due to space limitations, I shall not refer to this issue here.

5. Questionnaires administered to the 2004-5 volunteers at the end of the year, and to the 2005-6 volunteers at the end of the first term. Questionnaire items were partly open and partly closed, and related to various aspects of the tutoring process.

RESULTS

In this section I present selected results, referring to the research questions posed earlier, i.e., the effect of non-professional mathematics tutoring on students' achievements, and factors which were perceived by volunteers as most contributing to a successful tutoring process.

Students' achievements. Table 1 presents the mean change in students' achievements in each of the four tutoring periods, as obtained from comparing grades in mathematics before tutoring, and grades achieved in the final tests after tutoring. As can be seen, mean changes were high in all four periods, and exceedingly so in the first and second periods. The considerably high standard deviations reflect the fact that gains ranged from very high to almost no gain and in few cases even a decline in grades. For instance, seven students (2 in the first period, 3 in the second and 2 in the fourth) had received a "fail" grade (40 or under) throughout the year previous to the tutoring program, and attained a grade of a 100 (or nearly so) in the final test (which was by no means easy, as designed according to goals expected from all other students in the same age level). Of the seven, five had discipline problems which their math teachers found hard to cope with, and were frequently spending time outside the classroom. The other two had emotional problems that caused them to often sit in class and do nothing. All seven students developed close relationships with their tutors and worked seriously during sessions. In weekly meetings with the SHLAV counsellor, tutors often expressed their amazement at these students' rapid progress. On the other hand, there were seven students (2 in the second period, 3 in the third and 2 in the fourth) whose grades remained almost the same or declined after tutoring. Two of these students suffered from test anxiety, which - regardless of their progress during sessions, and in spite of efforts on part of tutors - they could not overcome during the final test. In another case a student was diagnosed as having a learning disability prior to tutoring, but the success of the first tutoring period had tempted the staff to include her in the program, hoping that the intensive attention will promote her understanding. This turned out as a mistake. Her tutor was frustrated by the fact that she could not remember basic procedures from session to session, and the student herself became dispirited since she continued to fail the weekly tests and eventually failed the final test. We learned in the hard way that non-professional tutoring is not recommended, and may even be damaging, for students with learning disabilities. For the other four students, lack of progress cannot be attributed to a single factor, as several problems were noted during their tutorial sessions, such as frequent absences and low level of involvement, as well as tutors' reported impressions that they were not explaining the material well enough or were not bonding well with these students.

Tutoring period	Mathematical content	No. of tutors	Grade level	No. of students*	Mean change in grade**	SD
1. Nov-Dec 2004	Linear functions	5	9th grade	10	53.2	23.25
2. Feb-mid April 2005	Equations of first degree in one or two unknowns	5	8th grade	13	33.3	25.65
3. May-mid June 2005	Quadratic functions; Quadratic equations	4	9th grade	10	13.1	10.98
4. Mid Sept-Nov 2005	Pre-algebra basis, simple linear equations	4	8th grade	12	24.92	20.83

* Excluding two students who were removed from the program

** Refers to the difference between grades before and after tutoring. Grades range from 0 to100

Table 1: mean gains in students' achievements in the four tutoring periods

Table 1 also shows that the third tutoring period was not as successful as the others. This may have happened due to the more advanced nature of the topic (indeed, one of the five 2004-5 tutors withdrew from participating in this specific period, asserting that she could not meet its mathematical demands). In addition, tutors had very little time to prepare for this period, which was also shorter than the other three.

Tutors' view of the tutoring process. As space is limited, I chose to refer herein only to results obtained from the questionnaire item requesting tutors to rank fourteen given factors by their impact on successful non-professional mathematics tutoring (1= most important factor, 14 = least important). The five factors perceived by tutors as most important were as follows (given are $\bar{X}$ and *SD* values, scale 1-14, n=9):

1. Volunteer's ability to maintain warm and supportive relationship with students ($\bar{X}$ =3.25, *SD* = 3.05).
2. Professional instruction regarding the mathematical content ($\bar{X}$ =3.85, *SD* =2.23).
3. Selecting students with no learning disabilities ($\bar{X}$ =5, *SD* =3.74).
4. Professional instruction regarding pedagogical issues ($\bar{X}$ =6.14, *SD* =3.35).
5. Volunteer's willingness to learn mathematical materiel ($\bar{X}$ =6.28, *SD* =3.84).

The two factors perceived by tutors as least important were:

13. The former mathematical knowledge of the volunteer ($\bar{X}$ =11.71, *SD* =1.48).
14. Selecting students with no acute behavioural problems ($\bar{X}$ =11.85, *SD* =1.64).

The general picture that emerges from this piece of data, confirmed also by other questionnaire items and field notes of discussions with tutors, is that volunteers viewed tutoring as a challenge they could meet by investing their affective-humane resources, provided that professional guidance was available. Tutors relied on their keenness to learn more than on their previous mathematical knowledge, and were much more concerned by the possibility of teaching a student with a learning disability than they were by the need to teach students with undisciplined behaviour.

Perhaps these views can be linked to the young age of volunteers, but this speculation needs to be supported by future investigations.

CONCLUDING REMARKS

The tutoring program presented in this paper appears to have a potential for advancing students, whose low attainments in mathematics could be attributed to social or behavioural circumstances. Subsequent research is needed in order to affirm and broaden the results of this pilot study. Questions about the long-term effect of such programs, the impact of group work as opposed to individual preparations of tutors, and the role of social organizations in monitoring tutoring models, are few of the issues that need to be further explored.

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MATHEMATICAL KNOWLEDGE FOR TEACHING: ADDING TO THE DESCRIPTION THROUGH A STUDY OF PROBABILITY IN PRACTICE

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In their description of mathematical work of teaching, Ball, Bass and Hill (2004) suggest eight types of mathematical problem solving that teachers do as they go about their work. In this paper we add to this description through our study of teaching of probability in a grade 8 multilingual classroom in South Africa. We use instances in teaching to highlight the mathematical problem solving that teachers face as they work with learners' ideas, both expected and unexpected. We discuss the issues of language and everyday knowledge which arise. We argue that mathematical work entailments for teachers, for example, being able to hear disconnects in mathematical terms, and being able to acknowledge and enable learners' intuitions to co-exist with mathematical notions, have not been elaborated in Ball et al's framework.

INTRODUCTION

Probability is a relatively new topic in South Africa's school curricula. When teachers are faced with a new topic, questions are raised about how to teach the topic effectively, as there is no previous experience of what learners find easy or difficult about the topic, what activities work well, and what misconceptions arise among learners. These are indeed the right questions to ask because for each topic or concept in mathematics, teachers need 'mathematical knowledge for teaching' (MKfT) that topic/concept (Adler, 2005). MKfT is more than just knowledge of the mathematical concepts, and how to solve relevant problems that entail mathematical processes and procedures. It includes knowing what to do, mathematically, in order to make that mathematics accessible to learners (Ball and Bass, 2000). For probability, we need to ask the question "what is it that teachers need to know and know how to do to teach probability well?" This paper discusses results of part of an ongoing research study that is attempting to answer this question.

MATHEMATICAL KNOWLEDGE FOR TEACHING

The idea that there is specialised knowledge used in and for teaching is not new; it has been discussed, debated and researched for at least two decades now. Shulman's seminal work (Shulman, 1986; 1987) points out that teaching entails more than simply knowing the subject matter. He suggests that besides content knowledge and curricular knowledge, teachers need 'pedagogical content knowledge' which "goes beyond knowledge of the subject matter per se to the dimension of subject matter knowledge for teaching" (1986: 9). Shulman argues that teachers need to know and

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 417-424. Prague: PME.

understand more of their subject than other users of the subject content because teaching entails transformation of knowledge into a form that learners can comprehend. Some researchers, drawing on Shulman's work, have attempted to identify and describe the knowledge required by teachers in order to teach a specific mathematics content area. For example, Marks (1992) worked on 'equivalent fractions'; Even (1990) has worked on 'functions'; Sanchez and Llinares (2003) have also worked on 'functions'. The research reported here is similar to these topic focused studies in that it focuses on the topic of probability. We contend that similarly to the above studies, the specificity of probability might bring additional aspects to the fore.

Our research question was "What mathematics do teachers need to know and be able to do in practice in order to teach probability in secondary school?" In this paper, we report on the problem-solving of a particular teacher as he introduced and taught probability to Grade 8 learners in a school in South Africa.

Adler (2005), drawing from Ball and Bass (2000), describes mathematics teaching as involving particular kinds of problem-solving – problem-solving that has mathematical entailments. In other words, teachers confront problems of teaching as they go about their work, the 'solving' (or action) of which requires mathematical thinking in action, in the practice of teaching. We argue that the teaching of probability, precisely because of its conceptual base, and its use of mathematical English, entails serious engagement with learners' everyday knowledge and meanings, As we develop our argument, we simultaneously elaborate the nature of the mathematical problem-solving teachers do. This, in turn, provides further description of and insight into MKfT.

THEORETICAL ORIENTATION AND ANALYTIC FRAMEWORK

The theoretical underpinning of the study is that mathematical knowledge for teaching is situated in the practice of teaching (Adler & Davis, forthcoming; Ball and Bass, 2000). Therefore, to study it entails an analysis of curriculum in both documentation and practice. In studying teaching, the study draws on Ball et al (2004) who suggest 8 types of problem-solving that mathematics teachers do as they go about their work. These are: (i) Design mathematically accurate explanations that are comprehensible and useful for students; (ii) Use mathematically appropriate and comprehensible definitions; (iii) Represent ideas carefully, mapping between a physical or graphical model, the symbolic notation, and the operation or process; (iv) Interpret and make mathematical and pedagogical judgements about students' questions, solutions, problems, and insights; (v) Be able to respond productively to students' mathematical questions and curiosities; (vi) Make judgements about the mathematical quality of instructional materials and modify as necessary; (vii) Be able to pose good questions and problems that are productive for students' learning; (viii) Assess students' mathematics learning and take next steps (Ball et al, 2004: 59).

We have condensed these into six, as follows: Definitions, Explanations, Representations, Working with students' ideas, Restructuring tasks, and Questioning.

It is this six-part analytic framework that we used to study some teaching of probability in relation to the mathematical problem solving entailments for teachers.

THE STUDY

The study involved working with and observing one mathematics teacher teaching probability. This was in grade 8 at a township secondary school in Johannesburg. Grade 8 was chosen because that is when probability is introduced at secondary schools in South Africa. Township is a context of interest in that it is similar to many schools across towns in Africa, and particularly because we work with teachers in similar contexts. The teacher was an opportunistic sample, known to the authors, and interested in exploring his own teaching of this new topic. An important point to note here is that the idea was not to evaluate this teacher's teaching but to learn from it, and particularly about the mathematical demands of teaching probability, and in this context. A total of eight lessons were observed and video recorded.

Analysis within and across the eight lessons revealed that each of the six aspect of mathematical problem solving by the teacher was evident, but in uneven ways. There were many instances of "working with students' ideas" and "restructuring tasks". Defining, explaining, representing and questioning were marked more by their absence than their presence. One explanation for this could be that these - how concepts might be variously and appropriately defined and represented , together with what might be productive questions and explanations for learners are - need to be attended to in planning.

In the rest of the paper we focus on 'working with students' ideas'. Our purpose is to illustrate the kind of problem solving that was demanded of the teacher.

WORKING WITH 'EXPECTED' AND 'UNEXPECTED' STUDENTS' IDEAS

We provide two examples from the study, one unexpected and one that perhaps could have been expected. In both cases, particular kinds of mathematical or mathematically related demands were made, and discussion of each provides for an elaboration of MKfT for probability in particular and mathematics in general.

Extract 1 – the unexpected – hearing disconnects

In the first lesson the teacher asked learners the question what is "probability", and whether they knew "what probability means". His aim was to find out if the learners had any familiarity with the idea of probability, or if they had used the word before. The extract below captures the discussion that followed in class: (T = teacher, L = learner, LS = learners)

T:	Our deal for the day is to do some mathematics with specific reference to this topic called probability (writes probability on the board)
T:	I don't know how far are you acquainted with the word probability …. do you know what probability means?
LS:	(inaudible)

T: you don’t know what it means?

LS: yes

T: okay, anyone who can give me a try … just give a try … you are allowed to guess, educational guess is good

(pause, class is quiet)

T: take a guess…what do you think probability could mean?

(points to one boy raising his hand up)

L1: (standing) it is about disabled people

T: it is about?

L1: disabled people

T: disabled people ….. he says probability is about disabled people. What are you saying (addressing the class) What do you have to say? (points to another boy raising his hand)

L2: (standing) it is about all things we can do

T: it is what?

L2: all things we can do

T: all things we can do

L2: yes

T: is probability

L2: yes

T: aha, somebody says it is all things we can do ... what are other people saying? Am going to take the last guess (pause) ... anyone to ... make an attempt? Lets give another person a second chance of ... eer… an attempt. Anyone to attempt?

T: (points to L3) L3?

L3: am still thinking

T: you are still thinking

L3: hhhmm

T: okay, so ... other people you don’t want to make an attempt ne? and it means you are hearing the word for the very first time ... probability

LS: (inaudible)

T: okay

The problem for the teacher here is that the responses from the learners were both unexpected and unintelligible in his terms. Talking with the teacher after the lesson, and as is apparent from the text, he said he did not expect the responses learners gave, and that he did not know how to make sense of the learners’ ideas. The source of the learners’ ideas is not the focus of this paper. However, it has particular relevance here, that is a function of learners in this class learning mathematics in English, where this is not their main language. From experience[1], we assert that learners who are not first language speakers of English often equate words that sound alike. In this case, the word ‘probability’ sounds like ‘disability’ or ‘ability’. From this perspective, the two learners’ responses of “disabled people” and “all things we can do” are a function of the sound of the word, rather than any experience of the use of the word.

On the face of it, an obvious move is to enquire into the strangeness of the learners’ responses. In the messiness of classroom life, it is precisely these way out meanings that are ignored. Yet, in the context of multilingualism, attentiveness to how words sound as well as mean is important. As Adler has argued (2001), different pronounciations, and so sound alike words, can become sources of confusion in mathematics (e.g. size, sides, sights were all used by learners in a trigonometry lesson to refer to the size of an angle). The mathematical work of teaching has linguistic entailments, and the problem-solving a teacher is required to do on their feet is to pay attention to what is said, how it is said and what could be meant, if they are to enable learners in multilingual settings to work with the language resources they bring to class. This linguistic aspect of problem-solving tasks of teaching mathematics is not highlighted in Ball et al’s more general framework, and is an important aspect of working with learners’ mathematics. The example here suggests that teachers need to be able to hear disconnects in mathematical terms, and reconnect these in mathematical ways – disconnects like ‘disability’ are indicative of what it is learners bring to the topic under discussion.

Exract 2 – the could have been expected – co-existing contradictory concepts

During Lesson 4 some learners expressed the belief that the number 6 on a die has less chance than each of the other numbers (1-5) of coming uppermost. The lesson started with an activity from a textbook (copied onto a worksheet). Learners in groups were asked to throw a die at least 30 times and record the frequencies of all the six numbers. One of the questions following the activity was “is it more difficult to get a 6 than any other number?” The teacher collected results of each group and displayed on a chart in form of a table as shown below.

Possible outcomes	1	2	3	4	5	6	Total
Group 1	1	5	9	7	4	7	30*
Group 2	8	10	5	9	11	7	50
Group 3	3	5	0	9	10	3	30
Group 4	3	9	5	3	3	7	30
Group 5	10	5	11	13	10	10	59

[1] We draw directly from Kazima’s experience in this regard. See Kazima (2005) for a discussion on Malawian learners’ meanings of some terms in probability.

* - we note the error, total should be 33

In the discussion that followed, most of the learners said that it is difficult to get a 6 than any other number on a die. After the teacher persuaded them to consider the table and the frequency of 6, some said 6 was easy to get in the class but is it harder outside class (transcript to be shown during presentation).

The teacher's intention was for the learners to reflect on the game they had just played and to use the results collected on the chart to answer the question of whether 6 is more difficult than the other numbers. At first the learners did not seem to pay attention to the game they had played in class, nor to the results but responded by reflecting on their everyday experience. The learners were assessing the probability of getting a 6 on a die by calling to their minds such instances from past experience. Elsewhere, this has been referred to as the 'availability heuristic' - the use of what is available in one's mind (Tversky and Kahnemann, 1982). Amir and Williams (1999), Green (1983) and Watson and Moritz (2003) all observed that many students, middle school age, think that the number 6 has less chance of coming up than the other numbers on a die, and that the thinking is influenced by the children's experiences with dice games.

The problem for the teacher here, and in our view this is specific to the challenges of teaching probability, is working with learners' everyday knowledge about probability that is at variance with mathematical knowledge. The teacher eventually managed to get the learners to use the information to respond to the question. However, the learners seemed to think that 6 was not difficult to get in the classroom but it is difficult to get outside the classroom, for example, when playing ludo where getting a 6 matters.

Others have demonstrated that and how it is possible for learners to hold two contradictory ideas simultaneously. Watson and Moritz (2003) give examples of students' statements such as "I know the chance of heads and tails are the same but I always chose tails because it comes up more for me" (page 272) and "some numbers come up more often, but all dice are fair" (page 296). Watson and Moritz conclude that many students hold beliefs that are idiosyncratic and contradictory throughout their years of schooling.

