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Research Reports

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Research Reports

THE ODDS OF UNDERSTANDING THE LAW OF LARGE NUMBERS: A DESIGN FOR GROUNDING INTUITIVE PROBABILITY IN COMBINATORIAL ANALYSIS

Dor Abrahamson and Rose M. Cendak

University of California, Berkeley

Twenty-eight Grade 4 – 6 students participated in 1 hr. clinical interviews in a design-based study that investigated: (a) probability-related intuitions; (b) the affordances of a set of innovative mixed-media learning tools for articulating these intuitions; and (c) the utility of the learning-axes-and-bridging-tools *framework supporting diagnosis, design, and data-analysis. Students intuited qualitative predictions of mean and variance, yet only through grounding computer-based simulations of probability experiments in* discrete–scalar, *non-uniform, multiplicative transformations on a special combinatorial space, the* combinations tower, *could students articulate their intuitions. We focus on a key learning axis, students' confusing likelihoods of unique events with those of classes of events.*

INTRODUCTION

Objectives

This paper reports on a design-based study in mathematics education. The study was designed to advance three interdependent lines of research: (a) *theory of learning*—probing late-elementary and middle-school students' intuition for probability; (b) *design*—examining the roles a set of innovative learning tools may play in supporting students' articulation of any probability-related intuitions they may have; (c) *design theory*—evaluating and improving a framework for mathematics education that guides a researcher from diagnosing a learning problem through to design and to data analysis.[1] The mathematical domain of probability was chosen as particularly auspicious for studying student articulation of intuitive interaction with specialized tools due to: (a) an "intuition gulf" created by this domain's ambiguous treatment of 'prediction'—the indeterminism of the individual sampling action as compared to the by-and-large predictability of aggregated results of sufficiently numerous sampling actions; (b) students' deep rooted and lingering confusion and even superstition that such ambiguity entails and the detrimental effect of these confusions on problem solving; and (c) the roles learning tools could play in enabling students to confront this ambiguity and reconcile it in the form of coordinated conceptual understanding (Abrahamson & Wilensky, 2004a; Liu & Thompson, 2002; Stohl Drier, 2000; Wilensky, 1997).

The study reported in this paper was the first stage of a larger design-based research project that includes: (1) one-to-one clinical interviews; (2) focus-group studies; and (3) classroom interventions. The objective of the interviews was to elicit students' *learning issues* (Fuson & Abrahamson, 2005) in this design, that is, potentially

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problematic aspects of the targeted mathematical content (probability) as embedded in, and emerging from, student interactions with the designed materials. In particular, we investigated whether students could build a coherent, if largely qualitative, account of the Law of Large Numbers—an account that recruits students' relevant, yet possibly conflicting, intuitions and bridges these intuitions. By 'intuition' we refer broadly to mental/perceptual action models that students *tacitly* bring to bear to interpret situations in the context of a mathematical problem (Fischbein, Deri, Nello, & Marino, 1985; Lakoff & Nuñez, 2000; Polanyi, 1967).

Theoretical Perspectives

Three related theoretical perspectives informed the design of this study, including its materials, procedure, and data analysis. One perspective, *learning axes and bridging tools* (Abrahamson & Wilensky, 2004b) characterizes learning as tackling and resolving pairs of conflicting ideas residing along conceptual "axes" [plural of 'axis']. Based on a domain analysis, a designer taking on a design problem can depict students' difficulty in terms of a lack of opportunities for addressing a set of concept-specific learning axes. These axes then frame a design plan. Once the axes are vested in actual tools, one speaks of 'learning issues,' i.e., the pragmatics of constructing new conceptual understandings within a particular design. Thus, the learning materials and activities are designed to embody, foreground, and juxtapose the axes to enable students to resolve the conflicts. Such juxtaposition is enhanced by embedding both conflicting ideas within a single ambiguous object. The second perspective, the *apprehending zone*, is that students learn through building connections within and between situations and mathematical objects—teachers model problem solving to facilitate the building of these connections (Fuson & Abrahamson, 2005). The third perspective positions mathematical objects as more than arbitrary epiphenomena aiding mathematical reasoning. Rather, conceptual knowledge may be embodied in learners' growing relations with artifacts supporting the construction of understanding (Gigerenzer, 1994; Pirie & Kieren, 1994; Vygotsky, 1978). Together, these perspectives suggest the criticality of the craft of design: Mathematical objects could be more than learning 'supports'—they could become internalized as permanent and inextricable imagistic vehicles of mathematical reasoning.

A key domain-analysis principle implemented in the design is that reasoning about probability from a complementarity of 'macro' and 'micro' perspectives is critical for deep and 'connected' understanding of the domain (Wilensky, 1997).

METHOD

In 9 visits spanning 3 weeks, 28 Grades 4-6 students from a K-8 suburban private school (33% on financial aid; 10% minority students) each participated in a one-to-one semi-clinical interview (Ginsberg, 1997; mean 56 min., SD 12 min.). From the pool of volunteering students, we selected students representing the range of mathematical performance in their grade levels as reported by their teachers. Also, we balanced for gender. All sessions were videotaped for data analysis.

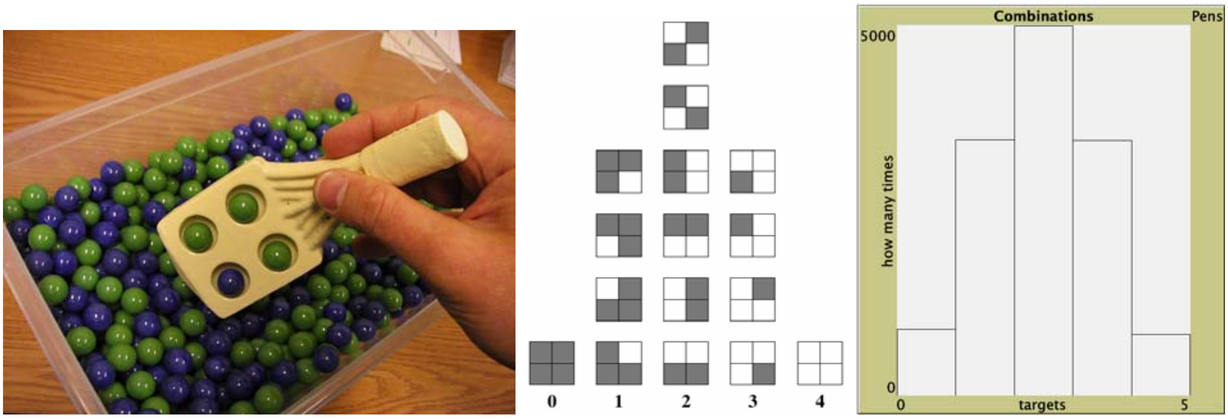


Figure 1. Selected materials: marble scooper, combinations tower, and 4-Blocks.

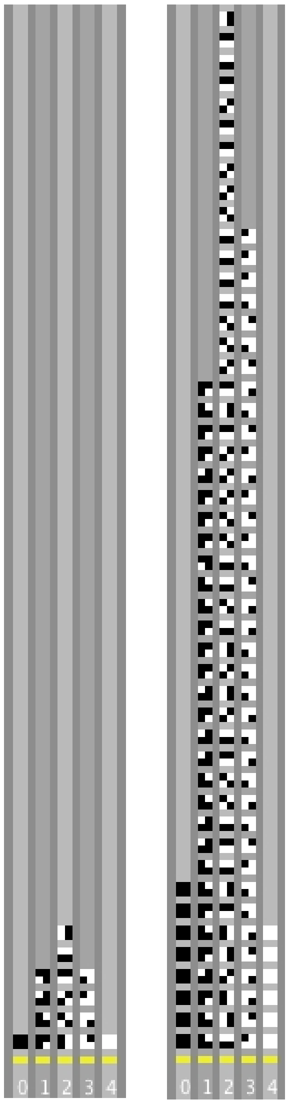
The materials were all embodiments of the *4-block* stochastic device from the ProbLab experimental unit (Abrahamson & Wilensky, 2002, 2005), implemented in different media (see Figure 1, above). The *marble scooper* device scoops a fixed number of marbles out of a vessel containing many, e.g., 100 green and 100 blue. The *combinations tower* is the combinatorial space of all 16 possible 4-blocks arranged by the number of light-colored squares in each. Students use crayons to build the tower from paper cards, each featuring an empty 4-block grid. Two simulations built in NetLogo, a multi-agent modeling-and-simulation environment (Wilensky, 1999), included: *4-Blocks*, where four squares independently choose randomly between two colors—the program dynamically records the blocks by number of green squares; and *4-Blocks Stalagmite* (see Figure 2, next page), in which 4-blocks are generated randomly, yet *the samples themselves* are stacked in a pictograph bar chart. The simulation can be run under various conditions, e.g., rejecting repeat samples (Figure 2a) or keeping them (2b).

Using microgenetic analysis (Siegler & Crowley, 1991), we studied the data to characterize properties of the learning tools, activities, and prompts that enabled students' to move from difficulty to understanding during the interview, where 'understanding' was operationalized as students' manifesting stable and coherent discourse about new content in terms of the tools and connections between them.

RESULTS AND DISCUSSION

Intuitive Judgements and Strategies

All but two students initially predicted a 2-green/2-blue 4-block as the most common scoop from the bin ("half–half"). Others said they had "picked this [idea] up" or that they did not know. Not a single student initiated an exploration of the combinatorial space as a means to warrant or validate that intuition. Even following prompts to construct the combinatorial space, all students asked whether they should include the permutations. Typically, the event *classes*, e.g., patterns with. Asked to support this guess, most students said, "It just seems so," pointing to the apparently equal numbers of green/blue marbles in the bin 1 green square, emerged only through actions of generating different patterns and assembling the combinations tower. For example, a student who had created only one pattern with a single green square realized that this square could be located in each of the four 4-block locations. Thus, attending to event classes emerged as a pragmatic principle for mathematical inquiry.



2a. *2b.*

Figure 2. Two appearances of a 4-block stalagmite.”

Drawing a Compound Event From a Hat

Students understood that each 4-block outcome resulted from the concurrence of 4 independent random outcomes. Yet, in referring both to the combinatorial space (the combinations tower) and to outcome distributions from repeated sampling (the simulated experiments), students came to treat the 16 possible compound events as though these were 16 independent events of equal likelihood—as though, for example, the experiment were analogous to drawing one of the 16 cards from a hat. Such “clumping” of compound events, supported by the combinatorial-analysis format (the cards), appears to have enabled students to reflexively apply to the set of compound events their intuition for a set of independent events. Albeit, this recursive strategy does not easily apply for $p \neq .5$.

Learning Axes: “My Mind Keeps Going Back and Forth”

The learning axis ‘specific event vs. event class’ (see Figure 3, next page), which we perceive as key to this design (see next section), posed great difficulty for several Grade 6 ‘High’ students, who required a mean of 9 min. until stability.

Student Discrete–Scalar Insight Into the Law of Large Numbers

Students understood that event classes have different likelihoods according to their relative sizes, with larger event classes having “better odds” in the experiments, e.g., it is more likely to randomly get any two-green 4-block than any three-green 4-block, because there are six items in the former group and only four in the latter. Although only two ‘High’ students could express these *inter*-class relations multiplicatively, students learned to use these relations to express what we have termed a *discrete–scalar* multiplicative model of the outcome distribution.

The traditional representation of binomial distribution (see Figure 4, next page, on the left) can be interpreted as an ambiguous figure enfolding theoretical and empirical constructs. These can be bridged by an *itemized distribution*, i.e., the combinations tower. This bridging tool illuminates stochasm as theoretical-to-empirical *transformation*, as follows. All events have equal opportunities to be repeated in the experiment, so outcome categories holding a larger variety of different events collect more groups of repeated events and therefore collect a larger total of outcomes. For instance, if each of the 16 different possible events appears in the experiment about 3 times, then a column collecting 4 types of outcomes (e.g., see Figure 4, next page, on the right) accumulates a mixture of 4 sets, each of about 3 outcomes, for a total of about 12 outcomes (see also Figure 2).

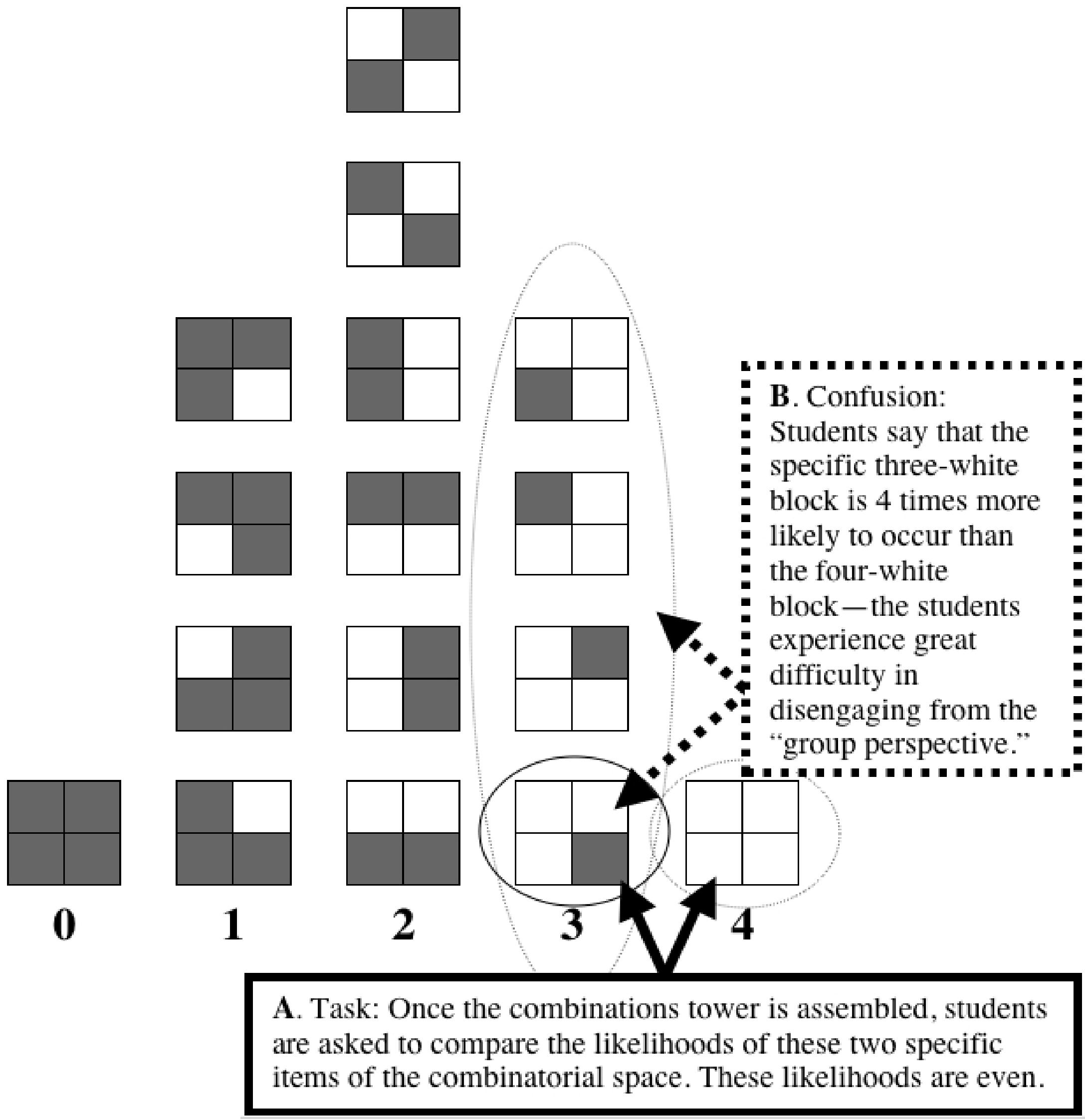


Figure 3. Students' difficulty with the learning axis 'specific event vs. event class.

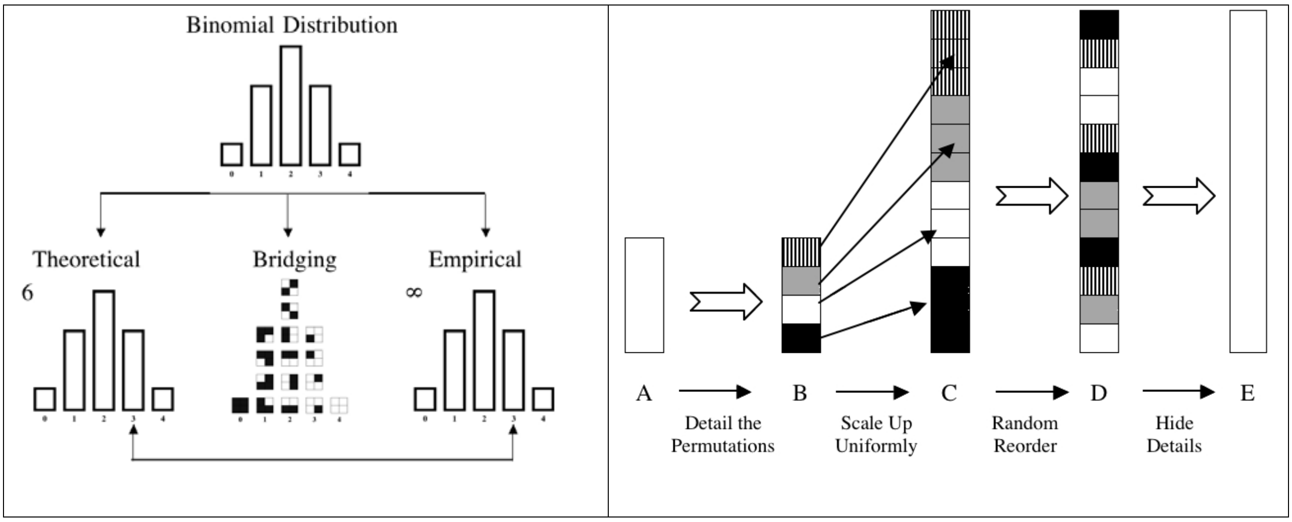


Figure 4. Bridging theoretical and empirical aspects of binomial distribution.

Students could investigate this line of reasoning within the 4-Block Stalagmite simulation, because, unlike in traditional representations, the samples *themselves* were all stacked in the columns (see Figure 2). Further, students could readily compare the outcome distribution and the combinatorial space, because these two structures were designed to appear similar (see Figure 1). We envisage such understanding as supporting a treatment of the sequence of column heights, e.g., 1-4-6-4-1, as a set of multipliers, i.e., as coefficients in the binomial function; the multiplicand 'scalar unit' would be the events' mean number of occurrences.

CONCLUSIONS

Design

The design was an example of a framework by which the designer first identifies a learning axis, then actuates this axis in the form of objects and activities that embody this axis as a learning issue; students must confront this issue and unravel it toward deep understanding of mathematical content. The demonstrated ubiquity of students' probability-related intuition together with the effectiveness of this design in enabling students to investigate this intuition suggest that this design could potentially be developed into a unit used in late elementary school and certainly in middle school. Whereas basic fluency with rational-number models appears to have helped a couple of Grade 6 students perceive the outcome distribution as a proportionally-equivalent scaling-up of the combinatorial space, such fluency is not necessary for appreciating the scalar–discrete and stochastic transformations explored in our design. Finally, in future development of the design we will explore its potential extension to cases in which $p \neq .5$ and also study its connections to normative, symbol-based, mathematical expressions.

Building on Intuition: "It's What I Was Trying to Say But Didn't Know How"

Just how intuition is grounded in teaching–learning contexts is a difficult yet important question to pursue, because not all students are able to recognize a resonance of their intuition within these contexts. Moreover, just because the combinations tower enabled students to *validate* and possibly *ground* their intuitive judgment, it does not necessarily follow that this validation was an *articulation* of the intuition itself. The intuitive judgment needn't have been combinatorial—it may have been some manner of proportional reduction or mapping of the marble population onto the marble-scooper 4-block template. This should be researched.

Teaching

Focusing classroom discussion on the learning axes is challenging, because students may hold fast onto their confusions. Nevertheless, intuitions should be recruited into learning spaces even if these intuitions initially appear vague or misleading, because they will persist anyway—students will achieve a sense of understanding only if the learning issues are faced head-on by probing and discussing interpretations of objects within problem-solving activity contexts.

This study has contributed an innovative design. The design framework outlined herein, too, could help education practitioners, both in the domain of probability and beyond. Finally, we call for further research on the nature of intuition and how it may be sustained through to deep mathematical understanding.

Note and acknowledgments

[1]Design-theoretical aspects are elaborated in a full report. The *Seeing Chance* project is supported by a NAE/Spencer Postdoctoral Fellowship to the first author. The participation of the second author was supported by UC-Berkeley's URAP. We are very grateful to the volunteering students, principal, and staff.

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IMAGINARY-SYMBOLIC RELATIONS, PEDAGOGIC RESOURCES AND THE CONSTITUTION OF MATHEMATICS FOR TEACHING IN IN-SERVICE MATHEMATICS TEACHER EDUCATION

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Zain Davis
University of Cape Town

We take as axiomatic (1) that in any pedagogical situation there are always two modes of identification at work: imaginary and symbolic (Lacan), and (2) today the imaginary dominates social relations. We focus on a model of in-service mathematics teacher education where the imaginary is rendered largely under the aspect of the symbolic, producing a pedagogic form that appears to generate a productive discursive space for the production of mathematics for teaching. Disruption occurs, however, when the notion of community is used as a pedagogic resource because, we reason, it renders the imaginary dominant in relation to the symbolic.

INTRODUCTION

In Davis et al (2005), we presented part of an emerging and challenging theme in our study[1], that of adequately capturing the pedagogic modalities that prevail in the field of mathematics teacher education but differ substantively in how *consciousness might be specialised* (Bernstein, 1996); in this instance, of mathematics for teaching. We analysed three cases of pedagogies circulating in mathematics education for in-service teachers in South Africa, and found that each recruited the image of the teacher (and teaching). In teacher education discourse, all, unspurprisingly, recruited some notion of ‘practice’. Learning to teach (whether in pre- or in-service) requires modelling (an image of the adept) and/or experience of the practice of teaching. What is interesting is how differently ‘practice’ manifests in three cases we are studying[2].

In this paper we focus in on one of those cases, as it exemplifies the constitution of a potentially productive discursive space where mathematics teaching is interrogated by examining records of practice with a set of symbolic resources that lift the practice beyond the specific. This imaginary-symbolic relation provides for a specialisation of consciousness that is at once practical *and* theoretical, empirical *and* principled[3].

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[1] Our study is concerned with *what* and *how* mathematics and teaching, as dual objects in mathematics teacher education, are co-produced. We contend that this will contribute to an elaboration of the notion *mathematics for teaching*. See Adler (2005) and Adler et al (2005).

[2] In Davis et al (2005) we point to questions that emerge about equity in the practice of teacher education, a point not in focus here.

[3] Implicit here is that this relation provides a theoretical resource for describing *mathematics for teaching*.

METHODOLOGICAL COMMENT

In any pedagogic practice, there are always two principle modes of identification at work: one is the identification of the pedagogic subject with an image; the other is an identification with one or more discursive fields. We refer to the first mode of identification as *imaginary identification*, where the term imaginary should be understood as referring to the *image* (in this case the image would be of the mathematics teacher). The second mode of identification is referred to as *symbolic identification* (cf. Lacan, 2002a, 2002b), and in this case identification would be with the fields of mathematics, mathematics education and teaching. We have elaborated these modes in Davis *et al*. (2005). We note briefly here that imaginary identification is produced through relations between subjects with the image of one in relation to that of the other. Symbolic identification is produced through symbolic relations entailing a subjection to social "institutions", which include discursive fields. Symbolic identification is predicated on the social existence of a legitimating field external to the individual subject, including s/he who holds any particular social/institutional mandate.

The relationship between the two modes of identification can be realised in a manner such that the symbolic largely appears under the aspect of the imaginary, and does not emerge in its full specificity. Here, the teaching is presented as largely a practical accomplishment. Alternatively, the imaginary can be rendered under the aspect of the symbolic, producing a distancing from the imaginary (cf. Žižek (2002: xi-xvii) for an extended discussion of the relations between the imaginary and the symbolic). In the latter, teaching is theorised and remains distant and abstracted from the realm of practice. Both of these pedagogic forms in teacher education have been criticised, suggesting resolution in the integration of theory and practice. As Bernstein reminds us, there is always ideology at work in any pedagogic practice. From this we can anticipate diverse forms of 'integration', of the imaginary and the symbolic. Our concern is how, in a study of instances of mathematics teacher education, we can adequately capture pedagogic forms in ways that reveal the mediation between the imaginary and symbolic, and that illuminate possible social effects.

Elsewhere (Davis *et al.*, 2005) we have elaborated how the manner in which actions and statements of subjects are authorised reveals the nature of the forms of identification at work; and, importantly, how it is that the symbolic *qua* discursive order is constituted in a given pedagogic context. For the coherence of this short paper we briefly detail aspects of our methodology. Our unit of analysis is referred to as an *evaluative event*: a teaching-learning sequence focused on the acquisition of some or other content. It derives from our use of Bernstein's (1996) proposition that the whole of the pedagogic device is condensed in evaluation. The purpose of evaluation is to transmit criteria for the production of legitimate texts. However, any act of evaluation has to appeal to some or other authorising ground in order to justify the selection of criteria. In the complex practice of mathematics teacher education, the interesting and challenging issue is that there is a spread of authorising social institutions (e.g. disciplinary mathematics, mathematics education and/or teaching as fields; curriculum texts) and agents (experienced teachers, teacher educators, researchers). In Davis *et al*. (2005) we have shown that what is taken as the ground for an appeal varies substantially within and across sites of pedagogic practice in teacher education, allowing us to examine the way in which identification is functioning, and to begin to pose questions about possible effects.

THE STUDY

From an initial review of formalised in-service math teacher education programmes across South African universities, we selected three sites of focus because of the continuum they offer with respect to the integration of mathematics and teaching (content and method) within courses. In two cases, the imaginary predominates. We described that as a prioritising of sensibility, which is experiential. Sensibility is an important feature of the teaching and learning, particularly of school mathematics, where some meaning remains absent for many learners. What then of intelligibility? Specialised knowledges, including mathematics and mathematics for teaching, in part aim at rendering the world intelligible, that is, providing us with the means to grasp, consistently and coherently, that which cannot be directly experienced. Consistency and coherence, however, demand a principled structuring of knowledge.

The pedagogic forms that predominated in the two cases mentioned are familiar: in one, the teacher educator models how Grade 7-9 algebra (for example) should be taught; in the other, the pedagogy centres on self reflection by the teacher through an action research project. We see these pedagogic practices as a function of ideologies and discourses in teacher education that assert the importance of teacher educators 'practicing what they preach'. This pressure is particularly strong when new practices (reforms) are being advocated, and so is a significant feature of in-service teacher education. At a more general level, these modelling forms are also explicable in relation to the discourse of the integration of theory and practice, in particular, that theories without investment in practice are empty. The pedagogic form in the case we discuss here is different, and illuminating. With the evaluative event as unit of analysis, we chunked course lectures and notes into a succession of evaluative events over the period of a complete course, and identified and coded the legitimating appeal(s) in each event. To capture, albeit briefly, the substance of the case, we start with a general description followed by the production of analytic statements supported by illustrations from selected teaching sequences. Following discussion of the case, we move to a more general discussion of the implications for the production of mathematics knowledge for teaching.

TEACHING AND LEARNING MATHEMATICAL REASONING

We have described the overall practice to be acquired by this course as the interrogation of records of practice with mathematics education as a resource (Davis et al, 2005). The course is structured by mathematics education texts on the nature, teaching and learning of mathematical reasoning as a mathematical practice. Teaching is presented through a variety of records of practice (videotapes of local teachers, student work, curriculum texts and tasks, as well as teachers' own experience). In other words Mathematics education texts are symbolic resources for interrogating practice – for studying teaching. Throughout the lecturer's interactions with students, the function of the academic texts is fore-grounded. After twenty minutes in the first session of this course, and during a discussion with students on the meanings of mathematical reasoning and mathematical proficiency, the lecturer states: "What is in the readings that help with our definitions – so that we look at these systematically?" In and across sessions, activity typically requires teachers to 'bring' examples from and/or descriptions of their own practice. They are then presented with a record of an other's practice (e.g., a mathematics classroom video extract; a transcript of a teaching episode), and a set of readings then function as the mechanism through which their experience and images of teaching presented are both interrogated. Thus, teaching, and in particular, teaching mathematical reasoning, is constructed as a discursive space. What we have is a relation set up between the imaginary and the symbolic in the form of images of teaching mathematics on the one hand, and a set of reasonably principled resources for interrogation those images on the other.

To elaborate: Teachers were required to prepare for their first session by reading three papers, one of which was on mathematical reasoning, and another on the five strands of mathematical proficiency. The latter is a chapter from the book "Adding it up: Helping children learn mathematics" by Kilpatrick and others. Teachers were also required to bring an example (written) of an observation of each of the five strands in one of their learners and were asked to "describe how you observed each strand (or lack of) in an interaction with a learner or in their written work and give reasons why you have identified that observation as a particular strand" (Course handout, Session 1). The discussion related to the fifth strand, productive disposition, is illustrative. One of the teachers (S1) says she "could not get something about productive disposition" from looking at her learners' responses. Another (S2) offers an example of productive disposition as "a learner who gets a hundred percent", and a third (S3) suggests "If learners can relate their mathematics to their everyday lives they have productive disposition". The short extract below is indicative of how the lecturer moves to interrogate these offerings: (L = lecturer; Sn = student)

L: (referring to the offering from other teachers) *S1,* do those responses help you?

S1: Yes, I think so especially the last one.

L: Would Kilpatrick agree with S3? What do they say about this in the text …?

Ss look at the reading, and L reads aloud from the text that "someone with productive disposition sees mathematics as useful and worthwhile" and asks for other key aspects of productive disposition. Discussion continues and she concludes this with:

L: It is not only belief in yourself it is also a belief about the subject – that mathematics can make sense. … and … it is difficult to see. … For me the important thing is whether you can see that the learner believes they can do it and they can do it. What Kilpatrick et al are arguing is that these *(the strands beyond procedural fluency)* are not developed. So this is what we need to be teaching, and so this is why we are not getting people going into higher mathematics. Their conjecture is that we need to be focusing on this, what it is and how to teach it? … In practice all the strands need to be done together … the image of interwoven strands is very powerful. And

> the interesting question in all of this is how to assess this? The argument in the paper is that you *(teachers)* should be able to recognise it and assess it.

Elsewhere (Adler *et al*., 2005) we have described another session in the course, focused on the identification of and engagement with mathematical misconceptions. The records of practice recruited here were a video-tape and transcript of a teaching sequence where a teacher in a local school grappled with errors and misconceptions as his grade 11 learners reasoned about the truth of the statement: $x^2+1 \neq 0$ if x is real. In addition, teachers were required to bring to class their own ideas about possible misconceptions and errors their students might produce in this case. These were interrogated drawing on mathematics education texts that located misconceptions within a theory of knowledge construction, and that offered conceptual distinctions between different types of misconceptions. Similarly to the first session, these texts were used to interrogate both what learners (in the video-tape) produced, how the local teacher interacted with these, as well as the participating teachers own experience of misconceptions and errors in their classrooms. Mathematical misconceptions (in relation to mathematical reasoning here) were produced as empirical and localised on the one hand (the imaginary), but subordinated to reasonably principled knowledge on the other (the symbolic).

There was an interesting disturbance to this structure during the fifth session of the course. While it was but one instance, we include it here as it provides additional insight into the imaginary-symbolic relation in the constitution of mathematics for teaching. The topic for the week was "communities of learners … creating a community in the classroom" and it begins with the class watching a video "of someone who it trying to do this". As in previous sessions the resources for interrogating practice here are a video extract (and an accompanying transcript), and three relevant readings. Four questions are provided to structure discussion: "1. What <u>mathematica</u>**l** work is the teacher doing? 2. How is he <u>teaching</u> them to be a community? 3. To what extent is he successful. 4. What would you do differently".

For the next twenty or so minutes teachers offer what they think the teacher in the video is doing to create a community. Teachers' responses include statements like: "he is encouraging them to participate … he says 'feel free to participate' "; "the teacher is a facilitator"; "the teacher is democratic"; "the learners are actively constructing knowledge". The lecturer responds by asking that they point to what they see as evidence of their claim or assertion in the transcript. The lecturer pushes teachers to invest their utterances, utterances which proliferate in new teaching and learning discourses, with meaning, and particularly practical meaning as revealed in another teacher's practice. For example:

> L: … that is not enough evidence for me – they could be talking about soccer – how do you know they are actively constructing ***mathematical*** knowledge – anyone else got other evidence?

Included in this lengthy discussion are sceptical voices, that there is "noise in the classroom", suggesting that the teacher is not in control of his lesson, that "some learners are not involved", that the discussion in the tape is "time-consuming" and learners appear "confused". These too are common in teacher discourses around curriculum reform, and the lecturer pulls all of this into focus:

> L: … developing a mathematical community takes time; having mathematical conversations takes time, there is no doubt about it, it takes longer. The argument is that it leads to better mathematics in the end … These learners spent fifteen minutes being confused about the mathematical concepts and that is not a long time (and she refers to mathematicians and how long they spend being confused about new ideas and continues…). You could just tell them, but will this remove or eliminate all confusion? …

> L: The point is, I would like you to be able to make a choice. OK … this mathematical conversation and mathematical community is not something we were trained to do. It is part of the new curriculum, … of the new order in mathematics, because people do believe it will lead to better mathematical learning. There may be many reasons why you can't do it, number one being a heavy syllabus and assessment, but then at least you are making a choice and you know why you are making it, you know what it looks like. And it is possible, perhaps not possible all the time, but it is possible, and I know there are people in this class who are doing this. He (the teacher in the video) is also by the way, a Grade 11 teacher, and this is … not a very strong class, and this teacher feels the same pressures, and he doesn't do this all the time. Time is an issue.

And she returns to focus on the teacher in the video, and how he is "building a community" and this is the point (time for the session is running out) at which the readings for the session are brought into focus:

> … he apologises to a learner for interrupting her … "sorry to break your word" …. He is modelling that it is not polite to interrupt someone. … he is trying to model what it is that he wants the learners to be doing with each other and with him. … The very, very important thing is that teaching learners to be a mathematical community requires mathematical work and that was Maggie Lampert's article … a long one … hello … the one you read for this week. The long one? (Laughter) "Teaching to establish a classroom culture". …

There is then a relatively brief discussion of Lampert's paper, and elements of a second paper read, with a focus on how Lampert describes her own teaching to build a community of mathematical learners. A pedagogic discourse that up to this point legitimated its utterances largely by reference to a discursive field (and categorised as 'mathematics education' in Table 1 below), is now predominantly focused on the image recruited for this session – the record of practice (and categorised as 'experience' in Table 1). It provokes a host of utterances from popular pedagogic discourses, and it is interesting that here the lecturer recruits the field of mathematics education only at the very end of the session. Two inter-related explanations follow. Firstly, the reform jargon that the teachers offer (participation, active construction …), and its oppositions (time consuming, confusing), needs to be engaged. Focusing on a recognisable image (a teacher in a familiar context), makes pedagogic sense: it offers the practical possibility of the proposed practice. Secondly, the field here (community of practice) is itself still weak, rendering it less effective as a discursive resource. The resulting pedagogy is a form where the imaginary is privileged over the symbolic. This is somewhat inevitable, and so a challenge to the practice of mathematics teacher education.

Table 1 categorises and summarises the appeals made for legitimating the texts within this pedagogic practice[4]. Evidence for our description of the practice to be acquired lies in the table. In the total of thirty-four events across the course, thirty-one (91%) direct appeals are made to mathematics education texts. We also note from the table that there is a spread of appeals across possible domains, reflecting the complex knowledge resources that constitute teaching for mathematics. Our analysis of this data leads to the following claims. Firstly, appeals to the metaphorical and the authority of the lecturer are, however, low, suggesting that mathematics is presented as a reasoned activity, and interrogation of practice is through the field of mathematics education. Secondly, the relatively high percentage of appeals to experience, together with appeals to mathematics education shows a different kind of identification in operation (contrasted with the other two courses we have studied). Finally, we noticed with interest that in this course, there are 95 appeals across 34 events. This is considerably different from the 45 appeals across 36 events, and 74 appeals across 36 events respectively in the other two cases. We suggest that the density of appeals is a key feature that reflects the different practices in these three courses, at the same time that all three recruit the image of the teacher.

	Mathematics	Mathematics Education	Metaphorical	Experience of either adept or neophyte	Curriculum	Authority of the adept
Mathematics	5	6	5	0	0	0
Proportion of appeals (N=16)	31,3%	37,5%	31,3%	0%	0%	0%
Teaching	15	25	0	23	10	6
Proportion of appeals (N= 79)	19%	31,7%	0%	29,1%	13,7%	7,5%
Mathematics & Teaching	20	31	5	23	10	6
Proportion of appeals (N=95)	21,1%	32,6%	5,3%	24,2%	10-,5%	6,3%
Proportion of events (N=34)	58,8%	91,2%	14,7%	67,7%	29,4%	17,7%

Table 1: Distribution of appeals in Case

[4] The six categories emerged from an interrogation of the data. We have already indicated the kind of appeals that have been coded as either *mathematics education* or *experience*. Briefly, a coding of *curriculum* was indicated when the legitimating resource was, for example, taken from a text book; *metaphorical* was the coding when, for example, everyday experience was the legitimating resource. An assertion like ‘this is a good example’, without any other justification was coded as an appeal to *Authority*. See Davis et al (2005).

CONCLUDING REMARKS

In the case presented here, the imaginary-symbolic relation is one where the image of teaching (the local and practical) is strong, but at the same time subordinated to an interrogative discursive space that is principled and theoretical: the image was always interrogated by way of appeals to mathematics education as a discursive field. We also noted the density of appeals in this course, a function, we propose, of a pedagogy where images of practice (of other teachers, and the teachers themselves), are constantly and explicitly interrogated, distanced from the lecturer, and positioned under the aspect of the symbolic.

Two final comments are apposite here. Firstly, the records of practice deployed in this course are not widely or readily available (relevant video tapes of local teachers in practice, with accompanying transcripts), with implications then for large scale education of teachers. Secondly, the disruption of the practice when the idea of a (mathematical) *community* was inserted, and the difficultly of preventing the imaginary from over-asserting itself, begins to throw a new light on the complex demands on pedagogical reform in through mathematics teacher education.

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RELATIONSHIP BETWEEN PRE-SERVICE MATHEMATICS TEACHERS' TEACHING AND LEARNING BELIEFS AND THEIR PRACTICES

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The purpose of this study was to determine pre-service mathematics teachers' teaching and learning beliefs and examine the relationship between their beliefs and practices. Qualitative and quantitative research methods were designed for this study. Survey, semi-structured interviews, observations, and pre-service teachers' written documents such as school practice portfolios were used to collect the data. Under the developed theoretical framework, it was found that some of the pre-service mathematics teachers' beliefs were consistent with their practices; while some of them presented different practices from their beliefs.

INTRODUCTION

Teachers' beliefs related to instruction have direct effects on their classroom practice; therefore, they have been a focus of attention in a large amount of research (Block and Hazelip, 1995; Hoban, 2003; Kagan, 1990; McDiarmid, 1995; Peterman, 1993; Thompson, 1992; Woolley & Woolley, 1999). Stipek, Givven, Salmon & MacGyvers (2001) emphasize that influencing teachers' beliefs is important to be able to change their classroom practice. If the purpose is to shape teachers' practices, their beliefs should be examined at the earliest stages in their professional development especially during their pre-service teacher training. Therefore, this study attempts to determine pre-service mathematics teachers' teaching and learning beliefs and examine the relationship between their beliefs and practices.

BELIEF AND BELIEF SYSTEMS

Kagan (1990), defines teacher belief as "the highly personal ways in which a teacher understands classrooms, students, the nature of learning, the teacher's role in the classroom and the goals of education" (p.423). Richardson (as cited in Woolley & Wooley, 1999, p. 3) gives three sources of teacher belief: a) personal life experiences which shape a teacher's world view, b) experiences as a student with schooling and instruction, and c) formal knowledge including pedagogical content knowledge. Gates (2005) emphasized the social dimensions of the sources of teachers' belief.

Fishbein & Ajzen (1975) define hierarchy of beliefs as a belief system. Green (1971) categorizes belief system as the following three dimensions: primary and derivative beliefs (primary beliefs are independent from other beliefs while derivative beliefs are the consequences of primary beliefs), central and peripheral beliefs (central beliefs are the ones that are most strongly hold and peripheral beliefs are the ones that are susceptible to change), and beliefs in clusters which might be isolated from each other.

According to Thompson (1992), belief systems are dynamic, permeable mental structures, susceptible to change in light of experience" (p. 149). In other words, "teacher beliefs and belief systems are grounded in their personal experiences and, hence, are highly resistant to change" (Block & Hazelip, 1995, p. 27). It can be derived from the literature that teaching and learning beliefs emerge from personal experiences and can be changed by having the related experiences.

RELATIONSHIPS BETWEEN BELIEF AND PRACTICE

There is ample research on the relationship between teachers' beliefs and practices (Kagan, 1992; Kane, Sandretto & Heath, 2002). Other research investigated pre-service and in-service teachers' mathematics related beliefs as they are central to the belief-practice relationship (Raymond, 1992; Andrews & Hatch, 2000). Research demonstrates the general inconsistency between pre-service teachers' pedagogical views of teaching and their classroom behaviour (Raymond, 1997). It is suggested that future studies should seek to elucidate the dialectic relationship between teachers' beliefs and practices (Thompson, 1992).

PURPOSE OF THE STUDY

The purpose of this study was to determine pre-service mathematics teachers' teaching and learning beliefs and examine the relationship between their beliefs and practices. Pre-service teachers' beliefs in relation to practice were investigated under the theoretical framework developed from the previous research.

THEORETICAL FRAMEWORK

In the literature, teachers' beliefs related to instruction are categorized mainly as traditional and constructivist. Traditional belief is "based on a theory of learning suggesting that students learn facts, concepts, and understandings by absorbing the content of their teacher's explanations or by reading an explanation from a text and answering related questions" (Ravitz, Becker & Wong, 2000, p.1). Constructivist belief, on the other hand, is "based on a theory of learning suggesting that understanding arises only through prolonged engagement of the learner in relating new ideas and explanations to the learner's prior knowledge" (Ravitz et al., 2000, p.1).

The theoretical framework of this study is based on the research done by Haney and McArthur (2002) where they investigated constructivist and behaviourist beliefs in relation to practice. They categorize constructivist beliefs as core beliefs which are enacted in the practice, and peripheral beliefs which are stated but not enacted in the practice due to external factors such as lack of resources in the schools. They present a further categorization of core beliefs as constructivist, conflict and emerging core beliefs. Constructivist core beliefs are the constructivist beliefs that are put into practice. On the other hand, conflict constructivist beliefs are those beliefs that are enacted in the practice, but are in opposition to constructivist theory (e.g. believing in hands-on student inquiry but relying on heavy lecturing). Emerging core beliefs are the ones that are both stated and put into practice but are not directly related to the constructivist practice (e.g. believing that good teachers are caring). Our theoretical framework which was extended from Haney & McArthur's (2002) framework is summarized in Figure 1 below. A category called transitional was considered to investigate beliefs in which neither constructivist nor traditional beliefs are dominant.

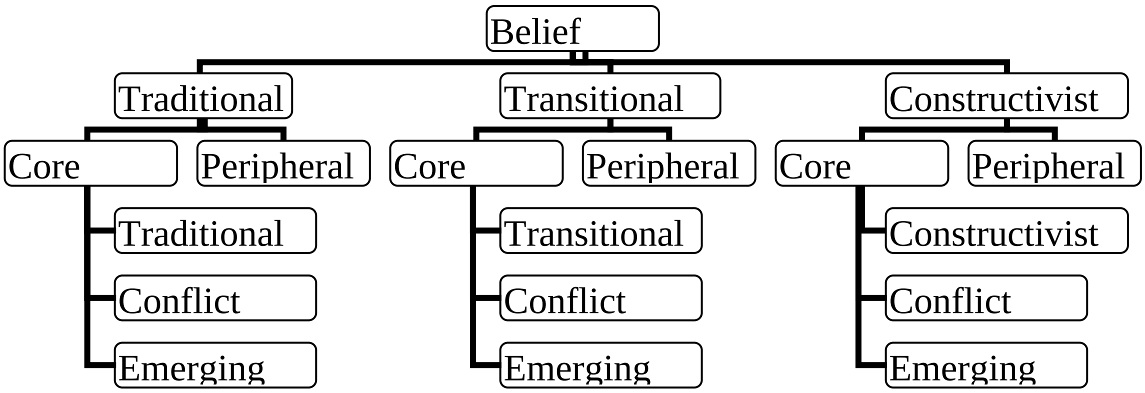

Figure 1. The categorization of beliefs in relation to practice

METHODOLOGY

Qualitative and quantitative research methods were designed for this study. The data was collected by using various instruments such as survey, semi-structured interviews, observations and pre-service teachers' written documents such as lesson plans and school practice portfolios.

Participants and setting

Participants of the study were 58 pre-service mathematics teachers attending the mathematics teacher education program. The age of the participants ranged from 20 to 25 and 45 % of them were female. Then, six pre-service teachers were selected to examine the belief-practice relationship deeply. The data was collected in "Instructional Methods in Mathematics-I" and "School Practice" courses.

Survey research

In order to examine pre-service teachers' beliefs, modified version of the TLC (Teaching, Learning, and Computing) survey developed by Becker & Anderson (1998) was used. As discussed in the literature review, teachers' beliefs are mainly categorized as traditional and constructivist. The new category called transitional beliefs was developed and pre-service teachers' beliefs were examined in five categories; traditional, close to traditional, transitional, close to constructivist and constructivist.

Interview

On the basis of the results of the survey, two participants from each belief category (constructivist, transitional and traditional) were randomly selected to be interviewed.

The interviews were semi-structured and had two purposes. The first purpose was to examine pre-service teachers' beliefs. The second purpose was to discover how pre-service teachers prepared for their teaching practices in schools. Therefore the structure of the interviews had two parts. The first part consists of stimulated-recalls which required pre-service teachers to talk about their preparation and evaluation of the lessons in the school practice. The second part included questions about classroom environment, planning of teaching activities, assessment, the role of a teacher in the classroom and instructional goals to reveal participants' beliefs.

Observation and Written Documents

In order to evaluate pre-service teachers' teaching practices, six participants were observed in their method courses and school placements as they were teaching. To analyze the data from the observations, Greer et al.'s (1999) Constructivist Teaching Inventory was used. This inventory is composed of 44 items in four clusters: community of learners, teaching strategies, learning activities, and curriculum and assessment. The data from the observations was triangulated with the written documents and interviews.

RESULTS

The analysis of the data from the survey, interviews and observations were summarized in table 1 below. Letters were used to refer to the participants who were selected from different belief categories considering the results of the survey.

	Beliefs		Beliefs in relation to practice	
Participant	Survey	Interview	Method course	School setting
A1	Close to constructivist	Constructivist	Constructivist core	Constructivist core
A2	Close to constructivist	Constructivist	Constructivist core	Constructivist core
B1	Transitional	Transitional	Transitional core	Transitional core
B2	Transitional	Close to constructivist	Constructivist conflict	Constructivist conflict
C1	Close to traditional	Transitional	Transitional conflict	Transitional conflict
C2	Close to traditional	Close to traditional	Traditional core	Traditional core

Table 1. Comparison of the participants' teaching beliefs and practices

The columns named as method course and school setting represent the relationship between belief and practice based on our theoretical framework. Categories of beliefs in these columns were determined by comparing the participants' beliefs revealed from the interviews to their practices determined by observing practices in the method and school practice courses. For instance, C1 has transitional beliefs as determined from the interview and his practices were observed to be traditional;

therefore, his belief in relation to practice both in the method course and school setting were determined as transitional conflict.

As seen in table 1, two of the pre-service teachers (A1 and A2) who were selected as being constructivist on the basis of survey results were also constructivist in the interview. They stated the following:

A1: I'd choose which method to use according to the topic. I would apply discovery, and computer assisted methods and use concept maps. Some topics are more appropriate for these methods such as functions and absolute value, but not polynomials and logarithmic functions…teacher's role in the classroom shouldn't be the leading one, he is like a secret hero.

A2: A teacher should be the facilitator when it's a suitable topic for students to discover for themselves… knowing why rules work is important…students should be active and investigate.

Both of them were also constructivist in their teaching practices. To illustrate why the practice of A1 was considered as constructivist, a brief account for his practice in the school setting will be given here. He taught absolute value in one of his lessons in the schools. He started his lesson by reminding prior knowledge such as number line and being non-negative. He tried to draw students' attention to the difference between the terms distance and length. He helped students to relate these to being non-negative. After giving the definition, he illustrated examples by asking students to express the algebraic expressions of absolute values in the colloquial language. He also encouraged students to express the algebraic expression on the number line.

The practice of A2 was also considered as constructivist. In one of her school placements she taught induction. She started her lesson with the story of Gauss and explained how he found out the sum of the numbers up to 100. She noted that generalization from specific cases may not always be true. She gave the example of Fermat's prime numbers as $F_n = 2^{2^n} + 1$ and mentioned that the induction method was needed to prove such statements. While explaining the method of induction, she gave the example of dominos. In her reflection report of the lesson, she wrote that:

A2: I had two choices to teach induction. One like the way the textbooks do with the definitions in formal notation, secondly by using colloquial language to give meaning to the notation. I chose the second way because if I had chosen the first way then it would have been too abstract.

Practices of these two pre-service teachers were constructivist which reflected their beliefs; thus, they were considered in the category of constructivist core.

B1 was selected as having transitional beliefs based on the survey results. Her beliefs were also considered as transitional based on the interview transcripts. For instance, in terms of classroom environment, she stated that:

B1: Students should listen and understand what is taught…they should participate and they can correct my mistakes.

Her practices in the method course and school practice reflected her transitional beliefs; hence, her beliefs in relation to practice were considered as transitional core.

Although B2 was selected as having transitional beliefs based on the survey results, his beliefs were analyzed as close to constructivist. In the interview, he stated that:

> B2: I would choose the teaching method according to the topic and the students' level. I would use group work, experiments, demonstration boards…I wouldn't teach directly. Students should try for themselves.

However, his practices both in the method course and school setting were not constructivist. For instance, in the school setting he heavily relied on giving definitions and rules followed by examples which aimed instrumental understanding as pointed out by Skemp (1978). Although, he tried to relate mathematics to real life, he did not assess whether students could make this relationship. Consequently, his beliefs in relation to practice were considered as constructivist conflicting.

Although C1 was selected as having traditional beliefs based on the survey results, his beliefs were determined as transitional in the interview:

> C1: I wouldn't want students be so quite or so noisy. I want them to participate….One to one interaction is important in the classroom….I want to share their problem…I would choose questions at different levels.

However, his practice was traditional. For instance, in the school setting, when he was teaching probability, he mostly relied on using rules. When students asked the reason why they multiplied the probabilities instead of adding, he mentioned that it was because there were rules for this. In the interview, when he was asked the reason for his answer to the students, he explained that he did not know the answer.

C2's belief was considered as traditional. In the interview she said that:

> C2: The teacher should teach thoroughly, not quickly…Group work becomes chatting…Meaningful learning is important but curriculum should also be followed.

Her practices reflected her traditional beliefs. For instance, when she was teaching probability in her school placement, she heavily relied on applying rules without reasons in the sense of instrumental understanding of Skemp (1978). She reacted negatively towards the different solutions from the students as she deleted a student's solution on the board and wrote her solution instead.

CONCLUSIONS

The following conclusions can be drawn from this research. First, results showed some inconsistencies between pre-service teachers' teaching and learning beliefs and practices. In this paper, these inconsistencies were described on the basis of our theoretical framework developed from Haney & Mc Arthur (2002). The data indicated that declared beliefs might not be enacted due to the various reasons such as lack of subject knowledge and the complexity of classroom environment. For example, one of the participants who believed in active participation of students could not put his beliefs into practice. This might be because surviving in a chaotic classroom environment requires pedagogical skills and experience.

As a second conclusion, pre-service teachers believe that teaching approach should be determined on the basis of the nature of the mathematical topic. As one of the pre-service teachers who held constructivist beliefs stated that functions and absolute-value could be taught using discovery methods and in a technology-rich environment while polynomials and logarithmic functions could only be taught by heavy-lecturing. More research is needed to investigate topic specific beliefs.

Finally, constructivist or traditional beliefs tend to be more consistent with practice. In other words, pre-service teachers who held constructivist or traditional beliefs have core beliefs in their practice.

The data revealed more comprehensive categories than the ones in the theoretical framework developed by Haney & McArthur (2002). Similar to the research done by Authors (2005), this study observed some of the categories (such as traditional core, transitional conflict and transitional peripheral) in our extended theoretical framework. Further studies need to be conducted for other categories which were not observed in this study. The theoretical framework might have significance for other studies which aim to change teachers' beliefs.

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TEACHERS' AWARENESS OF DIMENSIONS OF VARIATION: A MATHEMATICS INTERVENTION PROJECT

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This paper describes the findings of a 16-month longitudinal teaching intervention exploring how deliberate and systematic variation can be used to raise awareness in teaching and learning situations. The results indicate that intervention teachers attending to variation produce significant learning benefits for their students.

INTRODUCTION

This paper reports the findings of a 16-month longitudinal intervention called the *Dimension of Variation Programme (DVP),* investigating the *deliberate* and *systematic* handling of content and its consequences in the teaching and learning of mathematics. Informed by the findings of Runesson (1999), Leung (2003), Watson and Mason (2004), and Marton and Tsui (2004), this study widens the application of a theoretical framework to a new area of mathematics.

The work of Marton (e.g. Marton and Booth, 1997) has drawn on research results spanning over 25 years culminating in an outlined theory on learning and awareness called *variation theory*. The theory can be summarised as 'if one aspect of a phenomenon or event varies while another aspect or aspects remains unchanged, the varying aspect will be discerned'. The part of the content that varies is called the *dimension of variation*, hereafter referred to as *DoV*. Watson and Mason (2004) have defined the variation within a *DoV* as the *range of change*, hereafter referred to as *RoC*.

- For example, in the expression $x + 3$, one of the possible components that can vary is the addend (others include: the letter representing the variable, the operator and the order in which the variable and constant appear in the expression) hence this is a *DoV*. The values that the addend can take (i.e. natural numbers, negative numbers, rational numbers and so on…) would constitute the *RoC* of the *DoV*.

Mathematical content which appears superficially to be the *same* can have different characteristics for different students when being taught by different teachers (Marton & Tsui, 2004, p.35). Teachers teaching the same formal curriculum 'handle it' according to their ways of understanding the subject as a whole. The meanings that learners construct are inextricably linked to the teaching approaches used (Sutherland, 1987), and the material with which they interact (Brousseau, 1997). The qualitatively different ways situations can be experienced are related to the variation in the individuals' perception of the experience. It is therefore possible to gain new insight into areas of mathematics that are familiar to you when the situation in which you experience the circumstances differs with respect to time, place, approach

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(algebraic as opposed to geometric), discourse with others and so on. Hence, awareness has a dynamic structure and, consequently, the variation that students experience has a bearing on their learning. Deliberate and systematic use of variation can be seen as a teaching tool to raise awareness of mathematical structure.

JUSTIFICATION FOR THIS APPROACH

All teachers naturally use variation in their teaching, either deliberately or accidentally. For the purposes of this study, it was assumed that no single 'standard approach' that used variation in a deliberate and systematic way to handle the algebra content existed. Instead it was recognised that all teaching would exhibit variation which could be distinguished by frequency and type. Hence, in order to determine whether there were noticeable differences between this continuum of 'other' approaches and one where there was an awareness of *DoV*, it was not a simple matter of identifying and observing two sets of classrooms where contrasting approaches were being employed. Instead, a specific intervention, the *DVP*, was designed.

INTERVENTION

The *DVP* was not of the form of giving tasks, material or instruction, but of providing information, discussion and support opportunities around variation theory. The intervention teachers were committed to the use of variation as a means for handling the content of an elementary algebra lesson. The *DVP* was designed to encourage direct manipulation of expressions, making the process familiar and, as Lins (1994) would say, 'senseful'. This might lead to students appreciating generalities for themselves. The desire to then articulate these generalities creates the need for algebra (Brown and Coles, 1999). The field of algebra was chosen because the theoretical position that awareness is dynamic in structure had clear parallels to the following definition of algebra (Pimm, 1995):

> Algebra is about transformation. Algebra, right back to its origins, seems to be fundamentally dynamic, operating on or transforming forms. It is also about equivalence; something is preserved despite apparent change.

One reason suggested for the failure of students to acquire the desired understanding of algebra has been the inappropriate way in which the concrete and symbolic representations can been related to one another. The *DVP* can endow the elements of manipulative algebra with *meanings* that enable them to be applied to the solution of entire families of problems. For example, in $y = \mathbf{m}x + \mathbf{c}$ you could choose $\mathbf{c} = 1$, but until you augment this with other examples, the mathematical description of the classes of objects or relations between them remains latent for the learner. If the equation is handled so that **m** is varied systematically while **c** is kept invariant then the learner has an increased chance of attributing **c** as determining the y-intercept of the straight line. The intervention acknowledges that understanding develops and evolves relative to particular representations. Thus, for example, there is no *static* absolute meaning for the mathematical word 'variable' but rather a whole web of meanings built out of the many experiences of functions that student has had.

Students do not necessarily conceive situations in the same way the teacher does. For a student, recognition of the inherent properties of a particular system of representation significantly assists the understanding process. Properties must be recognised for their benefits to be available. Work by Stenning & Yule (1997) has shown that humans display substantially individual variation in their ability to recognise, and thus exploit, the benefits of particular representations. By deliberately controlling the variation offered in progressively developed examples the teacher focuses the learners' attention on particular aspects of the algebra, hence increasing the chances of common experiences. Carefully selected and presented generic examples can provide algebraic experiences that develop both manipulative abilities (Bell, 1996) as well as generalisations of structure rather than surface features (Bills & Rowland, 1999).

- Example 1: the series of equations: $x + 1 = 6$, $x + 2 = 6$, $x + 3 = 6$, $x + 4 = 6$ might illustrate to the learner that the variable in this case is an unknown and the value of this unknown can vary from question to question.
- Example 2: the series of equations: $x + x = 2x$, $x + 2x = 3x$, $x + 3x = 4x$, $x + 4x = 5x$ might illustrate to the learner how *like* terms are added together.

The *DVP* followed students through Year 7 and Year 8. It was a quasi-experimental design with 10 teachers and a total student cohort of 300 students. Stratified sampling was used to produce two groups of classes and to reduce the presence of confounding variables. While all teachers inevitably offered students dimensions of variation, the difference between the experimental groups was that the intervention teachers did so in a consistent, aware, *deliberate* and *systematic* manner while comparison teachers were not offered opportunities to develop this awareness. The longitudinal nature of the study produces multiple time-reference points. These provide an opportunity to explore how the temporal characteristics of the intervention influence the students learning. This feature is significant for two reasons. Firstly, the effects of an intervention may be more marked during certain periods than others (Jordan, Kaplan, & Hanich, 2002). Secondly, a student's *growth rate*, which is crucial to understanding learning and learning development, cannot be accurately ascertained when using only one time point measurement (Francis et al. 1994).

DATA

Quantitative data in the form of pre-, post- and delayed post-tests using standardised national exams was collected for each student. The average time between pre- and post-tests was 16 months. After a subsequent 2 months a delayed post-test was administered. It would be over-simplistic to assume the intervention only happened between the pre- and post-tests as the experimental teachers' understanding of variation theory evolved and continued to develop up to and beyond the delayed post-test. This data was augmented by two forms of qualitative data. First, each teacher was observed for a total of 8 lessons, 4 consecutive lessons at the beginning of the study and 4 consecutive lessons towards the end of the study. Second, two target students from each of the 10 classes were clinically interviewed. Each student was interviewed twice using the same questions from the CSMS study (Kuchemann, 1981), near the times of the pre- and post-tests. While the quantitative data helps to establish whether or not the intervention has ‘worked’, the qualitative data can be used to conjecture the reasons *why*, and through which *mechanisms* it has worked.

ROLE OF TEACHERS

Both the comparison and intervention teachers retained overall control of their teaching style as well as the content of what was to be taught. The intervention group together with the researcher formed a micro-research community, meeting regularly every few months during which time *variation theory* was introduced and discussed. Critical aspects of the content were collectively identified and through negotiation it was decided which *DoV* would be varied deliberately and systematically.

RESULTS

Table 1 below shows a summary of the adjusted post- and delayed post-test marks after controlling for the pre-test marks. Due to partial or missing data, the total number of students used in the analysis was less than 300.

Total marks on	Experimental Group	N	Mean	SD	Std Error Mean
Post-test	Comparison	89	71.81	27.89	2.96
	Intervention	138	77.68	17.51	1.49
Delayed post-test	Comparison	87	70.36	31.95	3.43
	Intervention	138	87.15	18.79	1.60

Table 1.

Two independent t-tests were carried out on the mean post- and delayed post-test scores. This analysis produced a significant difference between experimental groups for the post-test [$t(225) = -1.95$; $p < 0.05$], however there was only a small effect size $r = 0.13$. There was also a significant difference between experimental groups for the delayed post-test [$t(223) = -4.97$; $p < 0.001$], with a medium effect size $r = 0.32$. Two multiple regression models were generated in order to investigate the temporal effects of the intervention. The first used the post-test marks as the outcome measure with the pre-test marks and the experimental group as predictors. While this model did suggest that the experimental group had a significant effect on the outcome measure, at $p < 0.05$, the related effect size, d, was only 0.13. The second model used the delayed post-test marks as the outcome measure with the experimental group, the pre- and post-test marks as predictors. The results of this model are below.

Outcome measure	Marks on delayed-post test		
	Coefficient of linear model, *B*	SE *B*	Beta
Marks on pre-test	0.28	0.08	0.25
Marks on post-test	0.65	0.08	0.55
Experimental group	12.84	2.05	0.24

Table 2. Multiple regression in which the outcome measure was the marks scored on the delayed post-test, $p < 0.001$

In this model the experimental group again had a significant effect on the outcome measure, at $p < 0.001$, and represented a large effect size with $d = 0.51$. The respective outcome variables were chosen because they were considered those most linked to the developmental improvement of the students at those particular times during the intervention.

SYSTEMATICITY

Field notes recorded three types of data during each lesson; general contextual information such as: the number of students in the class, the time of the lesson, classroom layout, and so on; resources and tasks used; and occurrences of variation. The instances at which variation was offered and generated were the primary focus of the observation, and as such, great care was taken to record them with accuracy. Structured observation was used to record the variation in the lesson. Each lesson was audio-recorded and later transcribed. This data supplemented the field notes to ensure all the details pertaining to the variation in the lessons were captured. It provided both quantitative information on the frequency of the variation as well as qualitative information on its nature, type and degree.

The quantitative data showed that in both experimental groups there were some lessons in which the frequency of variation offered by the teacher was greater than that generated by their students; and other lessons where the reverse was true. A Kruskal-Wallis test was carried out on the mean variation offered by teachers. This analysis produced no significant difference between experimental groups [$H(1) = 0.046$; $p < 0.05$]. This evidence suggests that teachers across experimental groups do not use different amounts of variation and that there must be something else that the intervention teachers do to produce significantly better delayed post-test results. Any suspicion that this effect is due solely to the professional development atmosphere rather than their deliberate and systematic handling of the content and attendance to variation is dispelled when the qualitative data is considered. This data shows that in some cases the variation offered and generated seem to be independent of each other in nature. That is, students' sense of appropriate variation bears little relation to what teachers are trying to convey. In the intervention classes the variation offered and generated are 'in phase' with each other. They are more reciprocal in nature. This is perhaps because the intervention teachers are more aware that the variation generated by the students is indicative of what the students pertain to be critical in the learning situation. The excerpt below is presented as an illustration of this conjecture. The intervention teacher is explaining how to solve the simultaneous equations (1) $3y + 2x = 7$ and (2) $2y - x = 0$

1	T:	If the operator in (1) was a minus what would you have to do?
2	S1:	Minus them
3	T:	Why?
4	S2:	Because we are trying to get rid of one of the variables
5	T:	We are trying to get rid of either *x* or *y*
6	S1:	How do you know which one to cancel out?
7	T:	It doesn't matter, what's important is that you understand you can get rid of either

The teacher subsequently solves for y by eliminating *x*

8	T:	what do I do now?
9	S:	Sub into (1)
10	T:	We could sub. into any of them, but we usually sub. into one of the originals in case we made a mistake in the multiplying

The teacher then demonstrates this by substituting into both (1) and (2), in the process illustrating that it is easier to substitute into (2). This elicits:

11	S:	Subbing into (2) is easier because there is less work to do

In response to line 6, the teacher then solves for *x* by eliminating *y*. The teacher keeps control of the variation in a deliberate way by using the same equations but now solving for the other variable.

12	T:	You don't have to stick to multiplying just one equation. If necessary you could multiply both equations, what would I do if I wanted to get rid of the *y*'s?
13	S:	Times (1) by 2 and (2) by 3

The desire to describe how the type, nature and degree of variation impacted the handling of the content necessitated the development of a new analytical tool. The *systematicity* of the variation can be used as a tool to analyse beyond the existence of dimensions of variations. Considering how the systematic nature of presentation of variation differed from teacher to teacher gives us the ability to describe sets of instances in more detail, hence facilitating discrimination on a higher level. Systematicity is different from *RoC* here in that it is sensitive to the number of examples presented. One *good* example can illustrate the *RoC* along a *DoV* (in the sense that it can encapsulate the potential along a particular DoV), but a sequence of examples is needed to discern the *degree of systematicity*. The degree is the number of embedded variations that are actively or deliberately being illustrated.

Intervention teachers are not systematic with the *RoC* of *DoVs* that are already well understood as this would be time consuming and inefficient. The systematicity of intervention teachers is illustrated in two different ways:

- The systematic handling of the *RoC* of a few selected *DoVs* pertinent to the teaching aim
- When a new *DoV* is generated or the students ask a question showing confusion relating to a particular *DoV* or its *RoC* then the teacher responds with a deliberate and systematic exposition of it.

GENERAL DISCUSSION

In this study an intervention designed to raise awareness through the use of deliberate and systematic variation was introduced in order to measure the effects, if any, on teaching and learning. The students' performance on three standardised national tests and the relationship between this performance and classroom observations leads to two main conclusions.

The first conclusion, as evidenced by amongst others the Kruskal-Wallis test, is that the intervention teachers handle the content differently. This was also the conclusion of an earlier paper by Runesson (1999). Further, an increased awareness of variation amongst intervention teachers was reflected by an increase in the *systematicity* of their teaching. Systematicity seems to explain the relevant regularities in the handling of the mathematical content by them.

The second conclusion is that students involved in the intervention programme performed significantly better than non-intervention students. The longitudinal data showed that the relationship between increased awareness of variation and understanding (test performance being just one of the criteria used to infer increased understanding) holds over an 18-month period. The smaller difference between experimental groups in the first 16-months of the study suggests that other factors may have exerted an influence while the intervention was still taking root. The subsequent finding that this connection was more pronounced over the last 2-month period indicates that the intervention had a more powerful influence in the long run.

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THE STUDENT TEACHER AND THE OTHERS: MULTIMEMBERSHIP ON THE PROCESS OF INTRODUCING TECHNOLOGY IN THE CLASSROOM

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This paper presents part of a study on a pre-service mathematics teacher use of technology in the classroom during his initial year of teaching practice. It draws on the idea that the student teacher learning and developing process is situated, mainly in the ways of participation in communities of practice and in the need to articulate and negotiate different repertoires. From this point of view, the data presented offer a perspective on how the university, the school, the mathematics department and the group of pre-service teacher training (student teachers and co-supervisors) frame and account for the process of integrating technology.

INTRODUCTION

As a general background, this study encompasses the point of effectively introducing information and communication technologies in mathematics classes.

One consensual issue is the understanding of the crucial role that teachers play within the process of integrating technology in the classroom. This has been a strong argument to urge for further preparation of future teachers on the use of technologies. At the same time, several studies have shown strong resistance from a large number of teachers to the integration of technology in their classes (Kaput, 1992; Mariotti, 2002). The case of Portugal is paradigmatic considering the compulsory use of graphic calculators and the steady recommendation to introduce computers in mathematics secondary classes contrasting with recently collected indicators on considerable teachers' uneasiness and even rejection.

This study is focusing not only on the possible barriers that the beginner teachers face regarding the pedagogical use of technology but mostly on the changes and experiences they come across in their initial year of teaching practice. By endorsing the profound social nature of such a developing process, this work tries to highlight the transformations and bouncing moves that occur as part of perceiving membership, participating in a constellation of practices and becoming a mathematics teacher.

THE PROBLEM AND THE METHOD

The purpose of the research is to provide knowledge and plausible explanations on how pre-service mathematics teachers integrate technologies in their classes during their initial year of teaching practice, considering that they are both graduate students at a university and teachers at a school on a supervised training.

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In this paper, we are focusing on one student teacher and we offer a reading of the way in which his activity and his discourse has changed and evolved in time, concerning the use of technology in mathematics teaching.

The pre-service teaching team was composed of two student teachers and two supervisors, one being a senior teacher from the school and the other a teacher from the mathematics department of the university.

The data presented were selected among a much larger set of information gathered throughout the whole school year and also a few months earlier when the student teachers completed their fourth year of undergraduate studies at the university. The selection of data was guided by the intention of offering a perspective of change over time and allowing a view of the positions assumed by the student teacher within multiple communities of practice.

The research takes on an interpretative approach and follows an inductive method in giving sense to the data in the light of the conceptual framework developed, namely focusing aspects of the student teacher's actions, discourse and ways of participation in different communities. A variety of records were produced and combined: (i) videos of classes with and without the use of technology, (ii) interviews to the student teacher in the beginning, middle and end of the school year, (iii) written reports of the student teacher, reflecting on his views on the use of technology, (iv) teaching materials and plans of the classes produced, (v) researcher's field notes of class observations, meetings with other teachers and visits to the school.

The empirical data are displayed and analysed considering the identification of relevant events and the discourses and practices of each of the involved communities.

The researcher was the supervisor from the university. As one of the members of the pre-service teaching team, she had direct access to the different settings where the student teachers acted, that is, the university, the schools and their mathematics departments. Participant observation was conducted along the school year, including active involvement in some of the classes where computers were used.

THEORETICAL KEY ELEMENTS

A theoretical framework that draws on learning as situated in communities of practice informs our point of view. The learning trajectories of novice teachers can not be detached from the context in which their activity takes place. In their initial year of teaching student teachers are involved in different practices and taking part in several communities like the school and the mathematics department, among others that may be conceived as defining a constellation of practices.

From this theoretical perspective, we want to pinpoint and briefly discuss the key concepts that are more directly related and significantly connected to the research problem: (i) *learning as participation in communities of practice*, (ii) *shared repertoire*, *mutual engagement* and *joint enterprise* as distinctive features of such communities, and finally (iii) the notion of *constellation of practices*.

Learning as participation in communities of practice

In intersecting learning and participation (Clark & Borko, 2004), the concept of communities of practice incites to look at groups of people who interact, work and learn together in terms of developing meaning, sense of belonging and mutual commitment (Wenger et al, 2002). Practice in a community is not homogeneous and allows different levels of participation, ranging from core to periphery. Pre-service teachers' trajectories within communities of practice, having supervision, guidance and experience at its heart, tunes in with the idea the newcomers join the community by participating in its practice (Wenger, 1998).

Features of a community of practice

Much in articulation with forms of interpreting mathematics teaching and learning, some distinctive expressions integrate a particular form of discourse owned by mathematics teachers. The use of a specific language is both a symbol of alignment within the group and a sign that expresses membership. Communities of practice have their own repertoires. They include language but also ways of acting, routines, tools, gestures and shared stories and concepts produced and adopted by the community over time. The repertoire integrates the discourse by which and through which the members create ways of seeing the world that are inherent to the community. A repertoire is ultimately an expression of a common identity.

Participation in a community of practice is not tantamount to collaboration. Still practice exists and works as the source of coherence of a community because it rests on the mutual engagement of its participants. "Practice resides in a community of people and the relations of mutual engagement by which they can do whatever they do" (Wenger, 1998, p. 73). Members mutually engage as they develop a shared practice, giving and receiving, and contributing often in vary diverse ways to a common enterprise that is defined and negotiated by the participants in the very process of pursuing it. Divergence and difference as well as tension and conflict are part of mutual engagement. Finding a way to do something together motivates negotiation and requires coordination of aspirations and points of view among the community members.

Constellation of practices

In a professional domain it is possible to identify a variety of communities of practice that intersect and overlap, creating what may be defined as a constellation of practices. Seeing several communities as forming a constellation does not depend on a pre-established rule but emerges from existing common members, mutual purposes, linked practices, shared resources and spaces, confluent discourses as well as related enterprises. The styles and the discourses are the most immediately available elements to export and import across the boundaries around the several practices coexisting in a constellation. The process of diffusion of discourses is fundamental to the continuity of the constellation. As they move across boundaries, discourses are susceptible to combine to form broader discourses once negotiation, coordination and reconciliation of perspectives are produced.

TAKING THE PHENOMENON AS SITUATED

In addressing the phenomenon from a situated point of view, the more significant settings elected to produce an understanding of the pre-service teacher's learning processes were the university, the school, the school mathematics department and the pre-service teaching team. The following is a sketch of some of the prominent features of those settings, according to the student teacher whom we shall call Tiago.

The university

Tiago described his academic undergraduate studies as generally well organised but found the courses on didactics of mathematics the ones that provided more fruitful experiences to his coming future as mathematics teacher:

> We have learnt to be in front of a blackboard and to face our classmates (since we had no actual students) and we learnt to prepare lessons and to relate our lessons to technology!

The supervisor from the university had been one of Tiago's teachers in his first year at the university. Later, in his fourth year, they had the chance to participate in several activities related to mathematics education. This allowed the supervisor to realise how Tiago was keen on the use of technologies and how he was competent and committed to working with ICT. Therefore, at the beginning of his training at the school, he was closer to the supervisor from the university than to his school supervisor, whom he did not know before.

The student teachers' activity, during their first year of school practice, runs between the school and the university. At the university they take a course on planning and assessing educational practice and are also assigned to write an essay on a mathematics topic. Both activities require students to be at the university one day per week, a condition that stresses their twofold position of students and teachers.

The school

From the outside and also inside, the school has a nice appearance; it is well organised and the staff of teachers is quite stable and solid.

For the mathematics classes there is a laboratory equipped with graphic calculators and computers with a few programs suitable for mathematics teaching.

The school mathematics department

The mathematics department includes nine teachers with a permanent position and the two student teachers.

Early in the school year the mathematics teachers of the school enrolled in a continuous education course and they decided to invite the student teachers to join them. There were several schools represented in the course and most of the activities proposed were developed in small groups. Tiago came out as the chosen speaker of his group and all the participants noticed his remarkable performance, especially the colleagues from his school. The teacher educator who ran the course also distinguished his work as very positive. This event reinforced the school teachers' will to accept and embrace the student teachers as their legitimate fellow colleagues.

Moreover the dynamics of the mathematics department and the cohesion between the members have always surrounded the two beginner teachers. Such an attitude was sensed and much appreciated by the student teachers who, in turn, tried to correspond and give their contributions.

The pre-service teaching team

The school supervisor regularly used the graphic calculator in her classes but seldom the computer. In turn, the university supervisor had in mind to encourage and to stimulate every opportunity that student teachers could have to work with technologies in their classes. Although the two supervisors were acquainted, they had never had any experience of working together that could have informed about each other's views on the matter. Initially, there was a certain caution_and some natural retraction, both trying to figure out the other's intentions and positioning. At this early stage both the student teachers and the supervisors tried, not always in an explicit way, to understand the others' perspectives, aims and concerns. A crucial circumstance in overcoming this phase was the university supervisor's statement on her commitment to participate in a regular basis in the work developed by the pre-service teachers in the school. She expected to co-operate in planning lessons and to be an active element in the classroom whenever it was considered appropriate.

STUDENT TEACHERS, TECHNOLOGY AND THE OTHERS

Predisposition towards the use of technology in the classroom

The year before his school teaching training, Tiago's words were as follows:

> (...) A special attention to the use of technology in mathematics teaching and learning is needed. Graphic calculators and computers with adequate software are important tools for the work in the classroom, making lessons less conventional, more discernible to the students and raising their motivation.

At the university, before starting his teaching practice, Tiago had the chance to simulate lessons where he introduced technological tools, adopting a pedagogical perspective on its use. Those experiences seemed to have enhanced his convictions on the value of technology as a resource for the classroom. At the beginning of his work as a teacher he claimed:

> Technology must be wisely integrated in mathematics teaching. Sometimes calculators can be misused. A good use of technology is one that helps students to understand mathematical ideas.

The entrance: Taking part in and being part of

The course in continuous teacher education in which Tiago took part was a first shift from a student's position to one where he participated in a purely professional group composed of leading mathematics teachers. Although he was a newcomer his attendance was noticed by the senior teachers over his active and relevant participation. He was impelled to move to a position of more centrality inside a professional group and within the community of mathematics teachers of his school.

Tiago was able to persuade the more experienced ones of his competence as a teacher, which represented a fundamental step in his migration from peripheral participation towards the centre of the community. While acknowledging the curricular orientations, the mathematics teachers of his school were unanimous in the idea that the use of technology in the classroom was an obstacle to cover the entire contents in the curriculum. The overall determination consisted in sticking to one lesson with computers per term and only in case time was felt to be enough.

Tiago has rather promptly accepted the opinion of his more experienced colleagues. He absorbed the discourse of the community to which he aspired to belong. The reproduction of such a discourse becomes a means to get closer to and to act as a member of the community. His alignment with the type of statements that circulated and his sharing of a repertoire that was contradictory with his previous dispositions may be seen as a way of looking for acceptance. Newcomers join the community with the prospect of becoming full participants in its practice.

Act I: Expectations and disappointment

On his first observed class, Tiago used a data projector and a laptop computer. He handed out a worksheet having several activities and he selected one of those to work on. The problem situation consisted of finding the path of an electrical string that should connect two plugs in a room shaped as a rectangular prism, given the locations of the plugs and the dimensions of the room and the length of string available.

To illustrate the various possible solutions, he used a PowerPoint file along with questions that were placed to the class. On the board he made records of suggestions and calculations coming from the class. Students considered this an interesting and motivating lesson and above all, an unusual one. Yet, they were not given the possibility of working with the computer and either the students or the teacher's role did not undergo any change compared to a traditional lesson. Besides it seemed to be a kind of practice that was quite distant from Tiago's ideas on how technology should be a tool to help students understand mathematics.

In the meeting after this class the university supervisor showed her disappointment in face of the use of the technology that was adopted. It prevented students from think by themselves on the problem and attempt to find the solution. This was the first meeting where concrete and real facts of the classroom organisation gave the opportunity to clarify and discuss the purpose and the ways of integrating technology in the classroom. A few days later he asked to be observed again in a class.

Act II: Reaction and reconciliation

The subsequent class took place in the mathematics lab. Students were divided in small groups for the 12 computers available. Tiago used the software “Geometria” to work on sections of solids. He gave students a quick guide of the software and a worksheet with several tasks. Tiago had the immediate collaboration of his student teacher partner and also of the supervisor from the university. Together they circulated in the classroom, supporting students' work on the computers.

Students were most receptive to the type of lesson developed and showed enthusiasm and commitment, leaving Tiago nicely surprised:

> I had never seen my students that much interested and that much involved all at once!

By thinking over the encouraging outcomes, the student teacher recognised how this way of using computers contrasted with what he had done before. The effort to change his prior experience in tune with the comments he received on the previous lesson was rewarded.

The full involvement of the university supervisor was a relevant aspect in terms of developing a climate of co-responsibility and mutual engagement. It also made clear to the student teachers the role and the positioning of the supervisor in the process of introducing technology in the classroom. Other important elements underpinning a sense of mutual engagement were students' reactions to the activities proposed and the positive consequences they had on their learning. Tiago's former conceptions on the advantages of using technology in the classroom gained a new foundation. From then on his discourse is refreshed with the experience and his convictions re-emerged in reconciliation with his initial predisposition. At the end of the first term he stated:

> I think that technology should not be used once in a while. I may do a series of lessons with computers and then have two or three weeks without using them, but just once in each term I won’t do it. I would use them even more often if the school had more and better software.

CONCLUSIONS

From the presented data it is visible how Tiago's practice in what concerns the integration of technology in the classroom was a process that involved different communities of practice. We witnessed a bouncing movement concerning technology use that expresses the multiple positionings of student teachers education trajectories. From the early expectations and claims, clearly in favour of taking technology as a tool for the learning context, Tiago moved to a discourse aligned with the presumable more knowledgeable ones. Afterwards, already in the field of practice, he develops two very distinct experiences. Even with opposite consequences, these contributed to the construction of his own way of introducing technology in mathematics classes.

The trend and the willing to the use of technologies may help the student teachers to react against some resistance when in touch with more conservative practices. But the wish to become one of the others and to be part of the teachers group represents a very important impetus in many of their attitudes, actions and adopted or even imitated styles. Ultimately, the others are as much significant as the potential role of technology in mathematics classes. Among those diverse others we can name the members of the school mathematics department, the school supervisor, the university supervisor, the other fellow colleagues and the students.

In several occasions different repertoires, sometimes conflicting intermingled. Nevertheless, Tiago was able to reorganise his ideas on the use of technology as he became progressively more confident of its introduction in the classroom. In the field of teaching practice, the sharing of tasks and responsibilities between the student teachers and the supervisors together with students' reactions were determinant.

With technology a shift of the gravity centre from the teacher to the students' activity was clearly produced and opened the way for learning to everyone involved.

In the context of the student teachers training, the use of technology in the classroom turns into challenging work of reconciliation and negotiation of meanings.

As he experienced multimembership, Tiago's way of thinking, action and alignment illustrate how the work of reconciliation is not necessarily harmonious and entails struggling for ways of integrating participation in sometimes divergent coexisting practices. The student teacher's learning trajectories show how the introduction of technology in mathematics class became a profound socially constructed nexus across the landscape of practice.

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IMPROVING STUDENT TEACHERS' UNDERSTANDING OF FRACTIONS

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The research results presented in this paper are only a small part of an action research performed with the main aim of improving student teachers' understanding of mathematics. The teaching strategies were similar to those suggested for their future use in teaching children and involved the use of multiple modes of representation for most of the concepts and operations in the primary school curriculum. One of the most interesting children's activities performed by the student teachers were trading games which clearly expose relationships among two or more number concepts. The data collected indicated that playing these games was an important strategy to improve student teachers' understanding of, and attitudes to, fractions.

INTRODUCTION

My initial experiences as a novice mathematics school teacher in Brazil and later my experiences as a teacher educator led me to think that both mathematics student teachers and primary school student teachers (STs) do not have an appropriate understanding of the mathematics content they are supposed to teach. It did not take long, after I started teaching at schools, to notice that I did not have enough mathematics understanding to teach even the most basic curriculum contents to my first class of 11 year olds (5^{th} graders). I could present my students with correct procedures, but could not answer most of their questions concerning the reasons for using certain steps in the procedures (Amato, 2004). These experiences led me to undertake a research project with the main aim of investigating ways of helping primary school STs to improve their understanding of mathematics in pre-service teacher education.

SOME RELATED LITERATURE

According to Skemp (1976), relational understanding involves knowing both what to do and why it works, while instrumental understanding involves knowing only what to do, the rule, but not the reason why the rule works. Research tends to show that they often do not have sufficient relational understanding of the content they are supposed to teach and this is not a problem restricted to developing countries (e.g., Goulding, Rowland, & Barber, 2002). Research has also revealed that some primary school teachers and STs demonstrate negative attitudes towards mathematics (e.g., Philippou and Christou, 1998). They tend to blame instrumental teaching for these attitudes (e.g., Haylock, 1995). So most of the attempts to help STs improve their

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attitudes to mathematics in teacher education seem to involve improving their understanding of the subject. After a review of the literature about teacher education, the main research question of the present study became: “In what ways can primary school STs be helped to improve their relational understanding of the mathematical content they will be expected to teach?”.

The integration between the re-teaching of mathematics and the teaching of mathematics pedagogy is said to be a way of improving teachers and STs’ understanding (e.g., Bezuk and Gawronski, 2003) and attitudes to mathematics (e.g., Weissglass, 1983). Most of the literature reviewed concerning such integration suggests re-teaching mathematics to teachers and STs by using the same methods that could be used to teach mathematics in a relational way to school students. To develop positive attitudes to mathematics in children, primary school teachers must learn how to set up learning experiences that are enjoyable, interesting and give the learner a sense of accomplishment. In order to be able to do this, the teachers must have had such experiences themselves. Using children’s methods was also thought to help STs acquire some initial pedagogical content knowledge (Shulman, 1986) in an experiential or tacit way (Sotto, 1994) in initial teacher education.

According to the Swedish psychologist Claparede (in Tahan, 1965), playing does not finish with childhood. Even adults can not maintain themselves in an activity for long periods if the work involved is not done in a way that is amusing. It is very common to see adults enjoying themselves while playing with cards, dominoes and even 22 mature men happily chasing a ball in a football game. Ernest (1987) presents the rationale for the use of games in the teaching of mathematics. He argues that games provide active learning, enjoyment, co-operation and discussion. The enjoyment generated may result in an improvement in attitudes towards mathematics after a period of time. According to Orton (1994), through playing games students have mental practice which "it is not forced and it takes place in a natural and enjoyable way" (p. 47).

Games which clearly expose important relationships among two concepts can be called ‘relational games’ because they have the potential to develop students’ relational understanding of mathematics. They were thought to be particularly important in re-teaching rational numbers to STs. Research has shown that many school students’ (e.g., Stafylidou and Vosniadou, 2004; and Ni and Zhou, 2005) and even STs (e.g., Domoney, 2002) see fractions as two separate natural numbers and not a single number and develop a conception of number that is restricted to natural numbers. Ni and Zhou (2005) suggest that the teaching of fraction concepts should start earlier than is it is often recommended by curriculum developers in order to avoid the development of what they call ‘whole number bias’. Yet if teachers have not develop themselves an understanding that fractions are numbers, they can not help school students’ to avoid the development of such bias. Teachers must develop a strong understanding of rational numbers which, among many other things, involves the ability to differentiate and integrate natural numbers and fractions.

METHODOLOGY

I carried out an action research at University of Brasília through a mathematics teaching course component (<u>MTCC</u>) in pre-service teacher education (Amato, 2004). The component consists of one semester (80 hours) in which both theory related to the teaching of mathematics and strategies for teaching the content in the primary school curriculum must be discussed. There were two main action steps and each had the duration of one semester, thus each action step took place with a different cohort of STs. A teaching programme was designed in an attempt to: (a) improve STs' relational understanding of the content they would be expected to teach in the future and (b) improve their liking for mathematics. Four data collection instruments were used to monitor the effects of the strategic actions: (a) researcher's daily diary; (b) middle and end of semester interviews; (c) beginning, middle and end of semester questionnaires; and (d) pre- and post-tests. Much information was produced by the data collection instruments but, because of the limitations of space, only some STs' responses related to their use of games are reported. In the action steps of the research, the re-teaching of mathematics was integrated with the teaching of pedagogical content knowledge by asking the STs to perform children's activities which have the potential to develop relational understanding of the subject.

According to Thompson and Walle (1980 and 1981), teachers can help students to translate from concrete to symbolic modes of representation by asking them to simultaneously manipulate concrete materials and digits to solve word problems on a place value board (PVB). Orton and Frobisher (1996) also argue that students should not be asked to write symbols at a distance from the operations performed with concrete materials and proposes similar association of concrete materials and symbols on a PVB. The PVB I use with Brazilian school students and STs (Figures 1 and 2) is a cheaper version of the PVB proposed by Thompson and Walle. It consists of a sheet of white A3 paper (or 2 sheets of A4 paper glued together) folded into 4 equal lines and 3 unequal columns: large, medium and small columns which can be used to represent either (a) natural numbers with 3 digits such as 215 and 134 (Figure 1), (b) mixed numbers such as $15\frac{3}{5}$ and $34\frac{4}{5}$ (Figure 2), and decimals such as 27.8, or (c) decimals with 3 digits such as 1.56 and 3.48 using paper strips (Figure 3).

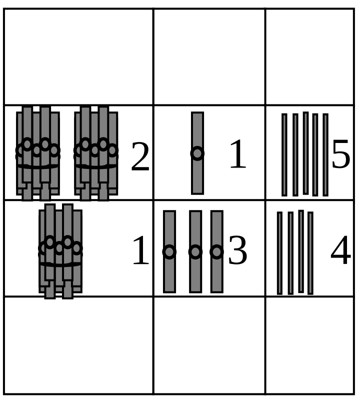

Figure 1

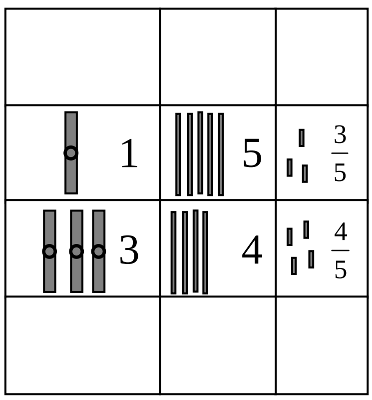

Figure 2

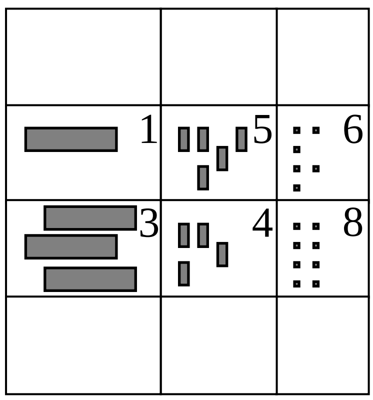

Figure 3

Pair work is used during the activities with the PVB to encourage STs' interaction and sharing of ideas. The pairs also interact among themselves if they get stuck. Some of the representations used for natural numbers were extended to fractions, mixed numbers and decimals in order to help students develop the concept of rational numbers as an extension to the number system (Amato, 2005). Versatile representations such as straws, part-whole diagrams, and number lines are often used in practical and written activities in an attempt to help STs relate natural numbers to fractions and decimals. The PVB is used to represent place value concepts and the four basic operations with natural numbers, proper fractions, mixed numbers and decimals. Therefore, the use of these versatile representations is a way of developing STs' learning of rational numbers in a meaningful way by relating rational numbers to their prior learning of natural numbers (Ausubel, 2000). Natural numbers (Figure 1) and fractions (Figures 2 and 3) are represented together with plastic drinking straws as follows: (a) units = loose whole plastic straws of any colour; (b) tens = bunches of ten straws gathered with a rubber band; (c) hundreds = bunches of ten tens gathered with a rubber band; (d) several pieces of straws: halves (red pieces), quarters (yellow pieces), fifths (green pieces), eighths (blue pieces) and tenths and hundredths (purple or pink pieces). The idea is to represent natural numbers and fractions together in order to make their relationships clear. For example, 15 whole straws and 3 pieces of $\frac{1}{5}$ to represent the mixed number $15\frac{3}{5}$ (Figure 2).