These contradictions are examples of inconsistency between learners' everyday out-of-classroom experiences (everyday knowledge) and in-classroom mathematical reasoning (mathematical knowledge). The mathematical problem solving demanded of the teacher in this case is first to understand cultural practices and related intuitions that learners have about dice drawn from their everyday experiences. Secondly, in Ball et al's terms, the teacher needs "to be able to interpret and make mathematical and pedagogical judgements" about the learners' ideas, and also to "respond productively" to them. It would clearly be helpful for teachers to know the results of previous research that teaching might not make any difference to learners' intuitions about dice. But what else might they need to do, mathematically, to move on? This example, and one that might be experienced in other areas of teaching probability, is the mathematical skill of acknowledging learners' intuitions and enabling these to co-exist with increasing experiences of mathematical notions. Asserting the mathematical case can be counter-productive as it could simply be experienced by learners as 'this is what you need to believe in the school mathematics class' rather than providing means for them to continue to engage the contradiction, and so strengthen their understanding.

CONCLUDING DISCUSSION

We have discussed two instances in the teaching of probability in Grade 8 in a township (and multilingual) school in South Africa, and brought into focus the kind of mathematical problem-solving a teacher faces. We focused on one problem identified by Ball et al (2004) as central in the mathematical work of teaching: working with students' ideas. The two extracts discussed both unexpected and expected student ideas and brought issues of language and everyday knowledge to the fore. Neither is elaborated in Ball et al's framework and suggest a specificity important for teachers to know and be able to act on. On the spot problem-solving is needed in multilingual settings when learners are working to understand both new concepts and the language in which these are being presented. A mathematical ear is needed to hear and then engage learner utterances that reflect sound-alike and not only mean-alike ideas. We have called this 'hearing disconnects'. On the spot mathematical problem-solving is also needed when learners' cultural knowledge and experience is in contradiction with mathematical knowledge, and these two competing ideas need to co-exist as the latter is strengthened. The assertion of mathematical correctness in this instance is likely to be counter productive to longer term depth appreciation of mathematical reasoning.

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MEASUREMENTS WITH A PHYSICAL AND A VIRTUAL QUADRANT: STUDENTS' UNDERSTANDINGS OF TRIGONOMETRIC TANGENT

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We investigate students' understandings of concepts related to the trigonometric tangent in a specially designed learning environment involving measurements with a makeshift Quadrant and its s/w simulation. Students' initially attached a linear relationship between angle and respective high as they used the physical tool. They subsequently negotiated on the kind of functional relationship between the two quantities after realising that their initial conjecture did not always work. We discuss the use of the tools in relation to the opportunities provided by the tangent for mathematizing measurement tasks.

FRAMEWORK

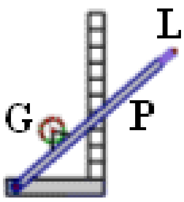
Research on children's difficulties in understanding trigonometric notions is rather limited (Da Costa & Magina 1998) and focused mainly in concepts related to sine and cosine functions. Trigonometric functions are not only related to the notion of periodic phenomena but are also involved in non linear functions that relate angles and sides of a triangle. Generally research on non-linear functions in Geometry has focused more on the relation between length and area (De Bock e.a. 2003). Further more, there are some references in modern curricula that address the tangent function in an algebraic way although there's no reference to its connections to authentic real world situations or problems. In this paper we present a two years' research in which we investigated the ways that 15 year old students construct meanings for the tangent function in a specially designed learning environment. The environment involved students using in combination, a manual tool for measuring the height of a distant object, and a simulation of the tool in a dynamic computational environment. We noted that primarily, the concept was invented and used in devising measuring tools for space as well as time to the extent that one could claim that the development of the operation of certain tools, such as Quadrants, sundials, and astrolabes, goes hand in hand with the development of the concept of the trigonometric tangent (Smith 1958). In the design of the learning environment, we adopted a guided reinvention approach through mathematizing activities in problem situations that are experientially real to students. (Gravenmeijer e.a. 2000). Hence, our intention was to involve students in activities through which they would use symbols, make and verify hypotheses in order to solve a particular real problem in a rich learning environment. We found useful the idea of two types of mathematical activity, such as "horizontal" and "vertical" mathematization, formulated explicitly in an educational context by Freudenthal (1991). Recently, researchers (Rasmussen e.a 2000) have used this idea and proposed a more detailed discrimination for each one of these two types of mathematical activity. Horizontal mathematization might include, but is not limited to, activities such as experimenting, classifying, conjecturing, and structuring. The activities, that are grounded in and build on these activities, such as reasoning about abstract structures, generalizing, and formalizing, are thought of as vertical mathematizing. We investigate mathematical activity through the use of physical and digital artefacts available to students because we perceive them as important representational registers with which students can mediate ideas for both practical and intellectual activities (Mariotti, 2000). We were interested to investigate student meanings formed through a functional, purposeful use of tangent, drawing from both physical tools and computational media serving as editable simulations of these tools. We prefer simulations by which students have access to the mechanism for constructing the simulation, since it would afford them the possibility of expressing, questioning, investigating and experimenting on the rules of the phenomenon that the simulation represents (Kynigos 1995).

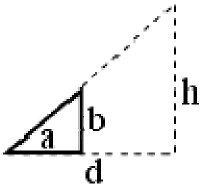
RESEARCH SETTINGS AND TASKS

In Greece where the study was set, the function of the trigonometric tangent is taught in the grade 11 (Senior High School) through the trigonometric circle. The emphasis lies on the manipulation of symbols for the concept making the transfer to problems and application in the real world obscure. We were interested to investigate ways in which the students would use formal symbolization to express mathematical meanings during the mathematization of this science-like measurement activity. We thus put together a makeshift device resembling the ancient 'quadrant' and designed a simulation of the quadrant by means of a programming language, leaving the parts of the code related to the mathematical mechanism of the tool open to the students.

A modern Quadrant

This makeshift device has been constructed based on the functional characteristics of the Quadrant and consists of an ironwork right angle. In the left-hand bottom corner,

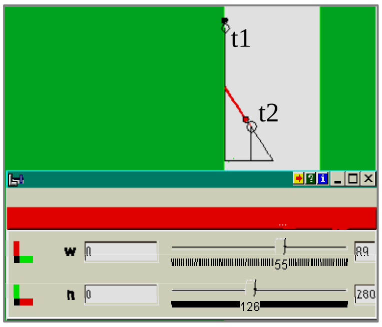


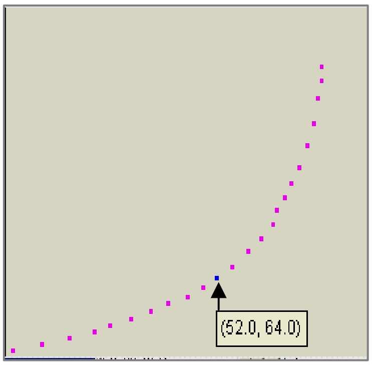


pointer P is attached which is able to rotate and, on the pointer, a small laser pointer L is affixed. The perpendicular side is graduated while the angle of P to the horizontal plane can be measured through the goniometer G. Hence the measurement of height h is relegated to the measurement of distance d of the tool and the application of the formula $h=\frac{b}{a}d$ since the dimensions a, b of the tool are known. If for any reason the device cannot be moved or the height that the focus point is found is too high, then the simplest way to focus at a certain height is to rotate the pointer. In this case, and if distance d is fixed and known, we will have to correlate the angle with the height, that is to investigate a trigonometric change, which, for the purposes of the particular problem, is located on the trigonometric tangent.

A simulation of the tool.