The forward and backward 'trading games' suggested by Thompson and Walle (1980 and 1981) to help students develop the concepts of place value with natural numbers were extended to fractions and mixed numbers. In the forward version of the trading game for mixed numbers, the pair of players select the materials needed for the game (a box with 40 divisible units such as coloured drinking straws, 20 fifths, and 4 rubber bands). Each player chooses one of the lines of the PVB to play the game and rolls a spinner with numbers such as 1, $\frac{5}{5}$, $\frac{1}{5}$, $2\frac{1}{5}$, $\frac{2}{5}$, $1\frac{4}{5}$, $\frac{3}{5}$ and $2\frac{3}{5}$ (for the version of game with fifths) written on it. The player who gets the bigger number starts the game. If both players score the same number, each player must roll the spinner again. Each player in his/her turn: (a) rolls the spinner and gets as many units and pieces as indicated by the spinner, (b) places the units and the pieces in the correct places in his line of the PVB (i.e., units in the medium column and pieces in the small column), (c) changes 5 fifths for 1 unit and places the new unit in medium column (the units' place) of the board. Players continue accumulating units and fifths until they get 10 units. Then the units are joined together with a rubber band forming a ten which should be placed in large column (the tens' place). The winner is the player who first gets two tens.

In the backward version of the trading game for mixed numbers, the materials needed, and the way of deciding who starts the game, are the same for the forward version of the game. Each player places the number defined by the teacher (e.g., 2 tens, 4 units and 3 fifths) in the correct places in his/her line of the PVB. Each player in his/her turn, rolls the spinner and removes as many units and pieces as indicated by the spinner. Players continue removing units and fifths from the PVB. If the player does not have enough units to remove the amount of units indicated by the spinner, he/she removes one ten from the tens' place and takes the rubber band from the ten to get more loose units. If the player does not have enough fifths to remove the amount of fifths indicated by the spinner, he/she removes one straw from the units' place of the PVB, and changes the straw for 5 fifths to get more fifths. The winner is the player who first removes all his/her whole straws and pieces from the PVB. When played with fractions and mixed numbers the trading games can help school students and STs visualise the relationship between fractions of the type n/n, $n \neq 0$ (e.g., $5/5$) and the natural number one (e.g., $5/5 = 1$) (Amato, 2005).

SOME RESULTS

One of my main worries related to the new teaching programme was the effect on the STs of asking them, as adults, to perform many children's activities along one whole semester. The data tended to show that most of the STs did not mind experiencing children's activities (Amato, 2004). They appeared to accept it as a normal strategy in a course component about teaching children. On the contrary, many STs mentioned that experiencing children's activities had been a positive aspect of the programme and had improved their understanding of mathematics. For example, "To experience the activities is very positive, as many times the teacher teaches the content to children without having understood it him/herself" or "The way mathematics was presented, through concrete materials and the relaxed way, led us to conclusions not previously understood".

A questionnaire asking the STs to evaluate the programme with respect to changes in their understanding of mathematics was administered in the last lecture of the semester (Amato, 2004). Question (1)(a) was: "What changes happened in your understanding of the mathematical content discussed in this course component? Give examples". The examples given by the STs were not prompted and so they could be said to be more convincing than their answers to the closed questions. All STs who answered the questionnaire said there had been improvements in their understanding of mathematics and/or in their pedagogical content knowledge of the content discussed in the course component.

However, the predominance of remarks related to improvements in topics related to fractions was clear. There were also a few responses to question (1)(a) about changes in their attitudes towards certain mathematical content. An example is Juliana's response: "The most meaningful changes were the ones about the rediscovering of mathematics. I learned, for example, that a fraction is not a beast of seven heads." Those responses to a question asking about changes in understanding tend to show that some relationship seems to exist between the affective and cognitive domains for those STs. Many other STs said in the interviews, and indicated in the post-tests, that their understanding of fractions had improved. Another example extracted from one of the interviews is:

> I am finding the fractions part [of the programme] excellent. I am overcoming many of my difficulties. You believe you know fractions but they were never well developed at school. You bring all those difficulties from the initial grades. ... After we manipulated the materials I could understand fractions without any fear.

One of the questions of another questionnaire administered at the end of the semester asked the STs to evaluate the activities in the teaching programme: ‘Write if you liked or not participating in the activities listed below. Tick your answer in the appropriate column according to the following code: (a) Much Disliked; (b) Disliked; (c) Indifferent; (d) Liked; and (e) Much Liked. Item (9) of the list of activities was: ‘to play the trading games with concrete materials on the PVB’. In the first semester, 100% of STs answered that they liked to play these games (13% liked, 65% much liked). In the second semester, 8% answered that they were indifferent and 92% answered that they liked to play the games (35% liked, 58% much liked).

During the trading games the classes of STs were as noisy as any children’s class. After playing the games, it was common to hear comments such as “It was very good” or “The children will love it”. It was also common to hear one or more of the STs asking “Are we not playing a game today?” when they noticed that the lecture was finishing and no games were played in that day. There were many other remarks about their enjoyment of the games in the interviews and in the end of semester questionnaires. Some STs explained that the games improved their understanding of, and liking for, mathematics. Some examples are:

> Maria (Interview) Games are great. They represent relaxing moments. When you relax it seems you broaden your insight. You say: Ah! I understood. It is nice to say: ‘I understood’. They are not a useless activity. Have you noticed how the child inside you teaches you about life? What does a child like to do? To play.

> Daniela (post-questionnaire about changes in attitudes) I continue to like mathematics as I did before. The fact is that after this course component I could see mathematics “with other eyes” because of the association with the playing aspect, games, etc.

Apart from helping STs relate fractions to natural numbers (e.g., $5/5 = 1$) (Amato, 2005), other important connections were facilitated by the use of the trading games on the PVB. In particular the connections between the fraction and decimal notations. During a whole classroom discussion after a trading game was played with quarters, I asked how many units we had in the number ¾. Several STs said ‘none’. Then I asked the class what digit I could write in the units’ place in order to say more clearly that there were no units in the number ¾. A ST replied that I could write a large zero in front of the fraction. Then I wrote on the blackboard a zero in front of the fraction (i.e., 0¾) and asked the class what was the difference between the number written with the zero and the one written before without the zero. Most STs replied none, but one ST said jokingly that I was wasting chalk. In other classes the STs enjoyed the idea of writing a zero in the units’ place and ‘wasting ink on paper’ to transform a proper fraction in a mixed number with zero units. Writing a zero in the units’ place became an important link between the notations for fractions and decimals, because the two notations become visually more similar. Interesting discussions were provided by asking STs to compare the two ways of writing a number such as 7 tenths ($0\,{}^{7}\!/_{10}$ and 0.7). One of these discussions were:

T (Teacher): Look carefully at both ways of writing 7 tenths [$0\,{}^{7}\!/_{10}$ = 0.7]. What are differences between writing a number like 7 tenths in these two ways?

ST B (Student Teacher B): We do not write the bar and the 10 when we use the point.

T: Where are the bar and the denominator 10 when we use the decimal point?

ST B: It is hidden in the point.

ST C: It does not make sense to me. The point is too small!

T: By the way, what does the decimal point mean?

ST C: It is used to separate the wholes from pieces such as tenths and hundredths.

T: Then where are the bar and the denominator 10?

ST C: It is inside our heads. It is in our imagination.

SOME CONCLUSIONS

Relational games proved to be a way of motivating adult learners, such as teachers and STs, to manipulate concrete materials which becomes the materials used in a game and not a tool just used by young children to help them understand mathematics. On the other hand, the knowledge of multiple modes of representation gained through these games was thought to be one of the most basic pedagogical content knowledge about teaching mathematics. With time and teaching experience, STs would be more able to use such knowledge in combination with more sophisticated teaching strategies. Using relational games was also considered an adequate strategy in helping STs' to improve their relational understanding of mathematics and their liking for the subject. Part of STs' dislike for mathematics was perceived as related to their instrumental understanding of the subject. Some STs suggested the inclusion of even more playing activities in the programme. More relational games and other children'339s activities involving rational numbers concepts and operations were included in the third and subsequent semesters. These changes proved to be quite effective in helping other classes of STs overcome their difficulties with, and dislike for, fractions.

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AUTODIDACTIC LEARNING OF PROBABILISTIC CONCEPTS THROUGH GAMES

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Pupils in grades 6-9 without a formal background in probability performed coin-toss game tasks, which led to inventing of probability terminology and understanding of concepts. The pupils were asked to answer questions dealing with the chances that an event would occur in different situations, and to convince their classmates of the correctness of their answers. The principal findings point to intuitive development of probabilistic concepts. Pupils constructed ways to quantify probability using fractions and percentages. Without the intervention of formal teaching, they created a linkage between sample size and probability of an event, and constructed a probabilistic language of their own for mutual communication purposes. This finding partly contradicts previous finding of others that claim there is a tendency to intuitively ignore the influence of the size of the sample when estimating probabilities.

THEORETICAL BACKGROUND

Probability learning among pupils is distracted by primary conceptions, wrong intuitions and misconceptions; affected by language, beliefs and daily experience (Amir & Williams, 1999; Van Dooren et al., 2003). When children solved probabilistic problems, there was a difference between the immediate (primary) answers and the given justification for the solutions. The meaning of the difference is that the primary answer expresses instant intuition, however the consequent justification does not necessarily reflect the rational thinking for choosing the answer, and it might be a delayed logical interpretation (Fischbein & Schnarch,, 1997). Learning activities based on games contribute to improvement of probabilistic understanding, relating decision-making in the context of fairness of games, and using sophisticated methods to justify and illustrate their thoughts (Alston & Maher, 2003; Amit, 1999). Tarr and colleagues (Jones et al., 1999) found that there is an intuitive intention to ignore the effect of the sample size while evaluating probabilities. The main idea of the "law of large numbers"—the larger the sample size, the greater the likelihood that a specific experimental result will be closer to the theoretical one—is misunderstood by pupils. Teaching by simulations can catalyze comprehension of existing concepts among pupils and develop their stochastic reasoning. Through simulations, it is possible to develop an understanding of particularly elementary concepts, and to learn that drawing conclusions should be based on large samples, whereas small samples usually lead to the wrong conclusions (Stohl, & Tarr, 2002).

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METHODOLOGY

The aim of this study was to investigate to what extent carefully chosen games, with "learning potential", can be an intuitive basis for the acquisition of probabilistic reasoning. The games were as follows:

Task 1 – "The coin game"

Examine four sets of outcomes of repeated tosses of a colored chip (the chip was painted white on one side and red on the other): (a) 3 of 4 whites, (b) 6 of 8 whites, (c) 12 of 16 whites, (d) 24 of 32 whites.

Questions:

1. Which of these four sets of outcomes do you think is the most likely to occur, or do you think all four sets are equally likely?
2. Which of these four sets do you think is the least likely to occur, or do you think all four sets are equally likely?

(Aspinwall & Tarr, 2001)

Task 2– Game "H" or "T"

1. A fair coin is tossed three times. The chance of getting heads ("H") at least twice when tossing the coin three times is: (a) less than; (b) equal to; (c) greater than the chance of getting "H" at least 200 times out of 300 times.
2. Two groups of children play a game tossing a fair coin. The likelihood of getting tails ("T") when tossing the fair coin is 50%. The first group of children tosses the coin 50 times, the second group tosses the coin 150 times. Each time the children toss the coin, they record the outcome. Which group of children is more likely to get 60% "T" when tossing the coin (explain why)?

(a) Group A, (b) Group B, (c) Neither group: both results will be the same

(Fischbein & Schnarch, 1997; Lamprianou & Williams, 2002)

<u>Population</u>: The population consisted of two groups of pupils who had no previous formal learning of probability. One group consisted of six talented pupils from grades 6–8 who participate in a mathematical club at Ben Gurion University of the Negev, called "Kidumatica for Youth". A second group consisted of six pupils from grade 9, who are high achievers but are not defined as talented and do not receive any math enhancement.

<u>Setting</u>: The pupils, working in triads, **actually experienced game tasks**. They documented the solution process and justified and negotiated their results with their peers.

During the experiment there was no formal instruction of any type and the pupils were not told whether their solution or strategies were right or wrong.

Data collection and analysis: The entire experiment, including group discussions, was videotaped. Classroom notes were taken by the researchers, and the pupils' notations were collected. Data was analyzed in a qualitative manner: Classroom observation of the process of "informal learning" revealed obstacles in the intuitive probabilistic reasoning process, and witnessed how the pupils' interactions helped overcome them.

FINDINGS AND INTERPRETATIONS

A. The quantification of chance and probability

One of the most important finding of this study is that in the gaming process pupils gained the insight and understanding that **it is possible to calculate likelihood of events.**

They gave a numerical meaning to probability and built it intuitively, using their familiarity with fractions and percentage. As stated above, prior to the experiment, the pupils had not learned that it is possible to calculate or to quantify chance. At the start of the experiment, the occurrence of an event was described by means of global comparisons, e.g., a better chance, an equal or smaller chance etc., without giving any numerical value that would be a measure of chance. This evolved as the experiment progressed.

From the following quotations, and others, we infer how pupils quantified chance: The coin game (task 1):

Rachel*: "In order to get a situation of 3 whites, how many tossing possibilities have to be checked?: There's 1 white and 3 red; 2 white and 2 red; 3 white and 1 red; 4 white and 0 red; 0 white and 4 red. There are 5 possibilities and we want on. That's why the chance is 1/4. Next time there are 8 tosses. I think there are 9 possibilities and the chance for one option is smaller, 1/8."*

For Rachel, the sample space is five for four coin tosses, and, accordingly, nine for eight tosses. She built a method to calculate the probability of an event as the ratio **between the desired "option"** (3 white and 1 red), **and the remaining four "possible results". Although** she is wrong in her calculation, the idea behind the step is important since **she is intuitively building an idea for calculating chances of events.**

Tamir (see Figure 1) supported this idea and calculated the chance in percentages for each "result" (event).

Tamir: *"With 1 to 4, the chance is 25%. With 8 tosses there are 9 options. The ratio is 1/8 and the chance is 12.5%."*

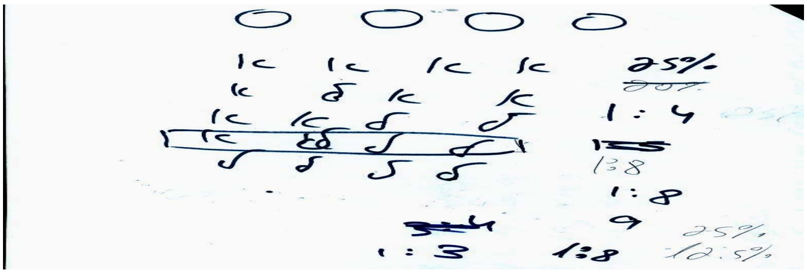


Figure 1 – The way Tamir represented all the different options of 4 coin tosses

Robert calculated and compared the chances, then drew conclusions based on his calculations:

Robert: *"When tossing 4 times, there are 5 options and 1 option must be received, so this is 1/5 which is 20%. When tossing 9 times, there are 9 options and 1 option is needed, so this is 1/9, which is approximately 11% – (loudly!) <u>And thus I have proved that not all the options are equally likely.</u>"*

<u>Gaming task "H" or "T"(Task 2):</u>

Shiran (see Figure 2 for tossing the coin three times): *"There are 4 options – all are "H"; 2 "H" and 1 "T"; 2 "T" and 1 "H"; all are "T". In this question we are required to choose 2 of the options: All of them "H" or 2 "H" and 1 "T", so the probability is 1/2. With 300 tosses there are 301 possible options that may occur, 200 of them give too small a number for "H". So if we omit 200 we have 101 options, which is the result for "H" out of 200 and more. The estimation for 101/301 is 1/3. Since 1/2>1/3, here there is a higher chance".*

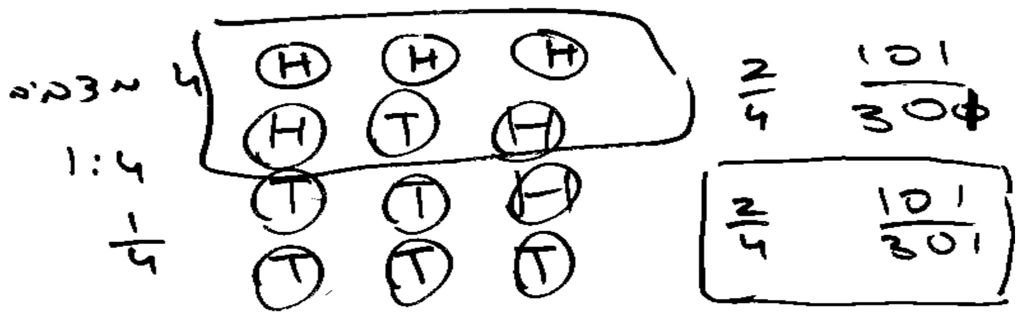
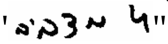


Figure 2 – The way Shiran presented and calculated chances (Note: the phrase "" in the top left corner of Figure 2 means "4 options").

In all the cases we described, the pupils quantified probabilities of different events, and based on the numerical result, they made conclusions, answered the questions and tried to convince their colleagues that they were right.

<u>B. Intuitive perception of empirical probability versus numerical probability</u>

From the following quotations it can be assumed that pupils developed an intuition that enables them to distinguish between the above two types of probability.

Dan: " the ratio is a constant thing that doesn't change, something (that is) mathematical, but chance is something else – chance is varying and not constant, it is more real and less mathematical.

Ariel: "Mathematically, chances are equal, but in our life not everything is mathematical. Chances [of getting the results of task 1] can change, becoming smaller when tossing many more times."

Interviewing the pupils after they performed the game tasks intensified our feeling that the pupils developed intuition that distinguished between numerical and empirical probability. Our interpretation is that when a pupil says that ratio is an unchanging "*constant* thing"—a mathematical "thing"—he intuitively perceives the meaning of numerical probability. When he talks about something "varying and *not constant* "—something that is "more real and less mathematical"—he intuitively perceives the meaning of empirical probability.

[Note: numerical probability of an event is determined analytically; empirical probability of an event is based on experimentation or simulation (Jones et al., 1999)].

C. Linking chance and a sample space –- the learning process

A two phase process leading to associating chance with sample space is described below.

Phase 1 –- Acquiring a tool for probability quantification

1. Primary perception: the pupils translated the data: 3 out of 4, 6 out of 8, to a form of simple fractions, then reduced the results and obtained the equality $\frac{3}{4}=\frac{6}{8}=\frac{12}{16}=\frac{24}{32}$. Thus they determined that a, b, c and d are equally likely events.
2. Primary thinking: In fair coin tossing, both sides have an equal chance of showing, thus repeated tossing of a coin would be expected to result in red and white showing an equal number of times.
3. First conflict: From the given condition in task 1, the ratio between red and white should remained constant (0.75) for any number of tosses. But in reality, all the trials had different ratios for white and red.
4. Second thinking: This conflict led to the second thought about probability regarding the number of coin tosses.
5. Wrong but significant attempt to link the number of coin tosses with the number that quantifies probability.

Phase 2- A repetition and re-examination of the game

In quotations 6 and 7 below, we see a connection forming between sample size, sample space and probability of a specific event.

6. "*With 4 tosses there are a total number of 5 different options from which only 1 is the result of the game, in 8 tosses there are 9 options, and so on…*".
7. *"The smaller the number of different options, the bigger the probability for a specific result".*
8. Second resolution**:** There is a contradiction **again** between data obtained from experience and data given in the task – a real dilemma.
9. Conflict solution: As a result of the dilemma, **a separation and distinction between ratio and probability is made** for the first time**:** *"Ratio is like 'how many out of how many', how many times in 4 tosses a white was received; probability is what is the chance that a white will be received when you toss 4 times".*
10. Intuitive concept formed: An intuition-based method for calculating theoretical probability of an event was formed thus: "*It can be expressed as a fraction form, with the number of options in the denominator and how many did we receive out of all the options in the numerator".*

D. Invention of probabilistic intuitive language

As a result of experiencing games, the pupils created a **probabilistic language of their own** (see next page). They connected words from their daily life to explain their perception of probability and constructed a probabilistic intuitive language. The following glossary contains the formal concept, the terminology the pupils used for this concept and the pupils' quotation with their own terminology in context.

DISCUSSION

Pupils in grades 6-9 without prior background in probability performed game tasks that included coin tossing in various situations (tasks 1, 2 above). During their experiences with the first task, a conflict arose from discrepancy between the events that were given theoretically as a datum in the task (see data for task 1), and the conclusions of the experiment. The theoretically determined relationship for 3 white showing among four coin tossing events (0.75) was accepted and the pupils' primary intuition led them to establish this relationship as the fixed probability for all events, a through d, in task 1.

The actual outcome of the trials during the game led to results that contradicted their first intuition. A heated debate ensued among them in a mutual effort to persuade, leading to surprising results. The pupils first created a way to calculate the probability using their prior knowledge of fractions and percentages. The most important result was in the insight that formed that there is a difference between theoretical probability and experimental probability, and intuitively comprehending the link between probability and sample size was not long in coming. This understanding can form the basis for understanding "the law of large numbers" in the future.

Glossary of probabilistic concepts:

Formal concept	**Concept created by pupils in gaming process**	**Quotations from pupils**
Event	Situation or specific result	*"More options of situations"* *"As the number of situations grows/ the chance that one of them will occur is smaller".* *"The larger the number of tosses , the smaller the chance for a specific toss"*
Sample space	All the different possibilities	*"Option of situations "* *"All the possible options"* *"All the different options for a coin tossed a few times".*
Ratio	Few out of few	*"Ratio is few out of few, e.g., how many times out of four tosses, a white will be received"*
Chance and Probability	Intuitive usage of the chance concept	*"…probability is: what's the chance to receive a white when you toss a coin four times".*
Probability of an event	The chance for specific toss	*"It is possible to express in fraction form, the number of options in the denominator on one hand and how many we got from the number of options, on the other hand".*
Sample size	Number of coin tosses	*"The larger the number of tosses the smaller the chances, because there are more possible situations".*

Table 2 – Concepts used by pupils versus formal probabilistic concepts

This experiment adds another layer in the study of probability learning and partly disputes the findings of previous studies which claim that, although the typical pupil in junior high school has no awareness of the connection between experimental probability and the sample size, correct cognitive activity focussing on simulations of random occurrences can advance the development of this connection (Aspinwall & Tarr, 2001).

This study is unique in that no formal teaching intervention occurred; instead, the pupils built their knowledge upon interactions among themselves. Because of this, they invented their own language for communication purposes, founding it upon everyday language (see Table 2).

In conclusion: Games rich in probability potential have added value for building probability concepts such as sample space and probability quantification, and for developing means of persuasion and rationalization. More importantly, we should not forget the pleasure that games and surprising outcomes provide.

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GRADUATE STUDENTS' PROCESSES IN GENERATING EXAMPLES OF MATHEMATICAL OBJECTS

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In this paper we report on a study about experts' strategies for producing examples of mathematical objects. Three strategies have been identified: by trial and error, by transformation and by analysis. These strategies and some related cognitive processes are analysed in detail. In particular, some meta-cognitive and meta-cultural remarks by the subject have been identified as fundamental elements for the activation of strategies as well as for transition between them. Finally we discuss the possibility of identifying links between the strategies for producing examples we expose here and processes of production of conjectures, argumentations and proofs.

INTRODUCTION

Generating examples is typical of a mathematician's work, in different activities and with many different objectives. The literature has underlined the importance of this activity also in mathematical education, as a learning and teaching strategy (Zaslavsky, 1995; Dahlberg & Housman, 1997; Watson & Mason, 2002), with relation to the construction of concepts (Hazzan & Zazkis, 1997; Zaslavsky & Shir, 2005), to the production of conjectures (Boero et al., 1999; Antonini, 2003; Alcock, 2004), and of proofs (Balacheff, 1987; Harel & Sowder, 1998).

In this paper we focus on the process of generating examples. Our point of view on examples generation strategies is that of Zaslavsky & Peled (1996) that claim that "*the state of generating examples can be seen as a problem solving situation, for which different people employ different strategies"*. The aim of the research presented here was identifying some experts' strategies and related cognitive processes for producing non-routine examples of mathematical objects.

THEORETICAL FRAMEWORK

In this study, according to Zaslavsky & Peled (1996), we have considered the construction of examples as a *problem solving* activity: as observed by Zaslavsky (1995), generate an example with some properties is an open-ended task, a problem with many answers that students can solve by various approaches. This point of view makes the study of both strategies for producing examples and the underlying cognitive processes meaningful. The protocols are analysed with particular attention to both *strategies* and subjects' *control* over the efficacy of the strategies, according to the role of these aspects emphasized in the studies about *mathematical problem solving* (see for instance Schoenfeld, 1992). As regards examples of mathematical objects, we often referred to *representations* of objects, in terms of the theoretical framework of *registers of semiotic representation* (Duval, 1995).

METHOD

The method used for investigation was that of clinical interviews. Subjects were asked to express their thinking processes aloud. Interviews were audio-recorded and subjects' notes and figures were collected.

The work presented here concerns experts' processes. We believe that observing experts' processes and strategies can be useful in order to identify those processes that are not activated in cases when non expert subjects get stuck. Seven subjects were interviewed, all of them postgraduate students in mathematics, with different research interests.

The following is a list of requested examples (in brackets we put the label identifying the problem within the paper):

1. Give an example of a real function of a real variable, non constant, periodic and not having a minimum period (*the periodic function* example)
2. Give an example of a function f: $[a,b]\cap\mathbf{Q}\to\mathbf{Q}$ (with $a,b\in\mathbf{Q}$) continuous and not bounded (*the function on Q* example)
3. Give an example of a binary operation that is commutative but not associative (*the operation* example, modified from a problem discussed in Zaslavsky & Peled, 1996)
4. Give an example of three natural numbers, relatively prime, whose sum is a number which is not prime to any of them (*the three numbers* example)

In order to stimulate the subjects' exploratory processes, the requested examples are either very peculiar or they are typical mathematical objects, whose properties requested in the task are rarely observed.

STRATEGIES FOR PRODUCING EXAMPLES

From protocol analysis we classified three strategies for producing examples. In order to make the reading easier we present an example from a protocol before presenting the related strategy.

Trial and error

Excerpt: Franco, the operation example

> Which operations do I know? Sum, multiplication,... but they are no good.... The product of matrices!... No, no, it is associative ... and it is not commutative at all. Let's see... division is not associative. No, it is no good, it is not commutative. ... The exponential! No, it is not a binary operation. ... Well, if I take a^b it is binary... but it does not commutate, so... Which other operations are there? [...]

In general:

The example is sought among some recalled objects; for each example the subject only observes whether it has the requested properties or not.

Transformation

Excerpt: Stefano, the function on Q example

Now… [sketching a graph, figure 1]… where c will be an irrational. Of course this one does not have [all] values in Q. Let's make it have values in Q.

I might take a sequence [in the rest of the interview it will be clear that the subject means a sequence of irrational numbers], so… [drawing, see figure 2]…

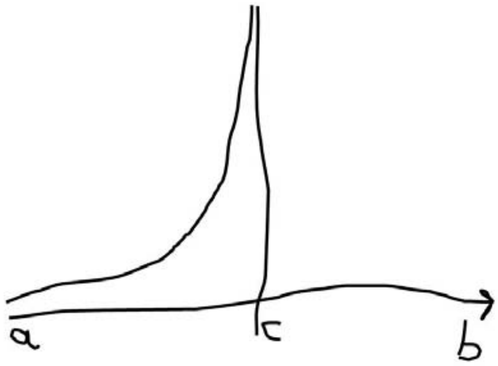

Figure 1

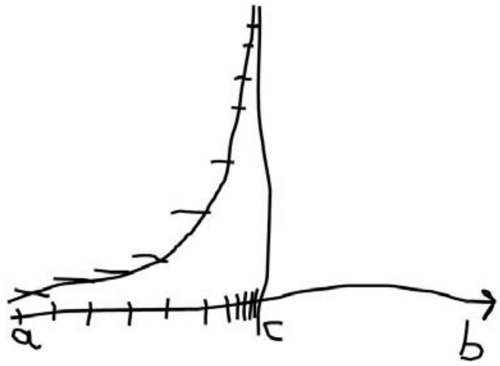

Figure 2

and there, in each little interval, taking a sort of maximum or minimum. Well, right, any rational number between the maximum and the minimum value. Is it continuous? […] Then on the other side [meaning in the interval between c and b], the same.

In general:

An object that satisfies part of the requested properties is modified through one or more successive transformations until it is turned into a new object with all the requested characteristics.

By *transformations* we refer here to a very wide class ranging from transformations made on the graph of a function, to the movement of a polygon's sides, to transformations of an algebraic formula into another, not necessarily equivalent, and so on.

Analysis

Excerpt: Sandro, periodic function example

It seems to me that if it is continuous it is no good ...or maybe I should make it on Q. Well, let's not complicate things... ... The examples I know are continuous enough periodic functions... and even if I adjust them I cannot get out of there … no, I must construct it from scratch. ... Example, a function that every 1/n is the same.

f(1/n)=f(2/n)......

Ah, so f(p/q) gets the same value! Now it will be enough to put another value for non rational numbers, for instance f(x)=0, if $x \in Q$ and f(x)=1, if $x \notin Q$.

In general:

Assuming the constructed object, and possibly assuming that it satisfies other properties added in order to simplify or restrict the search ground, further properties *are deduced up to consequences that may evoke either a known object or a procedure to construct the requested one.*

We named this strategy *analysis* due to the analogy with the equally named method used by ancient Greeks for both geometrical constructions and search for proofs: "i*n both cases, analysis apparently consists in assuming what was being sought for, in inquiring where it comes from, and in proceeding further till one reaches something already known*" (Hintikka & Remes, 1974, p.1).

ANALYSIS OF STRATEGIES

The interviewed subjects have produced the requested examples following one or more identified strategies. The *trial and error strategy* was almost always the first one, but subjects have promptly enacted other strategies. The *transformation strategy* is frequently found in protocols, whereas the *analysis strategy* seems to be activated when the other ones are considered as inefficient.

Let us start here a detailed description of the *transformation strategy*. One initial remark is that *transformations* are made on a *representation* of an object. To make this point clearer we refer to Duval theoretical framework (Duval, 1995) about *registers of semiotic representation*. Duval describes two types of transformations of *semiotic representations*: *treatments* and *conversions*. The former ones are transformations of representations within one single register, the latter ones are transformations of representations consisting of a change of register without changing the denoted object. Transformations described in this work, concerning strategies for producing examples, correspond to *treatments* within one particular register.

Enacting the *transformation strategy* necessarily requires a representation of an initial object and a set of transformations (*treatments*) to be carried out on this object's representation. In the examined example of the function on Q, Stefano operates on the graphical representation of a function: the subject plots the graph of an unbounded function and modifies it (he performs a *treatment*) in order to construct the graph of another function with only rational values. In the subsequent excerpt, in order to construct the commutative and non-associative binary operation, Sandro uses algebraic language to transform (*treat*) the initial operation into a new operation:

[...] So, a non-associative operation is division: a*b=a/b. Well, I should take out 0, I will adjust the definition set later. Now, the problem is that it is not commutative. Can I use it anyway? ... Ah! I can make it commutative by making it symmetrical! a*b=a/b+b/a ...

Sandro deals with a non-associative and non-commutative operation. Transformation of the considered operation into a new operation is performed within the algebraic register and seems to be caused by the fact that the subject translates the commutative property in this register into symmetry between representation's symbols and non-commutative property into non-symmetry. This translation seems to allow the subject to anticipate the possibility of constructing a new operation having the commutative property, by means of a *treatment* within the algebraic register that aims at “symmetrising” the symbolic writing so that the operation may become commutative (“*I can make it commutative by making it symmetrical!*”).

A detailed analysis of the processes that seem to guide subjects toward either the *transformation strategy* or the *analysis strategy* is needed. We identified some factors that seem to play a fundamental role in enacting either of these strategies and that seem to have a decisive role in the transition between them.

The transformation process seems to be enacted only if the subject has anticipated that the initial object can be efficiently modified through the available transformations. For instance, Stefano, constructing the function on Q, considers the graph of an unbounded function and modifies it so that it takes only rational values. It seems feasible to believe that through the graphical representation the subject anticipates the possibility of constructing a new function with the requested properties (“*this one does not have [all] values in Q. Let’s make it have values in Q*”). In a similar way, Sandro, in producing the binary operation, recalls known non-associative operations which are not commutative and tries to modify one of them to make it “symmetrical”. The subject seems to have anticipated the possibility to construct a new commutative operation. The operation’s algebraic representation allows this anticipation, through a translation of the commutative property into the “symmetrical” form of a formula.

On the contrary, in the periodic function example, Sandro is convinced that examples of periodic function known to him are continuous (or “*continuous enough*”, probably meaning piecewise continuous) and he conjectures, or knows, that periodic continuous, or “continuous enough”, not constant functions have a minimum period and therefore do not have the requested property. The most interesting aspect is that the subject anticipates that the transformations he might perform on the set of these periodic functions cannot transform them into non continuous or non “continuous enough” functions (“*… and even if I adjust them I cannot get out of there… no, I must construct it from scratch”).* Sandro’s is a *meta-cultural* control, since he claims that it is not possible to transform an initial object to construct another one having the requested properties. This is probably why he does not enact the *transformation strategy* but rather the *analysis* one. A different reason moves Marco not to activate the *transformation strategy*, in the function on Q example:

> [...] [It is] of the type $\frac{1}{x-c}$ but it is not in Q. How can I map it into Q? I don’t really know how I could handle this one. [...] Well, the typical one like this is $\frac{1}{x-\sqrt{2}}$. But how can I map it into Q?...... Well, let’s write what the problem asks …

The subject has an initial object to work on and seems to talk about his intention to search for adequate transformations. However he does not enact the transformation process because he thinks he does not have the instruments (i.e. transformations to be performed on the function) to do it (“*I don’t really know how I could handle this one*”). In this case, Marco’s is a *meta-cognitive* control (“*I don’t really know*”): the subject does not exclude the possibility to carry out transformations he does not know or he does not think about at the moment. In this case again the subject changes strategy, moving to *analysis*.

Based on the previous discussion we may formulate hypotheses on the factors that favour the activation of the analysis strategy. First of all we observed that this strategy seems to be enacted when the subject gets stuck with other strategies. In particular, it seems that the analysis strategy is enacted if transformations on the potential initial examples are considered insufficient (see Sandro's protocol) or rather, at that moment the subject does not have available methods he considers suitable (see Marco's protocol). Our hypothesis is that in these cases subjects might get stuck, as in the following protocol, concerning the periodic function example:

> [...] How shall I construct it, don't know..... […] starting from one that I know and modifying it, but how can I do that from scratch, I don't know where to start from. [...] Just to have a starting point. How can I make it up from scratch! [...] I am too focused on sinus. As if it were the typical periodic function, I am always stuck on that. [he gives up]

In this case, the subject claims his will to work by transformations but he does not have a suitable example to start from, and therefore he does not enact this strategy. But, differently to Sandro and Marco, in this case the subject does not enact the *analysis strategy* either and gives up the problem.

CONCLUDING REMARKS

The study presented here, considering the construction of examples as a problem solving activity, reports on strategies and cognitive processes in experts' production of examples of mathematical objects. Almost all the interviewed experts were able to manage all the identified strategies and checked in advance their efficacy. These controls seem to be decisive for a successful solution of the posed problems; in fact, the fundamental role played by control on strategies' efficacy in problem solving situations has been highlighted for example by Schoenfeld (1992).

Further studies on strategies we presented here are possible. For instance, an analysis from the point of view of embodied cognition (Lakoff & Nuñez, 2000) is extremely interesting and brings new elements characterising the identified process to the surface. During the transformation process we can actually observe that subjects widely use metaphors, like "handle", "adjust", etc. Subjects have also made use of many gestures, which were not recorded though, that seemed to point to a physical manipulation of objects. In accordance with the embodied cognition theoretical framework we may claim that for the subjects the initial mathematical example is conceptualised as an object on which adjustments can be made to transform it for one's objectives. Transformations and adjustments are physically carried out on one of the objects' representations, which works as provider of the raw material to be shaped in order to obtain the final object. Conceptualisation of mathematical objects as entities that can be constructed and modified seem to be crucial for the activation of the transformation strategy. In the analysis process, instead, we observe metaphors like "I must see (understand) how it is done", etc. Analysis of metaphors shows that the analysis strategy leads to identifying the object rather than constructing it. These remarks open up the way to a study on relationships between the strategies we have described and the conceptualisation of mathematical objects.

Further research is also needed to study how the identified strategies are intertwined with processes enacted in different situations where subjects produce examples. We also believe that these strategies may be useful to observe processes of production of examples in tasks involving a careful exploration in order to produce conjectures and proofs. The theoretical framework of Cognitive Unity (Garuti et al., 1996; Pedemonte, 2002), set up to study the relations between exploration, argumentation, and proof construction processes, is particularly suitable for deeper research in this regard. For example, it seems that the analysis strategy in the attempt to construct a potential counter-example to a valid mathematical statement may lead to the construction of a proof by contradiction. A first feedback comes from one of the interviewed subjects' words, at the beginning of the solution process for the periodic function problem: "I don't know whether it exists, but I suppose it does, so either I find it or else I prove it does not exist". The subject is not convinced that the requested object exists and believes that analysis may allow him to either find the requested function or prove that it does not exist. In fact, through the analysis strategy it is sometimes possible to get to deduce a property that may evoke the required object, but in other cases it might happen to deduce a contradiction. In the latter case, with analysis strategy one have enacted a process that can reveal elements for constructing a proof by contradiction: there are no examples having the requested properties, in fact assuming the existence of such an example one gets to a contradiction. In cases like this, it is easy to observe cognitive unity (in terms of Garuti et al., 1996) between exploration and proof construction processes, and structural continuity (in terms of Pedemonte, 2002) between argumentation and proof. In an ongoing work we are also dealing with the relationships between production of a potential counter-example to a mathematical statement through the transformation strategy and the proof to that statement. In these cases, we hypothesise that there might be a stop in the transition to proof, that can be explained in terms of structural gap (see Pedemonte, 2002) between argumentation and a proof. However deeper research studies are necessary.

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REASONING IN AN ABSURD WORLD: DIFFICULTIES WITH PROOF BY CONTRADICTION

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The study presented in this report is part of a wide research project[♣] *concerning proofs by contradiction. Starting from the notion of mathematical theorem as the unity of statement, proof and theory, a structural analysis of proofs by contradiction has been carried out, producing a model to be used in the observation, analysis, and interpretation of cognitive and didactical issues related to this particular type of proof. In particular, the model highlights the complex relationship between the original statement to be proved and a new statement (the* secondary statement*) that is actually proved. Through the analysis of an exemplar protocol, this paper discusses on cognitive difficulties concerning the relationship between the reference theory and the proof of the secondary statement.*

INTRODUCTION

The study presented in this report is part of a research project concerning difficulties that students encounter when are faced with proofs by contradiction, both at the high school and the university level. Although it was often observed that students spontaneously produce indirect arguments[1], or, at least with a structure similar to that of proofs by contradiction (Freudenthal, 1973; Thompson, 1996; Reid & Dobbin, 1998; Antonini, 2003a, Antonini, 2003b), current literature agrees on the fact that students show much more difficulties with indirect than direct proofs. Different aspects have been highlighted, from different points of view. First of all, some authors remarked that this issue does not find an adequate attention, neither at the high school (Thompson, 1996) nor at the university level (Bernardi, 2002). Some difficulties were identified in the negation of a statement, commonly required in proofs by contradiction: because of its peculiarities, this process of negation presents a specific complexity in the mathematical domain (Thompson, 1996; Antonini, 2003a; Wu Yu et al., 2003). In a historic epistemological study, Barbin (1988) raised the issue of acceptability, pointing out that students' attitudes towards this scheme of proof seem to echo ancient debates in the history of mathematics. According to Leron (1985) a specific difficulty seems to be related to the need of starting arguments with false assumptions: through these assumptions one enters a *false, impossible world*, and thinking in such an impossible world asks a highly demanding cognitive strain,

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[1] Polya (1945) describes the role that proof by contradiction can assume in the production of conjectures.

which may explain the difficulties observed. Moreover, at the end of the proof, as soon as a contradiction is deduced, this world has to be rejected, so that students feel deceived, and dissatisfied: they are faced with the unexpected destruction of the mathematical objects on which the proof was based (Leron, 1985). The research project (Antonini, 2003a), which this contribution is part of, is consistent with this direction of study and focuses on a cognitive and didactic analysis aimed to describe and interpret students' difficulties in proof by contradiction within the broader context of proving activity in mathematics. In the following, the results of such analysis will be briefly outlined and subsequently employed to explain specific difficulties related to developing arguments from false assumptions.

METHODOLOGY

Consistently with its aim, the project developed two main research lines: empirical and theoretical. Collection of data was carried out through various means, interviews, questionnaires, recording and transcripts of classroom activities, and involved students at the high school (12^{th} and 13^{th} grade) and at the University level (Scientific Faculties such as Mathematics, Physic, Biology, …). In both cases, it was reasonable to assume that students were acquainted, and even familiar, with mathematical proof, as well as with proof by contradiction. For more details on the experimental design see (Antonini, 2003a).

THEORETICAL FRAME

The analysis of proof by contradiction started from the general notion of mathematical theorem introduced in (Mariotti et al., 1997; Mariotti, 2000). According to the 'didactic' definition formulated by the authors, a mathematical theorem is characterized by the system of relations between a statement, its proof, and a theory within which the proof make sense. In particular, the definition refers to "the existence of a reference theory as a system of shared principles and deduction rules […]." (Mariotti et al., 1997, p. 182). As far as deduction rules are concerned, a clear difference emerges between direct proofs and proofs by contradiction. In fact, direct proofs, based on deduction rules, lead back to well established argumentation schema; on the contrary, an intrinsic structural complexity of proofs by contradiction emerges showing specific aspects that can explain specific difficulties. The following analysis aims at describing such a complexity.

STRUCTURAL ANALYSIS OF PROOF BY CONTRADICTION

We consider the proof by contradiction of a given statement, that we call *principal statement*. Such a proof consists in the direct proof of another statement, that we call *secondary statement.* The move from one statement to the other is commonly introduced by expressions like "prove by contradiction" or "assume by contradiction" that signal to the reader the change in the type of argument that is going to be developed. For instance, we consider a proof by contradiction of the following statement (*principal statement*):

let a and b two real numbers. If ab=0 then a=0 or b=0.

Proof: assume by contradiction that ab=0 and that a≠0 and b≠0. Since a≠0 and b≠0 one can divide both sides of the equality ab=0 by a and by b, obtaining 1=0.

Actually, this proof is a direct proof of the following statement (*secondary statement*):

let a and b two real number; if a≠0 and b≠0 and ab=0 then 1=0".

The hypothesis of this statement, "*a≠0 and b≠0 and ab=0*", is the negation of the principal statement (i.e. the conjunction of the hypothesis and the negation of the thesis of the statement to be proved) and its thesis is a false proposition, i.e. 1=0.

Thus, in order to prove the *principal statement*, that we indicate with E, one provides the direct proof of the *secondary statement*, that we indicate with E*.

Principal statement E	**Secondary statement E***
a,b real numbers If ab=0 then a=0 or b=0	a,b real numbers If a≠0 and b≠0 and ab=0 then 1=0

Table 1. Principal and secondary statement in a proof by contradiction

From the point of view of logic, a proof by contradiction of the principal statement can be considered accomplished if the *meta-statement* E*→E is valid; in fact, in this case from E* and E*→E it is possible to derive the validity of E by the well known "modus ponens" inference rule. But, the validity of the implication E*→E depends on the logic theory, i.e. the *meta-theory*, within which the assumed inference rules are stated. As it is commonly the case, i.e. in the *classic* logic theory, such a *meta-theorem* is valid, but it does not happen in other logic theories, such as the *minimal or the intuitionistic* logic.[2]

This analysis, although very brief, clearly shows the complexity of the argumentative structure of a proof by contradiction, and in particular highlights key elements of this complexity, such as the secondary statement E*, its proof in respect to the reference mathematical theory, the meta-statement E*→E, and its validity in respect to the assumed meta-theory.

We assume that specific difficulties may be related to each of these key elements and to their relationships.

In this paper we are going to address the problems related to the proof of the secondary statement E*; further research have been carried out and are still in progress, concerning the cognitive and didactic problems related to the validity of the meta-statement E*→E, and more generally to the acceptability of the proof by contradiction on the whole (Antonini, 2003a, 2004).

[2] For a definition in terms of rules of inference of the classic, minimal and intuitionistic logic, see Prawitz (1971).

FALSE HYPOTHESIS AND THE MATHEMATICAL THEORY OF REFERENCE

Consider statement E*, and the mathematical theorem which has E* as its statement; according to the definition given above, such a theorem is defined by the triplet (E*, C, T), where C is a direct proof of E* and T is the mathematical theory within which this proof is constructed and validated. One of the main characteristic of this theorem concerns the fact that both the hypothesis and the thesis of the statement E* are constituted by false propositions; although from the logic point of view this fact does not present any particular problem, from the cognitive point of view this peculiarity may have serious consequences. In fact, conflicts may arise between the theoretical and the cognitive point of view. From the logical point of view, we observe:

- in spite of the falsity of both the hypothesis and the thesis, the statement E* is logically well formulated. Moreover, in spite of the falsity of its hypothesis (and just because of it, according to the truth tables) the implication E* results to be *true*;
- the proof C constitutes a *valid* proof of the implication E*. That means something more than the fact that E* is logically *true*, it means that it is possible to construct a deductive chain within a mathematical theory, and this despite the fact that both the hypothesis and the thesis are false;
- deduction in a mathematical theory is independent of the interpretation of the statements involved, that means that axioms and theorems of a mathematical theory can be applied to objects which are mathematically impossible, and for this reason, *absurd*: for instance, two real numbers a and b different from 0 and such that ab=0, the rational square root of 2, parallel lines that intersect each other, ….

For example, let us consider the previous theorem and analyse its proof according to our discussion.

(Principal) statement: let a and b two real numbers. If ab=0 then either a=0 or b=0.
Proof: assume by contradiction that ab=0 and a≠0 and b≠0. Since a≠0 and b≠0 both sides of the equality ab=0 can be divided, respectively, by a and by b, obtaining 1=0.

This is a direct proof of the secondary statement "let a,b real numbers; if a≠0 and b≠0 and ab=0 then 1=0". The hypothesis of this statement is "a≠0 and b≠0 and ab=0": it is false because do not exist two real numbers a and b such that a≠0 and b≠0 and ab=0; the thesis is "1=0": it is false because 1≠0. The implication expressed by the statement is true, because the falsity of the hypothesis.

The proof is within the real numbers theory (or more generally the mathematical fields theory) and is based on the following two axioms:

1. If a number is not zero, it has a multiplicative inverse;
2. If both sides of an equality are multiplied by the same number, the equality relation is maintained.

In this proof, these axioms are used to make some deduction on impossible mathematical objects. Axiom 1 is applied to the non existing real numbers a and b such that $a\neq0$ and $b\neq0$ and $ab=0$; axiom 2 is applied to "$ab=0$", an equality formulated with the two non existing numbers.

In summary, while the truth of the secondary statement E* depends on the falsity of its antecedent, the validity of its proof C is based on the validity of a deductive chain within the mathematical theory T, that is applied to impossible mathematical objects.

From the didactical point of view, some authors, for instance Durand-Guerrier (2003), already pointed out students' difficulties in evaluating or accepting the truth-value of an implication with a false antecedent; the following discussion will focus on specific difficulties originated by a proof validating an implication where both the antecedent and the consequent are false.

ANALISYS OF A CASE

The protocol we are going to analyse clearly shows what kind of difficulties may emerge when a student is producing a proof by contradiction, and as a consequence of a 'false' assumption, he/she is required to manage the mathematical theory of reference in respect to impossible objects.

The subject of the interview, Maria, is an university student (the last year of the Faculty of Pharmacy) and as it is possible to notice, she is familiar with proof. In the following an excerpt of the interview is reported; in the transcript "I" indicates the interviewer, "M" indicates the student, and bold character indicates that the subject in some way emphasised her words.

Excerpt 1 (from Maria's interview)

1. I: Could you try to prove by contradiction the following: "if ab=0 then a=0 or b=0"?
2. M: [...] well, assume that ab=0 with a different from 0 and b different from 0... I can divide by b... ab/b=0/b... that is a=0. I do not know whether this is a proof, because **there might be many things that I haven't seen**.
3. M: moreover, so as ab=0 with a different from 0 and b different form 0, that is against my common beliefs (*ita. contro le mie normali vedute*) and I must pretend to be true, **I do not know if I can consider that 0/b=0. I mean, I do not know what is true and what I pretend it is true**.
4. I: let us say that one can use that 0/b=0.
5. M: it comes that a=0 and consequently … we are back to reality. Then it is proved because … also in the absurd world it may come a true thing: thus I cannot stay in the absurd world. **The absurd world has its own rules, which are absurd,** and if one does not respect them, comes back.
6. I: who does come back?
7. M: It is as if a, b and ab move from the real world to the absurd world, but the rules do not function on them, consequently they have to come back …
8. M: But my problem is to understand which are the rules in the absurd world, are they the rules of the absurd world or those of the real world? **This is the reason why I**

have problems to know if 0/b=0, I do not know whether it is true in the absurd world. […]

9. I: (The interviewer shows the proof by contradiction of "$\sqrt{2}$ is irrational") what do you think about it?

10. M: in this case, I have no doubts, but why is it so? … perhaps, when I have accepted that the square root of 2 is a fraction I continued to stay in **my world**, I made the calculations **as I usually do**, I did not put myself problems like "in this world, a prime number is no more a prime number" or "a number is no more represented by the product of prime numbers". The difference between this case and the case of the zero-product is in the fact that this is obvious whilst I can believe that the square root of 2 is a fraction, **I can believe that it is true and I can go on as if it is true**.In the case of the zero-product I cannot pretend that it is true, **I cannot tell myself such a lie and believe it too!**

Maria's arguments are firmly based on what she believes it is "true"; she seems to refer to numbers relations in a numerical world which she knows and is familiar with (integers, rationals, or perhaps real numbers). On the contrary, when she has to construct arguments in what she calls an *absurd world*, the world where there exist two numbers, different from zero and such as their product is zero, Maria looses the control, because in this world she does not know any more what is true and what is false (8). In Maria's opinion, when one assumes something false, everything can happen, it might even occur that 0/b≠0 (3).

Maria is not able to control the relationship between her argumentation and the mathematical theory within which such argumentation should make sense. The absurdity of the assumption, on which the deduction has to be based, upset the truth values of statements that for the subject are fundamental (3), so that she suspects that it might be possible the existence of a different theory, suitable for such an absurd situation. She opposes the "*real world*" and the "*absurd world*", each referring to its own different *rules* (5;8).

Thus Maria fails to grasp a fundamental point, which constitutes a key element of the proof scheme by contradiction: the fact that the secondary statement is *valid* in the mathematical theory of reference. She focuses her attention on truth rather than on validity, and she looks for truth in a world where there are two non zero numbers, whose product is equal to zero. Our interpretation of Maria's difficulties finds further support in her final remark; she states that she has no problems to accept the proof of the irrationality of $\sqrt{2}$, because in that case she finds easy to believe that $\sqrt{2}$ is truly a fraction; although *absurd*, the world where $\sqrt{2}$ is a fraction, is acceptable for Maria and in such a world, what are for Maria the fundamental truths are not upset (10).

CONCLUSIONS

The example discussed above clearly shows the crucial role played by the mathematical theory of reference in proving statements starting from false hypotheses. The example and its discussion confirms through empirical evidence what argued by other researchers, for instance Leron (1985), about the cognitive strain needed by reasoning in a *false world*, i. e. reasoning on the base of false assumptions.

At the source of the difficulties with proof by contradiction there seems to be the fact that in the *absurd world* some of the fundamental properties are upset, so that they no more can be true. It seems that false hypotheses may produce a shortcut in the system of beliefs of a subject and induce impasse or doubts on the proving process: the subject loses the control on the deductive steps of the proof, because he/she does not know what is or is not true.

This attitude is consistent with the well spread opinion that a theorem expresses the link between properties that are true, or at least assumed true. Using the words of Berenice, a 13th grade student of a scientific high school, during the interview:

Excerpt 2 (from Berenice's interview)

1. I: could you tell me what is a theorem for you?
2. […]
3. B: generally speaking, a theorem is … is something … and … that … that is proved on the base of things that I know for sure that are true and from them I can start to prove for sure other things, I mean … more or less
4. .[…] I mean, I do … I must come to say that one certain thing is true, I assume that one certain thing is true, or I assume one certain thing and … through some steps that I know I can do because I know that they are … are correct, I come to say that that thing is such as it was assumed.

According to what expressed by Durand-Guerrier (2003), implication with false antecedent are not accepted or however are considered as false by students: in order to accept a proof it seems necessary to start from true, or at least potentially true assumptions. In the *real world*, as Maria says, or at least in a world that can be assumed real (as the world where $\sqrt{2}$ can be expressed by a fraction), the theory and the rules to be applied are not put into question. On the contrary, assuming false hypotheses can block the deductive process because it may ask to apply the mathematical theory to absurd situations.

The structural analysis of a proof by contradiction, presented above, highlighted the complex system of relationships between key elements, such as the principal, the secondary statement, and its proof, that were subsequently employed in the discussion of an exemplar protocol; according to our general assumption, specific difficulties were interpreted focussing on the proof of the secondary statement. Further investigation have been carried out and are still in progress in this same direction; for instance, besides the problems related to the proof of the secondary statement, we drew our attention on the passage from the principal to the secondary statement and vice-versa and the problems that this passage can present, besides the difficulty coming from effect of mathematical negation on the correct formulation of the secondary statement.

We claim that this method of research can be generalized: in fact, the structural analysis provides an effective model for generating specific research hypotheses concerning students’ difficulties related to different elements involved in a proof by contradiction.

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WILL PENELOPE CHOOSE ANOTHER BRIDEGROOM? LOOKING FOR AN ANSWER THROUGH SIGNS

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Young students working in group have to solve a problem related to Penelope's legend. This paper analyses the students' process to the solution. In particular, it investigates the genesis of written signs starting from specific gestures, progressively shared within the group. These gestures have various functions: understanding the situation, looking for patterns or rules, anticipating and accompanying productions of written representations, drawings and symbols necessary to solve the problem. Gestures are constitutive part of the APC space, a theoretical notion introduced as crucial element to properly describe what happens in the activity of the group.

INTRODUCTION AND THEORETICAL BACKGROUND

Theoretical issues relative to recent studies conducted in the fields of Psychology, Neuroscience and Mathematics Education point out the relevance of perceptuo-motor ways of learning and of their multimodal features. They constitute the starting point of the present research study, which adopts a semiotic perspective as analysis tool. The notion of *APC-space* is introduced as crucial element to properly describe what happens in the maths classroom.

The existence of two fundamental kinds of learning modality has been pointed out in Psychology (Antinucci, 2001). The first one normally occurs in an abstract manner, through the interpretation and transmission of books. It is based on the encoding of symbols and on the mental reconstruction of what they refer to, and it is called *symbolic-reconstructive*. The second one is the so-called *perceptuo-motor* learning, which occurs through perception and motor action on the physical world. It "is not bounded to "practical" knowledge, to learning to do, as it is sometimes asserted" (Antinucci, 2001; p.12, English translation). The two modalities differ from each other not in terms of the nature of what is learnt, but in how learning occurs.

The relevance of action and perception has also been considered specifically in relation to mathematics learning (Arzarello et al., 2005; Nemirovsky, 2003). Taking into account recent findings in Neuroscience, Nemirovsky (2003) claims that the processes of thinking and understanding are constituted by *perceptuo-motor activities*, e.g. bodily actions, gestures, manipulation of materials or artefacts, acts of drawing, even eye motions, gazes, tones of voice, and facial expressions. Indeed, he sees thinking and understanding as perceptuo-motor activities. In his words:

> While modulated by shifts of attention, awareness, and emotional states, understanding and thinking are perceptuo-motor activities; furthermore, these activities are bodily

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> distributed across different areas of perception and motor action based on how we have learned and used the subject itself. This conjectures implies that the understanding of a mathematical concept rather than having a definitional essence, spans diverse perceptuo-motor activities, which become more or less active depending of the context. (Nemirovsky, 2003; p. 108)

An overall question coming from this discussion is then: *What is understanding?* Very recent discoveries in Neuropsychology can bring interesting results to such issue. The main result is that conceptual knowledge is mapped within the sensory-motor system of the brain (Gallese & Lakoff, 2005). Besides, the sensory-motor system not only provides structure to conceptual content, but also characterises the semantic content of concepts in terms of the way in which we function with our bodies in the world. This is consistent with the assumption of perceptuo-motor activities as a constitutive part of thinking. The sensory-motor system of the brain is *multimodal* rather than modular. Then, language, which exploits the character of the sensory-motor system, is inherently multimodal in the sense that it uses many different modalities linked together: sight, hearing, touch, motor actions, and so on.

Our study considers the learning process as a unique integrated system, composed by different modalities: gestures, oral and written language, symbols, and so on (Arzarello & Edwards, 2005; Robutti, 2005). In this approach, multimodality appears as a key-element. The notion of *Space of Action, Production and Communication* (APC-space in brief) allows to frame, within a realistic and multimodal-oriented picture, the learning process in the mathematics classroom (Arzarello, in press; Arzarello & Olivero, 2005). The APC-space is a 'space' in which cognitive processes develop through social interaction. It is like an integrated and dynamic set, acting as a whole, possibly fostered and shared in the classroom. Its main elements are: the body, the physical world and the cultural environment. The three letters **A**, **P**, **C** illustrate its dynamic features: students' **A**ctions and interactions (in a situation at stake, with their mates, with the teacher, among themselves, with tools), their **P**roductions (e.g. answering or posing a question, a new problem, and so on) and **C**ommunication aspects (e.g. when the discovered solution is communicated to a mate or to the teacher, using suitable representations). Such dynamic aspects involve the various components of the APC-space, i.e.: culture, sensory-motor experiences, embodied templates, languages, signs, representations, etc. These elements merged together shape a multimodal system, through which describing didactical phenomena.

In the APC-space we find the *semiotic representations* (Duval, 1999), namely "signs and rules of use that bear an intentional character" (Duval, 1999, p. 43): as *semiotic means of objectification* (Radford, 2003) they play a fundamental role in the construction of new mathematical knowledge. Typically, the semiotic representations can be transformed from one into another (Duval, 1993). This can occur within the same register, e.g. in algebraic manipulations, or from one register to another, e.g. translating an algebraic representation into a geometric one. Semiotic means of objectification do not include only written forms but also other elements, as oral and written languages (including sketches, drawings, graphs), gestures, gazes, and so on.

Our frame gives an account of this variety of mutually interacting components as a systemic whole, describing its dynamics and the way it produces new knowledge. Learning is considered as a result of the multimodal interactions among its different and lasting elements more than of the transformations of one into another.

The research focuses on the systemic interplay between gestures, speech and written signs produced during the solution of a problem. We describe how such elements act as means of objectification in the APC-space during the learning processes.

THE ACTIVITY

The Story. The story told to the pupils comes from the legend of Penelope's cloth in Homer's Odyssey. We modified the original text to get a problem-solving situation that allowed facing some conceptual nodes of mathematics learning (decimal numbers; space-time variables). The text of the story, transformed, is the following:

> … On the island of Ithaca, Penelope had been waiting twenty years for the return of her husband Ulixes from the war. However, on Ithaca a lot of men wanted to take the place of Ulixes and marry Penelope. One day the goddess Athena told Penelope that Ulixes was returning and his ship would have employed 50 days to arrive to Ithaca. Penelope immediately summoned the suitors and told them: "I have decided: I will choose my bridegroom among you and the wedding will be celebrated when I have finished weaving a new piece of cloth for the nuptial bed. I will begin today and I promise to weave every two days; when I have finished, the cloth will be my dowry". The suitors accepted. The cloth had to be 15 spans in length. Penelope immediately began to work, but one day she wove a span of cloth, while the following day, in secret, she undid half of it… Will Penelope choose another bridegroom? Why?

Methodology. When the Penelope's story was submitted to the students (Dec. 2004-Feb. 2005) they were attending the last year of primary school (5^{th} grade). One of the authors (B. Villa) was their teacher of mathematics. Later, in April-May 2005 in the same school six more teachers submitted the story to their classrooms, as part of an ongoing research for the Comenius Project DIAL-Connect (Barbero et al., in press). Students were familiar with problem solving activities, as well as with interactions in group. They worked in groups in accordance with the didactical contract that foresaw such a kind of learning. The methodology of mathematical discussion was aimed at favouring the social interaction and the construction of shared knowledge. As part of the didactical contract, each group was also asked to write a description of the process followed to reach the problem solution, including doubts, discoveries, heuristics, etc. Students' works and discussions were videotaped and their written notes were collected.

The activity consisted of different steps we can summarise as follows. First the teacher reads the story, and checks the students' understanding of the text; the story is then delivered to the groups. Different materials are at students' disposal, among which paper, pens, colours, cloth, scissors, glue. In a second phase, the groups produce a written solution. The teacher invites the groups to compare the solutions in a collective discussion; she analyses strategies, difficulties, misconceptions, thinking patterns and knowledge contents to be strengthened. Then, a poster with the different groups' solutions is produced. In the final phase, the students are required to produce a number table and a graph representing the story; they work individually using Excel to construct the table and the graph of the problem solution. Again, they discuss about different solutions and share conclusions.

The part of the activity analysed below is a small piece of the initial phase (30'); it refers to a single group in the classroom of B. Villa, composed by five children: D, E, M, O, S, all medium achievers except M, who is weak in mathematical reasoning.

ANALYSIS: A STORY OF SIGNS WITHIN THE APC-SPACE

The main difficulty of Penelope's problem is that it requires two registers to be understood and solved: one for recording time, and one for recording the successive steps of the cloth length. These registers must be linked in some way, through some relationship (mathematicians would speak of a function linking the variables time and cloth length). At the beginning, these variables are not so clear for the students. So, they use different semiotic means to disentangle the issue: gestures, speech, written signs. They act with and upon them; they interact with each other; they repetitively use the text of the story to check their conjectures; they use some arithmetic patterns. We see an increasing integration of these components within an APC-space they are progressively building and enriching. In the end, they can grasp the situation and objectify a piece of knowledge as a result of a complex semiotic and multimodal process. We shall sketch some of the main episodes and will comment a few key points in the final conclusion (numbers in brackets indicate time).

Figure 1

Figure 2

Figure 3

Figure 4

Episode 1. The basic gestures. After reading the text, the children start rephrasing, discussing, and interpreting it. To give sense to the story, they focus on the action of weaving and unravelling a span of cloth, which is represented by different gestures: a hand sweeping on the desk (Fig. 1), the thumb and the index extended (Fig. 2), two hands displaced parallel on the desk (Figg. 3 and 4). Some gestures introduced by a student are easily repeated by the others, and become a reference for the whole group. This is the case of the two parallel hands shown in Figg. 3 and 4.

Attention is focused on the action, and the gestures occur matching either the verbal clauses, or the "span", as we can see from the following excerpt:

(6'58'') **S:** She makes a half (*hand gesture in Fig. 2*), then she takes some away (*she turns her hand*), then she makes… (*again, her hand is in the position of Fig. 2*) […]

E: “It is as if you had to make a piece like this, it is as if you had to make a piece of cloth like this, she makes it (*gesture in Fig. 3*). Then you take away a piece like this (*gesture in Fig. 5*), then you make again a piece like this (*gesture in Fig. 3*) and you take away a piece like this (*gesture in Fig. 5*)”

Figure 5

O: “No, look… because… she made a span (*Fig. 4*) and then, the day after, she undid a half (*O carries her left hand to the right*), and a half was left… right? … then the day after…”

D: (*D stops O*) “A half was always left”

The dynamic features of gestures that come along speech condense the two essential elements of the problem: time passing and Penelope’s work with the cloth. Their existence as two entities is not at all explicit at this moment, but through gesturing children make the problem more tangible. The function of gestures is not only to enter into the problem, but also to create situations of discourse whose content is accessible to everyone in the group. The rephrasing of similar words and gestures by the students (see the dispositions of the hands in Fig. 4) starts a dynamics for creating an APC-space, a common substrate made of various interacting elements (gestures, gazes and speech at the moment), upon which the group starts to solve the problem.

Episode 2. From gestures to written signs. After having established a common understanding of what happens in Penelope’s story, the children look for a way to compute the days. S draws a (iconic) representation of the work Penelope does in a few days, actually using her hand to measure a span on paper. The previous gesture performed by different students (Figg. 3-5) becomes now a written sign (Fig. 6). As happened before with words and gestures, the drawing is also imitated and re-echoed by the others (Fig. 7): even these signs, coming from the previous gestures, contribute to the growth of an APC space. The use of drawings makes palpable to the students the need of representing the story using two registers. See the two types of signs in Figg. 7-8: the vertical parallel strokes (indicating spans of cloth) and the bow sign below them (indicating time).

Figure 6

Figure 7

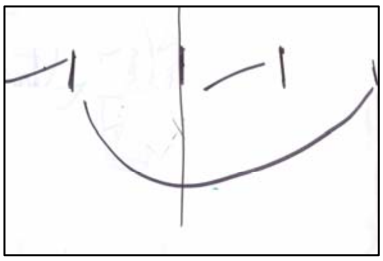

Figure 8

Episode 3. The local rule. In the following excerpts the children further integrate what they have produced up to now (speech, gestures and written representations) and use also some arithmetic; their aim is to grasp the rule in the story of the cloth, and to reason about it. They can now use the written signs as “gestures that have been fixed” (Vygotsky, 1978; p. 107) and represent the story in a condensed way (see Fig. 8); moreover they check their conjectures reading again the text of the problem:

(10’30’’) **S:** From here to here it is two spans (*she traces a line, mid of Fig. 8*). If I take half, this part disappears (*she traces the horizontal traits in Fig. 8*) and a span is left; therefore in two days she makes a span

O: No, in four days, in four, because…

S: In four days she makes two spans, because (*she traces the curve under the traits in Fig. 8*)…plus this

O: In four days she makes one, because (*she reads the text*), one day she wove a span and the day after she undid a half…

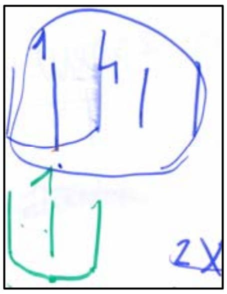

Figure 9

Figure 10

Figure 11

As one can see in Fig. 7, S tries to represent on paper Penelope’s work of weaving and also of unraveling, which causes troubles, because of the necessity of marking in different ways time and length. These two aspects naturally co-existed in gestures of Figg. 1-3. O finds the correct solution (4 days for a span), but the group does not easily accept it, and O gets confused. The drawing introduced by S (Fig. 8) represents the cloth, but with holes; due to the inherent rigidity of the drawing, students easily see the span, but not half a span. A lively discussion on the number of days, needed to have a span, begins. Numbers and words are added to the drawings (Figg. 9-10), and fingers are used to compute (Fig. 11). New semiotic means enter the APC space that is consolidating more and more, not by juxtaposition or translation but by integration of its elements: they all continue to be active even later, as we see below.

Episode 4. Towards a global rule. Once the local question of “how many days for a span” is solved, the next step is to solve the problem globally. To do that, the rule of “4 days for a span” becomes the basis (Fig.12) of an iterative process:

(13’30’’) **O, E:**… it takes four days to make a whole span (*E traces a circle with the pen all around*: Fig. 12)

D: and other four to make a span (*D shows his fingers*) and it adds to 8 (*D counts with fingers*)

S: so, we have to count by four and arrive at 50 days (*forward strategy*: Fig. 13) [...]

(14’25’’) **O**: no, wait, for 15 spans, no, 4 times 15

S: no, take 15, and always minus 4, minus 4, minus 4 (or: 4 times 5), minus 2, no, minus 1 [*backward strategy*: Fig. 14]

Two solving strategies are emerging here: a *forward strategy* (counting 4 times 15 to see how many days are needed to weave the cloth) and a *backward strategy* (counting "4 days less" 15 times to see if the 50 days are enough to weave the cloth). The two strategies are not so clear to the children and conflict with each other.

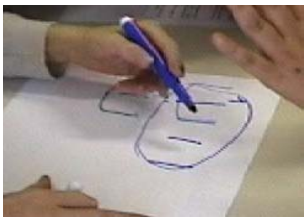

Figure 12

Figure 13

Figure 14

Figure 15

In order to choose one of them, the children use actual pieces of paper, count groups of four days according to the forward strategy and so they acquire a direct control on the computation. Only afterwards they compute using a table, and find that 60 days are needed for 15 spans of cloth. In this way, they can finally answer the question of the problem and write the final report: Penelope will not choose another bridegroom.

CONCLUSIONS

This is a story of signs, produced by children within a gradually developing multimodal cognitive environment, the APC-space, in which the signs (gestures included) live in a complex network of mutual relationships. The story starts with the gesture of the two hands displaced parallel on the desk (episode 1). This gesture generates later a written iconic representation (episode 2), successively enriched by numerical instances (episode 3) and by arithmetic rules (episode 4), expressed through speech and new and old gestures. Gesture, speech, written signs and arithmetic representations grow together in an integrated way. In so doing, they generate a richer and richer APC-space where the students can act and interact to grasp the problem, to explore it, and to elaborate solutions. All its elements (e.g. the initial gestures and written signs) are active in a multimodal and holistic way up to the end. This is even evident when the students discuss how to write the solution in the final report (Fig. 15: 27' 32"). Gestures and speech intervene first as thought means for understanding the story of the cloth; later as control means to check the conjectures on the rule. Information is condensed in gestures, entailing a global understanding of the story. Written signs make explicit what is condensed in gestures and perceivable the two registers that allow children to grasp the story separating its structural elements (time and cloth development). Speech objectifies the structure of the story, first condensing the local rule in a sentence (episode 3), then exploiting the general rule as an iterative process (episode 4). The three types of signs (gestures, written signs and speech) combined together produce the cognitive environment, within which the children can objectify their knowledge in relation to the problem. It is our contention that the semiotic objectification in this story consists mainly of the mutual interactions among the different registers. These interactions in our view do constitute an *integrated domain* rather than a sequence of transcriptions from one register to the other, as pointed out in other studies (e.g. Duval, 1993). As shown in this case study, objectification seems to happen in a holistic and multimodal way. Within our theoretical frame, the interpretation of the semiotic registers in the APC-space allows describing the didactical phenomena as a multimodal system developing in time.

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MOTIVATION AND PERCEPTIONS OF CLASSROOM CULTURE IN MATHEMATICS OF STUDENTS ACROSS GRADES 5 TO 7

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This study investigates changes in students' motivation and in perceptions of their classroom environment in mathematics across grades 5 to 7. The analysis of 488 students' responses to a four-scale questionnaire suggests that students' motivation in mathematics declines during the transition to secondary school. Elementary school students endorse more task and effort goals whereas middle school students endorse performance goals. Differences across grades were also found in students' perceptions regarding their classroom culture. Sixth grade students perceive their classroom goal structure to be rather task focused, the teachers to be more friendly and encouraging investigative processes, cooperation, participation, differentiation and personalization than seventh grade (middle school) students.

BACKGROUND AND AIMS OF STUDY

The period surrounding the transition from primary to secondary school has been found to result in a decline in students' motivation and achievement in mathematics (see e.g., Eccles et al, 1993, Midgley et al., 1995). Earlier studies have conceptualised motivation in very different ways, including various motivational constructs. In some studies motivation was operationalized in terms of a cognitive perspective, i.e. students' motivational orientation (Anderman et al., 2001, MacCallum, 1997), whereas other studies have adopted an affective perspective examining students' self-beliefs such as self-efficacy, self-esteem and self-competence (e.g., Pajares & Graham, 1999, Wigfield & Eccles, 1994).

The decline of students' motivation in mathematics across the transition from primary to secondary school was found to be related to certain dimensions of the school and classroom culture (e.g., Eccles et al, 1993, Midgley, Feldlaufer & Eccles, 1989a, 1989b). These studies suggested that there are developmentally inappropriate changes in a cluster of classroom organizational, instructional and climate variables. The dimensions of the school culture that were found to have an effect on motivation during the transition to middle school include the perceived classroom goal structure (Midgley et al., 1995, Urdan & Midgley, 2003), teachers' sense of efficacy and teachers' ability to discipline and control students (Midgley et al., 1989a), teacher-student relations and opportunities for students to participate in decision making (Midgley et al., 1989b).

A slightly different analysis of the possible environmental influences associated with the transition to middle school draws on the idea of person-environment fit. In this theoretical framework, it is the fit between the developmental needs of the adolescent and the educational environment that is important (Eccles et al., 1993), that is the fit

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between the preferred and the actual classroom environment. If it is true that different types of educational environments may be needed for different age groups to meet developmental needs and to foster continued growth, then it is also possible that some types of change in educational environments may be inappropriate at certain stages of development e.g. the early adolescent period during which students move to secondary school. In fact, some types of changes in the educational environment may be developmentally regressive. Exposure to such changes is likely to lead to a particularly poor person-environment fit, and this lack of fit could account for some of the declines in motivation seen at this developmental period.

The above-mentioned studies examined motivation using either an expectancy-value model or a goal approach to motivation. In the present study the Personal Investment Theory (PIT), as a goal approach to motivation, provided the conceptual framework for examining a number of different facets of motivation such as motivational goals, goal orientations, self-efficacy and perceived instrumentality of mathematics. PIT was further developed by a different formulation of the concept of culture (McInerney, Yeung and McInerney, 2000), leading to systematic consideration of whether there was an "optimum school or classroom culture" for personal development. In the present study classroom culture was operationalized as the classroom goal structure, teacher academic press, friendliness and personalization, teacher control (independence) and teacher instructional practices (participation, investigation and differentiation).

The purpose of the present cross-sectional study was to examine the developmental changes in students' motivation and in perceptions of their classroom environment in mathematics across grades 5-7 and especially across the transition from primary to secondary school. The transition to secondary school in the specific educational system where the study is conducted occurs after grade 6. More specifically, we sought answers to the following research questions:

1) Are there any developmental changes in specific aspects of students' motivation in mathematics across grade levels (5-7) and especially across the elementary (grade 6) and secondary school (grade 7)?
2) Are there any statistically significant differences in students' perceptions of their classroom environment in mathematics in terms of students' grade level (5-7) and especially across the elementary and secondary school?
3) Are there any developmental differences in the fit between the actual and the preferred classroom environment in mathematics across grade levels (5-7) and especially across the elementary and secondary school?

METHOD

Participants: The subjects of this study were 488 students from 3 elementary and 1 secondary school; 370 students were in elementary school (98 students in grade 5 and 272 in grade 6) and 118 students were in secondary school (grade 7).

Instrumentation: Data were collected through a self-report questionnaire comprising four scales. The first scale was an adaptation of the Inventory of School Motivation Questionnaire (McInerney, Yeung & McInerney, 2000); it included 53 items measuring students' motivational goals and orientations in mathematics (e.g. "I try hard to make sure that I am good at my math work" or "I want to do better than the other students in mathematics"). The second scale was an adaptation of the Patterns of Adaptive Learning Survey (Midgley et al., 2000) and included 30 items measuring students' perceptions of their classroom goal structure, teacher academic press, self-efficacy and instrumentality of mathematics (e.g. "In our class getting good grades in mathematics is the main goal", or "It is important for me to perform well in mathematics to reach my future goals"). The third scale was an adaptation of the Student Classroom Environment Measure (Eccles et al., 1993b); it included 17 items measuring students' perceptions regarding their teacher friendliness and practices such as cooperation and interaction, competition and social comparison (e.g. "The math teacher is friendly to us" or "We get to work in small groups when we do math"). The fourth scale was an adaptation of the Individualized Classroom Environment Questionnaire (Fraser, 1990) and included 23 items tapping students' perceptions on five classroom dimensions: personalization, participation, independence, investigation and differentiation (e.g. "Students give their opinions during discussions in mathematics" or "All students do the same work at the same time in mathematics"). The latter scale was completed by students in two different forms, measuring the perceived as actual and the preferred classroom environment in mathematics in each dimension.

The statements were presented at a six-point Likert-type format (1=strongly disagree, 6=strongly agree). The reliability estimates (Cronbach alphas) were found to be quite high for all scales, ranging from a=.71 to a=.88.

Data Analysis: Data processing was carried out using the SPSS software. The main statistical procedures used in this study were analysis of variance (ANOVA) and paired samples t-test.

RESULTS

To answer the first two research questions students' responses to the scales tapping motivation were analysed using one-way analysis of variance (ANOVA). Significant grade-level effects were followed up with the Scheffe multiple comparison procedure to assess the significance between each pair of means. The .05 level of significance was adopted for these paired comparisons.

Table 1 presents the means of the students in all the motivation variables. Similar numeric superscripts within each row indicate that the means in that row are not different from one another. The analysis of variance revealed that fifth and sixth graders' mean ratings in motivational variables (with the exception of praise motivational goal) are not significantly different. The overall grade difference effects between students in grade 6 and 7 are significant for effort, task, valuing and performance-approach motivational orientations and perceived efficacy in mathematics. More specifically, the analysis indicated that for effort and task motivational goals, for valuing/mastery motivational orientation and for perceived efficacy in mathematics, the sixth graders' mean ratings are significantly higher than those of the seventh graders' (for effort $F=3.180$, $p<0.05$; for task $F=3.052$, $p<0.05$; for valuing/mastery orientation $F=3.293$, $p<0.05$ and for efficacy $F=4.741$, $p<0.01$).

Students in grade 7 performed at a significantly higher level on performance-approach motivational orientation ($F=3.741$, $p<0.05$) than the students in grade 5 and grade 6, whereas students in grade 5 scored significantly higher on praise motivational goal ($F=4.769$, $p<0.01$) than students in grade 6 and grade 7.

	GRADE LEVEL								
	GRADE 5			GRADE 6			GRADE 7		
	N	M	SD	N	M	SD	N	M	SD
Motivational goals									
Effort	93	4.43^{1}	0.64	262	4.37^{1}	0.64	115	4.21^{2}	0.77
Praise	95	4.30^{1}	0.67	261	3.99^{2}	0.85	112	3.90^{2}	0.90
Token	94	2.68^{1}	1.03	364	2.61^{1}	1.06	116	2.51^{1}	1.02
Affiliation	96	3.39^{1}	0.90	206	3.39^{1}	0.89	117	3.21^{1}	0.88
Competition	94	2.92^{1}	1.00	265	2.91^{1}	1.12	115	2.83^{1}	0.98
Task	97	3.74^{1}	1.27	271	3.82^{1}	1.04	117	3.66^{2}	1.16
Social Concern	98	3.90^{1}	0.98	271	3.83^{1}	0.97	118	3.84^{1}	1.01
Mot. Goal orientations									
Valuing/Mastery	97	3.98^{12}	0.77	264	4.00^{1}	0.68	116	3.80^{2}	0.78
Performance-Approach	94	3.18^{1}	0.95	261	3.13^{1}	1.03	116	3.46^{2}	1.01
Performance-Avoid	95	3.00^{1}	1.07	257	2.82^{1}	0.98	115	3.03^{1}	0.97
Social	98	3.24^{1}	1.09	263	3.37^{1}	0.93	118	3.16^{1}	0.88
Instrumentality	97	4.40^{1}	0.68	266	4.28^{1}	0.66	115	4.26^{1}	0.69
Efficacy	97	4.15^{1}	0.66	266	4.08^{1}	0.64	117	3.89^{2}	0.78

Table 1: Mean level of motivational variables by grade level

Table 2 presents the means of the students' perceptions in all the classroom variables. The analysis reveals no statistically significant differences between fifth and sixth graders' mean ratings in any classroom variable. The overall grade difference effects between 6th and 7th grade students are significant for all the classroom variables except the teacher academic press ($F=1.948$, $p>0.05$). More specifically, 6th grade students perceive the goal structure of their mathematics classroom as more task focused ($F=5.973$, $p<0.01$) and less performance focused ($F=3.998$, $p<0.05$), whereas 7th grade students (middle school) perceive that the goal structure in their mathematics classroom stresses performance goals more and task goals less.

For personalization, investigation, teacher friendliness, participation, differentiation and cooperation, the sixth graders' mean ratings are significantly higher than those of the seventh graders', showing that in elementary schools teachers are friendlier ($F=4.628$, $p<0.01$), promote investigation processes ($F=3.102$, $p<0.05$), encourage cooperation ($F=6.265$, $p<0.01$), participation ($F=3.104$, $p<0.05$), differentiation ($F=4.215$, $p<0.01$) and personalization ($F=3.546$, $p<0.05$) rather than the teachers in middle schools.

	GRADE LEVEL								
	GRADE 5			GRADE 6			GRADE 7		
	N	M	SD	N	M	SD	N	M	SD
Classroom goal structure									
Mastery	92	4.27^{1}	0.65	261	4.23^{1}	0.63^{1}	116	3.99^{2}	0.80
Performance	90	3.07^{1}	0.88	253	3.11^{1}	0.87	106	3.38^{2}	0.81
Personalization	95	4.22^{1}	0.52	255	4.25^{1}	0.61	111	3.99^{2}	0.78
Teacher academic Press	96	3.44^{1}	0.74	257	3.40^{1}	0.67	113	3.27^{1}	0.77
Investigation	97	3.53^{12}	0.88	267	3.58^{1}	0.85	116	3.35^{2}	0.84
Teacher Friendliness	95	3.23^{12}	0.77	264	3.25^{1}	0.75	114	2.99^{2}	0.82
Cooperation	96	3.27^{12}	0.83	257	3.26^{1}	0.75	115	2.97^{2}	0.79
Independence	95	2.46^{1}	0.97	266	2.54^{1}	0.95	116	3.01^{2}	0.79
Participation	95	3.98^{1}	0.68	264	4.04^{1}	0.70	115	4.29^{2}	0.78
Differentiation	96	2.47^{1}	0.71	260	2.56^{1}	0.75	117	2.17^{2}	0.70

Table 2: Mean level of classroom variables by grade level

Students in grade 7 performed at a significantly higher level on the Independence variable than the students in grade 6 ($F = 11.421$, $p<0.01$), showing that students in middle school are feeling more free to control their own learning and behaviour in mathematics than the students in the elementary school.

As far as the developmental changes in the fit between the actual and the preferred classroom environment between elementary and middle school are concerned (third research question), paired samples t-test was conducted separately for students in each grade (5, 6 and 7) regarding students perceptions on the five dimensions of the classroom environment in mathematics (personalization, participation, investigation, differentiation and participation).

The analysis revealed that the differences between the actual and the preferred classroom environment in mathematics for the students in grade 5 and 6 are not statistically significant. On the contrary for the students in grade 7 the actual environment in mathematics that they perceive in their classroom is significantly different from their preferred classroom environment for personalization (t=2.690, p<0.01), for differentiation (t=5.307, p<0.01), for participation (t=-2.082, p<0.05) and for investigation (t=-2.252, p<0.05) but not for independence (t=-.523, p>0.05).

	GRADE LEVEL								
	GRADE 5			GRADE 6			GRADE 7		
	N	t	df	N	t	df	N	t	df
Differentiation	94	1.138	93	251	1.928	250	114	5.307**	113
Personalization	95	1.178	94	252	.311	251	112	2.690**	111
Participation	93	-1.692	92	256	.-183	255	113	-2.082*	112
Independence	89	-1.486	88	256	-1.931	255	112	-.523	111
Investigation	94	-1.898	93	262	-1.856	261	115	-2.252*	114

*p<0.05, **p<0.01

Table 3: T scores to classroom variables according to grade level

DISCUSSION AND RECOMMENDATIONS FOR FURTHER RESEARCH

The purpose of the present study was to explore the developmental differences in students’ motivation, in perceptions of their classroom environment in mathematics and of the fit between the actual and the preferred classroom environment across the elementary and middle school.

The results confirmed the conclusions of previous studies about the decline in students’ motivation in mathematics during the transition to secondary school and about the developmental differences in the way students perceive their classroom environment in mathematics across the elementary and secondary school (e.g., MacCallum, 1997, Urdan & Midgley, 2003). More specifically students in grade 7 endorsed more performance goals than the students in elementary school (grade 6) who endorsed more task and effort motivational goals and a valuing/mastery motivational orientation. Further more, 6th grade students perceive their classroom goal structure to be more task focused and less performance focused, whereas middle school students perceive that their classroom goal structure stresses performance goals more and task goals less. Lastly, students in the elementary school perceive that in their classrooms during mathematics the teacher is friendly, caring and helpful, and that the teacher encourages cooperation, investigation, differentiation and participation than students in the seventh grade in middle school. Given these differences in the perceived school culture, the goals teachers have for their students and the instructional strategies they use in their classrooms, it is not surprising that seventh grade middle school students adopt personal goals that are more performance focused than do fifth and sixth elementary students.

Seventh graders reported that they perceive their classroom culture as more independent than students in the elementary school. That is pretty logical taking into consideration that elementary school classrooms in Cyprus as compared with middle school classrooms are characterised by a greater emphasis on teacher control and discipline and fewer opportunities for student decision making, choice and self-management.

The findings of this study contribute to our understanding of the developmental differences between the actual and the preferred classroom environment in mathematics across the elementary and secondary school. Exposure to such changes is likely to lead to a particularly poor person-environment fit, and this lack of fit could account for some of the declines in motivation seen at this developmental period. Therefore, the environmental changes often associated with the transition to middle school seem especially harmful in that they emphasize ability self-assessment at a time of heightened self-focus; they emphasize lower level cognitive strategies at a time when the ability to use higher level strategies is increasing; and they disrupt social networks at a time when adolescents are especially concerned with close adult relationships outside of the home (Eccles & Midgley, 1989).

The findings of the present study highlight the developmental changes in students' motivation in mathematics and the differences in the perceived classroom culture across the elementary and secondary school. Longitudinal studies addressing these issues can assist in unravelling the complexity of motivational change during the transition from primary to secondary school. In these studies however, motivation must be studied as a multifaceted construct (with the inclusion of cognitive, affective and social constructs) and motivational change and its relation to the school and classroom structure must be viewed in different ways and not only for students as a whole group. Recent research in the area of students' perceptions of classroom environments adds credence to the view that students do not all perceive the same environment in the same way at least on some of its dimensions (MacCallum, 1997).

There is also a need to understand not only the effects of what is most prevalent in classrooms but also try to determine what the most facilitative environments are, even if they are uncommon, in order to test the effects of these environments on the nature of change in student motivation. Lastly, longitudinal research must address the developmental changes in the fit between the actual and the preferred classroom environment in mathematics and provide information of the dimensions of the school culture that influence motivational change. Such information will be useful for teachers, educators, counsellors and policy makers to make systemic transitions easier so that fewer students are lost. These preventive steps can include the identification of the dimensions of the school culture that have a positive or a negative impact on students motivation and the strengthening of the support structures provided to students either by their family or by the school (transition programs).

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DEDUCTIVE REASONING: DIFFERENT CONCEPTIONS AND APPROACHES

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This study examines the conceptions of, and approached to, deductive reasoning of people involved in mathematics education and logic. Data source includes 21 individual semi-structured interviews. Data analysis reveals two different meanings attributed to deductive reasoning. One approach focuses on the systematic process; the other on the essence of the inference as based on rules of formal logic. In addition, three conceptions regarding the relationships between logical rules inside and outside mathematics were identified, namely, unification, inclusion, and separation. Interconnections between the meaning attributed to deductive reasoning and the approach to logical rules inside and outside mathematics are discussed.

It is commonly accepted nowadays that one of the goals of mathematics teaching is to improve deductive (logical) reasoning. This is stated, for example, in curricula from all over the world (e.g., Ministry of Education and Culture, 1990; National Council of Teachers of Mathematics, 2000; Qualifications and Curriculum Authority, 1988). This study reveals that whereas people involved in various aspects of mathematics education and logic do believe that learning mathematics develops deductive reasoning, they have different conceptions and approaches regarding the meaning of deductive reasoning and its nature in mathematics and outside it.

THEORETICAL BACKGROUND

There are various sorts of thinking and reasoning. Among them are association, creation, induction, plausible inference, and deduction (Johnson-Laird & Byrne, 1991). Deductive reasoning is unique in that it is the drawing of conclusions from known information based on logical rules, where conclusions are necessarily derived from the given information and there is no need to validate them by experiments. A common type of deductive inferring method is the syllogism. The classic syllogism includes three statements: Two premises (or claims) and a logical conclusion, which is deduced from them. A well-known example (taken from Biletzki, 2002) is the following: It is known that all human beings are mortal (premise 1), and that Socrates is a human being (premise 2). Therefore, Socrates is mortal (conclusion). This way of deriving new information and gaining new knowledge is called the activity of deducing from the general to the individual (Morris & Sloutsky, 1998).

Since the early days of Greek philosophical and scientific work, deductive reasoning has been considered as a high (and even the highest) form of human reasoning (Glantz, 1989; Luria, 1976; Sainsbury, 1991). Already Aristotle, who laid down the foundations to this kind of thinking in the 4th century B.C., perceived a person who

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possesses logical-deductive ability as being able to grasp the universe in more profound and comprehensive ways. Similarly, more than two thousand years later, Luria (1976) views deductive ability as necessary for gaining new knowledge and Henle (1962) argues that without deduction we would not be able to have a dialogue of any worth. Throughout human scientific development, great scientists, such as Descartes and Popper, emphasized the importance of this kind of reasoning to science. A domain, which is mostly identified with deductive reasoning, is mathematics. And indeed, deductive reasoning is often used as a synonym for mathematical thinking. It is especially considered to have an essential role in building mathematical justifications and proofs (Ball & Bass, 2003; Reid, 1995; Yackel & Hanna, 2003). Still, deductive reasoning is considered important not only in science and mathematics but also in various other domains. For example, Johnson-Laird and Byrne (1991) claim that a world without deduction is a world without science, technology and a legal system. Wu (1996) says that its existence contributes to good citizenship by facilitating wise decision making related to politics and economy.