We use a piece of geometrical construction s/w called ‘Turtleworlds’ that combines symbolic notation through a programming language with dynamic manipulation of variable procedure values (Kynigos, 2002). Clicking on the trace of a variable procedure executed with a specific value activates the variation tool and the 2D variation tool. The former provides a slider for each variable, the dragging of which results in the figure’s DGS-like ‘continuous’ displacement as the value of the variable changes. The latter is activated when two variables are chosen on the former. A coordinate plane allows free dragging. Perpendicular dragging respectively changes the value of one variable. Dragging in any other direction changes both values at once, according to the coordinates at each position. When students clicked on the simulation of the tool, they could use the variation tool to change either of the values of the two variables w, h specifying at each moment the height h of the simulated wall, constructed by turtle1 as well as the angle w of the pointer P to the horizontal plane constructed by turtle2. The 2D variation tool was simultaneously available on the screen, providing a Cartesian plane where the calibration of the axes was specified by the extreme values of the variation tool. Clicking on a point on the 2D variation tool, the variables w, h received the values of the point’s coordinates so that both the representation of the device and turtle2 moved. Dragging leaves a linear trace of the hand movement. Finally, the students had direct access to the code through which the simulation was constructed, while certain procedures such as hypotenuse end leaser, were given as black boxes. The code used for the construction of these procedures contained mathematics beyond the scope of the research project, for example the arcsine function, or primitives and syntax unnecessary for the task at hand. The part of the code that was accessible to students, was describing the structure of the physical instrument, including mathematic notions familiar to students from previous grades and also permitting experimentation through simulation. The students also had at their disposal a measuring tape and a notebook in which they could keep notes and construct shapes. The problem posed to the students was to measure various heights in the physical environment with the manual tool and then try to work out whether they can predict measurements which were impossible to carry out by hand (too high up on the wall or higher than he ceiling), using either quadrant or simulator, or both.






```
to horizontalside :w
turtle2 left 90
make "a 40*cos(:w)
forward :a  right90
end
to verticalside :w
turtle2  make "b 40*sin(:w)
forward :b
end
to measure :w :h
turtle1forward :h
horizontalside :w
verticalside :w
hypotenuse :w  leaser :w
end
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RESEARCH QUESTIONS AND METHODS.

We had two research questions during our experiment a) Which were the specific

mathematization activities that would occur while students worked in the learning environment which combined a makeshift device and its computational simulation? b) How can these activities be useful in creating meanings related to the trigonometric tangent as function, and to its properties?

Design research was adopted where the researcher undertook the role of an observing teacher (Cobb et al, 2003). In this paper we report findings from a study involving 24 students aged 15. There were 10 different teams consisted of 2 or 3 students each and all teams worked for 7 hours at the school computer laboratory. The students came from a Greek public school and were members of a team that was already familiar with research projects and with DGS software but with no prior experience with Turtleworlds. Their performance in mathematics at school in the two previous years averaged B-. The students spent the first three hours, familiarizing themselves with the Turtleworlds software and the way the manual tool was put together, but without being aware how it would be used for the measurements. The whole course of the research was videotaped, audio taped and, later, the transcript was studied to locate and organize the data of mathematization that pertained to the core questions. The data were analysed in two phases. In the first phase episodes of meaning generation were identified (e.g. cases where the students expressed meanings related to the relation between angle and height or meanings related to the tangent function) for each group. In the second phase each episode was analysed in depth. The episodes were grouped in the areas of meanings which appear as section headings below. In this paper we report episodes from different student groups which were characteristic of the respective areas of meanings.

NON-LINEAR CORRELATION OF ANGLE AND HEIGHT.

The basic problem that students were asked to solve concerned the utilization of the makeshift device and it was explicitly formulated by the researcher; “How can we measure a specific height using the tool?” The mathematical problem involved the finding of a relation between the angle of the pointer and the opposite height. Students investigated different solutions using the makeshift device, the variation tool and the 2D variation tool. At the beginning the researcher asked each team’s students to zoom on a spot at the wall using their laser and then measure the angle using the goniometer. Then he asked them to rotate the pointer in order to double the height. All students’ spontaneous answer was that they should double the angle; therefore they had chosen a linear correlation in order to construct the problem situation. They soon realized that this choice was acceptable only if the initial angle was less than 20^0, in that case the doubling of the angle resulted the elevation of the laser point on the wall up to a height that seemed almost equal to the initial one. It is typical that some teams tried to explain their opinion with a brief written justification (team 2 or T2):

Peter: I've been thinking of something really simple, as long as 20^0 correspond to a height, lets say h and we want a 2h so $20^0/x=h/2h$. (he wrote this in his notebook)

Mary: It's logical…ok that is 40 degrees.

Students used proportions and tried, through vertical mathematical activities, to support the idea of linear correlation as a possible explanation to the observed phenomenon. The students concluded that their conjecture was not correct when they noticed that, on turning the pointer from 30 to 60 degrees, the spot appeared somewhere on the roof of the room. In that way students realized for the first time that linear correlation was not an effective strategy and as a result it couldn't be used to solve the problem of measuring a height using the makeshift device. The meaning of the non linear correlation for the students was attached to a phenomenon created in the physical environment and it resulted through the observation of the function of the device in this environment.

At that point students decided to investigate the relation between the angle and the height by using the variation tool on the simulation. Students were asked by the researcher to transfer the basic problem from the physical environment to the simulation environment and to propose an equivalent research question. The transformation of the problem from the real world to the simulation environment produced a vivid mathematical discourse among students and led them to the formulation of a new research question; "What are the necessary conditions in order to construct a bigger triangle?" Then students' activity involved the specification of extreme values for the angle w of the variation tool. These values had been noticed by students while they were experimenting with the physical and the simulation tool, so angles near 90^0 produced points at the ceiling of the room. Further more students reckoned that heights in real world couldn't have negative values. The mathematical discussion among members of team 4 is representative of that observation

John: At 90 laser becomes parallel…., it can't stop somewhere.

Jim: It can go as high as we want.

John: …So let's try 89,999…

We can see that students created an initial domain [0 90) for the values of the angle, this choice was based on the practical needs of the problem and the specific features and functionalities of the tools. This activity is a characteristic horizontal mathematization mediated by the tools. Then students decided to drag the variation tool and search for the angle values that could give an acceptable bigger triangle at the computer screen. In that case students again estimated that small angles could produce a linear correlation while simulation precision limited that angle up to 10^0. Angles more than 10^0 resulted heights that couldn't be solutions of a linear correlation so this observation was evidence that heights and angles weren't in proportion. Students engaged in dragging which gradually become more systematic and focused in their attempt to investigate arithmetical properties which concerned the construction of the big triangle on the screen.

Since students couldn't work out the relation between the two variables that constructed the bigger triangle, this difficulty led them to the use of the 2D variation tool where they searched for the appropriate points. Unfortunately students weren't able to transfer their later conclusions to the new representational framework. For example students from T2 and T5 located two points on the 2D variation tool they joined them with a straight line and searched for new points on that line. Students again tried to apply the linear model overlooking the fact that they had already agreed, while working with the two previous tools, that linear correlation was not a suitable choice. The ineligibility of the linear model was explicitly stated by students when at last they looked for points beyond 30 degrees and they realised that these points belonged to a curve; "beyond a certain angle, it appears that the linear relation may not work" (T5). The divergence of the points from the straight line was a crucial subject of negotiation among students of all teams. The way that the students articulated this finding is characteristic:

> "In the beginning the points are on a straight line, but then they confuse us because they deviate." (T9)

In other words, they felt that their initial conjecture was confirmed up to a point. So far students experimented with the tools and rejected the mathematical framework of linear correlation so again in that case their activities can be identified as horizontal mathematization. The activities with the 2D variation tool facilitated students to investigate geometrical aspects of the correlation between the angle and the height. Those aspects concerned the curve of the trigonometric tangent but for students it was only another representation of the non linear correlation.

THE TRIGONOMETRIC TANGENT AS A FUNCTION.

Students' activities led them to a dead end so they decided to read the logo code and there they observed that the two perpendicular sides of the small triangle, we represent them as :a and :b, were variables dependent on the angle w while the distance from the wall was constant and equal to 40 units. This information was utilized in two ways. Teams T2 and T5 drew the two similar triangles on their notebooks and used the proportion of the lengths of corresponding sides in order to find height;

> Peter: If this length is 40 then β/α=h/40 (he wrote on his notebook)
>
> Mary: Let's verify it with the code (T2)

Students worked on the code and replaced h by 40*:β/:α then they activated the new program and confirmed the construction of the big triangle. They used similarity of triangles, but while ratios of the lengths are constant on their notebook, in the code ratios represented a variable dependent on the angle. The rest teams discussed the angle's existence in the code, which led them to choose the definition of the trigonometric tangent so they wrote tan(:w)= :h/40 and :h=40.tan(:w). Children are familiar with tangent, cosine and sine definitions from grade 9. When they replaced, h with this formula in the code, they observed that the big triangle can be formed for every value of the angle :w in the variation tool. One of the teams (T1) noticed that there was a pass from the static schema of angle w on their notebook, to a different approach of angle w as a variable in the code.

Miria: Can we say that the angle is a variable?

Researcher: Why are you asking that?

Miria: We know that ratios and angles in similar triangles are constant.

Stella: Yes, but in that case we have variable triangles. (she points at the screen)

We can see that Miria practice is a vertical activity within the representational framework of static schemas on the notebook, using the mathematical framework of similarity of plane figures. Finally the trigonometric function of tangent resulted through a vertical procedure that took place while students were using mathematical symbols (symbolization), a fact that was facilitated both by the nature of the code representation and the dynamic nature of simulation. The big triangle was constructed by students in two different ways, first with the code formula :h=40.tan(:w) and second with the points of the curve in the 2D variation tool. The researcher asked students to discuss why this had happened; (T10)

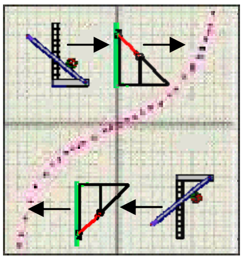
Anton: Perhaps… this curve might be the graphical representation of the formula!!(he points at the curve and the code formula)

Researcher: How can we name this function?

Stavros: Algebrogeometric.

Anton: Ooo.. trigonometry…… Trigonometric?

The starting point for the extension of the domain of the new function to negative values was the students' effort to measure the distance from a surface up to the floor. After they inverted the physical tool they discussed the values of the magnitudes that they could use in the simulation. The passing from positive to negative values was not easy for the students as they believed that: "There is no negative number in the physical environment". In 4 of the teams the students attempted to construct the inverted small triangle in the simulation. They also used the variation tool and they



pout the value -89 to the left utmost and the value 89 to the other. The students realised that the construction of the inverted small and big triangle was possible only when the points on the 2D variation tool were ordered on a curve. This curve seemed to be similar to the initial that corresponded to points with positive coordinates.

CONCLUSIONS

Certain interesting points arose since the students were involved with the activity in the particular learning environment. Students created meanings about the trigonometric function of tangent by using horizontal mathematical activities at the beginning and a combination of vertical and horizontal activities afterwards. The gradual overstepping of the linear correlation model happened separately in each representational framework. Students didn't manage to transfer their current conclusions about the non linear correlation of angle and height, from one framework to the other. These isolated results contributed to the creation of the notion of the non linear correlation in a broader sense. The variety of different tools, that is the different representational frames of the situated problem, gave students the chance to control their conjectures and revise their choices. The meaning that students attributed to the trigonometric tangent, as function, was established for the most part by the respective representation of this function in the simulation (construction of the big triangle). The function formula and its graph in the domain [-89, 89], were connected through their common result of the construction of the big triangle, in the simulation tool.We believe that this type of research could be fruitfully expanded both to the function of the sine and to the function of the cosine.

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LANGUAGE AND MATHEMATICS: TOWARD SOCIAL JUSTICE FOR LINGUISTICALLY DIVERSE STUDENTS

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This paper explores ideas drawn from Bilingual Education related to language use in connection to the learning and academic development of linguistically diverse students, or students who speak and/or represent language(s) other than the official one of instruction. The purpose is to situate a student's home language, discourse, and voice in mathematics education and to discuss the role and possibilities mathematics education has in creating educational democracy and social justice. The paper concludes with a set of questions for future research, teaching, and professional development.

INTRODUCTION

The increase in linguistically diverse or bilingual/bicultural students in schools in many countries has produced a common challenge among teachers and other educators to know how best to instruct students who are learning mathematics through a second language—a language that is officially designated for instruction and that in many ways is a student's weaker *academic* language. However, in the urgency to meet the instructional challenges presented by linguistically diverse students, educators may focus more on simply finding teaching strategies without considering deeply the broader social, political, and cognitive role of language (cultural and academic language) in learning. They may end up with less than effective quick fixes for teaching that really do not lead to student learning because they overlooked critical aspects of language use, aspects that can either promote or hinder student learning.

The argument I set forth is that there are key concepts from research in Bilingual/Bicultural Education that can inform mathematics education to ensure that mathematics pedagogy is as positive and effective as it can be for linguistically diverse students. Specifically, the paper will consider the role of students' home language, academic discourse, and voice in mathematics. I draw on the schooling context in the United States to consider what and why mathematics educators anywhere should care about language diversity and language practice in their own context, and why they need to consider the knowledge base from Bilingual/Bicultural Education. I also draw on my own work in and research related to teacher preparation and professional development for multilingual mathematics classrooms over the last decade and a half, work that has focused on one particular group in the United States, Latinos. While I refer to Latinos and Spanish in my discussion, my point applies more broadly to any multilingual context.

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 433-440. Prague: PME.

THE ROLE OF THE HOME LANGUAGE IN MATHEMATICS

Latinos in the U.S. have the most disturbing and persistent pattern of underachievement--especially in mathematics (NCES, 2004). They have the lowest completion rate of high school, have the lowest retention rate in higher education, and are the least represented in professional areas associated with mathematics (NCES, 2000). These demographic characteristics transcend generational status in that they do not apply only to children of immigrant parents but also to native-born children of native-born parents.

Over the last thirty-five years, Bilingual/Bicultural Education in the United States has produced a body of research that has been in response to civil rights issues for Latinos and other language minority students (e.g., Garcia, 1995). This area has been less about teaching language as about creating learning environments that empower students and that challenge the status quo of ethnic, economic, and language discrimination. The research has led to a set of principles for effective instruction for language minority students (see Dalton, 1998). Included in this area are findings that demonstrate that the highest academic achievement occurs among students with the *strongest* linguistic skills in their *primary or home language* (Thomas & Collier, 1997). Interestingly, the highest correlation with staying in school is enrollment in advanced mathematics (Cardenas, Robledo, & Waggoner, 1988). By implication, Latinos who are able to read, write, and communicate at a high level in their primary/home language—in this case, Spanish—do better in school where English is the medium of instruction, are more likely to enroll in advanced mathematics, and are more likely to complete school and enter higher education. Instruction in students' home language, and even the explicit valuing of it in classrooms, has been found to be part of what constitutes effective instruction in mathematics (Khisty, 2004; Khisty, 2001; Khisty and Morales, 2003). In other words, language and mathematics are intricately intertwined, and as such, together play a significant role in whether schooling is oppressive or liberating, and ultimately, whether Latinos have access to education.

Latinos have maintained their affiliation with Spanish regardless of proficiency in the language across generational status, On a national survey, only 37% of Latino students indicated that they speak only English in their homes (NCES 2000). These data point to the fact that language is more than simply words. The power of language extends to one's definition of self, one's relationship to a community, and one's status in a wider sociopolitical and cultural milieu (Cummins, 2000). The loss of proficiency in the home language has severe consequences for students socially and more; language loss impacts students' ability to effectively communicate with parents, family, and community. In essence, students loose access to the advice, guidance, and learning supports that parents and others can provide (Wong Fillmore, 1991). Language choices in schools, therefore, mediate how students perceive themselves, their parents and community, their place in the wider society, and also how much cognitive support they have access to.

Language also forms part of the context of learning. Language is at the heart of human interactions and communication. In classrooms, language is one of the primary mediums for constructing and conveying meanings, for presenting content and transmitting some forms of knowledge, for displaying what one knows and how she/he knows it, and for evaluating that knowledge. Language is also content in that schooling involves developing new forms of language (for example, writing) and new ways of using language (for example, writing genres). How these roles of language play out in mathematics has been well described in various works (e.g., Pimm, 1987).