As can be seen, deductive reasoning is considered important to various domains in life, as well as in mathematics. The development of deductive reasoning appears as a goal of mathematics teaching in many curricula from all over the world. But missing is research that examines how different people involved in mathematics education and logic view deductive reasoning. This kind of research is important because their conceptions and approaches influence how mathematics is taught, what kind of teaching and learning materials are developed, and how teachers are prepared to teach mathematics. The study reported here is part of a larger study that examines the conceptions that people involved in mathematics education and logic have regarding the connections between mathematics learning and the development of deductive reasoning. This paper reports the findings related to their conceptions and approaches regarding the meaning and nature of deductive reasoning.

METHODOLOGY

The research population includes 21 participants, who belong to at least one of the following groups: junior-high school mathematics teachers, mathematics teacher educators, mathematics curriculum developers, researchers in mathematics education, researchers in science education who study logical thinking, research mathematicians, and logicians. Individual semi-structured interviews were conducted with each of the participants. The interviews lasted between one to two hours, and focused on different issues related to the role of learning mathematics in the development of deductive reasoning. The interviews were transcribed. Using the Grounded Theory method (Glaser & Strauss, 1967) we coded the data from the interviews and generated initial categories, which were constantly compared with new data from the interviews. Based on refinement of the initial categories, we identified core categories, and used them as a source for theoretical constructs. Two of the main aspects that were developed through data analysis are discussed here. One aspect is the interviewees' approaches to the meaning of deductive reasoning.

The second aspect is their approaches to logical rules inside and outside mathematics. After a presentation of each of these aspects their interconnection is examined.

MEANING OF DEDUCTIVE REASONING

Two different approaches regarding the meaning of deductive reasoning were identified among the interviewees.

Approach 1: Focus on the systematic process

People holding this approach describe deductive reasoning as a process in which one develops a solution to a given problem in a systematic, step-by-step form. Each step of this process is derived from the previous one, and leads to the next. For example, an interviewee was asked what the term "deductive reasoning" meant for her. She answered:

> Deductive reasoning, I am talking about being systematic in thinking, thinking and developing ideas in an organized way (interviewee no. 11).

In some point later in the interview she emphasized again this meaning of deductive reasoning, while connecting it to problem solving:

> I think that deductive reasoning is, here, I found the word: being systematic in thinking. We have some problem that we need to solve; I want the students to have a systematic way of thinking that is built layer upon layer – he thinks about something, he draws a conclusion, which brings him to the next thing (interviewee no. 11).

Approach 2: Focus on the logical essence of the inference

People holding this approach describe deductive reasoning as an action of inference based on the rules of formal logic. A valid argument, according to these interviewees, is based on formal logic rules and is necessarily deduced from given premises. For example, an interviewee was asked what the term "deductive reasoning" meant for him. He answered:

> Deductive reasoning is when one starts with assumptions and proves something without including any additional considerations. Proving the conclusion in an entirely logical way (interviewee no.17).

Another interviewee responded to the same question:

> Inference and derivation are the essence of the work of logic. Deductive reasoning is seen when we make the transition from assumptions to a necessary conclusion, or when we examine the validity of an inference (interviewee no. 19).

In summary, Approach 1 focuses on deduction as a *process***,** which is built from successive steps, and is served as a means for solving a given problem. Approach 2, on the other hand, focuses on the *essence* of the inference. It focuses on the inference's validity as conditional to formal logic rules, and on its necessary derivation from its premises.

LOGICAL RULES INSIDE AND OUTSIDE MATHEMATICS

Three distinct conceptions regarding this issue were identified. It is worth noting that the meaning of the term 'logical rules' is not consistent among all interviewees. Some refer to logical rules as systematic "rules". For others, logical rules are the system of laws in formal logic (e.g., modus ponens). In the following we use this term according to the interviewees' approaches to it.

Approach 1: Unification

People holding this approach consider the nature of logical rules inside mathematics as identical to those of outside-mathematics deductive thinking. They do not see any differences in the ways we act according to these rules in the two domains. For example, an interviewee was asked to explain what she meant by saying earlier in her interview that deductive reasoning has an important role in our life. She replied:

> Look, the western world; it acts according to the same logical rules that are common in mathematics… The world demands from us the use of mathematical logical rules, meaning the use of systematic and organized ways of action, in order to make progress (interviewee no. 6).

Approach 2: Inclusion

People holding this view say that logical rules existing inside mathematics constitute the basis for the outside-mathematics deductive thinking rules. In real life, they say, we use the mathematical logical rules when, for example, we build arguments or validate other people’s claims. However, unlike mathematics, they add, in real life there exist different kinds of factors, which affect our deductive mechanism. Therefore we apply other rules, usually 'softer', in addition to the rigorous ones. Analysis of the data suggests a distinction between the types of factors that affect reasoning outside mathematics: Some people talk about internal conditions, such as emotions and beliefs. Others explain the distractive influence by external conditions, such as uncertainty and complexity of phenomena in nature and society. For example, an interviewee was asked whether she thinks learning mathematics contributes to the development of deductive reasoning. She replies positively, yet points at some obstacles. One of them is connected to the differences between the nature of deductions inside mathematics and outside it:

> The problem is that in life it is not always possible to use all these logical inferences [that are used in mathematics]. Sometimes the situations are very complicated and it is impossible to know for sure if something is correct or wrong. It means that not always one thing is derived deductively from the other. Besides, in mathematics there is no such thing as an exception, because then it is actually a counter example that refutes the argument. In life there are sometimes situations that do not conform to the rule and then we refer to them as exceptions. This means that it is impossible to apply a deduction to them… In life we use the mathematical deductive rules, but because it is not always possible, we use sometimes logic which is less strict, something like common sense (interviewee no. 12).

Approach 3: Separation

These people take the former presented approach even further. According to them, outside mathematical context, we do not or even cannot, use the logical rules existing in mathematics. This claim is based on three different points of views: a) the essence of thinking inside mathematics is entirely different from the one outside it, b) in life, as opposed to mathematics, one barely encounters suitable circumstances for using logical rules, and c) in life, in contrast to mathematics, the argumentative norms are such that the logic of an argument one builds is neither a necessary condition for understanding nor for accepting the argument. Following is a citation illustrating the last point:

> If you had taken segments from an everyday discourse in which people do derive things, and analyze them according to logical rules that you know from standard mathematics discourse, you would have said 'Oh my god'. There are infinite examples. A mother says to a child: 'If you don't eat, then you won't get sweets'. The child says: 'I ate, so I deserve some sweets'. It is obvious that that was the mother's intention. She meant to say that if he eats he would get some sweets. But it is not equivalent. It is a different logical phrase. And you know what, as a logician, even I could say to my child: 'If you don't do X you won't get Y'… and I would mean that if she does X she will get Y. These rules are not those rules… The whole thing is that logical rules are logical rules, but in daily life people understand each other even in the case of a rule that is wrong in the logical sense; like the example of the mother and the dessert (interviewee no. 1).

INTERCONNECTIONS BETWEEN ASPECTS

Examination of the different conceptions and approaches across the two aspects discussed above suggests interconnections between them. The interviewees who describe deductive reasoning as a systematic step-by-step approach are those who consider the logical rules inside mathematics to be identical to the rules outside-mathematics deductive thinking.

Similarly, the interviewees who relate deductive reasoning to an action of inference or justification using rules of formal logic are those who make a distinction between logical rules inside mathematics and the ones that are used outside mathematics. Four interviewees belong to the first group and 17 to the second. According to 13 of the second group, in daily life situations people indeed use mathematical logical rules, but different kinds of factors restrict their implementation. Six interviewees (out of the 13) talk about external conditions of uncertainty in life; seven relate the distractive influence to internal conditions like emotions and beliefs. The other four interviewees, from the group of 17 who focuses on the essence of the inference, make a more extreme distinction between the nature of logical rules inside mathematics and outside it. According to them, we do not, or even cannot, use the logical rules existing in mathematics in non-mathematical contexts. The Table below summarizes these findings.

<table>
<tr><th>Meaning of deductive reasoning</th><th colspan="2">Logical rules inside and outside mathematics</th></tr>
<tr><td>Focus on the systematic, step-by-step process (n=4)</td><td colspan="2">Unification
(n=4)</td></tr>
<tr><td rowspan="3">Focus on the essence of the transition from one step to the next, where a step is an inference (n=17)</td><td colspan="2">Inclusion
(n=13)</td></tr>
<tr><td>External
(n=6)</td><td>Internal
(n=7)</td></tr>
<tr><td colspan="2">Separation
(n=4)</td></tr>
</table>

Table: Summary of findings

Another interesting finding of this study is that the first group, which focuses on the systematic aspect and unifies the logical rules inside and outside mathematics, includes only people who are mainly involved in practical work (i.e., schoolteachers, teacher educators, and curriculum developers). All the theoreticians (i.e., researchers in mathematics, mathematics education, and logic) belong to the other group, which focuses on the logical essence of the inference and distinguishes between logical rules inside mathematics and outside it. Nonetheless, there are several practitioners in this group as well.

DISCUSSION

The literature usually approaches deductive reasoning as a process of reaching a conclusion in accordance with the rules of formal logic (e.g., Biletzki, 2002; Johnson-Laird & Byrne, 1991; Luria, 1976; Morris & Sloutsky, 1998; Simon, 1996). This common description fits the one suggested by the interviewees who focus on the logical essence of the inference, and is not compatible with the meaning offered by the interviewees who focus on the systematical characteristics of deductive reasoning.

The conception of some interviewees, that mathematical-logical rules are modified when used in daily life situations, can also be found in recent literature. Whereas in the past there was a widespread belief that logical rules are relevant to how people think in general (e.g., Inhelder & Piaget, 1958), reports of studies from the last decades challenge this claim. For example, there are numerous studies that point to an effect of subjects' beliefs regarding the production, as well as the evaluation, of a logical conclusion (Evans & Pollard, 1990; Markovits & Nantel, 1989; Oakhill, Johnson-Laird, & Gernham, 1989). Because outside mathematical contexts people tend to have considerably prior beliefs and opinions, the findings of these studies imply a significant difference between the nature of reasoning in mathematics and the one applied outside it. The approach of the 'separation' group, which distinctly discriminates between the nature of logical rules inside mathematics and outside it, emerges also in Leron (2003) and in Cosmides and Tooby (1997). These researchers' claim, which is also expressed in the 'separation' group, suggests that mathematical-deductive thinking is actually in a conflict with our natural thinking.

As opposed to the difference that the ‘logical essence of the inference’ group sees in the nature of logical rules inside and outside mathematics, the 'systematic' group identifies the logical rules inside mathematics with those outside it. This distinction between the two groups seems to be grounded in the meaning that each group attributes to deductive reasoning. Systematicness is a concept that people encounter in diverse situations in daily life. Thus, the interviewees who view the use of logical rules as acting systematically conceive these rules as identical inside mathematics and outside it. On the other hand, the strict, formal nature that the interviewees who focus on the essence of the inference assign to deductive reasoning and to logical rules could explain the complexity they, as well as the literature, attribute to the nature of inference in different areas of our life.

Considering the commonly accepted assumption that one of the goals of learning mathematics is to improve deductive reasoning, it is important to be aware that, as this study shows, deductive reasoning may not have the same meaning for all mathematics educators. Furthermore, in light of the finding that all theoreticians hold similar views, whereas practitioners have diverse views, it seems worthy to study more thoroughly whether there is a connection between the nature of people’s occupation and their approaches towards deductive reasoning. Moreover, the findings of this study support the need for empirical research that will examine the connections between teaching mathematics and the development of deductive reasoning.

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THE TENDENCY TO USE INTUITIVE RULES AMONG STUDENTS WITH DIFFERENT PIAGETIAN COGNITIVE LEVELS

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According to the intuitive rules theory, students are affected by a small number of intuitive rules when solving a wide variety of conceptually non-related mathematics and science tasks. The current study considers the relationship between students' Piagetian cognitive levels and their tendency to use intuitive rules. Hundred and fifty seventh graders answered two written tasks, each related to one intuitive rule: more A -- more B or same A -- same B. The findings indicate that the tendency to answer according to the intuitive rules varies according to cognitive level. Most surprisingly, a significantly higher rate of incorrect responses according to the rule same A -- same B, was found for the higher cognitive level. Additional findings and important implications for mathematics and science education are discussed.

INTRODUCTION AND THEORETICAL FRAMEWORK

Over the past decades, researchers, educators and educational psychologists dealing with mathematics and science education have explored students' conceptions and reasoning processes in a wide range of content areas (e.g., Fischbein, 1987, 1999; Greca & Moreira, 2000; Perkins & Simmons, 1988; Vosniadou & Ioannides, 1998). These studies highlighted the persistence of students' preconceptions, misconceptions, or alternative conceptions (i.e. conceptions that are in variance with currently accepted scientific notions). Although most of this research has been content specific and aimed for detailed description of particular misconceptions, several researchers have looked for common roots and have tried to build an extensive theoretical framework. One framework is the intuitive rules theory developed by Stavy and Tirosh (2000). The intuitive rules theory is largely indebted to Fischbein's pioneering work concerning the role of intuition in mathematics and science. Fischbein defines intuition as an immediate cognition that exceeds the given facts, as “a theory that implies an extrapolation beyond the directly accessible information” (1987, p. 13).

The current work aimed to explore if there is any relationship between students' Piagetian cognitive levels and their tendency to use intuitive rules when solving comparison tasks.

The intuitive rule theory

According to the intuitive rule theory, students are affected by a small number of intuitive rules when solving a wide variety of conceptually non-related mathematics and science tasks that share some common, external features. Interviews with children, adolescents and adults, and a recent reaction time study, suggest that these responses are immediate, self evident and coercive (Babai, Levyadun, Stavy, & Tirosh, in press; Stavy & Tirosh, 2000), meeting the major characteristics of intuitive reasoning (Fischbein, 1987). Responses in line with the intuitive rules are often correct. However, sometimes they are in variance with concepts and reasoning in mathematics and science, leading to incorrect judgments and students' mistakes.

So far, three types of responses were identified and, accordingly, three intuitive rules were defined: two relate to comparison tasks (*more A -- more B* and *same A -- same B*), and one to subdivision tasks (*everything can be divided endlessly*). The focus of the current study will be the two intuitive rules related to comparison tasks. An example and a general description for each one of these intuitive rules will be given below.

The intuitive rule *more A -- more B*

In a previous study, Azhari (1998) presented 100 children in grades 1, 3, 5, 7 and 9 (20 from each grade level) with the following rectangle to polygon task (Figure 1), related to the comparison of areas and to the comparison of perimeters.

Shape I Shape II

Figure 1: The rectangle to polygon (by square removal) task

Two identical rectangles are presented. A small square is removed from the corner of one rectangle (e.g., from the upper right corner of the rectangle depicted on the right hand side of Figure 1) to form a polygon (Shape II). Participants are asked to compare the perimeters of the two shapes, Shape I (the original rectangle) and Shape II (the derived polygon).

Azhari (1998) found that in each of these grade levels, at least 70% of the students incorrectly claimed that the perimeter of the rectangle is larger than that of the derived polygon because “the rectangle has more area”, “no corner was removed”, etc. These high percentages of *more A* (area or size of rectangle) -- *more B* (size of perimeter) responses at all grade levels suggest that this intuitive rule has a very strong effect on students' responses.

Responses of the type *more A -- more B* are observed in many mathematics and science tasks, including classic Piagetian conservation tasks and tasks related to intensive quantities. In all these tasks, relationships between two objects that differ in a salient quantity A are presented or described ($A_1>A_2$). The participants are then asked to compare the two objects with respect to another quantity B (B_1 is not bigger than B_2). In all the cases examined, a substantial number of participants respond incorrectly, according to the rule *more A* (the salient quantity) -- *more B* (the quantity in question), and claim that $B_1>B_2$ (Stavy & Tirosh, 1996; Zazkis, 1999).

The intuitive rule *same A -- same B*

In a previous study, Mendel (1998) presented eleventh grade students with the following rectangle to rectangle task (Figure 2).

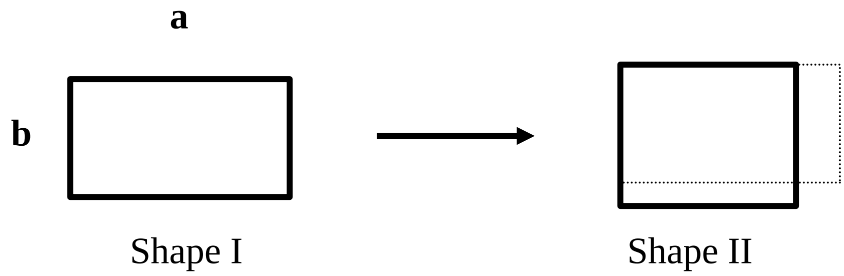

Figure 2: The rectangle to rectangle (by percentage manipulation) task

Consider the following rectangle (Shape I), seen on the left hand side of Figure 2. Side a of Shape I is reduced by 20% and side b of Shape I is increased by 20% to form Shape II (dashed lines represent the shape before the changes). Participants are asked to compare the perimeters of the two shapes, Shape I (the original rectangle) and Shape II (the derived rectangle).

Mendel (1998) found that only 8% of the students responded that the perimeter of Shape II is smaller because "side a is bigger than b, and decreasing 20% of it, is more than adding 20% to side b". The vast majority of the students (72%) claimed that the perimeter remains the same because "you added 20% and removed the same percentage, so they compensate each other". These high percentages of *same A* (same percent) -- *same B* (same perimeter) responses, suggest that that this intuitive rule has a very strong effect on students' responses.

Responses of the type *same A* -- *same B* are observed in many mathematics and science tasks. In all these tasks the presented two objects (or systems) are equal in respect to a certain salient quantity A ($A_1=A_2$), but differ in another quantity B (B_1 differs from B_2). In some of the tasks the equality of quantity A is perceptually or directly given, however in other tasks it may be logically deduced. When participants are asked to compare the two objects (or systems) with respect to quantity B, a substantial number of participants respond incorrectly according to the rule *same A -- same B,* claiming that $B_2=B_1$ since $A_2=A_1$ (Tirosh & Stavy, 1999).

The current study

Exploring factors that influence the tendency to respond in line with the intuitive rules could improve our understanding of the nature of this response. In addition, identifying such factors is essential when designing ways aimed to assist students in overcoming the effect of the intuitive rules, when they solve mathematics and science tasks. So far most of the research involving intuitive rules focused on investigating the tendency to respond incorrectly according to the rules, with regard to certain ages or grade levels. In the current study the relationship between students' Piagetian cognitive level and their tendency to respond according to the rules was investigated.

Piaget's theory showed that children's cognitive development progresses through well established stages (or levels), i.e., the sensorimotor stage, the preoperational stage, the concrete operations stage, and the formal operations stage. When they are at the concrete operations level, children tend to think more systematically and quantitatively than before. As opposed to preoperational children, children at the concrete operations level are able to take into account more than one perspective simultaneously. One of the landmarks of the concrete level is the ability to conserve in all forms (number, area etc.), which develops during this stage. Although they can understand concrete problems, children at this stage cannot yet perform on abstract problems, and they do not consider all of the logically possible outcomes. Only when they reach the highest level, the formal operations stage, can children apply logical and systematic thought functions to more abstract problems and hypotheses. The work of Shayer and Adey (1981) suggested that poor student achievements could be attributed to fact that the vast majority of high school students have actually not yet reached the Piagetian formal level. On the basis of these findings they developed a "cognitive acceleration program" aimed to improve students' academic achievements (for the recent CASE program see: Adey, Shayer, & Yates, 2001).

In the present study the relationship between the cognitive level and the tendency to respond according to the rules was examined by using two mathematics comparison tasks, each related to one intuitive rule: *more A -- more B* or *same A -- same B*. To neutralize the effect of age and degree of formal education the examined population included Grade 7 students (12-13 years old): in this junior-high school grade we can expect to find diverse cognitive levels - mainly early concrete to early formal, according to Shayer and Adey (1981).

METHDOLOGY

Participants

One hundred and fifty junior high school students from Grade 7 participated in the study. The students were from two schools in the center of Israel.

Intuitive rules tasks

To assess students' tendency to respond according to the intuitive rules two written tasks, related to comparison of area and perimeter, were constructed, each corresponding to one of the rules: *more A -- more B* or *same A -- same B*. These tasks were randomly mixed with an additional 15 tasks (not related to this study) to reduce interference. Students were asked in each task to compare the two shapes depicted (and described) and to decide in respect to each dimension (area and perimeter), whether one of the shapes was larger in size or whether it was identical.

***More A -- more B*: The rectangle to polygon (by square removal) task**. The task studied by Azhari (1998) - described above and illustrated in Figure 1 - was used to assess the tendency to respond according to this intuitive rule. In this task the salient property, A, is the area/size of the shapes, and students responding according to the intuitive rule claim that the perimeter of Shape II is larger than that of Shape I, although the perimeters of both shapes are identical.

***Same A -- same B*: The square to rectangle (by percentage manipulation) task**. This task had the same scheme as the task studied by Mendel (1998), described above and illustrated in Figure 2. Here, Shape I was a square (and not a rectangle). From the square (Shape I) the same percentage (20%) is added to one side and at the same time reduced from the other side. In this task there is an equality in the salient quantity A (same percent and same perimeter), and students responding according to the intuitive rule claim that the area of Shape II is identical to that of Shape I, although the area of Shape I (the square) is larger than that of Shape II (the derived rectangle after the percentage manipulation).

Cognitive level assessment test

To determine the cognitive level of each student we used the "Science Reasoning Task II" (Adey et al., 2001), described in detail and validated earlier by Shayer and Adey (1981). Administration of the test and assessment of the results were according to the instructions given by the test developers. This test incorporates many of the well-known Piagetian conservation tasks and it is based on Piaget and Inhelder (1974). The test covers the late pre-operational to early formal cognitive levels.

Procedure

Each student completed the two written tests (intuitive rule test and "Science Reasoning Task II") each in a different session that took about 45 min. To analyze the results the students were grouped according to their cognitive levels. In each cognitive group the rates of correct and incorrect responses, according to the intuitive rule, were calculated for each intuitive rule task. Statistical analysis was done by Chi-Square test using SPSS statistical software.

RESULTS

The cognitive level of the studied population extended from early concrete level (2A) and below, to the early formal level (3A). For the purpose of analysis we grouped the students into four cognitive levels: early concrete and below (<2A/B; 12 students); mid concrete, a sub-stage between early and mature concrete (2A/B; 25 students); mature concrete (2B; 65 students); and above mature concrete (>2B; 48 students).

Figure 3 depicts the rate of students' responses for each task.

The left graph in Figure 3 depicts the results obtained for the rectangle to polygon task, related to the intuitive rule *more A -- more B*. In this task the average rate of correct responses was 47%, similar to the average rate obtained for incorrect responses according to the intuitive rule. As seen in the graph, the rate of correct responses rises as cognitive level advances. In the lowest cognitive level (<2A/B) the rate was 25% climbing to 54% in the highest cognitive level (>2B). However, these differences in rates were not statistically significant even between these two extreme cognitive groups. Likewise, no significant differences were observed between rates of responses according to the intuitive rule in the different cognitive levels. Still, it was found that in the studied population, the rate of this type of response dropped down, from 58% to 42%, as cognitive level advanced.

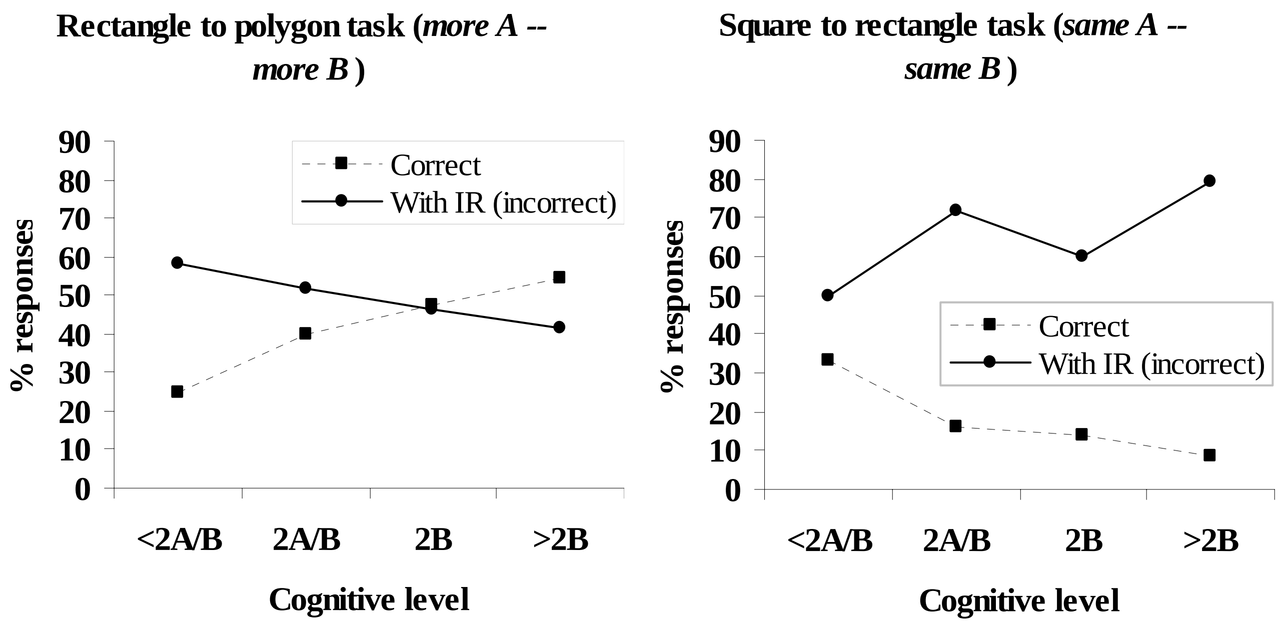

Figure 3: The rate of correct responses and incorrect responses according to the intuitive rule for each studied task

A different response pattern was observed for the square to rectangle task, related to the intuitive rule *same A -- same B* (right graph in Figure 3). For this task the rate of correct responses was very low with an average rate of 14%. At the lowest cognitive level (<2A/B) the rate was 33% dropping to 8% at the highest cognitive level (>2B). This more than four-fold decrease in correct responses was found to be statistically significant (*Chi-Square* [1] = 5.2*;* *p* = 0.02). Statistical differences were also observed for the rate of responses according to the intuitive rule *same A -- same B*. At the lowest cognitive level (<2A/B) the rate was 50% rising to 79% at the highest cognitive level (*Chi-Square* [1] = 4.2*;* *p* = 0.04). The rate of incorrect responses according to the intuitive rule at the mature concrete cognitive level (2B) also differed significantly from the rate observed at the higher cognitive level (60% and 79% respectively; *Chi-Square* [1] = 4.7*;* *p* = 0.03). The average rate of this type of response was 67%.

DISCUSSION AND CONCLUSIONS

The current research investigated the relationship between the Piagetian cognitive level of 7th Grade students and their tendency to respond according to the intuitive rules, related to comparison tasks. It should be noted that since only a single task was chosen for each rule, we should be careful in generalizing the results and further study is necessary, including different tasks belonging to other mathematical topics or other content domains. However, on the basis of the above findings and the tasks studied, it seems that cognitive level influences, in a different manner, the tendency to apply each rule.

In the case of the intuitive rule *same A -- same B* the rate of correct answers decreases as cognitive level advances. In addition, at the highest cognitive level the tendency to respond according to the rule was significantly higher. These observations suggest that advanced cognitive schemes reinforce the use of this rule. Such a scheme could be the ability to conserve that develops during the concrete level. In support of this, a higher tendency to respond incorrectly according to the rule with age was shown, recently for another task ("area and volume of cylinders" task) related to this rule (Stavy et al., in press). Furthermore, the rate of these incorrect responses was analogous to the rate found for answers showing the use of the scheme of conservation, for the salient property A, suggesting its influence on the use of the intuitive rule *same A -- same B*.

In the case of the intuitive rule *more A -- more B* the results did not supply a clear-cut picture. Although the results presented in Figure 3 indicate an obvious positive influence to respond correctly (or overcome the use of the rule) with the advance of cognitive level, the observed differences were not found to be significant. This could be attributed to the task itself and/or to the finding that the cognitive levels of the studied population did not significantly exceed concrete level. In support of the latter explanation, all three students who were below the early concrete level gave an incorrect answer according to the rule *more A -- more B*, and three out of four students at the early formal level gave a correct response (as opposed to the 54% of students at the >2B level, most of whom are in a transition stage between concrete and formal levels). This suggests that in order to clarify the influence of the Piagetian cognitive level on the tendency to use the intuitive rule *more A -- more B*, a larger population with variable cognitive levels (below and above the concrete level) should be examined.

Overall this paper explores, for the first time, the influence of the Piagetian cognitive level on the tendency to use two intuitive rules. The results show that this factor should be taken into account when dealing with incorrect judgments caused by the use of these rules such as when designing ways aimed to assist students in overcoming their effect. In addition, the study shows that advancement of the cognitive level, as suggested for improvement for academic achievements (Shayer & Adey, 1981), might not by itself be beneficial when considering mistakes related to the rule *same A -- same B*. Since this study included seventh graders only, it will be interesting to examine other grade levels in order to identify the interaction between different levels of education and cognitive levels.

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COMING TO APPRECIATE THE PEDAGOGICAL USES OF CAS

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An interview with one teacher in the early stages of teaching with a computer algebra system (CAS) demonstrated how his use of CAS moved from functional to pedagogical. He reflects on a successful lesson to highlight how access to CAS impacted on his teaching style. The teacher reported an increasing awareness of the possibility for pedagogical use of CAS. He found that considering CAS outputs resulted in students' development of a deep algebraic understanding, which he came to distinguish from better performance of by-hand algebraic manipulation.

Teachers with technology available in mathematics classrooms begin with views on its role for teaching, learning and doing mathematics, and these views will change as their experience of teaching with the technology changes. Their views are likely to be related to their views about how students learn mathematics, the teachers' knowledge of the technology, and their pedagogical content knowledge. Teachers' own preferences for solving problems may also impact on their views and teaching approaches. For example, it has been reported that teachers will privilege particular representations (symbolic, graphical or numerical) in their teaching (Kendal, Stacey and Pierce, 2005).

This paper reports how one teacher's classroom use of the technology of a computer algebra system (CAS) changed as his experience grew. In particular, we follow his reports of a move from viewing CAS in a purely functional use way (i.e. to solve problems) to coming to appreciate its potential for pedagogical use of CAS. As such, this study adds to a series of published case studies that trace teachers' developing practices for teaching with technology.

Pierce and Stacey (2001) reported on a four-year study of the teaching of a university course. In the first two years CAS was taught as a topic, with postgraduate students providing technical assistance to the mathematics students and little or no input from the lecturer. Students used CAS functionally to solve problems most likely outside the range of their by-hand skills. In later years, the lecturer began to make pedagogical use of CAS. CAS was now used to enhance mathematics learning.

Zbiek (2002) synthesised studies of pairs of teachers from three different counties. Teachers' previous experience of teaching with technology influenced the way they taught with CAS. Teachers lacking personal confidence with CAS restricted their use of CAS although the forms of restriction varied. They were more likely to focus on keystrokes and teaching approaches that enabled control of CAS use in their classrooms.

> …the combined data from these three teachers suggest more emphasis on CAS syntax and details may be natural for teachers with minimal CAS competence and with little experience teaching with technology in students' hands. (Zbiek, 2002, p. 134)

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 105-112. Prague: PME.

Two of these teachers restricted the use of CAS in their classrooms by providing students' worksheets. A third teacher, in a study by Heid (1995) controlled the use of CAS by avoiding CAS in her teaching. Doerr and Zangor (1999) reported on how a teacher who was confident and competent with graphics calculator also controlled students' use of the technology, directing how it would be used in the classroom, but for different reasons to the three teachers with limited technology skills. Lucy, an expert CAS user reported in Ball and Stacey (2005) also restricted use of CAS in her classroom, but her short-term tactical restrictions were based on wanting students to develop judicious use of technology.

These studies show that there are many individual variations in how teachers develop appropriate and comfortable teaching practices for new technologies, but that the general pattern is for novices to focus on using the machine, whereas using it to enhance and deepen learning comes later. Here is one inspiring example.

CONTEXT AND METHODOLOGY

The data reported here was from an interview with Neil (a pseudonym), an experienced mathematics department head, who had been teaching with a computer algebra system (CAS) for about 18 months. Teaching senior secondary school mathematics (Year 12) with CAS was permitted in this Australian state for the first time on a trial basis in 2001/2002 in three schools, one of which was Neil's. A new subject had been accredited, *Mathematical Methods (CAS)* (MM(CAS)), which allowed CAS for all mathematical work, including in examinations. Neil taught this subject with CAS to a Year 11 class in 2001, and to the Year 12 class in 2002. He was therefore a pioneer in teaching with CAS and was one of four teachers studied in the CAS-CAT research project accompanying the introduction of the new subject. Further descriptions and outcomes of the pilot and the CAS-CAT project are available in CAS-CAT (n.d.), Leigh-Lancaster, Norton & Evans (2003) and Stacey (2003). Neil was in most part a volunteer in the project, as the inclusion of his school in the first pilot had been influenced by the general strength of his teaching and his enthusiastic approach to his work. He was not especially regarded as a "CAS expert", by others or in his own opinion, but he had strong skills in teaching with a graphics calculator and reasonable command of its capabilities. Now, three years later, he is regarded as an expert in teaching with CAS.

Neil was interviewed by a member of the CAS-CAT project team in July 2002, midway through the school year, about 4 months before his students sat for the high-stakes end-of-school examinations. This was one of a series of interviews conducted throughout 2001-2002. His classroom was also observed by the project team (not reported here), which was in nearly daily contact with all project teachers. Three other project teachers were also interviewed. They all had individual responses to teaching with CAS, which will be reported elsewhere. The interview was audiotaped and transcribed. The brand of CAS used is not revealed here, to preserve Neil's anonymity.

In this interview, Neil provided comments on his views about use of CAS for teaching and for doing mathematics, reflecting on his own views from the start of his Year 12 class (6 months earlier) up to July. The particularly revealing section of the interview that is analysed in this paper began when Neil responded to the prompt: “Thinking back over the year, give an example of what has worked well”. He began his response by describing a classroom episode based on students’ solutions to the problem in Figure 1a. As will become evident in reading the interview, Neil’s responses are impressive because he bases them on classroom experiences about which he has reflected himself: there is no empty rhetoric parroting slogans about teaching with technology.

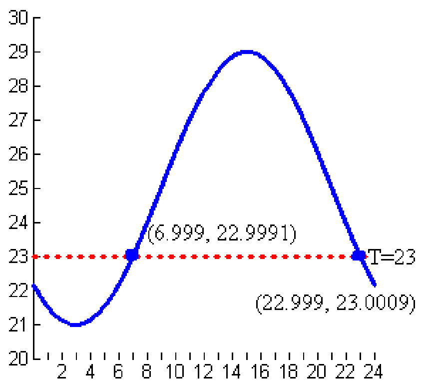
The problem: MM Exam 2 Qn 1 (abbrev.) (VCAA, 2001) The temperature, T degrees Celsius, in a greenhouse at t hours after midnight for a typical November day is modelled by the formula $T = 25 - 4\cos\frac{\pi(t-3)}{12}$, for $0 \le t \le 24$. Use this model to answer the following questions. c. At what times will the temperature equal 23ºC?	**Sample by-hand solution (abbreviated)** $23 = 25 - 4\cos\frac{\pi(t-3)}{12}$ $\frac{1}{2} = \cos\frac{\pi(t-3)}{12}$ $\frac{\pi(t-3)}{12} = \frac{\pi}{3} + 2k\pi$ or $\frac{\pi(t-3)}{12} = 2k\pi - \frac{\pi}{3}$ $t = 3 + \frac{12}{\pi}(\frac{\pi}{3} + 2k\pi)$ or $t = 3 + \frac{12}{\pi}(2k\pi - \frac{\pi}{3})$ $t = 7 + 24k$ or $t = -1 + 24k$ k=0 ⇒ t=7, t=23
Sample graphical solution  	**Sample CAS solution** Input: solve(23=25–4×cos(π×(t–3)/12) Output: ,t) t=24k–1 t=–24k+7 Substituting values for *k* gives *k*=0 ⇒ t=7, t=23

Figure 1 The problem and a sample by-hand, graphical and CAS solution

NEIL’S INTERVIEW

Below is an edited version of Neil’s interview. The original transcript is available from the authors. We will discuss two aspects of the interview: Neil’s perspectives on the multiple solutions offered by students to the question in Figure 1 (transcript Neil19 – Neil26, Neil32), and Neil’s changing views of CAS output and its potential for teaching (transcript Neil27 – Neil30) which provides evidence of his growing appreciation of pedagogical power of CAS and a subtle change in his view of the nature of mathematical understanding.

Interviewer: Thinking back over the year, give an example of what has worked well

Neil19: Probably that trig question [..]. I was amazed at how that worked […].I wanted to round off trig. “How will I do this?” And I grabbed from my cupboard last year's exam (*see Figure 1*) because I'd talked to the kids about the really typical analysis questions that are asked about trig. […]

Neil20: [..] And I talked about how these trig questions come up all the time and even though we are in a CAS course, these trig questions are very typical of what can be asked. So I grabbed last year's exam and […] I wrote it up on the board and I suppose that is an advantage of being an experienced teacher, you can think on your feet. I thought as I went along: 'I'm going to get kids to do it but I'm not going to give them any advice on what to use'. I did say: 'This is last year's exam (i.e. *CAS not permitted*). […] So, I want you to do it how you would do it by choice. Then I got the students to come up. [..]

Neil21: There was a very strong argument about the three different ways the students would solve this trig equation. One student came up, and he is the one who always complains about jumping from the CAS to non-CAS [*in another subject where CAS is not permitted*]. He came up and did it by hand, and it was a lovely solution, did it beautifully, no problems, no mistakes. It was quite a difficult trig equation to solve, there was a consideration of domain to look at. There was a translation [of the basic graph of *cos* function], so it was quite difficult.

Neil22: Someone else came up and showed how they would solve it on the graph, […] He talked about the fact that it had a decimal answer and [asked] "Did that matter?".[…]

Neil23: And then another student came up, who would be a weaker student […] and did a CAS solution (*i.e. used symbolic features of CAS*). Now the [CAS] does the parametric solutions very nicely. I've talked about saying to the kids "Get used to writing *k* as a parameter. Just get used to writing it down because it shows you're understanding what the value of *k* in the general solution is. Then look at your *k* values and get a feel for how many solutions you expect".

Neil24: And it raised fantastic discussions between the three of them as to which method they'd choose. And because it didn't specify an exact answer, each of the three were equally valid. In the end I got [the students] to vote and it was split three-ways, equally. I was really, really interested because I always thought, at the beginning of the year, that my teaching would have been more directive and that by the end of the year they'd have all chosen one method.

Interviewer: What would that method be?

Neil25: At the beginning of the year it would have been the by-hand method, but now I'd be using the CAS method. I've probably changed completely in six months. It's interesting isn't it? But then I'd probably would have had had the graph up, in my mind at least, to give me a feel for how many solutions to expect. I talk to them all the time about this general solution isn't much use to them unless you've got a real feel for how many solutions to expect. And I get them to look at how many solutions to expect either by looking at the unit circle or by visualizing the graph.

Neil26: I preferred by-hand solutions because I'm good algebraically. […] Security, safety, and familiarity in doing what I've always done. But with the CAS it was much quicker. [..]

Neil27: […] There's stuff that's come out this year that I'm enjoying too. Like when you solve a general equation in *sin* and *co*s, the [CAS] will give you two solutions whereas with *tan* it will only give you one. So I've talked to the students about why that's the case. And it's raised lovely, lovely discussions about the fact that *sin* and *cos* repeat themselves every 2π and *tan* repeats itself every π. So you can expect less solutions with the *tan.*

Neil28: […] It's not like the calculator's just going to give you an answer that means nothing. In fact, the students and I have moved to a feeling that the calculator gives you something that has got an intuitive feel to it, rather than just churning out a result that you use with no understanding.

Interviewer: Is the CAS result triggering a mathematical learning situation?

Neil29: Yes, yes. And I didn't think it was going to. I thought it was going to be just "This is just an answer, and manipulate it, and don't understand". And the fear is all the time that you're going to lose your algebraic skills because it's going to churn out an answer. I don't think that's the case because I think the answer means so much more when you've got the algebraic understanding underneath what's happening. And I don't mean the by-hand skills. I don't mean that. I mean that you've got the algebraic intuition into what this answer means and why there's two answers with *sin* and only one answer for *tan*. […]

Neil31: […] I'm fairly teacher-led in the way I teach, which probably reflects my personality […]

Neil32: [CAS will] affect my teaching in a totally different way than it's going to affect someone else because we've got different personalities and I think we'll see that across the project. But the fact that [students] voted that [they] were going to do it in three different ways, it surprised me but I was absolutely delighted. And I said to them. "This is terrific. Because what this means now is that teaching has moved from me telling you what to do into you making your own decisions that suits you and your personality and the way you think about things. And that each was equally valid. And I thought for me this was one of those WOW experiences of - "We've actually got somewhere" and that I could almost let them safely into the exam with the feeling of "They're making their choices" and they might each come out saying "We've done this different ways". But it actually doesn't matter. I was thrilled.

PEDAGOGICAL ADVANTAGES OF EXPLOSION OF METHODS

Neil selected this problem because a similar problem could be on the CAS examination at the end of the year. Three volunteer students demonstrated a by-hand algebraic (Figure 1b), graphical (Figure 1c) and CAS solution (Figure 1d). Neil was pleased that the three different solutions generated considerable discussion and that students expressed strong views about the different methods (Neil21-Neil24): he values these aspects of this lesson and places high importance on them. Neil's comment that the by-hand solution was *"…a lovely solution, did it beautifully, no problems, no mistakes. It was quite a difficult trig equation to solve…*" reveals his high regard for by-hand algebra, and perhaps pleasure in seeing these strong by-hand skills in a student permitted to use CAS. Neil is also pleased with the CAS solution, especially since the weaker student is able to work with the parameter. He also seems pleased that the CAS gave a parameter in the output and that interpretation of this was required to answer the problem.

Artigue (2002) suggested that an explosion of methods can cause difficulties for teachers. Neil took advantage of the different methods of solution and made them a feature of this lesson. He was excited about the explosion of methods and seemed pleased both that students could create different solutions and also that a variety of solutions were accepted in formal assessment. Neil used a similar approach to the teacher reported in the study by Guin and Trouche (1999) where different approaches were compared with a move towards institutionalisation of accepted approaches.

Neil was pleased that students had made individual choices about solution methods and his suggestion that he expected students would use the method he privileged at the start of the year showed a shift in his expectations of how students work. He was delighted at their growing independence (Neil32).

UNDERSTANDING, BY-HAND SKILLS AND CAS

The section of the transcript from Neil27 – Neil30 provides a self-appraisal of how Neil's view of CAS output has changed, how his understanding of how CAS can be used for pedagogical purposes has grown and how in this process his view of the "gold standard" for mathematical understanding has changed. Neil's comments about the role of CAS show an evolution in his thinking. At the start of the year he said he used CAS for speed (Neil26) and his comment in Neil29 suggests that he thought that CAS would be a purely functional tool – for "churning out an answer". He feared that students would get answers that they did not understand and maybe lose algebraic skills. In recalling this (Neil29), he seems to imply that other teachers may still feel this way – certainly we feel this is a common opinion.

Neil's comments suggested that as he learned more about CAS, and gained experience in the classroom, he started to appreciate the possibility for using CAS to promote students' understanding. He gives an insightful example about CAS outputs associated with the solution of trigonometric equations, which had promoted discussion in the classroom as Neil and his students interpreted the outputs and linked these to the period of trigonometric functions.

The phenomenon Neil discussed is illustrated in Figure 2. The inconsistencies evident here make it an instance where CAS output might be regarded as very unpredictable and maybe incomprehensible, which initial users of CAS, in our experience, often find very frustrating. Solving an equation *sin* or *cos* on Neil's CAS gives two families of solutions (see, for example, Fig. 1(d), and also Fig. 2). But solving an equation in *tan* gives only one family of solutions. Figure 2 shows different CAS outputs from the same machine, with the same settings, to effectively the same equation. Although they are mathematically equivalent they look different. Neil's strong mathematics has enabled him and his students to see firstly that both solutions can be correct, and secondly to understand why the machine might operate in this way. He is not frustrated by the unfamiliar and idiosyncratic CAS outputs, but now finds them an opportunity for “lovely, lovely discussions”.

Input:	Solve ($\tan x = \sqrt{3}$,x)	Input:	Solve ($\sin x / \cos x = \sqrt{3}$,x)
Output:	$x = \pi k + \frac{\pi}{3}$	Output:	$x = 2\pi k - \frac{2\pi}{3}$ $x = 2\pi k + \frac{\pi}{3}$

Figure 2. Two contrasting CAS solutions to one equation.

Neil enjoyed the discussions about mathematical understanding arising from CAS outputs such as this and appreciated the fact that CAS outputs acted like a third party in the classroom, stimulating discussion.

> So I’ve talked to the students about why that’s the case. And it’s raised lovely, lovely discussions about the fact that *sin* and *cos* repeat themselves every 2π and *tan* repeats itself every π, so you can expect less solutions with the *tan.* (Neil27)

As well as enjoying the discussions, Neil appears to attribute to such experiences, a change in what he sees as “understanding” mathematics. It is commonly observed that most teachers associate “understanding” with being able to solve problems by hand. Zbiek (2002) cites several examples. Neil came to believe that when students could use the outputs provided by CAS, this in itself was evidence of mathematical understanding (Neil29). His description of understanding is linked to an ‘intuitive feel’ that he and his students have developed for CAS outputs.

> The students and I have moved to a feeling that the calculator gives you something that has got an intuitive feel to it, rather than just churning out a result that you use with no understanding. (Neil28)

Neil contrasts the view that understanding is associated most with by-hand skills, with his new view that it can be also seen in the ability to predict and interpret CAS outputs:

> … the answer means so much more when you’ve got the algebraic understanding underneath what’s happening. And I don’t mean the by-hand skills. (Neil29)

CONCLUDING REMARKS

Initially Neil had few expectations of the capacity of CAS to impact on his teaching style and values. However, after 18 months, he had developed strong pedagogical use of CAS. He reflected on the positive way that CAS had impacted on the algebraic understanding of his students and how it had extended his view of what counted as understanding. Neil believed that discussion of CAS outputs promoted a deep understanding of algebraic concepts and he observed this for students of a range of abilities. For weaker students he noted that CAS had enabled a greater use of algebra, as they were able to perform algebraic routines successfully and focus on interpretation of the algebraic output. Neil was a project participant because of his strength as a teacher, but even so, he began with what he later considered naïve views of the potential of teaching with technology. His obvious pleasure in discovering creative ways to use CAS pedagogically was evident to the project team and beneficial to the mathematical growth of his students.

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STUDENTS' CONCEPTIONS OF *m* AND *c* : HOW TO TUNE A LINEAR FUNCTION

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This paper explores students' conceptions of the parameters m and c in the linear function y = mx+c. The multiple semiotic representations of linear functions lead to multiple mathematical and real-world meanings for m and c. The links between these are an important focus of teaching. The paper reports on responses from three classes of 15 year old students to 7 items requiring interpretation of information about a real-world context. Results confirmed that c (in all four aspects) is a conceptually simpler object than m, but identified that it is often omitted from students' verbal and symbolic descriptions, and treated as if it is not an integral part of the function. Interpretations of both c and m were affected by inadequate distinctions between time (the x variable) as a measure or as a duration of one event.

INTRODUCTION – FOUR DIMENSIONS OF *m* AND *c*

The linear function[2] is a concept particularly favoured in mathematics education, to support and accommodate a wide range of research. The multiple viewpoints from which one can tackle this mathematical concept and the diverse semiotic representations it carries are certainly among the principal reasons for its privileged use within our community. When considering the introduction of algebra from a functional approach, for example, linear functions appears to be the stage for many research studies interested in students' understanding of functions (see, for example, Nemirovsky, 1996). This mathematical concept has also been exploited as a particular case of modelling situations, shedding some light on students' ability to interpret and model realistic situations (see Yerushalmy, 2000). Also, there is abundant literature where this mathematical concept scaffolds research regarding graphic comprehension, such as in Moschkovich (1999) or Mevarech and Kramarsky's (1997) work. And ultimately, it has moulded many studies where the core is students' understanding and use of algebraic notation –and in particular of letters (see, for example, Janvier, 1996).

However, because in the introductory work in algebra what it is really important is to enable students to perceive one major characteristic of linear functions - namely that

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[2] A more correct term is affine functions (English), fonctions affines (French) and funções afins (Portuguese).

it is a relationship between two variables *x* and *y* - whichever domain in which linear functions appear as a scaffold or frame, the focus has legitimately, mostly been on the link (and on students' understanding of the relationship) between *x* and *y*. Among the research interested in students' conception of function, within the *structural-procedural* context for example, linking *x* and *y* values of a function is translated into adopting a process perspective (Knuth, 2000). Also, in research dealing with graphic comprehension, students´ understanding of how *x* and *y* are related to each other is often examined by investigating the phenomenon of *fixation* -privileging one variable over another (Bell et al., 1987).

If, on one hand, the study of the relationship between *x* and *y* is at the core of multiple research involving linear functions, few studies seem to have focused on the relationship regarding its other elements, namely the parameters *m* and *c* in the standard equation $y = mx+c$. For if making students understand that *y* and *x* are linked variables that take changing values, and if drawing their attention to the effects of such changes is crucial in the precursory work of algebra, the focus seems to shift elsewhere in our classrooms. From making students investigate the consequences of changes operated upon *x* and *y*'s, teachers tend to stress what happens when *m* and *c* change, instead. And what was originally meant to be a study where the variables *x* and *y* occupied the first role is deviated to a teaching where the emphasis is given to *m* and *c*, therefore swapping the very status of the objects of contemplation. Dealt with in the first case as unknown numbers, *m* and *c* themselves assume the role of variables. The present study is a response to this relocation of attention and therefore it will focus on students' conception of *m* and *c*.

A main characteristic that makes linear functions interesting for research in mathematics education is that several representations are intertwined. The multiple semiotic representations of linear functions can also help us organise our analysis of students' conceptions of parameters *m* and *c*. In parallel with work on functions in general, *m* and *c* can be considered from their symbolic, graphical, numerical (S, N, G) and real-world perspectives. It is from these different perspectives that *m* and *c* will here be considered. Study of the links between the 3 intra-mathematical representations (S, N, G) has been extensive, particularly stimulated by technology (see, for example, Kaput, 1986), but there has been less study of the links between the these 3 representations (S, N, G) and real-world contexts. We will extend work such as that of Moschkovich (1999), to give emphasis to the links between the real-world contexts and the graph. The seminal work of Janvier (1978) highlighted the importance of translation between real-world contexts and the graph (i.e. interpretation of graphs), but it differed from the present focus by dealing with complex situations without a pre-specified mathematical structure. The focus of the present study on situations modelled by a linear function is attributable to the desire of teachers to work on topics specified in the school curriculum.

To identify the different semiotic contents of *m* and *c*, each aspect will be labelled by the associated representation of the function, providing us with a total of eight "concepts" of *m* and *c* to examine. Table 1 summarizes the main features of each of these concepts. Note that in all eight cases, the value of *m* or *c* can be a number or an algebraic expression. “Symbolic” refers to work with equations and algebraic processes, and “Numerical” refers to calculations with values, whether known or unknown.

	SYMBOLIC	GRAPHICAL	NUMERICAL	CONTEXT
m	$\mathbf{m_S}$ The coefficient of x in the equation *y*=m*x*+c. It can either be a letter or a number.	$\mathbf{m_G}$ The gradient of the graph of *y*=m*x*+c.	$\mathbf{m_N}$ The ratio $\Delta y/\Delta x$.	$\mathbf{m_C}$ The meaning of *m* in a real world context (e.g. a speed, hourly rate of pay).
c	$\mathbf{c_S}$ The constant term of the equation. It can either be a letter or a number.	$\mathbf{c_G}$ The *y*-intercept. It is a co-ordinate of a point on the line.	$\mathbf{c_N}$ The value of *y* when *x*=0.	$\mathbf{c_C}$ The meaning of *c* in a real world context (e.g. taxi "flag fall", 32^{o}F in temperature conversion).

Table 1: The four dimensions of *m* and *c*

Reading Table 1 “by rows” or “by columns” gives rise to different questions. First, looking by columns, notice that belonging to the same representation does not necessarily mean having similar status. For example, even though m_G and c_G are both graphical objects, c_G is directly accessible (e.g. a value read from the axis) whereas m_G is not (one has to estimate or calculate slope from at least two points). In this sense, m_G is more complex than c_G, as suggested by Moschkovich (1999). In a physical situation, $\mathbf{c_N}$ may be an extensive quantity (e.g. 14 kilograms), whereas $\mathbf{m_N}$ is an intensive quantity (e.g. 6 kilograms per year). One question that then naturally arises is whether this hierarchy extends to all representations of *m* and *c*, psychologically as well as epistemologically.

Considering the links across the representations, notice that behaving similarly does not necessarily make the links - seldom evident - more transparent. For example, the symbolic m_S and c_S have similar status (at least if we consider their syntactical behaviours), which obscures their different graphical status. The question that we then address here is how, from the students’ point of view, do students “translate” a change of *m* and *c* input in one representation to another? Ultimately, it seems important to study students’ understanding of *m* and *c* separately and also by investigating connections students make between these two elements.

METHODOLOGY

This article analyses answers that Australian high school students provided to a written test. Three Grade 9 classes (about 15 years old) provided a total of 54 responses. The test was undertaken in the middle of the school year, soon after the students had been taught linear functions. In two classes teachers followed a real-world/graphical approach (Asp et al., 1995), where students worked on extended problems set in real world contexts and used graphics calculators (TI-83) throughout. The other class followed a “standard” approach, based on a commonly used textbook (Watson et al., 2001), with a few exercises set in real world contexts. The text included some use of graphics calculators at the end of the unit, but this was not done. The present article does not examine the impact the different approaches but explores students' conceptions students of *m* and *c* in the linear function.

Data from 7 items (see Figure 1) on the test are examined here. These items are set within the real-world situation of comparing charges of two plumbers, Bob and Chris.

To hire a plumber, you have to pay

- a fixed amount PLUS
- a cost depending on how long it takes to have the work done.

Bob and Chris are two plumbers. This graph (Fig. 1a) shows the cost of hiring each of them.

a) If you have $325, how long can you hire Bob for?

b) Is it cheaper to hire Bob or Chris? Explain your answer.

c) Explain in words how to work out the cost of hiring Bob if you knew he would be working for 14 hours, WITHOUT EXTENDING THE GRAPH.

d) Explain in words how to work out the cost of hiring Bob if you knew the number of hours he would be working, WITHOUT EXTENDING THE GRAPH.

e) Use algebra to write a rule to work out the cost in dollars with the number of hours of hiring Bob.

f) At Christmas, Bob gives all his customers a $10 discount but Chris gives all his customers a discount of $5 for every hour that he works. Add new graphs on the axes below to show the new hiring costs for Bob (Fig. 1b) and Chris (Fig. 1c).

[Item (g) presents piecewise linear graph (not shown here) for another charging plan, and asks students to describe in words, calculate a charge and sketch graph of hourly charge.]

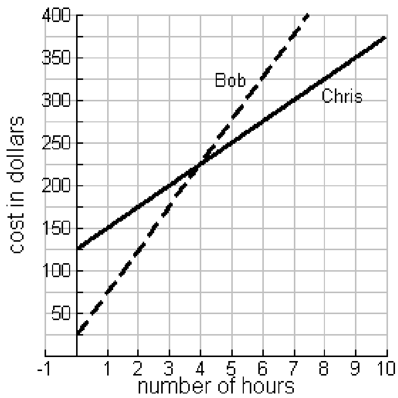

Fig. 1a: Cost of hiring Bob and Chris

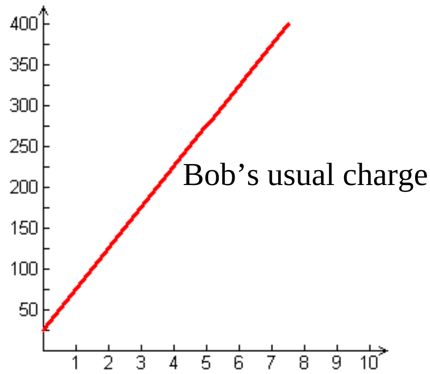

Fig. 1b: Graph of Bob's usual charge

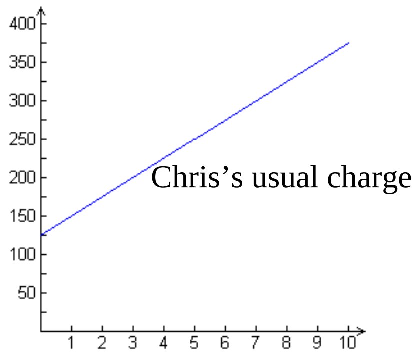

Fig. 1c: Graph of Chris' usual charge

Figure 1: First seven items of test, in the context of cost as a function of time.

The charges for each plumber (cost versus time) were provided on a graph, along with the (implicit) specification of the linear characteristic of each: "To hire a plumber, you have to pay a fixed amount PLUS a cost depending on how long it takes to have the work done". The items require students to move between symbolic, graphical, numerical and real-world representations of the situations.

RESULTS AND DISCUSSION

Students´ conception of *c* : A starting element too soon forgotten or a constant too boring to note?

When looking at *c* in each of the representations, it seems – and previous research (Moschkovich, 1999) has suggested – that it can be considered as a fairly simple object, especially when compared to its "functional partner", *m*. However, the data we have collected seem to indicate that students often do not perceive the genuine role this multi-faced object plays for the function. Despite the fact that the presence of a "fixed amount" in the hiring costs was specified in the introduction of the problem, this element seems secondary to the students, who often tend to focus on the "multiplicative feature" of the function only.

Regardless of the representation (symbolical, graphical, etc.) that framed the items or in which students' answers were provided, *c* was indeed often omitted. In item (d) of Figure 1, for instance, set in real-world context, students were asked to explain in words how to work out the cost of hiring Bob. Many students provided an answer such as the following, given by Sam: "look at how much he earns an hour and times it by how long he works". Omitting *c* in this item is often consistent with their answer to the previous item (c) which asked for a numeric cost (for 14 hours, beyond the graph). Students often consider that "7 hours is $375 and you double and you would get 14 hours $750", as Clayton did. These are both instances of false proportionality, studied in depth by Van Dooren et al (2005).

Also, if we consider *c* in the graphical context, c_G seems secondary for many students, who often refer only to *m* (either as m_C or m_G) to decide who is cheaper to hire for item (b). Anthony, who wrote "Chris is cheeper (sic) because his hourly rate is cheeper (sic)" and Clayton who wrote "It's cheaper two (sic) bye (sic) Chris because the line is less steeper" gave answers representative of this category.

The above may be attributed to students' poor language skills and possible impatience with constructed response items, but it is also evident symbolically. In item (e), many students, in their algebraic rules linking cost to time, stressed the multiplication factor, either by totally omitting the constant from the expected rule as in Sandra's answer "c=50×x" or, in an even less well-formed answer such as Jennifer's "c ×", still making explicit only the multiplicative part. Interestingly, one should mention that most students who tend to omit *c* were in the classes that followed the real-world/graphical approach. Indeed, about 60% of the students for whom linear functions were taught with the standard approach started their description by stating the fixed amount, such as the following response given by Clayton: "The fixed amount is $25 all you have to do is add $50 an hour". Only 15% of students having followed the real-world/graphical approach provided this kind of answer. This leads us to question the imprint of the contexts on students' conceptions of *c*. Students following the real-world/graphical approach often referred to *c* as a "starting fee", rather than the "fixed amount" mentioned by most of other students. Would interpreting a fixed amount as a starting fee rather than a characteristic of the function as a whole cause students to soon forget it?

Evidence from students' answers to other items of the test showed some consequence of privileging the real-world interpretation c_C when interpreting *c* and, in particular, consequences on students´ conception of c_N. Instead of assigning to c_N the value of *y* corresponding to x=0, students sometimes considered the value of *y* when x=1 as in: "start at $75 for the first hour and then just keep addin (sic) $50". For these students, it seems that *c* was, because of the context of the problem, understood as the cost for the hiring the plumber for one hour. This confusion between the cost for the first hour and the cost for one hour will also be seen in the discussion of *m*.

The data collected indicates that even when students seem to deal quite well with the different representations of *c*, the role *c* occupies in the function is often fuzzy. This is evident even for students who can describe in words the graphed functions and even provide its algebraic rule –which, for many teachers, remains the ultimate proof of mastering the concept. For example, Kristen wrote for Chris' cost: "Start at $25. Then add $50 for each hour" (item (d)) and gave the following rule: "H×50+25 = C" for item (e). However, when comparing the costs of the two plumbers (item (b)), she apparently disregarded *c* writing: "Chris because every hour Chris only goes up $25, whereas Bob every hour goes up $50". It appears that students often don't realize that *c* affects the function *as a whole* –especially when it is considered as a starting point- and the role of *c* in the function remains vague.

***m*: a rate per hour but for which time?**

As with students' conceptions of *c*, the context seems also to have misled some students in their understanding of *m*, and more precisely of m_N. However, the predominance of m_C over *m* is more subtle than the predominance of c_C over *c*. As noted in the previous section, the cost of hiring the plumbers *for the first hour* was interpreted as the fixed amount for some students. This same value is also sometimes associated with *m*. There seems to be a "sliding" when interpreting m_C, which generates an incorrect numerical value of *m*. The cost per hour has sometimes been understood by students as the cost of one hour, i.e. the value of *y* (cost) when *x* (the number of hours) equals to 1. Lionel's false proportionality answer is representative: "He charges 75$ one hour so you times the number of hours that you want him for".

A consequence of interpreting m_C as the cost of one hour of work can be seen elsewhere. In the last – and rather challenging - question of the test, students were asked to graph the hourly charge for a new plan, specified by a piecewise linear graph. Some students' sketches stopped at x=1, suggesting that what was relevant to them was the graphing of the cost for the first hour.

Behind such answers is a matter appearing to be essential for an appropriate understanding of *m* within this problem context. It seems that in order to grasp *m*, one has to be able to detach himself/herself from the conception of time as *duration* of particular work. Instead of considering the time reported in the *x*-axis for the duration of hiring the plumbers as an uninterrupted story, time has to be seen as possibly atomised (into intervals). Each atom, because it is an element of time, can be interpreted without giving emphasis on its duration feature. In this way, time is considered as a unit of measure. “One hour”, when accepted as an element of the whole (time) can correspond to a unit of measure and the concept *per hour* can appropriately be understood. It is then not interpreted as the first hour (not in the sense of its location on the time stripe) nor as the first part of a story that ended at the extremity of the *x*-axis. Not surprisingly, students who show a tendency to perceive the time as a continuum focus on the extreme values of the graphed interval; e.g. “It starts at the 50 and ends at 400”. Instead of seeing the graph as simultaneously giving the charges for many plumbers’ jobs of different lengths, they may see it as depicting only the accumulating cost during one job.

From situation to graph

The above analysis has been concerned with interplay between real-world context and graph, linked across to symbolic and numeric representations. Item (f) directly examined how students interpreted changes in the real-world situation that affected slope and intercept (i.e. for Bob, changes in c_C and m_C affecting c_G but not m_G , and for Chris, affecting m_G but not c_G). The omission rate on this question was high (44%), especially for the textbook class. Of the 31 students responding, 58% correctly identified the change in intercept alone for Bob, but only 32% identified the change in slope alone for Chris (with no students from the textbook class). About 8% of responses changed both intercept and slope; otherwise the most common error was to change the intercept c_G in both cases. This finding supports the proposition above that *c* is a simpler object than *m*.

CONCLUSION

An important component of instruction on linear functions is to understand and to link the multiple semiotic representations of the parameters *m* and *c*. There are links to be understood between three intra-mathematical dimensions (symbolic, graphical and numeric) of *m* and *c*, and also links to a wide variety of real-world situations where linear functions are applicable, of which this paper has explored only one relatively straightforward cost per unit time situation.

Close examination of students’ responses revealed that *c* is often seen as an accessory to the function, rather than a vital organ of it. In both students’ verbal and symbolic descriptions of the hiring charges, *m* triumphs over *c*. This is despite the fact that *c*, in all its representations except perhaps the contextual, is a simpler quantity than *m*, as noted in previous literature and in the findings reported in this paper.

An interesting finding was the consequences of fuzziness of distinction arising in the context between the cost per hour and the cost for just one hour (first hour, time when *x*=1, during the first hour; unit of measure). Consequences of the confusion were evident in wrong answers for both *c* (where it may have been expected) and *m*. To understand *m*, students have to delocalize this interval of one hour. In the context of cost as a function of time, *m* and *c* are far from steady, well-defined concepts in the students' point of view. Investigations using other contexts are recommended. If one considers the musical meaning of the term "affine" adopted in some languages such as Portuguese and derived from the Latin *affinis* – "afinar" means "to tune" – one could say that linear functions are, in this sense, for many students hard to tune.

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A CONTRADICTION BETWEEN PEDAGOGICAL CONTENT KNOWLEDGE AND TEACHING INDICATIONS

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Through an analysis of classroom observation and drawing upon Shulman's (1986) notion of pedagogical content knowledge, this paper examines a teachers conceptual indications when teaching the sub-concepts of function and his knowledge of the difficulties and misconceptions associated with them. The paper provides an assessment of the teacher's awareness of, and response to, these issues in the context of constant, inverse and piecewise functions. The evidence suggests that even if a teacher possesses a sophisticated understanding of specific conceptual obstacles and their causes, such awareness may not be prioritised during teaching.

INTRODUCTION

Though the knowledge base for teaching can be examined from a variety of perspectives the subject matter knowledge of the teacher would appear to make a fundamental contribution. It is generally agreed, even though there is no readily available mechanism by which teachers can deliver what they have in mind to the students, that strong subject matter understanding is essential, although not sufficient, to facilitate students' learning (Ball, 1991).

Shulman (1986) suggested that teachers need other types of knowledge to transform subject matter into a form that can be grasped by students of different ability and background. He considered that subject matter knowledge *"goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching"* and introduced the notion of pedagogical content knowledge (PCK) to embrace *"the amount and organisation of knowledge per se in the mind of the teacher"* (p.9). Such knowledge includes ways of representing and formulating the subject matter to make it comprehensible to the learners, entails an appreciation of the potential difficulties and misconceptions associated with a mathematical concept and an understanding of the possible ways of thinking that may lead to these outcomes.

The notion of PCK has inspired several studies. Marks (1990), for example, concluded that the teachers' understanding of their students' difficulties with equivalent fractions contributed to their selection of relevant tasks and the provision of alternative explanations. Hadjidemetriou & Williams (2001) considered students' difficulties and misconceptions with graphical conceptions through a diagnostic test which was later given to 12 experienced teachers who were asked to record their perception of the difficulties of the items on the test and suggest misconceptions that the students might have that would cause difficulty. An inconsistency between the hierarchy of difficulties and misconceptions the students revealed on the test and the easiest to hardest hierarchy, with associated misconceptions and errors, listed by the teachers, was noted.

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Part of a wider study exploring the relationship between teaching and learning in the context of functions (Bayazit, 2005), this paper continues the theme to report on the relationship between the pedagogic content knowledge and teaching of one teacher.

METHOD

Through a qualitative case study (Merriam, 1988) the paper examines a Turkish teacher's awareness of his students' difficulties and misconceptions with the function concept and considers whether or not this awareness is evidenced during his classroom practice. Burak, who had 24 years teaching experience, was deemed to be a highly qualified teacher of mathematics by his colleagues and his school principal. His pedagogical content knowledge was explored using semi-structured interviews and his teaching orientation was identified by observing 14/19 lessons associated with teaching the function concept. To eliminate possible effects of the interviews on his classroom practice, Burak was interviewed after teaching the course.

Data was analysed using cross-case analysis (Miles & Huberman, 1994) to allow for a comparison to be made between Burak's views of his students' difficulties and misconceptions with function and the ways he handled the associated challenges during his classroom teaching. Content and discourse analysis (Philips & Hardy, 2002) was employed to analyse transcripts of tape recorded interviews and classroom observations. The intention was not to interpret a specific instructional act in a single context but to construe the act in the context of the totality of the discourses.

THEORETICAL FRAMEWORK

Literature associated with the epistemology of the function concept provided a conceptual map for the study. Shulman's (1986) notion of PCK served as a framework to investigate the teacher's awareness of the students' difficulties and misconceptions and Breidenbach et al's (1992) notion of 'action-process' conceptions of function provided a framework to investigate classroom practice. Breidenbach et al suggest that the essence of an action conception of function is the step-by-step application of an algorithmic procedure with a typical indication being the ability to insert a number into an algebraic function and calculate the output through mental or physical manipulations in a step-by-step manner (ibid).

A process conception of function is suggested to be at a higher level of sophistication. The possessor not only illustrates characteristics associated with an action conception but is also able to rationalise these actions in terms of the concept definition, particularly in identifying that a function transforms an input to an output and at the same time maintaining sight of the univalence property of the function concept.

RESULTS

The results are presented in two ways. First we consider the teacher's classroom practices, and secondly we consider his pedagogical content knowledge.

Teaching approach in the context of inverse function

A process conception of inverse function entails possessing the idea that 'an inverse function undoes what a function does' without loosing the sight of 'one-to-one and onto' property. However, during the two lessons during which he taught the inverse function, Burak focused on reversing an algebraic function by going from the end to the beginning and inverting the operations in a step-by-step manner— an emphasis on inverse operation rather than inverse function. There was no evidence that he used conceptually focused and cognitively demanding tasks to encourage his students to reflect upon a function process and the inverse of that process in the light of the concept definition. Apart from one example, which involved a set-diagram (a negative case not meeting the 'one-to-one and onto' condition), Burak illustrated inverse functions through the use of algebraic functions and utilised several strategies to promote his students' mechanical skills. His most prominent strategy initially illustrated the idea of 'inverse operation' through functions with single-step operations, for example $f(x)=x-5$, and then this notion was given an additional emphasis through the use of functions containing multi-step operations, such as $f(x)=\frac{3x-5}{4}$. His supporting strategies included offering analogies from every day life and the provision of practical rules. For instance, he gave the following rule and emphasised it repeatedly when reversing linear functions:

> Leave the x alone on the one side of the equation by gathering the others on the other side, and then replace x and y with each other.

Vidakovic (1996) suggested that a process conception of an inverse function entails an understanding of why the inverse function can be obtained through the action of switching the independent and dependent variables in the original function and solving it for the dependent variable but Burak's explanation for this action was:

> Traditionally we represent a function by the notation *y*, that is why we should replace *x* and *y* with each other after leaving the *x* alone on the one side of the equation.

The relationship between the use of analogy, the provision of a practical rule and the implementation of a step-by-step procedure is seen through Burak's explanation of the problem, *Given the function* $f(x)=x^3+7$*; work out the rule of inverse function*:

> … We got up in the morning; we had a breakfast, put on our clothes; and then we locked the door when we left home… When we return back to home…first of all we shall open the door and then take off our clothes…and so on. … Yes, this is the logic we are going to implement… The last operation here is addition of 7; therefore in the first step we should subtract 7 from the *x*… The operation before the last one is the 3rd power of *x*, so we should take the 3rd root of *x-7*…

The premise of this episode is the idea of 'inverse operation' but the analogy does not appear to contribute towards understanding that 'an inverse function undoes what a function does'. Such a presentation may well constrain understanding to the idea of 'inverse operation' and be the source of a misconception for some.

Teaching approach in the context of constant function

The Turkish curriculum presents a constant function through a sub-definition:

> A function is called a constant function if it matches every element in the domain to one and the same element in the co-domain (*Çetiner, Yıldız, & Kavcar, 2000, p: 86*).

This is a specific form of the concept definition addressing 'all-to-one' matching (transformation). For a mathematician this is a model of simplicity but for students it can cause enormous difficulties. Tall & Bakar (1992) indicated that approximately half of their sample of college students rejected the idea that a line parallel to the x-axis represents a function claiming that '*y*' cannot be independent of the value of '*x*'. They suggest that such a misinterpretation stems from of an implicitly existing idea within school mathematics: *"the notion of function has variables, and if a variable is missing, then the expression is not a function of that variable"* (p: 109).

A process-oriented teaching approach that promotes students' conception of the algebraic and graphical aspects of function as a process transforming every input to one and the same output could avoid such difficulty. However, Burak's teaching emphasised two aspects of factual knowledge more in tune with action conceptions:

- a constant function does not involve an independent variable, *x*, and
- a graph of constant function is a parallel line to the x-axis.

Every instructional input, including the reinforcement of his students' previous knowledge system, addressed these two points. For example, consider Burak's explanation when addressing the problem:

Work out the precise form of the constant function, $f(x)=(4n-2)x+(2n+3)$.

> **Burak:** … We described it [the constant function] like a fixed minded person; did we not? [**1**[1]] … Whatever we say; he never changes his mind [**2**]. Yes, the constant function is like a fixed minded person; no matter whatever we put into the *x* we come up with the same image [**3**]. … Let's remember the algebraic form of the constant function… the algebraic form of a constant function involves just a number; this number would be an integer…or a rational number. Have a look at the expression, $f(x)=(4n-2)x+(2n+3)$. In this expression there are two terms; one is the constant term, $2n+3$, and the other is a term involving *x*, $(4n-2)x$. …So, first of all we should get rid of the term containing *x*; because if this is the constant function…it must not involve *x*(**4**).

At the beginning of the episode Burak offers an analogy [**1**] & [**2**] that does not communicate the essence and properties of the constant function. Though he provides a verbal descriptions of the situation, [**3]** his language is vague: he implicitly refers to inputs and output but does not explicitly refer to the notion of the function process as an 'all-to-one' transformation. More importantly, as the instruction develops Burak overemphasises, without meaning, the rule that *x* must be removed from the expression because a constant function does not involve *x* (**4**).

[1] Key statement in teacher's instruction that we refer to while commenting upon the quality of instruction in the subsequent paragraph.

During the teaching sequences reference to set-diagrams and ordered pairs were absent. Consequently the pedagogic power of the former to encourage the acquisition of ‘all-to-one’ transformations in algebraic and graphical situations was not utilised and any connection between these two forms of representation were not established.

Teaching approach in the context of piecewise function

A function described by more than one rule on the sub-domain does not violate the definition of function; however most students wrongly think that a function should be described with a single rule over the whole domain (Sfard, 1992). A graph made of branches or discrete points could represent a piecewise function on a restricted domain; nonetheless students usually reject such graphs because of their misconception that a graph of a function should be a continuous line or curve (Vinner, 1983). Dubinsky & Harel (1992) suggest that if students develop a process conception of function such difficulties and misconceptions would be overcome. A process conception of a piecewise function entails seeing a single function process behind the formulas or the segments of graphs on the sub-domains and interpreting this process in the light of concept definition.

Burak’s teaching of this topic included no inputs that would support his students’ conceptual growth towards a process conception of piecewise function and there was no implicit reference to the definition of function. His focus of instruction was on the selection of the right formula for each sub-domain. Again his teaching strategy involved analogy and, to reinforce the procedure, the comparison of the inputs with the extreme points of the sub-domains, features seen within the following example:

Given the function *f: R→R*, $f(x) = \begin{cases} \frac{3x+4}{9}, x < -2 \\ x^2 - 5, -2 \le x < 3 \\ 2^{x-4}, 3 \le x \end{cases}$; what is the value of *f(-5)+f(0)+f(4)*?

Burak: …the function is described by different rules for different values of *x*. Look at it; when the value of *x* is less than -2 we are going to use this rule ($\frac{3x+4}{9}$). We are going to use x^2-5 when $-2 \le x < 3$… and we shall use the rule 2^{x-4} when $x \ge 3$. Remember the example I gave at the beginning (of the lesson); we should dress up according to the weather conditions — if it is sunny and hot we dress up thin and relaxing clothes…; the logic is the same here … we are going to choose the formula according to the numbers we give to *x*… Let’s find the value of *f(0)*…0 is bigger than -2 and less than 3; therefore we should use the formula in the middle…take the square of 0 and then subtract 5…

Burak’s instructions on the piecewise function did not include the engagement of his students with the notion of piecewise function in a graph made up branches or discrete points. All the graphs used were smooth and continuous lines or curves — a feature of teaching that could encourage the ‘continuity misconception’.

Burak's Teaching: A Summary

Burak's teaching could be described as action-oriented practice. The focus of his instruction did not appear to be upon function-related ideas but on his student's acquisition of rules and procedures. There was generally little evidence of connections with underlying meaning. Such teaching appeared to suggest that on the one hand Burak had limited knowledge of his students' potential difficulties and on the other that it could lead to the development of misconceptions. However, evidence from interviewing Burak suggested a contradiction. He appeared to be well aware of his students' potential difficulties and misconceptions and to possess a sound understanding of the thinking that could lead to such obstacles

Burak's Pedagogical Content Knowledge

To compliment the evidence from his classroom practice, Burak's views on students' difficulties and misconceptions with the function concept were considered using three separate situations associated with inverse, the constant and piecewise functions.

During the interview Burak was invited to provide reasons for the following:

> When asked 'Is it possible to reverse the function $f: R \rightarrow R$, $f(x)=x^2$' a student says 'yes' and gives the answer: $f^{-1}(x) = \sqrt{x}$.

Burak identified the student's error as applying the idea of 'inverse operation' to the task and he interpreted this mechanical act as an indicator of the student's deficiency in conceptualising the process of $f(x)=x^2$ and the inverse of that process in the light of concept definition. He said:

> As they see the square of *x*, they automatically carry out the inverse operation and take the square root of *x*. … This student is not able see that this function produces the same output for two elements of the domain which have the same absolute value. … In one sense he/she is not able to move back and forth between the domain and co-domain…

The second task invited him to consider the situation: "*Suppose that a student does not accept the expression y=5 as a function*", and asked him to identify "What it was the student had in mind and why he/she could have rejected the situation". In Burak's view this was the student's inability to see the 'all-to-one' transformation in the situation. He also pointed out that the absence of *x* in the situation was a particular feature that would cause students to reject the expression as a function:

> … He is not able to evaluate an image [expression] and see the meaning behind it. In my view this student simply looks at the image and when he sees that there is no *x* in the situation, he thinks that *y=5* does not represent a function. … If this student has understood the notion of constant function, he/she would not make such a mistake. What is the constant function? It matches all the elements in the domain to a single element…; this is the point he/she is not able to see [in the situation]. …

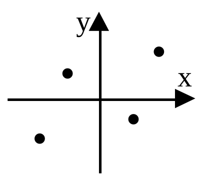

Figure 1: A graph made up four discrete points.

The third task was associated with the piecewise function. Presented with the graph (Figure 1) Burak was invited to consider the statement “a student claimed the graph did not represent a function”.

In Burak’s view it was the discontinuity aspect of the graph that caused the student to reject the situation. He considered that the student’s reaction was an inability to interpret the function process defined on the sub-domains and he suggested:

> … There are just four points, and they have not been linked with a curve… The student wants to see a smooth curve passing through the points…; they would join the points with broken-lines or curves. If they specify the domain set they would see that this graph does in fact a one-to-one matching from domain to co-domain. Of course the domain involves just four elements; the projections of the graph on the x-axis. …

DISCUSSION AND CONCLUSION

We conclude that Burak is well aware of his students’ potential difficulties and misconception with the functions, and he also indicates a good understanding of the sources of such obstacles. Burak:

- interpreted the students response to the inverse function as the indicator of an inability to move back and forth between the elements of domain and co-domain without loosing the sight of the ‘one-to-one and onto’ condition. This suggests that in Burak’s opinion the idea of ‘inverse operation’ does not enable one to handle this concept of inverse function in a more complicated situation.
- considered that the student rejected the constant function because of an inability to interpret the ‘all-to-one’ transformation behind the expression in the light of concept definition, explicitly addressing the absence of *x* as a particular source of misconception.
- considered that the discontinuity aspect of the graph of the piecewise function was the particular factor causing the students to reject the situation and predicted that the student would join the points by broken-lines or curves.

However, Burak made no effort to eliminate these obstacles during his classroom teaching. Indeed, it is possible that his teaching encouraged their development. He constrained his teaching of inverse function to inverse operation, emphasised, without connection to the concept, that a constant function does not involve *x* and that the graph of constant function is a parallel line to the x-axis and when teaching the piecewise function unit gave no illustration that encouraged his students to reflect upon the notion of piecewise function in a situation made up branches or disjoint points.

The evidence presented in this paper is in marked contrast with the results from other studies (Tirosh et al, 1998; Escudero & Sanchez, 2002). In our situation the issue is what is it that interferes between the teacher’s strong pedagogical content knowledge and his practice within the classroom? During the teaching observation there were several instances where Burak indicated to his students that examination or test success required a particular way of learning:

If you want to succeed in those exams you have to learn how to cope…

Do not forget…simplification. It is crucial, especially…[in] a multiple-choice test.

It would appear that his perception of success over-rode his deeper conceptions of thinking to provide an action oriented teaching practice in which his students' difficulties and misconceptions were peripheral to the rules and procedures that would lead to success in particular situations. It is an issue worthy of further study.

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IDENTIFYING AND SUPPORTING MATHEMATICAL CONJECTURES THROUGH THE USE OF DYNAMIC SOFTWARE

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This study documents the types of mathematical conjectures that high school students formulate within a problem-solving environment that promotes the use of dynamic software. In particular, we are interested in analyzing both questions that led them to the formulation of a particular conjecture and types of argument used to validate that conjecture. Results indicated that students consistently examined the viability and pertinence of a particular conjecture or relationship in terms of using the software to (i) identify the conjecture visually, (ii) examine whether the conjecture falls within a family of isomorphic objects (dragging test), (iii) build a macro that reproduces the construction and verify whether the conjecture was held in objects generated by the macro, (iv) quantify and verify properties of mathematical objects to detect patterns, and (v) present formal arguments to prove the emerging conjecture.

Formulating conjectures and developing mathematical arguments to support them are fundamental activities in mathematical practice. Harel & Sowder (1998) recognize how important it is for students to participate in the process of conjectures construction and the need to search for convincing arguments to validate the conjectures.

> The goal [of instruction] is to help students refine their own conception of what constitutes justification in mathematics: from a conception that is largely dominated by surface perception, symbol manipulation, and proof ritual, to a conception that is based on intuitions, internal conviction, and necessity (p. 237).

In this context, it is important to examine the role played by the use of specific tools in helping students to represent mathematical objects or problems that can be analyzed mathematically. What type of questions can students formulate and explore when they represent mathematical objects with the use of computer technology? What features of mathematical thinking are enhanced when students utilize computational tools in their mathematical learning experiences? These are relevant questions that need to be discussed in order to evaluate the actual potential of a particular tool in helping students to construct their mathematical knowledge meaningfully. It is recognized that diverse computational tools may offer distinct opportunities for students to identify, represent and explore relationships and properties of mathematical objects. Thus, when using a tool it becomes important to investigate features of mathematical thinking and ways of reasoning that students can develop as a result of representing and examining mathematical problems with the use of computational technology. In particular the use of dynamic software (Cabri-Geometry) seems to facilitate the process of searching for mathematical conjectures

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or relationships embedded in objects and problems that can be represented dynamically. In this perspective, we are interested in documenting types of conjectures and relationships that high school students formulate in a problem-solving environment that promotes the use of dynamic software. What types of representations of mathematical objects or problems do students construct with the use of dynamic software? What resources and strategies do students show during the process of searching for mathematical relationships or invariants embedded in problem representations? What type of arguments and proofs do students provide in order to support conjectures or relationships? We argue that the use of dynamic software not only helps students quantify and explore mathematical attributes of objects, but also influences the students' reasoning or thinking about supporting and presenting relationships or conjectures. In this study, we document the extent to which students use dynamic software to represent problems dynamically. In particular, to what extent do students formulate and pursue questions that might lead them to reconstruct or develop conjectures and basic relationships and to think of mathematical arguments in order to explain and justify them?

ELEMENTS OF A CONCEPTUAL FRAMEWORK

The development of mathematical knowledge and learning can be traced in terms of the type of representations used to think of mathematics. "Much of the history of mathematics is about creating and refining representational systems, and much of the teaching of mathematics is about students learning to work with them and solve problems with them" (Lesh, Landau, and Hamilton, 1993, cited in Goldin & Shteingold, 2001, p.4). In this context, we argue that dynamic representations (generated with the use of technology) of mathematical objects or problems offer students the possibility of detecting and examining patterns or invariants that emerge while quantifying mathematical attributes (lengths, perimeter, area, volume, angle, slope, etc) and moving particular objects (points, segments, perpendicular bisectors, etc.) within the representation. In addition, we recognize that a problem-solving environment provides learning conditions for students to participate actively in the construction of their own mathematical knowledge and conceptualize their mathematical learning as a continuous process in which the use of technological tools is seen as an opportunity for them to explore and examine problems or conjectures from distinct angles. Here, some questions or conjectures that are often difficult to verify with paper and pencil procedures can now be explored via the use of technological tools. As a consequence, with the use of different tools, students may be able to identify not only new relationships or conjectures but also, even different ways to support them. An overarching idea that students need to develop while using technological tools is that any conjecture or relationship that emerges during their interaction with the task needs to be supported with a clear mathematical argument. In this context, it becomes important to recognize not only the type of opportunities that the use of technology may provide for students; but also the challenges that they need to resolve while presenting their results to the learning community. As Schoenfeld stated:

> For the mathematician, dependence on argumentation as a form of discovery is learned behavior, a function of expertise. ..As one becomes acculturated to mathematics, it becomes natural to work in such terms. "Prove it to me" comes to mean "explain to me why it is true," and argumentation becomes a form of explanation, a means of conveying understanding (Schoenfeld, 1985, p. 172).

A relevant feature that emerges in students' ways to represent mathematical problems with the use of technology is the possibility of tracing "modeling cycles" in which their description of conjectures, explanations, and predictions are gradually refined, revised, or rejected – based on feedback from trial testing.

> [In problem solving] several levels and types of responses nearly always are possible (with one that is best depending on purposes and circumstances), and students themselves must be able to judge the relative usefulness of alternative ways of thinking. Otherwise, the problem solvers have no way to know that they must go beyond their initial primitive ways of thinking; and, they also have no way of judging the strengths and weaknesses of alternative ways of thinking –so that productive characteristics of alternative ways of thinking can be sorted out and combined (Lesh, and Doerr, 2003, p. 18).

RESEARCH DESIGN, PARTICIPANTS AND GENERAL PROCEDURES

The study took place in a problem solving class with eighteen senior grade twelve students, all volunteers, who were interested in using dynamic software to work on a set of problems and to think of different ways to solve and extend the original problems. The development of the course included two weekly sessions, two hours each, during one semester. In particular, we were interested in documenting the type of conjectures and arguments that students formulate during their interaction with the problems. Thus, students were encouraged to use the software to:

(i) Work on given problems that appear in regular textbooks, class material or web page problems and asked them to represent and examine those problems with the help of the software. The goal here was to encourage students to represent and solve the problem with the use of the software, and to pose and explore other questions that may lead them to identify conjectures or other relationships.

(ii) Construct a dynamic configuration with the software that includes simple mathematical objects (segments, lines, points, triangles, angles, etc.) and use it as platform to formulate questions that lead them to reconstruct or construct mathematical relationships or theorems. In both cases, students need to provide explanations and mathematical arguments to present and communicate their results. Thus, they become aware of the importance of using the software to visualize the behaviors of mathematical objects within the representation and the need to look for arguments to justify such behavior or relationships. A pedagogical approach that appeared consistently during the development of the problem-solving sessions included individual work, small group participation (group of three), presentation to the whole group, and general discussion led by the teacher regarding the distinct approaches and mathematical concepts and ideas that appeared during the process of solving the problems. So, our data included students' electronic files of their work and comments and explanations of the approaches, videotapes of students' small groups work and class discussions, and observation notes. We document results about problem solving rather than problem solvers since our examples are taken from students working individually, as well as from small groups and class discussions.

PRESENTATION OF RESULTS AND DISCUSSION

To illustrate the work shown by students, we focused on presenting students' ways to identify and deal with a conjecture that emerges while working on a problem that came from a web page (http://www.atm.org.uk/people/kath-cross.html#bear). We particularly discuss the kind of relationships and arguments that students exhibit during the development of two problem-solving sessions.

The problem: Cross's Theorem: Squares are drawn on the three sides of a triangle. Show that the areas of the four shaded triangles are the same (figure 1).

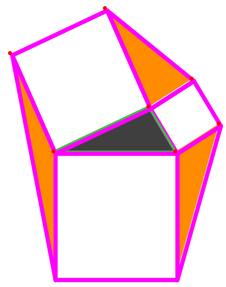


Figure 1: Are the triangles' shaded areas the same?

Faux (2004) posed this problem to readers of *Mathematics Teaching Journal* and received responses showing several ways to prove the theorem including those based on congruence of triangles and those that use rotation of figures (triangles) to explain the result. Students in the problem-solving sessions had opportunity to review those proofs and used the software not only to "verify" the theorem, but also to identify and explore other relationships.

An Emerging Conjecture. What about if we draw rectangles instead of squares on each side of the given triangle, how are the shaded areas? This was the initial question that students agreed to explore. However, they noticed that they could draw many rectangles taken as the base one side on the given triangle and as a consequence it was difficult to trace the area behavior of the shaded triangles. How can we relate the construction of the three rectangles based on the sides of the given triangle? Discussing this question led them to realize that it was important to introduce another condition to draw the corresponding rectangles. The condition was that the corresponding sides of the rectangles would share the same proportion. That is, they decided that $\frac{EC}{CB} = \frac{GA}{AC} = \frac{IB}{BA} = \frac{1}{2}$. They drew the rectangles keeping in mind this condition and observed that the areas of triangles CEF, AGH and BDI were the same (figure 2) and wondering if this relationship would hold for any triangle.