The essential point here is that "learning language" and "learning through language" are simultaneous (Halliday, 1993). What a student learns and how she/he learns it depends on the context in which learning occurs. However, linguistic choices (for example, use of the home language or the school official language) realize particular kinds of contexts. Classroom learning are socially constructed events, that is, contexts for learning created by the interactions of teacher and students (Gutierrez, 1995). Therefore, what language and how language is used constitutes both the context that mediates learning and the content of what is learned.

The challenge for language minority students in any context is to acquire proficiency in the dominant cultural language of instruction so that instruction is comprehensible while at the same time acquiring new content meanings and knowledge that are shaped by how, when, and why language, including that cultural language, is used. The challenge for schools is to assist and support students to navigate through this and to ensure they do not become marginalized or alienated because of any lack of value and validity given by schools to the home language. The use of the home language in teaching mathematics can provide a more comprehensible learning environment, maintain students' access to parents' knowledge, and give social and political status to the home language (e.g,, the language holds knowledge)

ACADEMIC DISCOURSE AS SOCIALIZATION IN MATHEMATICS

> Language acquisition is more than learning to speak; it is a process in which a child becomes a competent member of society by learning how to use language in a particular community. Language acquisition, then, is language socialization and involves the acquisition of discourse…It requires appropriating both linguistic and social knowledge. Social knowledge, however, is not acquired independent of linguistic knowledge. Members of communities are both socialized through language and socialized to use language… (Gutierrez, 1995, p.23).

The context and language of schooling are different from what one finds outside of school. In school, most children, and especially language minority students, must adopt new ways of using language to accomplish new tasks and to interact in new ways. They must convey information in new ways, with greater detail, and with an emphasis on specified relationships. Classrooms then become communities of practice--communities of academic practice--and speaking collectives. In schools, the communicative practice forms an academic discourse. Academic discourse relates not only to the language of a subject (in this case, mathematics) or its register and terminology, but *more* so to the nature of valuing, acting, and thinking that is associated with the subject (in this case, mathematics) and that is communicated through words, both written and spoken.

However, again, academic discourse involves much more than learning words, and to focus on words is to miss the crucial nature of actively use language. Each particular cultural or social group (e.g., mathematics) has its way of acting, talking, interpreting, and thinking, and discourse competence for the particular group means knowing when and how to use these characteristics. If we consider that meanings and knowledge are socially constructed and that this is accomplished heavily via language, then academic discourse forms the link between language and socialization into the ways of schools.

Academic discourse competence in this broader sense is acquired through active participation in the community that uses that discourse, and through interactions with a more capable other (Vygotsky, 1986). The lack of discourse competence suggests academic failings. Students who do well in school relate to, comprehend, and are able to use the various variations of academic discourses found in school. They display discourse competence and are evaluated by teachers as being academically proficient because of it (Gee & Clinton, 2000). Without the academic discourse or language, students are systematically excluded or marginalized from classroom curricula and activities.

However, students' academic discourse competence is a result of socialization processes where they are exposed to, have opportunities to engage with, and are taught academic discourse. Variation in academic discourse competence might be understood as the result of students' access to and participation in the kinds of activities and forms of discourse that lead to discourse competence. If students only have access to a teacher controlled script or lessons characterized by limited use of language or language that emphasizes expressions of lower level thinking, then students will appropriate this kind of discourse and not gain competence in more complex and advanced discourse. For example, if students spend much of their schooling doing basic skill mathematics, they will develop the discourse associated with this kind of mathematics. The talk that generally accompanies mathematics work that emphasizes procedural steps and correct answers tends to be simplistic with frequent single word utterances. It becomes what Setati and Adler (2000) call procedural talk. In this case, students will have little opportunity to develop the discourse associated with problem solving, conjecturing, generalizing, or the discourse we tend to accept as the academic discourse of a competent mathematics student by today's standards. Unfortunately, classrooms with language minority students tend to emphasize vocabulary learning (Garcia, 1995) and/or students passively learning mathematics and not engaging in much active language use (Brenner, 1998).

VOICE AND MATHEMATICS

McLaren (1989) describes "voice" as the cultural history and background experience a person uses to interpret and articulate experiences.

> "…when we talk we give expression to all we are—our class, gender, racial, and cultural identities, as well as our assumptions, values, and ideologies—all of which have been constructed in the social relations of our lived experiences….Depending upon our position in the social order and the specific context of the moment, our voices are given ample time and ways to be heard and therefore affirmed, or given few opportunities to be heard and therefore frequently silenced (Frederickson, 1997, p. 18)."

Voice is not how much one says or how loudly one says anything. It refers to whether what one says is seen to contribute to the discussion or to everyone's learning--that what one says has value and validity. For linguistically diverse students who have subordinated status because of history and/or because they are immigrants, voice is a critical element in instructional processes in mathematics. Students may be ignored or relegated to passive learning because they are not deemed to have sufficient proficiency in the second or school language to participate in active discussions or problem solving. This suggests that their solutions could not help others learn since they are not in the dominant language of instruction. Morales (2004) found that even in all-English classrooms, Latino students in advanced mathematics classes used Spanish as a resource for learning. These students moved relatively easily between Spanish and English to negotiate meanings, using whichever of the two languages best conveyed a meaning at a particular instant. Their use of the two languages was not because they did not know a word in a language but because the particular language at that moment better expressed their thinking. However, Morales also found that this language resource was not capitalized on by the teacher and was deemed inappropriate for whole-class presentations. In essence, Spanish was all right to use among the group but was not knowledge for the whole class.

In either of the above situations, it is implied that the home language is not appropriate for higher-level class interactions. In turn, the exclusion of the home language as a viable medium for participation in mathematics, expressions of thinking, or simply use in active learning, says that the home language (Spanish in this case) does not have knowledge or is useful for constructing knowledge. The denigration of the home language further alienates students from school and mathematics and silences students' voice.

On the other hand, bilingual/ bicultural forms of interactions and dialogues with others are the roots of a consciousness that develops the ability to think critically not only about learning but about the world in general and one's place in that world. This process is strongly related to having voice—and voice is related to self-determination and self-respect.

CONCLUDING THOUGHTS

Education for subordinated groups can mean self-determination, and this is intertwined with empowerment, self-respect, respect for one's history and community. From this perspective, understanding development in mathematics is to understand the relationship of a constellation of sociocontextual factors. Within this constellation is the nature of language use, the resultant discourse community in mathematics classrooms, and students' participation in this discourse community, especially when there is more than one cultural language. From these factors emerge the socialization of knowledge, social competence in the content, and ultimately, academic competence. Given this, a study of Latinos and other language diverse students in mathematics should not be simply a study of the development of mathematics content or a set of skills. Likewise it should not simply be a study of language in mathematics at the level of words. We instead should study which language is used and how, how students are initiated into a new discourse community, and how both of these processes contribute to students' voice.

My purpose in this paper has been to highlight a different, a political, role of language in learning mathematics especially among Latinos and other linguistic minorities—a role that can have significant impacts on this learning. For a significant portion of linguistically diverse students in many countries, first and second language proficiencies encompass classroom discourse processes which are intricately intertwined with academic competence and success, which in turn are critically linked to empowerment, enfranchisement, and simply a life out of poverty. From this we return to the beginning of our discussion and the findings by Thomas and Collier (1997) that students who have continued to develop the formal schooling of their home language *do better* academically in the dominant language of instruction. This raises questions about whether current reform curricula and instructional practices in mathematics contribute to or detract from students' formal schooling in their home language. It may not be enough to simply have home language versions of curricula. While the intent of home language versions should be to assist in making the content comprehensible, schools may see these as just materials students can use while they "transition" into the dominant language of instruction.

It also raises questions about how teachers and other educators position themselves relative to student's home language in mathematics. Do teachers and others understand and appropriately consider the political implications of which language is used and how? Do they view it as a learning resource or as something that does not have a place in mathematics classrooms, that should be ignored? Do they genuinely value the home language, do they recognize that differential status among students, including language status, is detrimental to students' learning, and do they seek ways to equalize language status? Do they seek ways to validate what students' have to say even when they do not speak the dominant language of instruction? In the U.S., as is likely in other contexts, very few teachers are members of ethnic or language minority communities, and few speak their students' home language. Given this, it might be asked how could teachers who are not bilingual/bicultural themselves actively use the student's home language in teaching mathematics? However, the real question is why are teachers et al not asking how they can capitalize on students' home language and make it a viable part of learning mathematics? If we ask this question, we will find answers to it.

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CONCEPT DEFINITION, CONCEPT IMAGE AND THE DISCRETE – CONTINUOUS INTERPLAY *

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This research deals with students' understanding of mathematical concepts that relate to the conceptualization of the continuous such as the notion of limit. The cognitive difficulties that accompany the learning of these concepts at the different stages of the mathematics education are well reported in the literature. We analyze the influence of activities based on the complementary aspects of discrete and continuous approaches, on students' conceptual understanding of the notion of limit in the derivative concept. We observe students' awareness of inconsistencies between their different intuitions, the way they develop and change conceptions during the course of relevant activities, and also the persistence of "treasured" intuitions.

INTRODUCTION

The cognitive difficulties that accompany the understanding of mathematical concepts that relate to the conceptualization of the continuous are well documented. Some of the difficulties might be a consequence of our intuitive thinking, for example, our intuition of infinity. Other difficulties might be related to visual representations or verbal descriptions of the concept that precede its definition. The intuitive thinking, the visual intuitions and the verbal description of a concept are necessary for its understanding. However, there might be a gap between the mathematical definition of a concept and the way one perceives it. In this case, we may say that there is a gap between the concept definition and the concept image. The term *concept image* describes *the total cognitive structure that is associated with the concept which includes all the mental pictures and associated properties and processes* (Vinner & Hershkowitz, 1980). *Concept definition* is defined as *a form of words used to specify that concept* (Tall & Vinner, 1981). The research study described in this paper is an effort to contribute towards the challenging aim to help students reduce the gap between the concept definition and the concept image of notions that relate to analysis. In particular, the discrete - continuous interplay will be used to help students understand the notion of limit in the definition of the derivative concept. We also aim to facilitate students' understanding of the need for a mathematical definition. We use the discrete – continuous interplay to create a situation of conflict where students are exposed to two different approaches to the same problem, namely, the discrete numerical approach and the continuous analytical approach. We describe students' conceptual thinking in such a situation of conflict, and characterize some factors that might

* This research was supported by THE ISRAEL SCIENCE FOUNDATION (grant No. 1340/05).

reduce the gap between students' concept image and concept definition. In the following section, we describe what we mean by concept image of the continuous.

CONCEPTUALIZATION OF THE CONTINUOUS. THE CONCEPT IMAGE

In previous studies concerning the way students conceived real numbers, Monaghan (1986) observed that students' mental images of both repeating and non-repeating decimals often represent *improper numbers which go on for ever*. Kidron & Vinner (1983) observed that the infinite decimal is conceived as one of its finite approximation " *three digits after the decimal point are sufficient, otherwise it is not practical"* or as a dynamic creature which is in an unending process- a potentially infinite process. In the literature, we read about the importance of the transition from processes to abstract objects in enhancing our sense of understanding mathematics (Sfard, 1991). Thus, the limit concept should lead to a new entity. In order to better understand the cognitive difficulties that accompany the concept of limit, we explain what we call concept image. The concept image may consist of students' mental images, intuitions and may also consist of figural models created by descriptions which precede the definition of the mathematical concept (like the geometrical representation of the derivative).

Intuition of Infinity: Fischbein (1978) found that students' intuitive conceptions of limiting processes tend to focus more on the infinity of the process than on the finite value of the limit. Fischbein, Tirosh, and Hess (1979) observed that the natural concept of infinity is the concept of *potential infinity*, for example, the non-limited possibility to increase an interval or to divide it.

Description and definition: Gian-Carlo Rota (1997) pointed out the complementary aspect of definition and description in Mathematics:"suppose you are trying to teach a new mathematical concept to your class. You know that you cannot get away with just writing a definition on the blackboard. Sooner or later you must describe what is being defined". Analyzing description and definition, we deal once more with the relationships between intuitive and formal knowledge. On one hand, the description is important and is sometimes accompanied by visual intuitions necessary to the understanding. On the other hand, it might activate some existing intuitions and might encourage new figural models by means of visual representations or verbal descriptions. Fischbein (1999) dealing with the relationships between intuitive and formal knowledge claims that the most interesting situation is that in which a conflict appears between the intuitive reaction to a given situation and the cognition reached through a logical analysis. We create such a situation of conflict by means of the discrete – continuous interplay.

THE DISCRETE – CONTINUOUS INTERPLAY

In this paper, we consider a specific use of the discrete-continuous interplay that **highlight the discrete** nature of the computations. Our aim is to highlight the difference between the analytical-continuous solution and the computational solution to the same problem by means of discrete numerical methods. We are especially interested in the situation of conflict which results by contrasting the two approaches. We deal with the notion of limit in the concept of derivative. If our aim is to describe what is being defined in $f'(x)$ as $\lim_{h\to 0}$ (f(x+h)-f(x)) / h, we may try to develop visual intuitions that support the formal definition. In this case we create the impression of a potential infinite process of (f(x+h)-f(x)) / h approaching $f'(x)$ for decreasing values of the parameter h. By means of animation, the students visualize the definition of the derivative. There might be other effect as well: The dynamic picture might reinforce the misconception that one can replace $\lim_{\Delta x\to 0} \Delta y/\Delta x$ by $\Delta y/\Delta x$ for Δx very small. How small? If we choose $\Delta x = 0.016$ instead of $\Delta x = 0.017$ what will be the difference? Will it be just one digit after the decimal point or something else?

There is a "treasured intuition" that *gradual causes have gradual effects* and that *small changes in a cause should produce small changes in its effect* (Stewart, 2001).

We were interested in a counterexample that demonstrates that one cannot replace the limit "$\lim_{\Delta x\to 0} \Delta y/\Delta x$" by $\Delta y/\Delta x$ for Δx very small, and that omitting the limit will significantly change the nature of the concept. The counterexample is taken from the field of dynamical systems. A dynamical system is any process that evolves in time. The mathematical model is a differential equation dy/dt = y' = f(t,y) and we encounter again the derivative $y' = \lim_{\Delta t\to 0}\Delta y/\Delta t$. In a dynamical process that changes with time, time is a continuous variable. Applying a numerical method to solve the differential equation, there is a discretization of the variable "time". Our aim is that the students will realize that in some differential equations the passage to a discrete time model, might totally change the nature of the solution. We also aim to help students realize that gradual causes do not necessarily have gradual effects, and that a difference of, say, 0.001 in Δt might produce a significant effect. In the following counterexample (the logistic equation), the analytical solution obtained by means of continuous calculus is totally different from the numerical discrete solution. Moreover, using the analytical solution, the students use the concept definition of the derivative $\lim_{\Delta x\to 0} \Delta y/\Delta x$. Doing the discrete approximation by means of the numerical method, the students use the intuitive view of $\Delta y/\Delta x$ for small Δx. We will see that the two solutions, the analytical and the numerical, are totally different. We aim to analyze the students' thinking processes in the situation created by means of the specific discrete–continuous interplay, and to characterize some factors which might help students reconstruct the formal definition. The cognitive difficulties concerning the limit notion motivated the design of the learning experiment.

The design of the learning experience

First year college students in a differential equations' course (N=60), were the participants in the research. The Mathematica software was used during the lectures for demonstrations and in the PC laboratories for the exercise sessions. The students were given the following task: a point (t_0, y_0) and the derivative of the function dy/dt=f(t,y) are given. Plot the function y(t). The students were asked to find the next point (t_1, y_1) by means of $(y_1 - y_0)/(t_1 - t_0) = f(t_0, y_0)$. As t increases by the small constant step $t_1 - t_0 = \Delta t$, the students realized that they are moving along the tangent line in the direction of the slope $f(t_0, y_0)$. The students generalized and wrote the algorithm: $\boldsymbol{y}_{n+1} = y_n + \Delta t\, f(t_n, \boldsymbol{y}_n)$ for Euler's method. They were asked how to better approach the solution. They proposed to choose a smaller step Δt.

The logistic equation dy/dt = r y(t) (1-y(t)), y(0) = $\boldsymbol{y}_0$ was introduced as a model for the dynamics of the growth of a population. An analytical solution exists for all values of the parameter r. The numerical solution is totally different for different values of Δt as we can see in the graphical representations of the Euler's numerical solution of the logistic equation with r = 18 and y(0)=1.3 .

In the first plot at the extreme left, the solution tends to 1 and looks like the analytical solution. In the second, third and fourth plot, the process becomes a periodic oscillation between two, four and eight levels. In the fourth plot, we did not join the points, in order that this period doubling will be clearer. In the fifth and sixth plot, the logistic mapping becomes chaotic. We slightly decrease Δt in the seventh plot. For the first 40 iterations, the logistic map appears chaotic. Then, period 3 appears. As we increase Δt very gradually we get, in the eight plot, period 6, and in the ninth plot period 12 so **the belief that gradual causes have gradual effects is false!**

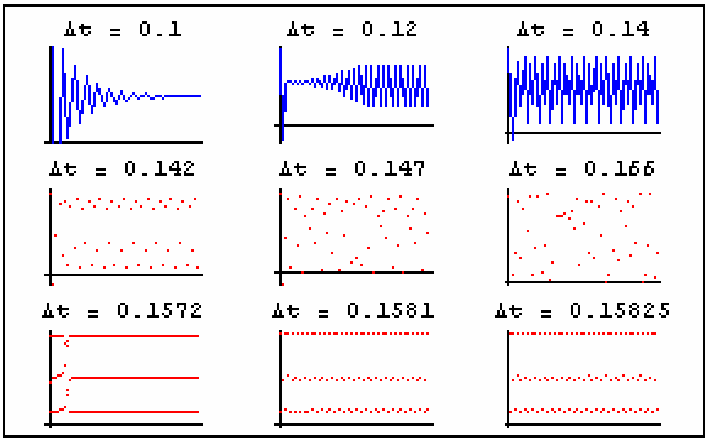

Figure 1 The numerical solution to the logistic equation

STUDENTS' CONCEPTIONS. METHODOLOGY, FINDINGS AND DISCUSSION

The students (N = 60) were given different questionnaires which were designed to elicit their thinking processes and intuitions concerning the limit in the definition of the derivative. Some of the students were also interviewed and invited to explain their answers. We present here some excerpts from the questionnaires and the interviews. The two first questions were given to the students before being exposed to the counterexample – the logistic equation. The reason for selecting these questions was to provide the background needed to better analyze students' answers to the third question in which the students were asked to characterize the source of error in Euler's method. By analyzing each student' answers to the different questionnaires, we examined the evolution of the student's thinking before and after being confronted with the counterexample. In the first question, the students were asked if a very small value for the step size Δt, for example - 0.02, will guarantee a good approximation to the solution. 88% of the students' answers expressed the claim that a small step size may not be small enough. Nevertheless, the students did not connect this claim to the definition of the derivative as a limit. Students used expressions like *"the smaller* Δt*, the better is the approximation"* and from their arguments we observe that the approximation is conceptualized as a potentially infinite process.

In the second question, the students were asked to express their opinion about the following statement:" If in Euler's method, using a step size $\Delta t = 0.017$ we get a solution very far from the real solution, then a step size $\Delta t = 0.016$ will not produce a big improvement, maybe some digits after the decimal point and no more". The substantial reason for introducing this test item was to observe how widespread is the students' belief that "gradual causes have gradual effects". This belief was expressed in 53% of the students' answers *"it seems to me that if with* Δt *= 0.017 we didn't get a good solution, then* Δt *= 0.016 will not produce a big improvement either"*.

Students' awareness of inconsistencies between their different intuitions

A student named Sarit reports her conflict between her potentially infinite process view and her intuition that gradual causes have gradual effects.

> In my opinion, if with $\Delta t = 0.017$ we didn't get a good solution, then $\Delta t = 0.016$ will not produce a big improvement even if it seems contradictory to what I wrote in the previous question that a smaller step size will improve the approximation. In my opinion, if a very small step size did not help and there is still a big difference then a smaller step size would not help. It looks like it depends on other factors than step size.

Sarit wrote her answer before being introduced to the counterexample. The counterexample will demonstrate that she was wrong. The counterexample will demonstrate unexpected results like the following one: the fact that <u>a little change in a parameter causes only a little effect</u>, does not necessarily imply that a little more change in the parameter will cause only a little more change in the effect. Once, a specific value of the parameter called the bifurcation value is passed, the effect could be a significant one! This is contrary to the belief that gradual causes have gradual effects.

Persistence of treasured intuitions

In the third question, the students were asked to characterize the source of error in Euler's method. We investigate whether the students realize that the source of error is the fact that in the numerical method the limit has been omitted in the definition of the derivative. Only a small percentage (19%) of the students wrote in their answers to the questionnaires that the source of the error resides in the fact that in the numerical method $\lim_{\Delta x \to 0} \Delta y / \Delta x$ is replaced by $\Delta y / \Delta x$ for Δx very small. 23% of the answers did relate the error to discrete - continuous considerations, but without a mention of the limit or of the formal definition of the derivative. The answers demonstrated well developed concept images of the derivative, well developed qualitative approach to differential equations, adequate to explain why there is an error, but inadequate to give a formal account how the discrete method employed the derivative concept. Analyzing the students' arguments, we noticed that for some students, only one representation of the derivative is active: the geometrical representation of the derivative as the slope of the tangent.

Even among the students who did succeed to give a formal account how the discrete method employed the derivative concept, we observed the persistence of treasured intuitions. This was demonstrated in Mira's answer:

> There is confusion in the way the concepts are used. The derivative in a point is defined for $\Delta t \to 0$ and in this case (the Euler's method) Δt does not tend to 0. Nevertheless, this answer is not a full answer. It is a wrong answer: Even if we attribute smaller values to Δt and we let it tend to 0, we still observe the crazy behaviour of the solution. This is strange!

In the interview, Mira expressed her potentially infinite process view of the limit and the way she views the limit as a monotonic process.

Reconstruction of the definition of the derivative as a limit

Next we follow the different phases of thinking of a student named Nurit while she reconstructs the definition of the derivative as a limit. In her answer to the first question, Nurit identified the limit as a process and that a small Δt may not be small enough. To the second question, Nurit's first reaction was:

> It seems to me that if with $\Delta t = 0.017$ we didn't get a good solution, then $\Delta t = 0.016$ will not produce a big improvement.

Then, she changed her mind:

> That was my first impression, but a second look at the expression for y_{k+1} in Euler's algorithm led me to the conclusion that the method is iterative, that is, on y_k we apply the algorithm in order to find y_{k+1} etc. etc. and after many iterations even a slightly smaller step size will produce a big improvement.

In this second stage, Nurit overcame the intuition that gradual causes have gradual effects by reflecting on the accumulating effect in the numerical solution. After being introduced to the logistic equation with the two different solutions, the analytical and the numerical, Nurit was asked to characterize the source of the error in Euler's method. At that point, she remembered an exercise on the sensitivity of some differential equations and realized that in the continuous approach too, small changes in a cause can produce large changes in its effect:

> We encountered this week an exercise that demonstrates that a change in the initial condition of a differential equation might cause a large change in the solution. When we choose an initial value $\mathbf{y(0) = 1 + \varepsilon}$ the solution curve tends to ∞ when $\mathbf{x \to \infty}$ but when the initial value was $\mathbf{y(0) = 1 - \varepsilon}$ then the solution curve tends to $-\infty$. All this happened due to the term $\mathbf{e^x}$ in the solution. Therefore in an expression like $\mathbf{c\,e^x}$ with $\mathbf{x \to \infty}$ it is very significant if c is positive or negative. Maybe, in our case, the small error made in the Euler's method induced big changes in the graph of the solution curve.

In this third stage, Nurit realized that small changes in a cause can produce large changes in its effect and this without the accumulating effect and not only in iterative processes. But at that stage, the reason for the small change in the cause in the case of the numerical solution to the logistic equation, was not clear to Nurit. Trying to identify the source of the error, Nurit's first reaction was that the error is due to the round - off effect and the fact that the error accumulates. Then she changed her mind:

> The source of the error in Euler's method is the way the derivative is defined $(y_{k+1} - y_k)/\Delta t = f(t_k, y_k)$ and by means of this definition we find y_{k+1} in Euler's algorithm. But we know that this definition is not precise. We have to add the condition that $\Delta t \to 0$ so we will know that we are not dealing with the secant to the graph of the function but with the slope of the tangent. Because of the numerical method Δt was chosen as a small number Δt =0.1; Δt = 0.12… but not small enough and in fact the derivative is defined for $\Delta t \to 0$.

Nurit reconstructed her knowledge about the definition of the derivative by means of interconnections with existing knowledge (the sensitivity of some differential equations) and intuitive ideas (gradual causes have gradual effects). Her process of construction led her to differentiate between the error due to mathematical meanings, namely, the fact that the limit was omitted in Euler's algorithm and the round off error. In her lengthy process of error analysis we distinguish different phases. In the later phases her attention is no longer distracted by the accumulating effect of the numerical method, nor by the round off effect. She is ready to seek 'the reason for a small change in a cause' in an error due to mathematical meanings. Now, Nurit is confident with her reconstruction of the limit concept, and is also resistant to challenges

> Now, it could be that there is also a round off error in the numerical method, but a round off error by itself could not have a so big influence on the graph of the solution, so that we will have a periodic oscillation between two levels instead of a solution that tends to 1. The error is due to the way the derivative is defined in the numerical method.

After the learning experience, Nurit confronted her figural model of the derivative, her concept image, with the formal definition. She prepared a figure of the secants tending to the limit curve, the tangent. In her geometrical representation the tangent was in red probably to differentiate it from the other elements of the sequence, the secants.

CONCLUDING REMARKS

The activities based on the specific discrete continuous interplay, helped some students to overcome treasured intuitions and to enrich their concept image with the refined intuition that gradual causes have no necessary gradual effects. This was possible for students who were able to link between different representations of the refined intuition and to make the interaction of the refined intuition with their existing knowledge. In this way, the students feel confident with the refined intuition. Nevertheless, the research study illustrates once more the never- ending struggle with the potential infinity (Tall and Tirosh, 2001). We also note that students are influenced by their figural models (for example, the geometrical representation of the concept of derivative) and we observe that students who have only one active representation of the concept, have difficulties with the formal definition. These students will need further help in their reconstruction of the formal definition.

The situation of conflict between the analytical and the numerical approaches motivated some students like Nurit to seek for the source of error, to understand the need for the formal definition, and to link their figural model of the concept with the formal definition. This last stage is an important stage towards their understanding that the formal definition is representation independent (Dreyfus, 1991).

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RECONCEIVING STRATEGIC KNOWLEDGE IN PROVING FROM THE STUDENT'S PERSPECTIVE

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Strategic knowledge in proof writing can be thought of as the heuristics which help a student to determine which strategies, theorems and definitions are helpful and when to use them while proving. Earlier Weber (2001) noted that students often lack the necessary strategic knowledge to prove statements; however, Weber's strategic knowledge does not take into account the goals of the prover. We propose a student-centered (or prover-centered) view of strategic knowledge, where the goals that the students are trying to achieve by proving are considered in the analysis of what constitutes strategic knowledge. We illustrate how viewing strategic knowledge in this way can allow us to interpret the proving behaviors of two students in an introductory real analysis course.

INTRODUCTION

Proving plays a pivotal role in many collegiate mathematics courses, but it is also an activity which causes undergraduates difficulty. There is a considerable amount of research on proof in collegiate mathematics education, much of it focusing on why students cannot construct proofs. Causes of difficulties with proof include, but are not limited to, students possessing different standards of justifications than those held by mathematicians (Harel & Sowder, 1998) and an inability to use, or a lack of appreciation for, definitions in formal mathematics (e.g., Moore, 1994; Vinner, 1991). A more comprehensive review of this literature can be found in recent articles by Selden and Selden (in press), Harel and Sowder (in press), and Weber (2003).

One specific reason that undergraduates cannot construct proofs is that they lack adequate proving strategies to do so (e.g., Schoenfeld, 1985; Weber, 2001). In describing this difficulty, Weber (2001) defines *strategic knowledge* as "heuristic guidelines that an individual can use to recall [mathematical] actions that are likely to be useful or to choose which action to apply among several alternatives" (p. 111). Weber includes knowing which theorems are important and under what conditions theorems are likely to be useful as strategic knowledge necessary for proving competence in undergraduate mathematics courses. His analysis of an expert-novice study on proof construction led him to conclude that mathematicians possessed strategic knowledge for proving that undergraduates lack. Weber suggests that this lack of strategic knowledge may be one reason that undergraduates have difficulty constructing proofs.

In Weber (2001), strategic knowledge is treated as a theoretical construct that an individual can use to achieve the goal of producing a formally correct proof. However, as many mathematics educators have noted, when writing a proof, mathematicians' goals typically extend beyond producing a formally correct argument, such as obtaining conviction and understanding (e.g., deVilliers, 1990; Hanna, 1990). It is both plausible and desirable for a student to have similar expectations. In this paper, we suggest that when analysing what strategic knowledge students are using in their proof construction, it is necessary to consider the goals that they are trying to achieve by proving. We will argue that such a viewpoint is desirable and consistent with constructivist perspectives of mathematics education. We will then illustrate how viewing strategic knowledge in this way can allow us to interpret the proving behaviors of two students in an introductory analysis course.

STRATEGIC KNOWLEDGE AND GOALS OF PROVING

In Weber's (2001) article, when students were engaged in the process of proving, it was assumed that they shared the same singular goal—obtaining a valid proof. However, deVilliers (1990) argues that their proving is an activity that mathematicians use to fulfil many different purposes, including providing conviction that a statement is true, explaining why a statement is true, communicating mathematical ideas to others, systematizing a theory, and providing an intellectual challenge. deVilliers and others (e.g., Hanna, 1990; Hersh, 1993) argue that in mathematics classrooms, proving should not be viewed solely as an exercise in which one produces a series of assertions that follow logical rules. Instead, proving should be used as a means to achieve a subset of the purposes listed above. In particular, proof should primarily be used as a pedagogical tool to understand why assertions are true (Hanna, 1990; Hersh, 1993).

Nunokawa (2005) asserts that what one learns from solving a mathematical problem depends upon the reasoning used to obtain that solution (see also Lithner, 2003). Similarly, the strategies that a student uses to construct a proof can influence the conviction and understanding that the proof provides for that student (Weber, 2005). For instance, Rodd (2000) argues that if students mechanically apply a proof technique that they do not fully understand, such as proof by induction, they will probably gain little by way of conviction or understanding from this experience. However, if students base their proofs on meaningful understanding of the concepts involved, they can gain a sense of why the assertion they are proving is true (c.f., Raman, 2003). Proving strategies that are effective for constructing proofs quickly and efficiently might not be appropriate for producing proofs that provide the prover with conviction and understanding. Likewise, some proving strategies that are optimum for maximizing learning might not be useful for producing proofs with a high degree of clarity or elegance. In summary, what proving strategies are effective depend upon the purpose of proving. Viewed this way, when delineating or assessing the strategic knowledge of a student, it is necessary to consider their goals of proving.

To illustrate the difference between viewing strategic knowledge solely as a means for efficiently producing correct proofs and conceiving of strategic knowledge as a means to achieve other goals, consider a student in a real analysis course who is completing a homework assignment on sequences. Suppose that when this student was asked to prove that the sequence {3 + 2/*n*} converged to 3, she began evaluating this sequence at small and larger values of *n* with the purposes of gaining insight into why this sequence converges to 3, searching for ideas on how this proof could be generated, and forming links between the proof that she produced and her understanding of sequences. If one only viewed this student's actions in terms of constructing a correct proof, it would seem that this student lacked an important piece of strategic knowledge. Weber (2001) observed that expert provers have "proving schemas"—i.e., systematic techniques or algorithms for proving classes of statements. When students do not apply these schemas in relevant situations, Weber (2001) interpreted this as a lack of strategic knowledge on the part of the student. Experts in real analysis have systematic techniques for proving sequences of the form {*a*+*b*/*n*} converge to *a*; for a mathematician, constructing such a proof is more an exercise than a problem. Students often learn to apply such an algorithm. In fact, some students can construct this type of proof with little understanding of sequences or the concept of convergence (e.g., Weber, 2005). Hence, this hypothetical student's failure to apply such an algorithm could be interpreted as a lack of strategic knowledge. However, when one considers this student's goals were to gain an understanding of sequences and to link her intuitive understanding of sequences, her actions are highly appropriate. Evaluating the sequence at small and large values of *n* would be an effective strategy for achieving her proving goals. Applying a proving algorithm that she did not fully understand would prevent her from achieving her goals. From the student-centered perspective, this student's decision *not* to apply such an algorithm would be an instance of an application of effective strategic knowledge. In the sections that follow, we will analyze two students' behaviors in real analysis, considering the relationship between their proving goals and proof strategies.

RESEARCH CONTEXT

The data for this paper is taken from a larger semester long workshop with students who were concurrently enrolled in Advanced Calculus at a large southwestern American state university. Students participated in weekly group proving activities, wrote reflective e-mails about their proof writing and discussed course topics on an internet discussion board. Each one hour class period consisted of students working in groups of 3-5 to solve tasks based in the Advanced Calculus curriculum topics. The teacher engaged the students throughout the class period by asking them to explain or justify their work.

Four of the nine students in the workshop were also interviewed throughout the semester in three hour long task based interviews. These case study students were upper level mathematics majors pursuing an undergraduate degree in mathematics or mathematics education at the secondary level. All four students worked hard in their courses and received an A or B in Advanced Calculus.

Initially the data was analysed for indications of the students' appropriation of proof writing strategies and strategic knowledge. Through the analysis it became clear that the students' focus on their own strategic knowledge was not necessarily consistent with the notion previously described by Weber (2001). Thus the data was recoded for students' proving goals and their strategies in an attempt to understand their view of the strategic knowledge which they view as important for proving.

RESULTS

For brevity the results here are presented for two students: Doug and Ben. For each student we report first their proving goals as described in reflective e-mails, workshop discussions and interviews. It should be noted that some students described various goals when asked to give attributes of a "good proof." In this case we note those goals which were emphasized in multiple settings. We then report on the proving strategies predominately used by each student to construct proofs. When appropriate, we note connections between their proving goals and proof strategies.

Doug

Doug's self-described goal of proving is to "derive a cohesive set of statements that resembles an argument." In reflective e-mails about proof, he often comments on the importance of a proof having a logical progression. He writes, "A good proof should have a logical transition. Each theorem used should be stated and expanded so that the reader has an idea of where you are going with the proof" (E-mail #12). While he is aware that proofs should have other attributes, in his proving activities he is "too lazy" to be concerned with these details as he explains below.