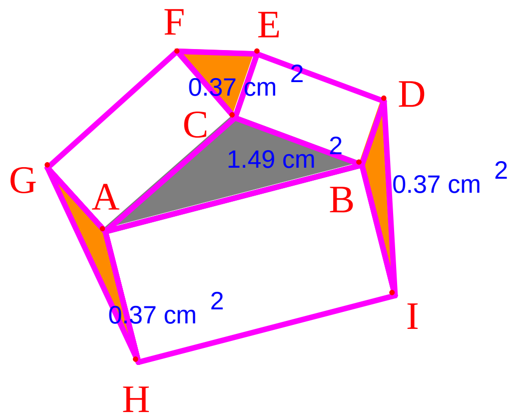


Figure 2: Drawing rectangles with proportional sides

At this stage, the goal for students was to first investigate if for other triangles with the same construction, the area relationship was maintained, and second to look for arguments to support the conjecture. The software became a powerful tool to explore both the plausibility of the conjecture and the search for ways to validate it. We illustrate the ways students showed to deal with this conjecture:

Visual Recognition. An important feature in using dynamic software is that mathematical figures can be drawn accurately. In this case, students drew triangle ABC and the corresponding rectangles (with the same constant of proportionality among their sides) and visualized that for this case, the area of triangles CEF, AGH, and BDI were the same. By determining the triangles' areas they confirmed the visual conjecture (figure 3).

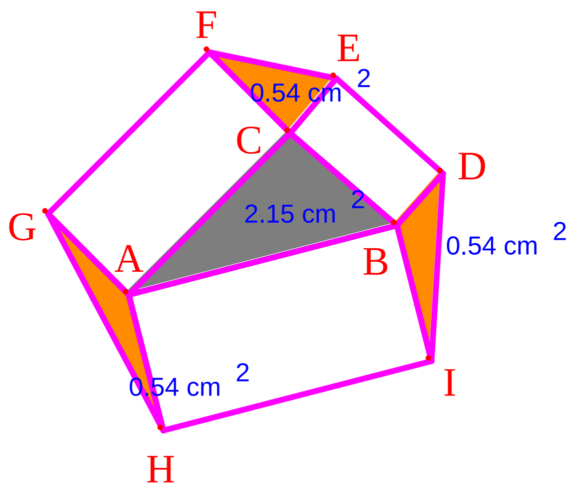


Figure 3: Visual recognition of a relationship in triangles CEF, AGH, and BDI

The Dragging Test. Here students explored the validity of the conjecture for a family of triangles. With the use of the software, they moved the position of the vertices of the given triangle ABC to generate a family of triangles with the same construction. They observed that when one vertex is moved, the family of triangles generated held that the area of triangles CEF, AGH, and BDI was the same (figure 4).

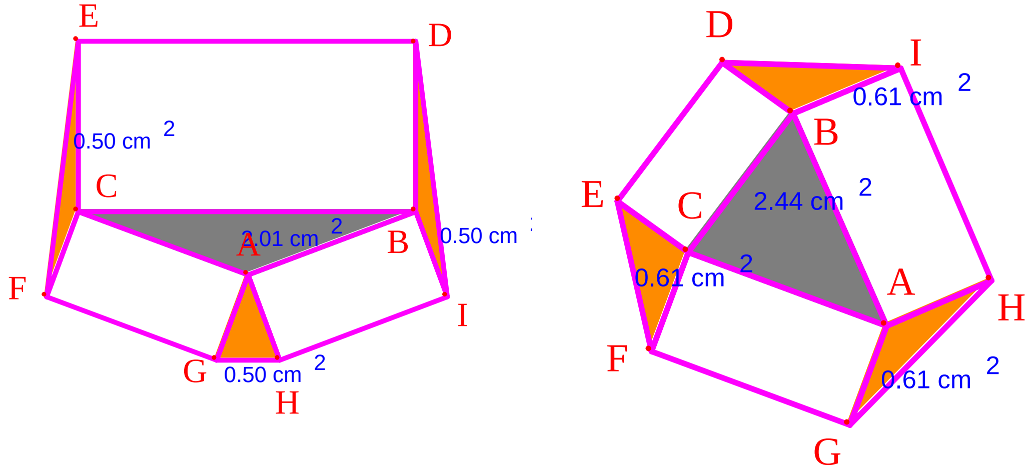


Figure 4: Verifying the conjecture for different positions of triangle ABC

Constructing a Macro. Another way to verify the conjecture was that students built a macro to reproduce the construction for any given triangle. That is, students identified initial objects (triangle ABC and ratio R of rectangle sides) and final objects triangles CEF, AGH, and BDI in order to reproduce the construction for any given triangle. By applying this macro to different triangles, students confirmed the conjecture, that is, in all triangles they applied that macro, they observed that triangles CEF, AGH, and BDI all have the same area. Figure 5 shows two of those triangles.

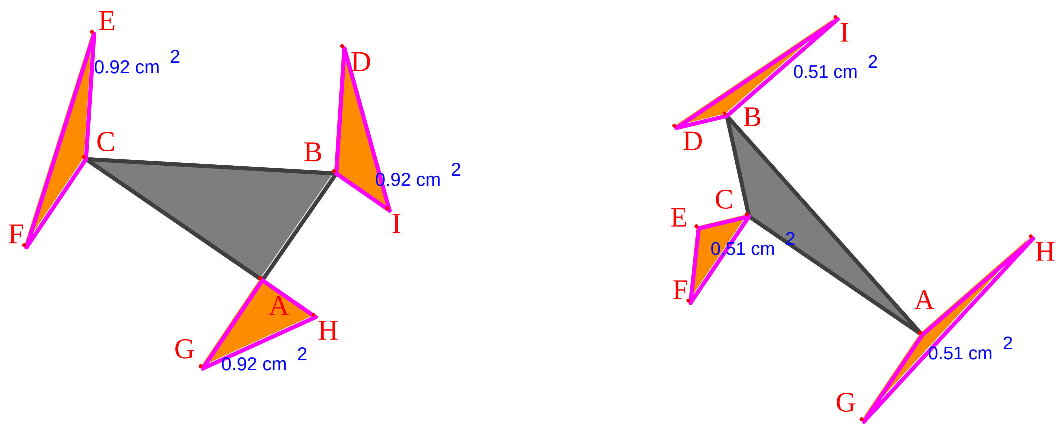


Figure 5: Applying a Macro to verify the conjecture

Quantifying Attributes and Patterns. It is easy with the use of the software to quantify attributes (lengths, areas, angles, etc.) of the figure and to observe their behaviors. For example, in addition to observing the behavior of the triangles areas, students focused on comparing (ration) areas of triangle CEF and triangle ABC for distinct values of the proportionality coefficient of the sides of the rectangles (figure 6). Based on this information, they notice that:

$$\frac{\textit{area} \text{ of } \Delta CEF}{\textit{area} \text{ of } \Delta ABC} = r^2$$

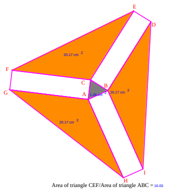
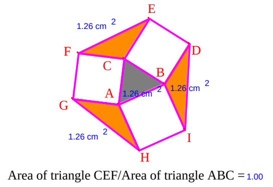
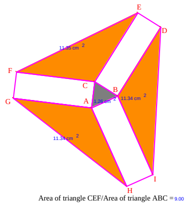
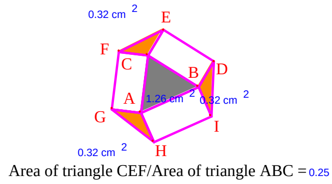
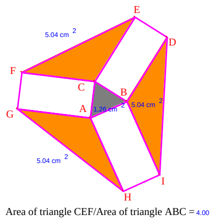
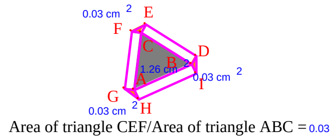
Figure	$\frac{Area\Delta CEF}{Area\Delta ABC}$	r	Figure	$\frac{Area\Delta CEF}{Area\Delta ABC}$	r
	16	4		1	1
	9	3		.25	.5
	4	2		.16	.4

Figure 6: Looking for patterns

Formal Proof. To prove both conjectures, students followed different approaches that were discussed with the whole class. Here, we present a proof that was constructed during the class discussion. The idea is to construct a triangle with one vertex on the origin of the Cartesian system, another on the X-axis, and the third one on a point in the first quadrant. That is, the vertices of the triangle will be A(0, 0), B (c, 0) and C (a, b) as is shown in figure 7.

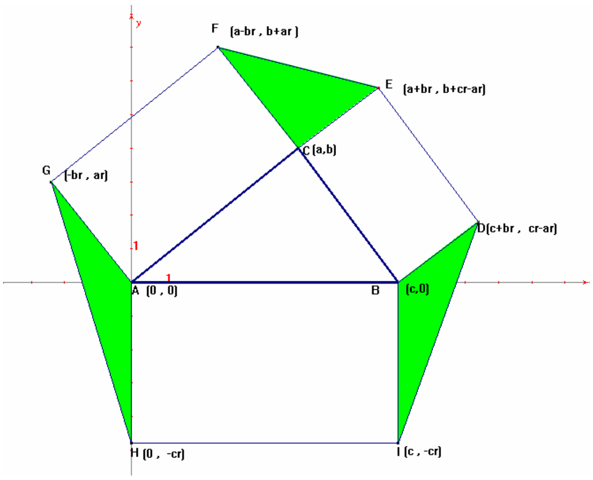


Figure 7: Identifying coordinates of vertices of the given triangle

What is the area of triangles AGH, BID, CEF and ABC? Given the coordinates of the vertices of those triangles, students recalled that the area of each triangle will be:

$$Area(\Delta AGH)=\frac{1}{2}\begin{vmatrix}0 & 0 & 1\\ -br & ar & 1\\ 0 & -cr & 1\end{vmatrix}=\frac{1}{2}bcr^2 \quad (1);\quad Area(\Delta BID)=\frac{1}{2}\begin{vmatrix}c & 0 & 1\\ c & -cr & 1\\ c+br & cr-ar & 1\end{vmatrix}=\frac{1}{2}bcr^2 \quad (2)$$

and $Area(\Delta CEF)=\frac{1}{2}\begin{vmatrix}a & b & 1\\ a+br & b+(c-a)r & 1\\ a-br & b+ar & 1\end{vmatrix}=\frac{1}{2}bcr^2$ (3), in addition, $Area(\Delta ABC)=\frac{1}{2}bc$ (4)

Based on the information given in (1), (2), (3), and (4) they concluded that:

$$Area(\Delta AGH)=Area(\Delta BDI)=Area(\Delta CEF)=Area(\Delta ABC)*r^2$$

Remarks. There is evidence that the use of dynamic software shaped and determined the way that students thought about the process of formulating conjectures and methods to examine and validate the conjecture. In particular, the dynamic representations of the problems and mathematical objects that students generated with the help of the software became an experience enhancing platform to identify and explore mathematical relationships. The easiness to quantify mathematical attributes and the exploration of visual representations of problems were fundamental activities that permeated the students' process of formulation of conjectures. To assess the validity of the conjecture, students relied on the use of the software to evaluate particular cases visually, to examine families of cases by dragging particular elements within the representation, to construct a macro in order to examine a family of cases, and to observe patterns that emerge as a result of exploring invariance in the behavior of particular data (changing the ratio coefficient of rectangles' sides, for example). In this context, the use of an analytical method to prove the conjecture came out naturally as a way to confirm the validity of the conjecture.

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STUDENTS CONSTRUCTING REPRESENTATIONS FOR OUTCOMES OF EXPERIMENTS

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This paper describes the process by which seventh-grade students constructed representations of outcomes of dice games from experiments. We report on the verbal, written, and physical representations during their problem solving investigations. They worked individually and together to produce lists, written mathematical statements, pictures, and graphs. Their representations were used to develop arguments for their solutions.

THEORETICAL FRAMEWORK

The building of a representation and mapping of the representation with earlier knowledge is a crucial step in solving problems (Davis & Maher, 1990, 1997; Kiczek & Maher, 1998). The representations that are built are frequently checked and modified by learners themselves as well as by others who collaborate in the learning. Sometimes, several trials are required to build a representation that is satisfactory and useful in solving a problem. Revisiting of earlier ideas makes possible opportunities for learners to reflect on these ideas and to refine, modify, and extend existing representations (Maher, 1998; Maher & Speiser, 2001). The tasks that are posed to students should be challenging and engaging to enable them to build on their previous knowledge (Francisco & Maher, 2005). This research describes the evolution of the representations produced by a cohort of seventh-graders and shows how these were elaborated during the problem-solving session. It builds on earlier work that describes the reasoning of the students who, as sixth-graders, explored problem tasks concerning the fairness of dice games. (Kiczek & Maher, 1998; Alston & Maher, 2003; Benko, 2006). The purpose was to investigate how the students used their representations and negotiated differences to convince themselves of the reasonableness of their ideas.

BACKGROUND

It has been well recognized for several decades that the learning of probability concepts is very complex (Fischbein, 1975). Garfield and Ahlgren (1988) recommended that longitudinal studies be conducted in order to better understand how students develop probabilistic and statistical concepts. The research reported here is a component of a longitudinal study of the development of probabilistic ideas over a seven-year period (Benko, 2006; Kiczek, 2000). Students worked on games of chance problems in small groups and the representations that they developed were captured on videotape. In 6th grade, during two problem-solving sessions, students worked on problems that involved hexagonal dice. In 7th grade students analyzed tetrahedral dice games during five sessions.

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 137-143. Prague: PME.

A qualitative design methodology was used. The cohort group was videotaped as they were engaged in probabilistic investigations in grades six, seven and twelve. The video portfolio includes videotapes of class sessions, smaller group sessions, group interviews, individual interviews, students' work and researchers' notes. Videotape analysis involved: view and describe video data, identify, transcribe and code critical events, write analysis, construct storyline and compose narrative (Powell et al., 2003).

RESULTS

We report here on data from 10/27/1994, which was the fourth session in grade seven (see http://www.gse.rutgers.edu/rbdil/site/resources/ for a list of the probability tasks). Shelly, Magda, Robert and Amy worked on the following task:

Contests 1: A hat contains 3 tetrahedral die, one white, one black, and one green. You win $900 if you roll a white 1 and a black 2 and a green 3

Contest 2: A hat contains 3 tetrahedral die all the same color. You win $900 if you roll a 1, a 2, and a 3. Is there a difference in your chance of winning for each contest? Why or why not? Explain.

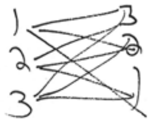
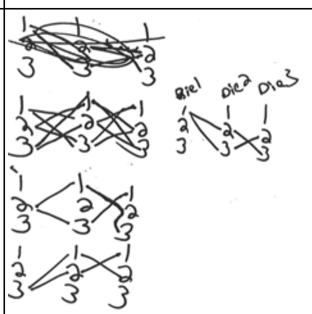
The report is organized so that the representations are summarized in Table 1. Text and diagrams accompany the respective episodes. Twenty-six episodes summarize the development and refinement of representations of group members.

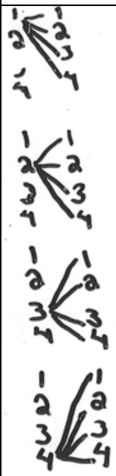
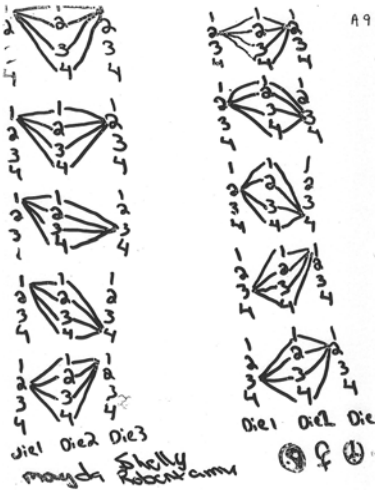
The episodes were divided into clusters. The first cluster describes students trying to make sense of how to roll a 1, 2 and 3 with three tetrahedral dice (Episodes 1-10). They started by listing the outcomes verbally (Episodes 1-3); then Shelly justified the outcomes by holding the first die as constant (Episode 4). Later, following Robert's attempt to make a chart (Episode 5), Shelly, with Magda's help (Episode 8) developed charts to represent the 6 outcomes (Episodes 9-10).

In the next cluster, students found 16 outcomes when rolling two tetrahedral dice (Episodes 11-14). Magda reasoned using the multiplication rule of counting (Episode 14) and Shelly developed a chart that was similar to her previous representation (Episode 10) to account for all 16 outcomes (Episode 14).

In the third cluster, students found 64 outcomes to roll 3 tetrahedral dice (Episodes 15-18 and 21-26). First Magda used the multiplication rule (Episode 15); then, Shelly attempted to use her earlier chart representation (Episode 10) to justify the 64 outcomes (Episode 16). Finally, Robert developed a chart representation (Episode 17) similar to Shelly's earlier representation to justify 6 outcomes of rolling a 1, 2 and 3 (Episode 10). Shelly indicated that she understood Robert's charts. She used her representation to justify the 64 outcomes and developed a tree representation (Episode 26) based on her earlier experiences with soccer tournament team pairing (Episode 21) and the trees of the tower problem (Episode 22).

In the last cluster students had no difficulty to find 256 outcomes of rolling 4 tetrahedral dice (Episodes 19-20). Robert outlined how to use his previous chart representation (Episode 17) to justify this finding.

No.	Student	Representation
1.	Magda	Pointed to different corners of the table and listed the following 3 outcomes to roll a 1, 2, and 3 with three tetrahedral dice: 1-2-3, 2-1-3 and 3-1-2.
2.	Shelly	Found three more outcomes and wrote down 6 possible outcomes in a list: 1-2-3, 2-1-3, 3-1-2, 3-2-1, 2-3-1, 1-3-2.
3.	Magda	Checked Shelly's list (Episode 2) by saying aloud the outcomes in the following order: "1-2-3, 1-3-2, ok, now 2-1-3, 2-3-1, 3-1-2, 3-2-1, yeah, that's it."
4.	Shelly	Used Magda's listing (Episode 3) to justify that she had all of the outcomes. She explained that if she held the first die constant, the other two numbers could be switched.
5.	Robert	Wrote out 2 columns with numbers 1 to 4 in each and drew a few lines between the columns.
6.	Shelly	 Followed up on Robert's idea (Episode 5); she made two columns of numbers of 1, 2, 3 and 3, 2, 1 and connected all of the numbers of the first column with all of the numbers of the second column. It was her first attempt to represent in the chart 6 outcomes to roll a 1, 2 and 3 with three tetrahedral dice.
7.	Shelly	Rewrote the two columns using the numbers 1, 2 and 3; she now connected 1 from the first column with all the numbers in the second column.
8.	Magda	She suggested making a third row. [Note that Shelly's diagram (Episode 7) represented using only two dice.]
9.	Shelly	 Made a third column of 1, 2 and 3 and represented with connecting lines the following 6 outcomes in her diagram: 1-2-3, 1-3-2, 2-1-3, 2-3-1, 3-1-2 and 3-2-1 to roll a 1, a 2 and a 3 with three tetrahedral dice.
10.	Shelly	 Made 3 diagrams of 3 columns of 1, 2 and 3 representing those two outcomes in each of the diagrams that start with the same number, representing 6 outcomes to roll a 1, 2 and 3 with three tetrahedral dice.
11.	Magda	Magda responded to R1's question of how many outcomes were possible if two tetrahedral dice were rolled and distinguished between outcomes 1-2 and 2-1 as she started to list them aloud.
12.	Robert	Said that there should be 16 outcomes for rolling 2 tetrahedral dice "because you can just reverse them around".

13.	Magda	Agreed with 16 outcomes to roll 2 tetrahedral dice (Episode 12) and reasoned that 4 times 4 is 16.
14.	Shelly	 Suggested making a diagram similar to the other diagram that represented 6 outcomes (Episode 10). She drew four times 2 columns with numbers from 1 to 4 in each column. In the first diagram she connected 1 from the first column with 1, 2, 3 and 4 from second column, etc. Created 4 charts to represent 16 outcomes to roll 2 tetrahedral dice.
15.	Magda	Magda responded to R1's question about how many outcomes 3 dice would have by saying that there were 64 outcomes when rolling 3 tetrahedral dice.
16.	Shelly	Placed 4th column of 1, 2, 3 and 4 next to three columns in chart in Episode 10.
17.	Robert	 Made two diagrams of 3 columns of 1, 2, 3, and 4. In his first diagram 1 (column 1) was connected to 1, 2, 3, 4 (column 2) and all those were connected to 1 (column 3). In his next diagram 1 (column 1) was connected to 1, 2, 3, 4 (column 2) and all those were connected to 2 (column 3). Robert continued to list 16 diagrams in such a way that each of the diagrams represented 4 outcomes. He found 64 outcomes to roll 3 tetrahedral dice. Magda questioned Robert's representation and Shelly explained it to her. All of the students in the group concurred that there were 64 outcomes.
18.	Robert	Robert explained that his chart *proved* that there were 64 outcomes since there were 16 charts and 4 outcomes in each chart. He said: "But this is a masterpiece. It proves that there is 64…There is 16 different ones and there is 4 in each one."
19.	Shelly	R1 asked the group how many outcomes there would be rolling 4 dice. Shelly multiplied 64 times 4 and the group agreed that there would be 256 outcomes.
20.	Robert	Robert used his previous chart organization (Episode 17) to show how many different ways it was possible to roll 1, 2, 3, and 4 with 4 dice and he outlined a plan how to write out the 256 outcomes rolling 4 dice.

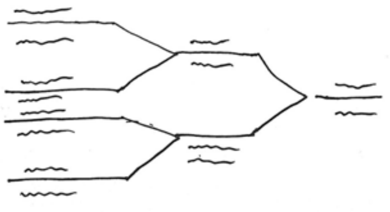
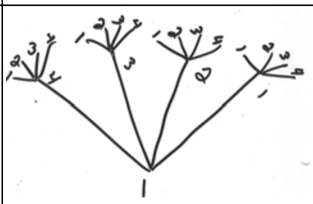
21.	Shelly	 Demonstrated that it was important for her to find her own organization to represent the outcomes and not only to use Robert's organization. She glanced at another group's work and saw a different type of diagram. She said that it reminded her of a soccer tournament pairing that her father used. She wrote out a horizontal soccer chart pairing starting with 8 teams, in which 2 winners would play each other; then the winner would be determined by pairing the winners of the previous games.
22.	Shelly	The soccer pairing (Episode 21) reminded her of trees that they used in the tower problem, a task from fifth grade.
23.	Robert, Magda	Reported that they also saw trees.
24.	Robert	Related trees to a problem about rabbits that made offspring and drew lines from a point. Later in 12th grade Robert again used the rabbit metaphor while he developed trees to justify the multiplication rule of counting. He said that he saw trees in his 5th grade science book referring to "rabbits making children".
25.	Robert	After Shelly's explanation about the soccer tournament (Episode 22), Robert indicated that he a saw similar chart in football team pairing and produced such a chart.
26.	Shelly	 Shelly developed 4 trees that represented the 64 possible outcomes. In her trees she placed a number at the root (1, 2, 3 and 4) and 4 branches went out from each number (marked with numbers 1, 2, 3 and 4), from each of those numbers 4 new branches branched out (labeled 1, 2, 3 and 4).

Table 1. Episodes of different types of representations

CONCLUSIONS

The students worked alone and together to share their ideas from their problem solving investigations. They contributed to each other's understanding by questioning their representations and sometimes using and refining them. For example Shelly used a similar chart representation in Episode 9 as on 10/26/1994 (a previous session) and Robert further built on this representation in Episode 17 to develop the sample space of rolling 3 dice. Robert and Shelly built on their earlier experiences. Shelly related the problem to a soccer tournament pairing and to the tower problem before she developed her tree representation (Episodes 21, 22 and 26); and Robert related trees to football pairing and the rabbit population (Episodes 24 and 25). When they found satisfactory representations, they were able to build on these to solve more complex problems. Shelly first represented, with a chart, the outcomes of rolling a 1, 2 and 3 (Episode 10); then she represented the sample space of rolling two dice (Episode 14). After Robert found his chart representation satisfactory, he was able to construct his 16 charts to represent 64 outcomes without any difficulty (Episodes 17) and he outlined how to use the same representation to find 256 outcomes of rolling 4 tetrahedral dice (Episodes 20). Later, on 10/28/1994, Robert used Shelly's tree representation (Episodes 26) to find 256 outcomes rolling 4 tetrahedral dice.

The students had the opportunity to think deeply about games of chance and build representations of the outcomes of experiments. They cycled back to earlier ideas, tested them, and explored in greater detail their own representations and those of others. By sharing, questioning and refining their work, there was a free exchange of ideas. In so doing, they developed deeper insights into the problem and showed growth in understanding. They used their representations to justify their solution in order to make sense of the problems for themselves and to convince others.

This study supports and extends earlier work in building probability ideas. By collaborating and justifying their solutions, and having the opportunity to think deeply about their ideas before receiving formal instruction in probability, the students built meaningful representations while working together on strands of problem tasks.

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LOGARITHMS: SNAPSHOTS FROM TWO TASKS

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Our study addresses the understanding of logarithms and common difficulties which high school students encounter as they study this topic. We focus on two tasks: one standard and one non-standard that involve logarithmic expressions or require the use of logarithms in a solution. For the purpose of analysis we have modified the interpretive frameworks developed by Confrey in her study of exponents and exponential expressions, to the study of logarithms. Our results indicate students' disposition towards a procedural approach and reliance on rules, rather than on the meaning of concepts. We conclude with pedagogical considerations.

The miraculous powers of modern calculations are due to three inventions: Arabic notation, decimal fractions and logarithms (Cajori, 1919). The first two of these inventions have been investigated in great detail by researchers in mathematics education, while logarithms have received very limited attention. This is rather surprising, given the centrality of the concept in applied mathematics, as well as in the secondary school mathematics curriculum. This article presents a part of an on-going study that aims to describe and analyse issues involved in the understanding of logarithms by high school students. It is best described as a series of snapshots highlighting students' perceptions, rather than an attempt to draw a comprehensive picture.

As an introduction, let us consider the following excerpt from a classroom interaction between a teacher and a group of grade 12 students. The conversation took place as part of a review after logarithms had been studied for several weeks.

Teacher: Can you find the exact value of $5\log_3 9$?

Ryan: You should calculate the log, the log is 2.

Teacher: Would anybody explain why $\log_3 9$ equals 2?

Bob: Because it equals log9 divided by log3, and it equals 2 (answer was given using calculator).

Teacher: Why does $\log_3 9$ equal log9 divided by log3?

Bob: Because of the change of base law.

Sharon: Somehow I got 1.2756.

Bob: O, you just forgot the bracket after 9.

Sharon: Why do you need that bracket?

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Bob: If you miss it, then you are finding a logarithm of 9 over log3, not a quotient of two logs.

Sharon: I see it works now. Thanks.

Ryan: But it is not fair, you've used a calculator!

Teacher: Is it possible to get the answer without a calculator?

Becky: I did. I can show it. First, I took 5 under the log, so it became $\log_3 9$. Then, I knew I had to find an exponent of 3 that equals 9 to the power of 5. Basically, I solved the equation $3^x=9^5$. Converting 9 to 3^2, I got x = 10.

Students: Cool, nice

Teacher: Can someone suggest a different approach?

Several observations from this exchange are warranted. The immediate and "trivial" solution – that involves 5×2 once the value of $\log_3 9$ is recognized – is missing. Further, there is unnecessary reliance on computational procedure and incompetent use of a calculator to implement this procedure. We wonder, what influences students' choice of approach? What is their understanding of logarithms? How can their understanding be enhanced beyond pushing the "log" button in a calculator? We address these questions in our study.

INTERPRETIVE FRAMEWORKS FOR LOGARITHMS AND LOGARITHMIC FUNCTION

The Frameworks used in this study are developed in analogy to the interpretive frameworks used by Confrey (1991) in the analysis of students' understanding of the exponents and exponential function. The three Frameworks – labelled below as A, B, and C – attend to logarithms considering numbers, operations and functions. Though presented in order of increased complexity, these Frameworks are to be viewed as a system, rather than a linear progression.

Framework A: Logarithms and Logarithmic Expressions as Numbers

In Framework A we investigate to what degree logarithms are understood as numbers and whether the value of a logarithm influences this understanding. In the traditional curriculum, the concept of logarithm is presented as an inverse of the exponent. A novice operating in this framework may correctly interpret, for example, the value of $\log_3 9$, by using the definition ($3^2=9 \Rightarrow \log_3 9=2$), but fail to interpret $\log_3 1/9$, or $\log_3 1$.

Framework B: Operational Meaning of Logarithms

The main issue explored in our second interpretive framework is the students' understanding of the operational character of logarithms. While focusing on operations with logarithmic expressions we were interested in students' awareness of the isomorphism between multiplicative and additive structures that determine the "rules" by which logarithmic expressions are manipulated, as well as students' ability to imply the isomorphic relationship in both directions. Furthermore, students' ability to imply the isomorphic relationship in both directions can be examined. For example, the approach that students take in simplifying $\log_3 90 - \log_3 10$ or in expanding $\log_c a^2 b$ may provide insight into the operational meaning students assign to these expressions.

Framework C: Logarithms as Functions.

In investigating students' understanding of logarithmic functions we consider how students' relate the definition of logarithm to logarithmic function, and how they use its properties and different representations in constructing graphs and solving problems. In the current analysis, this framework is mentioned only in passing.

RESEARCH SETTING

Participants and course

The participants in this research were 19 secondary school students who were enrolled in the Principles of Mathematics 12 course. It is important to mention that the course Principles of Mathematics 12 is not a required course for high school graduation in our site, so students enrolled in it are a self-selected and motivated group. Generally, the achievement level of these students ranges from middle to high. Many of them chose to enrol in this course because of their future plan to attend post-secondary programs in which this course is required for admission.

The unit *Logarithms and Exponents* was taught as a part of the course. In terms of recommended instructional time, this is the second largest unit accounting for about 17% of the course. While exponents and exponential notations were familiar to students from previous studies, this unit was their first introduction to the concept of a logarithm. The topics addressed in the unit include algebraic representations of exponents and logarithms, main laws and applications, logarithmic and exponential equations, the relationship between the graphs of the exponential and logarithmic functions, number *e* and natural logarithms. Further, the curriculum included modelling situations such as compound interest, radioactivity, continuous growth and decay.

Tasks

As a snapshot from our research, we focus here on two tasks:

(1) Simplify the following expression: $\log_3 54 - \log_3 8 + \log_3 4$.

(2) Which number is larger 25^{625} or 26^{620}? Explain.

These tasks were chosen as they illustrate a variety of tasks students faced in their learning of logarithms. Task 1 is considered "standard" as students approached similar tasks during their class sessions and in their homework. Task 2 is non-standard; it presents novelty in its level of difficulty and in providing no explicit reference to logarithms. Students' work on these tasks resulted in a variety of approaches and provided insight on how they view logarithms.

RESULTS AND ANALYSIS

Task 1: Simplify the following expression: $\log_3 54 - \log_3 8 + \log_3 4$.

This task was part of a quiz administered after the students completed the section on operations with logarithms (n=17). Table 1 presents a quantitative summary of students' solutions, where C, IC and PC indicate "correct", "incorrect" and "partially correct", respectively.

C/IC/PC	Examples of Solutions	#of students presenting this solution
C(1)	$\log_3 54 - \log_3 8 + \log_3 4 = \log_3 \frac{54}{8} \times 4 = \log_3 27 = \frac{\log 27}{\log 3} = 3$	9
C(2)	$\log_3 54 - \log_3 8 + \log_3 4 = \log_3\left(\frac{54}{8}\right) + \log 4 = \log_3 (6.75) + \log_3 4$ $= \log_3 (6.75 \times 4) = \log_3 (27) = 3$	3
IC(1)	$\log_3 54 - \log_3 8 + \log_3 4 = \log_3 (54 - 8 + 4) = \log_3 50 = 3.5609$	3
IC(2)	$54^3 \div 8^3 \times 4^3 = 19683$	2

Table 1: Quantitative Summary of Students' Solutions to Task 1

Students' responses to Task 1 are best viewed through Framework B – operational meaning of logarithms. Students' implementing IC(1) and IC(2) experienced difficulty in carrying out the operations. The IC(1) responses can be considered as a case of "misapplication of linearity" (Matz, 1982) or incorrect application of the distributive property. This tendency towards linearity is well documented in mathematics education literature, usually being exemplified with interpreting $(a+b)^2$ as a^2+b^2 or sin(a+b) as sin(a) + sin(b). According to Matz (1982) these errors may be explained as reasonable, though unsuccessful attempts of students to adapt previously acquired knowledge to a new situation. As such, these students performed symbol manipulation overgeneralizing familiar procedures.

IC(2) can be seen as a misinterpretation of the definition of logarithms. Indeed, logarithms are defined based on the exponential relation. It could be the case that the abbreviated phrase "logarithm is the exponent", which is often used in an attempt to interpret the definition, was memorized by these students and interpreted literally, by substituting 54^3 for $\log_3 54$. Considering Framework A, it appears that students presenting this solution did not view logarithms as numbers; as such, they attempted to isolate what they perceived as numbers in order to carry out a calculation. Considering Framework B, another interesting feature of IC(2) is the change of subtraction to division and of addition to multiplication. Though this transformation is appropriate in the context of logarithms, its implementation, as simple substitution, results in an error.

Students implementing C(1) and C(2) solutions demonstrated proficiency in manipulating an expression involving logarithms, as such it is reasonable to conclude that the operational character of logarithms was familiar to these students. In both cases the expression $\log_3 27$ appears in students' solutions, and it is of interest here to observe how students proceed from this point. In C(2), this expression is followed by the answer 3. As participants in this research were instructed to record every step in their solution, we believe that this answer was reached by recognizing that $3^3=27$. In C(1), we make a note of unnecessary application of change of base and the use of a calculator to reach the answer. This approach is similar to the approach recorded in our introductory vignette. It appears that the ability to manipulate algorithmic expressions overshadows students' ability to interpret them as numbers. Symbolic manipulation followed by calculation, wherever possible, is a preferred choice. Therefore the optimistic interpretation of the results will point out that 12 out of 17 students who attempted to solve this problem implemented correct procedures and reached a correct answer. A pessimistic interpretation notes that only three students were able to apply their understanding of the meaning of logarithm in their solutions.

Task 2: Which number is larger 25^{625} or 26^{620}? Explain.

This task was administered as part of a written questionnaire after the completion of the instructional unit on logarithms. To answer this non-standard question, students required a conceptual understanding of logarithms rather than memorization of a learned algorithm or technique. Table 2 provides a summary of students' responses.

C/IC/PC	Examples of Solutions	# of students presenting this solution
C(1)	26^{620} is larger because the logarithm of this number is larger (claim only)	5
C(2)	26^{620} is larger because the logarithm of this number is larger (claim followed by explanation)	2
C(3)	$26^{620} = \left(25^{1.012}\right)^{620} = 25^{632.6}$ $25^{632.6} > 25^{625}$	1
IC(1)	25^{625} is larger because it has a larger exponent	10
IC(2)	25^{625} is larger as it can be written as $5^{2^{5^4}}$	1

Table 2: Quantitative Summary of Students' Solutions to Task 2

Eight students gave the correct answer: however, only three of them – identified on Table 2 as C(2) and C(3) – presented mathematically sound solutions. Others followed their intuition, or just made a lucky guess. Eleven out of 19 students answered incorrectly.

The solution IC(1), produced by ten students, was justified by the claim that a larger exponent determined a larger number. Their guess was based on the premise that the exponent indicates the number of self-multipliers of 25 or 26, and so the "longer" product is the larger one. These participants did not connect this problem to the concept of logarithm whatsoever. Students' response IC(1) may be explained as the intuitive rule of the form "More A – More B" identified by Stavy and Tirosh (2000). A popular exemplification of this rule is in students' intuitive beliefs that a shape with a larger perimeter will also give a larger area, or that a taller container has larger capacity. "Larger exponent – larger number" is yet another example of this rule.

The result IC(2) was unique in this group. The number 625 attracted the participant's attention, since it is a power of 5, and the base of the first number is 5^2. The student tried to use this information, but his conclusion has not been justified. We believe that it was our unfortunate choice of numbers that created this distraction, since noticing powers of 5 does not help to reach the solution in this case.

Among the eight students who correctly identified the larger number, five presented the argument exemplified in Figure 1. They simply found the logarithm in base 10 for both numbers and concluded that a larger logarithm corresponded to a larger number.

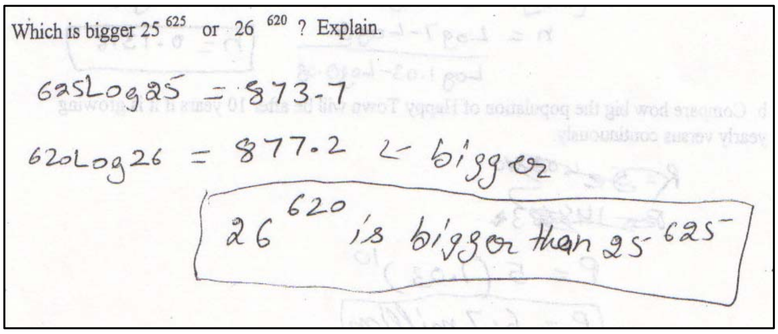

Figure 1: Example of solution C(1)

Since no additional explanation was provided as to why this was the case, it is hard to know whether the conclusion these students drew was based on their understanding of an increasing exponential function (within Framework C), or whether it was a "lucky" implementation of the intuitive "More A – More B" rule. After all, this rule is robust in people's intuition because experience shows that it "works" in a large number of cases.

As mentioned earlier, three students produced the complete and mathematically sound solutions that are shown in Figure 2. The solution in C(2) was used by two participants. Unlike their classmates who produced solution C(1), these two students explained why a larger logarithm corresponded to a larger number by looking at the exponents of 10. The solution C(3) was demonstrated by one student only, who presented both exponential expressions with the same base of 25. From the perspective of Framework A, these solutions illustrate that the students understand not only that logarithms are numbers, but also that any real number can be presented in the form of a logarithm.

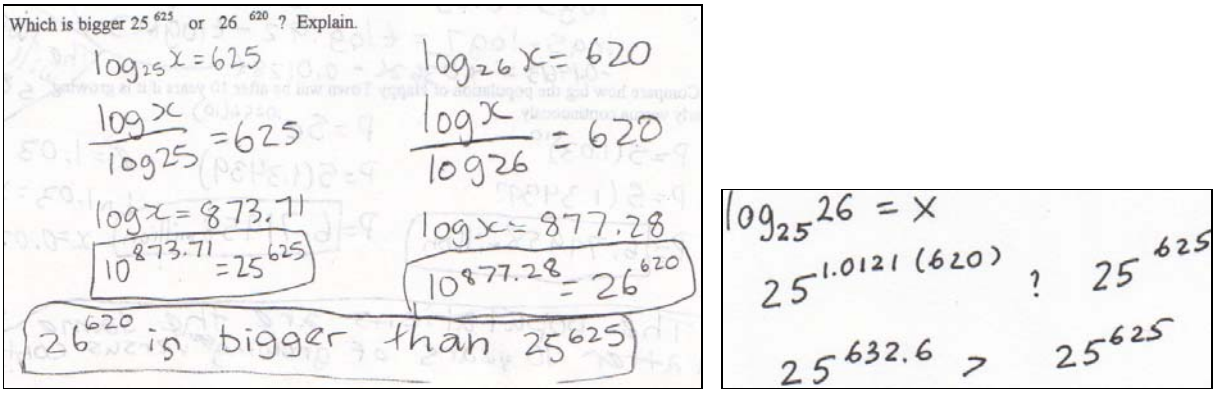

Figure 2: Left: Example of solution C(2), Right: solution C(3)

DISCUSSION AND PEDAGOGICAL CONSIDERATIONS

We believe that understanding the challenges students experience in a certain mathematical content, and determining the source of their difficulties are a necessary steps in an attempt to overcome these difficulties. In this study, we explored students' difficulties with logarithms by attending to a limited number of tasks that students performed. In an attempt to examine several issues involved in students' understanding, we proposed a system of interpretive frameworks. We also wondered whether the frameworks are helpful as the means to this end.

In considering Framework B, we observed that the ability to operate with logarithmic expressions should not be taken as understanding of their operational meaning. The introductory vignette, as well as solutions for Task 1 (labelled C(2)), provide a clear indication that operations can be performed successfully when the meaning is overlooked. The degree to which students' procedural fluency correlates with the operational meaning that students constructed requires the attention of further research. Framework B could be subdivided into "operational fluency" and "operational meaning".

Focusing on Framework A, our results suggest that students' ability to interpret a logarithmic expression and indicate its value does not indicate that logarithms are understood as numbers. We have reported instanced of overgeneralised linearity in working with logarithms, likely derived as an extension of previous experience with whole numbers. Understanding logarithms as numbers could present a greater difficulty. So, the pedagogical question is: Is it possible, and if so, how is it possible to help students understand logarithms as numbers?

It is reported in research that students often consider as numbers only standard decimal representations, and have difficulties in interpreting different representations of numbers as numbers. That is, while 25 is definitely a number, 27-2 or 5^2 are seen as exercises, operations or instructions to follow (Zazkis and Gadowsky, 2001). In order to treat a logarithm as number it should be perceived as an object. Treating mathematical concepts as objects supports the construction of corresponding mental objects in the mind of students (Dubinsky, 1991; Sfard, 1991). One possible way to treat concepts as objects is to involve them as inputs in mathematical processes: that is, to act on them or to perform operations on them. However, as our results show, following the prescribed curriculum and performing operations that implement the laws (for division, multiplication and change of base) does not necessarily serve this purpose. We wonder whether additional tasks integrated into students' experience could enhance understanding of logarithms as objects. For example, the task of ordering and placing on a number line the following set of numbers -log2, log5, log1/2, log1, -log3/4 may promote the understanding of these expressions as numbers. A further task may require the ordered placement of logarithms with different bases. Another example of a task that may support number/object construction is an equation of the form similar to $x\log_7 15 = (x+2)\log 20$. In our experience a task like this introduced confusion, and students attempted a variety of manipulations in order to present the expressions with a common base. However, once logarithms are perceived as numbers, the task in hand is just a linear equation.

As in any research that explores a novel area, we end up with questions rather than definite answers. Focusing on a series of snapshots is the first step in identifying the areas of further attention with the long-term goal of drawing a detailed and comprehensive account of the learners' conception of logarithms. As illustrative snapshots, we described students' work on one standard and one non-standard and challenging task, and provided several pedagogical considerations. Further research will examine the effect of implementing these suggestions on the understanding of logarithms, and will provide a more refined account of what this understanding does or might entail.

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TRYING TO REACH THE LIMIT – THE ROLE OF ALGEBRA IN MATHEMATICAL REASONING

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In this paper the role of algebra in students' mathematical reasoning about limits of functions in undergraduate mathematics is analysed, considering the students' educational setting. Data from a video study where six students worked in pairs to solve problems on limits of functions, and an analysis of their calculus lectures, indicate that algebra is at the same time a key and a lock to reach the limit in these problems. This double effect is related to the mathematical organisation taught, as well as the students' sense of authority as internal or external.

INTRODUCTION

The "method" introduced by Descartes significantly expanded the styles of mathematical practice and thinking. Algebra, by its integrated representational and operational features, became thereafter the dominating language for mathematics communication and reasoning, along with natural language. However, the Cartesian body-soul dichotomy in philosophy was thus paralleled by an intuitive-formal dichotomy in mathematics, turning into a major *problematique* of present day mathematics education. As many other false dichotomies, the problems it causes in students learning mathematics in institutional educational settings are likely due to didactical reasons. In this paper the role of algebra in students' mathematical reasoning about limits of functions in undergraduate mathematics will be analysed, while considering the educational setting where these students are situated.

There is a large body of research documenting the problems students have bringing together intuitive and formal conceptions of limits into a functional understanding (see e.g. Harel and Trgalova, 1996, pp. 682-686), as well as the different concept images they construct (Przenioslo, 2004). Accounts for the many difficulties refer to epistemological obstacles (Sierpinska, 1990), issues of language (Monaghan, 1991), models of infinity (Tall, 2001), APOS theory (Dubinsky et al., 2005), conceptual metaphors (Núñez et al., 1999), and epistemological analyses (Barbé et al., 2005).

In a recent study, Alcock and Simpson (2005) found an interplay between students' visual-algebraic preferences, their ways of working with formal definitions, and their beliefs about themselves as learners of mathematics. They conclude that 'visual' students with an "*internal sense of authority*, who seek to identify links among all of the representations, are likely to develop good knowledge of what objects satisfy certain properties" (p. 97, my italics), while those with an "*external sense of authority* tend to interpret the algebraic manipulations as related to procedures, and are unlikely to give much thought to the links between these and their images" (p. 97, my italics).

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 153-160. Prague: PME.

Critical for the learning of advanced mathematics is also the institutional and epistemological constraints within which the students are situated: what mathematics is intended to be taught, and what is actually taught in what didactical organisations? The consideration of such issues are needed to describe and understand student behaviour in an educational setting (Barbé et al., 2005).

STUDENTS WORKING ON LIMITS

As part of a larger study on teaching and learning limits of functions in undergraduate mathematics, six students volunteered for a video study where they were working in pairs to solve problems on limits of functions, with the presence of the author. Each session lasted for about 45 minutes. These beginning engineering students were engaged in their first semester calculus course, organised in large group lectures and classes with assisted individual work on task solving. At the time of the interview the lectures had covered the definitions and basic properties of limits and continuity, and introduced and proved theorems about standard limits such as $\lim_{x\to 0}\frac{\sin x}{x}=1$, $\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$, and $\lim_{x\to\infty}\frac{\ln x}{x^{\alpha}}=0$ $(\alpha>0)$, as well as worked out examples.

Mathematical organisation taught

To analyse these data it is necessary to display the epistemological framework within which the students were situated, i.e. the organisation of the actual mathematical work in the course. This is comprised by practical knowledge, i.e. types of problems under study and methods and techniques used, and theoretical knowledge, i.e. related technical tools and theory (Barbé et al, 2005). The main teaching format for the diffusion of this knowledge was large group lectures, while classes focused on individual task solving. A textbook provided a mathematically rigorous exposition of an elementary calculus course, based on the standard $\varepsilon-\delta$ definition of limits and continuity, including solved examples. In particular, standard limits were proved within this theory and used as theoretical tools to investigate the limit behaviour of functions given in algebraic form. Thus the type of problems studied were purely mathematical rather than applied, such as the following example discussed during one of the lectures (see Bergsten, 2006), also showing methods typically used.

- Investigate $\lim_{x\to 0}\frac{e^{\sin 7x}-1}{x}$; solved by $\frac{e^{\sin 7x}-1}{x}=\frac{e^{\sin 7x}-1}{\sin 7x}\cdot\frac{\sin 7x}{7x}\cdot 7\to 1\cdot 1\cdot 7=7$, $x\to 0$

Other methods taught include removing dominating factors, extension by the conjugate expression, and change of variable. The approach was algebraic and non-numerical. Diagrams were used to support an intuitive conceptual reasoning. Commenting on the goals set up for the lectures, the lecturer said (in Bergsten, 2006):

> I want to present, to make things seem true, the most important I think is that students believe they understand better what a concept means. To exemplify what you can handle practically, to illustrate the standard way of doing things. (p. 11)

In the lecture analysed in Bergsten (2006) mathematical rigour was maintained in proofs and solutions of examples, accompanied by the use of diagrams, metaphorical language and gestures in line with a more conceptually intuitive approach. Along with oral messages telling "how to do" when applying the technical tools, the lecturer thus established socio-mathematical norms within the course. In relation to the two mathematical organisations *algebra of limits* and *topology of limits*, observed as being disjoint in the Spanish high school curriculum as described in Barbé et al. (2005, p. 241-243), the lecture made efforts to integrate the algebraic treatment with basic conceptual ideas about limits and the behaviour of the elementary functions. The mathematics taught was predefined and presented in a closed and fully developed form, defining a didactic organisation where *everything is there* from the start. There was thus no distinct *process of study* to construct the target mathematical organisation (cf. Barbé et al., 2005).

The interview tasks

After an introduction about the concept of a limit and its definition, the task was to investigate the limits as $x \to \infty$ and $x \to 0$, respectively, for the three functions $f(x) = \dfrac{2x}{x^2 + \sin x}$, $g(x) = \dfrac{1}{x} - \dfrac{1}{x^2}$, and $h(x) = \dfrac{\ln(1 + x^2)}{x}$. The rationale for the choice of these tasks was to use the same type of problems as during work in class, and to enable an effort from the students. In one case, i.e. the function *g*, a visual key to the standard limits taught was deliberately avoided. To solve the tasks an analysis is needed to use the ideas, tools and techniques included in the mathematical organisation taught. Non-trivial standard limits were thus not directly applicable but available as technical tools out of the kind of reasoning practiced by the lecturer.

Results

An analysis of the work of each pair across tasks was performed along two dimensions, focusing on the cognitive behaviour and the didactic behaviour, respectively. In addition, a short epistemological analysis of the work on the tasks is given.

In the opening question all pairs express that the concept of limit is very important, and use an intuitive 'getting closer' description, drawing a standard graph, not offering a definition in formal algebraic terms. One pair mentions the complexity of such a definition, with its "many variables" to keep in control.

The style of work of pair A is strongly dominated by algebraic manipulations across all tasks, where the observed notations are used mainly as keys for performing procedures that hopefully will lead to a possibility to apply a standard limit. This is done immediately when starting a new task, without any prior reasoning between the students about how to attack the problem or what can be seen by considering properties of the functions involved. As a typical example of their work, after finding that $\lim_{x \to \infty} f(x) = 0$, from $f(x) = \dfrac{2}{x + \dfrac{\sin x}{x}}$ they look at the case where $x \to 0$:

Adam: Then we can’t do the same thing.

Anne: No, I have to scribble a little on my own [tears off apiece of the paper] for myself, am also not so sure about what to do.

Adam: You get zero divided by zero, you have to extend by something.

Anne: I don’t think it will help to extend by the conjugate, it will not.

Adam: No, then you would get sine two, but … can you make it two parts?

Anne: Yes, you’re right, it is two x divided by … no [writing, rubbing]

Adam: Then you have … no you don’t, that is not correct. It must be possible to remove some factor.

The work goes on along the same lines in all tasks, trying to remember what one can do and trying out different algebraic methods like changing variables, sometimes ending up in what could be called an *algebraic mess*: “this is just impossible”. Typical expressions are “then you get” and “no, that didn’t make it more fun”. They work this way 16 minutes on the task $\lim_{x\to\infty} h(x)$ without success. From the protocol it also shows that this way of working goes together with a need for an external authority to evaluate the result:

Anne: The question is if it is correct. Now I just want to know the right answer.

Interviewer: You don’t feel confident with the result?

Adam: I can’t say it should be another result, but this is a kind of task where I feel I could easily make a mistake.

Anne: Yes, me too.

Adam: By some change of variable it can be possible to make it tend to zero. /…/

Anne: I think it is zero in both cases. What was the answer?

For pair B the work proceeds in quite a different manner, dominated by conceptual reasoning about the sizes of the different parts of the given functions. Frequent expressions used are “a very small number” and “a very large number”. On the task $\lim_{x\to\infty} f(x)$ they note that the sine function is oscillating between -1 and 1 but after ending up at “two divided by something very big”, after cancelling $\frac{\sin x}{x}$, Bob says that “You can’t do it like this mathematically /…/ It can be done, there is a method”. But after repeated reasoning keeping the previously cancelled term they conclude:

Benny: Yes, sine x divided by x tends to zero, and x tends to infinity. Two divided by infinity plus zero is zero. /…/

Bob: Then we got what we were thinking.

This new solution is still as intuitive and non-formal as the previous, and in a more complex situation, as in the case $\lim_{x\to\infty} h(x)$, this kind of intuitive method proves insufficient to find the limit even after 15 minutes of work:

Bob: Zero times infinity is ok, almost zero times infinity is more tricky, it is not really zero but only tends to it. So it can be almost anything. Do we get anywhere? [looking at Ben]

Ben: No [Bob laughing]

Bob: Yes, but which one goes more, does that one go more to zero than that one to infinity? No it goes more to infinity than to zero, I think. [silence]

It seems as if algebraic methods, shown in the lectures, here are tried only when the conceptual approach does not produce an answer. However, when it does these students do not feel any need to verify the solution formally by the use of proven theorems on standard limits. They rely on internal authority. This is also evident for the task $\lim_{x\to\infty} g(x)$, where the dialogue opens up by Ben:

Ben: Well, this simply must be zero.

Bob: If you think straight on you get zero minus zero.

Somewhat surprisingly then, they make some attempt to "do" something by removing factors, but still stick to their infinitesimal way of reasoning:

Ben: But here we see that it tends to zero, it is simply so. You don't need to remove. Yes, my brain tells me so, very very small minus very very small, you get zero.

Internal authority is also evident by the use of the words "I think we are done" in the case $\lim_{x\to 0} h(x)$, after identifying and algebraically completing the application of a standard limit. But again no algebraic manipulations are performed on $\lim_{x\to 0} g(x)$, where they reason about approaching zero from the right or from the left. They conclude, after testing a numerical value, drawing a diagram and comparing infinities, that $g(x)$ tends to negative infinity. However, Bob is not fully satisfied:

Ben: So this [i.e. when approaching zero from the right] must also be negative infinity, don't you think so?

Bob: Yes, but it is kind of delicate when you take infinity minus infinity, it is kind of vague. But if we accept this way of reasoning with infinities of different size, then we have found that, if it is correct.

Thus, relying on internal authority may make you question the bases of your arguments and imply an uncertainty about the correctness of your result.

The students in pair C spend a lot of time thinking silently for each task, not jumping into algebraic trials and errors without first trying to get a conceptual idea about the behaviour of the function. The two tasks on $f(x)$ are solved quickly, intuitively as well as formally with algebraic transformations based on knowledge about the functions as $x \to \infty$ and, after some pondering, with a standard limit for $x \to 0$. In both cases the students express internal authority by the words "I'm fine with this". However, for the task $\lim_{x\to\infty} g(x)$ they are silent for some time until a common denominator is used, without giving a reason. It seems that the notation is used as a key for applying a known procedure when no conceptual solution is evident. In the same vein the factor x^2 is removed and the original expression returns, but they now realise that both terms tend to zero and express this by internal authority:

Carl: Both terms tend to zero, simply, as x tends to infinity. It feels as you have nothing in the numerator, it does not feel as if you can do very much.

Chris: I feel the same thing, hopeless. I feel fine with this solution. /…/ That you always have to make it more difficult than it is!

On the task $\lim_{x\to 0} g(x)$ these students remain silent for a long time, no ideas seem to show up, and they try some algebraic manipulations but get silent again.

Chris: Most often one can reformulate the expressions to something nicer but here it does not seem to be that easy. [silence] What do you say?

Carl: I don't know, I don't say so much.

Finally they realise that the second term tends to infinity faster than the first and state the result to be negative infinity, on an intuitive basis only. After some initial silent thinking on $\lim_{x\to\infty} h(x)$ Chris asks "What can you do here?" and "Shall we be so clever that we change the variable?" Instead long discussions about the "speed" of the logarithmic function lead them to the hypothesis that the limit should be zero. Meanwhile, Chris tries to work out the case $\lim_{x\to 0} h(x)$ by extending the fraction by x to apply a standard limit, and they both feel confident with the result zero. Returning again to the other case, they ask for a useful standard limit. When the interviewer asks what else they can do, Carl immediately responds: "Think! [laughter] To reason. It is when you shift between these methods that you sometimes can get stuck." Later, as the interviewer prompts them to look inside the parenthesis of the logarithm, Carl remembers an example where the logarithm was split up in two terms, by removing the dominating factor. By this method they finally solve the problem, and feel content. Carl concludes it was "good, we got the same result but in a correct way".

A posteriori task analysis

The tasks on $f(x)$ were problematic only for pair A, who did not realise how to use the "hidden" standard limit $\lim_{x\to 0}\frac{\sin x}{x}=1$, despite the fact that they discussed it. For the others, also in the task $\lim_{x\to 0} h(x)$, the notation provided a strong visual cue to successfully apply a known standard limit using the algebra of limits. However, when it was less obvious which algebraic technique to use, as in the case $\lim_{x\to\infty} h(x)$, the attempts to use algebra to reach the limit was not guided by a conscious strategy and often ended up in a non-conclusive situation or an algebraic mess. After setting $t=1+x^2$ pair B got the expression $\frac{\ln t}{\sqrt{t-1}}$ but did not link this to the standard limit $\lim_{t\to\infty}\frac{\ln t}{\sqrt{t}}=0$ to find a solution. Neither did any pair apply the squeezing technique for any of the tasks, though this was frequently used during lectures. Even if especially pair B based their reasoning on intuitive conceptions, it takes an experience that these students did not yet have, or a creative mind, to apply the inequality $1 < 1 + x^2 < x^3$, valid for $x > 2$, to get $0 < \frac{\ln(1+x^2)}{x} < \frac{\ln x^3}{x} = 3 \cdot \frac{\ln x}{x} \to 0$ as $x \to \infty$. The symbolic notation for the function $g(x)$ did not provide a visual cue to any of the standard limits used in the course, causing the students to either reason intuitively or engage in more or less random algebraic manipulations. This example turned out to be the most problematic one, from a formal justification point of view. No pair used the basic rules of the algebra of limits to give the solution $\frac{1}{x} - \frac{1}{x^2} = \frac{1}{x} - \frac{1}{x} \cdot \frac{1}{x} \to 0 - 0 \cdot 0 = 0$ as $x \to \infty$. Neither did they put $t = \frac{1}{x}$ and $u = t - \frac{1}{2}$ to find the solution $\frac{1}{x} - \frac{1}{x^2} = t - t^2 = \frac{1}{4} - \left(t - \frac{1}{2}\right)^2 = \frac{1}{4} - u^2 \to -\infty$ as $x \to 0\pm$. Such basic algebra of limits were not as ‘alive’ in the students as the more advanced standard limits, highlighted in the lectures.

DISCUSSION AND CONCLUSIONS

The data presented and analysed above indicate that algebra is at the same time a key and a lock to reach the limit in these problems. This double effect is related to the mathematical organisation taught, as well as the students’ sense of authority as internal or external.

Students in this study showing an external sense of authority tend to use mathematical notations as keys to apply algebraic procedures without first getting a sense for the mathematical objects involved or of where the manipulations will lead, a result in line with the study by Alcock and Simpson (2005). This way the algebra locks them out from reaching the limit. Algebra alone is void unless it takes you to known patterns, you also need an intuitive feeling for the mathematical objects involved and the links between these and the algebra.

On the other hand, students showing an intuitive approach, using images and quantitative reasoning, feel more secure in their work and show an internal sense of authority. However, knowing what the limit should be and providing conceptually based arguments for it, algebra is the key to reach it in a mathematically formal way. These students often “know” (by reasoning) what the limit is but still express a need to use algebra to really get there.

The technical tools taught are not all used by the students in this study, even when they would be useful. For example, the reason why the squeezing technique is not applied may be that it requires a familiarity to work with inequalities that these students have not yet acquired. The application of standard limits when $x \to \infty$ was also problematic, possibly due to not well established links between the intuitive and formal treatment of the concept of infinity within the taught mathematical organisation. When the students try to pave the way for applying a standard limit in the task $\lim_{x \to \infty} h(x)$, they get lost in random algebraic manipulations because they have not worked enough with tasks that could guide them to use the inequality $\ln x < x^{\alpha}$ for large *x* with an appropriate value for α, related to the standard limit $\lim_{x\to\infty} \frac{\ln x}{x^{\alpha}} = 0$.

We interpret the observations made in this study as effects of the mix or partial exclusion of the moments of the didactic process (see Barbé et al., 2005) provided by the lecture format of presenting the ready made full theory of limits and its tools and techniques, and the lack of a larger problematic within which the calculus of limits is taking place. To make students familiar with the behaviour of the elementary functions, work on algebraic manipulations, numerical tests, and inequalities is needed before attempting the types of isolated problems considered here. A need to explicitly teach why formal requirements are necessary is also apparent, as well as providing questions and problems where the calculation of limits is needed. We also note that the lecturer always knows what to do and how to do it – what effect does that have on the students' beliefs about doing mathematics, supporting an internal or external sense of authority, when they don't succeed themselves the same way?

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SEMIOTIC SEQUENCE ANALYSIS - CONSTRUCTING EPISTEMIC TYPES EMPIRICALLY

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Through the semiotic sequence analysis deep and detailed interpretative analyses of long lasting epistemic processes become manageable. This kind of investigation is presented by the use of an empirical research example. It consists of three data compressing steps. First, epistemic actions are reconstructed on the base of Peirce's sign concept. Secondly, these actions are used to get diagrams about the investigated episodes. Finally, comparison of these diagrams leads to even shorter phase diagrams, which can be divided into types of epistemic processes.

INTRODUCTION

Interpretative empirical research reconstructs structures of meaning. This kind of deep and detailed analyses increases the complexity of investigations considerably, in particular if comparisons of long lasting learning processes have to be done. The semiotic sequence is a systematic approach which makes comparisons of long lasting processes of constructing mathematical knowledge manageable.

This paper presents an example in which the semiotic sequence analysis is used. This example stems from the research project "interest in mathematics between subject and situation" (Bikner-Ahsbahs 2005, 2001) which was carried out to investigate special situations in a 6th grade class where learning about fractions took place. The following research question guided the analyses: Taking over an epistemological perspective, how is the emergence of special interest supporting situations in the class supported, how is it hindered? Focus of the paper is the methodological and methodical base. The content of research will just be used to explain the course of action.

METHODOLOGICAL ASSUMPTIONS

Background theory is the interpretative sociology. Its basic assumption is that reality is not given per se, but it comes into being by the individuals' interpreted actions which are mutually orientated among each other (Treibel 2000, p. 113). According to this assumption, mathematical meanings emerge through interpretations of actions. These interpretations orientate themselves mutually towards the other participants' actions and interpretations. Thus, mathematical meaning is a product of social interactions. Primarily, it is part of the interaction space and not of an individual. These meanings are the sequential steps which assemble together the process of social constructions of mathematical meanings. Mathematical meanings are taken as

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, pp. 161-168. Prague: PME.

pieces of knowledge if they are accepted. Taking over this view, they are not something which is true for ever, but they have proved to be viable in the local situation. Thus, pieces of knowledge are of local and situated existence. They develop and change in time and sometimes even are false. This is meant by the expression *a mathematical meaning is taken as shared through a process of negotiation* (see Krummheuer 1993).

THE SEMIOTIC SEQUENCE ANALYSIS - THEORETICAL BACKGROUND AND HANDLING DATA

Ten years ago, Steinbring (1993) has created a coding system in order to investigate epistemologically mathematical learning processes. He distinguishes three kinds of levels of meaning: a level of context, a level of mathematical signs and symbols and a level, on which students refer to symbols and objects alternately. This coding system can be applied, when a concept is developed from contextual references. However, this is not always the case in my data. In addition, Steinbring's coding system is developed to code different kind of meanings explicitly in order to earn insight into the process of building concepts. It is not designed to get information about learning processes through coding actions. This is just the aspect which distinguishes the semiotic sequence analysis from Steinbring's coding system. The first step of the semiotic sequence analysis is concerned with the reconstruction (see below) of epistemic actions. Among others, these actions are coded and diagrams, describing the process, are created. Two more compressing steps lead to the creation of theoretical types about the analysed situations.

Basic idea of this kind of analysis is the idea of reconstruction. Reconstructing means interpreting the interpretations of the students taking actions. Hence, reconstruction means interpretation of second order. In order to structure this interpreting process, Peirce's triadic sign relation and his distinction between the dynamic and the immediate object is taken as its basic pattern (Peirce 1906 see Hoffmann 2002, pp. 142).

Peirce's triadic sign relation

According to Peirce, a sign becomes a sign because it is included in a triadic sign relation among sign, object and interpretant. The sign stands for an object and induces an interpretant with a special view on the object.

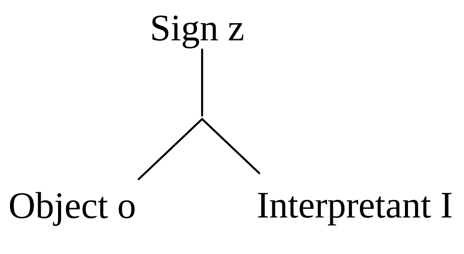

Figure 1: Peirce's triadic sign relation. (Hoffmann et. al. 2004)

Peirce defines a sign

> "as anything which on the one hand is so determined by an Object and on the other hand so determines an idea in a person's mind, that this later determination, which I term the

> Interpretant of the sign, is thereby mediately determined by that Object. A sign, therefore, has a triadic relation to its Object and to its Interpretant." (Peirce nach Hoffmann 2003, p. 198)

That means

> "A sign ... is something which stands to somebody for something in some respect." (Peirce see Zeman 1977, p. 24)!

First of all, only the sign z is visible. It mediates between object and interpretant. The interpretant itself occurs as a sign which itself is part of a triadic sign relation. If the person who puts a sign and the person who interprets the sign are different, then it is possible to analyse processes of negotiation of mathematical meanings in maths classes. If we would have to accept that the person who puts the sign and the person who interprets the sign have exactly the same view on the object, then this sign relation would not be an adequate base for an empirical analysis. However, Peirce distinguishes between the "dynamic" and the "immediate" object (Hoffmann 2002, p. 142). The dynamic object is a kind of limit concept. It is the object which would occur at the end of all possible interpretations and, thus, would be independent from all individuals. On the other hand, the immediate object is the one on which the interpretant refers in an actual situation. Immediate objects are neither obvious nor visible, but they can be reconstructed (as hypotheses) and verified through further data. In my research they are set together to find out the epistemic actions being in use in order to reconstruct the process of constructing mathematical knowledge.

Reconstruction of immediate objects

Cell no. / officer no.	1	2	3	4	5	6	7	8	9	10	until 20
1	**x**	**x**	**x**	**x**	**x**	**x**	**x**	**x**	**x**	**x**	
2		**x**		**x**		**x**		**x**		**x**	
3			**x**			**x**			**x**		
4				**x**				**x**			
5					**x**					**x**	
6						**x**					
7							**x**				
8								**x**			
9									**x**		
10										**x**	

Until 20 ....

Tab. 1: Crosses on the cell doors

The first three utterances of a scene in the class are taken as an example to show how the triadic sign relation structures the analysis in the first step. The teacher has posed a problem by telling a prison story: In a prison there are as much officers as there are cells. All the prison cells and the prison officers are numbered; every prison officer passes all the cells and makes a cross on the door if the cell number is divisible by his own number. The prisoners with cell doors that have exactly three crosses get free. Which numbers are meant? Written on the blackboard, table 1 shows the cell numbers, the officers' numbers, and their crosses.

1 T: well now let's look ,at o-u-r- table' ,which n-u-mbers have THREE crosses (..)

2 Sven: ei shall we write' them down' the numbers'

3 T: yeah we wish to NAME them at first' which we'll FIND those we wish to altogether SEEK 'em now here (.) Marcus.

The teacher puts the first sign z_1= "well now let's look ,at o-u-r- table' ,which n-u-mbers have THREE crosses (..)" in line 1. He says what has to be done, namely "to look at the table". The object of z_1 consists of the set of all cell numbers with three crosses in the table.

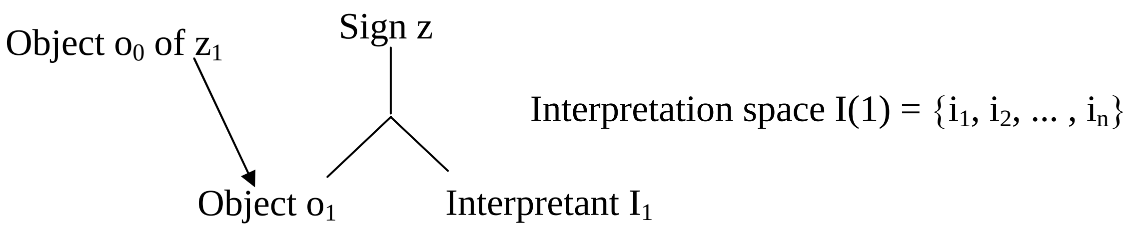

Figure 2: Triadic division of one interact

The interpretant I_1= "*ei shall we write' them down' the numbers'*" of the first sign is seen as Sven's utterance. The interpretation space I(1) consists of all thinkable interpretations i_k (k=1,...,n), g. e. Sven might regard the utterance of the teacher as a demand to act, for he proposes a possible kind of taking action, namely writing down the numbers. On the other hand, Sven probably does not know what to do because he answers hesitatingly. He might want to know more concrete what kind of action the teacher expects, because "looking" is not a mathematical kind to act. Otherwise, Sven might not be used to such an open kind of acting and, therefore, behaves confused. His proposition is expressed as a question. That indicates that Sven is not able to adapt his reactions to the expected expectations of the teacher: Sven does not know what to expect. He is somehow disorientated.

The immediate mathematical object o_1 concerning all i_k, is dealing with the numbers with three crosses. Now I_1 becomes the next sign z_2 and o_1 is the guideline object. The utterance of the teacher is the next interpretant I_2 which constitutes a new immediate object o_2. The interpretation space I(2) might consist of the following interpretation: The teacher begins with an affirmation, that means indeed: "yeah", the numbers shall be written down. But then he restricts his affirmation and develops an action program backwards: naming the numbers with three crosses, but finding them altogether at first, and more immediately "we wish to altogether SEEK 'em". Dealing with the numbers (o_1) is now more clearly defined as "o_2: seeking, finding and naming the numbers orally altogether".

The compressed process diagram shows phases

Through the first step of the course of analysis three main epistemic actions are reconstructed, namely *collectively gathering meaning*, *connecting meaning* and *structure seeing*. Signs, which indicate gathering meanings, just refer to single examples of the same kind. Signs, which indicate connecting meanings, refer to a small number of examples, which are connected somehow. And signs, which indicate structure seeing, refer to a set of examples of any (even potentially infinite) size, which all have the same structure in common. The following codes are used to compress transcripts of about fifty pages or more to one or maximum two pages:

gathering meaning:	•
connecting meaning:	
structure seeing:	
structure seeing and making them more concrete:	
structure seeing and reasoning:	

Table 2: Codes for epistemic actions

Teacher actions:		initiation
Student actions:	• • • • ⋗ ⋖	gathering examples and counter examples

Fig. 3: Beginning part of the compressed process diagram

In figure 3 we find the beginning of the compressed diagram from the scene described above.

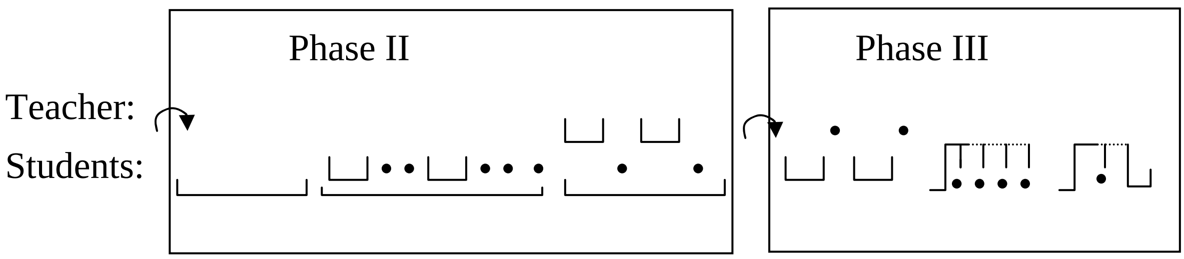

Fig. 4: Phase II and III from the compressed process diagram

The teacher initiates a gathering process and, altogether, the teacher and the students gather examples and counter examples for numbers with three crosses. Examples are 4, 9 und 25 and counter examples are 6 and 10. The compressed process diagram shows three different phases which all are initiated by the teacher. The first phase consists mainly of gathering actions, the second phase of connecting actions and the third one of seeing structures (fig. 4).

Phases represented by pictograms

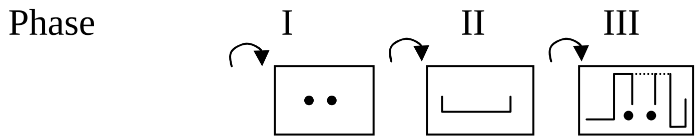

Fig. 5: Pictogram of the phase structure of the investigated episode

Figure 5 shows the next step of data compression. The investigated part of the lesson consists of three phases each of which is initiated by the teacher. The activity in every phase is mainly built by one of the epistemic actions. In the first phase the students gather examples and counter examples of numbers with exactly three crosses. In the second phase the examples are connected with each other and with other pieces of knowledge. During the third phase patterns and laws are sought, found and named, but also validated and founded: The result of the episode is that the numbers with three crosses are exactly the squares of prime numbers.

This phase structure describes a process in which the collective actions *gathering meaning*, *connecting meaning*, and *structure seeing* are built upon each other. Gathering meanings is a basic kind of action which provides the examples and pieces of knowledge on which construction of new knowledge is able to be built. During the phase of connecting meanings, gathered examples can be connected with each other. Finally, laws and patters are seen into the aggregated knowledge. They are validated and proved. We see: Every epistemic action provides a level of activity. Gathering activities prepare connecting activities and connecting activities prepare seeing, validating and proving structures. This *terrace-structure* of the epistemic process is represented by the pictogram in figure 5.

Types of epistemic actions

From every investigated episode a pictogram is extracted. All the pictograms are now used as a new sign which aggregates all the information from the analyses done before. This sign is taken as a representamen of a new object which is still not there: theoretical types which might distinguish the investigated episodes according their epitemic processes. The researcher now is the interpreter who has to construct a suitable interpretant. According to the aim to build theoretical types the pictograms have to be compared and reinterpreted. This step leads to the construction of three different ideal types (Bikner-Ahsbahs 2003): The *terrace*-type, the *spiral*-type and *confluence*-type.

The *terrace*-type is already presented above by figure 5.

The *spiral*-type of epistemic processes has got phases made of more than just one epistemic action. Data show that different kinds of activities shape different phases, in which gathering and connecting meanings are not separated but connected. Most of the phases are not teacher but student initiated. The beginning activities consisting of gathering-connecting actions (phases II, IV, V of figure 5) assemble and activate more and more mathematical meanings. Activities of this kind are repeated and, while repeating, further developed until the students are able to see structures (phases II, III, IV, V, VI of figure 6). In phase VI the students see, validate, and prove a structure. This provides new material which again is the basis for a next phase of seeing structures (phases VI to X of figure 6). This recursive process shapes a *spiral*-structured type of epistemic processes.

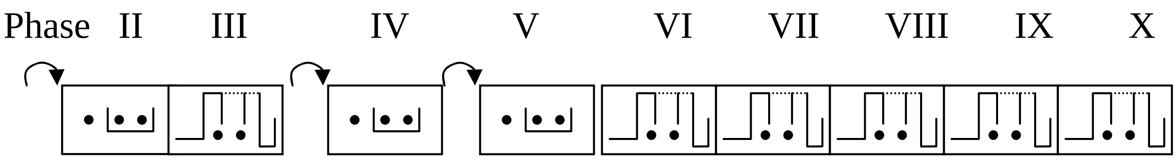

Fig. 6: Phase diagram of a spiral process

In *confluence*-structured processes (figure 7) the students start working alone or in small groups. Afterwards, insights the students have found, flow together. Like in the other types the epistemic process finishes with a phase of seeing, validating and proving structures.

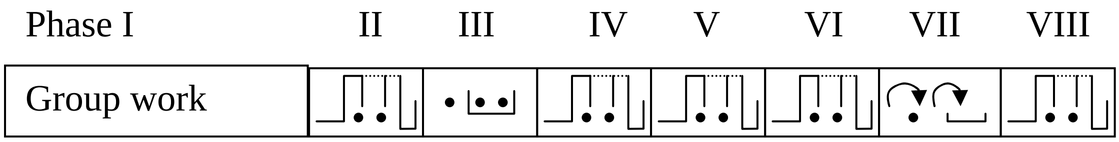

Fig. 7: Phase diagram of a confluent structured process

All the investigated situations lead to phases of structure seeing. This seems to be possible because the students are allowed to gather and connect mathematical meanings as long as and as deep as they need. Structure seeing, validating and founding are not able to occur before gathering and connecting activities have reached a saturation point.

A retrospective and a prospective view

In the presented example, the methodological perspective stems from interpretative sociology, including a semiotic approach: Signs are used to structure the analyses, additional signs are developed and used to represent the results of the analyses before in a compressed way and to approach building theoretical types (so called *ideal types*) at the end.

On the first step, a detailed analysis of a protocol leads to characterise main epistemic actions. Compressing signs give a clear and short overview about the process. These representations are taken as new signs for objects namely phase structured processes which are developed afterwards. In a next step of compressing the data, pictograms are created which represent the phases. These pictograms shape phase structured processes and lead to the construction of epistemic types.

What conclusions can be extracted?

The semiotic sequence analysis is a methodical approach which promises to get interesting results in the important research area of epistemological analyses of learning processes. Since only a special kind of interest supporting situations was analysed, this method can be used to investigate more and different situations, e.g. in the area of modelling or geometry.

The example above shows that all epistemic processes of the investigated interest supporting situations lead to phases of seeing structures. Thus, the theoretical types and the conditions behind them, increase the corpus of scientific knowledge about generalizing and constructing new structures. Used in teacher training, this theoretical knowledge can make teachers more sensitive to the epistemic processes in their own classes. In addition, it can assist with planning, analysing, and, hence, reflecting on constructing new mathematical knowledge within mathematics lessons.

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SERVICE TEACHING: MATHEMATICAL EDUCATION OF STUDENTS OF CLIENT DEPARTMENTS

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Tom Roper — University of Leeds, UK

This paper examines lecturers' views and practices on service teaching and factors that shape these views and practices. It reports on one aspect of a wider study which examined first year Mechanical Engineering and Mathematics students' conceptions of the derivative. The data comes from interviews with four mathematics and two physics lecturers supplemented with observations of calculus courses. With regard to teaching different groups of students the results suggest that lecturers: privilege different aspects of mathematics; set different questions on examinations; use different textbooks. We discuss influences on lecturers' decisions of what to teach to different groups of students.

INTRODUCTION

This paper concerns the teaching of mathematics to undergraduates who are not specialist mathematicians, 'service teaching' as it is often called in the English language. Service teaching is a significant part of the teaching practice of many mathematics departments and some form of mathematics is a compulsory part of the undergraduate studies of many students around the world. This paper focuses on lecturers' views and practices on service teaching and factors that shape these views and practices. This focus arose in research that examined first year Mathematics (M) and Mechanical Engineering (ME) undergraduate students' conceptions of the derivative. This paper has five sections after this introduction: a review of literature; an outline of the research study from which our data is taken; an outline of the research methods used with regard to this paper; illustrative interview excerpts on lecturers' privileging of examples, examination questions and textbooks; and lecturers' awareness of departmental needs. The discussion section considers factors which influence lecturers' teaching on service courses.

LITERATURE REVIEW

The majority of studies attending to teaching and teachers have concentrated on school teachers/teaching or pre/in-service courses for teachers. Substantially less attention has been paid to university teaching and lecturers in higher education literature (e.g. Hativa & Goodyear, 2002). The situation is no different in the mathematics education field and little research has been carried out on how lecturers actually teach at university level (Weber, 2004). With particular regard to service teaching in mathematics education Kent and Noss (2001) note that,

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 169-176. Prague: PME.

> compared to the corpus of work on undergraduate mathematics as a whole, the teaching of service mathematics remains relatively unexplored, and many of its fundamental assumptions (What is its purpose? What are the fundamental objects and relationships of study?) remain unexamined" (p.395).

Their subsequent work (Kent & Noss, 2003) also draws attention to the issue of service mathematics and suggests, with regard to engineers, that it should be re-examined in light of changes in the work of professional engineering with regard to technological innovation. The work of Sutherland & Pozzi (1995) also offers educational implications for service teaching of undergraduate engineers by drawing attention to the changing mathematical background of students entering engineering departments. Alongside these studies in mathematics education literature, however, perhaps the most well-known work on service teaching is the 1988 ICMI study series (see Howson et al., 1988). Studies in this series are mainly concerned with authors' personal considerations of the complexities of teaching mathematics in other disciplines rather than outcomes of systematic research.

Whilst empirical research on service teaching in mathematics education is scant, there is an abundance of suggestions by professional mathematical and engineering associations, e.g. Hibberd & Mustoe (2000). In a similar vein there are journals (see www.ltsn.gla.ac.uk) in which lecturers report on aspects of teaching mathematics including service mathematics. Much of this is undoubtedly useful and some certainly could be called teacher-research but, again, systematic research is missing.

THE CONTEXT OF THE STUDY

The study that informs this paper investigated first year ME and M students' (from a large university in Turkey) conceptual development of the derivative with particular reference to rate of change and tangent aspects and examined contextual influences of students' departments on their knowledge development (see Bingolbali, 2005). The data were collected using a range of quantitative and qualitative tools including tests, questionnaires, interviews and observations. The overall approach was naturalistic.

The M and ME calculus courses were distinct but both were taught by lecturers in the Mathematics department. The analysis of early data revealed an emergent theme: the ME lecturer (Ll) spent more time on rate of change aspects of the derivative and the M lecturer (L2) spent more time on tangent aspects (see Table 1).

	Rate of change		**Tangent**	
	ME	M	ME	M
Duration	≈133 minutes	≈11 minutes	≈10 minutes	≈85 minutes
Examples	9 examples	no examples	no examples	7 examples

Table 1: Analysis of ME and M calculus course' observations and student notes

We thought that the different emphases in teaching were not just a matter of personal preferences and that most lecturers would teach differently to students of different departments. To investigate this issue we interviewed two more mathematics lecturers (L3, L4) and two physics lecturers (L5, L6), all of whom had taught service courses.

RESEARCH METHODS

The data presented in this paper were obtained through semi-structured interviews with L1 to L6. L5 and L6 were selected for interview to gain the views of physicists on service teaching; both had taught physics to engineering and other departments.

Lecturers were asked: (1) if they teach students of client departments in a different way to how they teach students from their own departments; (2) if they set different types of questions on examinations for different students; and (3) if they use different textbooks for different departments. The semi-structured interviews allowed scope to pursue directions raised by the lecturers. Lecturers' responses were transcribed and analysed. Analysis consisted of repeated rereading of transcripts and categorising statements regarding lecturers' perceptions of teaching ME and M students.

Table 1 was obtained from observations of calculus lectures supplemented by an examination of students' course notes. L1 taught 20 45 minute lessons and L2 taught 36 40 minute lessons. Examination questions were also scrutinised.

RESULTS

The results are presented in two parts: lecturers' views regarding their 'privileging' (Kendal & Stacey, 1999) of particular aspects of concepts, setting different examination questions and using different textbooks; lecturers' perceptions of distinct departmental features. Selective interview extracts are interleaved with comments.

Lecturers privileging particular aspects, examination questions and textbooks

All lecturers stated that in teaching in different departments they emphasised different aspects of topics and noted that this principally occurs in delivering specific aspects of the topics and related examples. Both observed calculus course lecturers stated that they taught engineering students differently from mathematics students.

> L1:…In engineering departments it is more application-oriented.... I give examples regarding objects in motion with regard to time, pressure and so forth... I focus on rate of change aspects in teaching engineering students. But in our department (*maths*) we focus more on concepts rather than on application aspects, for example, tangent aspects.

> L2: If they were physics department students, lots of examples regarding physical meanings of the derivative would be given. But in the maths department the derivative as a concept is prioritised. For maths students to see just how it can be applied is enough.

These views accord with what was observed in their calculus courses (see Table 1). Further to this, although both lecturers presented 20 theorems over the course, mathematicians were given the proofs of 17 of these theorems whereas engineers were given the proofs of 10.

When asked if they reflected this difference with regard to concepts and examples in examination questions as well, all lecturers stated that they did so. For instance:

> L1: Mathematics students would be specialists in this area…they need to know this job's reason and logic. That is why you can ask them theorems in their examinations. This is their job. You can ask them some definitions as well. Nevertheless, if you do this in engineering departments, it would not do anything good to them but get them bored.

L3 and L4 also stated that they set theorems in Mathematics department but not in other departments' calculus examinations where they set application problems.

> L3: You have to ask theoretical questions to maths students anyway. Questions are all applications in engineering, 40% theoretical and 60% application in maths department.

Physics lecturers made similar remarks, e.g. L5 stated that he set more theoretical questions in physics examinations and "did the opposite" in engineering examinations.

An analysis of ME and M departments' mid-term and final calculus examinations showed that L1 and L2 reinforced the 'tangents for M, rate of change for ME' emphasis in examinations questions. M students were also asked to prove theorems in examinations questions whilst ME students were not asked to prove theorems.

With regard to the textbooks, both mathematics and physics lecturers stated that they made use of different textbooks or privileged different parts of the same textbook for different departments. When asked if they used different textbooks lecturers reported:

> L3: Of course, the books that we give our mathematics department students include more theoretical stuff and include theorems and proofs. But in engineering departments, books are more real life or application-oriented ones.

> L1: The books I use in engineering department are generally application-oriented ones…. But in our department I would use those books including more theorems and proofs…

Lecturers' awareness of distinct departmental features

The data presented so far show that both mathematics and physics lecturers stated that they privileged different aspects of a particular concept, set different types of questions and used different textbooks for the students of different departments in general and for mathematics and engineering students in particular. We now turn to stated reasons as to why they did this.

In responding to the question of whether they taught differently in client departments, L1, L4 and L5 referred to the departmental demands from 'higher authorities'.

> L1: They demand from us some stuff. It is like we use maths here and there, we want our students to know this and that so that they can be successful in the coming years' courses.

> L5:...We contact administrators of department and ask them what they want their students to get from physics and the physics we teach is based on their demands as well.

Some lecturers referred to students' needs with regard to mathematics. L3 stated:

> L3: The main aim is where maths and engineering students make use of maths. Maths students need to know everything but engineering students only need to know the parts which are useful for them…

Similarly when asked if he included theorems in M calculus examinations L1 stated:

> L1: Mathematics students will be specialists in this area, they need to know this job's (*studying mathematics*) reason and logic. That is why you can ask them theorems in their examinations. This is their job.

Like L1 and L3, L4 stated "the mathematics given in textile departments should meet the expectations of textile department". Common to all three lecturers' accounts is the perspective that students in different departments have different mathematical needs and it is reasonable to provide them with different aspects of mathematics.

Physics lecturers also mentioned students' needs in explaining how they taught.

> L5: …In physics department… topics are given in detail and their 'hows' and 'whys' are investigated… physics students need to get its fundamentals and essences. But in engineering or maths department you cannot get into detail much, it is not possible.

> L6: (of client department students)…we need to think how the physics we teach is going to be useful to them. What we teach them should be useful to them.

Some lecturers stated they taught the way they did due to 'departmental features'.

> L4: …This is done in accordance with the departmental features. That is why, different issues are more emphasised in different departments…

The Physics lecturers expressed similar statements:

> L5: In the maths department I tried to give examples concerned with the essence more, while in engineering departments it was more towards the application aspects … I tried to choose some typical questions which are peculiar to this or that particular department.

> L6: Topics are presented so that they are useful for the department's job…, are close to these departments features… so that they are useful to students.

Students' perceived needs and departmental features are, however, related conceptions; lecturers view students as students of a department. This inseparability of conceptions is evident in L3:

> L3:… We need to give theoretical aspects to maths students but for engineering students we need to explain to them where they can make use of what they are given. Maths students need to think in a broad sense… engineering students just need to know how to deal with engineering application problems.

DISCUSSION

Lecturers' actions shape and are shaped by the contexts in which they work. In this section we further explore factors, and their possible interrelations, that may have shaped lecturers' views and practices with regard to service teaching but begin with a review of what has been stated.

We noted three differences between the practices of L1 and L2: (1) L1 privileged rate of change aspects of the derivative whilst L2 privileged tangent aspects; (2) L2 provided proofs of more theorems than L1; (3) the privileging of rate of change aspects by L1 and tangent aspects by L2 was evident in the examinations and, further to this, L2 set theorems to prove in the examinations. We are quite certain that this privileging arises from lecturers' considerations of the students/departments the courses are for and, had they been teaching the other's course, that they would have privileged aspects as the other had. Further to this the interviews showed that L1 and L2 were aware of their privileging and that this was intentional, i.e. that their views and practices were compatible.

The other two mathematics and the two physics lecturers also stated that they adapted their instruction to suit students of client departments. They pointed out that in mathematics/physics departments, they privileged (though not in these words) theoretical aspects (the essence) but in client departments they privileged applications aspects or aspects which would meet the expectations of students. Similar comments were made by these lecturers in relation to examination questions they set and the textbooks they used. In a nutshell, the data consistently points to the fact that all lecturers privilege different aspects/particular concepts, set different questions on examinations and use different textbooks in teaching courses in client departments.

We take it as trivial that lecturers' privileging cannot be reduced to their personal preferences for way of teaching, so the question of interest is 'what leads these lecturers to teach students of client departments differently?' The interviews generated three factors that lecturers suggest influence their teaching: (1) departmental features or the department themselves (L2, L3, L4, L5 and L6); (2) departmental demands (L1, L4 and L5); (3) students' needs (L1, L3, L4, L5 and L6).

With regard to the first factor, all lecturers but L1 referred to distinct departmental features in explaining why they taught students of departments other than their own differently. They reported that the way they taught students was compatible with the departmental perspectives and the content they selected was "peculiar" (L5) to particular departments. Lecturers' accounts suggested that they perceived the departments as having distinct goals not only in terms of preparing students in line with their professions but also in terms of the courses they were teaching. We thus argue that lecturers' interpretation of the goals/nature/features of the departments impacted on what lecturers reported that they taught to the students of client departments.

With regard to the second factor, L1, L4 and L5 referred to the 'demands of the departments' in explaining what they teach to students of client departments. They reported that they had consultations with departmental administrators concerning what they would teach to the students. The key uncertainty here is the extent to which the demands of the administrators impacted upon what the lecturers taught. Given that it is, at university, largely left to lecturers to decide what to teach, we believe that although departments request that lecturers cover particular aspects of subject matter in particular ways, that lecturers do not 'mechanically' follow these requests but that this is bound of with lecturers' interpretation of departmental and student needs.