> Doug: Many mathematicians will argue that elegance is a key attribute of a good proof that the argument should flow to a logical conclusion. I tend to disagree with this statement, finding that the most elegant of proofs are often difficult to read for a 'lay' person or even someone with a fair amount of experience in the realm of proof theory. I find that a good proof will explain each transformation and give in between each step, the reasoning behind this, and if necessary an example of what is being done upon in terms of another problem. However, I'm much too lazy to do all of that personally (E-mail #5).

As Doug indicates he also thinks a proof should make sense. But his goal in proof writing, described in E-mail #12, is to get a proof which is "'good enough' for our purposes." Hence, although Doug appears to be aware that proofs can (and perhaps should) be used to provide convincing explanations for why assertions are true, his goal in this real analysis class is to efficiently produce proofs that are "good enough".

When Doug writes proofs, he would frequently look in textbooks that would help him with the proof construction. In the first interview, Doug was struggling to prove that all rational numbers have a terminating or repeating decimal expansion. He comments on the usefulness of a textbook he found in the library.

> Doug: This one is easier I would go to Edmond Landow's book it's foundations of analysis. He does an axiomatic approach towards the real number line and the rational numbers and he has a crap load of theorems and they are all laid out there nice and neat and they all refer back to earlier theorems he gives from the natural numbers with Peano's axioms and then he goes into the rationals. He goes into cuts he goes into irrationals and complex numbers so I am sure actually one of his

definitions with the rational numbers would actually link into the repeating and the terminating and I can link over to the rationals.

In the second interview Doug attempts to determine if all Cauchy sequences converge. Again, he does so by finding the information in his textbook.

Doug: I'm pretty sure they do. I thought they do. [looks in book]

Int: So how do you know where to look in the book?

Doug: yeah every convergent sequence is a Cauchy sequence and every Cauchy sequence is bounded, but I am pretty sure that every Cauchy sequence is convergent as well. Wait theorem 1.3 gives a necessary condition for convergence if -- How do I know where to look in the book?

Int: uh hmm.

Doug: remember. Plus like the chapters are nicely divided into convergence, limits, continuity and then integration. Ok if a sequence is convergent, convergent it must be Cauchy. If a sequence is not Cauchy then it is not convergent [reading from book] … Ok yeah every Cauchy sequence is convergent.

In each interview, Doug demonstrates that he relies heavily on those external resources to which he has access. Further, Doug has effective strategies for quickly locating the information that he desires, including choosing which textbook to use and knowing which chapter to turn to. These strategies allow Doug to produce proofs quickly and (usually) correctly. It should be noted that such strategies probably would not allow Doug to prove novel sophisticated statements nor do they seem effective for helping Doug develop conceptual understanding. Nonetheless, these strategies are appropriate for helping Doug achieve his goals of proving.

Ben

Unlike Doug, Ben's goal is to produce proofs which communicate; this might include producing proofs which are "smooth" or "elegant" as determined by his audience. Ben is focused on the structure and language at the surface of the proof. In a reflective e-mail he writes,

Ben: … a good proof is as simple as possible with the readability of anticipated audience. I do believe there is extra credit for creativity within reasonable excursions. If someone can give a unique look at what is happening then all who read the proof will be better for the diversified look it provides. I feel that a constructive argument is better than a contra-positive argument which is still better than a destructive argument just because definition is so much greater emphasized (E-mail #5).

During a class session the students were comparing two proofs written by the groups. The proofs differ in structure. One group used two cases to prove the limit existed, while Ben's group noticed the same delta was defined in both cases so they condensed the proof by cases to a more general direct proof. During the discussion Ben extolled the virtue of a short proof.

Dustin: Right. It depends on who's going to read it and why you're writing it.

Ben: I mean, I think this is nice [points to the two case proof], but this is a two liner [points to the other proof].

Dustin: It depends on what you're going to use it for.

Ben: You can't disagree that a two lined proof is a nice thing.

Ben further explained to the class the reason he preferred the "two lined proof."

Ben: If you have case one and case two, the only thing I don't like about having different cases, is that it's a lot more work. And so if you can do it in 1 case, it should be seen in one case and done in one case. But it is a lot more explicit. It is a lot prettier. It looks like it has a lot less holes in it.

Thus achieving a particular form and structure in the proof is goal for Ben. While Ben's proof writing strategies are varied: he often lists what facts he knows, and then writes where he needs to go. Ben is more concerned about *how* to write it. For example in the first interview when working on a proof that a function is bounded at a particular point he comments, "that sounds right, but I don't think that it diminishes maybe the strength of the proof but not the purpose of it." His concern is the wording of the particular mathematical phrase "about *a* except possibly at *a*." In his second interview, Ben is working on a proof that a uniformly continuous function maps Cauchy sequences to Cauchy sequences. The following conversation ensues:

Ben: A uniformly continuous function sends Cauchy sequence to Cauchy sequence so I can rephrase that to if from A to B is uniformly continuous then ahh every Cauchy sequence in A is – this isn't clear [erases] how about if $\{a_n\}$ a subset of A is Cauchy then $\{f(a_n)\}$ a subset of B is Cauchy.

I: ok I am going to stop you right here for a second and ask why did you choose to rewrite it and how come you rewrote it in that way?

Ben: I think it makes more sense. The idea of Cauchy means that sequences converge closer and closer to each other. If I think of $\{f(a_n)\}$ as a Cauchy sequence it's a little clearer than saying a Cauchy sequence sends a Cauchy sequence to a Cauchy sequence. I have $\{a_n\}$ I can show that a_n approaches each other far enough down the line and then if f is uniformly continuous I will use that somehow to show that there exists an N such that $f(a_n)$ get closer and closer to each other.

Ben's strategy of writing a sentence then erasing and rewording or rewriting the sentence several times before accepting the statement or phrase became an issue during the workshop sessions. Ben would erase statements before his group members could read or evaluate them. In one session the group threatened to take the eraser away from Ben. In his defense, Ben explained he was just wanting to write it clearer.

Since success in proof writing for Ben is based on the reader appreciating the form of the proof, his strategy of rephrasing and rewriting statements is a sensible way to achieve his goal. His concern to accurately word mathematical phrases and to write things in a nicer or "prettier" fashion is consistent with his proving goals. His strategy also reflects an emphasis from his professor on writing up "elegant" proofs.

DISCUSSION AND IMPLICATIONS

In Weber (2001), proving and strategic knowledge were viewed through an information-processing perspective. Such a perspective assumes that there exists a problem space with a set of initial conditions (assumptions, definitions), problem states (statements that can be deduced from the initial assumptions), operations that can be used to move from one problem state to another (the application of theorems), and a specific goal state (the statement to be proven). Further, it is assumed that this problem space is objective and independent of who is engaged in the task of proving. However, as Thompson (1982) notes, constructivists argue that there is no Platonic representation of a mathematical problem and, when analysing students' behavior, one must consider the ways in which students represent the problem. In this paper, we argue that with proving, what constitutes a "goal state" cannot be defined independently from the student constructing the proof. In many cases, students' goals are not simply to arrive at a correct answer, nor should they be. Thus we suggest reconceiving strategic knowledge for proving as strategies, heuristics, and techniques that can be used to attain students' goals of proving, and not the Platonic goal of getting a proof by any means. This interpretation of strategic knowledge sheds further insight into students' behavior.

While Doug's goal is to produce a proof, the knowledge he values (knowing the content of different textbooks) would not be considered under Weber's earlier interpretation as strategic knowledge; yet in this case his knowledge is an important heuristic which helps him to produce a "good enough" proof and hence allows him achieve his goal. This also explains the many hours Doug reports spending in the library "researching" for his course. Likewise, including Ben's refinement strategy as strategic knowledge also broadens the definition. Although Ben's erasing and rewording is sometimes premature, he continues to develop this skill since it helps him produce proofs which meet his qualifications.

Doug and Ben's goals and strategies are influenced by their advanced calculus classroom experiences. This leads to a discussion of the pedagogical implication of our work. As strategic knowledge is a necessary component of proving competence, this is knowledge that students should develop in their proof-oriented mathematics classrooms. Other researchers have noted that when students struggle with proofs, their teachers often provide them with heuristics or algorithms that can allow students to construct proofs, but not understand the proofs they are constructing (e.g., Fukawa-Connelly, 2005). The analysis in this paper implies that when deciding what proving strategies should be taught to students, one must consider what the students should gain from proving. Often, strategies students can use to display ostensible proving competence may not be useful for achieving more valuable pedagogical goals.

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PATTERNS OF MIDDLE SCHOOL STUDENTS' HEURISTIC BEHAVIORS IN SOLVING SEEMINGLY FAMILIAR PROBLEMS

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We report repeated clinical interviews with 12 Israeli middle school students. The 8th graders of different aptitudes were instructed to think aloud while solving word problems and geometry problems that looked like familiar tasks, but, in fact, were not. Based on principles of constant comparison method, four patterns of the students' heuristic behaviors—naïve, progressive, circular and spiral—are distinguished. We also found that the interviewees demonstrated multiple heuristic behaviors both in algebra and geometry contexts and that the weight of naïve heuristic behavior decreased from the first to the last interview.

INTRODUCTION

The goal of this paper is to present patterns of problem solving behaviors that emerged from three rounds of clinical interviews with middle school students of different mathematical aptitudes. In the interviews the students were given algebra and geometry problems that looked like tasks recently solved in class, but, in fact, were much more challenging. We call such tasks *seemingly familiar problems*.

Building on past research, we argue that *heuristics* are useful organizational units in modelling problem solving and present an empirical definition of *heuristic behavior*. That multi-attribute definition enables us to make inferences regarding similarities in some solutions, and, in turn, to distinguish four patterns of the students' heuristic behaviors that cut across algebra and geometry problem solving domains.

THEORETICAL BACKGROUND AND CONCEPTUAL FRAMEWORK

The mathematics education community has expressed keen interest in cognitive mechanisms of problem solving for more than 45 years, yet many aspects of problem solving still do not appear to be understood (Cai, Mamona-Downs & Weber, 2005; Lester, 1994). Early research produced several progressive multi-phase models of mathematical problem solving. Polya's famous (1945/1973) four-step procedure "understand—plan—execute—look backward" or Mason, Burton and Stacey's (1982) formula "entry—attack—review" represent such linear models.

More resent research tends to appreciate the complexity of problem solving, which manifests itself in overlapping the problem-solving phases in many ways. For example, Verschaffel (1999) suggested a five-phase model "understand—construct a model—rearrange the model—evaluate—communicate", and noted that the phases have to be considered as cyclic, rather than as a linear progression from a given state to a goal state.

Lately, Carlson and Bloom (2005) suggested a multidimensional framework that emerged from clinical interviews with professional mathematicians. Building on past research, they considered four phases in problem solving: orientation, planning, executing and checking. Embedded in the framework are two cycles, each of which includes at least three of the four phases. The model also includes a sub-cycle "conjecture—test—evaluate" and operates with various problem-solving attributes, such as conceptual knowledge, heuristics, metacognition, control and affect. The reported interplay of the problem-solving attributes is fairly sophisticated and even overwhelming as some of the attributes are not (or, perhaps, cannot be) operationally separated. For instance, the roles of heuristics, metacognition and control seem to overlap in the framework. That could be expected: Goldin (1998, p. 153) described heuristics as "the most useful organizational units and culminating constructs" in a representational system of planning, monitoring and executive control.

In our study, the broadly defined concept of *heuristics* is a central component in modelling problem-solving behaviors. Consolidating many definitions (in particular, by Goldin, 1998; Schoenfeld, 1985 and Verschaffel, 1999), we refer to *heuristics* as a systematic approach to representation, analysis and transformation of mathematical problems that solvers of those problems use in planning and monitoring their solutions. Some heuristics are narrow and domain-specific, whereas others are universal and cut across many problem-solving domains. In actual problem solving, a particular heuristics can come as an enduring or as a transient way of thinking. It can govern a relatively extensive and structured attempt to solve a problem or trigger just a short-lived problem-solving step. We refer to the former ways of thinking as *macroheuristics* and to the latter ones as *microheuristics*.

Heuristics used in our study are chosen on the basis of the problem solving literature (e.g., Schoenfeld, 1985; Larson, 1983) and have been tried out in two preliminary studies on experts' strategic behaviors (Koichu, Berman & Moore, 2003a; 2003b). Ten heuristics (or 21, including sub-categories) are considered:

(1) Planning, including (1a) Thinking forward, (1b) Thinking from the end to the beginning and (1c) Arguing by contradiction. (2) Self-evaluation, including (2a) Local self-evaluation and (2b) Thinking backward. (3) Activating a previous experience, including (3a) Recalling related problems and (3b) Recalling related theorems. (4) Selecting problem representation, including (4a) Denoting and labelling and (4b) Drawing a picture. (5) Exploring particular cases, including (5a) Examining extreme or boundary values and (5b) Partial induction. (6) Introducing an auxiliary element. (7) Exploring a particular datum. (8) Finding what is easy to find. (9) Exploration of symmetry. (10) Generalization.

These heuristics are described in detail elsewhere (Koichu et al., 2003a; 2003b). Building on the above list of heuristics, we operationally define *heuristic behavior* in solving mathematical problems while thinking out loud: It is a problem-solving behavior, which, from the observer's viewpoint, is characterized by 5 attributes:

Attribute 1: Number of different heuristics used in the solution of a problem.

Attribute 2: Heuristics used at the beginning of a solution.

Attribute 3: Intention to continue solution in awkward situations with or without asking for assistance.

Attribute 4: When there is more than one attempt to solve a problem, whether or not there is a tendency to use new macroheuristic(s) that have not been used in the previous attempt(s).

Attribute 5: Typical combination(s) of microheuristics used in succession.

Having the conceptual framework outlined, we now formulate the main research question, as follow: What are the *heuristic behaviors* of middle school students in solving seemingly familiar algebra and geometry problems? By answering this question, we seek to validate (or modify) the existing models of mathematical problem solving when applied to middle school students of different aptitudes.

THE STUDY

The study reported here is a part of a larger research project, in which two 8th grade classes from two urban Israeli schools took part in a 5-month experiment, aimed at developing the students' heuristic literacy (Koichu, 2003; Koichu, Berman & Moore, 2004). Since 7th grade the participants in that study took the accelerated mathematics curriculum MOFET, in which mathematics is taught eight hours a week, 5 hours of algebra and 3 hours of geometry (Schneiderman et. al, 2003).

Interviewees

Twelve students, 7 girls and 5 boys, took part in interviews conducted at the beginning, in the middle and at the end of the classroom intervention. Before the intervention, the students were tested by means of Scholastic Aptitude Test – Mathematics (SATM) and Raven's Progressive Matrix Test (RPMT). In terms of these two standardized instruments, the interviewees were of very different aptitudes. The SATM scores were between 5 and 29 (out of possible 35); the RPMT scores were between 20 and 29 (out of possible 30). The average of SATM scores was 13.58 (SD=6.09); of RPMT 24.42 (SD=2.57).

Interview procedure

The length of the interview was 30 to 90 minutes, depending on the persistence of the interviewees in solving the interview problems. The interviewing method was adapted from Erickson and Simon (1993) and Clement (2000). In short, the students were instructed to think out loud while solving the given problems. If the interviewee remained silent for more than 15 seconds, the interviewer (the first author) prompted him or her in a neutral manner (e.g., "keep talking", "don't be silent").

Interview problems

In each interview the students were given two *seemingly familiar problems*: a word problem concerning whole numbers (Problems 1N, 2N and 3N in Table 1) and a geometry open-ended problem concerning quadrilaterals (Problems 1Q, 2Q and 3Q).

1st interview	2nd interview	3rd interview
Problem 1N: The sum of the digits of a two-digit number is 14. If you add 46 to this number the product of digits of the new number will be 6. Find the two-digit number.	Problem 2N: Represent the number 19 as a difference of the cubes of two positive integers. Find all possible solutions.	Problem 3N: The first digit of a three-digit number is 1. If you carry the digit 1 to the end of the number, you will get a new number. It is given that the difference of the new and the original numbers is divisible by 11. A. Find the original number. B. Find all the possible original numbers.
Problem 1Q: Check the following statement: If a quadrilateral has two congruent opposite sides and two congruent opposite angles then it is a parallelogram. If you think that the statement is correct, prove it. Otherwise, disprove it by counterexample or by any other method.	Problem 2Q: Check the following statement: If a quadrilateral has two right angles and two congruent diagonals, then it is a rectangle. If you think that the statement is correct, prove it. Otherwise, disprove it by counterexample or by any other method.	Problem 3Q: Given a quadrilateral ABCD. Point E bisects AB, point F bisects CD, $EF = \frac{BC + AD}{2}$. What can you say about the quadrilateral ABCD? Formulate your conjecture and prove it.

Table 1: Interview problems

At first glance, Problems 1N and 3N naturally involve a classical solution by means of equations, and the students learned such solutions in the classroom. However, composing equations appears ineffective at a second glance. At first glance, Problem 2N may be solved by trial and error. Indeed, it is possible to find one pair of positive integers that fit the problem. In order to find all possible solutions, one can use equations. Problems 1Q, 2Q and 3Q also looked like problems that had been recently discussed in the classroom, but they were not. For example, a full solution of Problem 3Q presumes discovery and proof of a converse of the theorem of median of a trapezoid. The problem was given in the interview a week after discussing this theorem in the classroom. It was expected that the interviewees could try to adapt the learned proof of the direct problem to its converse, but, to our knowledge, this approach was hardly helpful. Problem 3Q is better handled by means of auxiliary constructions and arguing by contradiction.

Analysis

The data analysed consisted of 72 ($12\times2\times3$) videotaped problem-solving episodes. The protocols were segmented into content units and coded by the first author. Content units were determined as the largest unbroken parts of the transcripts that bear a particular heuristic interpretation. Commonly, content units consisted of several words up to several sentences, which represented a particular problem solving step. The following 30-second excerpt from the 3rd interview with Alon (Problem 3Q) illustrates the coding. The categories are presented by numbers that correspond to the numbers in the above list of 10 heuristics.

> Alon: So, how can we prove that? Assume **(1)**… No **(2a)**. Auxiliary construction…**(1+6)**. Let's try DE and EC, DE and EC **(1+7)**. What we have now is four triangles **(8)**… Just a second, no, I don't need the auxiliary construction; I won't use it **(2)**! Let's build **(1+6)**…No **(1a)**…how can we prove that the sides are parallel? **(3+1)”**

An extended coding procedure was applied to a sample protocol. It was independently analysed by two coders: the first author and a trained mathematics educator, who was not involved in the rest of the study. An agreement rate of 84% was found; all cases of disagreement were resolved in discussion. Next, based on the coding, each problem-solving episode was characterized in terms of the 5 attributes listed in the above definition of heuristic behavior. Then solutions to the same problems by different students were compared using the following criteria:

> Two solutions of the same problem with numbers H_1 and H_2 of the different heuristics are *similar with respect to Attribute 1* if $|H_1 - H_2| \le 2$. Two heuristic behaviors are *essentially similar* if similarity is indicated on 4 or 5 attributes and *fairly similar* if a similarity is indicated on 2 or 3 attributes.

Note that by these criteria two mathematically different solutions can be found heuristically similar. On the other hand, mathematically similar solutions in most cases are heuristically similar but not necessarily *essentially similar*.

For each interview problem, the above criteria of similarity were applied to all pairs of the students' solutions. Based on the essential similarities found (for more details, see Koichu, 2003), the 72 problem-solving episodes were reduced to 12 prototypical ones, from which four different patterns of heuristic behaviors were generatively extracted.

RESULTS

The following brief descriptions of the inferred heuristic behaviors, to which we refer as *naïve*, *progressive*, *circular* and *spiral*, along with the information about frequencies of their appearance in the interviews is a précis of an answer to the research question.

<u>Naïve heuristic behavior</u>: There are 1-2 attempts to solve a problem with 1-2 macroheuristics. Typically, "Activating a previous experience" and “Exploring particular cases” are in use. There is a tendency to non-critically rely on available experiences. At the end, the student hopes that she or he has solved the problem, but is uncertain and asks for evaluation. Assistance is requested also at the “orientation” phase. “Planning” can hardly be indicated and “checking” is not done without prompting. At the level of microheuristics, separate acts of "Thinking forward" and "Local self-evaluation" are observed among many non-heuristic problem solving steps.

Progressive heuristic behavior: There are 1-2 attempts to solve a problem with 1-3 macroheuristics. Typically, "Activating a previous experience", "Planning", "Selecting a problem representation" or "Exploration of a particular datum" can be observed. At the end, the student adequately evaluates his or her success or failure. The problem-solving phases “orientation”, “planning” and “executing” do not overlap. The phase “checking” is rarely indicated. At the level of microheuristics, there are occasional cycles “thinking forward—non-heuristic step—local self-evaluation”.

Circular heuristic behavior: There are 3-10 attempts to solve a problem with 1-3 macroheuristics. The attempts become shorter and closer towards the end of a solution. Typically, the first attempt starts from "Activating a previous experience," then "Selecting a problem representation" along with "Finding what is easy to find" are in use. "Thinking backward" is the main strategy toward the end of a solution. At the end, the student is sure that she or he cannot solve a problem. The phases “orientation”, “planning”, and “checking” overlap; “executing” is missing in some of the attempts. At the level of microheuristics, there are systematically observed cycles “thinking forward—non-heuristic step—local self-evaluation”.

Spiral heuristic behavior: There are 3-10 attempts to solve a problem with 4-8 macroheuristics. New attempts continue and advance the previous ones. Typically, the first attempt starts from "Activating a previous experience,” then "Selecting a problem representation" along with "Planning" and "Finding what is easy to find" are in use. "Arguing by contradiction" appears towards the end of a solution, if relevant. At the end, the student adequately evaluates his or her progress. The phases “orientation”, “planning”, and “checking” overlap. At the level of microheuristics, there are systematically observed cycles “planning—other heuristics—local self-evaluation”. These (local) cycles can include many different intermediate heuristics.

Table 2 contains frequencies of the heuristic behaviors’ appearance in algebra and geometry contexts.

Heuristic behavior	Algebra context	Geometry context
Naïve	8 (22%)	14(39%)
Progressive	16 (44%)	4(11%)
Circular	4 (11%)	7(19%)
Spiral	8(22%)	11(31%)

Table 2: Frequencies of the heuristic behaviors’ appearance

In algebra, the most frequently indicated heuristic behavior was *progressive*, while in the geometry context it was the *naïve* one. Ten students demonstrated more than one type of heuristic behaviors either in algebra or geometry contexts. The heuristic behaviors of the stronger (with respect to SATM scores) students were more stable than those of the weaker ones. The weight of the naïve heuristic behavior decreased from the first to the third interview: from 67% to 25% in the algebra context, and from 25% to 8% in the geometry context. Conversely, the weight of spiral and progressive heuristic behaviors increased. We suggest that this change is in part due to promoting heuristic literacy during the intervention (Koichu et al., 2004).

DISCUSSION AND CONCLUSIONS

In our opinion the important contribution of the reported study is the exposure of the central role of heuristics as useful organizational units in modelling problem solving. Although past studies have identified many heuristics used mainly by expert problem solvers (e.g., Carlson and Bloom, 2005; Larson, 1983; Schoenfeld, 1985), the heuristics are rarely seen as entities of one's (finite) set of available problem-solving resources that govern the entire solution path. Consistently with Goldin (1998), we argue that such a view on heuristics is a fruitful way to discover some hidden problem-solving phenomena.

Based on principles of the constant comparison method (Glaser & Strauss, 1967), four across-domain patterns of heuristic behaviors were found. Noticeably, naïve and progressive heuristic behaviors complement the early linear models of problem solving (e.g., Polya, 1945/1973; Mason et al., 1982), whereas spiral and circular heuristic behaviors are closer to the more recent models that stress the cyclic nature of problem solving (Verschaffel, 1999; Carlson and Bloom, 2005). At this point, let us note that the interview problems used —so called *seemingly familiar problems*— seem to have a great potential both as research tools and as instructional materials.

Students of different mathematical aptitudes were repeatedly interviewed in our study, and we found that most problem solvers demonstrated multiple heuristic behaviors. This finding consolidates the early and the recent models of problem solving as follows: linear models are applicable not only for novices, and cyclic models are not only for experts. Apparently, one's heuristic behavior is a function of many variables, either internal or external to a problem solver. Also, in the interviews naïve and circular heuristic behaviors were more frequently demonstrated by weaker (with respect to mathematical aptitude) problem solvers, but some of the students tended to enrich their heuristic arsenals and to advance to more effective problem-solving behaviors. This observation may also have important pedagogical implementations: well-organized problem solving experiences may enrich students' heuristic behaviors. This suggestion is in line with current pedagogical experimentation (e.g., Harel, in press) and, hopefully, will be empirically tested in the near future.

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IMPROPER PROPORTIONAL REASONING: A COMPARATIVE STUDY IN HIGH SCHOOL

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In this paper, we compare the appearance of the illusion of linearity among 2nd and 3rd year junior high school students, as well as the impact of the geometrical shape (square, rectangle, circle and irregular figure) on the solution of non-linear problems with reference to different solution strategies. The results revealed that while students' performance at the proportional tasks improved from one grade to the other, their performance at the non-proportional tasks remained static. This is indicative of the persistent character of the linearity misconception, which it even resists the instructions of teaching.

INTRODUCTION

Proportional relations are an important subject which is present in all stages of mathematics' education. The attention given to proportional relations is to a large extent owed to the fact that they are the basic model for solving a great variety of mathematical and scientific problems (De Bock, Verschaffel, & Janssens, 1998). Moreover, the linear model constitutes an easy and sufficient tool for handling many real life situations (Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2003).

However, the great attention given to the linear model in our everyday life in general, and more specifically in school mathematics, can create the illusion that it can be applied everywhere (Gagatsis & Kyriakides, 2000). Various researchers studying the wide application of proportional reasoning in different situations (De Bock, Van Dooren, Janssens, & Verschaffel, 2002; Modestou & Gagatsis, 2004a) indicate a strong tendency of applying the proportional model in non-proportional situations, even among high school students. The results of these studies show that this strong tendency towards the application of the proportional model everywhere, constitutes a phenomenon which resists every attempt of change and affects many students within a wide age group and on different mathematical occasions (De Bock et al., 2002).

A well known example of improper use of proportional reasoning appears in the field of geometry, concerning relations between the side's length and the reduced or enlarged figure's area or volume. Recently, the misconception that the area and volume of a geometrical figure is enlarged x times when the dimensions are enlarged x times, has been studied to a large extent by various researchers (De Bock et al., 1998; De Bock et al., 2002; De Bock, Verschaffel, & Janssens, 2002; De Bock et al., 2003; Modestou & Gagatsis, 2004a; Modestou & Gagatsis, Pitta-Pantazi, 2004; Modestou & Gagatsis, 2004b; Van Dooren et al., 2003).

Compendiously these studies found that the majority of students (even 16 year old students) fail in non-proportional problems because of their powerful tendency to apply proportional reasoning everywhere. Even with the benefit of important help, as for example the supply of visual representations, metacognitive stimuli or problems' authentication, only certain students were led in solving non-proportional problems. In some cases, when students, because of the auxiliary intervention, discovered that some of the problems were not-proportional, they began to apply non-proportional methods even in proportional problems (De Bock et al., 2002; Van Dooren et al., 2003).

It is obvious that numerous studies have been made for overcoming the vast and erroneous application of linearity at non-proportional problems. With this study, however we do not attempt to reduce the effect of the linearity misconception. The character of our study is comparative with references to four elements that may or not affect students' performance at proportional (perimeter) and non-proportional tasks (area). Therefore, purpose of the present study is to compare (1) the appearance of the illusion of linearity among 2nd and 3rd year junior high school students, (2) the impact of the geometrical shape (square, rectangle, circle and irregular figure) on the solution of the non-linear problems, (3) the use of self-made representations for supporting the solution of the tasks and (4) the appearance of different strategies at the solution of the non-proportional tasks.

METHOD

The analysis was based on data collected from 268 students attending 2nd and 3rd year of junior high school, in different schools of Cyprus. Specifically, the sample of the study consisted of 134 13-year old students (2nd year); 67 boys and 71 girls and 134 14-year old students (3rd year); 58 boys and 76 girls.

A written test was used to collect the data, which was given to all 268 students. The structure of the test is presented in Figure 1. The test consisted of 8 different geometrical problems, four proportional (perimeter) and four non-proportional (area) tasks. In each category there was a problem that referred to a different type of figure (square, rectangle, circle and irregular figure). The problems involved enlargements of different types of figures, with indirect measures for perimeter and area. The following problem (pr.1 of the test), is an example of the way the test's problems were formulated: "*Mr. Marios needs 4 days to dig a ditch around a square field with a side of 100m. How many days does he need to dig a ditch around a square field with threefold dimensions?*". It is obvious that the above problem is indirect as no reference to the word perimeter is made. Therefore, students have to proportionally enlarge the square (4x100) in order to find the answer.

No special instructions were given. All possibly relevant elements that could affect students' performance, such as the degree of familiarity with the problem's context, the grammatical complexity and the nature of the given numbers, were controlled as much as possible. Therefore, only one digit natural numbers were used as scale factors (twofold or threefold), so that all required computations had the same degree of difficulty. Finally, the response sheets could be used for computations, drawings and other comments concerning the solution of the problems.

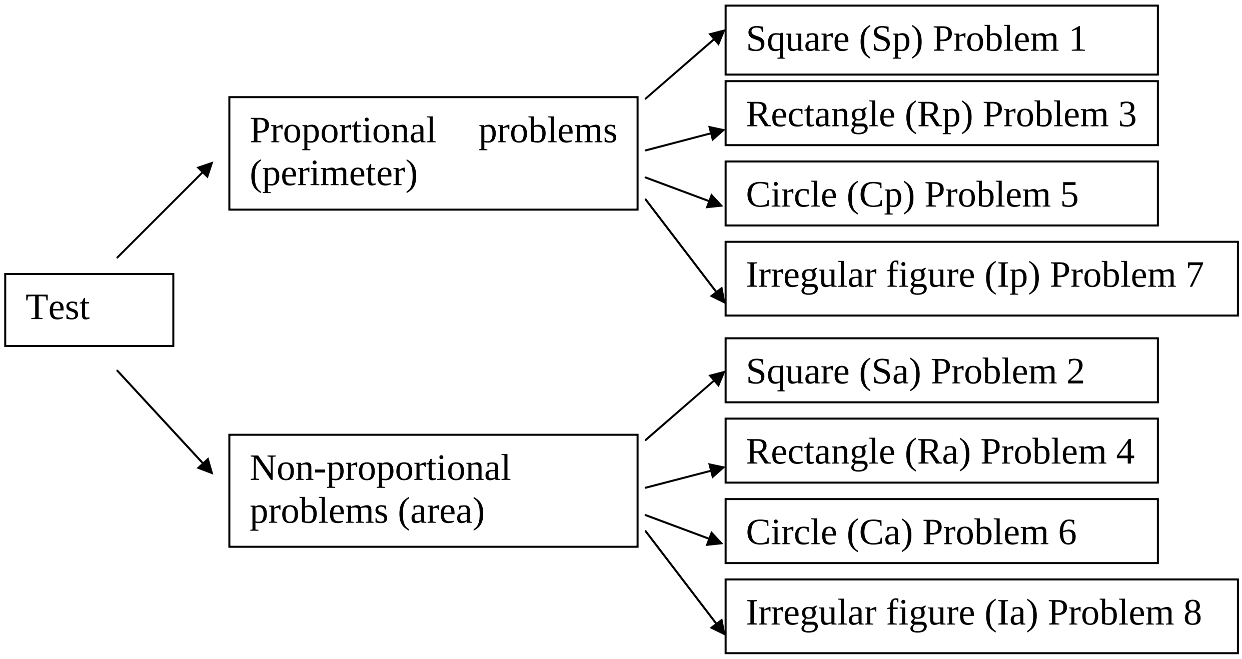

Figure 1: The test structure

The results concerning students' responses to the above problems were codified in four ways: (a) *The type of the figure*: Square (**S**), Rectangle (**R**), Circle (**C**) or Irregular figure (**I**), (b) *The type of the task*: area (**a)** or perimeter (**p**), (c) *The use of a representation* (**r**) and (d) *The type of students' solution strategies* in non-proportional problems (**1**), (**2**), (**3**) or (**4**). These numbers correspond according to De Bock et al., (1998) to: (**1**) Direct application of the linear model (x2, x3, rule of three), (**2**) Finding the area of a plane figure by paving it with small, similar figures (paving), (**3**) Finding and applying the appropriate mathematical formula for the area (formula) and (**4**) Applying the general rule "length x *r*, thus length x r^2 " (x4, x9).

For instance, the variable ***Ca*** refers to the finding of the **c**ircle's **a**rea; the variable **Spr** indicates the use of a **r**epresentation while finding the **p**erimeter of a **s**quare, whereas the variable **Ra1** refers to the use of the linear model (strategy **1**) in order to find the **r**ectangle's **a**rea. All variables were codified as 0 and 1. Therefore, the correct solution was assigned the score of 1 and each wrong solution the 0. In a similar way, the use of a representation or a particular strategy was codified as 1 and the non use as 0.

For the analysis and processing of the data collected, the statistical package of SPSS (t-test for independent groups) was used as well as an implicative statistical analysis using the computer software CHIC (Bodin, Coutourier, & Gras, 2000). From the statistical package CHIC only the similarity diagram was used. A similarity diagram represents groups of variables which are based on the similarity of students' responses at the test's tasks.

RESULTS

Table 1 presents an overview of students' performance in proportional and non-proportional problems as it is shown in both grades. As expected students' performance in proportional problems was higher, with the irregular figure causing the most difficulties for both 2nd ($\overline{X}$=0,60) and 3rd ($\overline{X}$=0,73) graders. The circle ($\overline{Xb}$=0,81) and square problems ($\overline{Xc}$=0,90), were the easiest for both grades. Concerning the non-proportional items, both grades students' performance was very low, with the circle problem being the most difficult ($\overline{Xb}$=0,01 and $\overline{Xc}$=0,01), and the rectangle ($\overline{Xb}$=0,13) and square problems ($\overline{Xc}$=0,07) appearing as the easiest for both grades.

A more detailed comparison of students' performance at both grades, reveals that while a statistically significant difference exists amongst students' performance in proportional tasks (t=-3,32; p<0,01), with 3rd grade students performing better ($\overline{X}$=0,84) than 2nd graders ($\overline{X}$=0,73), there is not such difference for non-proportional problems. The results lead to the conclusion that students' performance at the non-proportional tasks is independent from their grade.

Grade			Square (S)	Rectangle (R)	Circle (C)	Irregular Figure (I)	Total
2nd	Proportional	Perimeter (p)	0,79	0,70	0,81	0,60	*0,73*
	Non-proportional	Area (a)	0,07	0,13	0,02	0,01	0,06
3rd	Proportional	Perimeter (p)	0,90	0,87	0,87	0,73	*0,84*
	Non-proportional	Area (a)	0,07	0,03	0,01	0,01	0,03

Table 1: Students' mean performance in all test items

Figure 2 illustrates the similarity relations which are formed among both grade students' responses at the 8 tasks of the written test. Students' responses to the tasks are responsible for the formation of two clusters (i.e., groups of variables) of similarity. The first group consists of all proportional tasks [perimeter (p)], while the second group of all the non-proportional tasks [area (a)]. This indicates that students handled, as expected, proportional problems in a different way compared to the non-proportional tasks. However, between each group, students deal with the tasks in a systematic way, which is independent from the type of the figure. This observation is strengthen by a statistically significant similarity relation between the non-proportional problems Ra and Ca in the second similarity group, as the students deal with the two area problems in a similar way, despite the fact that they correspond to different figures (rectangle and circle).

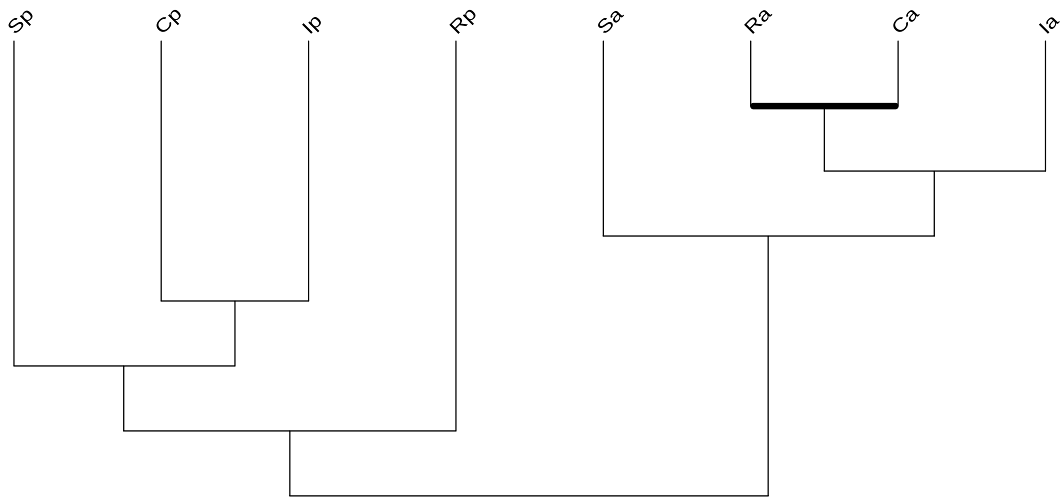

Figure 2: Similarity diagram of both grade students' responses in all test items

Students of both grades did not use representations often in their effort to solve the problems, as it is shown from the means of representation use in each problem (Table 2). This was may be due to the fact that students did not consider the construction of a representation useful for the solution of the problem, or even they were not familiar with such use. Noteworthy is however the fact that a reduced use of representations was observed in the 3rd grade compared to the 2nd grade. This difference is statistically significant in the case of the proportional tasks of perimeter ($t=2,63$; $p<0,01$), as well in the case of the non-proportional tasks of area ($t=3,01;p<0,01$). The reason for this difference probably lies to the fact that 3rd graders are more familiar and more experienced with the use of formulas compared to the 2nd graders and therefore, show a preference to their application for the solution of the problems.

Grade			Square (Sr)	Rectangle (Rr)	Circle (Cr)	Irregular Figure (Ir)	Total
2nd	Proportional	Perimeter (p)	0,22	0,17	0,13	0,04	0,14
	Non-Proportional	Area (a)	0,16	0,19	0,11	0,02	0,12
3rd	Proportional	Perimeter (p)	0,08	0,08	0,04	0,07	0,07
	Non-Proportional	Area (a)	0,05	0,05	0,04	0,05	0,05

Table 2: Students' means of representation use in all test items

Regarding the strategies used by students while solving the non-proportional area tasks, the majority of students (71%) systematically applied the proportional strategy, while the other three strategies appeared rarely. It should be noted that concerning the use of the proportional strategy, statistically important differences were observed between the two grades ($t=-4,56$; $p<0,01$). In particular, more 3rd grade students (80%) used it when solving non-proportional area items, compared to the 2nd grade students (62%), observation that reinforces the fact that the errors that occur in the non-proportional tasks are not occasional; they persist and even resist instruction.

Figure 3, presents the similarity relations between the strategies used by the students of both grades, at all the test's tasks. Students' strategies are responsible for the formation of three clusters of similarity. In the first cluster the use of the proportional strategy as it is presented in the four non-proportional problems is grouped together. This indicates that the children systematically apply the proportional strategy while solving this type of problems, irrespectively from the type of the figure. In the second cluster, the use of the paving strategy in the square and rectangular area items is grouped, but with very low rate of appearance.

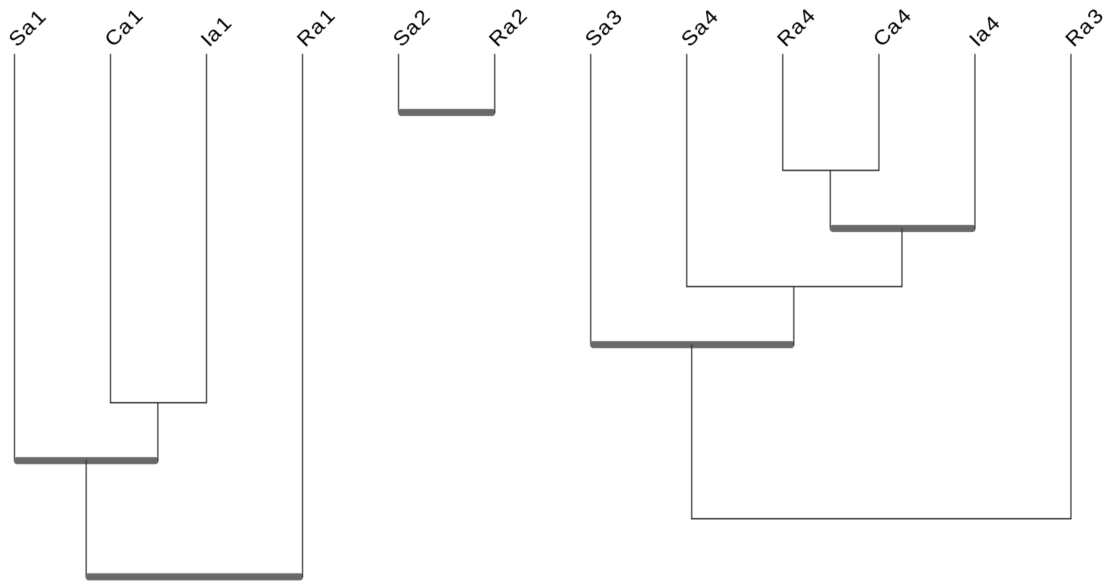

Figure 3: Similarity diagram of the strategies used by the students of the 2nd and 3rd grade at the non-proportional items

Note: Similarities presented with bold lines are important at significant level 99%.

Finally, in the third cluster, the strategies of using the general rule (x4, x9) and of applying the correct formula are grouped together. In this cluster, there is a statistically important similarity between the variables Ia and Ra, Ca, which refer to the application of the general rule in the non-proportional area problems concerning the rectangle, circle and irregular figures. Moreover, the area formula in the non-proportional square item is grouped with all the non-proportional problems in which the general rule was used. Therefore, these two strategies where handled similarly by the students as for them the direct application of the general rule -multiply by 4 in the case where the figure's dimensions are doubled and multiply by 9 when the dimensions are tripled in order to find the new figure's area – is a short form of the area formula.

CONCLUSIONS