With regard to the third factor, all lecturers but L2 referred to 'students' needs' in their discussions of their teaching. These lecturers' accounts suggested that they were quite conscious of differing students' needs for students of different departments. This perception of student need appeared to be particularly marked with regard to mathematics and engineering students, and the need to teach them differently (see L1, L3 and L6). L3, for instance, clearly differentiates between engineering and mathematics students in terms of their mathematical needs. He considers that since mathematics students are going to be mathematicians, they need to know 'everything' but the same thing cannot be expected from engineering students because they are perceived to be 'consumers' of the mathematics and hence they just need to know, or should be taught, the parts which will be useful to them in engineering.

Although we present these three factors separately, we see them as very closely, inseparably, interrelated. Having said this, however, lecturers' perceptions of departments appears, to us, as the dominant factor. This is a difficult proposition to substantiate but we can draw implications from and to it. Although all but one of the lecturers referred to students' needs in explaining what impacts upon their teaching in client departments, lecturers' perceptions of the needs of the students cannot be separated from their perceptions of the departments or their distinct features. In a similar vein, although departments undoubtedly make 'demands' upon lecturers as to what to teach to their students, we think, as noted above, that what lecturers teach to students (from any department) is largely influenced by their perceptions of the departments (and their students' needs).

If the argument that lecturers' perceptions of departments is the dominant influence on their teaching is correct, then a crucial question is, 'what is it about departments that leads lecturers to perceive them in a way in which they can envisage their students' needs and teach them accordingly?' We think that there are two different but complementary ways of addressing this question. The first is related to lecturers' material experiences of teaching in particular departments; that when lecturers teach in particular departments, they come to know norms, values, rituals as well as the overarching goals of these departments (or 'institutions' or 'communities of practice') and that these norms and values enable lecturers to make judgements as to what the students' (mathematical) needs are. Our second consideration concerns the ways in which departments are historically and culturally perceived, and that 'allows' (affords and constrains) lecturers to perceive departments in particular ways and to interpret students' needs and, unlike the first consideration, this is the case even if the lecturers have not had any teaching experience in the department.

Both considerations relate to the positioning of an individual with respect to an institution. As Castela (2004, p.41-42) states:

> When persons 'enter' an institution, their life in this institution is submitted to collective *constraints* and *expectations* that regulate their actions. These constraints and expectations specify their position as *subjects* of the institution. Several subject positions exist in a given institution: for example, students, lecturers and assistants at a university.

Mathematics departments, for example, are historically seen as 'pure' and 'abstract' subject areas whereas mechanical engineering is seen as 'applied' and 'related to real life' (Biglan, 1973). We think that the titles of these departments *per se* can evoke particular meaning for lecturers with regard to the departments' nature and their students' (mathematical) needs. We argue, then, that lecturers' perceptions of departments, which might come from their own experiences or simply from their perceptions of the departments, can lead lecturers to position themselves in terms of what to privilege in their teaching in particular departments.

In conclusion it appears to us that lecturers' teaching is influenced by their perceptions of the departments in which they teach and this particularly manifests itself when lecturers teach in client departments. The changes lecturers make in their instruction in teaching in client departments cannot be reduced to their personal preferences but, rather, is related to their position as subjects of that department.

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STUDENTS' THINKING ABOUT THE TANGENT LINE

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The aim of this study is to trace students' conceptions about the tangent line of a curve. For this purpose we investigated the characteristics of students' concept images, as well as their ability for symbolic manipulation. A questionnaire with 15 tasks was administered to 196 12th grade students. Through Latent Class Analysis of the data three latent classes of participants were constructed indicating a scaling of students' performance. These levels describe a gradation of students' concept images from an elementary level, closer to the Euclidean context, to the more advanced level of Calculus context, as well as the involvement of symbolic manipulation abilities in this developmental process.

INTRODUCTION

Mathematics education researchers have called for more detailed investigations into students' conceptualizations of calculus concepts (Hauger, 1999). More specificly, understanding how students learn and come to know calculus concepts will help inform calculus teaching practices. This paper is part of a larger study which focuses on the understanding of advanced mathematical topics.

There has been comparatively little research on students' learning and understanding of the concept of the tangent line, which is an essential mathematical concept and is central for many collegiate mathematics courses (e.g. geometrical interpretation of derivative, applications in physics). Thus, it is important not only to study the way students think when they work on tangents and other related topics in mathematics, but also to provide a theoretical framework that may explain students thinking about the tangent line and their misconceptions. In this paper, we report a study that investigates students' conceptions of tangents; we hypothesize that these conceptions are strongly influenced by students' previously developed concept images; and, we explore the characteristics of this influence. In order to do this, we first provide the theoretical background of the study, then we describe the method and results of our empirical study, and finally we present some conclusions.

THEORETICAL BACKGROUND

Students enter high school with many experiences that are bound to shape their learning about new mathematical concepts (Tall & Vinner, 1981). The notion of a tangent is a complex concept that students encounter in a different context. According to the Greek mathematics curriculum, students by grade 10 are taught about the tangent in the context of Euclidean geometry. Then, 11th graders are introduced to the tangent lines of conics with their particular algebraic expressions

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and properties in the context of Analytic geometry. Finally, 12th graders are introduced in Calculus courses to the tangent line of a function curve which is a line with a slope equal to the derivative of the corresponding function at the point of tangency. We assume that the specific context in which the tangent line is presented affects students' cognitive structures relative to the concept of the tangent. Following Tall and Vinner (1981), we consider these structures, which include all the mental pictures and associated properties and processes, to be students' concept images of the tangent line. According to Tall and Vinner, when ideas are presented in a restricted context, the concept image may include characteristics that are true in this context but not in general. In the case of the tangent line, Tall (1987) and Vinner (1982, 1991) observed that early experiences of the circle tangent contribute to the creation of a generic tangent as a line that touches the graph at one point only and does not cross it. These images contain coercive elements inappropriate when more extreme cases are considered, such as the tangent at an inflection point, where it does cross the curve, or the "tangent" at a point where the function is continuous and not differentiable.

In our study we investigate the development of students' understanding of the tangent line from grade 10 to first year university students. In this paper we refer only to the results regarding students at grade 12 as they have dealt with the tangent line in the context of Euclidean and Analytical geometry, as well as in Calculus.

We hypothesise that initial conceptualizations of the tangent line within a given context stand in the way of students' understanding in subsequent contexts (from Euclidean and Analytic geometry to Calculus). Following Winicki and Leikin (2000), we assume that students who have studied tangents in different contexts may perceive as defining conditions properties that are not generally valid. Therefore, we expect to diagnose intermediate levels of understanding of the tangent line, reflecting students' use of relevant and irrelevant properties of the tangent within a specific context.

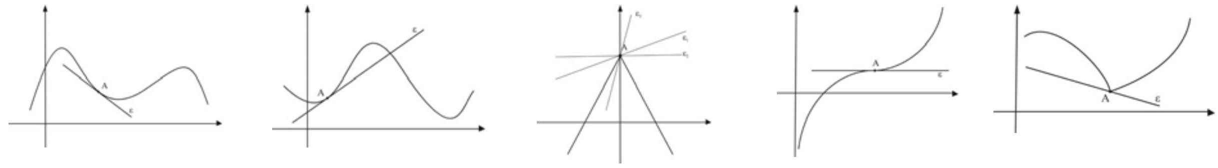
THE STUDY

Method

The participants in this study were 196 12th grade students (125 male) from nine different Greek high schools. All of these students had mathematics as a major subject since they were candidates for science, medical or polytechnic studies but they had various levels of performance. By the time the research took place, all participants had been taught elementary Calculus, including functions, limits of functions, continuity, derivative, tangent line and the applications of derivative to monotone functions, extreme values, concavity, inflection points and curve sketching.

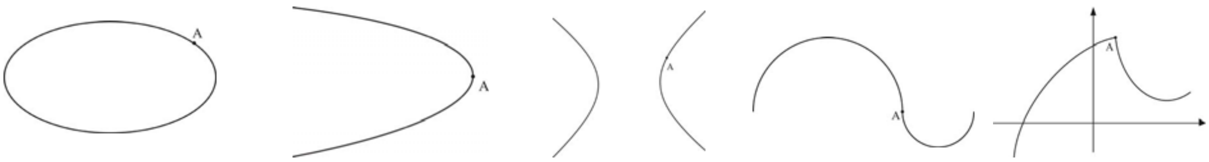
The data were collected in 2005 with a questionnaire administered to the participants in the middle of the second semester. The questionnaire consisted of 15 tasks regarding geometrical properties and symbolic representation of the tangent line (Figure 1).

1. Which of the lines that are drawn in the following figures are tangent lines of the corresponding graph at the point A? Justify your answers.

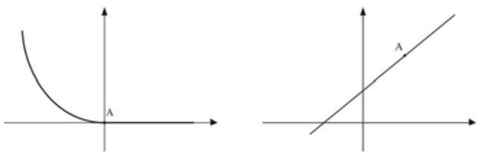


q1.1 q1.2 q1.3 q1.4 q1.5

2. Sketch the tangent lines of the following curves at the point A, if they exist. Justify your answers.

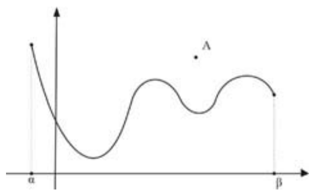


q2.1 q2.2 q2.3 q2.4 q2.5



q2.6 q2.7

3. In the following figure, draw as many tangent lines of the drawn curve as you can passing through the point A.



4.1 Let f be a function and a point A(x_0, $f(x_0)$) of its graph; write the equation of the tangent line of the graph of f at the point A if it exists.

4.2 Calculate the equation of the tangent line of the graph of the function f with $f(x)=(x-2)^3+3$ at the point A(2, $f(2)$).

Figure 1: Questionnaire

In the questionnaire[1] there are three types of tasks: recognition, construction and symbolic manipulation. In the tasks q1.1- q1.5 (figure 1) the students have to recognize if the drawn lines are tangent lines of the corresponding graphs. In the

[1] The questionnaire and the data presented in this study have been translated from Greek

construction tasks the participants have to draw, if they exist, the tangent lines through a specific point A on the curve (q2.1- q2.7) or outside the curve (q3) (figure 1). Especially, in q3 there are three different tangent lines of the curve through the point A: one that "touches" the curve only at the point of tangency (q3.1), one that intersects the curve in its extension after the point of tangency (q3.2) and one that has to intersect the curve in order to reach the point of tangency (q3.3). In the symbolic manipulation tasks, students have to write the formula of the tangent line of the curve of a function f at a specific point (x_0, $f(x_0)$) in general (q4.1) and in a concrete case (q4.2).

The tasks q1.1 – q3.3 are in a graphic context. Through them we aimed to investigate students' ability to recognize and construct a tangent line of various designed curves, where no formula is given, as well as justifying their choices. Tasks q1.1, q1.2, q2.6, q2.7 and q3 indent to examine students' beliefs about the number of common points of the tangent line and the curve: no other common point with the curve except the point of tangency (q3.1); no other common point in a neighbourhood of the tangency point (q1.1, q1.2, q3.2, q3.3); and infinite number of common points in any neighbourhood of the tangency point (q2.6, q2.7). Tasks q1.3, q1.5 and q2.5 are referred to tangency at "edge points". We call "edge point" a point in which the derivative from the left and the right exist without being equal. Tasks q1.4 and q2.4 are chosen for the case of tangency at an inflection point. Finally, q2.1, q2.2 and q2.3 are curves of conic sections already known to the participants by an Analytic geometry course at the grade 11. The tasks q4.1, q4.2 are in a symbolical context and presented in order to examine students' competence in utilization of appropriate symbols and processes to express and calculate the equation of the tangent line.

Data Analysis

Each response is characterized as correct or false. For the needs of this paper an answer is regarded as correct if the student has chosen or drawn as tangent the right line regardless his/her justification. Otherwise, the answer is considered as false. Particularly in task 3, three variables were created: q3.1, q3.2 and q3.3 for each one of the different potential tangent lines.

In order to investigate different models of students' thinking we based our analysis on Latent Class Analysis (LCA) which is a special case of Finite Mixture Modelling. LCA provides classification of individuals to their most likely latent classes measured on observed variables. The model was tested for two, three, four etc groups of individuals and as the best fitting model was chosen, this with smallest AIC and BIC indices. For the above analysis we used a widely known structural equation modelling software MPLUSv3.11 (Muthen & Muthen, 2004).

Results

The model was tested through LCA for up to seven groups of students and as the best fitting model was chosen this with three groups (AIC 2906.693, BIC 3060.764). Group A with 78 students, group B with 60 students and group C with 58 students. These three groups are distinguished. The probability that a student belongs to the group where s/he was classified is for group A 0.902, for group B 0.926 and for group C 0.959. In table 1 we present the tasks with corresponding percentages of success for all students (second column) and in each group separately (third, fourth and fifth column). The numbers above the bold zigzag line, for each group, correspond to the tasks that more than half of the members of the corresponding group had answered correctly.

Task	Total success	Group A (78)	Group B (60)	Group C (58)
q3.1	0,94	0,94	0,97	0,93
q1.1	0,94	0,99	0,97	0,84
q3.2	0,91	0,95	0,98	0,79
q2.1	0,90	0,99	0,83	0,86
q2.2	0,87	0,99	0,82	0,78
q2.3	0,87	1	0,80	0,78
q1.2	0,76	0,95	0,82	0,43
q1.3	0,75	0,91	0,97	0,31
q4.1	0,75	0,96	0,73	0,48
q1.5	0,72	0,88	0,98	0,24
q4.2	0,65	0,87	0,70	0,31
q2.5	0,58	0,73	0,85	0,10
q1.4	0,54	0,94	0,20	0,36
q3.3	0,50	0,85	0,47	0,07
q2.4	0,33	0,64	0,03	0,21
q2.7	0,32	0,56	0,17	0,16
q2.6	0,32	0,53	0,12	0,24

Table 1: Results

A general observation is that students had better results in the recognition tasks than the corresponding construction tasks. For example 72% of the participants answered correctly the task q1.5 compared to 58% who answered correctly the corresponding task q2.5. Moreover, in q1.4 and q2.4 the difference increases as the first one is more familiar to students from their school experiences. Although the tasks q1.1, q3.2, q1.2, q3.3 are similar relatively to the number of common points, the results of the first two tasks are much better than the results of the other two. An explanation of that could be that in q1.1 and q3.2 the other common point, except the tangency one, is not sketched or is not necessary to be sketched.

About the performance of each group, we observe different combinations of correctly answered tasks that reflect different categories in students' conceptions of the tangent line. Each one of these categories reveals the extent of influence of relevant or irrelevant property/ies of tangent line to the students' concept images. These properties are related to the numbers of common points that the line could have with the curve; the related position of two figures; and, the way that the one touches the other. In addition, the students of these groups demonstrate different levels of ability in symbolic manipulations that are requested in order to calculate the formula of the tangent line. According table 1 the success of any task by more than 50% of the students in one group is associated with success by more than 50% in the preceding groups.

To be specific, participants who are classified in group C have more than 78% success in the first six tasks of the table 1 (q1.1, q2.1, q2.2, q2.3, q3.1, q3.2) but fail in the rest of them with less than 48% success rate. This combination of success/failure in the tasks reveals a concept image about the tangent line which is strongly influenced by the circle case. According to this concept image the tangent line has only one common point with the curve and leaves the curve in the same semi - plane. Moreover, no "smoothness" is demanded at the point of tangency, as they accept tangent lines at "edge points". Also their ability in symbolic manipulation is weak. A representative student with this concept image answers in the task q1.2: "The line (ε) is not a tangent of the graph at the point A because the line (ε) cuts the graph at another point" and rejects the line (ε) in q1.4 by saying: "... it cuts the curve at A". But in q1.3 he accepts (ε_1) and (ε_2) and rejects (ε_3). Despite this, in this group there are students who have a concept image according to which the tangent line has only one common point with the curve independently of their relative position. For example, in q1.1 and q1.2 a student answers: "ε is not a tangent line because it cuts the figure at its extension" and in q1.3: "ε_1, ε_2 and ε_3 are tangents because they have only one common point with the function". Moreover, in q1.4 she accepts (ε) because "... as far as it could be extended it will not cut the function". Some of the students of this group fail in tasks q2.1, q2.2 and q2.3. The most common error is the drawing of more than one tangent lines passing through A. This usually is accompanied with the same error in q2.5 or the selection of two or all three drawn lines in q1.3. It appears that their responses are connected to their beliefs about the possibility of existence more than one tangent line at a point of a curve.

The students, who are classified in group B, besides being successful in the same tasks as the students of group C, succeed in the next six tasks (q1.2, q1.3, q1.5, q2.5, q4.1, q4.2). They fail in the remaining tasks with less than 20% correct answers. The majority of the students of this group has a concept image with the following three characteristics: there is a neighbourhood of tangency point where there is not another common point between the line and the curve; there is no tangent line at an "edge point"; and, the tangent line can not intersect the curve (e.g. at an inflection point). Despite the success in q1.2 task (82%), 47% of the students of this group fail in the similar q3.3, as the former requires only recognition of the tangent line contrary to the last that requires the construction of it. A student from this group responds correctly in q1.1 and q1.2 and answers in q1.4: “the line isn’t [tangent] because it cuts the graph and separates it in two parts” and in q2.6: “There is no tangent because it would coincide with the graph”. Some students in this group have a concept image with the following characteristics: the tangent line has only one common point with the curve; leaves the curve in the same semi - plane; and, there is no tangent line at an “edge point”. This corresponds to a concept image strong related to the circle. The difference of this concept image in comparison to the similar one of students of group C is the conservation of the figural representation of the circle as a “smooth” curve. A representative student with this concept image rejects the line (ε) in q1.2 because “... it crosses the curve”. In addition in q1.4 he answers: “It cross the curve, so it is not the tangent”. But in q1.5 he rejects (ε) and justifies: “It does not touch; (ε) has only one common point with the graph.” It would appear that the students of this group have created more sophisticated, but still inadequate, concept images of the tangent line. However, they show sufficiently good results in the symbolic manipulation tasks (q4.1 and q4.2).

The students who are classified in group A performed all tasks successfully. Each one of the tasks was answered correctly by more than half of them. It appears that their concept images include the tangent line at an inflection point and for at least half of them the coincidence of two curves is acceptable.

CONCLUSIONS

The findings of this study support our hypothesis, that initial conceptualizations of the tangent line within a given context stand in the way of students’ understanding in subsequent contexts (from Euclidean and Analytic geometry to Calculus). It appears that students create concept images, grounded on Euclidean and Analytic geometry, sufficient to deal with specific cases, but usually inadequate in more general situations. A careful analysis of the results brings to light different students’ concept images. A detailed description of them reveals the spectrum of students’ misconceptions regarding tangent line and the way that these are influenced by prior knowledge. As expected, there is a gradation in students’ performance in the tasks. These findings seem to support the hypothesis that there are at least three developmental levels of concept images which characterize students’ thinking about tangent line. The students’ abilities concerning symbolic manipulation, which are cultivated during the upper levels of high school, is accompanied by a more sophisticated concept image of the tangent line, but these abilities do not entail an adequate concept image.

The different models of thinking about the tangent line as well as the levels of students’ performance reveal similarities with the historical development of the notion. It seems that the image of the tangent line in Euclidean context as a line that has specific geometrical properties with the whole of the circle acting as an epistemological obstacle (Brousseau, 1997; Sierpinska, 1994) to the process of understanding the general meaning of this notion.

The next step of our research is the further investigation of these aspects of tangent line in first year university students of a Mathematics department in order to examine which of them remain active, and to what extent, a few months after the introductory instruction at secondary level and before any instruction at tertiary level.

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HABERMAS' THEORY OF RATIONALITY AS A COMPREHENSIVE FRAME FOR CONJECTURING AND PROVING IN SCHOOL

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In this paper I will argue that Habermas' theory of rationality, conveniently interpreted and adapted, can become a comprehensive frame to deal with complex educational and cultural problems related to the teaching and learning of theorems in school. Focus will be on conjecturing and proving as processes, in relationship with statements and proofs as products. An example, taken from an activity of in-service teachers' preparation, will offer elements to support my position and address future theoretical work.

INTRODUCTION

In the last decades, mathematics education has adopted several tools and theoretical constructs from other sciences (psychology, linguistics, sociology, etc.). The need for adopting further theoretical constructs must be motivated by showing how they can be useful to handle relevant problems in mathematics education, which are difficult to face by using existing tools. However, theoretical constructs adopted from other sciences cannot be used in the way they were elaborated in their disciplines of origin; they must be interpreted and adapted according to specific mathematics education specific needs (depending on the teaching and learning of a specific subject, namely mathematics, and its peculiar status in culture and school).

In this paper I will present an example (concerning mathematical proof) in order to show how it is difficult to deal with it in a comprehensive, satisfactory way with the existing tools elaborated in the mathematics education domain; then I will propose an interpretation of Habermas' theory of rationality as a possible answer to the need of framing complex educational and cultural questions regarding conjecturing and proving in mathematics education. Finally I will discuss possible integrations of Habermas' frame in order to make it more suitable for our needs.

AN EXAMPLE

Let us consider the following problem, posed as an individual task in a 7th-grade class (students had never used algebraic language for proving; they did not know models concerning with the way a text of proof must be organised; but they had already made an experience on how to justify an arithmetic statement in general):

> Find the Greatest Common Divisor (CGD) of all the products of three consecutive integer numbers.

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Anna, a brilliant student, produced the following written solution:

> *1x2x3* makes *6*, so the GCD cannot be bigger than *6*. I must see if *6* is a divisor of the product of three consecutive numbers in every case. I try with *2x3x4=24*, it is divisible by *6*. *3x4x5=60*, it is divisible by *6*. *9x10x11=990*, it is still divisible by *6*.
>
> I try to understand why.
>
> *2, 3, 4.* *3, 4, 5.* *9, 10, 11.*
>
> *1, 2, <u>3</u>, 4, 5, <u>6</u>, 7, 8, <u>9</u>, 10, 11, <u>12</u>.*
>
> If I take three consecutive numbers, one or two out of them are even, so they are divisible by *2*; one out of them must be divisible by *3*, because the multiples of *3* follow each other with two other numbers in the middle.
>
> So the product of three consecutive numbers must be divisible by *2* and by *3*, namely by *6*. Hence *6* is the CGD of all the products of three consecutive integer numbers.

This solution was proposed to 36 high school mathematics teachers at the very beginning of an in-service teacher training activity concerning the teaching and learning of mathematical proof. No information was given about Anna's mathematical background; only the information that she was a 7th grade student was given. The individual task for the teachers was:

> Evaluate the acceptability of Anna's solution from the mathematical point of view, and motivate why it is acceptable or not.

7 teachers refused to answer because they thought that the *"mathematical point of view"* was an ambiguous expression, and/or for the lack of information about the educational context (some wrote that *"if letters had been introduced by Anna's teacher, the solution was unacceptable, otherwise it was acceptable"*; some said that *"it would have been necessary to know what was a good solution for the teacher"*).

11 teachers answered that Anna's solution was acceptable, because her reasoning was substantially correct (the lack of an algebraic treatment of the problem was considered by 5 out of them, but justified by saying that *"Probably, Anna was not sure in the use of letters"*, or *"the use of letters had not been taught enough to her"*).

18 teachers answered that the solution was not acceptable as a mathematical solution to the problem with the following kinds of motivations:

- *"Anna remains at an intuitive level"* (5 teachers);
- "*The use of examples is wrong in mathematical proof"* (6 teachers)
- "*Anna's first statement is OK, but then she should move to an algebraic reasoning with letters in order to show that 6 is a common divisor"* (2 teachers)
- *"Anna's reasoning should have been translated into an appropriate mathematical formalism"* (4 teachers).
- *"In order to make the solution more acceptable, the second part of the text had to be made more general, by saying that '6 is a divisor of the product of three consecutive numbers in every case, because if I take three consecutive numbers, one or two out of them are even, so they are divisible by 2; one out of them is a multiple of 3, so the product is divisible by 2x3=6'"* (one teacher, Lorena).

Before starting a discussion, teachers had to deal with the following task:

> Many of you answered to the previous question by saying that the solution had to be written with the use of letters. Show how to do it.
>
> Do you confirm or not your evaluation about Anna's solution?
>
> If you are still not satisfied with Anna's solution, how would you behave in receiving students like Anna in your class, at the entrance of the high school, in order that they improve their proving?

Translation of the problem into algebraic terms (namely, $n(n+1)(n+2)=n^3+3n^2+2n$, or $(n-1)n(n+1)=n^3-n$) resulted in a general, frustrating standstill situation. Only partial results were got (for instance, the divisibility by 2 through the second formula). Five teachers wrote some vague sentences about modular arithmetic as a possible tool to solve the problem in advanced algebra.

4 teachers out of the first group (of 7 teachers), and 7 teachers out of the third group, changed their opinion and told that Anna's solution was acceptable. The motivations for their change were that the algebraic solution was too difficult (5 cases).

According to 11 teachers Anna's solution was still unacceptable (two of them made a comment of this kind: *"although it is not easy to find a mathematical solution, Anna's solution is unacceptable"*). These teachers (and 8 out of the others) answered the last question writing that *"I would show how to solve problems with algebra*"(9 out of 19); *"I would explain that examples have no value for proof"*(5 out of 19); *"I would propose easier problems, in order to show how to perform a proof with algebra, avoiding problems difficult like that"*(4 teachers). Lorena confirmed her position *("to help a student like Anna means to help her to make her text more general").*

Some comments about the example

The example can be considered under different points of view.

If we focus on the *student*, we can recognize that Anna was not aware that her text does not conform the requirements of the written product "proof" in today mathematics; indeed she writes down her proving process, she does not engage in writing a standard proof. We can be sure that she was not aware of the epistemological distinction between argumentation and proof (Duval, 1991). And surely she was not aware that it was not easy to reach the solution by alternative ways (i.e. by using algebraic formulas). But a professional mathematician might recognize in her text both a valid solution process, and a good style of informal communication of why 6 is the GCD of all the products of three consecutive integer numbers: four colleagues of mine, researchers in the field of Algebra, valued the solution and two of them were enthusiastic about the style of communication. One of them observed that the fact that divisibility by 2 and 3 implies divisibility by 6 should have been further elaborated *("but Anna is an VII-grade student; her performance is remarkable!").*

If we focus on teachers, we can recognize signs of the crucial importance for most of them of the use of algebraic language in proving (but note that proving arithmetic theorems by using the algebraic language is a marginal subject in the Italian high school curriculum). After having experienced their personal difficulty to reach a solution through an algebraic treatment of the problem, several teachers move to a better evaluation of Anna's solution only because an algebraic solution would have been too difficult for her. Their *belief* (Thompson, 1992) concerning the value of algebraic language as a universal tool for proving is probably reinforced by the fact that learning algebraic language is one of the crucial goals of high school mathematics teaching. As a consequence, most of them are unwilling to follow Anna's process in depth and recognize the general validity of some of her statements, as well as the efficient enchaining of them. Moreover, the fact that 'to teach proof' means (for most Italian high school teachers) to teach proofs to be understood and repeated by students could have contributed to make them unaware of the value of Anna's effort in *constructing a proof* for her conjecture.

If we focus on the aim of improving the teaching and learning of mathematics, we can consider two related issues: how to build a long term educational strategy to develop Anna's relevant resources in order to meet the requirements of mature mathematical performances in conjecturing and proving; and how to help those teachers to meet this kind of educational challenge (starting from a correct evaluation of Anna's resources revealed by her solution).

A PANORAMA OF EXISTING TOOLS AND PERSPECTIVES

Conjecturing and proving are in the core of mathematical activities. We can consider them under different perspectives, either borrowed from other disciplines or elaborated within the field of mathematics education. Each of these perspectives provides the researcher (and, in some cases, the teacher too) with some useful tools to analyse an example like that of Anna (in the previous Section the comments concerning Anna's performance have used more or less explicitly some of those tools). However, I think that no existing perspective can account for the complexity of the situation concerning Anna's performance in a coherent and comprehensive way. We must consider a plurality of roles (Anna, and her present and future teachers), the importance of the cultural side and its controversial epistemological aspects, the cognitive complexity of Anna's performance. Complex problem solving research (see Sternberg and Frensch,1991) and mathematical problem solving research (see Schoenfeld,1992) can provide us with useful tools to describe and value what Anna does and writes (in particular, her efficient top-down and bottom-up control mechanisms, her adaptive strategy based on the production and evaluation of appropriate examples, etc.). But the problem of the cultural value of her performance, and the relative value of her final product according to different epistemological positions and cultural constraints, remains out of the reach of problem solving research. The theoretical construct of community of practice (Lave and Wenger,1991) and its developments in mathematics education might account for the teachers' relatively homogeneous values and evaluations. Research on students' and teachers' beliefs could provide us with tools to better situate Anna's performance and the teachers' positions. But both perspectives cannot cover the epistemological and cognitive sides of Anna's performance.

Coming to specific theoretical constructs in Mathematics Education, the theoretical construct of "Theorem" as "statement, proof, and reference theory" (see Bartolini et al, 1997) could allow to analyse and evaluate Anna's product in a rather precise way, specially if we interpret 'Theory' in a broad sense (including inference rules and meta-knowledge). But the cognitive quality of the process (an aspect neglected by most teachers in their evaluation of Anna's performance) cannot be valued in that frame. Moreover, in that frame it is not easy to account for, and put into question, the teachers' criteria of evaluation of Anna's performance (so strictly related to their values and their practice of teaching proof). The frame proposed by Duval (1991) (concerning the cognitive distance between argumentation and mathematical proof) can be useful to estimate the distance between Anna's argumentative performance and the model of proof as formal derivation, and to discuss the opportunity that teachers or students, or both of them, become aware of that distance (for a critical position on it, see Boero, Douek, Ferrari, 2002). But Duval's frame cannot account neither for the quality of Anna's problem solving strategy, nor for teachers' evaluations. Harel's “proofs schemes” (Harel and Sowder, 1998) are useful to describe and value some aspects of Anna's performance; they could help to understand the origin of some high school students' “authoritarian schemes”. But they are not suitable to deal with the problem of the distance between Anna's performance and the usual requirements of a “proof” text; and cannot account for the teachers' beliefs.

We need a frame to deal in a comprehensive and coherent way with the epistemological issues inherent in the analysis of Anna's product and teachers' evaluations (the implicit epistemological assumptions of Anna, the explicit evaluations of the teachers), as well as with her high quality problem solving strategy and her communication performance related to her community of practice. Such a frame should take into account a long term educational perspective, where a text like that of Anna can be accepted and valued at her school level, while it needs to result in a standard "proof" text at further school levels.

HABERMAS' RATIONALITY

According to Habermas (1999), the adjective "rational" can be attributed to a person that performs an activity, when he/she is able not only to behave according to his/her intention in order to achieve the aim of the activity, but also to account for his/her choices according to validity criteria and communications constraints. As a consequence of this approach, rationality (of a person that performs an activity) consists of three interrelated components: *the epistemic rationality; the teleologic rationality; the communicative rationality.*

The *epistemic rationality* is related to the fact that we know something only when we know why the statements about it are true or false (otherwise our knowledge remains at an intuitive or implicit-pragmatic level). According to Habermas, it may be that an opinion is rational even if some statements about it are not true (or reveal to be not true afterwards). The crucial requirement is that the person has elaborated an evaluation of propositions as true, and is able to use them in a purposeful way and to account for their validity.

The *teleologic rationality* is related to the intentional character of the activity, and to the awareness in choosing suitable tools to perform the activity and orient it to the aim to be achieved.

The *communicative rationality* is related to communication practices in a community whose members can establish communication amongst them. Three elements constitute communicative rationality: the subject, the content of communication and the interlocutor. *Rational* means that the subject has the intention of reaching the interlocutor in order that he/she can share the content of communication, with an adequate and conscious choice of tools to make it possible.

We can interpret Habermas *rationality* not as a requirement of a cultural product (a text, a hypothesis, an argument) or a conceptual construction, but as a peculiarity of some human intentional problem solving behaviours related to socio-cultural context. From this point of view, the object of Habermas' elaboration is different from the object of other contemporary elaborations concerning "rationality" (cf. Lerouge's "rationality frame" for mathematical conceptual constructions: Lerouge, 2000).

Back to Anna (and her future teachers) in Habermas' perspective

According to what we know about her community of practice, Anna behaves in a rather rational way: she is able to justify quite well her steps of reasoning (*epistemic rationality*), she is able to develop a "rational" line of reasoning (in the sense of *teleologic rationality*) and account for it in an acceptable way for her teacher and her schoolmates, and even for some mathematicians (*communicative rationality*).

Anna should move towards an increasing and conscious knowledge of the tools that can be used to validate a mathematical statement, of different legitimate methods of proof (e.g. proof by induction) and of the cultural constraints of the products "statements of a theorem" and "mathematical proof". In particular, Anna should learn to distinguish the text that relates about her process from the text that presents her product according to the mathematicians' style of formal communication (but Thurston, 1994 observes that such a style is usually kept only in written communication in mathematical journals; the style of oral communication in the informal occasions is completely different, and very near to Anna's style, as observed also by two colleagues of mine!). Perhaps, at some stage of her preparation, after a consistent experience of processes of conjecturing and proving in different fields of mathematics, and of writing their products, it could be useful to introduce Anna to the model of proof as formal derivation (proofs developed through algebraic transformations might offer an occasion in that direction) (for a discussion, see Boero, Douek, Ferrari, 2002. See also Hanna, 1989).

In any case, we note that to value and enhance Anna's coherent effort to keep an intention till the (supposed) attainment of the goal, her accounting for her reasoning, her care for communicating in clear and understandable terms her intellectual activity, can be an important contribution to the intellectual maturation of Anna, far beyond its relevance in the perspective of her preparation in mathematics.

The first group of teachers (those who refused to evaluate Anna's performance according to an abstract "mathematical point of view"), or at least one part of them, might be sensitive to this perspective. Many of the other teachers conform their judgements to an absolute perspective of exterior quality of a mathematical performance of proof, and probably without a suitable in-depth intervention they would not be able to recognize and promote Anna's *rationality*. Even their behaviour in evaluating Anna’s performance seems rather far from requirements of *rationality*.

DISCUSSION

Habermas' theory of rationality seems to be an adequate, general frame to analyse a complex situation, like that of Anna's performance and its evaluation by high school teachers, in the perspective of the development of Anna's competencies in conjecturing and proving. However it would be good to further elaborate Habermas' "components" of a rational behaviour in the perspective of improving the students' approach to conjecturing and proving in school and better evaluate their performances and the teachers' choices. This could be made by using two concurrent strategies: to develop some points of Habermas' elaboration further (in the spirit of it), in order to get ad hoc precise theoretical tools; and to integrate other theoretical constructs, already developed in the field of mathematics education, in the frame of Habermas' elaboration (taking care of the local coherency of the integration).

One of the crucial points to be dealt with is represented by the nature of the awareness in the perspective of the *epistemic rationality*. It would be good to identify some crucial steps to be reached at different school levels, in order that students can gradually move towards the awareness of the fundamental cultural aspects of theorems' *statements* and *proofs* (for instance, the fact that in mathematical proof reference must be made to *valid* propositions within a *reference theory*).

The *teleologic rationality* should be further elaborated, keeping into account the specificity of conjecturing and proving processes. The central aspect remains the *intentionality* related to the aim to be attained. The fact of experiencing possible resources (for instance, different semiotic systems, or the different uses of examples in different fields of mathematics) and progressively becoming aware of them might become an object of specific activities in the classroom.

The *communicative rationality* should give an importance to different models of communication of conjectures and mathematical justification, according to different mathematical domains and levels of treatment. Different models could be presented to students and discussed (through elementary examples) by the teacher as *voices* to be *echoed* by students in the classroom (cf Garuti et al., 2002).

In teachers' preparation, Habermas' theory of rationality, conveniently adapted and integrated, could be useful as a tool to address teachers' attention towards some aspect of students' mathematical activities (like the *teleologic rationality* and the *communicative rationality*, usually neglected in the classroom) that are relevant not only for the improvement of their mathematical performances, but also for their intellectual growing up. It could also be useful to help them to distinguish between the fact that a student's mathematical performance may result in an unsatisfactory solution, and the quality of his/her performance in terms of *rationality*.

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EXTENDING STUDENTS' UNDERSTANDING OF DECIMAL NUMBERS VIA REALISTIC MATHEMATICAL MODELING AND PROBLEM POSING

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The quasi-experimental study presented in this report involves a teaching experiment based on a sequence of classroom activities aimed at enhancing the understanding of some aspects of the structure of decimal numbers in upper elementary school. Besides the use of suitable cultural artifacts and the application of a variety of complementary, integrated, and interactive instructional techniques, the teaching-learning environment designed and implemented in this study is characterized by an attempt to establish a new classroom culture. The focus is on fostering a mindful approach toward realistic mathematical modeling and a problem posing attitude.

There is considerable evidence from studies involving both school students and adults that the system of decimal numbers is neither simple to learn nor generally understood (e.g. Hiebert, 1985; Stacey & Steinle, 1998). In previous studies we have analyzed some difficulties regarding this topic, in particular the conceptual obstacles elementary and middle school students encounter in mastering the meaning of decimal numbers and in ordering sequences of decimal numbers. A central problem seems to be that few connections are made between the form students learn in the classroom and understandings they already have. Thus it is important that teachers recognize the numerical culture acquired outside the school in order to offer children the opportunity to develop new mathematical knowledge preserving the focus on meaning found in everyday situations (Bonotto, 2005).

The quasi-experimental study presented in this report involves a teaching experiment based on a sequence of classroom activities in upper elementary school aimed at enhancing the understanding of the structure of decimal numbers. This study is part of an ongoing research project aimed at showing how an extensive use of suitable cultural artifacts, with their incorporated mathematics, can play a fundamental role in bringing students' out-of-school reasoning experiences into play, by creating a new tension between in and out of school mathematics (Bonotto, 2005). In this study the artifacts used are some pricelists and menus of restaurants and of pizza shops. As in our other studies the classroom activities are also based on a variety of complementary and interactive teaching methods, and on the introduction of new socio-mathematical norms an attempt to create a substantially modified teaching/learning environment. This environment is focused on fostering a mindful approach toward realistic mathematical modeling, i.e. both real-world based and quantitatively constrained sense-making (Reusser & Stebler, 1997), and a problem posing attitude. To reach the learning goals it was decided to use also estimation processes in order to favour a connection between solutions and their reasonableness (Bonotto, 2005).

THEORETICAL AND EMPIRICAL BACKGROUND

We deem that an early introduction in schools of fundamental ideas about modelling is not only possible but indeed desirable even at the primary school level. Further we will argue for modelling as a means of recognizing the potential of mathematics as a critical tool to interpret and understand the reality, the communities children live, or society in general. Teaching students to interpret critically the reality they live in and to understand its codes and messages so as not to be excluded or misled, should be an important goal for compulsory education (Bonotto, 2006).

In this contribution the term mathematical modeling is not only used to refer to a process whereby a situation has to be problematized and understood, translated into mathematics, worked out mathematically, translated back into the original situation, evaluated and communicated. Besides this type of modeling, which requires that the student has already at his disposal at least some mathematical models and tools to mathematize, there is another kind of modeling, wherein model-eliciting activities are used as a vehicle for the development (rather than the application) of mathematical concepts. This second type of modeling is called ‘emergent modeling’ (Gravemeijer, 2004). Although it is very difficult, if not impossible, to make a sharp distinction between the two aspects of mathematical modeling, it is clear that they are associated with different phases in the teaching/learning process and with different kinds of instructional activities (Greer, Verschaffel & Mukhopadhyay, 2006).

To implement an early introduction in elementary schools of fundamental ideas about realistic mathematical modeling changes must be made. In particular the type of activity delegated to create interplay between mathematics classroom activities and everyday-life experience, substantially the word problems, must be replaced with more realistic and less stereotyped problem situations. These should be more closely related to children’s experiential world and meaningful. We deem that an extensive use of suitable cultural artifacts could be a useful instrument in creating a new link between school mathematics and everyday-life with its incorporated mathematics. The cultural artifacts we introduced into classroom activities (Bonotto 2003 and 2005) are concrete materials that are relevant to children because they are part of their real life experience. This enables children to keep their reasoning processes meaningful and to monitor their inferences. Finally I believe that certain cultural artifacts lend themselves naturally to favor problem posing activities.

Recently many mathematics educators realized that developing the ability to pose mathematics problems is at least as important, educationally, as developing the ability to solve them and have underlined the need to incorporate problem-posing activities into mathematics classrooms. Given the importance of problem-posing activities in school mathematics, some researchers started to investigate various aspects of problem-posing processes (Silver & Cai, 1996; English, 1998 and 2003).

Several studies have reported approaches to incorporate problem posing in instruction. Some studies provided evidence that problem posing has a positive influence on students' ability to solve word problems and provided a chance to gain insight into students' understanding of mathematical concepts and processes.

In this paper we consider mathematical problem posing as the process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems. It, therefore, becomes an opportunity for interpretation and critical analysis of reality since: i) they have to discern significant data from immaterial data; ii) they must discover the relations between the data; and iii) they must decide whether the information in their possession is sufficient to solve the problem. This process is similar to situations to be mathematized that students have encountered or will encounter outside school.

THE STUDY

Participants

The study was carried out in two fourth-grade classes (children 9-10 years of age) in Desenzano del Garda, a lakeside resort in the north of Italy, by the official logic-mathematics teacher, in the presence of a research-teacher. The first class consisted of 17 pupils, the second class of 19. As a control, two fourth-grade classes were chosen in the same town (the first class of 16 pupils, the second class of 18). All pupils, in both the experimental groups and control groups, had previously tackled the subject of decimals. Nevertheless, as the results of the pre-test show, their understanding of the subject was rather poor and superficial and needed further attention.

Materials

In order to choose an artifact which i) allowed learning trajectories on decimal numbers as part of the syllabus planned by the teacher, ii) really formed part of the children's experiential reality (in order to favor significant and motivating learning) the teaching experiment was preceded by a survey. It was found that among the objects examined the pricelists and menus of pizza shops, fast food and restaurants were part of the children's real life experience (most of the local population are involved in tourism and catering, and the children's parents were for the most part either owned or worked in restaurants, bars, ice-cream parlors or pizza shops).

Procedure

The teaching experiment took place from February to March 2005. It was subdivided into six sessions, each lasting two hours, at weekly intervals. The first session was devoted to the administration of the pre-test and the introduction of various kinds of artifacts (a menu from a typical pizza shop, an ice-cream parlor and a fast food restaurant). Sessions 2 to 6 concerned five experiences involving different opportunities offered by the artifacts. In the second session, the children read and interpreted the data and information in the various menus (products on offer, prices, ingredients, cover charges etc.). Working individually on one of these menus, the children compiled a hypothetical order, which they themselves chose according to their experience outside school. In so doing, they had to follow the structural features of a blank receipt (description of goods, quantity, cost etc.) provided by the teacher. Finally, the children had to calculate how much they would have to pay, by adding up the bill. In the third session, the children working in pairs had to: i) decide what to order (choosing from the data on the menu in question) and tell the other student their choice; ii) compile the meal order for two people; iii) make out the bill for the meal for two people; iv) calculate how much each person would spend (deciding whether or not to pay separately). In the fourth session, the children were asked: i) to read and interpret a meal order which had already been completed and make up a problem based on the data contained therein; ii) to write the bill for that order and try to calculate the total in their heads; iii) to calculate how much each person would have to pay. In the fifth session, the children were given a more complex menu (including, for example, various “supplements”). They were asked: i) to analyze the menu and to read and interpret all the data and information contained therein; ii) to choose what to order knowing that they had only 15 euros to spend; iii) to make a mental estimate of what they would have to pay and whether it would come within their budget; iv) to write out the bill in full to check their estimate.

Data

The research method was both qualitative and quantitative. The qualitative data consisted of students’ written work, fields notes of classroom observations and mini-interviews with students after the experiences. Quantitative date was collected by means of pre- and post-tests, administered to the two experimental classes as well as the other two control classes. Both the pre- and post-tests were organized in such a way as to evaluate students’ knowledge of “pure” decimal numbers in the first part and their knowledge of the Euro in the second part.

Research questions and hypotheses

The first general hypothesis was that the teaching experiment class fostered i) the understanding of decimal numbers, in particular of what Hiebert (1985) calls site 1 (“symbols and their referents”), ii) the understanding of some aspects of the additive and multiplicative structure of decimal numbers in a way that was meaningful and consistent with a disposition towards making sense of numbers; iii) the connection between results of operations with decimal numbers and their reasonableness through creating a link at site 3 of Hiebert (“solutions and their reasonableness in light of other knowledge”). Furthermore, we hypothesized that, contrary to the practice of word-problem solving documented in the literature, children in this teaching experiment would not ignore the relevant, plausible and familiar aspects of reality, nor would they exclude real-world knowledge from their observations and reasoning (hypothesis II). Finally children would also exhibit flexibility in their reasoning, by exploring different strategies, often sensitive to the context and quantities involved, in a way that was consistent with a sense-making disposition and closer to the procedures emerging from out-of-school mathematics practice (hypothesis III).

SOME RESULTS

Regarding the quantitative results, our first hypothesis would appear to be confirmed. In the case of the experimental group, the overall percentage of errors in the post-tests was less than that of the pre-tests, whereas in the case of the control group, it actually increased. More specifically, in all 10 exercises of the post-test, the percentage of errors diminished for the experimental group, whereas in 6 of the 10 exercises the percentage of errors increased for the control group [both in 3 exercises on "pure" decimal numbers and in 3 concerning the euro]. In particular, for one exercise involving the ordering of a sequence of decimal numbers, the percentage of errors dropped from 78% to 31% in the experimental group, whereas in the control group it rose from 78% to 84%.

The problem posing activity was a new and positive experience for the children; for the first time they were asked to "construct" their own problem and not just to solve one given to them. The children read the various menus and orders, and had no difficulty translating typical everyday data and information into problems suitable for mathematical treatment. They also showed remarkable originality due to their wealth of experience outside school. By presenting the students with activities which are meaningful because they involve the use of material familiar to them increased their motivation to learn even among the less able ones. A good example is the case of an immigrant child with learning difficulties related chiefly to linguistic problems. For her, as for many others, being confronted with a well-known everyday object with "few words and lots of numbers" acted as a stimulus. Indeed, it led her to say "It's easier than the problems in the book because we already know how things work at a restaurant!". This confirms what was stated in Bonotto (2005):

> Roughly speaking using a receipt, which is poor in words but rich in implicit meanings, overturns the usual buying and selling problem situation, which is often rich in words but poor in meaningful references.

The idea of "pretending to be at a restaurant" and "acting like grown-ups" ["grown-ups don't take a calculator or a pencil and paper with them to see if they can afford to order this or that …. They work it out in their heads" said one child] helped the children, including those with greater difficulties, to reason more freely and adopt calculation strategies they had never used before. Many examples of written works demonstrated that the children showed flexibility in their reasoning processes by exploring strategies often sensitive to the quantities involved, in a way that was closer to those arising from out-of-school practices (hypothesis III confirmed).

The classroom discussions at the end of the sessions were of particular interest. Here is an example: in the sixth session, the children were asked: i) to make out the bill for the order they had made; ii) to answer the questions: "How many people were eating together?; and "If they decide to pay separately, how much would each person pay?". Here are some excerpts from the class discussion which followed.

Paolo:	I think three people were eating because there are three things to eat and three things to drink. But there's a problem. I tried to do the division like this: 20.50:3=6.833333 [he shows the whole algorithm, editor's note] That still makes 3... So I thought that perhaps since 3 is a small number I could round it up and have everyone pay €6.83. [Like Paolo, all the other children reached the same solution]: The number is infinite but the money has to be counted down to the cents and so you have to round it up.
T.	So... we have seen that when you divide a decimal number, you sometimes get a recurring decimal number. Would anyone like to try and explain what this means?
Paolo.	Does it mean that the result is never-ending?
T.	OK. You're right... but there's something else... what about the part that never ends?
Chiara:	It's always the same.... 3 for example!
T.	What do you think? Where can we place this number in the line of numbers?
Marco:	I think that we can put 6, 83333333333 here, between the 6 and the 7 but nearer the seven.
T	OK Let's try and be a little more precise... Between the unit 6 and the unit 7... How about the decimal?
Marco:	It's 83 cents of a euro and so 8 tenths and 3 hundredths...
T	Very good... but we saw that it has infinite numbers ... so we have to try and continue... There's another 3 after that. Do you remember what that indicates?
Chiara:	It means a little bit of a thousand parts.
T	Good... it's called a thousandth. How did we draw a thousandth before? Do you remember?
Anna:	We divided the hundredth into 10 parts.
T	But we said that if we wanted to we could carry on... How can we do that? Do you remember?
Anna:	I think we can make a bigger drawing... a gigantic one... so we can divide the thousandths into 10 parts...
T	Very good... Do you realize we've now reached the fourth number after the decimal point... if we carry on we can divide again and again...
Anna:	If we had an enormous piece of paper, we could divide and divide to infinity!

…

T	What do you think we should do with these recurring numbers... in our case for example? [referring to the question "If each person pays separately, how much would each one pay?", editor's note].
Paolo.	I rounded it up and so everyone had to pay € 6.83...
Anna:	But if each of them pays € 6.83 it comes to € 20.49 and the bill is for € 20. 50... So I think one of them has to put in an extra cent.
Giulia:	I don't think that's right... they all have to put in the same amount of money and so they all put in € 6.84 and that makes € 20.52. There are two cents extra and in restaurants the change is usually left as a tip.
Matteo:	I think that's the best way too and if they don't want to leave the extra two cents as a tip they can put it away for the next time they eat together.

This discussion shows, on the one hand, the emergence of a learning process that Freudenthal (1991) defined as "anticipatory learning" (in this case regarding the concept of recurring decimal numbers), and on the other, how the children have by no means ignored the relevant and familiar aspects of reality. Nor have they excluded real-world knowledge from their observations and reasoning (hypothesis II confirmed). In this session and others, they showed a great deal of originality in the solutions they proposed. If this problem had been given in the traditional form using only words, the children would almost certainly have given the solution as € 6.833333333.... without any concern for the plausibility of the solution. What is more, by upsetting the rule, which says that every mathematical problem has only one solution, the children came closer to the more complex reality of everyday life where there are problems which have a number of different solutions.

CONCLUSION AND OPEN PROBLEMS

From the results it appears that the teaching experiment had a significant positive effect on achieving learning goals, in particular enhancing an understanding of the decimal numbers in a way that is meaningful and consistent with a sense-making disposition. This was not the case in the control group where an increase in errors was found in the post-test. In our view, the positive results can be attributed to a combination of closely linked factors: (a) the use of suitable cultural artifacts that allowed children effective control of inferences and results, and fostered a grasp of the connection between symbols and their referents; (b) the use of estimation processes that allowed a connection between solutions and their reasonableness; (c) the introduction of particular socio-mathematical norms ; (d) an adequate balance between problem posing and problem solving activities; (e) a systematic attention being paid to the nature of the problems and the classroom culture.

In future research, we will take a further look at the role of cultural artifacts not only as mathematizing tools that keep the focus on meaning found in everyday situations and as tools of mediation and integration between in and out-of-school knowledge, but also as possible interface tools between problem posing and problem solving activities.

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DIFFERENT MEDIA, DIFFERENT TYPES OF COLLECTIVE WORK IN ONLINE CONTINUING TEACHER EDUCATION: WOULD YOU PASS THE PEN, PLEASE?

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This paper presents some findings regarding the interaction between different computer interfaces and different types of collective work. We want to claim that design in online learning environments has a paramount role in the type of collaboration that happens among participants. In this paper, we report on data that illustrate how teachers can collaborate online in order to learn how to use geometry software in teaching activities. A virtual environment which allows that construction to be carried out collectively, even if the participants are not sharing a classroom, is the setting for the research presented in this paper.

INTRODUCTION

Authors such as Noss & Hoyles (1996) and Borba & Villarreal (2005) have, to different degrees, stressed how different media change mathematics. As different software designs become available, different mathematics can be developed. The classic examples are related to the teaching and learning of geometry (Laborde, 1998), functions and calculus (Confrey & Smith, 1994). Availability of dynamic geometry software can transform the type of tasks that one can develop in the classroom. For instance, problems can be presented to students in such a way that they try different constructions and see if there are invariants after dragging a figure. Coordination of multiple representations, as a way of knowing different aspects of functions and other calculus topics, were also transformed as different software were available. For instance, Confrey & Smith (1994) have argued in the past about the importance of the software design for learning. In the case they analysed, she showed how the design of Function Probe, which allows the graph of a function to be dragged, offered students new possibilities for coordinating representations. Being able to drag a graph and see the change in a given table of x-y values of the original function was a way of bringing transformation of functions to the forefront. The influence of humans in a piece of software made it possible for it to shape the way other humans would learn.

More recently, Borba (2005) extended these ideas with the conjecture that mathematics was also transformed as we change from a face-to-face context to online distance courses for teachers. For instance, there are online environments in which writing in a chat or in a forum is the only way for participants to communicate among themselves. Analysis was developed showing how writing shapes the mathematics discussed in this course offered to mathematics teachers. The analysis was carried out based on the idea that when technology changes, the possibilities of mathematics are also altered. This is the main idea behind the notion that a collective of humans-with-media (Borba & Villarreal, 2005) is the basic unit that constructs knowledge. Knowledge is always produced by humans, but also by different media such as orality, writing, or the new languages that emerge from computer technology.

In this paper, we will explore a different facet of this very idea that knowledge is produced not solely by humans, but by a collective of humans-with-media. We want to stress that the way teachers collaborate is shaped by the possibilities of different e-learning platforms. We will present an example from an online course, based on a platform that allowed for videoconference, among other features. In this case, it was not writing, but a combination of images, orality and the possibility of sharing a geometrical construction that shaped the mathematics produced. We claim, therefore, that different media shape different kinds of collaboration.

THEORETICAL AND METHODOLOGICAL FRAMEWORK

Although computer technology has been discussed intensively throughout PME conferences for at least fifteen years, online mathematics education has not been a theme of discussion. Although there are exceptions to this statement (e.g. Brown & Koc, 2003), this gap in the discussion regarding different aspects of online education should be filled, in particular since there is an increasing number of courses, and other practices, being offered to teachers as part of their continuing education.

Many models and different approaches have been offered in such courses. In one model of online course, content is uploaded onto the Internet, in a format that resembles a book, and little interaction is expected. Teachers who take courses like these are expected to download material and learn by themselves, and are then assessed through some kind of standard test. Many teachers can be enrolled in courses like these, thus generating profit for the organizers, and they dismiss the role of interaction in teachers' professional development. They are the prevalent type of courses offered in mathematics education, according to Engelbrecht & Harding (2005). At the other end of the spectrum are courses, which use the Internet as a means of generating quick feedback among participants. For instance, forum is used for asynchronous relations in which participants express their ideas and doubts and post solutions to problems. Synchronous relationships, that use chats, videoconference, and other tools, make it possible to share ideas in real time, even if people do not share the same space. Courses such as these are based on the idea that media should not be domesticated, which means that we, as mathematics educators, should always try to design curricula that fully explore the possibilities of new media or interfaces. In this case, we would explore the possibility of quick feedback and collaboration as a landmark that distinguishes online distance education from models developed in the past.

Courses like the ones we offer are based on psychological theories regarding the way computers reorganize thinking (Tikhomirov, 1981), as they play a qualitatively different role than language does in writing and orality. This "new language", based on a combination of images, other kinds of visual feedback, writing and sound, possibilities of sharing ideas in multimedia environments, becomes a co-actor in the production of knowledge, as discussed by Borba (2005) and Borba & Villarreal (2005). This is why exploring dynamic aspects of computer media and communication possibilities are paramount for courses based on the idea that we should not reproduce in a new media patterns that come from another.

Interaction and collaboration can also be defended drawing on the literature on teacher education, where authors such as Hargreaves (1994), Larraín & Hernández (2003) and Fiorentini (2004) claim that collaboration and sharing are powerful actions that generate new knowledge. If we bring these ideas to online courses, we have a strong argument for generating courses that emphasize interaction not only with the leader of the course, but among participants.

This is why our research group, GPIMEM, has been developing and researching models for courses that aim to create virtual communities in which participants can expose their doubts, feelings and insecurities about the use of technology in the face-to-face classroom, or about the use of other trends in mathematics education to which they are not accustomed. In a cyclic model, we researched different types of courses offered, chose one that emphasizes collaboration, and have been researching how different interfaces of platform for such courses shape the knowledge that is produced, or in other words, how different collectives of humans-with-media produce different kinds of mathematics and collaboration.

To approach this research question, we have been using qualitative procedures in which we analyze discourse and other data that is recorded in automatically by most platforms we have used. In this sense, the dialogues and the sequence of use of a given software (e.g. a dynamic geometry software) is easily retrieved. We can triangulate data as we can use recordings from the interactions we were all involved in, and at the same time, we can "interview" teachers through e-mail or other means to check conjectures we may have about a given aspect of our findings. In this way, we practice "member checking" (Lincoln & Guba, 1985), which means that we make sure that the participants agree with our interpretations, increasing the trustworthiness of the results.

THE CONTEXT

A course, entitled "Geometry with Geometricks", was developed in response to a demand from mathematics teachers from a network of schools sponsored by the Bradesco Foundation spread throughout all the Brazilian states. The teachers from their 40 schools, which include some in the Amazon rainforest, have access to different kinds of activities, such as courses that are administrated by a pedagogical center based in the greater São Paulo area. Following the improvement of Internet connections in Brazil, they realized that online courses were a good option, since sending teachers from different parts of Brazil (which is bigger in area than the continental U.S.A.) to a single location to take courses was neither cost nor pedagogically effective. The cost factor is related to the size of the country, and the pedagogical consideration is related to the fact that teachers would usually participate in the courses for a short period of time, with little or no chance of implementing new ideas while still taking the course.

The first type of online courses offered to teachers by the Foundation was based on a model involving little interaction between the leader of the course and participants, as criticized before in this paper. This is one reason we had to overcome some initial resistance when we began teaching the course, as the “didactical contract” that the teachers were accustomed to in online courses like this was a more passive one. Our model, based on online interaction, and applications in their face-to-face classes in middle and high school, and new discussion, gained respect gradually.

The pedagogical headquarters of this network of schools approached us, asking for a course about how to teach geometry using Geometricks, a dynamic geometry software originally published in Danish and translated into Portuguese. It has most of the basic commands of other software such as Cabri II and Geometer Sketchpad, and it was designed for plane geometry. As we know from extended research on the interaction of information technology and mathematics education, just having a piece of software available, and a well-equipped laboratory with 50 microcomputers, as is the case of these schools, is not enough for them to be used, even if the teachers are paid above average compared to their colleagues from other schools.

We designed a course using an exploratory problem solving approach (Schoenfeld, 2005) divided in four themes within geometry (basic activities, similarity, symmetry and analytic geometry). Problems usually have more than one way of being solved, and they could be incorporated at different grade levels of the curriculum according to the degree of requirements for a solution, and according to the preference of the teacher. Both intuitive and formal solutions were recognized as being important, and the articulation of trial-and-error and geometrical arguments was encouraged. We “met” online for two hours for eight Saturday mornings over a period of approximately three months. Besides this synchronous activity, there was a fair amount of e-mail exchanged during the week both for clarification regarding the problems proposed, technical issues regarding the software or pedagogical issues regarding the use of computer software in the classroom (e.g. should we introduce a concept in the regular classroom and then take the students to the laboratory, or the other way around?). Pedagogical themes were also discussed during the online meetings, in particular in one session in which, instead of the students working on problems during the week, they had to read a short book about the use of computers in mathematics education (Borba & Penteado, 2001). We encouraged teachers from one given school, or from different ones, to solve problems together in face-to-face or online fashion.

The headquarters of the Bradesco Foundation had already purchased an online platform that allowed participants to have access to chat, forum, and e-mail, videoconference and that allowed the download of activities, as well. In our course, participants could download problems, and they could also post their solution if they wanted to, or they could send it privately to one of the leaders of the course (the authors of this paper). The platform allowed the screen of any of the participants to be shared with everyone else. For instance, we could start showing a screen of Geometricks in our computer, and everyone else could see the dragging that we were performing on a given geometrical construction. A special feature, which was important for the purpose of this paper, allowed that we could "pass the pen" to another participant who could then add to what we had done on a Geometricks file. As will be described, there were times when which one teacher would have the pen, and another would be commenting or giving instructions on how to proceed with a given problem.

RESULTS

In this section, we will describe an episode in which problems were solved collaboratively. This example can illustrate how the convergence of different ideas generates the collective construction of knowledge about geometry (content knowledge) about use of a given geometry software in the classroom (pedagogical content knowledge), and about use of the geometry software itself (technological knowledge). This problem involved symmetry. A Geometricks file had already been given to them with the figure MNOPQ, presented below (figure 1). Teachers were asked to find the symmetric figure, in relation to axis "q". Teachers were reminded in the text that the symmetric figure had to remain symmetric even after being dragged.

As mentioned earlier, teachers were given each problem before the synchronous sessions and could interact with one of the teachers of the course by e-mail or other means. This allowed us to sometimes choose issues to start the debate and, at the same time, we could also limit undesired exposure of some errors. For this problem, the vast majority of solutions used a "count dot approach" in which they counted how

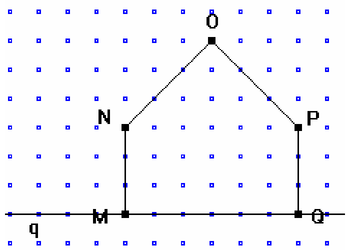

Fig. 1. Picture of the file given to them

many dots a given vertex was distant from "q". The result can be visualized in figure 2. We invited a volunteer to make a construction. We passed the pen to one participant who offered to do so, who in turn was helped by another who was acting like a sports narrator. After the construction was done, we asked questions such as: is MVWZQ symmetric to MNOPQ? The "quarrel" was quite intense, and the participants were divided. The issue about dragging emerged, and we came to the conclusion that if the dragging of a vertex is considered to be essential, that solution would not generate a symmetric figure (see figure 3). Different solutions were developed by different actors who took over the pen and/or whose voices emerged including the one presented in figure 4 – where circumferences are used playing the role of a compass in dynamic geometry – that would still be symmetric after dragging since, as we pull a point away from the axis, for instance, the symmetric point will do the same, since it is part of the circumference. This issue is relevant and for authors

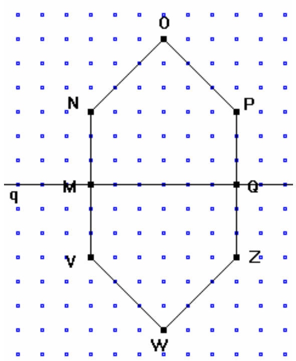

Fig. 2. The first solution of the teachers

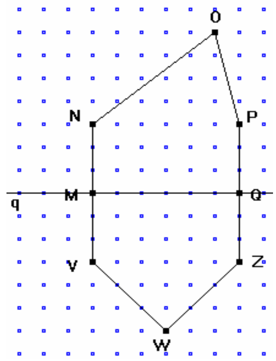

Fig. 3. Dragging and the first solution

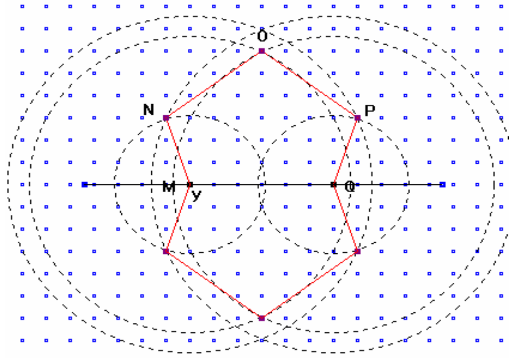

Fig. 4. The second solution that passes the test of dragging

such as Laborde (1998) and Olivero et al (1998), a straight quadrilateral with four equal sides and four angles is only considered a square in a dynamic geometry software if it passes the "test of dragging", which means, in this case, that it is still a square after different kinds of manipulation are performed. For these authors, if the resulting figure is no longer a square, we did not have a construction of a square but just a drawing. We agree with these authors that this is an important issue, even though in our course, we also emphasized the relevant role of "drawing solutions" depending on the grade level, the complexity of the problem, or as a path for more complex solutions. Other examples such as these were developed during this course. Teachers were overwhelmingly positive about the idea of collective problem solving during the synchronous videoconference sessions, especially when they compared the experience with ones in which just we, the leaders, would present our solution or present one of their solutions. Our own assessment, as leaders, was that this virtual collaboration created a bonding that was not experienced when we did not use this technical feature. The issue about drawing versus construction is not new in the literature; what we believe to be original in the episode reported is the fact that it came from an online course, and it was a result of collaboration among teachers.

DISCUSSION

Hargreaves (2001) proposes that teaching is a paradoxical profession. Everyone expects from teachers, even in Third World countries, efforts to help the construction of communities of learning in the new "knowledge society". Even though they were not educated when microcomputers were extensively available, they are required to teach using computers and software in the classroom. On the other hand, there is a discourse in governments and segments of society that claims we should have a "shrinking" of the state due to the needs of this knowledge society, and teachers become one of their first victims with funding cuts for sectors such as education. Teacher should enhance the knowledge society, even though they are some of their first victims. Whether they are aware of this dilemma or not, teachers feel the need, and there is institutional pressure, to constantly be working on their professional development. Integrating software into the face-to-face classroom is still a challenge as teachers enter a risk zone (Penteado, 2001) in which the students sometimes know better how to manipulate computers and often come up with original solutions to a given problem, or even create new problems which are not easy to handle. Soon the demand to incorporate the Internet in the classroom should rise as it seems like we are moving towards a "blended" approach, in which virtual and face-to-face interaction will happen "in" the classroom.

It makes sense, therefore, that teachers experience online courses, in particular if one considers that, in the case under scrutiny, there was also the economic factor. Reimann (2005), in his plenary at PME 29, emphasized the need to build artifacts on the Internet that could foster collaboration. We agree with him, and believe that we are helping to construct collaborative practices within online continuing education courses offered to mathematics teachers. This type of collaboration was possible due to a design in the platform that allowed us to do so. Of course, having the possibility is just one step, and a pedagogical approach that enhanced participation should always be pursued. But we want to emphasize that we should also be creating demands for technical developments in online platforms, and this is an area in which our research group, GPIMEM, has started working in the last two years.

Finally, we would like to say that, once design features are incorporated either into software, such as dragging, or into online platforms, like the "pass the pen" option, they become co-actors in the process of creating and recreating knowledge. Different media, different people, imply different perspectives in the knowledge constructed. That is why we believe that using the construct of humans-with-media is useful to analyze educational practices that use technology, since they foster the search for specific uses of an available interface. "Pass the pen" was a unique characteristic of this platform that made possible the co-construction of mathematical knowledge in an online course.

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REFORMULATING "MATHEMATICAL MODELLING" IN THE FRAMEWORK OF THE ANTHROPOLOGICAL THEORY OF DIDACTICS

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We will start by introducing some aspects of the theoretical framework used in the paper: the Anthropological Theory of Didactics (ATD). Then, we will focus on the research domain commonly known as "modelling and applications". We will briefly describe its evolution using the ATD as an analytical tool. Finally, we propose a reformulation of the modelling processes from the point of view of the ATD, which is useful to identify new educational phenomena and to propose and tackle new research problems.

INTRODUCING THE ANTHROPOLOGICAL THEORY OF DIDACTICS.

The works of Chevallard (1999), Chevallard, Bosch, Gascón (1997), Gascón (2003), Barbé, Bosch, Espinoza, Gascón (2005) and Bosch, Chevallard, Gascón (2006) show the way in which research in Didactics of Mathematics has evolved in recent years and how the Anthropological Theory of Didactics (from now on ATD) emerged, considering the incapacity of other theories to explain some aspects of educational phenomena. This new modelling also allows the emergence of new educational problems which could not be set out in other theoretical frameworks.

The mathematical activity: mathematical praxeologies.

One of the basic ATD axioms is that "toute activité humaine régulièrement accomplie peut être résume sous un modèle *unique*, qui résume ici le mot de *praxéologie"*. (Chevallard, 1999, p. 223). The concept of *praxeology* is the main tool proposed by the ATD to model the mathematical activity. Two levels can be distinguished:

- The level of *praxis* or "know how", which includes different kind of problems to be studied as well as the required techniques to solve them.
- The level of *logos* or "knowledge", of the "discourses" that describe, explain and justify the used techniques and even produce new techniques. This is called *technology*. The formal argument which justifies such *technology* is called *theory*.

Mathematics, as any other human activity, is something that is produced, taught, learned, practised and diffused in social institutions. It can be modelled in terms of praxeologies called *mathematical praxeologies* or *mathematical organizations* (from now on, MO). In order to have the most precise tools to analyze the institutional didactical processes, Chevallard (1999, p. 226) classifies mathematical praxeologies as *specific, local, regional* and *global*. The nature of a MO depends on the institution where it is considered:

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 209-216. Prague: PME.

- A *specific* MO is generated by a unique type of task and is characterized by a unique technique to deal with the task.
- A *local* MO is generated by the integration and articulation of several specific praxeologies. A local praxeology is characterized by a technology that justifies, explains and articulates the diverse techniques of each specific praxeology.
- A *regional* MO is obtained by coordinating, integrating and articulating several local MOs in a common mathematical theory.
- A *global* MO emerges when adding several regional MOs to the integration of different mathematical theories.

In a simplified way, we can say that what is learnt and taught in an educational institution are mathematical praxeologies. Praxeologies are rarely individual: they are shared by groups of human beings organised in institutions. Cognition is thus institutionally conceived.

The process of study: didactic praxeologies.

Mathematical praxeologies do not emerge suddenly. They do not have a definite form. On the contrary, they are the result of a complex and ongoing activity, where some invariable relationships which can be modelled exist. There appear two indivisible aspects of the mathematical activity:

- The process of mathematical construction; the *process of study*.
- The result of this construction; the *mathematical praxeology*.

Chevallard (1999, p. 237) places this *process of study* in a specific space characterised by six didactic stages or *moments*[1]: (1) *the moment of the first encounter* with a specific type of tasks , (2) *the moment of the exploration* of the type of tasks, (3) *the moment of the construction of the technological-theoretical environment*, (4) *the moment of working* on the technique (which provokes the evolution of the existing techniques and the creation of new ones), (5) *the moment of institutionalization* and (6) *the moment of evaluation* of the praxeology constructed.

Once again, this process of study, as every human activity, can be modelled in terms of praxeologies, which are now called *didactical praxeologies* (Chevallard, 1999, p. 244). As every *praxeology*, didactical praxeologies include a set of problematic educational tasks, educational techniques (to tackle these tasks) and educational technologies and theories (to describe and explain these techniques).

A new conception of didactic of mathematics appears, in which *didactic* is *identified* with everything which can be related to *study* and *helping to study*:

> "Didactic of mathematics is the science of study and helping to study mathematics. Its aim is to describe and characterize the study processes (or didactic processes) in order to

[1] The idea of *didactic moment* is defined not in a chronological or linear sense, but in the sense of different dimensions of the mathematical activity.

provide explanations and solid responses to the difficulties which people (students, teachers, parents, professionals, etc.) face when they are studying or helping others to study mathematics" (Chevallard, Bosch y Gascón, 1997, p. 60).

The levels of determination

The ATD postulates that it is not possible to separate the mathematical knowledge from its process of construction in a specific institution. Thus, Chevallard (2001) proposes a hierarchy of co-determination levels between mathematical organizations *living* (or able to live) in an institution and the possible ways of constructing those MOs in this institution, i.e., the didactical praxeologies. This hierarchy is as follows:

Society → School → Pedagogy → Discipline → Domain → Sector → Theme → Subject

Thus, the structure of the MOs on each level of the hierarchy conditions the possible ways of organizing its study. Reciprocally, the nature and the function of the *didactical tools* existing in each level determine, to a large extent, the type of MOs that could be reconstructed in this educational institution.

Every question Q that generates a didactic process in an educational institution, is embedded in a theme, belonging to a sector, included in a domain of a discipline. If the discipline is *mathematics*, we will refer to these levels as *mathematical levels*. In contraposition, the levels beyond the "discipline" are called *pedagogical levels*.

It is important to notice that if a question Q is studied in an educational institution, it is because this hierarchy of levels has been constructed. Otherwise, the study of this question Q would be almost impossible.

For example, in García (2005) we showed that in the Spanish Secondary Education there exist two different hierarchies related to the study of the proportional relation between magnitudes. The first one places the proportional relation in the sector of "Proportionality" and in the domain of "Numbers and measures". This implies that proportionality is conceived as a static relation modelled in terms of *rations and proportions*. The second hierarchy places the proportional relation in the sector of "Characterizing relations between magnitudes", which is a part of the domain called "Functions and its graphical representation". This implies that proportionality is conceived as a dynamic relation modelled in terms of *lineal functions*. The existence of the two hierarchies gives raise to the reconstruction of two different praxeologies in the present Secondary Education, studied at different moments of the school year and deficiently articulated.

Traditionally, the work of the teacher has been limited to the "Theme → Subject" levels. The higher levels have been determined by the official curriculum and the educational authorities. This phenomenon, identified by Chevallard (2001) as the "*phenomenon of the teacher's confinment*", has made the mathematical themes and questions studied at school lose the reasons that motivated their presence in the educational system. It seems that these themes and questions have always existed and that they have been always the same. In other words, Chevallard (2001) says that they are *dead questions*, because the institution ignores where they come from and where they lead us to. It is the *monumentalization of the mathematical organizations*: the students are invited to *visit* them.

Research in mathematic education also seems to be restricted, in many cases, to the "thematic" level, centred in studying the "appropriate" way of introducing some mathematical content in a specific educational institution, without no deeper reflection upon the way the mathematical knowledge is structured and without taking into account the conditions and restrictions imposed by the different co-determination levels beyond the mathematical level.

RESEARCH ON MATHEMATICAL MODELLING

Researchers in Mathematics Education have a growing interest, since the mid eighties, in the role that modelling processes can play in the teaching and learning of mathematics at all levels of the educational system. For a long time, "modelling" has been restricted to the application of a mathematical knowledge, already constructed, to a specific "real" situation. Nowadays, this use of the term "modelling" persists.

However, "modelling" is considered in a richer and more fertile way by the research in Mathematics Education. It forms a large research domain that it is constantly growing. Perhaps one of the first uses of modelling in education research was done through the design of "manipulative materials" and "visual models". Nowadays, it is common to hear about the necessity of linking the mathematical contents to certain aspects of real life and about the necessity of developing the "modelling competence" as a basic mathematical competence in students.

To explore the evolution of what we may call the "problem of modelling", we propose to start from two problematic questions, initially separated. On the one hand, the *idea* of the processes of mathematical modelling as a "powerful didactical tool", that is, *how could modelling processes improve the teaching and learning of mathematical contents*?

On the other hand, the necessity of an explicit teaching of modelling processes as another mathematical content, linked to the students' specific formative needs (for instance, students of biology, chemistry or engineering).

If we try to reconstruct a hypothetical evolution of the domain of research commonly known as "modelling and applications" (not necessarily historical), we can consider that a first step would be the search of *appropriate* "real systems and models" to introduce the desired mathematical contents. From the beginning, the notion of "modelling" was taken from the "pure mathematics", and summarized in the well known "modelling cycle" (see, for instance, Blum & Niss, 1991). Nowadays, this kind of research continues and produces, by reflection and experimentation, good examples of modelling and mathematical applications ready to be performed in the classroom.

However, the evolution of the research in "modelling and applications" led to a growing interest in the modelling processes. As Blum & Niss (1991) established, "there is no automatic transfer from a solid knowledge of mathematical theory to the ability to solve non-routine mathematical problems". They consider that "problem solving" and "modelling" have to be explicitly taught.

That is the point of departure of a new domain of research centred in studying how students can learn to solve "applied mathematical problems". Nowadays, that research domain is large and heterogeneous, including a lot of theoretical frameworks.

Not trying to be exhaustive but trying to organize the existing literature, we can consider that there exist two main trends (not separated):

- On the one hand, and linked with the distinction made before, a trend that considers "modelling" as a *means* (or a *tool*) to teach and learn mathematics. Thus, "modelling" is conceived as a *didactic technique* justified by some *didactic theory*.
- On the other hand, a trend that considers "modelling" as a teaching goal, linked to the *general mathematical skills* or *competences* that every student should develop.

Although there exist a great variety of papers dealing with those topics, it is possible to organize them using the co-determination levels explained before.

In that way, a great part of the existing research places "modelling" on a *thematic level*. According to that, we can synthesize their research problem as follows: *how does a person act when s/he is trying to solve a "real problem" that implies the necessity to model a system?* The formulation of the research problem on the *thematic level* leads to the creation of isolated "real problems" designed to introduce the desired mathematical contents. The isolated nature of that type of problems often makes them become anecdotic: the student ignores where they come from and where they lead her/him to.

On the other hand, the research problem is established in the *discipline level* when the main aim can be resumed in the following terms: *How can a person achieve the modelling skill or competence?* (linked to the *problem solving* general competence).

Between both, there exist other theoretical frameworks that try to place the "modelling problem" in a *sector level* (trying to construct local MOs). For instance, the research domain known as "Realistic Mathematics Education", that considers mathematics as a human activity (see Freudenthal, 1973), tries to build *didactic trajectories* that aim to construct mathematical knowledge starting from "real situations"[2] through the process of horizontal/vertical mathematization (from a "model of" to a "model for").

It is important to notice that, in most of the educational research carried out in "modelling and applications", the description of the modelling processes, using the "modelling cycle", remains almost the same. At the most, there is a questioning of the cognitive processes activated in each step of the modelling cycle or in the

[2] "Real" not in the sense of "real life", but real for students.

transition between steps. This has led to an enriched version of the modelling cycle (for instance, the one proposed in Blum, 2005). In other words, the notion of "modelling" is not problematic in research in mathematics education (as it is not problematic, for instance, in biology or engineering). The built "patterns" of the *modelling processes* are very close to those suggested by mathematics itself. They are seldom modified or extended from the considered experimental facts.

Although it is undeniable that great progress has been made, it is true that the results obtained are far from the desired ones. As Blum (2002) says:

> "While applications and modelling also play a more important role in most countries' *classrooms* than in the past, there still exists a substantial gap between the ideals of educational debate and innovative curricula, on the one hand, and everyday teaching practice on the other hand." (p. 150)

We consider that it is necessary to continue working on the study of the modelling processes and on their relevance to the teaching and learning of mathematics. One way of doing that, which is not developed enough in the existing research, would be to establish a general epistemological framework of mathematics, as the one proposed by the ATD, and try to reformulate the modelling processes within this general theoretical framework.

REFORMULATING MATHEMATICAL MODELLING IN THE ATD.

The research paradigm known as "didactic of mathematics" or "epistemological approach of didactic", originated in Guy Brousseau 's first works of the 70s, places the question of the epistemological model of mathematics in the core of educational research. The main hypothesis can be resumed as follows: each *didactic phenomenon* has an essential mathematical component and, reciprocally, each *mathematical phenomenon* has an essential didactic component (Chevallard, Bosch & Gascón, 1997). From this new point of view, the "didactic facts" and the "mathematical facts" are inseparable, and inaugurate a new way to tackle the research in didactic of mathematics: the question of the epistemological model of mathematical knowledge.

Centred in the ATD, one of its main axioms is that "most of the mathematical activity can be identified (…) with *a mathematical modelling activity*" (Chevallard, Bosch and Gascón, 1997, p. 51). This does not mean that *modelling* is just one more aspect of mathematics, but mathematical activity is a *modelling activity* in itself.

First, this statement is meaningful if the idea of *modelling* is not limited only to "mathematization" of non-mathematical issues, that is, when the intra-mathematical modelling is considered as an essential and inseparable aspect of mathematics. Second, this axiom will only be meaningful if a precise meaning is given to the *modelling activity* from the own epistemological model proposed by the ATD.

As Bosch and Gascón (2005) establish, in the framework of the ATD, what is relevant is not the specific problem proposed to be solved (unless in *life or death situations*), but what can be done with the solution obtained. The only interesting problems are those that can reproduce and develop in wider and more complex types of problems. The study of those *fertile problems* provokes the necessity of building new techniques and new technologies to explain these techniques. In other words, the research should focus on those *crucial questions* that can give rise to a rich and wide set of mathematical organizations. Sometimes, those *crucial questions* have an extra-mathematical origin, sometimes not.

The important starting point to design a *process of study* should not be the realness of the situations, but the possibility of creating a *chain* of well articulated and integrated praxeologies that would allow the development of a wide mathematical activity in any educational institution, taking into account the restrictions and conditions imposed by that institution. So, we propose to reformulate the *modelling processes* in the framework of the ATD as processes of reconstruction and integration of praxeologies of increasing complexity (*specific* → *local* → *regional*), that may start from *crucial questions* for the individual of the institutions in which the study process will take place. Thus, the modelling processes are not described only in terms of system-model relations, but in terms of praxeologies and relations between praxeologies.

CONCLUSION

The reformulation of the modelling processes in the previous paragraph constitutes a new way to identify and propose new educational problems and to tackle them. “Modelling” is not a particular field of research in mathematics education but a central topic in the identification of educational phenomena and problems and in their possible solutions.

For instance, in García (2005) we worked on the *problem of the disarticulation of Mathematics in Secondary School*. Focusing on the problem of the disarticulation in the study of the functional relations between magnitudes, we propose a *modelling process*, designed as a set of praxeologies of increasing complexity, which tries to develop a wide and complete process of study starting from a *crucial question* for the individual of the Compulsory Secondary Education institution: the necessity of planning how to save money.

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STUDENTS' IMPRESSIONS OF THE VALUE OF GAMES FOR THE LEARNING OF MATHEMATICS

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The use of mathematical games in primary classrooms is commonplace in Australia. This paper reports on key findings from a larger investigation exploring the impact of games on mathematical learning, student attitudes, and behaviours. 222 Grade 5 and 6 children were taught multiplication and division of decimal numbers using calculator games. This paper raises questions about the students' attitudes towards games as a vehicle for learning mathematics. One aspect reported in this paper is an apparent difference between students' attitudes to games usage when data were collected quantitatively compared with qualitatively.

INTRODUCTION

Games can ignite the interest of children through anticipation of competition, challenge, and fun (Owens, 2005). An assumption underlying this doctoral research was that a novel pedagogical approach may have a positive affect on students' attitudes towards mathematics and classroom engagement. The research questions were (a) "Does the use of games contribute to mathematical learning?" and (b) "What are the relationships between games, learning, and student responses?" This paper focuses on the latter question. It is an important question because understanding varied pedagogical approaches that assist mathematics education and improve students' attitudes and behaviours is vital. The motivation of students is a general concern because of society's acceptance of poor attitudes to the potential for success in mathematics (Kloosterman & Gorman, 1990).

A great deal of research and interest has surrounded the area of attitudes and mathematics (e.g., Leder, 1987; McLeod, 1992; Reynolds & Walberg, 1992). For the purpose of this paper McLeod's (1992) definition of attitudes is adopted: "affective responses that involve positive or negative feelings of moderate intensity and reasonable stability" (p. 581). McLeod attributes time elapsed to the development of an attitude, and notes that changes in students' attitudes may have a long lasting effect.

Children enter school with an attitude towards learning that is derived from their home environment but once they start school, attitudes become performance related as success or failure impact approaches to subsequent situations (Lumsden, 1994; Reynolds & Walberg, 1992). Other factors include motivation, the quality of instruction, time-on-task, and classroom conversations (Reynolds & Walberg, 1992, Hammond & Vincent, 1998). Students' also develop responses through interaction

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with peers (Fishbein & Ajzen, 1975; Reynolds & Walberg, 1992; Taylor, 1992). Learned responses to school activities form from this interaction and understanding. This may then impact on students' attitudes as they get older, when positive attitudes towards mathematics appear to decline (Dossey, Mullis, Lindquist, & Chambers, 1988). These factors informed the research when considering the second research question: What are the relationships between games, learning, and student responses?

METHODOLOGY

The research was conducted with 222 Grade 5 and 6 students from eight classes in three Melbourne schools. The children undertook two experimental teaching programme sessions per week for four weeks, a total of eight sessions.

Two forms if data collection were used: 5-point attitude scales (administered pre-, post- and delayed post-intervention) resulted in qualitative data, and semi-structured student interviews (conducted post-intervention) with 18 randomly-selected children resulted in quantitative data that gave greater insights into the impact of game-playing on both concept development and attitudes. 121 students successfully completed all three attitude scales. Therefore, only the data from these students were considered. Other data were collected via achievement tests, and researcher observations, however, this paper focuses on the attitudes scales and interviews, and mainly on the statement "Maths games help me to learn maths". Themes were identified from the interview transcripts on the basis of the comments being either representative of the interviewed group or their giving a differing perspective.

RESULTS AND DISCUSSION

Table 1 presents the frequency of responses to the rating scale for each question on the pre-intervention scale. Overall, at this stage the students had a positive attitude to mathematics games as a vehicle for learning mathematics, suggesting positive prior experiences with mathematical games.

Table 1

Pre-intervention Attitude Scale Results for Game-Playing Students (n = 121)

Attitude Statements	Strongly agree	Agree	Neutral	Disagree	Strongly disagree
Maths games help me to learn maths	46	45	19	9	2

Table 2 illustrates the shifts in the children's attitude immediately after the teaching programme period. The change column represents the difference on the Likert scales between the pre-intervention and the post-intervention. The students' pre-intervention results were subtracted from their post-intervention results to gain an indication of any shifts in attitudes.

Table 2

Interval Changes (Pre-intervention and Post-intervention) in Responses to: Maths Games Help Me to Learn Maths (n = 121)

Change	Frequency	Percent %
-4	5	4.1
-3	6	5.0
-2	17	14.0
-1	24	19.8
0	48	39.7
1	15	12.4
2	6	5.0

Nearly 40% of the students in the game-playing groups did not exhibit a shift in attitude between the pre-instructional period and immediately after it. Seventeen percent of students felt that games helped them learn maths, whilst 43% more felt that games were not helpful. This latter response was not anticipated.

To provide an alternative perspective on the shifts in attitude, longer-term changes were examined through the delayed-instructional scale. Table 3 illustrates the shifts in the children's attitude 10 weeks after the teaching programme period.

Table 3

Interval Changes (Pre-intervention and Delayed Post-intervention) in Responses to: Maths Games Help Me to Learn Maths (n = 121)

Change	Frequency	Percent %
-4	8	6.6
-3	8	6.6
-2	15	12.4
-1	28	23.2
0	45	37.2
1	13	10.7
2	3	2.5
3	1	.8

Over 37% of the students in the game-playing groups did not indicate a shift in attitude over the whole 14-week period, and 14% responded with a more positive attitude, whilst 49% indicated that the mathematics games did not help them to learn. It appears that the games employed seemed to have resulted in less positive attitudes towards learning mathematics through the use of games - a surprising result as the students had seemed to enjoy the games and demonstrated a developing understanding of key concepts.

Some possible explanations for the negative trend in the attitude scales are:

- The games were addressing both content and process that were quite advanced for these students.
- The effect of these games was disequilibrating, in that it was creating cognitive conflict with previous issues of multiplication; e.g., multiplying always results in bigger number, and so created a desirable but disconcerting uncertainty.
- It was observed that some children became bored with playing the same types of games twice a week over the four-week period.
- The students may have become fatigued with answering the same questions on the Likert scale and in turn answered more negatively.
- Likert scales may not provide an accurate measure of the students' attitudes and need to be read in conjunction with other data, such as interviews.

In order to gain insights into the students' perceptions of their learning during the game-playing sessions, students were asked the following question, "Has there been a time during playing the games that you thought, 'Hey, I am learning this?' If so, tell me about it." The examples of learning during game-playing presented below included: recounting the effect of key mathematical concepts addressed in the games; using problem-solving strategies; and the use of tools to assist learning.

Several children referred to learning about the effect of multiplication and division of decimals, and commented on related strategies they had used; e.g., "Yeah, a couple of times, when I thought, 'Oh, this number gets it down to this and then that number will get it up higher'" (Andrea).

A number of children offered more detailed responses to the question; illustrating the problem-solving strategies that they used to assist their learning. For example:

Frazer: When I played the games I didn't know what it would do if I times it by a point, like just a point and a tenth, or just a whole number and a whole number and a tenth or hundreds or a thousandth. That's what I didn't know. When I experimented with it then I found that point 1 is 10 to the 100 or something. It goes lower than with a whole number and a whole number and a tenth or a hundredth … and with just a whole number would go higher anyway. And, so that's how I learned how to times and what would happen. That's how I knew, that's how I got it first shot, because I learnt how to do it when I was experimenting.

It seems that Frazer used trial and error to develop an understanding of the effect of multiplying fractions. Frazer's teacher considered him to be a highly motivated and proficient mathematics student, but similar understandings were developed by those who lack confidence and view themselves as poor mathematicians, as in the case of Katie.

Katie had indicated many times throughout the study that she was "hopeless at maths". However, it appears that the use of calculators in the games helped free Katie to experiment, test theories, and build the key mathematical concepts. For example:

> Katie: Oh when I figure out that I can actually bring it up and down without going, "Oh, how do I do that?", and sort of wondering without asking somebody. Because you can try anything with a calculator. Which is good because you can always put it back, it's not like writing it. And you can sort of like trial, I can just keep trying, it just makes it easier without anybody watching you. Yeah, without everyone going "That's wrong", got to rub it out now or keep it there because you know like every time in class you know I rub out the right answer sometimes. … if you hear anyone coming up saying you can't put it like that you can just clear it and put it back in again and work it out more without writing it down. You're using your head.

Although Katie was using a computational tool in the form of a calculator, she also used mental computation to assist in playing the game, rather than simply trial and error on all occasions. The game atmosphere appears to have been a less threatening setting than the typical mathematics classroom environment for Katie, enabling her to engage openly in developing an understanding of mathematical content.

In summary, it appeared from the interview data that the students felt comfortable with game-playing and were interested in developing strategies for winning, and that the desire to win encouraged them to grapple with mathematical concepts that were beyond the scope of the prescribed curriculum for their level. In fact, the students interviewed were able to share strategies they had developed that highlighted their emerging understanding of the relevant mathematical concepts. Further, it was clear from some of the students' responses that many were still attempting to create meaning from the concepts addressed in the games. From a constructivist perspective, disequilibrium seemed to prove to be an important step towards engaging students in mathematical learning. Some of the positive reasons given for liking the game-playing were a shift from traditional style teaching, a challenging and enjoyable activity, and potential for social interaction with peers. Some children reported that the games supported learning through the conceptual feedback inherent in each turn, and from engaging in teacher or peer dialogue about the mathematics involved in the activity as the games progressed. The games also appear to have assisted in developing an awareness of other skills such as problem-solving.

Thus the less than positive response to the games displayed in the attitudinal scales was in contrast to the students' interview data. (It also contrasted with the informal observations of the researcher and class teachers.) This was puzzling but also very interesting in terms of the methodological choices that researchers make.

It is possible that the use of games previously as a warm up activity or as a reward may have undermined the significance of games as a teaching tool. Baker et al.'s (1981) study found a negative relationship between reward games and students' completion of class work. Students presented poor quality work in a hurried effort to be rewarded with game-playing. It is possible that the students in this current study had had a similar experience of games being used as a reward or viewed games as a warm up activity before the "real" learning took place. It is also possible that the positioning of the attitude scales on the front of the achievement test may have produced a negative attitude towards the games. Many students have performance anxiety towards the completion of the achievement tests (McDonald, 2001). Therefore, completing the attitude scale, whilst feeling anxious about the impending test, may have caused the students to respond negatively on the scales. The problematic nature of the research methodology is a consideration for future research in this area.

CONCLUSION

One of the aims of this study was to explore students' attitudes towards games as a vehicle for learning. One barrier for the employment of games as a pedagogical tool may be possible negative attitudes held by students towards the likelihood of the games' effectiveness in assisting mathematical learning. As the students appeared to appreciate and enjoy games that provided them with a positive learning experience, perhaps the usefulness of games as a tool for learning needs to be made more explicit to the children. Allowing the students an opportunity to communicate the benefits of the game beyond the key mathematical concepts may also draw their attention to the potential of games to provide a positive learning experience.

The games in this current study were viewed as too difficult for some students and lacked challenge for others. Employing activities that cater for all may be problematic for many teachers. In recent years, there has been a movement towards providing open-ended tasks that cater for the differing capabilities of students (Sullivan, Mousley, & Zevenbergen, 2005). A way for teachers to adopt an open-ended approach to games is to consider using games that have different levels built in and challenges that the students can review and adapt while playing. Adapting games to make them less or more challenging can be a whole-class activity. The teacher may ask the students to brainstorm for ways to alter the objectives or mechanics of the game to vary the game's complexity. This approach to developing the games further provides the students with a sense of ownership of the games and potentially acts as a vehicle for promoting student involvement in the game-playing experience.

In summary, it appeared that the game-playing negatively affected attitudes based on the results of the attitude scale. However, this was not necessarily the case in the interviews and other communications. One of the implications from this is that it is very difficult to get clear indications of people's attitudes from only one type of data source. It is recommended when seeking data about attitudes that a variety of data collection methods are employed.

On balance it appears that assumptions that students will see the usefulness of mathematics games in classrooms are problematic. Specific links should be made to connect the content of the game, and the concepts to be learnt, to the curriculum and to other aspects of mathematics such as problem-solving. Teachers can be encouraged to continue to use games, but should be aware that they need to take specific actions to make sure that they maximise the opportunities of games in the classroom for supporting students' own knowledge of their mathematical learning.

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THE TRANSITION FROM ARITHMETIC TO ALGEBRA: TO REASON, EXPLAIN, ARGUE, GENERALIZE AND JUSTIFY

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This paper considers year eleven students' solutions of a specific task, which includes both a numerical part and a generalization of the task context that needs a transition from arithmetic to algebra. We investigate if students can solve the given problem, explain a solution and justify why the generalized problem always can be solved. Students prefer to explain in rhetoric rather than in symbolic algebra. Seven different ways of explaining are traced among the written answers. Few students are able to give an acceptable justification for the solution of the generalized problem, even after a course including work with this kind of problem. There are signs of students being unfamiliar with meta-cognitive activity.

INTRODUCTION: ON THE ROLE OF ALGEBRA IN EDUCATION

The place and role of algebra in school mathematics today is set in focus from many perspectives, and traditional algebra seems to have a need for a fundamentally new way of thinking (see for example Stacey, Chick & Kendal, 2004). Students have problems seeing connections to other areas inside or outside of mathematics, and they often see it as a formal, isolated system where manipulation of symbols and rules are dominating (Kieran, 1991; Brekke, Grönmo & Rosén, 2000; Wagner & Kieran, 1989). Upper secondary teachers find that students' knowledge from lower secondary school is weak. Algebra has been a route to higher mathematics, but at the same time a barrier for many students forcing them to take another educational direction (Kaput, 1995). Also students struggling against the barrier may perceive little meaning in algebra.

Three categories of skills or competencies will be activated by students when moving from competence in arithmetic, number handling, and calculation to competence in algebra (Kaput, 1995):

* To be able to talk about numbers, and magnitudes without calculating anything.
* To be able to describe the calculations one wants to do or get others to do. The description gives a recipe that we may say in words, by abbreviations, or by formula.
* To be able to change the way in which numbers or magnitudes are calculated.

The purpose of the current study is to examine how students handle one specific task, intended to reveal whether students can take the step from arithmetic to algebra. We will present the research questions after we have formulated the task below.

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THEORETICAL FRAMEWORK AND RELATED LITERATURE

As part of research in mathematical education, the teaching and learning of algebra has received a great deal of interest the last decades. As early as 1922 Thorndike published *The Psychology of Arithmetic* where the ”bond-theory” is used, which has been influential within traditional teaching: Algebra means for many students to establish bonds or links of the type $a \cdot ab = a^2b$ or $a(a+b) = a^2 + ab$, $a = 1a$, or links between a given quadratic equation and the formula for its solution. In the last decades, the focus has moved from behaviouristic perspectives to a deep analysis of cognitive competencies involved in learning algebra, This research has revealed problems students have, and why they are enduring and fundamental, and how they may relate to the transition from arithmetic to algebra (Booth, 1984; Sfard & Linchevsky, 1994; Linchevski & Herscovics, 1996; Kieran, 1991, 1992). The research has been related to an analysis of the field of algebra in a didactical context (Sfard, 1995).

The abstractness of algebra is one reason for students' problems. The generality of the algebraic ideas makes it semantically weak (Lee & Wheeler, 1987). Since ideas from algebra connect to all areas of mathematics, and to many contexts outside mathematics, it does not *belong* to any. Thus, a deep engagement by students in many examples seems to be necessary to develop their algebraic reasoning skills. Blanton and Kaput (2005) found different categories of algebraic reasoning, spontaneously or planned in a classroom as early as grade 3: Generalized arithmetic, functional relationship, properties of numbers and of operations, and algebraic treatment of number.

Research reports that secondary students may work with variables without a deep understanding of the power and flexibility of the algebraic symbol system. They may develop skills in routine manipulations, based on the knowledge, that variables follow the same basic structures as do number calculations. What is special for algebra, and different from arithmetic, however, is so fundamental that it may represent a leap in the education (Kieran, 1991, 1992; English & Halford, 1995; Wagner & Kieran, 1989; Bell et al, 1988; Bell, 1995; Wagner & Parker, 1993).

The complexity of algebra has been attributed to syntactic inconsistencies with arithmetic, such as the following: a variable may simultaneously represent many numbers, the letter may be chosen freely, the absence of positional value, equality as an equivalence relation, the invisible multiplication sign, the priority rules and use of parentheses.

The type of task we have used, is proposed and solved by Diophantus, and solved again by Viète to demonstrate the strength of his new symbol system (Harper, 1987). Diophantus solved equations using a letter for unknown, and not for parameter. He identified the unknown as a number. A letter was thus at this stage of history used for an *unknown*, having the potential of being determined. Viète also used letter as *givens*, as *parameters*, assuming them to be representing a range of values. This transition was a watershed in history (Sfard, 1995), and in this respect the task is chosen to give a view into this transition in education. Task **a** is a task on special, simple numbers and can be approached by a possible strategy for trial and error, by some calculation, or setting up one or two equations. Task **b** is intended to give insight into students' awareness of the structure of the problem and the strategy they apply. Task **c** might reveal any transition to handling the concept of a parameter, and express any solution in terms of these. Thus, it goes into algebra as a tool to explain the method, to generalize to a class of problems, to justify the solution and solution strategy. Ability to think algebraically, by understanding the structure of the problem, and of one's own solution, is investigated in task **b** and **c**.

In the analysis of students' answers we apply the APOS-theory (Dubinsky, 1991), which relates to action, process, object and scheme. Actions become processes by interiorization, and objects are created by an encapsulation of processes. A scheme is a network of processes and objects. We also use the levels of algebraic development described by Sfard & Linchevski (1994): rhetoric, syncopated, symbolic and abstract algebra.

PROBLEM FOR THE STUDY AND METHODS USED

The current study is part of an ongoing larger study on Learning Communities in Mathematics, where didacticians work together with teachers in eight schools (Jaworski, 2005; Andreassen, Grevholm, & Breiteig, 2005). In this paper we focus on a micro-study of one single problem, which has been used to assess students' mathematical knowledge at the beginning of the project. The task we will analyse is one item in a comprehensive test including other items on algebra and numbers.

The task

Eva is thinking of two numbers. The sum of them is 19. The difference between them is 5.

a Find the numbers.

b How can you find the numbers?

c Why is it always possible to find the two numbers when we know their sum and difference?

Data has been collected from all students in the first year of the two participating upper secondary schools (in a larger Norwegian town). This means grade 11 in the Norwegian school system. The students, who take the general course (the most academically oriented) in mathematics in grade 11, make up about 40-45 % of the year-group. The number of students included was 236 in autumn 2004 and 220 in spring 2005 (a few students were absent each time). An identical test, comprising 15 items, was used on both occasions. 209 students were present on both. Students were allowed sufficient time to complete all items. The test was given by the regular mathematics teacher of the class, in a normal mathematics lesson in the classroom, but evaluated by the researchers in the study. Feedback was given to the teachers. In order to indicate a wider picture over time we also use results on the same item from two other groups, where data was collected in other studies in 1993 (154 students in grade 9, part of a whole year-group in compulsory school) and in 2002 (150 students in grade 11, taking the general course).

The evaluation includes interpretations in task **b** and **c**. The interpretations were checked by both researchers until unanimous results were reached. First, a categorisation of correct, fail, and no answers was inserted on all tasks. After that the answers in **a**, **b** and **c** where categorised according to the lists in the tables below.

The research questions are:

1 Can students solve the numerical problem in task **a**? What kinds of answers do they give and what thinking do they reveal?

2 Can students explain their own method in **a**? Can students answer the generalised problem with any sum and difference? What kinds of explanations do they give in **b** and **c**?

ANALYSIS AND RESULTS

First we present the data regarding correct, failure or no answer. This will give an overview of the situation. Results from the students in 2004 and 2005 show:

	a		**b**		**c**	
	2004	*2005*	*2004*	*2005*	*2004*	*2005*
Correct	64	73	39	28	3	4
Fail	15	9	32	48	56	58
No answer	21	18	30	24	41	38

Table 1. Solution frequencies (percent) of task **a**: numerical task, task **b**: explaining one's own solution and **c**: general solution, algebraic task.

The solution frequency is rather high for the concrete numerical task in **a,** much lower for the explanation in **b** and justification in **c**. We will relate the results to data from 1993 and later (Bjørnestad, 2002, Andreassen, 2005). Then we go into the ways of thinking that may be represented, what kind of strategy is used, and what developmental level is indicated by the single response. The outcomes at the four occasions (1993, 2002 and 2004-5) are consistent in that the overall picture is the same. We will now take a closer look at students' answers.

Categories of answers in task a other than the correct one

A closer look at the unsuccessful strategies might show some pattern of thinking.

Response	*Example of response*	*1993*	*2004*	*2005*
14 ($s - d$) is one component	14 and 5; 14; 12+x2	15	5	5
Using the average ± 5	4,5 and 14,5	2	0	1
One condition, s=19 or d=5	13 and 6; 11 and 6;	2	5	1
No pattern suggested	10 and 4; 1 and 4; 6 and 9;	4	2	1
Just one number	$7\frac{1}{2}$; 28	2	1	1
More than two numbers	4+9+10;	1	1	0

Table 2. Unsuccessful strategies for Task **a**: Numerical task. Frequencies in %.

A high percentage in 2004-5 and 1993, namely 5 and 15 %, respectively, give one of the numbers as (s – d). They take a step towards a solution. Although this difference might be an interesting number for a general solution, the solution is not carried through. Other unsuccessful strategies are not very frequent, including the strategy of using $s/2 \pm d$ to find the two numbers.

Acceptable explanations in different ways of expression

The students' use of acceptable explanations and the degree of awareness of their own thinking, will influence the general solution on a task like this. Hence, it is of interest to investigate their fruitful strategies. How did they find their numbers? Here we find three categories of explanations. They can be referred to the use of rhetoric expressions, or numerical and with algebraic symbols (Sfard & Linchevski, 1994).

Using verbal statements: Example: I found the numbers by taking the total sum – difference and divide by two. Then I found the smallest number. The other I got by adding the smallest to 5.

Using numbers: Example: 19 – 5 = 14; 14 : 2 = 7; 7 + 5 = 12. (Remark: In Norway the symbol : is used as division sign instead of /). - This may be taken as a generic example to illustrate the general solution. However, there is a step from this process to regarding the numerical structure as an object (Dubinsky, 1991).

Using algebraic symbols. Example: $x + (x + 5) = 19$. - By setting up this equation, is it possible to see the role played by 5 and 19, which is needed in **c**?

This analysis may be summarized in a table:

Strategy used	2002	2004	2005
Explains the solution mainly verbally	32	36	21
Shows the calculation by numbers	24	3	6
Shows the calculation by using symbols	12	1	1

Table 3. Explanations in task b. Frequencies in %.

The students prefer to use mainly verbal solutions, more so in 2004/5 than in 2002.

The result signals that students can handle the numerical task, but that it is much more difficult for them top explain how they solve it and even more difficult to justify why the generalized problem always can be solved. The step from arithmetic to algebra is hard to take and still in grade 11 most students have not come that far.

Explanation or justification in tasks b and c

In task **b** (and **c**) students expose seven different ways of explaining or justifying the solution. They are listed in the following table below:

*Type of explanation or justification used by students, who gave a correct answer in **a***	*2004*	*2005*	APOS level[1]
1. Guess and check	79	72	Action
2. Table search: two conditions	30	13	
3. Calculate (19 – 5)/2, add 5 for the 2nd number	6	10	Process
4. Halving: 19/2 + 5/2 and 19/2 - 5/2	9	17	
5. One equation $x + (x + 5) = 19$ and solved	7	7	
6. Two equations $x + y = 19$; $x - y = 5$ and solved	3	11	
7. General solution $x = \frac{s+d}{2} \quad y = \frac{s-d}{2}$	7	9	Object
Others, no explanation	16	16	

Table 4: Ways of explaining in task **b** and **c** (when **a** is correct), number of students.

Here the first two ways of explaining can be regarded as actions (Dubinsky, 1991). Students do not have a mathematical algorithm, they act out on numbers in order to find the result. Methods number 3 to 6 are on the level of process. A mathematical process is carried out in order to solve the problem. The process can be numerical or algebraic. Sometimes it is expressed in a mixture of verbal and symbolic language (syncopated). If students are able to generalise and set up the equations for any sum and difference, we consider them to be on the object-level. They can see the structure of the problem and use equations as objects. In all cases the way of expressing the explanation can be rhetoric, syncopated or symbolic, with numerical or algebraic symbols (Sfard & Linchevski, 1994).

The dominating way of explaining is to refer to guess and check at both occasions. It seems as if use of table decreases over the year and use of calculations increases. Use of two equations increases much, which can be explained by the fact that students worked with systems of equations during grade eleven. Only 7 and 9 students, respectively, can give the general solution. There is a trend to choose more sophisticated methods at the end of grade eleven but the development seems to be slow.

DISCUSSION AND CONCLUSIONS

We note that about 3 out of four students give a correct solution for the numerical problem. With trial and error there are few cases to test as with use of table, and there is an absolute possibility to check if the answer is correct. Thus one could expect more success. During grade 11 in Norwegian school, the curriculum contains work with systems of linear equations. Thus, all participants in 2005 have worked with this

[1] We are aware of the fact that we are using the APOS-terminology in a slightly different way than Dubinsky, but we found it useful to characterise our findings here.

kind of problem during spring 2005. The effect of this work seems to have an influence on the results, but still only about every tenth student uses this method. The different kinds of answer and ways of thinking in task **a** are shown in table 3.

Another observation is students' limited competence to communicate verbally in mathematics. It is likely that it is unusual for them to write down their explanation for the solution. We conclude that meta-cognitive activity of this kind in the classroom must be uncommon. Students also show an attitude of not seeing the need for an explanation, when they say things like: ”It is just like that”. Some even try to make jokes like answering: ”It is because Fred has green shoes.” Another observation is the fact that nearly no student answers in exactly the same way in both 2004 and 2005.

The facility of task **b** a low (less than 40 %). It indicates that the goals of curriculum are not achieved for more than a minority of the students. Both correct rhetoric answers, mixture of rhetoric and symbolic and pure abstract algebraic answers are used but rhetoric answers dominate. In task **c** a still lower result is evident (3-4 %), indicating that the step from a numeric problem to a general problem with parameters, which demands algebra, is still too great for most of the students. Less than one out of twentyfive students can generalize and justify the solution in this case. The dominating way of explaining is to guess and check or use a table. Use of calculations or systems of equations seems to increase over grade eleven.

To overcome problems and to enhance the development, it seems valuable to introduce writing explanations into mathematics lessons and use contexts where students have to explain to each other how they are thinking. Discussion of alternative solutions could be one way of creating awareness of one's own way of explaining. The application to problem-solving situations of knowledge of how to solve a system of linear equations seems also to be lacking from students' experience. It is possible that mathematics teachers could learn much from an analysis of their students' work on a task like the one we used here.

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RESISTING REFORM PEDAGOGY: TEACHER AND LEARNER CONTRIBUTIONS

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Classroom conversations are a key feature of reforms in mathematics education. Many challenges in conducting these conversations have been identified, but explicit analyses of learner resistance to such conversations are rare. In this paper, I provide one such analysis, from the perspectives of learners and the teacher. I argue that resistance is a complex phenomenon and needs to be understood as such if teachers are to make progress in conducting mathematical conversations.

MATHEMATICAL CONVERSATIONS

A key aspect of reform pedagogy in many countries is supporting learners' conversations with each other about their mathematical thinking. There are a number of reasons for supporting learner-learner conversations. Developing and transforming thinking can be seen as a process of making connections between two previously separate ideas, or differentiating a previous single idea into a number of more nuanced ideas (Hatano, 1996). Articulating one's thinking supports such differentiation and integration of ideas (Mercer, 2000; Vygotsky, 1986). Communicating mathematics is an important mathematical practice, which supports other mathematical practices such as connecting, generalizing and justifying ideas (RAND Mathematics Study Panel, 2002). As learners and the teacher consider, question and add to each other's thinking, important mathematical ideas and connections can be co-produced. Hearing learners' ideas can also give teachers a window into the learners' thinking, which can help her understand the conceptions and misconceptions underlying learners' contributions.

Instances of successful mathematical conversations, in which learners seriously engage with and transform each others thinking (Sfard, Nesher, Streefland, Cobb, & Mason, 1998) are extremely rare, across a range of contexts (Brodie, Lelliott, & Davis, 2002; Chisholm et al., 2000; Stigler & Hiebert, 1999). A number of challenges in supporting successful conversations have been identified. These include: supporting learners to make contributions that are productive of further conversation (Heaton, 2000; Staples, 2004); respecting the integrity of learners' mistaken ideas while trying to transform them and support the development of appropriate mathematical ideas (Chazan & Ball, 1999); seeing beyond one's own long-held and taken-for-granted mathematical ideas in order to hear and work with learners' ideas (Chazan, 2000; Heaton, 2000); maintaining a "common ground" which enables all learners to follow the conversation, its mathematical purpose and to contribute

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appropriately (Staples, 2004); and creating appropriate norms for talking and interacting (Cobb, 2000; Lampert, 2001; McClain & Cobb, 2001) in the classroom.

Shifting classroom norms is seen to be an important aspect of mathematics reforms. In traditional teaching, teachers explain ideas, ask questions to which they require correct answers, and strongly evaluate learners' responses as correct or incorrect. In pedagogy that tries to establish and maintain conversations, teachers are encouraged to refrain from "telling" too much and to rather probe for learners' ideas and meanings[1]. Norms that are appropriate for such pedagogy are that mathematical reasoning, justification and communication are as important as correct answers; partially correct or incorrect contributions are helpful and often illuminate important ideas; and learners are expected to listen to and comment on their classmates' contributions in respectful ways. Teachers who develop successful mathematical conversations do continuous work on developing these norms (Boaler & Humphreys, 2005; Lampert, 2001; Staples, 2004).

THE CONTEXT OF THE RESISTANCE

This episode took place in a Grade 11 classroom in a racially diverse school in Johannesburg, South Africa. Most learners came from middle or lower-middle class backgrounds. The teacher had as an explicit goal to develop learners' mathematical reasoning through conversation. I videotaped two weeks of lessons in this class, one week in March and one in May, and interviewed the teacher five times over the three months. This episode of resistance occurred during the second week.

The theoretical frame from which the resistance was analysed is a broadly social constructivist frame, which includes the notion of situated learning and communities of practice (Hatano, 1996; Lave & Wenger, 1991; Wenger, 1998). This literature underlies the notion of mathematical conversations discussed earlier. In addition, my analysis of the resistance assumed that people's talk and actions are informed by their own perspectives on the situation and by what happens as they interact in the situation. So in my analysis I attempted to understand how the ways in which individual learners and the teacher made sense of the conversation as well as the ways in which they interacted, led to the learners' resistance.

The learners were working on the question: What changes as the graph of $y = x^2$ shifts 3 units to the right to become $y = (x-3)^2$, and four units to the left to become $y = (x+4)^2$. A learner, Winile, was reporting back on some of her group's observations. As Winile finished her report back, a conversation began which established an important question: Why does a negative sign in brackets correspond to a shift to the right and a positive turning point; and a positive sign correspond to a shift to the left and a negative turning point. This question was co-produced in conversation between two learners, Michelle and Lorrayne in relation to Winile's contribution. A number of other learners indicated that they were grappling with the same question.

[1] For critiques of this idea see Chazan and Ball (1999) and Brodie and Pournara (2005).

The teacher worked hard to facilitate discussion around this question. First, he tried to establish a common ground (Staples, 2004), making sure that all learners understood and were interested in the question. He did not directly tell or explain ideas, rather he probed for learners' thinking and tried to get them to respond to each other's ideas. He also did substantial 'norm' work, communicating how he expected learners to listen and talk to each other and respect each other's contributions. For example:

> That's why it's important for you guys to listen. We need to learn to listen to each other and talk to each other to help each other to learn. That's the point of being in this class in the first place

The conversation about this question lasted for over 30 minutes. During the conversation, many learners made extensive contributions, articulated their ideas and developed their thinking substantially. For example in her initial presentation, Winile presented a rather confused argument, suggesting that she did not understand the relationship between the equations and the graphs. Michelle and Lorrayne had picked up on this in their question, which attempted to get at the relationship. After listening to the conversation for some time, Winile contributed:

> the +4 is not like the x, um, the x, like, the number, you know the x *(showing x-axis with hand)*, it's not the x, it's another number … you substitute this with a number, isn't it, like you go, whatever, then it gives you an answer … we're not supposed to get what x is equal to, we getting what y is equal to, so we supposed to, supposed to substitute x to get y

Here she argued that they could not make a direct link from the superficial features of the equation, the +4, to a feature of the graph, the turning point. Rather, they had to take account of the underlying relationships between the variables that gave rise to the graph, and that the equation transformed the numbers in ways that were not obvious. This suggests a deepening of her thinking about equations and graphs – she had differentiated the direct equation-graph relationship and now saw a more complex relationship, between the variables in relation to each other and in relation to the graph.

Although much learning took place through the conversation[2], and many learners openly expressed enjoyment of the interaction and their learning, a point came in the lesson where a number of learners began to openly express resistance to the conversation. An analysis of this resistance from the perspectives of the learners and the teacher is illuminating of some of the difficulties in conducting conversations in classrooms.

LEARNER RESISTANCE

The extract below shows some of the resistance that was openly expressed by the two most vocal learners: Michelle and Melanie. The extract followed an exchange where

[2] A more detailed analysis of how learners contributed to each others' learning through this conversation is available in Coetzee (2003; 2004).

the teacher asked a learner to explain his thinking. Many learners were unclear about his explanation and so the teacher asked him to explain again. Michelle interpreted this as a delaying tactic:

Michelle:	If you had the chance you would keep us here until tomorrow
Teacher:	I will keep you here until tomorrow
Learners:	Aah, sir (*talk simultaneously) (Michelle sits back and folds her arms, other learners, including Melanie have their hands up)*
Teacher:	Because, because, ai, this is what I want, I want you people to talk to each other, I want you to talk about the mathematics and it’s happening, so I’m happy about this, very
Melanie:	But when I go home what did I learn *(laughter)*
Teacher:	You’d be surprised
Melanie:	Excuse me, sir, I, okay, sir, you can see we are so battling with this but, you refuse to just give us the answers
Learner:	Because we just get more confused
Learner:	Ja, but don't give me that
Michelle:	How long are we going to play this game *(learners talk simultaneously)* man, I can play the game until it’s over
Teacher:	Sh, guys, listen up, *(inaudible)*, I think I need to say something here, this is not a game, okay, this is mathematics that you need to understand, okay, there’s, there is mathematics that you need to understand, there is mathematics that is in your syllabus and you need to make sense of it, I just don’t want to give you answers, which means nothing to you, If I give you the answer what will it mean to you, really, without an understanding of why the answer is that, and that is the point here, and I like, Winile is making sense of it, David is trying to make sense here, we’ve got Lorrayne here that contributed, Maria, Ntabiseng had wanted to say something, (*Melanie indicates herself)*, you, exactly, that’s what I want. So let’s, let’s
Michelle:	Some of us aren’t that intelligent as these guys
Teacher:	You don’t need to be
Michelle:	This guy that comes with the answer out of the blue, I mean, I never did, look at that, I can’t just sit here fiddling, and I don’t even understand what’s going on, no man

The two main “resistors” in this episode, Michelle and Melanie, showed interesting similarities and differences in their orientations to the conversation and their learning. They both contributed to the discussion, and were clearly engaged in trying to make sense of the ideas throughout the lesson. When the teacher listed a number of contributors, Melanie eagerly indicated that she too was one. Both girls also found their classmates' explanations confusing, as did many other learners. Melanie's response to her confusion was to request clarity from the teacher, in the form of answers. In the next extract we will see that she asks the teacher whether an explanation is right. So Melanie was working with the norms of traditional teaching, looking for right answers rather than explanations and understanding.

In contrast, Michelle seemed more interested in understanding the mathematics. She asked the question that began the discussion and was clearly engaged in trying to work out a justified answer. In the episode above, she was frustrated that an answer "came just out of the blue" and that she could not understand it.

In an earlier exchange, when a classmate had suggested as an answer to her question: "that's just the way it is", she was unwilling to accept the response without a proper justification. Michelle's reference to time suggests that it was not the struggle to make sense that bothered her, but the fact that it took so long. It most likely seemed to her that little progress in understanding the issues surrounding her question was being made.

THE TEACHER'S CONTRIBUTIONS

The teacher tried to work against the resistance, but also may have inadvertently contributed to it. In the extract above, some of his contributions attempted to make clear what his expectations were and how they differed from what is usually expected in mathematics classrooms; the 'norm' work that I referred to earlier. At the same time the teacher may have unintentionally undermined some of what he was trying to do by making flippant comments.

In response to his remark "I'll keep you here until tomorrow", Michelle sat back with folded arms looking angry. It is not clear why the teacher made this flippant remark, nor the next one: "you'd be surprised". His first statement suggests that he may have been trying to reclaim some of his authority in the face of obvious resistance. When teachers are faced with these kinds of difficulties, they may fall back on well-worn expressions of authority and certainly might make counter-productive statements while trying to deal with issues in a productive way.

A second aspect of the teacher' pedagogy contributed to this resistance. Because he was trying to get learners to talk to each other and to express and articulate their own ideas, he often claimed not to understand what a learner was saying. Sometimes these claims were genuine; he did not understand the learner's idea and needed more explanation himself. At other times, his claims were not genuine; he feigned ignorance in order to get more articulation and justification from learners.

The extract below followed immediately after Winile's contribution above. A number of learners had responded positively to her explanation and the teacher asked others to comment on it or to say what they thought:

Teacher:	Can somebody else try, because I saw a lot of people said, oh-ja, now we see, can I just see, Lorrayne, maybe you can put it another way
Learners:	*(General chatter – inaudible)*
Teacher:	Sh, okay, you want to try, Maria, sh
Melanie:	Sir but is the explanation right
Teacher:	I don’t know, what do you think
Melanie:	I dunno, that’s why I’m asking you
Teacher:	That’s why I’m asking, that’s why I’m asking Maria what she thinks
Melanie:	No but you know sir
Teacher:	Maybe she can explain it.
Maria:	I'm gonna try, sir, I think here, on the equation, y equals *(inaudible)*

Before Maria could make a contribution, Melanie asked whether Winile’s explanation was right. The teacher threw her question back to her, claiming not to know and asking what she thought. The teacher did know that parts of the explanation were incorrect and later he directly challenged Winile on her mistake. It may have been the case that Melanie suspected a problem with Winile’s explanation, which was why she asked the question. From her perspective, the teacher knew the correct answer and seemed to be deliberately withholding it. In this case, it may have been more productive for the teacher not to feign ignorance but to explain why he wanted others to contribute. It also may have been useful for him to indicate that there might be problems with Winile’s explanation and ask learners to comment specifically on these. At the same time it needs to be noted that other learners did not have a problem with the teacher feigning ignorance. In the above extract, Maria was certainly eager to try to explain Winile’s idea.

CONCLUSIONS: THE DYNAMICS OF RESISTANCE

The literature on creating norms of participation for reform pedagogy and mathematical conversations suggests that when learners have difficulties in participating it is because they do not understand what is expected of them, and that they struggle to shift from traditional to new pedagogies. My analysis in this paper suggests that this is only part of the story. First, different learners will have different reasons for resisting, in this case, related to their orientations to learning. It is significant that a learner who was engaged in the mathematical practices of asking questions, relating different representations to each other, and attempting to make sense of mathematical ideas, began to resist the conversation, precisely because she was not making progress on making sense, and found some of the discussion difficult to follow. How to enable all learners to follow each other’s, sometimes muddled, articulations, without constraining their thinking and participation, is a demanding task for teachers. Second, the idea of probing learners' ideas rather than explaining clearly might seem dishonest and a shirking of the teacher's responsibility. Additional norm-work might be required to help learners to see this as part of the teacher's responsibility and a way of challenging learners to think more deeply. Finally, it might be the case that while doing appropriate norm-work to enable learners to participate, teachers inadvertently slip back to well-worn expressions of authority, which undermine what they are trying to achieve. This is to be expected when undertaking such a difficult task – teachers cannot shift all of their habits and routines overnight.

In this paper, I have given a small taste of some of the dynamics of resistance and suggested that both learners and teachers contribute to them, in complex ways. Teaching in reform-oriented ways and using conversations requires much more of teachers than going beyond "not telling" and establishing appropriate norms. It also requires some understanding of where resistance might arise, and the range of reasons for this resistance, thus adding to the difficulties and demands of such teaching and explaining its rarity.

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MANIFESTATIONS OF AFFORDANCES OF A TECHNOLOGY-RICH TEACHING AND LEARNING ENVIRONMENT (TRTLE)

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This paper reports an instrumental case study of the perception and enactment of affordances in a Year 11 mathematics classroom where students were studying functions. Manifestations of affordances of the TRTLE were identified from several data sources. The conditions that enabled and /or promoted perception and enactment of affordances of the TRTLE to facilitate students' understanding of, and working with, functions were identified using thematic matrices and Valsiner's Zone Theory. Factors both internal and external to the TRTLE impacted on the students' Zone of Free Movement influencing the effectiveness of the teacher's Zone of Promoted Action.

The transformational power offered by technology in mathematics classrooms has been discussed by many. Myriad affordances that would be useful in the teaching and learning of function are offered by classroom environments involving these technologies. Within many classrooms across Australia, and in particular within the state of Victoria, electronic technologies such as graphing calculators, are an assumed part of the mathematics curriculum. Use of these is increasingly expected by statutory authorities overseeing the upper secondary mathematics curriculum. The focus of this paper is the manifestation of affordances during the study of functions in one particular *technology-rich teaching and learning environment* (TRTLE) where electronic technologies were readilyavailable to both teacher and students.

The term *affordance* has multiple meanings in academic literature. In the research reported here, the definition follows that of Gibson (1979) and Scarantino (2003). Gibson invented the term in his study of what is now known as perceptual ecology to explain what animals perceive. More recently, Scarantino elaborated on the notion of affordances. Applying Scarantino's elaboration to *affordances of a TRTLE*, these are the offerings of the environment for facilitating and impeding teaching and learning. Affordance bearers are those specific objects within the environment that enable an affordance to be enacted. The manifestation of an affordance in a TRTLE in background circumstances [C] involves an event [E] in which both the affordance bearer [AB] and the actor [A] are involved (e.g., a student [A] in confirming her conjectured function model of collected data sees a view of a graphical representation of a function [E] on a graphing calculator [AB], given the function has been entered in the function window, the viewing window is set appropriately, and the GRAPH key is pressed [C]). Affordance bearers can be described in general terms, as in the previous example, or as specific features of the technology being used. Affordances

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are opportunities for interactivity between the user and the available technology for some specific purpose (Brown, 2005). The affordance described here is check-ability.

The existence of an affordance does not mean that interactivity will occur. Before either the teacher or student can act, the affordance needs to be perceived. To take up affordances of a TRTLE, both teachers and students must "learn to perceive a perceivable affordance, that is, learn to become attuned to" (Scarantino, 2003, p. 954) what specifies it, but what conditions make this possible? Perception may result in the rejection of the affordance rather than its enactment and this may be the appropriate action, however, within classrooms students and teachers need to become attuned to particular positive affordances of the environment and being able to reject other negative ones. Following from this, the existence of a TRTLE does not imply that the technology, nor its richness, are necessarily embraced or result in a transformation of the environment.

The research reported here is part of a larger study investigating the perception and enactment of affordances by teachers and students (Years 9-11, approximately 14-17 year olds) involved in the study of functions in TRTLE's. This paper reports one TRTLE and addresses the following research question: What conditions exist in a TRTLE that enable manifestations of particular affordances of the TRTLE to facilitate students' understanding of, and working with, functions?

VALSINER'S ZONE THEORY

Valsiner (1997) uses a theoretical framework based on three abstract zone concepts "that are viewed as organisers of development, both interpsychologically (between persons) and intrapsychologically (in the semiotic regulation of one's own thinking, feeling, and acting)" (p. 188). These zones have been applied to mathematics teacher education (e.g., Blanton, Westbrook, & Carter, 2005) and are applied here to teaching and student actions in a TRTLE. The *Zone of Free Movement* (ZFM) is described by Valsiner as "the set of what is available (in terms of areas of environment, objects in those areas, and ways of acting on these objects) to the child's acting in the particular environmental setting at a given time" (p. 317). In a TRTLE this includes parts of the classroom, technologies and other learning artefacts, affordances and allowable actions, available at any given time. Within this environment interaction can occur between learners, between the teacher and learners, and between either of these and the artefacts present including technology.

The *Zone of Promoted Action* (ZPA) characterises "the set of activities, objects, or areas in the environment, in which the person's actions are promoted" (Valsiner, 1997, p. 192). In the TRTLE these include both formal and informal learning activities involving the use of electronic technologies and learning artefacts utilised by the teacher to promote the learning of functions. Valsiner's third zone, a modification of Vygotsky's Zone of Proximal Development describes possible learning states, something not directly observable and hence, the focus here will be on the other zones. Zone theory is being used to characterise conditions existing in TRTLE's that enable or promote perception and enactment of affordances.

METHODOLOGY

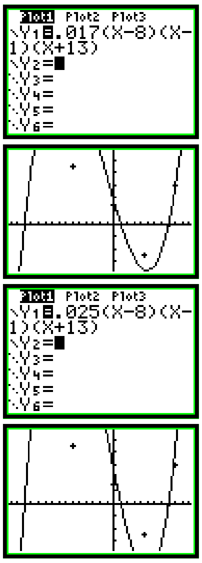
A challenge exists to “capture and interpret characteristic forms of technology use exhibited by students and teachers” (Galbraith & Goos, 2003, p. 364). By looking closely at what is occurring in a TRTLE, a clearer picture will be gleaned about what is occurring in mathematics classrooms where teachers and students have ready access to various electronic technologies. Only through increasing this understanding will others be able to become increasingly effective users of technologies in education systems where their use is expected. A qualitative approach will provide this comprehensive picture of what is occurring across and within TRTLE’s.

A collective case study (Stake, 1995, p. 4) using an instrumental approach (p. 3) is being used in the larger study. The purpose is to study each case to “understand the phenomena or relationships within it” (p. 171) rather than the cases themselves being of primary interest in order to construct a grounded theory establishing what conditions enable students and teachers to perceive and enact affordances offered by the TRTLE for the teaching and learning of functions. The data from the one case presented here detail the manifestations of the phenomena being studied, affordances within the TRTLE, and the conditions that enabled or promoted their enactment. In this study the environment, that is, the TRTLE, includes electronic technologies and other objects, ways of acting with these objects by the teacher and students, and the teacher and students themselves. The case being studied, included one teacher, James and his Year 11 Mathematical Methods students (16-17 year olds). Function is one of the main study areas in Mathematical Methods. Content includes the graphical representation of linear, quadratic, and cubic functions (VBOS, 1999, p. 68). The class comprised of 22 students, 2 female and 19 male. The students and teachers had daily access to Texas Instruments graphing calculators (83/84 Plus models) and laptop computers both at home and in class. The teacher regularly used a data projector to display Power Point presentations to the class.

Data were collected from a variety of sources to maximise complementarity (Clarke, 2001). These included field notes, audio and video recordings, photographs, student scripts, graphing calculator key recordings, post-task student interviews, teacher interviews and reflections, and documentary materials from lesson observations of nineteen 50 minute lessons, teacher designed task implementations (*Quadratic Function Task* [QF], *Cubic Function Task* [CF]), and a researcher designed task implementation, *Platypus Task* [PP]. The teacher designed tasks involved students taking physical measurements involving flexible stick/cord shaped into several functions (QF and CF), hanging chains (QF), chromatography (QF), and a pulley system (QF). Students were required to relate coordinates of key features to particular forms of the given function type and then to link the algebraic and graphical representations, refining the final unknown (dilation) parameter to identify the equation of the function being modelled. Regression was then used to check the value of this final parameter. Both tasks involved repetition of the same sub-task, albeit for different general forms of the function. The ZPA for students when undertaking these two tasks was well contained within their ZFM with students being told the methods to use. In the researcher designed task, students were presented with two sets of data representing a platypus population before and after a ‘Save the Platypus’ project and asked to find a model to represent platypus numbers over time for both data sets. Students were asked to consider such questions as, would the platypus become extinct without intervention, what was the predicted population after a further decade, and when would the population return to the initial value? The PP required students to make use of functions once identified and promoted the use of a broader understanding of functions, including various manifestations of function calculate-ability, and function view-ability. In promoting student choice of actions for this task, the ZPA was significantly closer to their ZFM.

ANALYSIS AND RESULTS

Initial analysis occurred through preliminary coding after the data were entered into a NUD*IST database (QSR, 1997) to identify manifestations of affordances, affordance bearers, and the circumstances in which these occurred. The focus then turned to the examination of both the coded data and re-analysis of the case record to identify conditions enabling and promoting enactment of affordances. A thematic conceptual matrix (Miles & Huberman, 1994, p. 131) was developed to show manifestations of affordances, affordance bearers, and *conditions enabling perception* or *promoting enactment* of particular affordances of this TRTLE for student understanding of function (see Table 1).

Manifestations of the Affordance	Affordance Bearers	Conditions	
		Enabling Perception	Promoting Enactment
Editing parameters S95: I think the *a* value is wrong. S108: Let us work on the *a* value then. [He accesses the function window and edits the *y* value to .025. S101: Should it be smaller or larger? [CF7Apr05 364-366] CF [S101, S108, S111 & S95] 7Apr05 354-402	*y=, GRAPH*  CF [S101, S108, SB111 & S95] 7Apr05 365-392	*Lesson element focused on effect of parameters* James: Take a guess at what you think the equation should be. Then fit the graph of the equation over your data points. If the graph doesn't fit the data points well then adjust the values in your equation. [QF2Mar05 257] *Teacher Scaffolding* James: What do you think *a* might be? SX: I don't know. James: Try 1 then. Now is that spot on? Can you improve on that? SX: $y = 1.1x^2$. I want it smaller. [QF2Mar05 284-286, 295-296]	*Function task where data points given (PP) or collected (CF) where students have to determine an appropriate algebraic model* James: What are the letters in a polynomial, *a*, *b* and so on. But some of these forms don't need that many. [CF 7Ap05 28] *Teacher Promotion* James: You can use QuadReg on your data. Which is fine when you don't understand what you are doing. But you see this [task] actually helps you understand, oh, what if I change this. What effect will that have? [QF2Mar05 323]

Note: CF = Cubic Function Task, PP = Platypus Task, QF = Quadratic Function Task, SX = student, LE = Learning experience

Table 1: One manifestation of the affordance: Function Identify-ability

Table 1 shows one row of the matrix, one manifestation of the particular affordance: *Function Identify-ability*. This affordance includes identification of the type of function to best fit given data, (e.g., quadratic rather than cubic), or the specific algebraic representation of a given function (e.g., $y = 5x^3$). For illustrative purposes only the manifestation resulting from *editing parameters* is reported. Other manifestations of this affordance included those resulting from *use of regression, testing of multiple function types and selecting the function type which best matches real world aspects*, *visual identification*, and *searching for the 'perfect' fit*.

The first column in the matrix includes summary phrases describing the manifestation of the affordance. Representative illustrative quotes or actions from interviews or observations are included to make as explicit as possible the meaning of these phrases. To indicate the extent and nature of further evidence for the manifestation, student or teacher codes and text units from the data are shown (e.g., [Obs16Mar05 18], where 18 represents the NUD•IST text unit number). The second matrix column indicates the affordance bearers utilised in the enactment of the affordance. In the example shown, these are the function window (y=) and the GRAPH key as the WINDOW had been set previously. The final two columns describe the conditions existing either during or prior to the manifestation of the affordance being considered. The conditions are described as either those *enabling perception* (column 3) or *promoting enactment* (column 4) of a particular affordance.

A condition *enabling perception* is a circumstance where a teacher or student action allows a particular affordance to be perceived. Conditions enabling perception include those where a learning experience is provided during which the student experiences a particular affordance. For example, during a teacher led class discussion, whilst acknowledging other possibilities for the enactment of the affordance: function zero determine-ability, the teacher may suggest an alternative, utilising the affordance bearer, CALCULATE Zero. Subsequently the teacher carefully steps the class through the use of this new calculator feature to facilitate enactment of the offered affordance. In other situations, the enabling condition is more focused on the reasons for enactment, as in the following classroom interaction. Concurrent student graphing calculator screens are shown in Figure 1.

> James: You obviously haven't discovered the secret of using WINDOWs, of using the WINDOW. Can you see on the way this is graphed, there is a whole lot of blank space? Can you see that? Look at your WINDOW. See here it is -10, 10. You might change that. Make it 5 say. Now, have a look at the graph. [Obs 16Feb05 160].

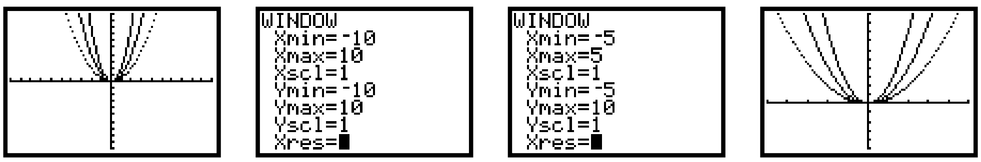


Figure 1. Graphing calculator screens before and after editing the WINDOW settings

Here James attunes students to experiencing editing WINDOW settings to enhance function view-ability, a learning experience that may enable these students to enact the offered affordance at their own instigation in the future. Conditions enabling perception may be of a detailed specific nature, as are both cases in Table 1, where attention is drawn to technical aspects. In other circumstances, conditions may be more global in nature with attention drawn to the rationale for enactment rather than to the detail (e.g., WINDOW example). Conditions enabling perception may also result from serendipity as occurs when a student presses GRAPH on the graphing calculator and a global view of a function is displayed merely because previous WINDOW settings used are suitable. In these circumstances the researcher relied on student conversations or reports for evidence of their occurrence.

A condition *promoting enactment* is a circumstance where a teacher or student action promotes enactment of a particular affordance. The final column of Table 1 is indicative of the ZPA. The teacher's organisation of the students' ZPA canalises (Valsiner, 1997) the students' current and future thinking about the concepts and methods being taught. In other words, the ZPA canalises the students' future cognitive actions as they internalise the conditions orchestrated by the teacher for their learning. Promotion may be of a preferential or an excluding nature as is the case in Table 1 where James asks his students not to use regression, or could be promoting a choice of affordances or affordance bearers. For example, James, provided his students with a choice of affordance bearers to enact the affordance: function value determine-ability, "Note there is a local maximum here. This can easily be found using the TI83. 2nd CALC. Maximum or TRACE" [Obs9Mar05 581].

To consider data at a more conceptual level, content analytical summary tables were constructed. Table 2 shows an example for the affordance function identify-ability.

Manifestations of the Affordance	Conditions	
	Enabling Perception	Promoting Enactment
Editing parameters	Lesson element: focus on effect of parameters Teacher scaffolding: effect of a particular parameter	Teacher Promotion: varying parameters to identify a function helps understanding
Using of regression	Lesson element –regression to check parameters found using 'trial and error' Lesson element: minimum information required for cubic regression	Function task where data collected or given
Multiple function types tested, best matching real world aspects selected	Did not observe	Functions task where data given but function type unknown
Visual identification	Lesson element – form an idea of what the graph looks like	Functions task where data given but function type unknown
Searching for 'perfect' fit	Lesson element – find equation which best fits data	Functions task where data given

Table 2: Manifestations of the affordance function identify-ability

Impact of Task on the Zone of Free Movement

Actions that students could undertake in their solution of the CF were more restricted than for PP. For example, in the PP the post intervention data set was exactly quadratic, hence students could use second differences in their identification of the function type and/or its specific equation. This was not an option on the CF where students were focused on collecting data identifying key features, not on collecting information regarding coordinate pairs where Δx was constant. Furthermore, although CF promoted use of 'trial and error' to identify one unknown parameter after key coordinates had been measured and other parameters subsequently identified, this is not necessarily what occurs. The teacher may be "the leading organiser (but not the sole determiner) of the ZFM" (Valsiner, 1997, p. 189) and this is clearly the case in this task as the adolescents acted contrary to their teacher's direction and simply used regression to identify the given function. The reality of the ZFM/ZPA complex is that it includes "'forbidden'—yet possible—actions" (Valsiner, p. 193). Whilst regression was used by most students in their solution of the PP, finding the model was only part of the task. Moreover, lack of an obvious general form, or identifiable key features of the function may have made this a more appropriate manifestation of the affordance: function identify-ability, than was the case in the QF or CF.

DISCUSSION

It is not only the teacher of a particular TRTLE who influences what happens. The dynamic ZFM is also influenced by students in the TRTLE as well as by other teachers (including those from previous years, of a second mathematics subject studied, and of the same subject) and students outside the TRTLE (including those studying the same mathematics in another TRTLE). Only partial evidence of this is available through the interviews, so in this sense the enabling and promoting conditions identified here are not exhaustive even for the particular TRTLE. As a result of these myriad influences, the ZFM available to students can be broader than expected by the teacher. Subsequently, when carefully restricting the ZPA for learning purposes, teachers need to be explicit not only in the technical aspects required by the manifestation of a particular affordance, but also in promoting the reasons for enacting a particular manifestation in the circumstances and comparing from a mathematical learning viewpoint, the relative advantages of one manifestation compared to another. Data presented here included learning tasks where James was deliberately promoting the manifestation editing parameters for the affordance function identify-ability. Although the teacher stated the reason for this was to gain an understanding of the effect of the parameters and that use of regression would not facilitate understanding, it was apparent that some students still resorted to the use of regression as a means to identify the given functions. At the upper secondary level, in situations where past experiences have provided students with a broad ZFM, and where students are easily capable of ignoring teacher promoted actions, tasks and learning experiences need to be developed that ensure that particular actions orchestrated by the teachers for learning purposes are enacted by students. By actually experiencing particular manifestations of an affordance students are able to interiorise these experiences such that canalisation of their future cognitive actions can occur, again broadening their future ZFM. Hence, when situations occur where manifestations such as regression are not appropriate, students' ZFM will have other affordances available and also information as to which affordance to select and for any given affordance what manifestation may be the most appropriate. To enable the richness of the available technology to transform the learning environment students need to enact multiple manifestations of a range of affordances and develop an understanding of when given manifestations are appropriate. Only in this way will the richness of the TRTLE fulfil its potential to transform the learning environment for students.

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TYPES OF REPRESENTATIONS OF THE NUMBER LINE IN TEXTBOOKS

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This article shows the results of a study undertaken to discover how the number line model is used in Compulsory Education by three different Spanish publishers of mathematics textbooks. We noted why and how the number line is used: number concept, operations, order, and different representations. The results show that there are differences between publishers with regard to the frequency and way in which they use the number line. The number line is mainly used when new number systems (whole, integer, rational or real numbers) are introduced, while it is less often connected with basic operations.

THEORETICAL FRAMEWORK

The number line is a basic representation when learning numbers. Ernest (1985) states that in primary education the number line can be used: (1) as a model for teaching the ordering of numbers; (2) as a model for the operations of addition, subtraction, multiplication and division, and (3) as part of the contents themselves of the mathematics curriculum. While the first two ideas (1) and (2) form part of methodological decisions that teachers have to take, idea (3), which is that various types of numbers be represented on the number line, should form part of students' mathematical knowledge, as thus indicated in curricula.

Various research works regarding children's understanding of the operations of addition and subtraction with whole numbers have used items where the pupils are asked to represent those operations on the number line (Carr & Katterns, 1984; Rathmell, 1980; Ernest, 1985). The results of this type of research show that most primary school students do not understand the principles on which the representations of operations of addition and subtraction are based. Pantsidis et al. (2004) and Gagatsis & Elia (2004) state that the number line is a geometrical model that implies continual interchange between a geometrical representation and another arithmetical one. Geometrically, the numbers given on the number line correspond to vectors and to a set of discreet points shown on this line. Arithmetically, the points on the number line are numbered so that measurement of the distance between the points represents the difference between the corresponding numbers. These researchers believe that the simultaneous presence of these two ideas conditions students' success when it comes to solving arithmetical tasks. Janvier (1983) also believes that as a model for representations the number line gives rise to difficulties because numbers have a double representation as they can be positions on the number line or movements on it.

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 249-256. Prague: PME.

The number line is not an obvious model for students and requires a process of instruction in the classroom for its correct use. This conclusion is reflected in works on both positive and negative numbers (Bruno & Martinón, 1996; Bruno & Cabrera, 2005; Gagatsis & Elia, 2004). Students usually commit mistakes both in concept and procedure (Bruno & Cabrera, 2005; Gallardo & Romero, 1999).

The importance of the number line, both as model and mathematical contents, lies in the fact that it is a common representation for all number systems, and is a connecting thread in numerical knowledge. The number line should be treated in a unified way throughout the process of students' familiarisation with the different types of numbers. However, the reality is that its use depends on the individual teacher and sometimes on the curricular plan that is followed, especially regarding textbooks.

One way of noting the importance given to the number line in schools is by analysing its use in textbooks, as these are a sign of the various focuses that knowledge can be subject to. Most studies on textbooks are undertaken through global analysis of the text. Van Dormolen (1986) makes a distinction between *a priori*, a *posteriori* and *a tempo* analyses of learning textbooks. Van Dormolen defines *a priori* analysis as the study of the textbook as a possible means of instruction before its use in the classroom; *a posteriori* analysis compares the results of learning with the text, and *a tempo* analysis studies the way that students and teachers use the textbook during the teaching and learning process. *A posteriori* and *a tempo* analyses lead to intervention in the classroom as they depend on teachers and students. Our research is an *a priori* analysis of textbooks. We study in what way and for which number sets textbooks use the number line throughout primary and secondary school education in relation to the three ideas outlined by Ernest: as mathematical contents (which we call *number concept*), as a model for operations and as a model for ordering.

We pose the following questions: In the ideal case that a student might follow textbooks from one single publisher throughout his or her compulsory schooling, would this student follow the same use of the number line as he or she goes from one year to another? Would this student use the number line for all number systems? And, in what way?

OBJECTIVES AND METHODOLOGY

We carried out a study of the treatment given to the number line in primary schools (6 levels) and secondary schools (4 levels) with regard to three widely sold publishers.

Publisher 1 (P1): Edebé   Publisher 2 (P2): Santillana   Publisher 3 (P3): SM

We analysed 30 books, 10 from each publisher (one book for each year of primary and secondary schooling) as used in the current education system (see years of publication in Appendix 1).

Analysis was carried out by registering each appearance of the number line when the textbooks covered the themes of numbers and operations. The objectives set out in this work were as follows:

- To study for what numbers (whole numbers (N); rational numbers (fractions and decimals) (Q); integer numbers (Z); real numbers (R)) and how often the number line is used in primary and secondary school education.
- To analyse the aspects for which the number line is used: concept, operations or order.
- To differentiate the types of representations. For the *number concept*: points, arrows, direction of number lines (horizontal, vertical or Cartesian axes). For addition and subtraction: points-arrow, three arrows. For the multiplication and division: *repeated additions* or *Cartesian product*.

Number concept

By *number concept* we understand the use of the number line as mathematical content, that is, isolated representations of numbers without any association with operations or order. It is to be expected that this happens when a new number system is introduced, but not only in this instance.We studied whether the representations on the number line are used to "place the number" (that is, to represent the numbers in an isolated way) or as an aid for "estimating". Also, we differentiated between whether the numbers are represented with points or with arrows, and whether the number lines used appear horizontally, vertically or with Cartesian axes.

Addition and subtraction

For addition and subtraction we differentiate whether the representations used are of the *points-arrow* type (see Figure 1) or *three arrows* type (see Figure 2).

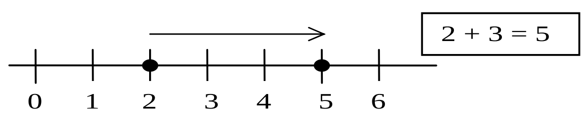

Figure 1: Representation of an addition: *points-arrow*

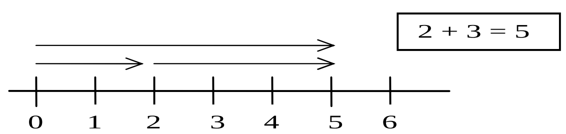

Figure 2: Representation of an addition: *three arrows*

Multiplication

For multiplication we differentiated between whether the representations used are *repeated additions* (see Figure 3) or rather *Cartesian product* (see Figure 4).

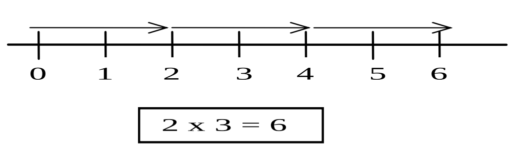

Figure 3: Representation of multiplication: *repeated additions*

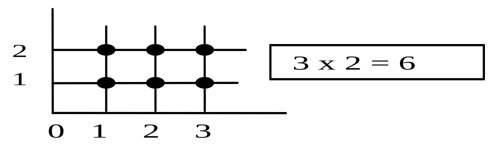

Figure 4: Representation of multiplication: *Cartesian product*

RESULTS

Table 1 shows the frequency with which the number line appears throughout the 10 school years analysed and for each publisher. This frequency is broken down for each of the number aspects. Percentages were calculated with regard to the number of appearances of the number line for each publisher.

There are some differences between the three publishers with respect to the number of number lines appearing over the 10 school years. P2 is noteworthy with a total of 268 appearances of the number line, as opposed to 164 for P3.

The three publishers are similar insofar as the fact that the number concept is the aspect for which the number line is most used, far more than their use of the number line for representing the four operations and order. P2 uses the number concept most often (155), while P1 and P3 achieve higher percentages regarding the number of times they use the number line (73% and 76%, respectively).

Publishers	P1		P2		P3	
	Frequency	%	Frequency	%	Frequency	%
Nº of number line	196		268		164	
Number concept	143	73	155	58	125	76
Addition	23	12	73	27	24	15
Subtraction	5	2	19	7	7	4
Multiplication	2	1	7	3	6	4
Division	0	0	0	0	0	0
Order	23	12	14	5	2	1

Table 1. Frequency and % of appearances of the number line (three publishers)

The percentages of representations of the number line descend notably with regard to operations. The number line is used most for addition, while the percentages are much lower for subtraction and multiplication. None of the three publishers use the number line for division. Therefore, as a model for explaining all of the four operations the number line is not used by any of the three publishers.

Also, the number line is hardly ever used for ordering, P1 being the publisher which most uses it in this way (12%). We noticed that ordering is only ever carried out on horizontal axes and only P1 orders with all the number sets on the number line.

Figure 5 shows the number of number lines used by the publishers for the different number sets. P3 uses the number line more homogeneously for the different types of numbers. Publishers P1 and P2 are similar in their use in this respect; that is, they use the number line more for whole and integer numbers but seldom use it for fractions, decimals and real numbers. That the number line is used so infrequently with real numbers is a result of the fact that it is not used as a model for operations but merely for the number concept.

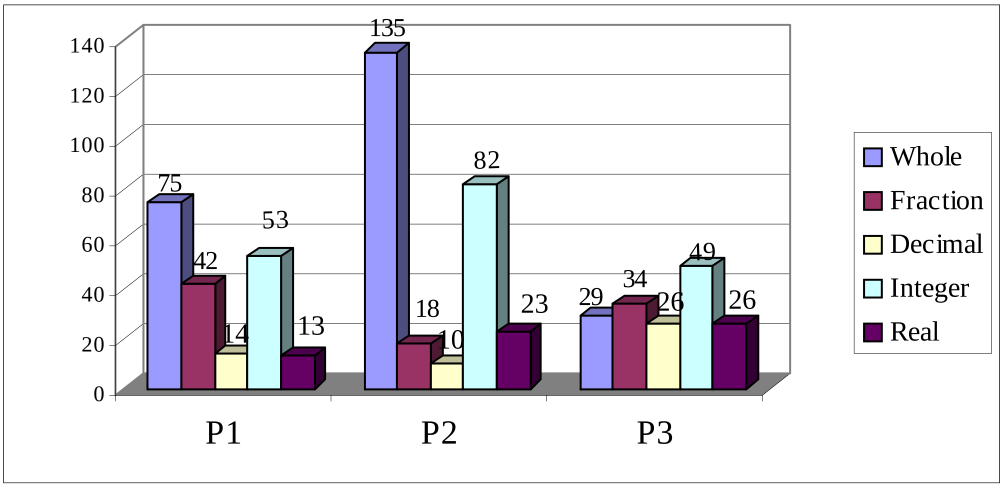

Figure 5. Number of number lines used for types of numbers

Types of representations of the number line

Table 2 shows the characteristics of representations of the number line for the *number concept*. The number line is mainly used by the three publishers to place and situate numbers (there being similar percentages for the three publishers: P1, 92%; P2, 92%, and P3, 90%), while the number line is hardly ever used to help estimate the position of numbers. The number line is mainly used to estimate the position of rational numbers (especially, decimals) or real numbers, while integer numbers are never estimated in this way.

Numbers are represented by points, arrows being scarcely used for representations of isolated numbers. The wide use of points can lead to difficulties in understanding representations of operations, as these also require arrows. With regard to the direction of the number line, the horizontal direction is mainly used, followed by Cartesian axes and finally vertical number lines.

	Funtion of the number line		Representation		Direction		
	Place	Estimate	Point	Vector	Vertical	Horizontal	Axis
P1	92%	8%	92%	8%	3%	85%	12%
P2	92%	8%	96%	4%	10%	75%	15%
P3	92%	10%	94%	6%	6%	81%	13%

Table 2. Types of representation for the number concept

Table 3 shows the types of representations for addition and subtraction operations. *Other representations* cover the number lines are left empty when given to the students for them to make a representation of a given addition or subtraction.

P1 only uses the *points-arrow* representation for these two operations and never the *three arrows* type. P3 uses both types of representation for the two operations. Both of these publishers are coherent with regard to the types of representations. However, P2 uses both representations for additions, but only the *points-arrow* type for the other operation. This leads to a breakdown in knowledge of the number line model.

	P1		P2		P3	
Representation	Addition	Subtrac.	Addition	Subtrac.	Addition	Subtrac.
Points-arrow	18	3	44	16	6	2
Three arrows	0	0	9	0	11	4
Other representation	5	2	20	3	7	1

Table 3. Types of representations of the addition and subtraction (three publishers)

The number line is rarely used for multiplication (see Table 1). The only representation used by all three publishers for multiplication is the representation of *repeated additions*; so *Cartesian product* does not appear even once. It is to be noted that multiplications are only carried out on the number line with whole and integer numbers and that other types of numbers are not used even once for multiplication on the number line.

CONCLUSIONS

The textbooks studied demonstrate the three uses of the number line indicated by Ernest (1985). However, most representations occur when a new number system is introduced; use as a model for basic operations and ordering of numbers is far less. The number line is treated differently by each of the publishers, although there are some similarities between them that lead us to question the coherence of the use of this type of representation.

The publishers fail to make coherent use of the number line throughout students' schooling; in other words, depending on the school year and the author of the school textbook, a student will have a greater or lesser knowledge of the number line. Two of the publishers use the number line with whole numbers but far less so with fractions and decimals, only then to use it again in secondary schools for integer numbers. In general we see that none of the three publishers use the number line with real numbers to any great extent. Perhaps this is because they consider that students of these ages when such types of numbers are introduced have greater powers of abstraction and do not require so many representations of the number line. However, the idea that real numbers complete the number line is one of the basic tenets of number knowledge that many students fail to construct fully (Robinet, 1986).

There are also considerable differences between the various operations. Through their textbooks students can learn to represent additions on the number line but subtractions and multiplications are treated in this way to a lesser extent. Representations of divisions are never used. All of this gives the idea that the number line is an incomplete representation. The three publishers under study fail, then, to provide an overall perspective of the model of the number line. We believe that curricular material should be subject to previous planning as to when and how the number line is introduced throughout the years of compulsory education.

Horizontal directions are used predominantly over vertical ones in the number lines used and the scarce use of the vertical number line is not taken into account when initiating representations with Cartesian axes.

Greater problems can arise owing to the fact that numbers are mainly represented with points. Use of the number line as a model for operations requires that students be familiar with representations of numbers with both points as well as with arrows. Representations with arrows are not obvious for students, as shown in Bruno & Cabrera (2005). It is difficult for students to understand representation of additions and subtractions with point-arrows or three-arrow types if there is no previous work done on representing numbers with both points and arrows.

This *a priori* study carried out on textbooks demonstrates that a more coherent treatment would mean completing the activities set out in the textbooks with other activities that allow students to understand that the number line can be used to represent all number systems and can reflect operations and order. The success of the number line as a model depends on students' familiarity with its use, including the various manifestations of that use.

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Appendix 1. Textbooks analysed. Year of publication

Level	P1: Edebé Barcelona	P2:Santillana Madrid	P3: SM Madrid
1º Primaria	1996	1997	1995
12º Primaria	1996	1997	1995
3º Primaria	1993	1997	1996
4º Primaria	1993	1998	1993
5º Primaria	1994	1998	1994
6º Primaria	1995	1999	1994
1º Secundaria	1996	1996	2002
2º Secundaria	1997	1997	2002
3º Secundaria	1995	1995	2002
4º Secundaria	1996	1995	2002

EDUCATIONAL NEUROSCIENCE: NEW HORIZONS FOR RESEARCH IN MATHEMATICS EDUCATION

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Simon Fraser University

This paper outlines an initiative in mathematics education research that aims to augment qualitative methods of research into mathematical cognition and learning with quantitative methods of psychometrics and psychophysiology. Background and motivation are provided for this initiative, which is coming to be referred to as educational neuroscience. Relations and differences between cognitive and educational neuroscience are discussed. This is followed by descriptions of methods, along with some of the kinds of expertise required for educational researchers to use such methods more effectively. Overall, this paper points to new horizons and opportunities for research in mathematics education, with some associated pitfalls in areas that appear to be particularly ripe for opening up with these new methods.

BACKGROUND AND MOTIVATION

The 26th annual PME conference in Norwich was a watershed moment in my thinking about research in mathematics education, where our field currently seemed to be, where it was heading, and how I would most like to and might best contribute. A colleague recently captured well the impressions and frustrations I had at that time: “Theories are like toothbrushes,” he said, “everyone has their own, and no one wants to use anyone else’s.” I didn’t feel that our various theories necessarily lacked in insight, or were wanting in intrinsic value, or practical applications. My frustrations were grounded in my lack of ability to discern which amongst a plethora of theories would best survive the test of time. I value theoretical speculation, but I seek empirical grounding. I had recently conducted studies in mathematical cognition and learning in which I had sought greater observational grounding through video capture of preservice teachers’ overt behaviour as they engaged in problem solving activities working with computer-based learning environments, simultaneously with video capture of their on-screen activities (Campbell, 2003a). This method of “dynamic tracking” led me to believe I was capturing bona fide “aha! moments” of learners in the very act of conceptual understanding. As I attempted to analyse my observations from a perspective of schema theory (Campbell & Zazkis, 2002), I was deeply frustrated by the speculative nature of my interpretations about what was happening in the minds of these learners. My methods were much more tangible to me than my speculations. From a theoretical perspective of embodied cognition (Campbell, 2003b; Campbell & Dawson, 1995), I was left wishing I had some means to also simultaneously observe and record brain behaviour and various other embodied responses of my participants, such as eye-movement, pupil dilation, heart rate, and galvanic skin response. I wanted better grounding for my speculations.

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So it was, that I was enveloped with an excitement serving to dissolve my despair when I attended talks in Norwich by Wolfgang Schlöglmann (Schlöglmann, 2002), and Judith Sowder (Philipp & Sowder, 2002). Schlöglmann addressed some interesting psychometric results in connection with results from neuroscience regarding affect and cognition, while Sowder discussed the use of eye-tracking methods in tandem with qualitative interview methods. I knew there and then, that if I were to continue with empirical research in mathematics education, this direction, toward more embodied observational methods, was the clear way (for me) to go.

One might wonder why such approaches would have much import or relevance to the study of the minds of learners. More specifically, with regard to brain research, Byrnes (2001) presents three main arguments *against* the relevance of brain research to the psychology of cognition and learning. Roughly, the first argument pertains to a computer analogy, whereby brain is identified with hardware, and mind is identified with software. Accordingly, as educational researchers interested in matters of mind, we can restrict our consideration to the software/mind, independently of the hardware/brain. Byrnes counter-argues that this computational view is "anti-biological," as embodied views of cognition and learning naturally entail. Further, he suggests, the computer analogy notwithstanding, that interdependencies between software and hardware are much greater than commonly supposed. A second argument against the relevance of neuroscience to psychology is that they address different levels of analysis, and as such, they provide very different answers to the same questions. Byrnes illustrates this argument through the different kinds of answers that a physicist, physiologist, and psychologist attending a baseball game would provide to the question "Why did (the pitcher) throw a curve ball?" Educational researchers are typically loath to reduce psychological questions to matters of physiology, let alone physics. Byrnes suggests there are important insights to be gained from studies seeking understandings of interfaces between different levels of analysis, and between psychology and physiology in particular. One need not be a reductionist to maintain that such interfaces must interact in coherent ways. The third argument Byrnes poses for ignoring relationships between psychology and physiology, and neurophysiology in particular, is that too little is known about the brain at this point, and as brain science is in such flux, psychologists should just "forge ahead alone." Byrnes rightly emphasises that psychology and the neurosciences have much to offer each other. This is a key point: psychologists have cognitive and psychometric models that can help guide physiological investigations, and the results of those investigations can help substantiate and refine those models.

Indeed, to paraphrase Byrne, collaborations between cognitive psychologists and neuroscientists have been "forging ahead together," resulting in the vibrant and rapidly expanding new field of cognitive neuroscience. Driven by new imaging methods, coupled with decades of lesion studies, cognitive neuroscience is making great strides in correlating cognitive function with brain and brain behaviour. Research in mathematics education can benefit greatly from these developments.

FROM COGNITIVE TO EDUCATIONAL NEUROSCIENCE

Cognitive psychologists, computer and neuroscientists, psychophysiologists, geneticists, and others, have made substantive advances in understanding mental function, brain structure, and physiological behaviour. Furthermore, substantive progress is also being made in our understanding of the relations between these traditionally diverse and separate realms of disciplined inquiry (Gazzaniga, 2004). These interdisciplinary efforts have been fuelled by at least two major developments — an increasing knowledge base from lesion studies, and advances in brain imaging.

Brain lesions, i.e., neural damage, can result in various ways from developmental abnormalities, impact injuries, surgery, strokes, or disease. Lesions, be they local or widespread within the brain, typically result in altered or compromised mental functioning of those who suffer them. Lesions can have rather bizarre implications for cognitive function, some of which have been widely popularised by authors such as Oliver Sacks (e.g., 1990, 1995). Yet, in many cases, the mental life of those with brain lesions can be remarkably robust and quite adaptable as well (e.g., Sacks, 1989). The bottom line here is that there is a broad and multidimensional range of correlations between local and widespread damage to neural assemblies with specific and general aspects of mental functioning. Although innovations in brain imaging are providing new insights, neuroscientists are still working hard to understand various mechanisms behind such correlations, and psychologists can still find themselves at odds with neuroscientists regarding various fundamental assumptions about the nature of those correlations (e.g., Uttal, 2001).

Brain imaging techniques have opened new windows on brain structure and brain behaviour. From hemodynamic (blood mediated) techniques such as functional Magnetic Resonance Imaging (fMRI) and Positron Emission Tomography (PET), to electromagnetic techniques such as magnetoencephalography (MEG) and electroencephalography (EEG), great strides are being made in our understandings of correlates between brain anatomy, brain behaviour, and mental function (Gazzaniga, 2004). Of particular interest here, as will become more evident below, are brain oscillations in human cortex and cerebellum which are closely, if not causally, associated with mental phenomena characteristic of mathematical thinking ranging from profound insight to deep aversion. Such oscillations are readily detectable by EEG, within certain experimental conditions and thresholds of signal and noise.

Concerned as it is with understanding psychological, computational, neuroscientific, and genetic bases of cognition, cognitive neuroscience is now recognised as a well-established interdisciplinary field of study with its own society and annual meetings. The Cognitive Neuroscience Society (CNS) presents itself as a “a network of scientists and scholars working at the interface of mind, brain, and behavior research” (CNS, 2005). As such, it would seem, then, that cognitive neuroscience shares many areas of common interest with educational researchers, especially with regard to educational psychology and psychometrics.

Yet, at the same time, the CNS sees its members as “engaged in research focused on elucidating the biological underpinnings of mental processes” (ibid.), thereby suggesting that their approach is more fundamentally reductionist than interactionist in nature. Educational researchers such as myself want to be informed by biological mechanisms and processes underlying learning, and we also want to have access to the methods of cognitive neuroscience. As an educational researcher, however, my primary focus is not strictly on biological mechanisms and processes underlying cognition and learning. Rather, it is on the lived experiences of teaching and learning, along with the situational contexts and outcomes of those experiences.

Neuroscience, approached from a “hard” scientific orientation, has the luxury of focusing on various aspects of brain behavior solely in terms of neural structure, mechanisms, processes, and functions. On the other hand, neuroscience approached from a more humanistic orientation has the luxury of not having to be concerned with trying to explain, or explain away, the lived experience of learners solely in terms of biological mechanisms and processes underlying brain behavior.

The above considerations suggest the possibility of an *educational* neuroscience, as a new area of educational research that is both informed by the results of cognitive neuroscience, and has access to the methods of cognitive neuroscience, especially conscripted for the purposes of educational research into the lived experience of embodied cognition and learning. As such, educational neuroscience can be accurately described as a bona fide and full-fledged neurophenomenology (cf., Varela, 1996; Varela & Shear, 1999; Lutz & Thompson, 2003).

Indeed, educational neuroscience is a fast emerging and potentially foundational new area of educational research. A general consensus is emerging on two basic points. First, educational neuroscience should be characterized by soundly reasoned and evidence-based research into ways in which the neurosciences can inform educational practice, and, importantly, also vice versa. Secondly, educational research in cognitive psychology informed by, and informing, cognitive neuroscience should constitute the core of educational neuroscience (e.g, Berninger & Corina, 1998; Bruer, 1997; Geake & Cooper, 2003). New centres and labs to this end have recently opened in England, Germany, the United States, Canada, and elsewhere. This appears to be a very timely development, as there has been increasing emphasis on informing educational practice through advances in the neurosciences (NRC, 2000), along with increasing concern that much educational research, especially of the qualitative ilk, is lacking in a scientific “evidence-based” foundation (NRC, 2002). There is much to be said and much has been gained from qualitative research, and I would not wish to appear neglectful of these facts. Having said that, however, when protocols and data obtained from such research consists of “talk-aloud” reports, often of questionable reliability and validity, and cognitive models of learners’ thinking remain essentially analytical or speculative in nature, one must ask: how robust and generalisable will our qualitative research into subjective mentalities ultimately prove to be? There are methods available for research in mathematics education to address these concerns.

METHODS AND EXPERTISE

What is gained from using methods such as EEG, are means for operationalising the psychological and sociological models educational researchers have traditionally developed for interpreting the mental states and interactions of learners in the course of learning mathematics. This statement holds for qualitative educational researchers and quantitative educational psychometricians alike. It bears emphasis that educational neuroscience can augment traditional qualitative and quantitative studies in cognitive modelling in general, and particularly, in research in mathematics education. McVee, Dunsmore, & Gavelek (2005) argue compellingly that schema theory, the mainstay of cognitive modelling, remains of fundamental relevance to contemporary orientations towards social and cultural theories of learning. Holding fast to a humanistic orientation, educational neuroscience concerns both psychological and sociological dimensions of learning, only now, using methods of cognitive neuroscience, all the while guided by, and yet also serving to test and refine, more traditional educational studies. No comprehensive treatment of these matters can be provided here, just brief indications of some possibilities and pitfalls.

Consider, for example, what kinds of detectable, measurable, and recordable psychophysiological changes are occurring in learners' brains and bodies during mathematical concept formation — that is, when various mental happenings coalesce into pseudo or bona fide conceptual understandings of some aspect of mathematics. For instance, what changes occurred in a student working with Geometer's Sketchpad™ as she came into a realisation that all right triangles inscribed in a circle must pass through the centre of that circle? And what of the student at ease with graphs, who cringed at the sight of mathematical symbols on the screen? (see Campbell, 2003a) Capturing moments such as these in some psychophysiological manner would provide a rich venue for analysis. But how best to go about it?

Of particular interest for educational researchers are EEG and Eye-tracking (ET) systems, and for a variety of reasons. First, relative to most other methods, EEG and ET instrumentation fall within the realm of affordability. Secondly, they are relatively easy and safe to use, involving minimal risk to participants. Thirdly, with sampling rates in the millisecond range, both EEG and ET are well suited for capturing the psychophysiological dynamics of attention and thought in real time. Both methods basically offer temporal resolution at the speed of thought and place fewer spatial constraints on participants than other methods. Furthermore, as evidence of increasing confidence in both the reliability and robustness of these methods, many “turnkey” acquisition and analysis systems are now readily available, placing fewer technical burdens on researchers venturing to use such systems.

Eye-tracking studies have commonly used methods severely limiting head movement (e.g., Hutchinson, 1989). More recently, less constraining, non-intrusive, methods have been developed for remotely measuring eye movements in human-computer interactions (e.g., Sugioka, Ebisawa, & Ohtani, 1996). These remote-based methods have become very reliable, quite robust, and easy to set up (e.g., Ebisawa, 1998).

Most instructional software today can be offered through computer-based environments. Remote-based eye-tracking, therefore, is bound to become an important and well established means for evaluating the design and usage of computer-based mathematics learning environments.

With EEG, cognitive neuroscientists have developed a viable approach to studying complex cognitive phenomena through electromagnetic oscillation of neural assemblies (e.g., Niebur, 2002; Klimesch, 1999). The key to this approach is the notion of event related desynchronization/synchronization (ERD/S) (Pfurtscheller & Aranibar, 1977). In the course of thinking, the working brain produces a fluctuating electromagnetic field that is not random, but rather appears to correlate well within distinct frequency ranges with cognitive function in repeatable and predictable ways.

As previously noted, brain oscillations in human cortex and cerebellum can be reliably correlated with mental phenomena characteristic of mathematical thinking ranging from insight (Jung-Beeman, et al, 2004) to aversion (Hinrichs & Machleidt, 1992). There have been increasing efforts to tease out a "neural code" for such correlates of affect and mentation (such as emotional response, working memory, attention, anxiety, intelligence, cognitive load, problem solving etc.) of synchronic brain behaviour in distinct frequency bands, variously identified as Delta (< 1-4 Hz), Theta (~ 4-8 Hz), Alpha (~ 8-13 Hz), Beta (~ 13-30 Hz), and Gamma (~ 30-60 > Hz).

A prerequisite to understanding and using this method is a basic mathematical understanding of signal processing, such as sampling, aliasing, Nyquist frequencies, and spectral analysis. There are two fundamental pitfalls in signal processing. The first is mistaking noise for signal, and the second is mistakenly eliminating meaningful signals. The first pitfall is typically a matter of faulty interpretation, whereas the second is typically a matter of faulty data acquisition and/or analysis (Campbell, 2004). Gaining an elementary level of expertise in such matters should be relatively straightforward for researchers in mathematics education with a mathematics, physics, or engineering degree. For those researchers in mathematics education with insufficient prerequisite expertise, or who would simply rather not delve into such matters, there is always the option of seeking out cognitive neuroscientists with expertise in EEG, and especially in other, more sophisticated methods as well. While there are many benefits to interdisciplinary collaboration, there are some drawbacks — in that, allocations of lab time and equipment usage would typically not be under the direction or control of the educational researcher.

As powerful as the tools and methods of cognitive neuroscience are, however, and as promising as the prospects for bridging gaps in our understanding regarding the relation between brain and mind may be, some philosophical problems appear as intransigent and recalcitrant as ever. What are we to make of a "mindbrain"? What does such a thing look like? Well, it looks like a brain. And how does it think? Well, it thinks like a mind. Like quicksand, questions like these can quite readily draw the unwary back into classical dualist conundrums (Campbell, 2002).

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VARIABILITY IN A PROBABILITY CONTEXT: DEVELOPING PRESERVICE TEACHERS' UNDERSTANDING

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While recent and ongoing research has begun to reveal ways that precollege students think about variation, little research has been done with the preservice teachers who will eventually serve such students. Specifically, more research is needed to understand what are the conceptions of variation held by elementary preservice teachers (EPSTs), and also how to shape the university courses where those preservice teachers learn. This paper, sharing an excerpt from an exploratory study aimed at EPSTs, describes changes in class responses to a probability task where variation is a key component. Overall, going from before to after a series of instructional interventions, responses reflected a more appropriate sensitivity to the presence of variation.

INTRODUCTION

The purpose of this paper is to report on research aimed at elementary preservice teachers' conceptions of variation. Other research has already begun to illuminate precollege student thinking about variation in several contexts, such as sampling, data and graphs, and probability situations (e.g. Reading & Shaughnessy, 2004; Watson & Moritz, 1999; Shaughnessy & Ciancetta, 2002). However, as the picture begins to get painted about how precollege students reason statistically, the research on how teachers reason about variation remains thin. In particular, there is a paucity of research about how preservice teachers think about variation, or variability in data.

Therefore, doctoral research was undertaken to explore what are the components of a conceptual framework that help characterize elementary preservice teachers' (EPSTs') thinking about variation, how EPSTs' conceptions of variation before an instructional intervention compared to those conceptions after the intervention, and what tasks were useful for examining EPSTs' conceptions of variation in the contexts of sampling, data and graphs, and probability.

The specific research question derived from the larger study and addressed in this paper is: How do EPSTs' responses concerning variation in a probability context compare from before to after an instructional intervention? After describing the conceptual framework and methodology for the study, the results will next be presented, followed by further discussion.

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CONCEPTUAL FRAMEWORK

Three key aspects of understanding variation that governed the overall study focused on how students were *expecting, displaying,* and *interpreting* variation. In dealing with expectations, students need an opportunity prior to conducting statistical investigations to express both what they expect and why. After gathering or analysing data, they can then go back and discuss their a priori expectations in light of their emergent understanding, with particular attention paid to the extent to which their intuitive expectations had emphasized centers to the neglect of spread (Reading & Shaughnessy, 2004). With displays of data, students need to create their own graphs to either highlight or disguise variation, depending on the context of the situation. They also need to evaluate displays and compare distributions in ways that take an aggregate view of data, considering shape and spread in addition to centers (Shaughnessy, Ciancetta, Best, & Canada, 2004). From discussions about probabilistic and statistical situations, students' interpretations of variation emerge as they speculate on both causes and effects of variation and also on ways of influencing variation and expectations. Intuitively, physical causes are often the easiest for students to imagine, but an understanding of randomness as an inherent cause of variation can be a more tenuous concept. Effects of variation were seen as a part of the conceptual framework in terms of the effects on students' perceptions and decisions. For example, if students perceive that the presence of variation precludes any kind of confidence in making inferences, they may likewise be reluctant to make decisions based on data. Influences on expectation and variation generally reflect the sizes of samples used in conducting investigations, or the numbers of trials performed in simulations.

METHODOLOGY

The thirty subjects in the study of EPSTs (24 women, 6 men) were enrolled in a ten-week preservice course at a university in the northwestern United States designed to give prospective teachers a hands-on, activity-based mathematics foundation in geometry and probability and statistics. During the first week of the course, prior to instruction in probability and statistics, subjects took an in-class survey (called a PreSurvey) designed to elicit their understanding on a range of questions about sampling, data and graphs, and probability. The probability question (PreSurvey Q7c) that relates to the current paper concerned six sets of fifty flips of a fair coin. For each of the six sets, students were asked how many times out of the fifty flips the coin might land heads-up. They were also asked why they had chosen the numbers they did. Following the PreSurveys but prior to the class instruction on probability and statistics, individual interviews were conducted with ten subjects to allow further probing of their thinking. After instructional interventions took place in class, a similar PostSurvey question (PostSurvey Q1c) was asked concerning six sets of fifty spins of a fair half-black and half-white spinner. For each of the six sets, students were asked how many times out of the fifty spins the pointer might land on black, and also why they had made the choices they did. Finally, after the PostSurveys the same students who had been earlier interviewed were interviewed once again.

The class interventions were a series of small-group and whole-class activities and simulations that engaged all three contexts of data and graphs, sampling, and probability situations, and were designed to elicit discussion about variation. The two activities comprising the Class Intervention for the context of data and graphs were called “Four Questions” and “Body Measurements”. The “Four Questions” activity offered a good opportunity to discuss both average and spread in data sets, and Sam started the class exploration of statistics in the fifth week by having the entire class gather data from one another in response to four questions: (1) How many pets do you have? (2) How many years have you lived in this city (to nearest half-year)? (3)How many people are in your household? (4) How much change (in coins) do you have today? After graphing the data in different ways, the class had a discussion about levels of detail provided by each type of graph and about what were “typical” values for an individual student or for the whole class. The tension between centers and spread of data was one theme to emerge from the discussion over graphs from the “Four Questions” activity. For the second activity in the class intervention focusing on the context of data and graphs, “Body Measurements”, everyone’s own armspan, height, handspan, head circumference, and pulse rate per minute were recorded. Also, all students in class measured a designated person’s armspan, to gather data from a repeated-measurements experiment. Again, we had a class discussion about the data and graphs for the body measurements, this time focusing more on causes of variation.

In the seventh week of class, the two activities “Known Mixture” and “Unknown Mixture” were done with Sam’s students. Prior to the “Known Mixture”, we started with a general discussion of what samples were, who uses samples, and what samples were good for. Then the following scenario for the Known Mixture Activity was given as a part of a handout: The band at Johnson Middle School has 100 members, 70 females and 30 males. To plan this year’s field trip, the band wants to put together a committee of 10 band members. To be fair, they decide to choose the committee members by putting the names of all the band members in a hat and then they randomly draw out 10 names. The class discussed initial expectations for this scenario, especially focusing on what would happen if the random draw of 10 names were to be repeated thirty times. After students talked about predictions for drawing thirty samples each of size ten, we simulated this activity using chips in a jar. Actual data was gathered and graphed. Then we had a discussion about how the graphs of the predicted data compared to one another, how the graphs of the actual data compared to one another, and also how the predicted graphs compared to the actual graphs. We then made a transition into the second activity in this intervention, the Unknown Mixture. Now we had larger jars, each containing 1000 chips of yellow and green with the same mixture. However, the exact mixture was not known to the class (it was actually 550 yellows and 450 greens). The students were asked to decide in their groups what sample size they wanted to use and how many samples they wanted to draw. Then they were to carry out their plans, do the sampling, graph the results, and make some conjectures about the true mixture in the jar. After the simulation was carried out, we had a class discussion about the different choices made in sampling, the class results, and we tried to forge a class consensus about what the true mixture was.

There were two activities that made up this intervention, “Cereal Boxes” and “The River Crossing Game”. These were chosen specifically because of the probability aspects involved in the activities. Cereal Boxes relies on the use of spinners and River Crossing on the use of dice as random generators, and these two activities were the main ones done in class involving random devices. The first activity for this intervention (Cereal Boxes) actually took place in the first class session of week 2, just before we gathered data for Body Measurements. As explained earlier, there was considerable overlap in the three contexts, and Cereal Boxes is a good example of this overlap. Cereal Boxes is sample-until scenario, assuming that any of five different stickers can be obtained within each box of cereal, and that the five stickers have equal chances of being obtained. The question is about how many boxes would need to be opened to obtain all five stickers, and the situation was simulated by using an equal-area five-region spinner. Cereal Boxes brings together probability, sampling, and data and graphs in a way that highlights variation. The second activity for this intervention (River Crossing Game) involved finding the sum of two dice. Both the Cereal Boxes activity and River Crossing Game are part of the *Math and the Mind’s Eye* curriculum (Shaughnessy & Arcidiacono, 1993). Using two players, each player receives 12 chips to place on their side of a “river”, along spaces marked 1 through 12. After configuring their chips in an initial arrangement along the spaces, players take turns tossing a pair of dice. If either player has any chips on the space showing the total for the dice, one chip can “cross the river” and be removed from the board. The winning player was the first one to remove all the chips on his or her side. As with Cereal Boxes, in the River Crossing Game we made predictions, gathered and graphed data, and discussed results.

The activities in the all the interventions were designed to elicit discussion about variation. For instance, the intervention on data and graphs included different types of graphs and the amounts of variation they showed. Body Measurements got at the ideas behind repeated measurements, as did the muffin weight questions on the PostInterview. The Known and Unknown mixtures had students actually draw chips from a container to experience drawing candies from Large and Small Jars. Cereal Boxes and the River Crossing Game had students use traditional random generators such as spinners and dice to get a sense of what was likely in a probability context. The Fathom software (Finzer, 2001) was used to aid graphical representations and to extend the simulations that the class had already participated in manually.

RESULTS

Both parts of the probability question (*what* students expected and *why*) were taken into consideration for coding purposes, primarily to retain consistency with an analogous rubric derived for a similar question asked in a sampling context (Shaughnessy et. al., 2004). The rubric places a higher value on responses that integrate proportional reasoning as well as variation. The codes and class results for this subquestion are presented in Table 1. Only inappropriate choices for listing *what* was expected (or blank answers) were coded at Level 0. Deciding what would constitute an appropriate choice for the results on six sets of flips or spins involves making a judgment call, and the subcodes used for this subquestion question help identify inappropriate choices as (W)ide, (N)arrow, (H)igh or (L)ow. Of the 30 students enrolled in the class, 27 were in attendance to complete the PreSurvey and 29 completed the PostSurvey.

Code Level	Description of Category	Number of Students (Pre)	Number of Students (Post)
L3	Appropriate choice & Explanation explicitly involves proportional reasoning as well as variation	2 (7.4%)	9 (31.0%)
L2	Appropriate choice & Explanation reflects proportional reasoning or notions of spread	10 (37.0%)	15 (51.7%)
L1	Appropriate choice & Explanation left blank or lacks any specific reasons relating to details of the distribution	4 (14.8%)	3 (10.3%)
L0	Inappropriate choice (Regardless of Explanation) W(ide) = Range > 19, N(arrow) = Range < 2, H(igh) = Choices > 24, L(ow) = Choices < 26	11 (40.7%)	2 (6.9%)

Table 1: Results for PreSurvey Q7c & PostSurvey Q1c

Of the eleven inappropriate PreSurvey responses, one was narrow, one was high, one was low, and four were wide (the remaining were left blank). Of the two inappropriate PostSurvey responses, both were wide. A few of the Level 0 exemplars are:

Alice (Q7c) {25, 25, 25, 25, 25, 25} I don’t see how the chances of getting heads will change if he does more sets of 50 flips

Brita (Q7c) {7, 21, 23, 25, 29, 31} I chose numbers close to 25 because I think with a 50% probability, the results would come out pretty close to 25. I put the oddball 7 in for fun, because there is always that element of chance

Susie (Q1c) {5, 15, 30, 40, 45, 50} It is chance

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Alice narrow response is obviously over-influenced by the expected value, but it seems surprising that more subjects did *not* put all 25s for their choices in the PreSurvey, given results discussed by other researchers (e.g. Shaughnessy et. al., 1999). Brita's choice of 7 is extremely unlikely and makes her overall range too wide, although her upper bound of 31 is plausible. Susie's choices are too extreme at both the upper and lower ends. Level 1 responses had appropriate choices for *what* was expected but the reasons *why* did not specifically reflect distributional thinking:

Carrie	(Q7c) {22, 23, 24, 25, 26, 27} It's usually not the same
Maria	(Q1c) {20, 23, 25, 25, 26, 30} I think he will hit 25/50 one time. The rest of the times, he will be close, but not exactly on. Also I think he will be controlling the way he hits the spinner more on the second day, which accounts for no 23 or 28.