The analysis of the comparative data collected supplements the research on non-proportional reasoning. Students' performance at the proportional tasks was clearly higher compared to the one at the non- proportional tasks; a finding which confirms the findings of previous research (Modestou & Gagatsis, 2004; De Bock, Verschaffel, & Janssens, 1998). In particular, comparing students' performance in relation to their grade, statistically important differences were observed at the proportional perimeter items with the 3rd grade students outperforming 2nd graders. However, such differences between the two grades were not observed at the non-proportional items. A small but significant difference was observed at the application of the popular proportional strategy at the non-proportional tasks, with more 3rd graders preferring it for the solution of the tasks. These findings reveal that the phenomenon of the illusion of linearity is not a common error which appears occasionally. It corresponds to an epistemological obstacle (Brousseau, 1997) which resists to any effort of improvement through teaching.

The type of the figure affected students' performance regardless of the character of the task (proportional or not proportional). In particular, the most difficult problem for both grades, in both proportional and non-proportional items, was the irregular figure item, probably because of the absence of some formula which could facilitate and direct students' actions. These results are consistent with the findings of De Bock et al. (1998), according to which the students had higher performance at the non-proportional items concerning the enlargement of a regular figure, compared to an irregular figure.

The use of self-made representations did not affect students' performance in both proportional and non-proportional items, as only a small number of students used them. It is however, worth noticing the fact that the degree in which the students used self-made representations was related to their grade. Thereby, the fact that 3rd grade students deal more with perimeter and area formulas, led to a reduced use of self-made representations compared to that of 2nd grade students.

In a subsequent research, a long-term classroom intervention, which promotes at the same time students' conceptual understanding of proportional reasoning and simultaneously takes into consideration the social, cultural and sentimental frame of learning, can be investigated. It is preferable that this intervention takes place very early in students' school career in order to prevent the appearance of the phenomenon of the illusion of linearity. When students study proportional relations, they could at the same time be confronted with counterexamples; that is cases where linearity is not applicable. In this way, the creation of the perception that all multiplicative comparison problems are characterized by linear relations might be limited.

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INTEGER INSTRUCTION: A SEMIOTIC ANALYSIS OF THE "COMPENSATION STRATEGY"

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A realistic instruction of integer addition and subtraction was conducted through the "dice games" featuring a "compensation strategy" (after Linchevski and Williams). We analyse the semiotic processes in the dice games and apply Radford's theory of objectification to the compensation strategy. In our case in contrast to Radford's, however, semiotic contraction occurs during factual generalization, which we analyse as a multistage process of semiotic objectification. Also, children's formulations of contextual and symbolic generalization were different from Radford's account, particularly in the transition from factual to symbolic generalization. Nevertheless, this case study suggests the value of this analytical framework.

INTRODUCTION TO FACTUAL, CONTEXTUAL AND SYMBOLIC GENERALIZATION

An intuitive instruction for integer addition and subtraction was implemented through replication of the "dice games" approach (Linchevski & Williams, 1999). We argue the significance of semiotic processes within these games because they are underinvestigated in instructional method. Generally, in *Realistic Mathematics Education* (RME) on which the games are based, *symbolising* is underinvestigated (Gravemeijer, Cobb, Bowers, & Whitenack, 2000) and investigation of semiotic mediation may clarify the transition to *reification* (Sfard, 1991), which is to date insufficiently explained (Goodson-Espy, 1998). In the theory of semiotic mediation, we particularly value *objectification* (Radford, 2002, 2003), within *factual, contextual* and *symbolic generalization* (Radford, 2003).

> … A factual generalization is a generalization of actions in the form of an operational scheme (in a neo-Piagetian sense). This operational scheme remains bound to the concrete level (e.g., "1 plus 2, 2 plus 3" …). In addition, this scheme enables the students to tackle virtually any particular case successfully. (Radford, 2003, p. 47)

In Radford's (2003) investigation of students' processes, factual generalization occurred when the students had to calculate the toothpick number of figure 25 in a pattern. The aim was achieved through the introduction of an operational scheme, capitalizing on deictic gestures (like pointing), linguistic terms (i.e. "next" and "always") and rhythm, enabled via face-to-face communication (ibid).

Contextual differs from factual generalization by the following two new elements:

- "The social-communicative element" (ibid, p. 50): the students explain to a "generic addressee" (p. 50) how they find the toothpick number for any figure. "Implicit and mutual agreements of face-to-face interaction (e.g.,

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 473-480. Prague: PME.

gestures, clue words) need to be replaced by objective elements of social understanding demanding a deeper degree of clarity" (p. 50).

- "The mathematical element" (p. 51): "a new abstract object has been introduced into the discourse" (p. 51).

When Radford's (2003) students had to find the toothpick number of any figure, the operational scheme "25 plus 26" for figure 25 became: "You add the figure and the next figure" (p. 52). Having to refer to a generic non-specific figure, the students needed a means of reference to it. The solution was to address this figure as *the figure* and the following as *the next figure*. These "generic and locative terms" (p. 53) are crucial because they "call our attention to certain objects" (p. 53) and "the various signs used in social intercourse allow the individuals to go beyond what is offered visually and to create conceptual worlds" (p.53). Concluding, "the new scheme does not operate on the level of concrete numbers, as factual generalizations do. … specific figures (like the fifth, sixth, etc.) have been displaced and put in abeyance… The new generalization encompasses an abstraction from actions and … from specific figures" (p. 52).

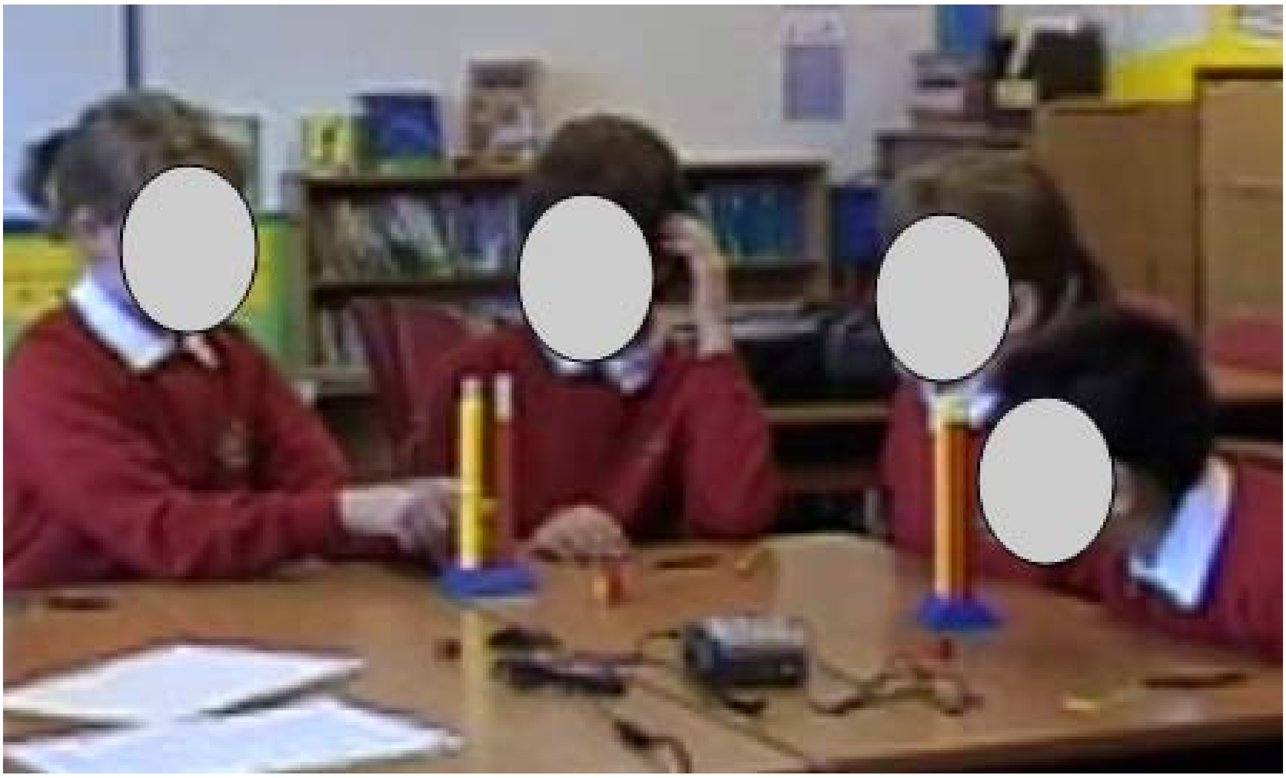
In symbolic generalization (in Radford, 2003, an algebraic one) the objects have to become "nonsituated and nontemporal" (p.55). Moreover, the students should "not have access to a (figurative) point of reference to "see" the objects. … the crucial term, the next figure, in the … contextual generalization supposes that the individual has a privileged view of the sequence, a point of reference: She or he sees the figure (in a figurative way), and this allows him or her to talk about the next figure" (p.55).

Radford's students fulfilled the above parameters firstly by "the insertion of a speech genre based on the impersonal voice" (p. 56). The generalization "You add the figure and the next figure" (p. 52) becomes "*n* plus *n* plus 1" (p. 56). Also, they replaced "the general deictic objects (e.g., "this figure")" (p. 56), which allowed "the emergence of objective scientific and mathematical discourse" (p. 56). Also in the expression (n + n) + 1 "the sign n appears as an abbreviation of the generic linguistic term *the figure*" (p.60). Based on this, Radford (2003) argues that "the sign n can be seen as an index in Peirce's (1955) sense (Radford 2000b, 2000c)" (p. 61), as the symbols in the two expressions remained indexical to the situated, temporal objects of the previous generalization. Conclusively, symbolic signs are indexical to the general deictic objects of contextual generalization, which in turn were indexical to the actions in factual generalization. Without these connections, the symbolic signs would have been meaningless.

THE COMPENSATION STRATEGY IN THE DICE GAMES

Linchevski and Williams (1999) presented an instruction method for integer addition and subtraction, based on the RME instructional framework and underpinned by the theory of reification. This instruction method was based on "dice games", which take place in groups of four students, arranged in two teams of two. The students of the two teams throw dice and collect points for and against each team in each throw, which they record on abacuses. On each abacus there is a column for each team.

Therefore, the points of each team on the two abacuses add up to give the team points. A team wins a game if it gets 8 points ahead of the other team. In consequence of this rule, it is not actually important how many points a team has



collected on the two abacuses, but how many points "ahead" a team gets. Thus, a *cancellation strategy* allows for cancellation of "points for" and "points against" or even "red team points" and "yellow team points".

Often a team's column on an abacus is full and team points cannot be added, or is empty and team points cannot be subtracted. According to the *compensation strategy*, if you cannot add/subtract a number of team points, you can subtract/add the same number of points from/to the other team, hence ensuring the correct difference of team points. This strategy allows the game to continue, when it would have been stuck, as well as the intuitive construction of equivalences like: $+ (+2) \equiv - (-2)$ and $+ (-2) \equiv - (+2)$. Despite being an important strategy, the compensation strategy has not been investigated semiotically before.

COMPENSATION STRATEGY: THE FACTUAL GENERALIZATION

Though factual generalization for Radford is seemingly a clear-cut process based on action on physical objects and deictic activity, we find a complex multi-step or multi-stage process. This complexity may begin to be appreciated in the following episodes, which we consider as co-constituting factual generalization as a multi-stage process of reification. The students in the episodes ran out of space to add points/cubes on the abacus and had to introduce in the discourse the concept of compensation of yellow and red points. In episode 1, Umar had to add 1 yellow point/cube but the yellow cubes column was full. Umar was stuck and Fay tried to help him: she proposed taking away 1 red cube instead.

Episode 1: Minutes 14:30-14:50, lesson 1. "“…” indicates a pause of 3 sec or more, and “.” or “,” indicate a pause of less than 3 sec" (Radford, 2003, p. 46).

Fay: You take 1 off the reds [pointing to the red column on her abacus]. […] Because then you still got the same, because you're going back down [showing with both her hands going down at the same level] cause instead of the yellows getting one [raising the right hand at a higher level than her left hand] the red have one taken off [raising her left hand and immediately moving it down, to show that this time the reds decrease].

Here Fay articulated the process of compensation of the addition of a yellow point with the subtraction of a red point. We may say at this point the *interiorization* (Sfard, 1991) of a new process is taking place. This process is based on the idea that it is fair to subtract a red cube instead of adding a yellow cube on the abacus. The detailed gestural justification of the equivalence of actions, which allows the construction of the process, is especially noticeable.

In episode 2, as the yellows were full and there was space for 1 red, compensation was employed. Zenon could not understand and Jackie explained:

Episode 2: Minutes 20:15-20:43, lesson 1.

Jackie: It's still the same, like … [a very characteristic gesture: she brings her hands to the same level and then she begins to move them up and down in opposite directions, indicating the different resulting heights of the cubes of the two columns of the abacus] because it's still 2, the yellows are still 2 ahead [she does the same gesture while she talks] and the reds are still 2 below, so it's still the same… [again the gesture] … em like… [closing her eyes, thinking hard] … I don't know what it's called but it's still the same… score [the gesture again before and while articulating the word "score" – meaning same score on her abacus].

We noticed the repetition of the phrase "it's the same" and Jackie's persistent gesture, while trying to find an appropriate linguistic term for the compensation intuition. Later the term "score" was introduced, verbalised simultaneously with the same gesture. The importance of the term "score" is revealed by Jackie's persistence in finding an appropriate articulation. We believe both the term "score" and the associated gesture achieved the *semiotic contraction* (Radford, 2002) of the process. This contraction, taking place inside factual generalization, reminds us of *condensation* (Sfard, 1991), in the sense that it was used in the theory of reification. We propose that the generic linguistic term "score" facilitated the semiotic contraction of the compensation strategy.

Episode 3: Minutes 21:27-21:57, lesson 1. There's only space for 2 yellow cubes, but Fay has to add 3 yellows and 1 red.

Fay: Add 2 on [she adds 2 yellow cubes] and then take 1 of theirs off [she takes off a red cube] and then for the reds [pointing to the red dice] you add 1, so you add the red back on [she adds 1 red cube].

Researcher: […] Does everybody agree? (Jackie and Umar say "Yeah").

Now the students no longer need to justify the use of the compensation strategy. Moreover, from this point on they simply use the compensation strategy when the abacus is full and no space is available for extra cubes to be added: compensation seems to be the intuitive thing to do. This consensus without need of explanation suggests the establishment of the *reification* (Sfard, 1991) of the compensation strategy.

In factual generalization the students used deictic gestures in the form of pointing, touching or moving cubes. Also, they used team names (reds, yellows) deictically: i.e. "take 1 off the reds" (episode 1). Still, the concept of "difference" of yellows and reds has not been introduced explicitly into discourse until episode 2. In this episode the concept of "difference" was introduced through the repeated use of the explanation "it's the same", along with Jackie's gesture, eventually matched with the general linguistic term "score". The use of the combination of Jackie's gesture with the term "score" introduces a new abstract characteristic to the discourse. Finally, in episode 3 the use of deictics is very limited, as the students now need to focus (and indicate) only on the resulting action. A similar analysis of factual generalization can be found in Koukkoufis and Williams (2005).

COMPENSATION STRATEGY: THE CONTEXTUAL GENERALIZATION

For contextual generalization, a generalization to language or *logos* (Radford, 2003) needs to arise. The students must understand that whenever they cannot add a number of yellow/red cubes, they can subtract the same number of red/yellow cubes instead. The above group of students did not spontaneously articulate the compensation strategy through language alone, though they applied the strategy for any number of cubes. We believe the lack of the need to articulate was the result of it being so obvious that it did not need to be said. In contrast, the same group spontaneously articulated a contextual generalization of another important strategy, the *cancellation strategy* (Linchevski & Williams, 1999). For the cancellation Fay said (minutes 38:17-38:40, lesson 1, 5 reds and 2 yellows) "you find the biggest number, then you take off the smaller number". In this extract, as in Radford's (2003) contextual generalization, new abstract objects enter the discourse: "the biggest number" and "the smallest number". We believe in this case the contextual generalization was needed because the cancellation strategy was not so intuitive for these students.

COMPENSATION STRATEGY: THE SYMBOLIC GENERALIZATION

Symbolic generalization in Radford (2003) resulted from the replacement of the general deictic objects of contextual generalization with indexical symbolic signs. In our case, symbolic generalization is first indexical.

Indexical Symbolic Generalization

Symbolic generalization begins with the intuitive introduction of formal mathematical symbols. We replaced the two dice with a single die with -3, -2, -1, +1, +2, +3: "+" are yellow points, while "–" are points taken from the yellows, thus they are red points. Episode 4 illustrates the resulting transition to + and – symbols.

Episode 4: Minutes: 20:45-21:55, lesson 3.

Researcher:	+1. Who is getting points?
Jackie:	The yellows
Researcher:	[…] Who is losing points?
Jackie, Umar:	The reds
Fay:	[…] reds are becoming called minuses and then the yellows are becoming called plus.

As a result of this connection of the formal mathematical symbols to the informal pre-symbolic signs, the "+" and "–" come to indicate yellow and red points, acquiring in this way a contextual meaning.

In Radford (2003), symbolic expressions like *n, n +1* indicated general deictic objects of contextual generalizations. In the *indexical symbolic generalization* in the dice games, a sign like "-2" does not replace a general deictic object, but a specific concrete object, in this case the object "2 red points", which on the abacus is 2 red cubes. Thus, it is not general deictic objects of a contextual generalization that are replaced, but the specific objects of the factual generalization. Moreover, as we can see below in episode 5, symbolic signs like "-2" are being manipulated through concrete objects (abacus cubes and the die) and deictic activity is associated with their manipulation, just as in factual generalization.

Episode 5: Minutes: 33:15-33:53, lesson 3.

Researcher:	[…] you get -2. What would you do? (Fay takes 2 yellow cubes off) […] What if you had +3?
Umar:	You take away 3 of the reds.
Zenon:	… or you could add 3 to the yellow.
Fay, Jackie:	… add 3 to the yellow.
Researcher:	Oh, 3 off the reds or 3 to the yellows. (All the students agree)

Above, though the question is in terms of *minuses* and *pluses* (in formal symbolic form), the proposed action is in terms of team points. If the symbolic signs were used non-indexically, Umar would probably say take away "-3" or "3 minuses" and the others would say add "+3" or "3 pluses". The realistic context and the model allowed the transition from *factual* to *indexical symbolic generalization*, without the completion of contextual generalization. We ascribe this difference from Radford's case to the intuitive use of the formal symbolic signs on the die and abacus, afforded though RME modelling.

Yet, we do not consider indexical symbolic generalization simply a replication of factual generalization with formal symbolic signs. The first difference is the instantaneous generalization of the compensation strategy in analogy to its factual generalization operational scheme: no reification of a new scheme is needed. Simply and intuitively the pre-symbolic signs (yellows and reds) are replaced in the existing operational scheme with the symbolic signs of positive and negative numbers. The students understood that if you can't add -2 (the abacus is full), you can subtract 2 yellow cubes, or if you have +3, you can either take 3 off the reds or add 3 yellows.

Non-indexical Symbolic Generalization

The next step would be non-indexical use of the + and – sign, which means that the students would be able to use symbols like -2 as autonomous entities. We do not claim that the formal mathematical symbols will lose the connection to their game meanings. On the contrary, we consider it vital for the students to be able to go back to the realistic context whenever they need to do so. What we *do* mean is that the students will also be able to treat the formal mathematical symbols non-indexically. Practically, the discourse *should* eventually include reference to specific numbers of "minuses" and "pluses" (negative and positive integers) and even the compensation strategy with pluses and minuses in general (general abstract objects).

Non-indexical symbolic generalization was targeted by urging students to verbalise the "+" (*plus*) and "–" (*minus*). We believe verbalization is important to enable the students to connect the non-contextual names *plus*/*minus* to the symbols +/–. Initially, the students needed often reminding to verbalise *plus*/*minus*. Gradually, these names were attached to the symbols and the students could play the games referring to *pluses* and *minuses* instead of yellows and reds and then do the appropriate actions on the abacus. The students managed to do so when an extra die giving *add* or *subtract* (*add/sub die*) entered the game: now the students had to add or subtract integers. E.g. (my brackets): Fay: *add [minus 3], subtract [2 of the minuses];* Zenon: *add [2 to the pluses]*; Jackie: *add [minus 2]*. Only Umar still felt uncomfortable and sometimes said *[minus 1] add* or *add [subtract 2]* etc.

Finally, we checked whether students spontaneously produced a further verbal-generalization of the form "when you can't add minuses, subtract pluses" or vice versa. This group of students did not produce such a generalization of compensation. We hope that further data collection will allow the observation of a symbolic generalization of the compensation strategy in this form.

APPRAISAL OF THE OBJECTIFICATION ANALYSIS

Though contextual and symbolic generalizations require further investigation with additional data and closer attention to the semiotic means, we believe it was useful to analyse semiotically the objectification of the compensation strategy. We consider significant the reification in the factual generalization and the lack of necessity of contextual generalization before symbolic generalization when there is a strong intuition involved. The direct connection of symbolic signs with the specific objects of factual generalization was also different from Radford's (2003) analysis, as well as the use of deictics in symbolic generalization. Also the distinction between indexical and non-indexical symbolic generalization helped us focus on the transition from indexical symbolic signs.

We have shown that (i) Radford's theory of semiotic objectification has been applied to a very different mathematics education context; (ii) in our case it was necessary to analyse factual generalization as a multistage process of reification; and (iii) differences in children's formulation of contextual and symbolic generalization were found, in particular the transition from factual to symbolic generalization without the completion of contextual generalizations.

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