Maria points to *causes* of variation in noting the physical manipulation of the spinner, and other subjects also seemed to indicate that spinners are not viewed as true random devices because the user can ostensibly control outcomes by altering the way the pointer is spun. The Level 2 responses included an indication of reasoning using an average, proportion, or a measure of spread:

Sofia	(Q7c) {20, 20, 24, 25, 26, 27} Because they average to about 25
Sally	(Q7c) {22, 23, 24, 25, 26, 27} They are all close to 25, ½ of 50
Leila	(Q1c) {23, 24, 24, 25, 25, 26} The numbers are pretty close to half or 50%
Rocky	(Q1c) {20, 22, 23, 27, 28, 30} These numbers represent a distribution across a range of likely results

Note how Sally's Level 2 response includes the same choices as in Carrie's Level 1 response shown earlier. However, Sally gives more specificity than Carrie in describing her reasoning, which is proportional in Sally's case. Rocky doesn't include the expected value in his choices, but feels he has given a likely range, and his sophisticated language borders on a Level 3 response. What distinguished the Level 3 responses was an indication of reasoning using *both* centers and spread:

Maya	(Q7c) {23, 24, 25, 25, 25, 26} Because there should be variation around the mean. The average should be 25
Ross	(Q7c) {22, 23, 24, 26, 27, 28} While 25 flips are likely to be heads, in reality some variation is likely. My numbers represent a range that averages 25
Sally	(Q1c) {21, 24, 25, 26, 28, 29} All numbers are 25 or close to 25 (1/2 of # of spins). Not all are 25 in order to account for variation
Daisy	(Q1c) {18, 21, 24, 26, 28, 31} Because they are close to the 50% chance to get 25 hits of black allowing for variation due to random spinning hits. But none of the #'s are too high or too low (far from the 25) which would be hard to hit based on the 50% odds

There were more Level 3 responses in the PostSurvey than in the PreSurvey, and the relative sophistication is apparent as subjects reconcile the tension of having results be close to an average value while also acknowledging the presence of variation. Also, there were more subjects in the PostSurvey than in the PreSurvey whose choices did not include the expected value of 25 (such as Susie, Rocky, and Daisy), suggesting that the class experiences helped counter the natural tendency to pin expectations solely to a theoretical average without an appreciation of the variation in repeated trials. Average class performance on the task also increased, from a mean of 1.11 in the PreSurvey to 2.07 in the PostSurvey.

DISCUSSION

It was clear from subsequent discussions in class that students initially felt uncomfortable venturing a guess for six results, often expressing their perception that it was difficult to guess correctly, or that “anything could happen”. One interesting feature in the results was the low incidence of narrow responses from the EPSTs, which contrasts with research involving 93 high schoolers and a sampling task, whereby almost 26% of responses were narrow (Shaughnessy et. al., 2004). Another interesting feature in responses was the tendency avoid repeating choices when making predictions for multiple trials in the PostSurvey. For example, in giving choices for on the PreSurvey, most students gave some repeated values for their choices, such as James’ (20, 22, 25, 25, 26, 27) or Daisy’s and Emma’s (23, 24, 25, 25, 26, 27).

There is nothing wrong per se with having repeated values in six conjectured results, but most of the PostSurvey choices contained no repeated values. Ross’s choices on the PostSurvey were (22, 23, 24, 26, 27, 28), and he commented how his choices still were “similar, but not identical” and how “there’s no repeats.” Sandy said of her PostSurvey choices (20, 23, 24, 26, 28, 29) that “they could repeat, but I just did a range – from 20 to 30, just to choose... different numbers, but still somewhere in that range”. Ross and Sandy, like many others, also seemed to deliberately avoid including the expected value among their choices in the PostSurvey.

One reasonable hypothesis for why the class as a whole seemed to shift to a greater awareness of variation in results stemming from a probability experiment is that their collective engagement in the activities, simulations, and subsequent class discussions made them more expectant of variability in data. More than half of the students referred directly to class activities or simulations in explaining their thinking, and comments like these are representative:

Dixie: In our class experiments, when I repeated an experiment you’d often have some new variations pop into the picture but the central probability remains the same

Rosie: Because we had the same activity in class, the same concept: The more chances or tries you have more different answers you can get

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Frida: I based it on the activities we have done in class w/ computer program as well as hands-on activities

Sheila: I know this because we saw it on the computer program in class.

CONCLUSION

If a goal is for teachers to provide students with authentic, inquiry-based tasks meant to develop children's reasoning about variation, then a natural step in achieving this goal is to improve teacher training courses. Thus, by discerning components of preservice teachers' reasoning, teacher educators can better design university experiences that promote an understanding of variation for preservice teachers, as well as an understanding on how precollege students come to learn this topic. As research in the field of statistics education advances, one goal is that teacher education can improve not only the subject matter knowledge of EPSTs, but also the pedagogical content knowledge of teaching about variation. Steps toward improved pedagogical content knowledge can certainly be informed by recent research about how precollege students learn. Meanwhile, steps toward improved subject matter knowledge can be informed by a consideration of what are the conceptions of variation held by preservice teachers as they enter university programs. Collective discourse in the class, bolstered by activities and simulations targeted at eliciting conceptions of variation and developing these concepts, hold promise as ways of building EPSTs knowledge while also reflecting the kinds of practice they themselves will want to demonstrate in their own classrooms.

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IMPLEMENTING A REFORM-ORIENTED MATHEMATICS SYLLABUS: A SURVEY OF SECONDARY TEACHERS

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This paper presents part of an on-going study into how secondary teachers in New South Wales are implementing a reform-oriented syllabus. A self-report survey was used in a preliminary investigation to identify how teachers interpreted the aims of the syllabus and the impact it had on their classroom practice. Teachers appeared to understand the reform agenda of the new syllabus as set out in the Working Mathematically strand of the curriculum but were divided in their response to it. Three groups of teachers emerged from the data analysis: Dissenters who oppose any change, Aspirants who want to change but feel unable to do so, and Supporters who welcome the reforms and are willing to adopt them in their classrooms.

INTRODUCTION

Recent years have seen increased and sustained efforts to engage all students in their mathematics learning. The National Council of Teachers of Mathematics [NCTM] in its Standards (NCTM, 2000) articulated and promoted a view of mathematics teaching and learning where students solve real-world problems set in meaningful contexts, communicate their ideas in appropriate mathematical language and symbolism, make conjectures and justify their solutions. In addition, students should have opportunities to make connections between what they do in the classroom and their everyday world, and connect different representations of mathematical concepts so that they view mathematics as an integrated whole rather than a series of seemingly unrelated ideas.

These principles are reflected in the recently developed mathematics syllabus for secondary students in New South Wales [NSW] from the Board of Studies [BOS] (BOS, NSW, 2002). The Working Mathematically strand of the syllabus incorporates five interrelated processes, namely questioning, applying strategies, communicating, reasoning and reflecting. Working mathematically is about students developing the ability to think for themselves as they engage in problem solving activities. In addition, the syllabus document encourages alternative assessment tasks to complement pencil-and-paper tests, outcomes-based reporting procedures and greater use of ICT resources such as spreadsheets and graphics calculators.

THEORETICAL FRAMEWORK

The role of the teacher is critical in successfully implementing the ideals of the reform movement (Reys & Reys, 1997). The teacher needs to choose rich tasks that permit the use of a variety of problem solving strategies and encourage higher-order thinking skills (Spillane & Zeuli, 1999), allow students to work together in small groups to solve problems (Schoen, Cebulla, Finn, & Fi, 2003), and help students make connections among various mathematical representations (Kaput, 1989).

Mathematics teachers are often resistant to change and in spite of repeated calls by reformers to shift from teacher-centred explanations that essentially tell students what to do, there is evidence that little has changed in the classroom (Hollingsworth, Lokan, & McCrae, 2003). Teacher practices are partly due to their beliefs about how students best learn mathematics (Cooney, 1994), the need to prepare students for examinations (Norton, McRobbie, & Cooper, 2002) and their own content and pedagogical knowledge (Brown & Borko, 1992).

Manouchehri and Goodman (1998) studied how 66 middle school mathematics teachers used standards-based curricular materials over a two year period and found a number of obstacles that prevented these teachers from fully implementing the reform ideals. These included the need for more time to plan lessons, an incomplete understanding of some mathematical concepts, inadequate knowledge about how to combine teaching for understanding and mastery of basic skills, and a lack of support and leadership. Perry, Howard and Tracey (1999) investigated teachers' beliefs about teaching and learning mathematics by analysing 233 self-report surveys and interviewing 8 head teachers. They found that many teachers held a traditional view of mathematics as a static body of knowledge that is learned by transmission from teacher to student and that this view of mathematics could limit the impact of a reform agenda. Anderson and Bobis (2004) examined survey responses from 40 teachers and reported that teachers' knowledge and beliefs about the role of working mathematically in learning mathematics is a critical factor in determining how willing they are to implement changes. Anderson and Bobis (2004) also noted that teachers may be confused about what working mathematically might actually look like in the classroom.

This paper explores how secondary mathematics teachers are responding to the introduction of a reform-oriented syllabus. In particular, the research questions for the study include:

- How familiar are teachers with the working mathematically reforms of the new syllabus?
- What kinds of classroom activities proposed in the new syllabus do teachers report using?
- How do teachers respond to the requirements of the reform agenda contained within the new syllabus?

Previous studies suggest that the use of self-report survey instruments can provide useful evidence for determining whether standards-based teaching has been implemented (e.g., Ross, McDougall, Hogaboam-Gray, & LeSage, 2003). However, other studies (e.g., Mayer, 1999) indicate that the reliability of self-report data can be improved if it is supplemented by other means. For this reason, the present study employs a combination of survey, interview and classroom observation data. Only the results from the survey are reported here.

METHOD

A self-report survey organised in three parts was used to determine teachers' views about the new Years 7-10 Mathematics Syllabus (BOS, NSW, 2002) and how they were implementing it in their classrooms. Part A covered background information about the teacher and the school. Part B contained a series of yes/no response items designed to collect preliminary data about the teachers' familiarity with the syllabus (9 items covering activity such as attendance at professional development sessions or conference workshops on the new syllabus) and the kinds of classroom activities and resources they regularly used (13 items such as use of ICT, alternative assessment tasks or groupwork).

Part C consisted of three open-ended questions used to probe teachers' attitudes to the new syllabus and its impact on their classroom practice. Part C first required respondents to explain an important difference in the new syllabus compared to the old. This question was included as another means of identifying teachers' familiarity with the reform objectives of the new syllabus. The second question called for a description of one major impact of the syllabus on the teachers' classroom practice in order to ascertain how they were implementing the syllabus. Finally, teachers could make any additional comments if they wished to do so.

The aim of the survey was to form a preliminary understanding of how teachers were implementing the new syllabus and to recruit teachers for the subsequent interview and observation phases of the study. Approximately 480 secondary schools across NSW were randomly surveyed in 2005, the second year of the implementation of the new syllabus. A statistical investigation of the quantitative data from the first two parts of the survey was conducted, and the free-response questions in Part C were organised into themes.

RESULTS

There were 193 surveys returned. The background information provided in Part A indicated a reasonable distribution of teachers in government and non-government schools across both metropolitan and rural districts. The median secondary mathematics teaching experience of the respondents was 17 years.

The results for Part B were determined by the percentage of teachers who responded in the affirmative to each of the items. It should be noted that the information gathered from the items in Part B formed only a preliminary understanding of the teachers' practices and was used mainly as source material to be explored further in the subsequent interview and observation phases of the study.

The first 9 items dealt with teachers' familiarity with the syllabus document and all teachers indicated a positive response for at least two items. The greatest number of positive responses occurred with statements such as "Read the syllabus support materials" (68%), or "Attended a professional development course on the new syllabus" (66%), and "Read a journal article on the new syllabus" (65%). The remaining 13 items investigated the kinds of classroom activities and resources teachers were using regularly in conjunction with the new syllabus. The items with the highest positive response rates were those referring to classroom use of "Concrete materials" (76%), "Groupwork" (72%), and "Alternative assessment strategies other than traditional tests" (61%). The three items referring to ICT resources (Spreadsheets, Dynamic geometry software, and Other computer software) all scored a positive response rate of below 25%. The reasons for the low scores on these items will be examined in the follow-up interviews.

Part C of the surveys indicated that the respondents were familiar with some of the important elements of the new syllabus but reactions to it varied considerably among the teachers and, in analysing the responses, three distinct groups emerged. Some teachers were opposed to any change and could see no valid reason to abandon traditional teaching techniques they regarded as having served them well in the past. Others expressed a desire to change their practice but were unwilling or unable to do so for a variety of reasons. The third group comprised those who appeared to have embraced the reforms and reported that they were beginning to make changes to the way they taught.

Dissenters

Teachers in this group expressed some antagonism to the new syllabus and were dismissive of the "fancy approaches" of its reform agenda. As one teacher wrote,

> Maths is maths. We are teaching maths and changing the syllabus doesn't change that. Maths hasn't changed in the past 100 years – a new syllabus won't change that fact in any way.

Despite an extensive consultation period prior to the introduction of the syllabus, many dissenters regarded the changes with suspicion because they were "forced on teachers" by people who "should spend more time in the classroom".

> Curriculum changes should include more classroom teachers in the development of new courses – too many theorists and non-teaching personnel make inappropriate or unrealistic changes.

For some older teachers, the revised syllabus was another impost in what seemed an endless series of documents that came too often and allowed little time for reflection or consolidation. These teachers had "seen it all before" and regarded the process of change as essentially "re-inventing the wheel" so if they waited long enough the latest phase would pass and they would be back where they were before it began.

Instead, they preferred to keep to traditional teaching methods which they described as “tried and true”.

The dissenters often focused on the difficulties they foresaw in implementing the syllabus and claimed it would “result in a massive increase in workload”, particularly because it involved writing new teaching programs for each class. For these teachers, the pressure to produce programs, maintain adequate records of each student’s progress and adopt outcomes-based reporting procedures “does not allow for implementation in the spirit of the new syllabus”.

Aspirants

Some teachers used a language of intent in responding to Part C of the survey and wrote comments like “I would like to use technology more” or “I need to move away from the textbook” and “I will try to make my lessons more student-directed”. These teachers were more open to the reform agenda of the new syllabus than the dissenters and were hopeful of seeing change in their classrooms but were unsure about how to achieve it. Aspirants identified obstacles they would need to overcome if the syllabus was to be fully implemented in their classrooms. They acknowledged that the process of change would take each teacher “out of my comfort zone” and involve “more work than chalk and talk”. However, the critical factor for many of these teachers was time.

> [The new syllabus] Has made me want to teach differently and assess in different ways. However, this is not always happening as I don’t seem to have time to think through and research new ideas.

When aspiring teachers felt pressured and did not have enough time to prepare more interesting or creative lessons they were likely to “fall back on old habits” and use the textbook rather than develop new teaching strategies.

The teachers in this group recognised the importance of working mathematically and wanted to find ways of introducing it into their lessons but they had never seen this kind of teaching before and were unsure about how to change their practice. They called for “more support”, “more time for professional development” and “more examples of teaching practices” so they could get a better idea of how working mathematically activities would look in a classroom setting. These teachers were also apprehensive because they imagined that management problems might arise if they allowed students to take a more active role in their learning.

Another impediment to working mathematically identified by the aspirants was the requirement to cover the syllabus content and prepare students for high-stakes external examinations. These teachers saw a syllabus crowded with content that they felt obliged to cover so that students were well prepared for senior studies. Only after this priority had been achieved could the teachers begin to think about working mathematically.

> I wish there was less content to cover. If there was, I could make up and do so many interesting activities and tasks with my students, to encourage them to discover and enjoy the beauty of maths. Unfortunately this is not possible because there's so much to cover by certain times.

Supporters

The third group of teachers wrote enthusiastically about the new syllabus which they described as "a step in the right direction" and "a significant improvement" on the old. They welcomed the key reforms associated with working mathematically, student-centered learning and the use of technology, and described how they had begun implementing these approaches in their lessons. Comments such as "I am not driven by the textbook as much as in the past", or "my teaching has become more creative and includes more practical work" and "I use a more student-centered style with more open-ended investigations" are typical of supporters' responses.

For many of these teachers the introduction of a reform-oriented syllabus was "an opportunity to share ideas" with colleagues and the new syllabus acted as a catalyst for change in their practice because it encouraged them to try approaches they had been "hesitant to use in the past" but were now willing to embrace. For others, it was an affirmation of the teaching practices they had been using for some time and one teacher was pleased as it was "good to know I'm on the right track". In particular, head teachers commented favourably on the new syllabus because it supported their efforts to implement change in their departments. As one head teacher observed, "It's there in the syllabus so that makes it easier for me when I want teachers to try something new".

FURTHER RESEARCH

The evidence gathered here by means of a survey instrument offers a convenient starting point for examining how secondary teachers deal with the introduction of a reform-oriented syllabus. However, it is necessary to confirm these self-report findings by other means and the next step in the research project is to conduct teacher interviews and lesson observations with survey respondents in order to discover precisely how teachers understand and interpret the aims of the working mathematically strand in the new syllabus and how they implement working mathematically in their classroom teaching.

The classification of *dissenters*, *aspirants* and *supporters* provides a useful framework for investigating different reactions to the reform agenda of the new syllabus. The next stage of the research will seek to identify the particular characteristics of each group and consider a number of important questions raised by the results of the present study. For example: Why are dissenters so reluctant to change their teaching style? What can be done to assist aspirants become supporters of the syllabus reforms? Are the changes espoused by supporters reflected in their classroom teaching?

It is also important to identify and describe good examples of working mathematically activities used in the classroom. Sharing illustrations of best practice has the potential to equip teachers better as they implement the syllabus reforms and therefore improve the quality of working mathematically outcomes for students.

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STUDENT'S MODELLING WITH A LATTICE OF CONCEPTIONS IN THE DOMAIN OF LINEAR EQUATIONS AND INEQUATIONS

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We present a student's modeling process in algebra which is situated in the framework of the deployment of the Aplusix system, a learning environment for algebra. The process has two phases. The first phase is a local diagnosis where a student's transformation of an expression A into an expression B is diagnosed with a sequence of rewriting rules. A library of correct and incorrect rules has been built for that purpose. The second phase uses conceptions for modeling students more globally. Conceptions are attributed to students according to a mechanism using the local diagnoses as input. This modeling process has been applied to data (log files) gathered in France and Brazil with 13-16 years old students who used the Aplusix learning environment. The results are described and discussed.

INTRODUCTION

Student's modelling in arithmetic and algebra is an old research field of the AI-ED community (Wenger, 1987). Several frameworks have been adopted, based on elementary items like procedures or rules: issues-and-examples (Burton and Brown, 1979), procedural networks (Brown and Burton, 1978), mal-rules (Sleeman, 1983), intermediate representations (Twidale, 1991). Automatic diagnoses of the students have been realised in these research works. Researchers in mathematic education often model students with non elementary items like conceptions or competences (Artigue, 1991 ; Balacheff and Gaudin, 2002). Diagnoses have often been realised by hand, but a few ones have been obtained automatically, in collaboration with computer science teams (Jean et al., 1999).

Since 2003, we are engaged in a research work[1] devoted to automatic student's modelling in algebra. Our general model is based on rules allowing to interpret the local behaviour of the student, and on conceptions allowing to model the student in a more global way. A detailed description of this study is available in (Nicaud et al. 2005).

For gathering the data, we have used the Aplusix learning environment (Nicaud, 2004) which allows students to freely make calculation steps, as they do in the paper environment, and which records all the students' actions in logs (cf. figure 1). Data have been gathered in France and Brazil.

[1] This work is funded by the programme 'ACI, Ecole et Sciences Cognitives' of the French Ministry of Research.

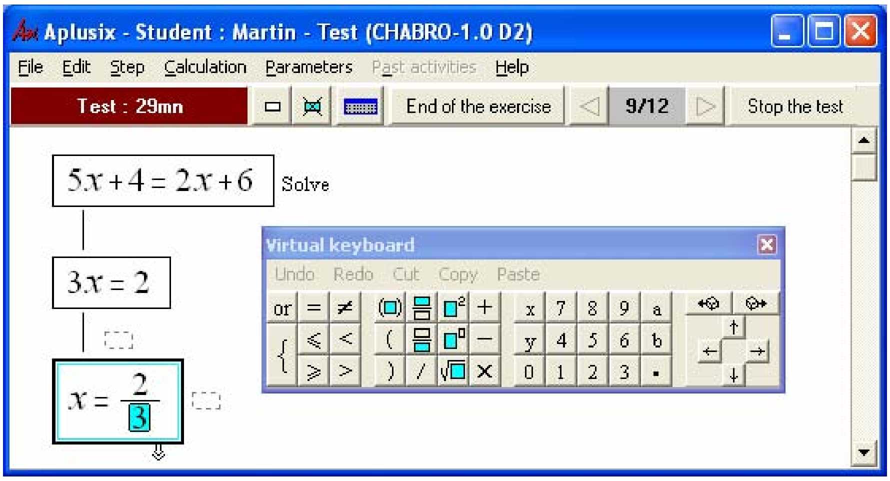

Figure 1. Screen copy of APLUSIX. The student solves an equation using the editor of algebraic expressions. The Test mode where no feedback is provided has been chosen to allow all the possible errors.

The automatic process for computing the students' models has been implemented in a computer program called Anaïs. This program runs off-line and uses the logs produced by Aplusix.

The main goals are: (1) to have a map of possible students' conceptions that can be published in a mathematics education journal, with statistics concerning conceptions of real students from several countries and grades; (2) to inform the teachers of the conceptions of their students after a few training and test sessions with Aplusix in order to help them to take didactical decisions (re-teaching of a part of algebra or choice of new activities) at an individual or at a class level; (3) and to allow Aplusix choosing adequate exercises for favoring a self-correction of misconceptions.

Conceptions

According to (Artigue, 1991), a conception is related to a concept and is characterized by three components: (1) a set of situations which give meaning to the concept; (2) a set of significations (mental images, representations, symbolic expressions); (3) tools (rules, theorems-in-act, algorithms). Our current focus concerns theorems-in-act. An example in algebra is the following: *When a sub-expression is moved from one side to the other side in an (in)equation, its sign is always changed*. Note that this is correct for additive movements (e.g., $x–3 = 5 \rightarrow x = 5+3$) and incorrect for multiplicative movements (e.g., $–3x = 5 \rightarrow x = 5/3$ or $x/2 = 5 \rightarrow x = –2*5$). Such theorem-in-act is not a rewriting rule: it applies to all the movement rules in (in)equations. In the rest of the paper, we will use the term *conception* instead of the *theorem-in-act* (which is a component of a conception) for fluidity reasons. With regard to reference knowledge, a conception has a domain of validity (the domain where it performs correct actions). A *misconception* (term often used in the AI-ED community (Dillenbourg & Self, 1992) is considered here as a conception (theorems-in-act) which is not 100% correct.

MODELLING STUDENT'S CALCULATIONS WITH REWRITING RULES

From the log files containing all the students' actions, we have extracted the steps. In a step, an expression A is transformed into an expression B. A rule diagnosis of a step consists of providing an adequate sequence of correct or incorrect rules that transforms A into B. For that purpose, we have built a library of correct and incorrect rules and design a diagnosis algorithm. Our methodology for building the library of rules combines an epistemic approach, which consists of analyzing the algebraic elements, and a cognitive approach, which consists of studying the students' behaviors.

Linear equations and inequations solving

The main strategy for solving linear equations and inequations consists of expanding both sides, if necessary, then isolating the variable, using theorems which carry out identical operations on both sides. The corresponding rewriting rules are:

Rule 1: $A=B \rightarrow A+C=B+C$ Rule 2: $A=B \rightarrow A-C=B-C$

Rule 3: $A=B \rightarrow AC=BC$ $(C\neq 0)$ Rule 4: $A=B \rightarrow A/C=B/C$ $(C\neq 0)$

Rules 1 and 2 can be combined with reductions, leading to more efficient compiled rules (Anderson J. R., 1983):

Rule 5: $A+C=B \rightarrow A=B-C$ Rule 6: $C=B \rightarrow 0=B-C$,

Rule 7: $A=B+C \rightarrow A-C=B$ Rule 8: $C=B \rightarrow C-B=0$.

In these rules, C is moved from one side of the equation to be placed on the other side: that is why we call them "movement rules", more precisely "additive movement rules". There are 4 additive movement rules.

In the same manner, rule 4 produces:

Rule 9: $AC=B \rightarrow A=B/C$ $(C\neq 0)$ Rule 10: $C=B \rightarrow 1=B/C$ $(C\neq 0)$

We call these rules "multiplicative movement rules". These rules must be augmented by rules in which the left side is a fraction, C being the numerator or a factor of the numerator. In a similar manner, rule 3 introduces multiplicative movement rules for C as a denominator of a fraction. There are 12 movement rules for equations (4 additives and 8 multiplicative). The inequations with $\neq$ have analogous rules.

For the other inequations ($<$ $\leq$ $>$ $\geq$), multiplicative movement rules are duplicated according to the semantic sign of C.

A unique movement rule

We have realized a particular work concerning the movement concept in (in)equations leading to a unique abstract rule that move a sub-expression (the argument of the movement) from one side to the other. This rule is particularized by several features described in table 1. For example, the incorrect transformation $2x-4 < 5 \rightarrow 2x > 5-4$ is represented by a movement with the argument –4 and the vector of features (<, LefToRight, NumToNumerator, InitAdditive, FinAdditive, SignUnchanged, OrientationChanged). This unique movement rule represents 648 correct and incorrect detailed rules. Note that the cardinality does not decrease so much, because the description of an application of the movement rule needs to indicate the features that concern some goal, but the description level is less syntactical.

Our library contains 260 correct and incorrect rules expressed in the SIM language (the rule language used by Aplusix). For example, there are 8 SIM rules for the abstract unique movement rule and a unique SIM rule for the correct *collect like terms* actions.

Dimension	**Possible values**
Symbol of relation	$= \ \neq \ < \ \leq \ > \ \geq$
Horizontal orientation	LeftToRight, RightToLeft
Vertical orientation	NumToNumerator, DenoToNumerator, NumrToDenominator, DenToDenominator
Initial position of argument	InitAdditive, InitMultiplicative
Final position of argument	FinAdditive, FinMultiplicative
Change sign of argument	SynSignChanged, SynSignUnchanged
Change inequality orientation	OrientationChanged, OrientationUnchanged

Table 1. The seven dimensions of the unique abstract movement rule.

The design and diagnosis of conceptions

Design of conceptions. Rule diagnoses are not enough to model students in a usable way. There are too many rules or too many rule features. As a consequence, statistics built with these rules are poor because the elements of the students' conceptions are distributed over too many rules or features.

In order to model students with conceptions, we have studied in detail the rule diagnoses of many students. At the present time, we have worked on the movement concept in (in)equations. We have built a model for this concept containing three aspects: the *sign aspect* (whether the syntactic sign[2] of the argument is changed or not), the *inequality orientation* for inequations (whether the orientation of the inequality is changed or not) and the *operator evolution* (what happens to the operator linking the argument to the (in)equation in the movement). For each aspect,

[2] The syntactic sign is the visible sign of the number or the monomial (it is "–" for –2 and –x).

we have defined conceptions corresponding to many students' behaviors. They are presented in table 2.

Aspect	**Conception name**	**Description**
Sign aspect	CorrectSign	Correct treatment of the argument sign
Sign aspect	AbsoluteValue	Change the argument sign if it is "–"
Sign aspect	SemiAbsoluteValue	Change the argument sign if it is "–" and the argument is multiplicative; or if the argument is additive
Sign aspect	SaveSign	Never change the argument sign
Sign aspect	ChangeSign	Always change the argument sign
Orientation	CorrectOrientation	Correct treatment the inequality orientation
Orientation	UnifiedOrientation	Change the inequality orientation if the argument sign is "–"
Orientation	SaveOrientation	Never change the inequality orientation
Orientation	ChangeOrientation	Always change the inequality orientation
Operator evolution	CorrectOperator	Correct treatment of the operator
Operator evolution	AdditiveOperator	Final position of the argument is always additive
Operator evolution	NumeratorOperator	Final position of the argument is always numerator
Operator evolution	DenominatorOperator	Final position of the argument is always denominator
Operator evolution	SaveOperator	Final operator of the argument is always identical to initial

Table 2. The defined conceptions of the 3 aspects of movement. Only 3 conceptions are correct. The other conceptions produce correct or incorrect calculations depending on the context.

A conception is attributed to a student for a *domain of application* which can be global, or limited to equations, or limited to equations and additive arguments, etc. We have built a lattice of conceptions and domains for representing the main possibilities. A conception is attributed to a student if it is verified by several behaviors and contradicted by none or just a few ones.

Diagnosis of conceptions. In order to automatically diagnose conceptions, we have built the lattice of conceptions introduced above such as the lowest nodes correspond to precise behaviors called Local Behavior Vector (LBV). An example of LBV for the sign aspect is: the movement occurs in an equation; the argument has an additive position; the argument has a "+" sign; the sign of the argument is changed. Such LBVs have three important properties: they are 100% correct or 100% incorrect; each LVB has an opposite LVB (the opposite of the example has the same three first elements and a last element which is "its sign is not changed"); they can be indicated by rules. Having defined this lattice, we have built an algorithm for determining conceptions from the rule diagnosis.

At the end of this process, we obtain a list of conceptions for a student. Each conception has a level in the lattice (from 1 for the global ones to 3 or 4 to the most contextual ones). This level is important: level 1 means a large application field and a real conception; at the opposite, level 3 or 4 means a small application field and not a real conception. Of course, the ideal student has the 3 level 1correct conceptions.

The design and diagnosis of conceptions

We have applied the diagnosis of conceptions to a part of our data: French and Brazilian classes of grades 8, 9 and 10. It produced a description of each student with a list of conceptions, and a summary table containing the number of occurrences of each conception.

Brazilian results. A group of 342 Brazilian students of grade 9 used Aplusix with 20 minutes familiarization and 1 hour in the test mode (where no feedback is given). The modeling process has been applied to the data of the test. The analysis of the distribution of the conceptions shows what follows:

○ Type of exercise: 56% of conceptions concern equations (97% correct and 3% incorrect); 44% of conceptions concern inequations (71% correct and 29% incorrect).

○ Initial position of the argument: 62% of conceptions concern an additive position (95% correct and 5% incorrect); 38% of conceptions concern a multiplicative position (68% correct and 32% incorrect).

Most of the conceptions concern equations with an additive position of the argument. The high rate of correct conceptions cannot be viewed as certitude of a good result because the level of the conceptions has to be taken into account. Actually, we had only 2% of level 1 correct conceptions (e.g., CorrectSign). At the other levels, we have 32% correct conceptions for level 2 (e.g., CorrectSign in equations), and 66% correct conceptions concerning very specific contexts of levels 3 and 4. These results are coherent with hand analysis made for a part of the students and with the general opinion of the teachers of the classes.

The distribution of the conceptions with respect to the aspects of the movement concept is shown in table 3. There are many correct conceptions for *Sign aspect* and *Operator evolution*, but just a few of them are at level 1. There is an important amount of incorrect conceptions at level 1 for *Inequality orientation*. This is coherent with the fact that these students have had many exercises about equations and not many about inequations in the preceding school year.

Level of the conception	Sign aspect		Inequality orientation		Operator evolution	
	Correct	Incorrect	Correct	Incorrect	Correct	Incorrect
1	20	0	0	37	0	0
2	158	0	85	4	158	0
3	129	11	0	0	186	0
4	0	0	0	0	335	9

Table 3. Distribution of the conceptions with respect to the aspects of the movement concept.

Discussion and future work

This work is a significant step towards the achievement of the goals we have presented in the introduction. The obtained results are coherent with opinions of teachers and with analyzes by hand of a part of the data. However, we need to analyze in depth the data that are not captured by the process. For example when we find that a student has 3 conceptions, we have an interesting result, but we would like to have an opinion about the behaviors of this student that do not participate to these 3 conceptions. Some of them may be random behaviors, others rational behaviors not captured by the model.

We aim at covering the domain of algebra for grades 8 to 10. We are currently working on conceptions for expansion and reduction. When this coverage will be achieved, we will be able to produce statistics for various situations (we currently have experimentations in Brazil, France, India, Italy, Vietnam). At this moment, we will be also able to add a new module to Aplusix for the teacher's use. These two objectives have requested to produce conceptions understandable by common teachers, teacher who are not involved in research projects, and who are just willing to teach the curriculum. Of course, pioneer teachers and mathematic education researchers will also be concerned by the results.

Our third goal, consisting of automatically choosing adequate exercises for favoring students' self-correction of misconceptions, will be pursue by an automatic indexation of exercises with conceptions and adequate feedbacks. At the present time, we do not envisage giving access to students to their model. We will think of that when our map of conceptions will be achieved. In algebra, many concepts are *in action*, so it is not sure that the description of conceptions, correct and incorrect, will help students. It may be more efficient to provide good feedbacks in specific situations.

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USING READING AND COLORING TO ENHANCE INCOMPLETE PROVER'S PERFORMANCE IN GEOMETRY PROOF

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The result of a national survey in Taiwan in 2002 shows that there are one third of 9th grade students who have just learnt formal proof in geometry lessons are able to recognize some crucial elements and finish their proof but missing one deductive process in a 2-step unfamiliar question. We named these students as incomplete prover. In this study, we develop a learning strategy named reading and coloring (RC) for helping incomplete provers to complete their proving. This strategy is based on students' cognitive characteristics and modified from regular teaching method. A small group teaching experiment in a traditional teaching style is conducted. The results show that RC strategy is effective in non-visual-disturbed questions and is helpful to retrieve suitable theorem for reasoning.

INTRODUCTION

The context of learning and teaching geometry in Taiwan

The learning context concerning geometric shapes and solids in Taiwan is considerably abundant in the elementary and junior high school. In the elementary school, students learn to recognize shapes, find some properties of angles and sides by manipulative, and apply these properties to calculate perimeter, area and volume of geometric shapes. The textbooks introduce the formal term as the name of shapes and use these terms to formalize the computational process of perimeter, area and volume. In junior high schools, the geometry lessons introduce the properties of parallelogram, rhombus, trapezoid, parallel lines, circle, symmetry figure and congruence and similarity of triangles and quadrangles. The manipulative approach is allowed in geometry properties. After introducing the congruent conditions of two triangles in the second semester of grade 8, the way of verifying a geometry property is basically deductive. In the final chapter of geometry lessons in the first semester of grade 9, the students learn how to construct a formal deductive proof with geometry properties. If we say that inferring a conclusion by one property is one step of deduction, the number of steps of geometry proof task in our lessons is at most three. The teaching style in Taiwan junior high school is basically lecturing. Most of the teachers teach geometry lessons by exposition to about 30 students in one classroom.

Taiwanese students' performance on geometry proof

In December 2002, the National Science Council (NSC) conducted a nation-wide survey to investigate Taiwanese junior high students' competences of mathematical argumentation. The survey asked the grade 9 students, while they had just learnt formal proof in geometry lessons, to construct a proof in a 2-steps unfamiliar question (as Fig 1). Students' proof was analysed and evaluated by the project team members including math educators, mathematicians and school teachers. The results of the national-wide survey showed that there is about one quarter of them can construct acceptable proof. More than one-third of them do not have any responses in this question. And approximately one third of them are able to recognize some crucial elements to prove but missing some deductive process such as missing AC=BC, or missing AB=AC, or missing the concluding step (Lin, Cheng and linfl team, 2003).

A is the center of a circle and AB is a radius.
C is a point on the circumference where the perpendicular bisector of AB crosses the circle.

Write your answer in a way that would get you the best mark you can.

Triangle ABC is always equilateral.

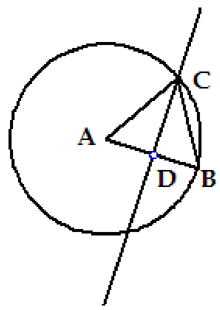


Fig. 1. The 2-steps unfamiliar question in the national-wide survey

The aim of the study

The results of the national-wide survey showed that there are about one third of grade 9 students whose geometry proof performance are missing one necessary deductive step in the 2-steps question when they had just learnt formal geometry proof. If they can finish the missing step then their proof is acceptable. The main purpose of this study is to explore an effective learning strategy in order to enhance incomplete provers to perform one more necessary step towards acceptable proof.

COGNITIVE CHARACTER OF INCOMPLETE PROVERS

Constructing a mathematical argumentation can be seen as a transformation process from initial information to new information with inferring rules (Tabachneck & Simon, 1996). Healy & Hoyles (1998) propose that the process of constructing a valid proof involves sorting out what is given, the properties already known or assumed, from what is to be deduced, and organizing the transformation necessary to infer the second set of properties from the first into a coherent and complete sequence. Duval (2002) propose a two level cognitive features of constructing proof in a multi-steps question. The first level is to process one step of deduction according to the status of premise, conclusion, and theorems to be used. The second level is to change intermediary conclusion into premise successively for the next step of deduction and to organize these deductive steps into a proof. In summary, in order to construct an acceptable proof, the students have to recognize all the necessary information from given conditions and intermediary conclusions, to use necessary information as premises to process deductive steps with applying appropriate geometric property, and to organize their deductive steps into a sequence from given conditions to the wanted conclusion.

The incomplete provers are those who missed one necessary deductive step in multi-steps proof task. According to the cognitive analysis mentioned above, these students may not be able to retrieve a necessary property from known information and/or to keep all intermediary conclusions in attention for the successive deduction.

DESIGNING A LEARNING STRATEGY FOR INCOMPLETE PROVERS

The principle of designing

Designing a learning strategy to help incomplete provers is essentially a kind of diagnostic teaching: it focuses on students with specific cognitive status. Bell(1993) proposed some principles for designing diagnostic teaching, such as the task should be related to students' experience, the teacher should provide operative tool such as immediate feedback of correctness and activities for consolidating new concepts. The incomplete provers may not be able to retrieve a necessary property from known information and/or to keep all intermediary conclusions in attention for the successive deduction. So we have to provide them an operative tool to highlight crucial information and keep all useful information in attention. Bell (1993) also proposed that the task for learning must be familiar to students. This is also important to the school teachers because it is not easy for them to apply a completely new method in regular teaching. We should introduce them a new strategy that is easy to fit into their typical teaching approaches. So the strategy should to be acceptable and effective in the way of exposition and exposition. In conclusion, we design a learning strategy based on two principles. One is the strategy have to provide an operative tool to students for highlight necessary information, the other is the strategy can keep teachers' regular teaching approach.

The study of learning strategy in geometry proving

There are many researches concern about the issue of enhancing students' proof skill. The experiments took place in different society with different strategies. Lin (2005) reviewed the research reports about this issue in PME and founded out that there are many effective strategies proposed in recent years. These strategies include: (1) arranging the context of proof situations (Garuti et al., 1998), (2) interactive discursion to create students' cognitive confliction (Boufi, 1998; Krummheuer, 1996; Douek et al., 1996; Sackur et al, 1996; Antonini, 2000), (3) learning within conjecturing (Miyazaki, 1996; Gardiner, 1994; Hoyles et al, 1995; Sanchez, 1999; Hadas, 1994), (4) questioning as scaffolding (Balnton et al, 1999; Douek et al, 1999), and (5) using metaphors for setting target goals (Sekiguchi,1996). In addition, Reiss (2005) reported a successful experiment that using the 'heuristic worked-out examples' which guided students from clarifying the task settings, exploring the possible inference, to finishing complete proof step by step.

The strategy used in the study

According to the principles of designing in this study, the learning strategy should provide an operative tool to students for highlight necessary information, the other is the strategy can keep teachers' regular teaching approach. By exercising a "thought experiment"(Gravemeijer, 2002) between each possible strategies reported on the literatures and our practical teaching context, one strategy which can be observed in many regular geometry lessons is chosen and modified in this study. The strategy ask students to read the question and draw or construct given conditions and intermediary conclusions on the given figure by coloured pens, where the congruent configurations in same colour. We name it as reading and colouring (RC) strategy. The RC strategy is modified from a typical strategy used in Taiwanese junior high geometry lessons named 'labelling'. The teachers usually use the short segments, arcs or signs to label out the equal sides or angles between subfigures. The RC strategy modify labelling to be coloured drawing.

Byrne (1847) used coloured diagrams and symbols instead of letters to present the formal geometry proof in 'Elements'. He proposed that the coloured diagrams are easier to understand the formal deductive process of Euclidean proof. The function of this kind of visual aids was mentioned in recent studies. Mousavi, Low & Sweller (1995) showed that a suitable visual presentation may integrate all the information necessary in problem-solving task, reduce the cognitive load and increase memory capacity, and then the students may perform better in problem-solving task. Stylianou & Silver (2004) also find out that the difference between experts and novices when solving advanced mathematical problem is the use of visual representation. The experts always construct an elaborate diagram to include all the literal information and thinking on this diagram. Besides these findings from different cultures, Cheng and Lin (2005) showed that the colouring the known information was effective in a highly interactive instruction. This study conducted an interview to 9 incomplete provers. The researcher ask students to colour known information if the student fail to prove. The results showed that the intervention of colouring enhance 12/20 of not-acceptable proof in three different unfamiliar 2-steps items to be acceptable. And the study also reported that colouring might be noneffective if the one coloured angles are included in the other (Cheng and Lin, 2005).

By reviewing the textbook used in our junior high schools, we find out that the proof task with included angles is seldom. Although the effectiveness of RC is limited, we select it in this study because it may be effective to most of proof tasks in our geometry lessons.

STUDY DESIGN

The samples

A questionnaire with four multi-step items were developed (as Fig 2) and tested as pre-test in 2 classes of grade 9 students, while they had just learnt formal geometry proof. The item I2 is designed as a 'disturbing' question that the RC will colour almost of segments and angles. We think that this kind of task setting will produce visual confusion and use this item to examine the limitation of effectiveness of RC. A student is identified as 'incomplete prover' if there is at least one item of incomplete type and at most two items of acceptable type in this test. There are 8 students identified as incomplete prover selected for experimental teaching.

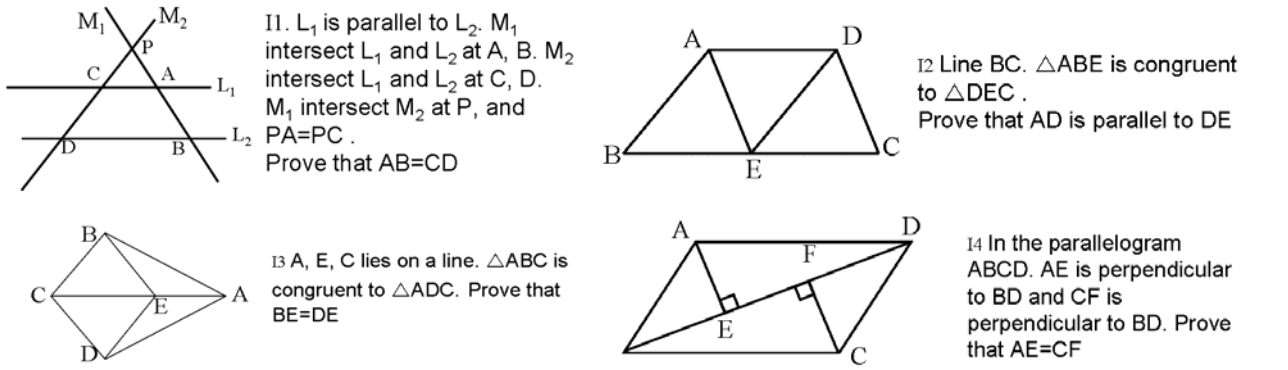


Fig 2. The items of pre-test

The process

In the experiment teaching, the teacher demonstrates the RC process about 10 minutes by a 2-step question different to the pre-test. Then the students are asked to complete a proving task in this way individually without any intervention. The task is to prove the items used in pre-test again. The performance in this task is used as post-test in our data analysis. Two months later, a delay post-test is conduct for these students. The items are all 2-steps unfamiliar questions to these 8 students. In the delay post-test, we provided the coloured pens but did not ask them to use. There is no more geometry lesson during post-test and delay post-test.

THE RESULTS

Effectiveness of RC

We show the results in number of items. There are 32(4×8) items. The change of students' performance on not-acceptable (not-acc) items in pre-test and in the post-test is shown in table 1.

The item / From not-acc	I1 (n=8)	I2* (n=8)	I3 (n=2)	I4 (n=4)	totality (n=22)	totality except I2 (n=14)
to acc.	8	1	2	4	15	14
Enhanced but not-acc	0	2	0	0	2	0
noneffective	0	5	0	0	5	0

Table 1: The change of number of not-acceptable items in pre-test and post-test.

Table 1 shows that the RC strategy can help students to construct acceptable proof in 15/22 items. If we except out the disturbed item I2, 14/14 of not-acceptable items enhance to be acceptable. This shows the RC is effective in non-visual-disturbed setting but is not effective when colouring causes visual disturbance. In item I2, almost of the given conditions are coloured and then disturbed the information in visual. 7/8 of students use the relation of AD=BE or AD=CE. The typical reason is "*the figure become colourful and I can not see anything in this colourful figure, …*" (S9113).

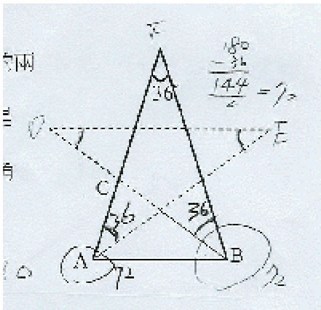
All 8 students finish the delay post-test after two months. The result of delay post-test is shown in Table 2. Table 2 shows that 23/24 of items are acceptable in the delay post-test. Since there is no geometry lesson during this period, the result of delay post-test validates the effectiveness of RC strategy. RC strategy is effective to enhance the incomplete prover's performance of geometry proof towards acceptable.

Item / Type of proof	1	2	3	Totallity
acceptable	8	7	8	23
incomplete	0	1	0	1
improper	0	0	0	0
no response	0	0	0	0

Table 2. The number of proof type in delay post test

Transference from colouring to labelling

Observing students' proving process in the delay post-test, we find out that none of the students coloured the given figures. Instead, all of them labelled: they use one single pen to label the information: short segments, arcs and other recognizable sign to show the correspondent equal sides and angles(as Fig3). Even the coloured pens were provided and allowed to use, the students did not use them.



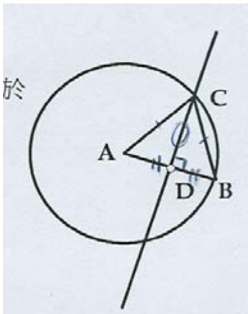


Fig 3. The labels used by students in delay post-test

The interview after the delay post-test shows that the students have some insights about the RC strategy such as:

> '*Coloring is to show the relations between figures. I know it. It is the same to use label. As our teacher did in the lessons.*' (S9103)

> '*Coloring can keep all information on the figure and I can prove the question easily by observing the figure. It is not so important to color it or not.*' (S9108)

> '*I know that our teacher always labels the figure but I dot not. When I color the figure last time, I found out that it is easier to prove because I can see more relations and use correct theorem. And I do everything by observing the figure. I then understood that why the teacher labeled the figure. I then copied the teacher's way. It works. So, it is not necessary to color. To label it is enough.*' (S9411)

The students' responses in the interview show that they can recognize some essential function of RC from the learning activity. Colouring is a way to show the relation between configurations. It is helpful to retrieve suitable theorem for reasoning. It transfers literal information and intermediary conclusions into visual information on figures. It is also helpful to reduce the memory loading when organizing several steps into a proof sequence. The students recognize these functions and they abandon the troublesome colouring for easier labelling actively.

CONCLUSION AND DISCUSSION

RC is an effective but limited strategy for incomplete provers

This study shows that in the non-visual-disturbed multi-steps questions, 14/14 of not-acceptable items change to be acceptable by using RC strategy and 23/24 of items are acceptable in the delay post-test. It shows that RC strategy is an effective strategy to enhance incomplete provers' performance towards acceptable proof in geometry. But the function of RC is limited because RC is helpless if the coloured figure causes visual disturbance. Although this kind of proof task is seldom in our geometry lessons and the limitation of RC seems to be acceptable, more study of investigating the limitation of RC and large samples of experimental teaching are both necessary.

Colouring shows the subfigure associate to a geometry theorem and keeps all information visible and operative

In the delay post-test, none of the students coloured the given figures. Instead, all of them use one single pen to label the information. The students' responses in the interview show that they can recognize the essential function of RC. According to their opinions, colouring transfers literal information and intermediary conclusions into visual information on figures. So RC is helpful to retrieve suitable theorem for reasoning and also helpful to reduce the memory loading when organizing several steps into a proof sequence. The essential functions of RC are showing the subfigure which associate to a geometry theorem and keeping all information visible and operative. The students can recognize these functions and transfer colouring into labelling actively. Duval(2002) proposed that retrieving the suitable theorem is one of the key processes in geometry proof and this process is highly depend on the theorem mapping. This study shows that the theorem is easier to retrieve when the correspondent subfigure is highlight by colouring.

Designing an effective learning strategy in consideration about continuity of teaching is possible

The effectiveness of RC in this study shows that teachers can keep their teaching approach: exposition and demonstration. What different is to change labelling to be colouring in 10 minutes. And the change which do not destruct teachers' traditional approach really help many students. It shows that the continuity between a more effective teaching strategy and the traditional teaching strategy is existent. This kind of teaching strategy may be easier to be accepted by teachers and applied in classroom teaching. As a conclusion, the students may enhance their learning efficiency. This study suggests that we should pay more attention to our own teaching context. In the existent teaching context, modifying some elements of typical teaching ratter transplantation from other context may be a better way to popularize a new teaching strategy and benefit students' learning efficiency.

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ASPECTS OF TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE FOR DECIMALS

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Attention on teachers' pedagogical content knowledge (PCK) has highlighted the many facets of knowledge contributing to teachers' decision-making. In this paper we propose a framework for investigating PCK, and apply it to the content domain of decimal numbers, using data from a group of 14 Grade 5/6 teachers who responded to a questionnaire item and interview. The PCK framework allows us to investigate aspects of the teachers' understanding about decimals and the teaching of decimals. Many of the teachers had well-developed PCK, but there were also some weaknesses.

LITERATURE AND THEORETICAL FRAMEWORK

We begin by reviewing some literature on Pedagogical Content Knowledge (PCK) in order to develop a PCK framework. We then consider issues in decimal numeration that are particularly relevant for PCK, before establishing our research questions.

A framework for Pedagogical Content Knowledge

Teacher knowledge is complex and multifaceted. Shulman (1987) suggests seven categories that might make up a knowledge base for teaching, including knowledge of content, pedagogy, curriculum, students, educational contexts, and the purposes of education. His term *pedagogical content knowledge* describes the mix of content and pedagogy that is the domain of the teacher, involving "an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (Shulman, 1987, p. 8). Elements of PCK include knowledge of models for key ideas, understanding what makes topics easy or hard to learn, and knowledge of common student conceptions (Shulman, 1986). Other important aspects are knowledge of deep and complete explanations, the ability to select appropriate representations to convey key ideas, and an awareness of why confusions and misconceptions are likely to occur (Leinhardt, Putnam, Stein, & Baxter, 1991). Ball (2000) notes the importance of "the capacity to deconstruct one's own knowledge", and to make "critical components … accessible and visible" (p. 245). This skill helps teachers work with students whose understanding is still incomplete, make sense of student thinking, and construct appropriate explanations (Ball, 2000). Ma (1999) describes *Profound Understanding of Fundamental Mathematics* (PUFM) as understanding elementary mathematics with depth, breadth, and thoroughness. Teachers with this understanding make connections among concepts and procedures; approach concepts and problems from multiple perspectives; demonstrate explicit awareness of "simple but powerful basic concepts" (p. 122); and have an understanding of the curriculum as a whole, rather than a knowledge of the parts they are required to teach. Ma links this understanding with student performance. This link is supported by Askew, Brown, Rhodes, Johnson, and Wiliam (1997), who found that a connectionist approach, implying knowledge of the structures and connections within mathematics, was a predictor of higher gains in student learning.

Based on this and other literature, we propose a framework for PCK, shown in Table 1, that makes explicit these facets. The elements of PCK are grouped in three categories. Elements may be "clearly PCK", where pedagogy and content are completely intertwined. Examples include knowledge of teaching strategies for mathematics (such as suitable explanations and activities), knowledge of student thinking (such as knowledge of misconceptions and individuals' learning styles), knowledge of a range of alternative models and representations, and knowledge of resources and curriculum. Other elements are categorised as "content knowledge in a pedagogical context", including the ability to deconstruct knowledge to its key components, awareness of mathematical structure and connections, and PUFM. The third category, "pedagogical knowledge in a content context" covers situations where teaching knowledge is applied to a particular content area, and includes knowledge of strategies for getting and maintaining student focus, and knowledge of classroom techniques. Some elements within the framework have been further subdivided as it seemed appropriate to distinguish, at times, between knowledge demonstrated about a general situation or teaching principle, and an explanation of a more specific activity, task, or student's thinking. In addition, knowledge of *mis*conceptions is a significant component of knowledge of student understanding so we included subcategories addressing this. Table 1 describes the evidence that identifies each aspect of PCK.

It is not suggested that the framework is complete, and it is acknowledged that there may be overlap among the aspects. As an example, in talking about a method for solving a problem a teacher may show procedural knowledge and may also provide evidence of having deconstructed the content into key components. That said, however, the framework provides categories, based on the literature, within which we can investigate PCK. The intention is that the framework can be applied to data from teachers about teaching and content, including actual teaching events, written data, and data from discussions and interviews. This will be tested, in part, by this paper.

Understanding decimals

To apply such a framework to a particular topic, we must know the complexities of a topic from a teaching point of view. Research on decimal understanding (e.g., see Steinle & Stacey, 1998, for a good overview) has provided researchers with a greater understanding of student conceptions, misconceptions, and suitable representations for teaching about decimals. One of the challenges for teaching decimals is to find appropriate representations, and ways of bringing out the key aspects of place value. The volume-based Multibase Arithmetic Blocks (MAB) model, which is used widely for teaching whole number place value, can also be used to model decimal numbers. However, students seem to have difficulty adjusting from regarding the small cube as a unit to regarding some larger component as the unit. The Linear Arithmetic Blocks

PCK Category	Evident when the teacher …
Clearly PCK	
Teaching Strategies – General (explicitly maths related)	Discusses or uses general strategies or approaches for teaching a mathematical concept
Teaching Strategies – Specific (explicitly maths related)	Discusses or uses specific strategies or approaches for teaching a particular mathematical concept or skill
Student Thinking - General (excludes misconceptions)	Discusses or responds to possible student ways of thinking about a concept, or recognises typical levels of understanding
Student Thinking - Specific (excludes misconception)	Identifies a particular student's specific level of understanding or ways of thinking about a concept
Student Thinking - Misconceptions – General	Discusses or addresses typical/likely student misconceptions about a concept
Student Thinking - Misconceptions – Specific	Identifies a particular student as having a specific misconception about a concept
Cognitive Demands of Task	Identifies aspects of the task that affect its complexity
Appropriate and Detailed Representations of Concepts	Describes or demonstrates ways to model or illustrate a concept (can include materials or diagrams)
Knowledge of resources	Discusses/uses resources available to support teaching
Curriculum Knowledge	Discusses how topics fit into the curriculum
Purpose of Content Knowledge	Discusses reasons for content being included in the curriculum or how it might be used
Content Knowledge in a Pedagogical Context	
Profound Understanding of Fundamental Mathematics	Exhibits deep and thorough conceptual understanding of identified aspects of mathematics
Deconstructing Content to Key Components	Identifies critical mathematical components within a concept that are fundamental for understanding and applying that concept
Mathematical Structure and Connections	Makes connections between concepts and topics, including interdependence of concepts
Procedural Knowledge	Displays skills for solving mathematical problems (conceptual understanding need not be evident)
Methods of Solution	Demonstrates a method for solving a mathematical problem
Pedagogical Knowledge in a Content Context	
Goals for Learning – Mathematics Specific	Describes a goal for students' learning directly related to specific mathematics content
Goals for Learning – General	Describes a goal for students' learning not directly related to content
Getting and Maintaining Student Focus	Discusses strategies for engaging students
Classroom Techniques	Discusses generic classroom practices

Table 1: Framework for analysing Pedagogical Content Knowledge.

model (LAB) has recently received emphasis because it highlights the factor-of-10 relationship between consecutive places, and represents decimals as a length, thus facilitating connections to number line representations. There is some evidence for the efficacy of the LAB model. (For more details about MAB and LAB see Stacey, Helme, Archer, & Condon, 2001). Also of relevance to our study are misconceptions such as thinking that the magnitude of a decimal depends on the number of digits after the decimal point. There are two variants of this: “longer-is-larger” and “shorter-is-larger”, the former exemplified by the “whole number thinking” of students who regard the digits after the decimal point as if they are whole numbers. There is also a specific misconception in which only the first two decimal places are considered, either by rounding or truncating, perhaps as a result of associating decimals with money (see Steinle & Stacey, 1998).

Research questions

Two research questions underpin this study. The first concerns what pedagogical content knowledge teachers bring to the topic of decimals, and if this knowledge appears to be adequate. The second, more general question, is to determine whether or not the PCK framework is appropriate for examining teachers’ PCK.

METHODOLOGY

As part of a larger study examining primary (elementary) teachers’ mathematical PCK (see also Chick & Baker, 2005), 14 Australian Grade 5/6 teachers completed a questionnaire about various aspects of teaching mathematics and then undertook a follow-up interview with one or both of the first two authors. The decimal number task that was the focus of this study asked the teachers to decide which of 54.1978, 54.775, 54.8102, 54.9189, and 55.87 is closest to 54.87 and explain why. They were then asked if they would use the same explanation for a student having difficulty, and invited to give an alternative explanation. Teachers completed a written response to the questions in their own time, after which the researchers, having examined each teacher’s written responses, elicited more detail and explanation in an interview. The interview questions were influenced by each teacher’s written answers.

Both the questionnaire and transcribed interview data were then analysed to identify the occurrence of the different types of PCK, as categorised in Table 1. Data were coded by two of the authors and then checked by the remaining authors, and discrepancies resolved. A total of 348 PCK instances were identified and coded. The data were then sorted by PCK type and examined—both quantitatively and qualitatively—to determine the extent of the teachers’ PCK about decimal numbers.

RESULTS

In demonstrating their procedural knowledge for determining which decimal number is closest to a given number, some teachers presented explanations as if for a class, which often gave additional evidence about aspects of their PCK. Not all categories from the framework appeared in the data, and some are combined to save space.

Teaching Strategies. Four teachers spoke generally about their strategies, such as using concrete materials, relating concepts to real life, and organising activities. Twelve teachers outlined more specific strategies, including number ordering and calculator games; using place value charts, number lines and LAB; and emphasising correct language. Two expressed doubt about the approach they would take, and three used inappropriate representations (see "Representations of Concepts" section).

Student Thinking. As the data comprised responses to hypothetical questions, we did not expect much reference to specific students and teachers' own classrooms; however, three teachers referred to their own students. One teacher stated that most of her students felt decimals and fractions caused them the most problems, even after additional support. In contrast, another teacher stated that some of her Year 6 students would be able to do the comparison question, even though they had not yet covered this topic. Ten teachers discussed general elements of student thinking, with five recognising decimals as one of the most challenging areas, and three pointing out that students have problems with place value beyond the hundredths. One teacher also noted that the nature of the ragged decimals made the task harder. Three teachers discussed some student misconceptions regarding decimals, including whole number and longer-is-larger thinking (Steinle & Stacey, 1998), and confusing decimal and negative numbers. Two of these discussed the need to adopt strategies to address such misconceptions. One teacher also mentioned that, when finding the closest number to a target, some students believe that larger numbers can never be the closer number.

Cognitive Demand. Eleven teachers identified aspects of this question that affected its cognitive demand. Most identified the number of decimal places as important, and suggested teaching strategies or task modifications to deal with this. Only one teacher explicitly recognised the ragged decimals as adding complexity. The four teachers who suggested using number lines made little mention of the difficulty of representing hundredths after tenths, and one teacher actually struggled to use the number line correctly. Finally, one teacher discussed the need to modify tasks to cater for students' individual differences.

Representations of Concepts. Of the 14 teachers, 11 discussed representing decimals using concrete and visual aids, although not all were appropriate. Two teachers mentioned the use of LAB to represent decimals, with one teacher emphasising its value for showing thousandths; no one mentioned the use of MAB. Four mentioned number lines, although one teacher had difficulty using it to solve the "closer number" problem. Problematic representations included a pie diagram, and three teachers related decimals to money.

Knowledge of Resources. A range of resources was mentioned by nine teachers, including LAB, hundreds charts or hundreds squares to link decimal ideas to fractions, the classroom text, number cards, a huge number line, flash cards, and flip books that highlight place value.

Curriculum Knowledge. Eight teachers mentioned how decimals fit into the curriculum, with one relying on the classroom text as it contained activities that supported curriculum outcomes. Four teachers stated that they knew the curriculum does not require students to work beyond three decimal places at the Grade 5/6 level, but another admitted that she did not know the curriculum recommendations. One considered subtraction using a calculator to be inappropriate for children; other teachers talked about an emphasis on integrating different sub-strands of the mathematics curriculum. Finally, one teacher felt that her university course did not prepare her well enough for planning mathematics teaching and so she relied heavily on her colleagues for support.

Profound Understanding of Fundamental Mathematics. Eight teachers clearly displayed aspects of PUFM, largely demonstrated by the fluency with which they discussed decimal place value comparison and/or using number lines. One teacher successfully demonstrated that decimals with a different whole number part could be closer to the target. Another gave a detailed explanation of games for enhancing decimal understanding, with appropriate adaptations to suit different levels of knowledge. Unfortunately, 10 teachers—including some who had demonstrated aspects of PUFM—gave responses that indicated problematic understanding of some key concepts or a significant, and evidenced, lack of understanding. Three teachers experienced difficulty or uncertainty with their final answers or explanations. Three teachers, including one who admitted needing assistance from a colleague to complete the task, did not take full account of all four decimal places. These teachers used only two decimal places—possibly with misconceptions about rounding—or money representations. Another teacher repeatedly said "tens of hundredths" for "ten thousandths" and, finally, one relied on using "tricks" to convert decimals to equal fractions but demonstrated only superficial understanding.

Deconstructing Content to Key Components. Twelve teachers demonstrated some deconstruction to key components among their responses. Half clearly stated the importance of place value understanding for decimals. The importance of the factor of 10 in subdividing was emphasised by three teachers: one mentioned counting activities with tenths, another made decimals with LAB emphasising the 10 equal parts, and another discussed subdividing while working with number lines. Three mentioned renaming decimals or reading their names correctly as important.

Mathematical Structure and Connections. Seven teachers made connections among mathematical concepts and topics, most commonly between decimals and fractions. Two teachers discussed the relationship between decimals and the whole number place value system. One teacher was quite explicit in using number lines to make connections among whole numbers, negative numbers, and decimals.

Procedural Knowledge *and* Methods of Solution. All teachers outlined at least two procedures when solving the "closest number" problem, sometimes combining the methods described below. Procedures used by at least half of the teachers included using place value to compare numbers from left to right one column at a time, using a process of elimination to locate possible answers, and using subtraction to find the smallest difference. Less common approaches included equalising the digits after the decimal point to make comparisons easier, using fractions for the subtraction, and using number lines (either drawn or visualised) to look at the distance between numbers. Problematic procedures that were, unfortunately, adequate for this particular problem included thinking of the decimal numbers as money or rounding to one or two decimal places. These appeared to be used by only one or two teachers.

Getting and Maintaining Student Focus. Nine teachers discussed strategies for engaging students, with five describing various decimal ordering games. Another encouraged participation and discussion among her students by creating a huge number line, while others described activities using flash cards, bingo, and LAB.

DISCUSSION AND CONCLUSIONS

There is no doubt that the ideal way to gather data about teachers' PCK would be to observe lessons (e.g., McDonough & Clarke, 2002), but this is not always possible when the study involves many teachers and/or the focus is on a number of mathematical topics. From a methodological point of view, it is pleasing that this simple questionnaire-and-interview task has revealed as much as it has. It should be noted, though, that the research design meant that teachers could resolve any difficulties they might have had with the task, by whatever means they wanted, *before* responding to the questionnaire. One of the teachers admitted that she could not do the comparison problem on her own and so sought help from another teacher. We cannot tell whether this occurred for other teachers, which needs to be borne in mind when interpreting the results. That said, however, the research design was deliberate because in preparing to teach, teachers have the opportunity to revise topics and procedures themselves, and although it might be desirable for all teachers to have PCK instantaneously to hand, it is usually sufficient if at least they can find it when needed. Moreover, knowing when it is needed also demonstrates PCK.

In terms of PCK for decimal numbers, the results are mixed. It is not our purpose here to "measure" PCK for individual teachers, although we can identify teachers with stronger and weaker (and even problematic) PCK. As a group, the study shows that many of the teachers had good knowledge of resources and representations, such as LAB and number lines, although there were also some problems. In particular, the occurrence of "money thinking" or similar among the teachers is of concern. Most teachers recognised that decimal numeration is a challenging topic, but only a few mentioned typical misconceptions or explicit difficulties. It should be noted, however, that the task did not specifically ask for these. On the other hand, most of the teachers were able to recognise what made the given task challenging for students and could suggest suitable adaptations for students of different abilities. The data also revealed the complexity of PUFM; it is not an all-or-nothing phenomenon. For example, some of the teachers spoke accurately about place value, yet struggled in other areas, such as number line representations. Most identified place value as a key underlying principle for decimal understanding, but very few articulated the fundamental factor-of-10 iteration that underpins place value. Finally, on the actual comparison task itself, most—but not all—teachers had no difficulty with it, indicating that in most cases teachers had appropriate content knowledge. Furthermore, they could discuss alternative methods for solving the problem.

The framework seems to provide an appropriate set of lenses through which to examine teachers' PCK. Not all of the categories occurred in this context, but as yet there is not compelling evidence for additional categories. It will be of interest to test the framework in other mathematical contexts and in other situations such as classroom lessons to see if it is adequate.

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COLLABORATIVE ACTION RESEARCH ON IMPLEMENTING INQUIRY-BASED INSTRUCTION IN AN EIGHTH GRADE MATHEMATICS CLASS: AN ALTERNATIVE MODE FOR MATHEMATICS TEACHER PROFESSIONAL DEVELOPMENT

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This paper reports on a collaborative action research exploring an in-service and a pre-service mathematics teachers' professional development with implementing inquiry-based instruction. The research period covered two semesters. The collaboration the two teachers includes: (1) before-class discussion for preparing the lesson plan; (2)in-class teaching and classroom observation; (3) after-class reflection and modification; (4) weekly meeting with a mathematics educator for university based support. Through three phases of exploration, the in-service teacher participant had developed from a novice at inquiry-based instruction to an experienced teacher who had strong confidence and intention to popularise the benefits of the teaching strategy. As for the pre-service teacher participant, he also had gained a deeper understanding of the complex role of a mathematics teacher and had more confidence to conduct inquiry-based teaching on his own. Besides, the mathematics educator played a crucial role as a strong support for the two teacher participants behind the scenes.

INTRODUCTION

During the last quarter century, one of the goals of education that has been taken into account was to foster students to become independent learners (Darling-Hammond, 2000; Feiman-Nemser, 2001). It is particularly true for mathematics education as National Council of Teachers of Mathematics [NCTM] stressed the aim of mathematics education is to enable students to think creatively and flexibly about mathematical concepts and solve mathematical problems with understanding (NCTM, 2000). In order to reach this purpose, teachers need to teach mathematics through engaging students in problem solving, mathematical argumentation and reflective communication. In other words, mathematics teachers should create an environment in which students can get involved in exploring, making conjectures, and justifying their reasoning to mathematical ideas in the classroom. In fact, Calls for instructional reform in mathematics have been accompanied by demands, in many countries, for radical changes in teaching practices (Cobb & McClain, 2001). Taiwan is also in the trend of instructional reform as it is stressed in our new curriculum standards (Ministry of Education Taiwan, 2003).

The term "inquiry-based instruction" in the literature has been closely associated with other teaching methods, such as problem-solving, laboratory and cooperative learning, and discovery instruction. These methods are commonly referred to as inquiry approaches (Mao & Chang, 1999).

National Science Education Standards considers inquiry as the "central strategy for teaching science" (National Research Council [NRC], 1996, p.31). However, lots of documents and research studies indicate that inquiry is not only central to science but also to mathematics instruction (American Association for the Advancement of Science [AAAS], 1993; Jarrett, 1997; NCTM, 1991; NRC, 1996; Wubbels, Korthagen & Broekman, 1997).

When instructional reform is considered, teacher's professional development ought to be taken into account simultaneously. Some neo-Vygotskian studies of human learning and development seem to offer a rather firm foundation for action research of teachers (e.g. Crawford, Gordon, Nicholas & Prosser, 1994; Vygotsky, 1978). Under such perspective, the actions of people and the meanings and purposes that they attach to an activity, the relationships between people and the arena in which they think, feel and act, and the presence of culturally significant artifacts, all become important as determinants of multiple consciousnesses. Therefore, the terms 'teaching' and 'research' have meanings that are experienced and formed through social activity. And also learning is considered as an outcome of both teaching and research activities.

To sum up the above discussion, one of the best approaches of conducting a radical change of teaching practice in mathematics classroom seems to be combining teaching and research together by using inquiry-based instruction as the teaching strategy. On the one hand, as for the in-service teachers, even though they might have the desire to implement action research for making teaching improvement, many practical obstacles, such as technical problems (e.g. collecting data by manipulating the video camera), lack of theoretical backings, ……etc., make them hesitate to move forward.

On the other hand, as for the pre-service teachers, they have more chances to be equipped by the training of new teaching skills and instruction theories from teacher preparation programmes. Nevertheless, they might not receive enough hands-on experience in the mathematics classroom. It seems that there are different dilemmas and constraints of the in-service and pre-service teachers respectively. But, on the contrary, it could be a great opportunity to promote the collaboration between in-service and pre-service teachers since their advantages and disadvantages appear to be complementary. Hence, in this study, it is planned to integrate a senior in-service teacher Chien and a pre-service teacher Ko to conduct alternative collaborative action research on implementing inquiry-based instruction in the eighth-grade mathematics class, intending to make a bridge to bring pre-service and in-service teacher education together.

THEORETICAL FRAMWORK

Collaborative Action Research

Collaborative action research [CAR] is becoming an increasingly recognized field of educational research (Cardelle-Elaqar, 1993; Miller & Pine, 1990). It is a systematic process that helps teachers, administrators, counselors, parents, and students work together to (a) reflect on what is happening and what is significant; (b) take action to make schools and classrooms more powerful places for learning (Gadotti, 1996). The participants could systematically examine their own educational practice using the techniques of research (Caro-Bruce, 2000). In this study, the definition of the alternative collaboration action research refers to the viewpoints of Miller and Pine (1990) and Clift, Veal, Johnson and Holland (1988) of collaborative action research, that CAR is an on-going process of systematic study in which in-service and pre-service teachers work together to examine their own teaching and students' learning through descriptive reporting, purposeful conversation, peer discussion, and critical reflection for the purpose of improving classroom practice. It emphasises on professional development and support for collaboration between the two teachers under the supervision of a mathematics educator.

INQUIRY-BASED MATHEMATICS INSTRUCTION

Borasi and Fonzi (1998) formulate eight major categories of inquiry-based mathematics instruction that are adopted as the framework of the study to enhance the in-service teacher's teaching: (1) <u>M</u>odeling (M). (2) Making <u>e</u>xplicit the <u>p</u>urpose of an inquiry learning experience (EP). (3) <u>S</u>ynthesizing and <u>r</u>eflecting on the results of an inquiry experience (SR). (4) <u>O</u>rchestrating and <u>f</u>acilitating students' inquiry when working in small <u>g</u>roups (OFG). (5) <u>O</u>rchestrating and <u>f</u>acilitating students' inquiry when working <u>i</u>ndividually (OFI). (6) <u>O</u>rchestrating and <u>f</u>acilitating "sharing" sessions where students <u>c</u>ommunicate the results of their inquiries (OFC). (7) <u>O</u>rchestrating and <u>f</u>acilitating students' inquiry when working as a <u>w</u>hole class (OFW). (8) <u>R</u>esponding to the learning <u>n</u>eeds of diverse students in inquiry maths classes (RN). These eight categories of teaching practice for supporting students' inquiry were used as the criteria to examine whether the in-service teacher Chien was able to implement inquiry-based teaching strategy in his own maths classroom.

RESEARCH METHOD AND PROCESS

This study was taken place in Chien's classroom (eighth grade) at an urban junior high school in Taiwan during two semesters from the fall 2004 to the spring 2005. Chien is a senior teacher with 17-year teaching experience. He was studying in a summer master course for in-service teachers since he was willing to learn some new teaching and learning theories in order to improve his own teaching. Ko was pre-service teacher and also a master student majored in mathematics education. When taking the teacher preparation programmes, Ko was nurtured by the relevant theories under constructivist perspective of education. Even though he had stronger theoretical background compares with Chien, he lacked of the abundant practice experience which Chien had. Therefore, the design of the study is that they could complement each other as implementing the collaborative action research. The collaboration between Chien and Ko was: (1) before each maths class (there were four classes a week), they discussed the curriculum design and made necessary modification; (2) during each maths class, Chien led the teaching whilst Ko manipulated the video camera for recording some qualitative data as an objective observer; (3) after each maths class, they reviewed the video and exchanged their opinions for sharing, criticising and reflecting. In addition, they had weekly meetings with Dr. Chin, who is a mathematics educator and offered theoretical support for them. Basically the model of CAR in this study is based on Carr and Kemmis' (1986) four-step cycle of planning, acting, observing, and reflecting.

The data collection for the study included curriculum designs, video-taped classroom observations, student's worksheets, Chien and Ko's reflective journals, and audio-taped meeting discussions. For identifying Chien's quality of inquiry-based instruction, the criteria referred to Borasi and Fonzi's eight categories of recommended teaching practices. Within each category there are some details interpreting the exact content and required techniques of the teaching practice. For example, in the "modeling" category, the recommended contents are: (a) genuinely engaging in an activity as a learner and making one's thought process explicit; (b) directing the class through a new process and then articulating key steps; (c) making explicit a process that someone in the class has illustrated; (d) articulating the key steps of the modeled process in writing (Borasi & Fonzi, op. cit.).

RESULTS AND DISCUSSION

This study is focused on in-service and pre-service teachers' professional development through an alternative CAR mode which can be elaborated as three distinct phases as follows:

Phase 1: Preparing for the inquiry-based instruction

Before conducting inquiry-based instruction, Chien had been using the teaching strategy of cooperative learning in groups for a year. However, he seemed to have the traditional transmission perspective as his students were expected to learn the mathematics contents in its authorised or legitimate forms and he took learners systematically through a set of tasks that lead to mastery of the content. Ko's classroom observation recorded in his reflective journal provided some evidence:

> Chien spent a lot of time preparing today's lesson, assuring his mastery over the content to be presented. He said that the learner is like a 'container' to be filled. (02/10/2004)

Therefore, in this phase, the main task was to help Chien to develop his conception of mathematics inquiry and inquiry teaching strategy. Under this purpose, the weekly meeting with Dr. Chin offered them the opportunity to reach the relevant inquiry theories. In fact, Ko played an rather crucial role as he had more time to study the literatures so that he could share what he learned with Chien which would help Chien get the ideas more quickly. Besides, Chien and Ko started to discuss how to design adequate materials for supporting student's inquiry.

Phase 2: Undertaking the inquiry-based instruction

At the beginning of undertaking the inquiry-based instruction, Chien's great ambition made him urgently apply all the recommended practices to his maths teaching. However, because unfamiliarity of the details of these practices, he could not grasp the essence of inquiry-based teaching strategy yet.

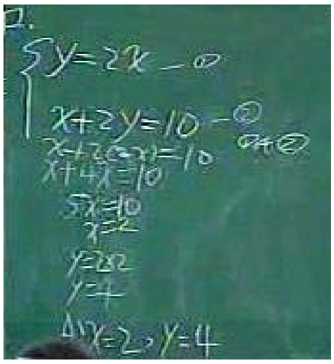
Since Chien had the experience of involving students working in small group when teaching mathematics, he seemed able to apply some of the strategies for advancing student's inquiry and learning in his teaching, such as offering tasks for group work, providing directions for group discussion, giving group members opportunities for individual thinking, requesting each group to present its findings, and providing opportunities for individuals to write personal reports,……etc. Nevertheless, there was still space for Chien to elaborate and orchestrate these skills.

In this phase, there were two major problems happened with Chien. Although Chien would design the worksheet for each lesson under Ko's assistance, He still always faithfully followed the sequence of activities in textbook as he thought it is more convenient and time-saving. However, the contents of the textbook might not be consistent with the spirit of inquiry. Hence some conventional questions, which were not in a proper form for promoting student's inquiry, might appear in the worksheet. The second problem was about Chien's questioning skill that is one of the basic tools for inquire-based instruction. The following conversation and picture offer the episode of a practical situation. (the symbol S_ab means student "b" in group "a").

Chien: Please solve linear equations y=2x and x+2y=10, discuss with your group partners.

(after about two 90 seconds)

Chien: Who can present your answer on the platform? (When no one responded, Chien requested $S_2$5 to write his answer on the blackboard.)



$S_2$5: x+4x=10, 5x=10, x=2, y=2×2, y=4, A: x=2, y=4

Chien: Raise your hand if you and your partner agree with the answer. (after about one minute) Ok, does anybody want to revise his answer?

$S_5$3: He skipped a procedure.

Chien: Ok, Can you ($S_5$3) rectify his solution?

($S_5$3 added "x+2(2x)=10" in the first line)

Chien: Anybody else?

($S_6$2 went on the platform and added "putting (1) in (2)")

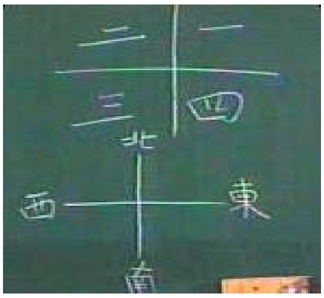
In this conversation, Chien neglected to ask probing questions to provide students opportunities to process information by justifying or explaining their responses – dealing with the "why", "how", and the "based upon what" aspects (Jarrett, 1997; Wood & Turner-Vorbeck, 2001). This kind of questions may promote student's reflective and critical thinking. But it can also be one of the most difficult questioning skills since it requires teachers to respond quickly in the moment (Jacobsen, et al., 1993). Therefore, after several pear discussions between Chien and Ko, as well as weekly meetings with Dr. Chin, it was concluded that Chien had to try to design more adequate worksheets with concretely illustrated lesson plans. That meant the tasks offered to students should be contextualised with their former experiences, and the details of how the lesson might proceed, which should contain teacher's guidance, probing questions, …etc, should be anticipated in advance and included in the lesson plan.

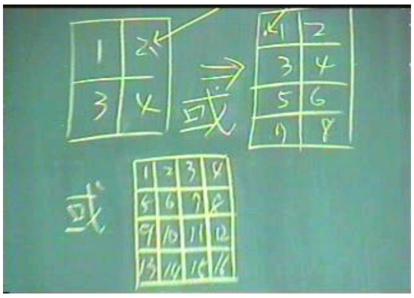
Phase 3: Sophisticating the inquiry-based instruction

Instead of relying heavily on the textbook, Chien started to develop each lesson plan and worksheet intending to motivate his students to explore important mathematical ideas by themselves. In order to design more interesting and adequate tasks, Chien had to refer to more references for improving his own content knowledge. Chien and Ko also spent more time discussing on the curriculum design before each maths class. Students were expected to deal with the given tasks in various ways and to explain their ideas in public. The correctness of the students' answers was no longer under Chien's main concern. Now Chien was concerned about how his teaching practices influenced student's thinking and vice versa, rather than their examination performance.

The following episode illustrates how Chien conducted an inquiry-based activity in this phase.

> Chien: Look! I put a black magnet on the blackboard. How will you describe its position?
> $S_1$1: Use "area"!
> Chien: Good! Can you explain how it works on the blackboard?
> $S_1$1: Divide the blackboard into 4 areas and mark area 1, area 2,…, respectively. For example, I can say the point is in area 1. (the left picture)
> Chien: Great! Can anybody offer a more precise description?
> $S_3$2: Well, you can use more areas, such as 8 areas…16 areas…etc. (the middle picture)
> Chien: OK! Each group take your graph paper to examine his method, or you can find out a better way.
> (Students started to work in groups. Under Chien's guidance, they progressively elaborated their ideas and finally the idea of quadrants was developed by some groups. The right picture shows $S_4$4 explaining their findings on the blackboard.)





From the above episode, it can be noticed that firstly Chien posed an open question to the whole class and invited students to share their ideas. Based on what students had shared, Chien provided suitable tool (graph paper) and direction for the students to develop the target concept on their own. Finally, he invited a student to illustrate his group findings to the other students. Chien expressed that most of the maths teachers, including himself, were like sole purveyors always presented the final answers directly, and students were just passive receptacles. Inquiry-based teaching provided students with opportunities to regain the happiness of finding the answer of a problem on their own which should be the right belonging to them but deprived of by teachers.

By the end of the study, Chien, Ko and Chin tried to make a careful assessment of Chien's teaching to see the progress he made in the last two phases. The criteria were based on Borasi and Fonzi's recommendations of the teaching practices that were illustrated earlier in the paper. Table 1 summarises the results which seem to show that Chien did make satisfactory progress and almost could grasp the essentials of inquiry-based instruction after taking this study with Ko's assistance. ("×" means having no enough evidence to infer; "○" means having enough evidence.)

	M	EP	SR	OFG	OFI	OFC	OFW	RN
Phase 2	○	○	○	○	×	×	×	×
Phase 3	○	○	○	○	○	○	○	○

Table 1: Categories observed during the Phases 2 & 3

CONCLUSION

In-service and pre-service teacher education has long been recognised as two independent systems. This study intended to make a connection between them. According to the research results, both of the in-service and pre-service teachers grew up in the collaborative action research. From a novice at inquiry-based instruction to an experienced teacher willing to popularise the benefits of inquiry-based teaching with colleagues and in in-service teachers workshops, Chien expressed that he could not complete the action research without Ko's support and assistance. As for the pre-service teacher Ko, through the intensive cooperation with Chien, he developed a deeper understand of the complex role of a maths teacher and had more confidence to conduct inquiry-based teaching on his own. In addition, Dr. Chin who offered university based knowledge and support for Chien and Ko played an important role behind the scenes.

The issue of linking theory with practice is always likely to be problematic for teacher education. It is not claimed in any way to have solved this problem, or even to have produced a perfect model for others in this study. We are simply recognising the issue, and exploring through our study. Nevertheless, we feel that we have evidence emerging that the pre-service teacher's initial pedagogical knowledge is unlikely to become the most useful or reliable when it has to form a base for real classroom practice. Hence it seems worthwhile to promote the collaboration between in-service and pre-service teachers for teacher's professional development as this study provides a feasible alternative mode. In addition, this exploration also has led to university based support having a quite crucial influence on the classroom practice, and that this, in turn, has contributed to an improvement in that practice.

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ROUTINE AND NOVEL MATHEMATICAL SOLUTIONS: CENTRAL-COGNITIVE OR PERIPHERAL-AFFECTIVE PARTICIPATION IN MATHEMATICS LEARNING

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The study aimed to identify influential variables that determined students' routine and novel responses to a creative mathematical problem in the TIMSS 2003 study. A special focus was placed on comparisons between the five Asian outperforming countries and the five English-speaking Western ones. The Western countries had more novel solutions than their Asian counterparts. Routine solutions were related to a centrally cognition-oriented context of mathematics learning; novel solutions came from a peripherally affect-oriented context. Asian novel solvers experienced a far more peripheral and negative affective participation in their mathematical learning communities than Western novel solvers.

INTRODUCTION

There has been a growing concern on affective issues in mathematics education (McLeod, 1994; Grootenboer, 2003). Most studies on the affective domain of mathematics education, however, aimed at students' attitudes toward mathematics as a whole (e.g. Di Martino & Zan, 2003; Hannula, 2002). There are relatively fewer studies focusing on the affective issue in relation to mathematical problem-solving, especially to performance on actual mathematical problem-solving tasks (e.g. Schoenfeld, 1989; Whitebread & Chiu, 2004; Chiu, 2004). The study reported in this paper was a further endeavor in the study of 'the intersection of the cognitive and affective domains' (Schoenfeld, 1989, p. 338) in mathematics education. Using student responses to TIMSS 2003 for five Asian performing and five English-speaking Western countries, the present study compared students' routine and novel solutions to a creative mathematical problem in relation to their affective issues and learning contexts.

LITERATURE REVIEW

Mathematical problem-solving is not only a cognitive issue but also an affective one (Mason et al., 1996). With the trend toward constructivist mathematics in the US, McLeod (1994) indicated that students' affective responses to solving non-routine mathematical problems deserve researchers' attention. This is because, in order to solve non-routine problems, students need to invest a significant amount of time and

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intense feelings are likely to occur. Middleton & Spanias (1998) also indicate that realistic mathematical problems tend to 'provide more avenues for failure' (p. 68). This suggests differential affective responses to different types of mathematical problems.

Researchers have identified a variety of types of mathematical problems, such as word and calculation problems (Vermeer et al., 2000), routine and non-routine problems (e.g. McLeod, 1994), well-structured and ill-structured problems (Nitko, 1996), and creative/construction problems (e.g. Schoenfeld, 1989). Relating problem types to affective issues, Schoenfeld found that students tended to solve proof tasks with already-known procedures and to solve construction ones with a trial-and-error approach. Vermeer et al.'s study showed that children made a higher appraisal of computation problems than application problems; they also had higher motivation to solve computation than application problems. Boys tended to have more confidence than girls; girls tended to attribute their failure to ability and the difficulty of tasks. Boys had higher achievement than girls for application problems but there was no difference between boys and girls for computation problems. Boaler's (1998) research indicated that boys had more confidence, enjoyed more in mathematics and had higher achievement for close-ended problems than girls. However, in the process-based, mathematics-project teaching, this gender difference had been eliminated. Whitebread & Chiu's (2004) study identified four distinct patterns of students' affective response in relation to problem types, with involvement students preferring ill-structured/challenging problems, rebellion students preferring well-structured problems, conformity students preferring easy, diverse problems, and avoidance students preferring easy well-structured problems. The above literature suggests that there are complex interactions between students' diverse performances on complex mathematical problems and affective issues, in relation to learning contexts, such as gender, teaching, and learning resources, as was explored in the present study.

METHOD

Participants

The participants were 4,198 13-year-old students in five Asian outperforming countries and five English-speaking Western countries (Table 1 shows the student number of each country); they filled in the student questionnaires and took the Booklet 3 mathematics tests in the TIMSS study in 2003. As there were some missing data for the student variables, these cases were deleted and ended with 3,269 students for the analysis of influential variables (Table 2).

Indicators

Four kinds of indicators were taken from the TIMSS 2003 study. (1) *Country mathematics achievement*. The individual country's mathematics achievements were indicated by their average scale scores of each country, as provided by on Page 34 in the TIMSS 2003 International Mathematics Report. (2) *Student mathematics achievement*. One set of plausible values of student mathematical achievement provided by the TIMSS database were used as the indicators of individual students' overall mathematical achievements. These values were good estimates of parameters of student populations. (3) *Student responses to the focused problem.* The focused problem in the present study was ‘Write a fraction that is less than 4/9’. This was the only ‘creative’ mathematical problem, in the TIMSS 2003 study, which had endless correct answers. In addition, these correct answers can be clearly classified as ‘routine correct solutions’(3/9, 1/3, 4/10, 2/5), ‘novel correct solutions’ (other correct solutions), and ‘incorrect solutions’. (4) *Student self-report results*. There were 16 other variables selected from the self-report TIMSS student questionnaire (Table 3). The variables based on the Likert-scale were derived from factor analysis on the sample of 4,198 students in the present study.

RESULTS

There were significant differences between the Asian and Western countries in the four kinds of mathematical achievements, as presented in Table 1. The Asian countries outperformed the Western counterparts in the aspects of the overall mathematical achievements (t (8)=14.67) and the percentages of routine responses to the focused problem (t (8)=6.49). As a reasonable result, the Asian countries had less incorrect responses than the Western countries (t (8)=-4.66). However, the Asian countries had fewer novel ‘correct’ responses than the Western ones (t (8)=-5.18). This result was a contrary to the above trend, as it is sensible to predict that different samples should have similar distribution patterns of ‘correct answers’ of many kinds. The result also implied that there are different meanings of ‘novel responses to mathematical problem-solving’ between the Asian and Western students.

Discriminant function analyses were performed, in order to determine the influential student variables in defining the three kinds of responses. The analysis was conducted for the three samples of all the students, Asian students and Western students respectively, given the possibility that there were differential meanings of novel responses between the Asian and Western students. As can be seen in Table 2, Function 1s for the three samples can best distinguish routine responses from incorrect ones. This implies that Function 1s address issues of achievements. Function 2s, on the other hand, can distinguish novel responses from the other two types of responses, addressing issues of novelty.

Table 3 reveals that, for all the students, the most significant influential variables that distinguish routine responses from incorrect responses are mathematics achievements and aspiration to higher education. Novel solvers were distinguishable by their test-language use at home, computer-use for learning, computer availability, positive mathematical affect (self-efficacy and value), and low disposition toward schooling. In addition, most novel solvers were boys and had fewer extra mathematics lessons.

Both Asian and Western routine solvers were distinguishable by their high mathematical achievements, self-efficacy in mathematics, and aspiration to higher education (Table 3). The two groups of students, however, were different in their perceptions of mathematical teaching. Transmission- and constructivist-oriented teaching approaches were both positive variables in influencing Asian students’ achievements. Western high-achievers experienced fewer extra mathematical lessons or tutoring.

	Asian countries					Mean	Western countries					Mean	T
	Hong Kong	Ja-pan	Ko-rea	Singa-pore	Tai-wan	(N=5)	Aust-ralia	Eng-land	New Zea-land	Sco tlan d	United States	(N=5)	
Number of students[1]	413	403	439	499	445		403	241	323	297	735		
Country math achievement[2]	586	570	589	605	585	587.0	505	498	494	498	504	499.8	14.67**
Routine responses[3] (%)	62.7	65.3	76.3	76.6	68.3	69.8	48.9	52.7	40.6	39.4	44.9	45.3	6.49**
Novel responses[4] (%)	15	14.9	4.3	5.4	9.0	9.7	24.8	19.5	21.7	22.9	22.7	22.3	-5.18**
Incorrect responses (%)	22.3	19.9	19.4	18	22.7	20.5	26.3	27.8	37.8	37.7	32.4	32.4	-4.66**

[1]For each country shows the number of students who solved the focused problem in Booklet 3, TIMSS 2003.
[2]Math achievements are the average scale scores as indicated in Exhibit 1.1, TIMSS 2003 International Mathematics Report, p. 34.
[3]Routine responses to the focused problem include 3/9, 1/3, 4/10, 2/5.
[4]Novel responses are the correct solutions to the focused problem, except the above routine responses.
** Significant at the .01 level

Table 1: Achievements by countries, and test results

Country group		% of Variance	Wilks' lambda	Routine responses	Novel responses	Incorrect responses	F (ANOVA test)	LSD test results
All students	Function 1	90 %	.78**	.386	-.279	-.773	399.04**	R>N, R >I, N>I
	Function 2	10 %	.97**	-.041	.387	-.137	45.42**	N> R, N>I, R >I
Asian students	Function 1	92 %	.83**	.252	-.170	-.828	163.47**	R >N, R >I, N>I
	Function 2	8 %	.99*	.030	-.394	.070	13.84**	R >N, I>N
Western students	Function 1	92 %	.80**	.424	.030	-.673 [b]	159.24**	R >N, R >I, N>I
	Function 2	8 %	.98*	-.082	.262 [a]	-.070	14.52**	N> R, N>I

* Significant at the .05 level
** Significant at the .01 level

Table 2: Discriminant analysis: Functions at Group means

Compared with their Western counterparts, most Asian novel solvers' parents were from other countries. They has low disposition toward schooling and negative mathematical affects (including self-efficacy, deep approaches, and value). Although

Asian novel solvers had more learning resources (such as speaking test-use languages at home and having computers at home and school), they perceived few opportunities of computer-use for learning purposes. They also experienced ‘freer’ mathematical teaching approaches or strategies: few extra mathematics lessons or tutoring, few transmission-oriented teachings, and little mathematical homework.

Discriminant analysis	All students		Asian students		Western students	
Student variables	Function 1	Function 2	Function 1	Function 2	Function 1	Function2
Affects						
1. Self-efficacy in math	.18	.37 [a]	.39 [a]	.37 [a]	.42 [a]	-.06
2. Deep approach to math	.12	-.07	.21 [a]	.36 [a]	.04	.02
3. Value of math	-.05	.29 [a]	.14	.31 [a]	.02	.28 [a]
4. Disposition to schooling	.02	-.30 [a]	.07	.46 [a]	-.06	-.30 [a]
5. Aspiration to higher education	.25[a]	.01	.30 [a]	-.00	.23 [a]	-.22 [a]
Math teaching approaches						
6. Content-complexity	.02	.19	.18	-.08	-.02	.05
7. Constructivist teaching	-.02	.18	.21[a]	.10	-.06	.05
8. Transmission teaching	.09	.06	.24 [a]	.25 [a]	.06	-.05
Teaching strategies						
9. Math homework	-.00	-.03	.02	.23 [a]	-.01	.06
10.Extra math lessons/tutoring	.17	-.51 [a]	.13	.32 [a]	-.22 [a]	-.10
11.Computer-use for learning	-.09	.21 [a]	.07	.26 [a]	-.07	.35 [a]
Social & economic status						
12. Mother born in the country	.07	-.00	.09	.53 [a]	.03	.38 [a]
13. Father born in the country	.07	-.10	.10	.40 [a]	-.01	.15
14. Test-language use at home	-.05	.52 [a]	.08	-.36 [a]	.11	.38 [a]
15. Computer availability	.14	.46 [a]	.19	-.22 [a]	.23 [a]	.50 [a]
Others						
16. Gender	.12	.20 [a]	.15	-.03	.13	.32 [a]
17. Math Achievement	.97 [a]	.01	.95[a]	.04	.97[a]	-.05

[a] Absolute correlation between the student variable and the Function 1 (or Function 2) equal to or larger than .20

Table 3: Student variables by discriminant analyses

Western novel solvers were distinguishable from their Asian counterparts by their abundant learning resources and high regard for mathematics. Most Western novel solvers were boys and their mothers were natives. They, however, had slightly low disposition toward schooling and low aspiration to higher education. A comparison between the three samples in the influential variables is presented in Table 4.

DISCUSSION

Although some Asian countries outperformed other countries on several international competition tests in mathematics and science, such as TIMSS and PISA, the present study highlighted their significant lack of novel problem-solutions in mathematics, even by analyzing responses to one ‘creative’ mathematical problem in TIMSS 2003 study. While routine solutions were related to their general mathematics achievements and aspiration to higher education, novel solutions was more related to positive affects about mathematics and plentiful learning resources. Novel solvers also appeared to have negative disposition toward schooling and did not rely on extra teaching. This result implies routine and novel mathematical solutions are cultivated by different contexts of mathematics learning, with ‘routine solutions’ determined by mathematical achievement and performance/ability goals, and ‘novel solutions’ determined by strong learning-resource supports, positive mathematical affects, and detached schooling/teaching experiences. In other words, routine solutions were developed by an achievement-centered learning context or centrally cognition-oriented context of mathematics learning; novel solutions came from a self-centered learning context or a peripherally affect-oriented contexts of mathematics learning. Mathematics has long been regarded by students as full of routine problems that can be solved by routine procedures (Schoenfeld, 1989). In order to become the ‘formal member’ of mathematics learning, i.e. becoming a high-achiever, providing routine solutions are likely to be the most significant means. Given the critical aim of high-achievement in mathematics learning, novel solvers are the ‘peripheral’ (Lave & Wenger, 1991, p. 29) members of mathematics ‘classroom communities’ (Hamm & Perry, 2002, p. 135).

The tension between central-cognitive and peripheral-affective participation tends to be stronger for the Asian students than for the Western students. Both Asian and Western routine solvers have high mathematics achievement, self-efficacy in mathematics and high aspiration to higher education. Asian routine solvers benefited from both transmission and constructivist approaches of mathematical teaching, while Western routine solvers experienced few extra mathematics lessons/tutoring in school. In other words, Asian routine solvers were at the very center of mathematical learning, in terms of both achievement orientation and participation in teaching activities; Western routine solvers were at the center of learning, in terms of achievement orientation, but less in terms of teaching.

Asian novel solvers experienced a far peripheral and negative affective participation in their mathematical learning communities. The only positive factor for Asian novel solutions was learning resources of test-language use and computer availability. Asian novel solvers’ parents were not natives; they had various kinds of negative affective responses to schooling and mathematics; they experienced loose mathematical teaching. This picture is a mirror image of their Asian routine solvers’, a rather positive one. On the other hand, except for low disposition toward schooling and low educational aspiration, Western novel solvers appeared to possess advantages of strong support for independent learning from abundant learning resources, high regard for mathematics, and native mothers. The narrowly achievement goals fail to be their focus, Western novel solvers had positive or no less social/economic and affective support. This might encourage them to create their own novel/unique solutions for the sake of their own and mathematics. The differential achievements between Western and Asian novel solutions on the international competition test, or later achievement of mathematics expertise, are likely to be explained by the differences between the positive learning contexts of Western novel solvers and the negative ones of the Asian solvers.

Student group	Influential variables of ***routine*** solving	Influential variables of ***novel*** solving
All students	(+)Math achievement (+)Aspirational affect	(+) Learning resources (language, computer use/availability) (+) Math affect (self-efficacy, value) (-) Schooling affect (-) Teaching (extra math) (+) Boy
Asian students	(+)Math achievement (+)Math affect (self-efficacy) (+)Aspirational affect (+)Teaching (transmission, constructivist)	(-) Nationality (mother, father) (-) Schooling affect (-) Math affect (self-efficacy, deep approach, value) (+) Learning resources (language, computer availability) (-) computer use (-) Teaching (extra math, transmission, math homework)
Western students	(+)Math achievement (+)Math affect (self-efficacy) (+)Aspirational affect (-)Teaching (extra math)	(+) Learning resources (computer availability, language, computer use) (+) Math affect (value) (+) Nationality (mother) (+) Boy (-) Schooling affect (-) Aspirational affect

Table 4: Student groups by influential variables of routine and novel solving

Although some studies have shown that the difference between mathematics achievements of boys and girls are minimal and only in the top bands of 10% to 20% (e.g. Askew & Wiliam, 1995), the present study revealed that boys tended to produce novel solutions more than girls, especially in Western countries. In comparison with Vermeer et al.'s (2000) study, which indicated that boys tended to be better at complex 'application' than girls, there appears a need to study in depth gender differences in mathematics achievements in relation to problem types and affective issues.

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THE ROLE OF SELF-GENERATED PROBLEM POSING IN MATHEMATICS EXPLORATION

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The purpose of the study was to examine the role that problem posing plays in the solution of open-ended mathematics problems. In this case study, subjects were interviewed as they solved open-ended problems. Drawing from episodes of two students, the analysis explained solvers' solutions in terms of varying degrees of problem posing and solving that co-evolved in the course of their on-going solution activity. The results contribute to the research of how solvers use problem posing to build up their mathematical knowledge.

INTRODUCTION

The mathematics education community has asserted that observation, experiment, discovery, and conjecture are important processes in the practice of mathematics that must be included in our school mathematics curriculum (NCTM, 1989, 2000). For example, the National Council of Teachers of Mathematics has advocated that "the essence of studying mathematics is exploring, conjecturing, examining and testing, all aspects of problem solving. Students should be given opportunities to formulate problems on their own and to create new problems by modifying the conditions of given problems" (NCTM, 1989, p.95). Although studies have analyzed the reasoning processes of students while they solve well-structured problems, studies are needed that examine the processes that students use as they solve open-ended problems in which some aspect of the task is unspecified and must be re-formulated by the solver.

Advocates of problem posing cite several reasons for its inclusion in the mathematics curriculum. Problem-posing activities may help lessen students' anxiety and foster positive dispositions towards mathematics (Brown & Walter, 1993). Problem posing involves self-generated activity and thus may stimulate the students' overall abilities to mathematize and develop understanding within new situations (NCTM, 2000; Schoenfeld, 1992; Silver, 1994). Finally, there is optimism that having students engage in problem posing may help them become better problem solvers. As a result, researchers have called for studies to investigate the various aspects of problem posing. Several researchers have focused on identifying possible links between problem posing and problem solving with the view that having students engage in problem posing may help them become better problem solvers (Cai & Hwang, 2002; English, 1997; Silver & Cai, 1996). For example, Silver and Cai (1996) analyzed the problem posing of middle school students, asking them to pose questions based on an automobile driving situation. The researchers found that good problem solvers generated more mathematically complex problems than did less successful problem solvers. These findings were confirmed and extended in a recent cross-national study (Cai & Hwang, 2002).

While these studies suggested possible links between problem posing and solving, the researchers did not probe the nature of the interconnections. In a recent study, Cai and Cifarelli (2005) examined how two college students posed mathematical problems as they explored an open-ended problem situation. This study took the dynamic view that problem posing evolves from the solver's attempts to make sense of on-going results throughout the solution of the problem and found that students used problem posing throughout their problem solving; their on-going reflections on their solution of the problem often suggested to them additional questions and problems to formulate, reflect on, and solve. This continual pushing forward of problem formulation followed by problem solution and verification, constituted a problem posing-solving chain of mathematical explorations for the solvers. While the study indicated the importance of problem posing in problem solving, several unanswered questions remain. For example, does the general structure of the posing-solving chains always follow the phases as suggested by Cai and Cifarelli (2005) even as different tasks are considered? In addition, none of the studies cited here considered individual differences of solvers' reasoning and how they may impact upon the different problems that solvers pose. Hence, studies are needed that consider different ways that solvers make sense of situations and the role that informal actions have on the solvers' problem posing.

The purpose of the current study was to examine problem posing in the following contexts: 1. The role of posing in the overall structure of solution activity to determine if the findings of our earlier study (Cai & Cifarelli, 2005) could be confirmed and extended to other tasks; and 2. The role of posing as a meaning-making process that helps solvers develop increased understanding of the problem situation. Of interest here is to examine the informal reasoning that students might employ as they pose problems.

METHODOLOGY

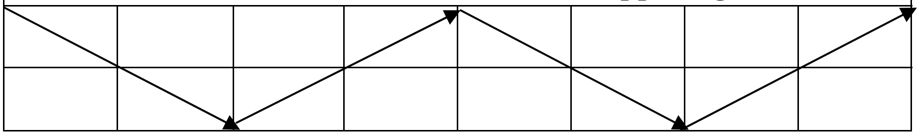
The subjects were two secondary math education majors, Sarah and Gavin. Our studies suggest that observing college students solving problems can be an effective way of modeling problem solving processes (Cai & Cifarelli, 2005; Cifarelli & Cai, 2005). The students worked individually and were interviewed as they solved a Billiard Ball task and a Number Array task; interviews were videotaped and transcribed for the analysis.

<u>Billiard Ball task</u>. The students developed mathematical relationships within a computer microworld, BOUNCE, that simulates the path of a billiard ball on a pool table. Once numbers for the height and width of the table are entered, the ball starts in the upper left-hand corner and travels at an angle of 45 degrees. The number of bounces (B) includes the initial contact of the 'pool stick' to set the ball in motion followed by 3 'banks' and a final hit when the ball drops in the upper right pocket.

The path length (L) is the number of squares traversed by the ball. For a 2X8 table, the number of bounces is 5 and path length is 8 (Figure 1). BOUNCE will tabulate and maintain the results of all cases that are investigated.

Figure 1: Billiard Ball Table for Dimension 2 X 8

H=2, W=8, B=5, L=8, Ball finished in upper right corner



Number Array task. The Number Array task requires subjects to generate mathematical relationships from a square array of integers (Figure 2).

Figure 2: Number Array task

The following table was produced in with a certain rule and has many rich relationships. Study the arrangement of numbers in the table and find as many relationships as possible among the numbers.

Number Array Task									
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**
3	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**
4	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**
5	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**
6	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**
7	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**
8	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**
9	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**
10	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**

Analysis. The data included written transcripts of the videos, the researchers' field notes, and the subjects' written work. The protocols were parsed into episodes of activity and transition points between episodes (Schoenfeld, 1992). Episodes consist of a set of the solver's individual actions, oriented towards a goal. Transitions identify activity where the solver has appeared to change his/her goals and purposes. Since we hypothesized that problem solving in open-ended tasks involves varying degrees of problem posing and solving, transitions are indicators that the solver may be making conceptual progress towards a solution by altering his/her view of the problem.

RESULTS

By inputting values for the table's dimensions (h and w) and viewing the corresponding path of the ball, the solvers explored their evolving ideas about relationships involving table dimensions (of height h and width w) and the corresponding path of the ball. For example, both students generated the relationship that Bounces= ($h+w$) if h and w have no common factors. However, only Sarah was able to generate the more advanced relationship that the number bounces could be found by considering the ratio of the dimensions: if $h/w=p/q$ in reduced form, then Bounces=($p+q$). Table 1 contains the total number of relationships that each student generated from his/her solution activity.

The data in Table 1 indicate that Sarah explored a greater range and total number of relationships than did Gavin. While Table 1 gives an overall sense of each solver's breadth of mathematical exploration, we will draw from episodes of each student's activity to illustrate and explain how these explorations evolved and how problem posing helped extend conceptual boundaries for the solvers in their problem solving.

Table 1: Mathematical Relationships Generated by Solvers (Billiard Ball Task)	Sarah	Gavin
Relationships between the number of bounces and table dimensions	7	6
Relationships between the length of path and table dimensions	4	0
Total	11	6
Mathematical Relationships Generated by Solvers (Number Array Task)		
Relationships about the arrangement of the numbers	4	4
Relationships about the sums of the numbers	7	2
Relationships about the products of the numbers	3	2
Relationships about number sequences	5	0
Total	19	8

Problem posing and solving for the Billiard Ball task. Sarah's approach was to consider an initial idea about a relationship (initial posing), explore the idea by carrying out specific cases on the computer (achievement of goals) and then reflect on the results to consider new questions to explore (problem re-formulation). She continually monitored her evolving ideas and invited new questions whenever the results of her actions did not agree with her current ideas and conjectures. Sarah first used the computer to generate some data. From these, Sarah developed a solution for the number of bounces (B=h+w) and the path length (L=h•w) that appeared to work for all but one of the cases.

> ***Sarah***: I should be able to figure bounces and length based on dimensions. ***(she generates a table of values and reflects on the following cases: 8x2, B=5, L=8; 5x3, B=8, L=15, 7x3, B=10, L=21; 8x1, B=9, L=8; 8x3, B=11, L=24). (reflection)*** Ah, for all but 8x2, B is the sum of m and n. What is different about that one? So B=m+n, L=mxn, but neither is true for that one (8x2)!

Sarah realized the limitation of her solution for the 8x2 table and re-structured her explorations accordingly:

> ***Sarah***: How about <u>when both are odd, it seems to work</u> ***(points to cases 7x3, 5x3 in table)*** or when one is even and one is odd ***(she points to 8x1 , 8x3 cases in table)***, it works. Let's try <u>both even</u>. So 8x6 ***(she generates an 8x6 table)***. Ah, so Bounces is … not 14 but 7, so – it looks to be 14/2, so, the rule is (8+6)/2. I bet it is the same for length, yes, length is 24, (8x6)/2. We have to divide by 2.

Sarah had re-structured her goals to consider the 8x6 case and thus developed increased awareness of the applicability of her solution to tables having dimensions that are even numbers (B=(h+w)/2, L=(h•w)/2). As in the previous episode, Sarah's refined solution was limited in that it did not correct answers when the dimensions had common factors other than 2: in the case of a 12x18 table, her solution would predict the number of bounces by B=(12+18)/2=15, and the path length by L=(12•18)/2=108. In comparison, the correct answers for a 12x18 table are B=5 and L=36. Sarah did not realize this though she was able to extend her solution somewhat by considering the ratio of the table dimensions. In particular, Sarah explored the idea that she should consider the ratio of the dimensions in order to compute bounces and path length.

Sarah: How about an odd and even like 6x3? Bounces should be 9. ***(she generates the case on computer, gets B=3)*** my rule is wrong! ***(reflection)*** they are multiples! So, it may have to do with if 'one goes into the other'. What if they are both even like 8 and 2? Let's try 8 and 4; for 6 and 3 we had B=3 and L=6. Ah, B is always 3 when h/w = ½, then B=1+2=3, L=6, ah, 6 is the larger of the 2 dimensions. For 8 and 4, we should get B=2+1=3 and L=that larger of 4 and 8, or 8.

She then considered fractions in general and made a conjecture that her solution needed to consider the ratio of the table's dimensions.

Sarah: So, I had the ½ in both of these ***(points to 3x6 and 8x4 cases in her table)*** I think I need to think of them as fractions!! So, all 1/2s have the same, 2/3 the same. ... Let's play with this idea.

So, Sarah had extended her rule to include cases where one dimension is a multiple of the other: $h/w=p/q$ with B=$p+q$ and L=max (h, w). However, her conjecture that tables having equivalent ratios in the dimensions such as 3x6 and 4x8, 4x6 and 2x3, will have the same number of bounces and the same path length values is only partially correct: it is correct for the number of bounces <u>only</u> and is not correct for path length. She never fully explored this implication for situations such as tables of 2x3 and 4x6.

In contrast to Sarah's strategy, Gavin focused on solving the problem of finding the number of bounces of the ball only and never considered the possibility that there he might also be able to determine the path length of the ball based on table dimensions. He did not use the computer to generate new information as did Sarah; rather, he manually iterated each case by tracing the particular path on the computer screen with his fingers. Instead of using the computer to tabulate his results, Gavin developed his own representational diagrams to examine and keep track of the number of bounces and only used the computer as a check after he had performed manual counting. Gavin's problem posing focused on generating patterns from these representations. For example, he noticed that successive quantities in his table differed by a constant of +2 and stated a relationship between the number of bounces and the corresponding table dimensions:

Gavin: For 9 and 4, you get ***(he traces the path on the screen)*** I get 13 bounces ***(runs the case on the computer)*** yes, that's right. ***(he traces results for cases of 9x4, 11x4, 13x4 and makes a table of values; 9/4 goes to 13, 11/4 goes to 15, and 13/4 goes to 17).*** Okay, so in the first 3 I had a constant change of +2 **(*points to his calculations: 9/4=13, 11/4=15, 13/4=17)***, so that is what I think the pattern is. So, ***(reflects on pattern of differences)*** 15/4 should go to 19.

Gavin developed his solution to cases where one dimension was a multiple of the other.

Gavin*:** When this number divides into the other ***(points to 8x4), we get 3 bounces; so 8/4=2+1=3. <u>I see it now</u>! We divide by 4 and add +1 to get bounces. I sum the 2 numbers and divide by the factor if one is a multiple of the other. For 6 and 3 we get (6+3)/3=3 bounces. This is the pattern.

Since he focused on finding relationships between dimensions and bounces only, Gavin did not accomplish as much as Sarah. Nevertheless, Gavin's problem posing followed a structure similar to Sarah's: he posed the problem of finding the number of bounces and then developed and carried out his goals, modifying his goals whenever results did not agree with his current solution. However, Sarah's problem posing appeared to be hypothesis-driven in the sense that her re-formulations of the original problem each appeared to extend her current understandings; the problem kept getting 'bigger' for her in scope and she was more willing than Gavin to entertain novel questions that went beyond particular results. In contrast, Gavin's solution activity remained within a more narrow conceptual focus. His ideas did not advance beyond simple patterns of differences in bounce totals. We interpreted his activity as data-driven in that his conjectures remained fixed on simple differences in the number of bounces and never considered what might happen in other more novel cases.

Problem posing and solving for the Number Array task. In noting the different structures and contexts between the Billiard Ball and Number Array tasks, we expected that students would be required to generate more information on their own in solving the Number Array task than for solving the Billiard Ball task. For example, they could explore spatial relationships about the array of the numbers, the effects of performing arithmetic operations on the numbers, and sequences of numbers from their operations.

Sarah appeared to evolve and stretch her solution activity in ways that Gavin did not. For example, she used an informal 'skipping' method to find the sums of the entries of all NxN blocks containing the square numbers on the diagonal. We interpreted this as Sarah invoking a metaphor to pose her problems; this approach that helped her organize and structure her solution activity. The following episodes provide a brief overview of this development. (In order to refer to various blocks of numbers in the array, we use a notation that lists the top-to-bottom rows of the block. For example, the 3x3 block in the upper left position of the array is denoted by [1,2,3:2,4,6:3,6,9].)

> ***Sarah***: So, for a 1x1, I get a sum of 1. For a 2X2 ***(Points to [1,2:2,4])*** I get a sum of 9 … but what happened to 4? It has been skipped! ***(reflection)*** Okay, let me try this, I will write down the sequence of squares of all numbers, all in a row ***(She writes the sequence: 1, 4, 9, 16, 25, 36, 49, …, 2251).*** The first number, 1, is the sum of the first matrix, a 1x1. And the first 2x2 has a sum of 9. So, I skipped over 4 to get the sum for the 2X2 ***(crosses out the 4 in the sequence of squares)***, going from 1x1 to a 2x2, a sum of 9. The 4 is skipped! Interesting! So, going from a 2x2 to 3x3 (***Points to [1,2,3:2,4,6:3,6,9])***, we go from 1, to 9, to 36, so we skipped over the next two numbers, 16 and 25 ***(crosses out 16 and 25 in the sequence)***, a skip of 2 in this sequence! Okay, then we skip over the next 3 square numbers, and that tells us the sum for a 4x4 should be 100 ***(crosses out the next 3 in the sequence after 25: 36, 49, 81)*** – that is what I have here. Cool! So, for a 5X5, we skip over the next 4 numbers in the sequence, the number 121, 144, 169, 196 and get 225 – yes!

Sarah had developed an informal 'skipping' method for computing the sum of the entries of NxN matrices down the diagonal. She generalized a more powerful algorithm that involved operations of the row and column numbers of each NxN block of numbers.

> ***Sarah***: I wonder why this skipping works? For 6X6, we add rows of the block, 21+42+…+126= 21(1+2+3+4+5+6)= 21x21=441. Do we get 441 by skipping the next 5 in this square sequence? ***(She checks her original sequence, crosses out the 'skips', and***

gets 441 as the next number) I notice that 21 ***(she points to the factored form 21(1+2+3+4+5+6)*** is the sum of the 6 numbers in that first row. Yes! To find the sum of NXN blocks, you just need to look at sum of 1 to N and then square the total. For an 8x8 it would be 1+2+…+8=36, and then I take 36^2 …1296. Does it check with my skipping sequence? I skip 6 over 21 to get 28^2 for 7x7, and then skip 7 to get the one for 8x8, so 7 more is 35 and the next one is 36! So my algorithm works! How about a 100x100 grid?

In contrast, Gavin's solution activity for the Number Array task mostly was based on his noticing some general relationships about the numbers in the array, including basic symmetries among the numbers and simple sums of rows and column entries.

Gavin then considered relationships involving sums and products in 2x2 blocks.

> ***Gavin***: A 2x2 square ***(circles [1,2:2,4])***. If I add and subtract the sum of the opposite corners, the difference is 1. That is true here but what about everywhere else? **(*checks [30,36:35,42] and [56,64:63,72]*)**. Yeah it works. If I multiply those corner numbers? **(*points to [1,2:2,4]*)**, I get 2x2 and 1x4, so the numbers are equal. Let's try bigger ones **(*[30,36:35,42] and [56,64:63,72]*)**. 30x4 is 1260 and 35x36, yes, I get 1260. So products of these corner numbers in any 2x2 are equal.

He then considered multiplication properties and developed the idea of 'pivot' numbers:

> ***Gavin*:** This 3x3 **(*points to [2,4,6:3,6,9:4,8,12]*)**, that middle 6 looks interesting, maybe it relates to sums and products. **(*reflection*)** I notice that 6 is ½ times (4+8) or (3+9), … so two times that middle number is the sum of the other two in the row or column. Let's check one of these other 3x3 **(*circles [16,20,24:20,25,30:24,30,36]*)**. So 25 is middle number. So 20+30=50 for the column and it is 2x25. For the row, I have same sum, 20+30=50. So, all 3x3 blocks, that middle number times two gives the sum of the other two numbers in the row or column.

SIGNIFICANCE OF THE STUDY

The findings suggest important roles that problem posing plays in solving open-ended situations. First, the general structure of solution activity was similar for each student: each student posed initial problems, developed and carried out goals to explore the idea by carrying out specific cases and then reflect on the results to consider new questions to explore. This finding provides some confirmation of the results of our earlier study (Cai & Cifarelli, 2005) and is also compatible with contemporary models of mathematical problem solving that posit a cyclic development of understanding in problem solving situations (Carlson & Bloom, 2005). Second, the individual differences that were identified between the students in their posing indicates the need for us to be cautious and vigilant when students are solving open-ended problems: the problems they pose may not make sense to others and may appear unsophisticated. In solving the Number Array problem, Sarah's evolution of her solution activity from informal methods to a more sophisticated method illustrated important ways that solvers stretch their conceptual boundaries in solving open-ended problems. However, Sarah's initial use of 'skipping' to find simple sums of NxN blocks did not appear to us to be sophisticated. It was only through her sustained persistence in making sense of and further develop her results that she was able to transform her 'skipping' method into a more sophisticated method. Her development appeared to involve a type of multi-faceted generalization that involved generalization from simple to more complicated numerical cases as well as generalization from informal to formal levels of operation. While educators have touted the conceptual benefits of having students solver open-ended problems (Becker & Shimada, 1997), this finding indicates that students sometimes do things that we as teachers cannot foresee or predict yet may turn out to be mathematically sophisticated.

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A LONGITUDINAL STUDY OF CHILDREN'S MENTAL COMPUTATION STRATEGIES

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Using a 40-minute, task-based, one-to-one interview, 323 students were assessed on nine occasions from their arrival at school as five year-olds to their final weeks in primary school nearly seven years later. The data provide a substantial picture of developing understanding and strategies over time. We report on students' understanding and use of addition and subtraction and multiplication and division strategies over the seven-year period. An important feature of this study was the detailed nature of the data, the large number of students involved, and that it was collected using the same process over seven years. Key findings included the significant number of students who after seven years of schooling were not using desired mental strategies, relying upon counting strategies in solving problems mentally. The timing of the teaching of conventional written algorithms is discussed.

The Early Numeracy Research Project (ENRP) was conducted from 1999 to 2001 in 35 project ("trial") schools and 35 control ("reference") schools, and involved 353 teachers and over 11 000 students, in Victoria, Australia (Clarke, Sullivan, & McDonough, 2002). There were three main components to the project: a framework of research-based growth points as a means for *understanding* young children's mathematical thinking through a hypothesised learning trajectory; a one-to-one assessment interview used by all teachers at the beginning and end of the school year as a tool for *assessing* knowledge and strategies for particular individuals and groups, and a multi-level professional development program geared towards *developing* further such thinking. As part of a follow-up study, 323 students were "followed" through their next four years of schooling (2002-2005), with regular assessment interviews. The longitudinal data (1999-2005) form the basis of this paper.

THEORETICAL BACKGROUND

A research-based framework of "growth points"

It was decided to create a framework of key "growth points" in mathematics learning. Students' movement through growth points could then be tracked over time. The project team studied available research on key "stages" or "levels" in young children's mathematics learning (e.g., Carpenter & Moser, 1984; Fuson, 1992; Mulligan & Mitchelmore, 1996; Wright, 1998), as well as frameworks developed by other authors and groups.

Within each mathematical domain, growth points were stated with brief descriptors in each case. There are typically five or six growth points in each domain. To illustrate the notion of a growth point, consider the child who is asked to find the total of two collections of objects (with nine objects screened and another four objects). Many

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young children "count-all" to find the total ("1, 2, 3, …, 11, 12, 13"), even though they are aware that there are nine objects in one set and four in the other. Other children realise that by starting at 9 and counting on ("10, 11, 12, 13"), they can solve the problem in an easier way. *Counting All* and *Counting On* are therefore two important growth points in children's developing understanding of addition.

The six growth points for the domain of *Addition and subtraction strategies* are shown in Figure 1.

1. Count-all (two collections)
Counts all to find the total of two collections.
2. Count-on
Counts on from one number to find the total of two collections.
3. Count-back/count-down-to/count-up-from
Given a subtraction situation, chooses appropriately from strategies including count-back, count-down-to and count-up-from.
4. Basic strategies (doubles, commutativity, adding 10, tens facts, other known facts)
Given an addition or subtraction problem, strategies such as doubles, commutativity, adding 10, tens facts, and other known facts are evident.
5. Derived strategies (near doubles, adding 9, build to next ten, fact families, intuitive strategies)
Given an addition or subtraction problem, strategies such as near doubles, adding 9, build to next ten, fact families and intuitive strategies are evident.
6. Extending and applying addition and subtraction using basic, derived and intuitive strategies
Given a range of tasks (including multi-digit numbers), can solve them mentally, using the appropriate strategies and a clear understanding of key concepts.

Figure 1: ENRP growth points for the domain of Addition and subtraction strategies.

These growth points informed the creation of assessment items, and the recording, scoring and subsequent analysis.

We do not claim that all growth points are passed by every student. For example, growth point 3 involves "count-back", "count-down-to" and "count-up-from" in subtraction situations, as appropriate. There appears to be a number of children who view a subtraction situation (say, 12-9) as "what do I need to add to 9 to give 12?" and do not appear to use one of those three strategies in such contexts. This student is using a "fact family," one of what we call "derived strategies" (see Growth Point 5).

The growth points should not be regarded as necessarily discrete. As with Wright's (1998) framework, the extent of the overlap is likely to vary widely across young children, and "it is insufficient to think that all children's early arithmetical knowledge develops along a common developmental path" (p. 702).

A task-based, assessment interview

A one-to-one interview in Number, Measurement and Geometry was developed to be used with every child at the beginning and end of the school year, over a 40-minute period. The disadvantages of pen and paper tests have been well established by Clements and Ellerton (1995) and others, and these disadvantages are particularly evident with young children, where reading issues are of great significance. The face-to-face interview was an appropriate response to these concerns. Many writers have commented on the power of the one-to-one assessment interview as providing powerful insights into student thinking (Schorr, 2001). Bobis, Clarke, Clarke, Gould, Thomas, Wright, and Young-Loveridge (2005) reported on the major role of the interview in key numeracy projects in Australia and New Zealand.

Although the full text of the ENRP interview involved around 60 tasks (with several sub-tasks in many cases), no child moved through all of these. The path was specified, in accordance with a student's response to each task. Figure 2 shows a question, involving little plastic teddy bears, from the section on *Addition and subtraction strategies*. Words in italics are instructions to the interviewer. In normal type are the words the interviewer uses with the child.

18) Counting on
a) Please get four green teddies for me.
Place 9 green teddies on the table.
b) I have nine green teddies here *(show the child the nine teddies, and then screen the nine teddies with the ice-cream lid).*
That's nine teddies hiding here and four teddies here *(point to the groups).*
c) Tell me how many teddies we have altogether... Please explain how you worked it out.
d) *(if unsuccessful, remove the lid).* Please tell me how many there are altogether.

Figure 2: An excerpt from the addition and subtraction interview questions.

Question 18 provided information on whether the child was able to count-on or use a known fact, needs to count-all, or was unable to find the total by any means. Our aim in the interview is to gather information on the *most powerful strategies* that a child accesses in a particular domain. However, depending upon the context and the complexity of the numbers in a given task, a child (or an adult) may use a less powerful strategy than they actually possess, as the simpler strategy may "do the job" adequately in that situation.

The key focus of the interview was on *mental* computation strategies, reflecting our commitment to the important role of mental strategies, and data (e.g., Northcote & McIntosh, 1999) that almost 85% of all computation by Australian adults is mental.

THE LONGITUDINAL STUDY: METHOD

The ENRP was conceived originally as a project focusing on the first three years of school. However, in light of the high quality data which was emerging from the use of the interviews and the deliberately high "ceiling" in the original creation of the interview, particularly in the number domains, it was agreed that a sample of students would continue to be interviewed in subsequent years. The data reported in this paper involved 323 students, and reports achievement at the beginning of school as five-year olds (Prep or "Grade 0"), and at the end of Grades 0, 1, 2, 3, 4, and 6 (Grade 5 interviews were not possible). Data therefore covers ages 5 to 12, our primary years.

The schools and students are broadly representative of Victorian students, on variables such as school size, location, language background, and socio-economic status. We note that these students remained at the same school for seven years, and so it could be hypothesised that their average achievement may be slightly higher than typical achievement, given that research on student mobility suggests that the frequency of moves, particularly at the primary level, is associated with lower academic performance (Blane, Piling, & Fogelman, 1985).

Tasks in the mathematical domains of Counting, Place value, Addition and subtraction, and Multiplication and division were administered in the student's own school, with interviews following a strict script for consistency, and using a standard record sheet to record students' answers, methods and any written calculations or sketches. The starting point within each domain was determined by what was achieved in the previous interview. In general, students were required to start at items which enabled them to "re-establish" the growth point achieved on the previous occasion.

A trained team of coders took the data from the record sheets, assigned growth points for each domain, and entered the data into SPSS. A process was developed to enable the growth point data to be converted to an interval scale, to enable comparisons between groups of students (see Horne & Rowley, 2001). Key findings are provided in the next section for tasks and growth points relating to whole number computation.

RESULTS

Addition and Subtraction data

The percentage distribution of the highest growth point achieved by students at seven different stages of their primary schooling is shown in Table 1.

Table 1: Percentage distribution of the highest Addition and Subtraction Growth Point achieved by students (n = 323) across years of school.

	Start Gr0 1999	End Gr0 1999	End Gr1 2000	End Gr2 2001	End Gr3 2002	End Gr4 2003	End Gr6 2005
GP 0 not apparent	65.3	11.5	0.6	0	0	0	0
GP 1 count all	26.6	47.1	13.3	1.5	1.2	0.9	0.3
GP 2 count on	8.0	31.9	37.2	20.4	12.7	8.4	1.2
GP 3 count back, down to, up from	0	8.7	18.9	12.4	9.6	4.0	1.2
GP 4 basic strategies	0	0.6	20.4	26.9	29.1	24.5	9.0
GP 5 derived strategies	0	0.3	9.3	36.8	40.2	40.4	28.8
GP 6 extend & apply	0	0	0.3	1.9	7.1	21.7	59.4

These data show that, on arrival at school ("Start Gr0"), few students were counting on, although around 32% could do so by the end of the first year. There was steady growth over the primary years. To achieve the highest growth point, students needed to have demonstrated a variety of mental strategies (e.g., "100–68", "double 26"), and to solve problems involving tasks such as *estimating* the answer to 642-376 within the range 200 to 300, and finding the *exact* answer by mental or any pen and paper method. Table 1 indicates that 21.4% of students could do so by the end of Grade 4, with 59.4% of the sample doing so by the end of Grade 6. One child was even capable of this at the end of Grade 1 and six could solve these problems at the end of Grade 2. This growth point and related tasks provided a very high "ceiling," and the development of basic and derived strategies was actually of greater interest to the research team.

These same data are also shown in a "region graph" (adapted from the style of Carpenter & Moser, 1984) in Figure 1.

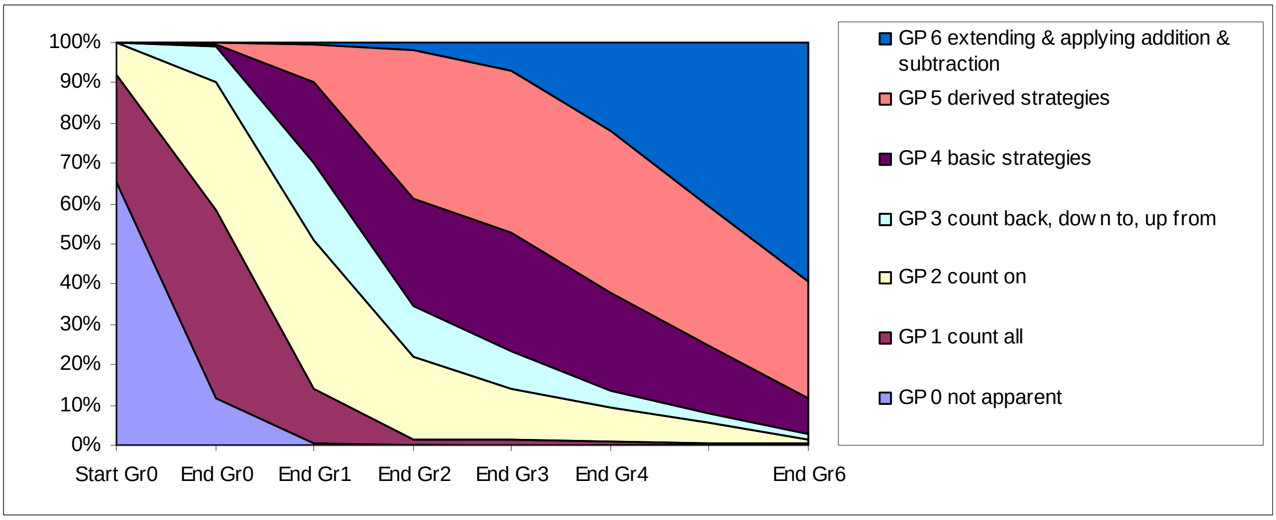

Figure 1: Percentage distribution of the highest Addition and Subtraction Growth Point achieved by students (*n* = 323) across years of school in "region style."

This graph takes some interpreting, but we have found it to be a helpful picture of the data. As colour is not possible, the different regions from left to right correspond to increasingly "higher level" growth points. In understanding the graph, begin on the "x axis," and move vertically. For example, looking up vertically from "End Gr 2" gives the distribution of the highest growth point achieved at that stage of schooling.

Looking from the lower left corner to the top right corner provides an indication of the relative time that students spend at each growth point. So, for example, in addition and subtraction, students spend relatively little time at Growth Point 3 ("Count-back/ count-down-to/ count-up-from").

By considering the slope of the lines separating each region, we also gain a sense of rate of growth over time. For example, the Grade 3 year (between the end of Grade 2 and the end of Grade 3) "flattens off," indicating less rate of growth across that year compared to other years. The data also show that at any grade level, there is a considerable range of understanding and acquired strategies to address.

Multiplication and division data

Sullivan, Clarke, Cheeseman and Mulligan (2001) discussed the ENRP Growth Points and related assessment tasks for this domain in considerable detail, and so these will not be revisited here. However, of greatest interest is the transition from Growth Point 2 to 3, i.e., moving from *Modelling multiplication and division (all objects perceived)* to *Abstracting multiplication and division*. As Sullivan et al. explain, "a key stage in learning multiplication and division is a capacity to move beyond reliance on physical models of problem situations and to form mental images to seek solutions" (p. 233). This study extended this research for five more years.

Table 2: Percentage distribution of the highest Multiplication and division strategies Growth Point achieved by students (n = 323) across years of school.

	Start Gr0 1999	End Gr0 1999	End Gr1 2000	End Gr2 2001	End Gr3 2002	End Gr4 2003	End Gr6 2005
GP 0 not apparent	66.9	19.8	3.7	0	0.6	0.3	0
GP 1 counting group items as ones	25.4	26.0	3.4	0	0	0.3	0
GP 2 modelling multi.& division	7.7	52.9	74.0	46.4	34.4	20.8	5.3
GP 3 abstracting multi. & division	0	0.9	13.6	29.4	20.4	15.8	5.3
GP 4 basic, derived & intuitive strategies for multiplication	0	0	5.0	13.6	29.4	28.9	17.3
GP 5 basic, derived & intuitive strategies for division	0	0.3	0	7.7	8.7	25.2	20.1
GP 6 extending & applying multi. & division	0	0	0.3	2.8	6.5	8.7	52.0

Once again, steady growth is evident. The relevant region graph is shown in Figure 2.

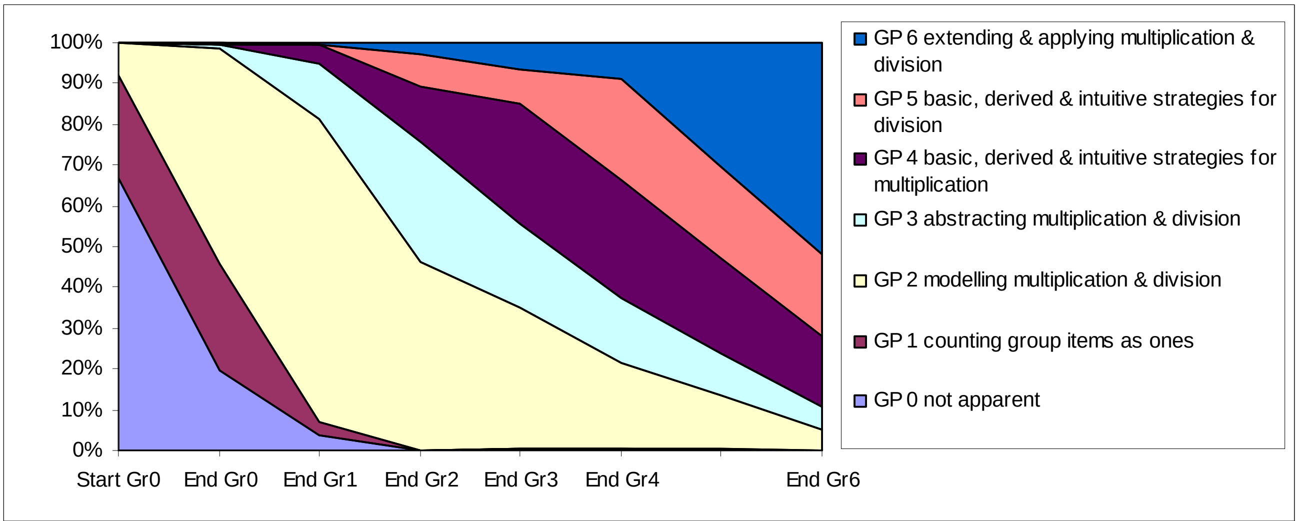

Figure 2: Percentage distribution of the highest Multiplication and division strategies Growth Point achieved by students (n = 323) across years of school in "region style."

Two important observations are evident. First, as with Figure 3, we see a considerable range of achievement at any grade level. Second, we note the considerable period of time for which students are typically *modelling* before moving on to *abstraction*, continuing the pattern reported by.

DISCUSSION

It is clear that for most students there is a steady progression in skills and understanding over time. The slower rate of growth in Grade 3 may be explained by two factors. First, when the students were in Grade 3, they were being taught by teachers who had not participated in the ENRP professional development program, and were therefore less likely to emphasise basic and derived mental strategies in their teaching. Second, careful examination of these and other data also indicates that the common major emphasis in Victorian schools on *written algorithms* and rote memorisation of *number facts* in Grades 3 and 4 (often to the exclusion of an emphasis on further developing mental *strategies*) may be premature and inhibit the development of basic and derived strategies in addition and subtraction, and abstraction in multiplication and division (see also, Narode, Board, & Davenport, 1993). At the end of Grade 4, over 21% of students had not moved beyond modelling in multiplicative situations, and over 37% did not demonstrate the use of desired basic and derived strategies in addition and subtraction tasks.

The data presented in this paper provide further evidence of the huge range of understanding and achievement in mathematics classrooms, and also a helpful basis for discussions on the timing of intervention program for students experiencing difficulty in whole number learning (Gervasoni, 2004). Is it best to intervene early or to provide additional support to students experiencing difficulty in Grades 3 and 4?

We recommend a delay in presenting conventional written algorithms to students and rote memorisation, while encouraging the development of their own invented mental and written methods, and join with Sullivan et al. (2001) in recommending greater experience for students in activities which prompt visualisation in multiplicative situations such as groups, arrays, and multiplicative comparisons.

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ASSESSING FRACTION UNDERSTANDING USING TASK-BASED INTERVIEWS

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Following a literature review that established key components of a sound understanding of fractions, a range of assessment tasks was developed for use in one-to-one interviews with 323 Grade 6 students in Victoria, Australia. In this paper, we summarise briefly the research literature on fractions, describe the process of development of assessment tasks, share data on student achievement on these tasks, and suggest implications for curriculum and classroom practice. A particular feature of this report is that one-to-one interview assessment data were collected from a larger number of students than is typically the case in these kinds of studies. Recommendations arising from these data include the importance of teachers understanding and presenting a wider range of sub-constructs of fractions to students in both teaching and assessment than is currently the case, using a greater variety of models, and taking available opportunities to use the interview tasks with their own students.

THEORETICAL BACKGROUND

Fractions are widely agreed to form an important part of middle years mathematics curriculum (Lamon, 1999; Litwiller & Bright, 2002), underpinning the development of proportional reasoning, and important for later topics in mathematics, including algebra and probability. However, it is clear that it is a topic which many teachers find difficult to understand and teach (Post, Cramer, Behr, Lesh, & Harel, 1993), and many students find difficult to learn (Behr, Lesh, Post, & Silver, 1983; Kieren, 1976; Streefland, 1991). Among the factors that make rational numbers in general, and fractions in particular difficult to understand are their many representations and interpretations (Kilpatrick, Swafford, & Findell, 2001).

There is considerable evidence that the difficulties with fractions are greatly reduced if instructional practices involve providing students with the opportunity to build concepts as they are engaged in mathematical activities that promote understanding (Bulgar, Schorr, & Maher, 2002; Olive, 2001).

In the Early Numeracy Research Project (Clarke, Sullivan, & McDonough, 2002), a task-based, interactive, one-to-one assessment interview was developed, for use with students in the early years of schooling. This interview was used with over 11 000 students, aged 4 to 8, in 70 Victorian schools at the beginning and end of the school year, thus providing high quality data on what students knew and could do in these early grades, across the mathematical domains of Number, Measurement and

Geometry. There was equal emphasis in the teachers' record of interview on *answers* and the *strategies* which led to these answers.

The use of a student assessment interview, embedded within an extensive and appropriate inservice or preservice program, can be a powerful tool for teacher professional learning, enhancing teachers' knowledge of how mathematics learning develops, knowledge of individual mathematical understanding as well as content knowledge and pedagogical content knowledge (Clarke, Mitchell, & Roche, 2005; Schorr, 2001).

The success of the interview and comments from middle years' teachers prompted the authors to consider extending the use of the assessment interview to the middle years of schooling (Grades 5-8). As a first, major step in this process, it was decided to focus the interview on the important mathematical topics of fractions and decimals. This paper reports the process and findings from this work, with particular emphasis on fractions.

FRACTIONS: CONSTRUCTS AND MODELS

Much of the confusion in teaching and learning fractions appears to arise from the many different interpretations (constructs) and representations (models). Also, generalisations that have occurred during instruction on whole numbers have been misapplied to fractions (Streefland, 1991). Finally, there appears to be a void between student conceptual and procedural understanding of fractions and being able to link intuitive knowledge (or familiar contexts) with symbols (or formal classroom instruction) (Hasemann, 1981; Mack, 2002). The dilemma for both teachers and students is how to make all the appropriate connections so that a mature, holistic and flexible understanding of fractions and the wider domain of rational numbers can be obtained.

Kieren (1976) was able to identify several different interpretations (or constructs) of *rational numbers* and these are often summarised as part-whole, measure, quotient (division), operator, ratio and decimals. For the purpose of this review these interpretations will be explained in the context of fractions.

The *part-whole* interpretation depends on the ability to partition either a continuous quantity (including area, length and volume models) or a set of discrete objects into equal sized subparts or sets. The part-whole construct is the most common interpretation of fractions and likely to be the first interpretation that students meet at school. Lamon (2001) suggests "mathematically and psychologically, the part-whole interpretation of fraction is not sufficient as a foundation for the system of rational numbers" (p. 150).

A fraction can represent a *measure* of a quantity relative to one unit of that quantity. Lamon (1999) explained that the measure interpretation is different from the other constructs in that the number of equal parts in a unit can vary depending on how many times you partition. This successive partitioning allows you to "measure" with precision. We speak of these measurements as "points" and the number line provides a model to demonstrate this.

A fraction (a/b) may also represent the operation of *division* or the result of a division such that 3÷5 = 3/5. The division interpretation may be understood through partitioning and equal sharing. These two activities have been the focus of much research (Empson, 2003).

A fraction can be used as an *operator* to shrink and stretch a number such as ¾ x 12 = 9 and 5/4 x 8 = 10. The misconception that multiplication always makes bigger and division always makes smaller is common (Bell, Fischbein & Greer, 1984). It could also be suggested that student lack of experience with using fractions as operators may also contribute to this misconception.

Fractions can be used as a method of comparing the sizes of two sets or two measurements such as; the number of girls in the class is 3/5 the number of boys, i.e., a *ratio*. Post et al. (1993) claim "ratio, measure and operator constructs are not given nearly enough emphasis in the school curriculum" (p. 328).

While these constructs can be considered separately they have some unifying elements or "big ideas". Carpenter, Fennema and Romberg (1993) identified three unifying elements to these interpretations and they are: identification of the unit, partitioning and the notion of quantity.

METHOD

Focusing on the rational number constructs of part-whole, measure, division and operator, and the "big ideas" of the unit, using discrete and continuous models, partitioning, and the relative size of fractions, a range of around 50 assessment tasks was established, drawing upon tasks which had been reported in the literature, and supplemented with tasks which the research team developed. These tasks were piloted with around 30 students in Grades 4-9, refined, and piloted again (see Mitchell & Clarke, 2004).

Using a selection of the set of tasks, 323 Grade 6 students were interviewed at the end of the school year. The schools and students were chosen to be broadly representative of Victorian students, on variables such as school size, location, proportion of students from non-English speaking backgrounds, socio-economic status. A team of ten interviewers, all experienced primary teachers, with at least four years' experience in one-to-one assessment interviews of this kind, participated in a day's training on the use of the interview tasks, including viewing sample interviews on video.

The tasks were administered individually over a 30- to 40-minute period in the student's own school, with interviews following a strict script for consistency, and using a standard record sheet to record students' answers, methods and any written calculations or sketches. Each actual response to a question was given a code by the authors, and a trained team of coders took the data from the record sheets, coded each response, and entered it into SPSS. Key findings are provided in the following section.

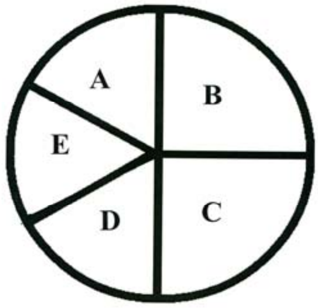
RESULTS

In this section, data from the 323 Grade 6 students are provided on a sample of the tasks, organised around relevant sub-constructs of fractions (Kieren, 1976). In each case, the task is outlined, the mathematical idea it was designed to address is stated, the percentage student success rate is given, and common strategies and solutions, including misconceptions are outlined.

Part-whole

Three tasks focused on part-whole thinking.

1. *Fraction Pie task* (adapted from Cramer, Behr, Post, & Lesh, 1997). Students were shown the pie model, and asked:

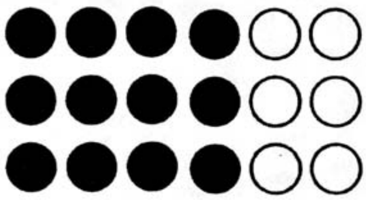


a) What fraction of the circle is part B?

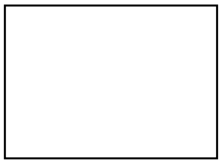
b) What fraction of the circle is part D?

Part (a) was relatively straightforward, with 83.0% of students giving ¼. 3.3% offered a correct equivalent fraction, decimal or percentage answer, while 5.6% and 1.9% answered “1/5” and “1/2”, respectively. Part (b) was more difficult, with only 42.7% giving a correct answer, with 13.6% answering 1/5 (presumably based on “five parts”). The same percentage answered 1/3, probably focusing only on the left-hand side.

2. *Dots Array task*. Students were shown this array, and asked, “what fraction of the dots is black?” They were then asked to state “another name for that fraction.” 76.9% gave a correct answer, with the three most common answers being 2/3 (35.6%), 12/18 (30.7%), and 4/6 (8.7%). The most common error was 3/4. Only 53.5% of students were able to offer another correct name for the fraction, with 4/6 being the most common response (17.0%). These data indicate that students generally showed a flexible approach to *unitising* (Lamon, 1999).



3. *Draw me a whole task*. In assessing students’ capacity to move from the part to the whole, acknowledged by Lamon (1999) and others as an important skill, students were shown a rectangle, and asked, “if this is two-thirds of a shape, please draw the whole shape,” explaining their thinking. 64.1% were able to do so successfully, with 28.5% of them dividing the original shape into two equal parts first, and 35.6% showing no visible divisions. Students were then presented with a different rectangle, told that it was “four thirds,” and asked to show the whole. In this case, 40.5% drew a correct shape, with just under half of these breaking the original rectangle into four parts, indicating three of these as the whole.

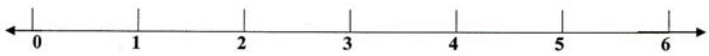


Fraction as an operator

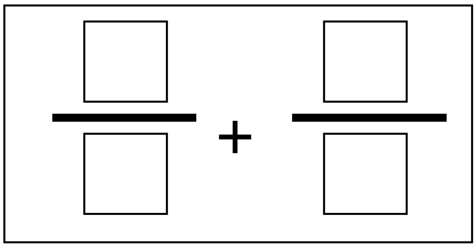
Students were asked four tasks, with no visual prompt, which required students to work out the answer in their heads. They were as follows: "… one-half of six?" (97.2% success); "… one-fifth of ten?" (73.4%); "… two-thirds of nine? (69.7%); and "… one third of a half?" (17.6%). The data on the last item, by far the most difficult, is interesting in light of the relative difficulty with the related pie task.

Fractions as measure

Students were asked to "please draw a number line and put two thirds on it." If students didn't choose to indicate where 0 and 1 should be in their drawing, they were asked by the interviewer, "where does zero go? … where does 1 go?" Only 51.1% of students were successful in correctly locating 2/3 on the number line. A common error was placing 2/3 after 1, or two-thirds along some line, e.g., at 4 on a number line from 0 to 6, or two-thirds of the way from 0 to 100.

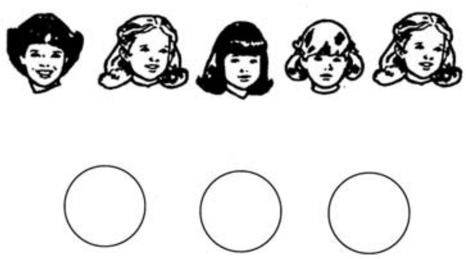


Given a number line as shown, students were then asked to mark, in turn, six thirds and eleven sixths (Baturo & Cooper 1999). Only 32.8% and 25.4% were successful, respectively. Many placed 6/3 on 6 or 3. Several students located 11/6 well to the right of 6.



In a task designed to get at students' understanding of the "size" of fractions, we used the Construct a Sum task (Behr, Wachsmuth & Post, 1985). The student is directed to place number cards in the boxes to make fractions so that when you add them the answer is as close to one as possible, but not equal to one. The number cards included 1, 3, 4, 5, 6, and 7. Each card could be used only once. The capacity for students to move cards around as they consider possibilities is a strong feature of this task. Only 25.4% of students produced a solution within 0.1 of 1, the most common response being 1/5 + 3/4 (5.3% of the total group). 24.5% of students chose fractions at least 0.5 away from 1, and most of these included an improper fraction. The best possible answer (1/7 + 5/6) was chosen by only four students.

Fractions as division



Children were shown this picture, and told, "three pizzas were shared equally between five girls. … How much does each girl get?" Students were invited to use pen and paper if they appeared to require it.

A total of 27.0% of students gave a correct answer, with 11.8% calculating the answer mentally and 15.2% using a drawing. A further 3.4% gave 1/2 + 1/10 as their final answer. Although students were not asked to elaborate on their answer in this task, it is clear that no more than 11.8% immediately recognised that three things shared between five (3÷5) must result in shares of 3/5. It is possible that many of these students did not make this connection, but are forming a picture in their minds of the pizza being broken up, eventually leading to 3/5.

DISCUSSION

Despite the strong recommendations from researchers that school mathematics should present students with experience with all key sub-constructs of fractions, and the many useful models that illustrate these sub-constructs (Lamon, 1999; Post et al., 1993), it is clear that a large, representative group of Victorian Grade 6 students do not generally have a confident understanding of these and their use.

Generally, performance on *part-whole* tasks was reasonable, although when the object of consideration was not in a standard form and not broken into equal parts (e.g., in the Fraction Pie task), less than half of the students could give a correct fraction name to the part. The teaching implications here are clear. Students need more opportunities to solve problems where not all parts are of the same area and shape. On the other hand, the dots array task showed that students handled this discrete situation well, *unitising* appropriately, and usually had access to fractions that were equivalent to a given fraction.

Although simple fraction as an *operator* tasks were straightforward for most students, it seems that only around one-sixth of students being able to find one-third of a half indicates that students may need more encouragement to form mental pictures when doing such calculations. The second part of the Fraction Pie task was closely related, and it is interesting that of the 138 students who solved the pie task correctly, only 47 could give an answer to “one-third of a half.” On the other hand, of the 59 students who were successful with the mental task, 47 could solve the related pie task. Once again, the importance of visual images in solving such problems is clear.

The experience of the authors is that Australian students spend relatively little time working with number lines in comparison to countries such as The Netherlands. Given that only around half of the students could draw an appropriate number line which showed 2/3, it is clear that fraction as a *measure* requires greater emphasis in curriculum documents and professional development programs, as many students are clearly not viewing fractions as numbers in their own right. In light of these data, the performance on locating six thirds and eleven sixths was relatively high. The Construct a Sum task revealed similar difficulties with understanding the size of fractions, particularly improper fractions, and a lack of use of *benchmarks* in student thinking. Emphasising these aspects instead of fraction algorithms may be wise.

From our experience, few Australian primary school and middle school teachers and even fewer students at these levels are aware of the notion of *fraction as division*. Just over a tenth of our sample could confidently state that 3 pizzas shared between 5 people would result in 3/5 of a pizza each, without a drawing. Presumably an even smaller percentage knew the relationship automatically. This supports the data of Thomas (2002) that 47% of 14 year-olds thought 6÷7 and 6/7 were not equivalent.

In summary, our data indicate clearly that Victorian students (and probably their teachers through appropriate professional development) need greater exposure to the sub-constructs of fractions and the related models, as noted by Post et al. (1993) and other scholars. We would also encourage teachers to use some of the tasks we have discussed in a one-to-one interview with their students, as our experience is that the use of the interview provides teachers with considerable insights into student understanding, common misconceptions, and forms a basis for discussing the "big ideas" of mathematics and curriculum implications of what they have observed.

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EVALUATION OF A TEACHING CONCEPT FOR THE DEVELOPMENT OF PROBLEM SOLVING COMPETENCES IN CONNECTION WITH SELF-REGULATION

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Technical University of Darmstadt

The objectives of the present intervention study are the implementation of a material-based teaching concept to achieve problem solving competences in connection with self-regulation in math lessons in secondary school level I and the investigation of related effects on teachers and students. The design and some results of longitudinal and processual evaluation are presented.

OBJECTIVES AND BACKGROUND OF THE STUDY

This article is focussed on the investigation of the effects of problem solving learning which is organized on the basis of a specific teaching concept. Several investigations on the learning of mathematical problem solving are already available. Our intention was to develop a comprehensive concept which can be adopted effectively by especially trained teachers and lead students to sensible and lasting learning effects. The content of it differs insofar from other concepts as it aims at the integration of self-regulated learning elements in maths lessons and to connect them with problem solving learning. We are speaking about a mathematical problem when a student encounters requirements in the maths lesson which he/she is not prepared to cope with so that a purely schematic proceeding is not possible. Learning of problem solving means getting to know heuristic tools, principles and strategies (heurisms) which can be individually helpful to approach such mathematical problems. This kind of problem solving learning beyond the general maths context is attributed an important educational function of the math lessons, cf. Winter (1995). The knowledge and ability to use heuristic tools and strategies prove to be beneficial to master the four basic steps for the solution of mathematical problems (understand the problem, make a plan, carry out the plan, review) defined by Polya (1949). According results were shown in out-of-school training courses as well as after the integration of training concepts for problem solving and self-regulation in regular maths lessons by teachers who participated in the test, cf. Komorek, Bruder & Schmitz (2004). The purpose of the present study is the large-scale implementation and evaluation of a material-based teaching concept to gain problem solving competences in connection with self-regulation. The connection of problem solving learning and self-regulation was also reached with a material-based homework concept. In preparation of the present study the developed teaching and homework concepts for the promotion of problem solving competences in connection with self-regulation were tested and evaluated successful in the first phase of the teacher training at the Technical

University of Darmstadt as well as during further teacher traineeship. The central idea of this teaching concept is:

Mathematical problem solving competences are achieved by promoting different forms of intellectual flexibility (reduction, reversibility, consideration of aspects and change of aspects) and the development of partial activities of problem solving in connection with a more and more inconscious use of heurisms.

Against this background, problem solving learning may be considered as a long-term teaching- and learning process which includes four phases:

- Intuitive familiarisation with heuristic methods and techniques (reflexion).
- Making aware of special heurisms with an example (gain of strategy)
- Period of conscious training
- Enlargement of context of the applied strategy

In the first phase the students get intuitively used to heurisms by supportive impulses and questions. In the next phase these heurisms are specified and brought to mind with significant examples. The third phase is a training period to get accustomed to the new heurisms by the individualised practice of different requirements, e.g. long-term homework. The last phase aims at flexibilisation and more intuitive application of the new heurisms to make students gradually reach their own pattern of problem solving. The developed homework concept to promote self-regulated learning is integrated in phases 3 and 4.

In the following the research design of the implementation study is described and some results of the evaluation are presented. One of the main aims of the study is the improvement of specialist and self-regulative competences of students in maths lessons in secondary level I. In preparation of this the development of problem solving and self-regulation competences of students in class 7 and 8 was observed during one school year, according to the contents and methods of the training courses followed by the teachers. A pre/post comparison of questionnaires and tests was used to measure the effects on the students.

The question if and how subjective ideas of the teachers concerning the learning potential of maths problems had changed after completion of our further training program was analysed by means of the Repertory Grid Inquiry, cf. 3.2. The instruments teacher questionnaire and standardised lesson report were also used in the study, their evaluation is not yet finished. Moreover the work products of participating teachers like own tasks and lesson projects were analysed and assessed.

The central hypothesis of the research is:

The application of our teaching concept for the learning of problem solving in conncetion with self-regulation by especially trained teachers during one year leads to differentiated learning improvement with respect to the mathematical abilities of the students and, in particular, their problem solving competences.

Problem solving competences mean the ability to ask suitable questions for the solution of a task and the flexible use of heurisms acquired in the learning process.

According to Bruder (2003) intuitive problem solvers are characterised by high intellectual flexibility. A lack of intellectual flexibility in the mathematical context can be partially compensated by learning heurisms and self-regulation stategies by means of the above-mentioned four-step learning process. Therefore measurable performance increases are expected above all for medium and low achievers.

RESEARCH DESIGN

The fieldwork covers 50 classes of the levels 7 and 8 from 9 schools with 50 teachers.

Different teacher groups were classified according to the intervention contents and methods of the training program. The intervention content experienced by the teachers is either problem solving learning or self-regulation, or both. The two applied intervention methods are either a single compact training with four coaching units to be included in the lessons or a web-based training with online-coaching support.

As only a small extract can be presented from the complex study design we renounced to go in every detail of the intervention contents and methods.

Used instruments

In order to evaluate the development of student performance depending on the different training approaches run by the teachers the students had to undergo, at the beginning and at the end of the school year, a 45 min test developed by us. The student questionnaire, used as additional tool, allows to gain insight into the maths image, the subjective perception of learning effects, the interest in problem solving, the learning attitude of students etc. with regard to the pre- and post test results.

The learning and teaching concepts of the teachers and their dealing with maths problems was analysed with the instruments teacher questionnaire, standardised lesson reports and a Repertory Grid Inquiry (cf. 3.2).

Design of the student performance test

The student performance test consists of 17 (class 7) respectively 18 (class 8) tasks which meet the requirements of the Hessian secondary-school curriculum. It includes basic tasks as well as tasks of higher complexity from the issues figures, geometry, dimensions and functions. The processing time is 45 minutes. The test is conceived in a way that, because of the open-end design, not all students are able to solve all tasks successfully within the given time. Chosen exercise types were open tasks and multiple choice tasks.

The student performance test concentrates more on the learning improvements of students with regard to their mathematical abilities and problem solving competences than on performance increase in the treated subject. By means of a matrix-model for education standards developed by the German Conference of the Ministers of

Education (2003) the test tasks can be classified according to their central mathematical ideas and competences in connection with three requirement fields (reproducing, connecting, generalising and reflecting). The test tasks are so constructed that they check basic knowledge before they ask for more demanding problem solving competences. The tasks are arranged in a way that they produce an oscillating pattern of difficulty. Each task can be assigned a theoretical difficulty and a difficulty proved by experience. Because of the three requirement fields only the values 1 (requirement field I), 2/3 (requirement field II) and 1/3 (requirement field III) can be assigned to the theoretically supposed task difficulty, whereas the empirically observed task difficulty can assume all values between 0 and 1 as relative solution frequency and is evaluated with a quotient of the attained and attainable marks per task. Low values represent high task difficulty. We talk about good correspondence of theoretical and empirical task difficulty where the two graphs reveal “similar” results.

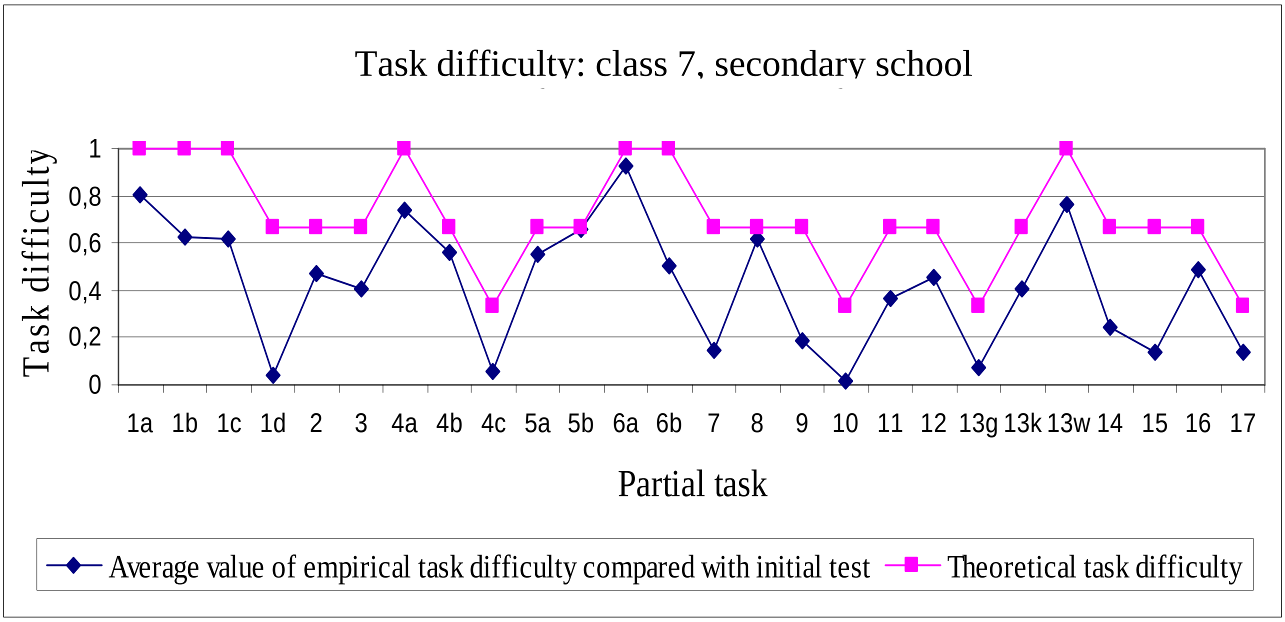


Fig. 1: Theoretical task difficulty in comparison with empirical task difficulty.

Class 7, secondary school: N=370.

In the following we will only go into the results of class 7. Similar results and effects in mathematical performance and problem solving competences were found also in class 8. The use of anchor tasks from TIMSS and PISA allowed approximate comparison of the tested students and the TIMSS- and PISA-students with respect to their performance level. One task taken from TIMSSII (class 7, task 9 product/sum, cf. complete tests in www.math-learning.com) was solved by 18% (secondary school type Gymnasium) of our population. In TIMSSII the German students of class 7 achieved a solution rate of 21% for this task (international: 31%).

Two test version concepts (group A and B), differing only in the context of the given tasks, however not in content and possible approach methods, allow direct comparison of the results. Students working in group A at the beginning of the school year worked in group B at the end of the school year and vice versa. The student performance test was intended to show if students are able to solve a task by using specific heurisms. In order to encode the different ways of solution prime numbers were assigned to the different heurisms. Consequently, as each heurism corresponds to a prime number, the evidence of decomposition into primes indicates the use of heurisms by the students. The sample size of 7th class students participating in both performance tests was N=370 (secondary school type Gymnasium [school years 5 to 13] respectively N=193 (secondary school types Hauptschule [school years 5 to 9] and Realschule [school years 5 to 10]). In total 30 classes of class level 7 were tested. The following results refer to the students who participated in both tests.

RESULTS

Results of the student performance test

In class 7 of the secondary school types Haupt- and Realschule a performance increase of 14,4 percent was reached in the course of the school year. The secondary school type Gymnasium showed an improvement by 10,4 percent. These results support the hypothesis that sensible performance increases can rather be achieved with medium- and low attainers.

However, learning of problem solving also leads to improved learning performance of good achievers. By splitting up the tested students, according to their school types, into three performance categories (low, medium, high), the performance increase can be considered separately for each group. The classification of the groups was made on the basis of the results reached in the preparatory test. As an example we present below the results achieved in class 7 of the secondary school types Haupt- and Realschule. Similar effects can be observed also for the three groups in class 7 of secondary school type Gymnasium and class 8.

The below picture demonstrates a performance increase in every performance group. Eye-catching is the clear performance increase of low attainers by 14 percent which corresponds to three additional partial tasks successfully solved in the post test. Moreover the picture shows that in the post test the low and medium attainers were able to cope with the starting level of the medium- and high performance group.

In the post test the students used heurisms such as informative figure, forward working (FW), backward working (BW), combined FW and BW, systematic trying-out and specific heuristic principles like the principles of decomposition, symmetry, extremum and invariance.

When analysing the student test with respect to the heurisms used it becomes obvious that the number of heurisms used by the students to solve problems almost tripled for both school types in the course of the school year. The frequent use of heurisms in the post test can be considered as a clear performance increase of the students.

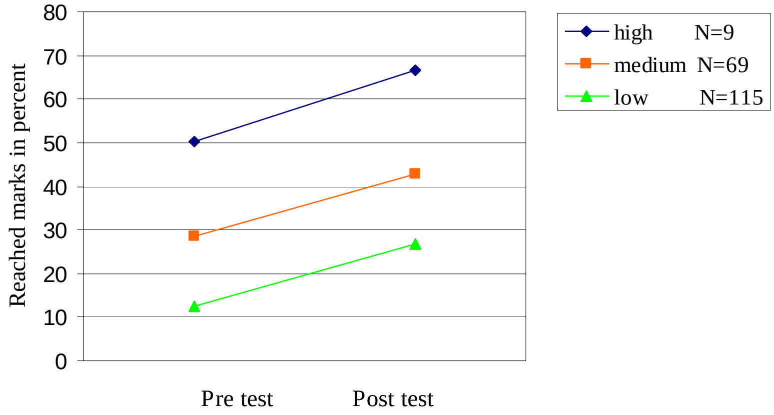


Fig. 2: Secondary school types Haupt- and Realschule class 7

The following table gives an overview on the correlation between test score and number of used heurisms. This data is exemplary and refers only to the secondary school type Gymnasium. The correlation between the number of used heurisms and the average value reveals that the performance increase of the students can be explained with the increased use of heurisms.

Class 7		Average value	Standard deviation	N	Correlation according to Pearson (p<0,01)
Pre test	Heurisms	2,0	1,8		0,52
	Marks	15,1	5,0		
Post test	Heurisms	6,0	2,9	370	0,77
	Marks	19,3	6,3		

Table 1: Class 7, Gymnasium. Correlation between test score and number of used heurisms.

The following test task for class 7 may serve as an example to show how the implemented teaching concept contributed to further development of problem solving competences.

> The tyres of motorbikes and cars (without spare tyre) are changed in a garage. During one day tyre changes were accomplished for 20 vehicles with 70 tyres mounted. How many motorbikes got a tyre change?

For this task the class 7 students in secondary school type Gymnasium ameliorated their results by 7,7 percent (+52,4%), the students of the secondary school types Haupt- and Realschule by 9 percent (+123,3%). An analysis of the heurisms applied by Gymnasium students reveals that the number of heurisms used to solve this task increased from 42 to 102 in the post test. In this task 74 Gymnasium students improved their performance, 40 students declined. 256 Gymnasium students showed neither amelioration nor deterioration. The increase of mathematical performance with particular respect to problem solving competences can be explained with the increased use of acquired heurisms. In the table are listed the most frequently used heurisms for the solution of the above task in the pre- and post test of Gymnasium students.

	Systematic trying-out	Extremum principle	Equation	Table	Invariance principle	Combined FW and BW	Informative figure	FW
Pre test	14	11	7	5	3	1	1	0
Post test	27	34	1	7	1	23	5	2

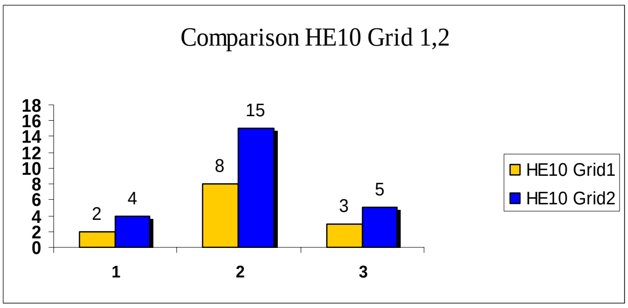
Table 2: Number of heurisms used most by Gymnasium students to solve test task.

Results of the Repertory Grid Inquiry

The Repertory Grid Inquiry of teachers mentioned at the beginning represents a questioning to register subjective theories on maths problems. The Repertory Grid Inquiry was carried out at the beginning and at the end of the further training program, cf. Bruder, R., Lengnink, K. & Prediger, S. (2003). Distinctive task features stated by the teachers were classified in six categories: Exterior features of the task, degree of difficulty, potential for student activities, explicit mathematical content, didactic function and potential of solution strategies.

For further evaluation the categories were split up in three levels:

- Level 1: Exterior features of the task, explicit mathematical content
- Level 2: Structure of the task with respect to the target of activity, degree of difficulty, activity of the student (basic activity/reflected activity)
- Level 3: Didactic function in the learning process, solution strategies



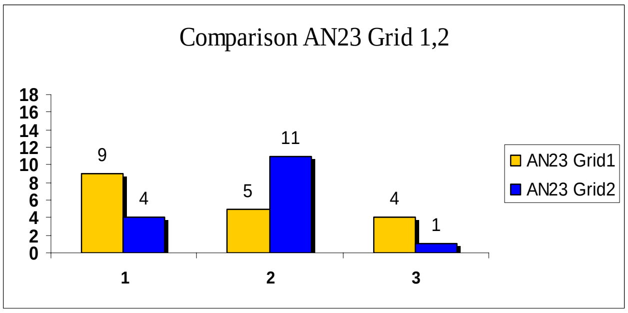


Fig. 3: Repertory Grid: Number of mentioned features per level

A comparison of the profiles of a pre- and post Repertory Grid questioning of two teachers shows that on the one hand the number of different features had increased on one level and, on the other hand, there had been a shift within the levels.

CONCLUSION AND PROSPECT

The intervention in the field reveals that the learning of problem solving following the developed teaching concept leads to increased learning effects of students with respect to their mathematical performance and, especially, their problem solving competence. This performance increase may be attributed, above all, to the increased use of learned heurisms. In the present phase of evaluation 2006 the instruments mentioned at the beginning are furtherly evaluated. Of special interest are correlations between student questionnaire and student performance test as well as those between teacher questionnaires, lesson reports, Repertory Grid and their effects on the students. The Technical University of Darmstadt meets the claim for general education in maths lessons not only by the development of a training and further training concept to promote problem solving learning in connection with self-regulation but also by the development of a database of problem solving tasks which can be applied in the lessons (www.madaba.de). It offers the teachers a wide range of tasks with variations as well as potential ways of solution, possible heurisms and further task classifications. As a parallel to the maths database a public learning platform on problem solving learning and self-regulation was established (www.problemloesenlernen.de).

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DEVELOPING PROBABILITY THINKING IN PRIMARY SCHOOL: A CASE STUDY ON THE CONSTRUCTIVE ROLE OF NATURAL LANGUAGE IN CLASSROOM DISCUSSIONS

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The aim of this paper is to analyse some mechanisms through which social interaction resulted in knowledge construction in a long term teaching experiment concerning random events and probability from grade I to grade IV. Performed analyses put into evidence a peculiar function of natural language in classroom discussion as a tool to transform the content of the discourse through interactive mechanisms of linguistic expansion based on key-expressions.

THEORETICAL PERSPECTIVE, AND PURPOSE

In the last decades, an important trend of research in mathematics education has been the increasing attention paid to language and semiotics aspects in the construction of mathematical knowledge, both in an individual and in a social construction perspective. This occurred in relationship with research advances in other domains (psychology, linguistics, hermeneutics). Let us consider the perspective of the "constitutive character" of natural language (see Bruner, 1986, Chapt.4): on one side, it suggested to consider whether other semiotic systems (in particular, algebraic language) share the same potential, and how students can approach the "mathematical realities" inherent in the specific expressions of those systems (cf Sfard, 1997, and Radford, 2003); on the other side, it opened the way to study how the "mathematical realities" are "constituted" during verbal activities in the classroom (cf Sfard, 2002).

With reference to these streams of research, we can take into account previous studies that are related to the issues considered in this report. Boero (2001) and Consogno (2005) consider how mathematicians deal with algebraic or natural language written expressions. In the case of algebraic expressions (Boero, 2001) crucial steps of a mathematician's activity consist in the reading of the algebraic expressions produced by him/her: sometimes this reading suggests ideas that go far beyond what the reader thought during the writing phase. The novelty can consist in the discovery of a possibility to simplify the expression, in the discovery of a new meaning, or in the anticipation of some moves that can allow to achieve the goal of the activity. In the case of natural language expressions, Consogno (2005) considers the flow of the writing/reading phases during individual activities of conjecturing and proving performed by undergraduate mathematics students. The *Semantic-Transformational Function* (STF) of natural language is the construct that accounts for some advances of their conjecturing and proving process. The student produces a written text with an

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 353-360. Prague: PME.

intention he/she is aware of; then he/she reads what he/she has produced. His/her interpretation (suggested by *key expressions* of the written text) can result in a *linguistic expansion* and in a *transformation of the content* of the text that allow advances in the conjecturing and proving process.

Douek (1999) is concerned with the analysis of the role of argumentation during classroom discussions aimed at the construction of mathematical concepts in activities of elementary mathematical modelling of physical phenomena. She identifies *lines of argumentation* whose development and crossing contribute to the enrichment of concepts both in terms of reference situations, operational invariants, linguistic representations (according to Vergnaud’s definition of concept: Vergnaud, 1990), and in terms of maturation towards the level of *scientific concepts* (Vygotsky, 1990, Chapter VI). The analyses show how a *line of argumentation* in some cases develops through someone’s interpretation of linguistic expressions produced by some others far beyond their intention in producing them.

The aim of the study reported in this paper is to see if the STF of natural language (see Consogno, 2005) can account for the development of a *line of argumentation* during a classroom discussion (by focusing on those phases when oral productions by some students are interpreted by other students), and how it works.

This report addresses two questions:

I) Can classroom social construction of mathematical meaning be interpreted in terms of STF (i.e. of semantic *transformations* that happen through *linguistic expansions* produced by someone, and suggested by *key expressions* uttered by some others)?

II) Can a student profit, in classroom discussion, by others' interventions (in order to develop his/her intuitions) through mechanisms that involve the linguistic transformations of his/her own expressions?

METHODOLOGY

This paper refers to a long term teaching experiment performed by the same teacher over four years (from grade I to grade IV) in the same class (18 students). The teaching experiment was part of a wider research project concerning the development of probabilistic thinking and the approach to elementary probability in primary school. Dealing with probability (in terms of the ratio between the number of favourable outcomes and the number of all possible, equally likely outcomes) was a point of arrival, in Grade III, of a sequence of didactical situations intended to promote the emergence of the idea that it is possible to forecast (on the basis of suitable analyses of objective elements) if an event is more likely to happen than another (for more details on the Project, see Gazzolo and Massi, 2003). The sequence of didactical situations from Grade I to Grade IV was designed around some steps considered as crucial in the individual and/or historical development of probabilistic thinking (cf Piaget, 1951, Fischbein et al., 1971; Fischbein and Gazit, 1984; Hacking, 1975). The steps related to the episodes reported in the following Subsections are:

A: The approach to the notion of "probability of an event" as the ratio between the number of favourable outcomes and the number of all possible equally likely outcomes. In the early history of probability, this construction meant the separation between the magic or fatalistic view of random events, and their evaluation in terms of objective measures of probability (Hacking,1975).

B: The distinction between a complex outcome (e.g. 4 as the result of the sum of the numbers obtained by casting two dice) and the elementary equally likely outcomes that bring to it (e.g. 4 as 1+3 or 2+2 or 3+1), in the perspective of developing the "combinatorial" approach to probability. According to Hacking (1975), this was a crucial progress in the early history of probability: Galilei struggled with it in the problem of determining the number that is more likely to result between 9 and 10, and between 11 and 12, if one casts three dice and adds their digits.

C: The reflection on the notion of random event (only depending on chance, not on the luck or the will or ability of the gambler). This step is relevant in young students' construction of a rational "objective" view of random phenomena.

Selected steps have been considered either as objects of specific tasks designed to promote the students' direct encounter with them (cf Episodes 2 for step B and 3 for step C); or as objects of attention for the teacher, in order to catch the opportunities offered by tasks designed to "go near" to the aimed point (cf Episode 1 for step A).

The whole teaching of mathematics and other disciplines in the classroom conformed to general educational choices that are usual for teachers engaged in our research team. From Grade I on, classroom discussions orchestrated by the teacher (Bartolini Bussi, 1996) follow individual productions, and prepare the elements for individual and then collective syntheses of the outcomes of the discussion (see Douek, 1999. Exhaustive oral and written wording of students' thoughts is strongly encouraged by the teacher as a tool to allow productive discussions.

The teaching experiment provided the research team, which the teacher was a member of, with the set of all students' individual written productions and all audio-recordings of discussions concerning probability and random events.

We identified as *salient episodes* those classroom episodes, where students (under suitable tasks designed according to the a-priori analysis), for the first time in the classroom history, produced ideas that (conveniently mirrored by the teacher and discussed under her guide) represented relevant advances in the probabilistic "manner of viewing" of the classroom. *Relevance* of those advances depended on their relationship with the crucial steps selected in the planning of the teaching experiment.

Salient episodes will be analysed according to the STF hypothesis, namely we will try to detect possible links between some verbal expressions produced by someone, and the development of the content of the discussion performed by some other(s) by expanding those expressions and transforming their meaning. In the last Section we will discuss possible limitations of this kind of analysis, as well as some possible developments.

SOME SALIENT EPISODES

Episode 1 (cf Step A)

Grade III: students approach the idea of ratio between the number of favourable cases and the number of all cases as a measure of probability of an event. Students are asked to make a choice between two games: the game with a coin (by betting on heads or tails), or the game with a dice (by betting on one of its digits). Giulia writes:

> In the case of the coin there are much more possibilities. For instance, suppose that in two labyrinths there are 2 paths (in the former) and 6 paths (in the latter). The 2-paths labyrinth offers more possibilities to get out, if in each labyrinth there is only one exit.

The text produced by Giulia is chosen by the teacher to feed a classroom discussion, because it can help the students to compare on the same, neutral ground (labyrinths) two different random situations. Note that in the text produced by Giulia (as well as in all the other texts) there is no trace of reasoning in terms of "ratio" *("more possibilities"* concerns only the comparison of 5 against 1). Note also that Giulia orients the discussion towards a simplified, yet abstract model of labyrinth. The teacher asks to take position on Giulia's text and to evaluate if her last sentence (*"The two paths labyrinth…"*) was necessary, or could have been omitted.

Anna: I agree with Giulia that in the 2-paths labyrinth you get out earlier, in the case of the 6-paths labyrinth you must try all the paths and you spend a lot of time.

Matteo: But in the 6-ways labyrinth you do not need to try all the paths, because for instance the first time you fail the exit, but then at the second or third trial you may find the good way to escape… You don't need to try all the paths!

Giovanni: It is necessary to consider the condition posed by Giulia, namely that there is only one exit, otherwise all the paths might have an exit, and it would not be a labyrinth any more!

Mattia: If a labyrinth would have more exits than paths with no exit, practically it would be very easy to escape, on the contrary if the labyrinth has the same number of paths and exits, …it would be easier but the exits must be more than one half of the number of the paths.

Some voices: less than one half!

Teacher: I would like Mattia to repeats his sentence - please, listen to him, then we will discuss what he said

Mattia: Can I make an example? In the 2-paths labyrinth there is one exit, while in the 6-paths labyrinth there are 3 exits; in order to make the 6-paths labyrinth easier than the 2-path labyrinth, you must put exits to more than one half paths, because if in the other labyrinth there are two paths and one exit, it is one half.

Anna's and Matteo's considerations seem to suggest Giovanni the reason why the condition posed by Giulia is necessary: the expression *"all the paths"* (be it necessary to try all of them, or not) can suggest the fact that if *"all the paths have an exit"* then it is sure that one can escape from the "labyrinth" in each trial. The situation is transformed by passage to a non-labyrinth limit situation. Then the extreme cases of one exit and six exits seem to open the way to Mattia to consider the number of exits in the 6-paths labyrinth as a variable that can take values between one and six. He tries to express the idea that the right comparison with the 2-paths situation must be made by considering *"one half of the number of the paths"* as the discriminating case. In terms of the STF, he performs some linguistic expansions (*"more exits than paths with no exit (…) the same number of paths and exits"* in his first intervention, and then *"an exit to more than one half paths"* in his second intervention) of the limit situations uttered by his schoolmates, which results in a transformation of the situation: the number of exits becomes a variable related to the number of paths. From Mattia's second intervention on, several more and more precise interventions will concern *"3 exits out of 6", "one half of the exits"*, and so on, till to the explicit comparison between 3 out of 6 and 1 out of 2 as *"one half"* in both cases.

Episode 2 (cf step B)

Grade III: in couples, students will throw two dice; they will bet on *odd* or *even* according to the number got by adding the digits of the dice. Before playing the game, the question is: *"Is it better to bet on odd or even?"*. Individual answers follow, and a discussion takes place. At the beginning of the discussion, students consider odd outcomes: 3, 5, 7, 9, 11; and even outcomes: 2, 4, 6, 8, 10, 12. *Even* seems more likely to come out because the number of even outcomes is bigger. But…

Elisa: I agree with Mattia, as he considers the results.

Giulia: Mattia has considered all possibilities, because he has considered the two dice and has put the results and (I think) has looked at all possibilities

Teacher: Is it the same thing to think of the result or to think of the two dice?

Mattia: It is the same thing … no… yes!

Giulia: If you think of dice… to the digit shown by your dice… because the result is one digit plus another digit that makes a result. Before adding them, those two numbers are alone, they are not together… because if one casts 3 and the other 4

Roberto: for instance, 4 is a number and 3 is another number, as Giulia told, if you add them, they make 7, but before putting them together, 4 is a solitary number and 3 is another solitary number, then when they go together we get a number formed by smaller numbers

Giulia: yes, but before getting the result, the two numbers can be other numbers.

The teacher asks Giulia to make an example, then she invites the other students to produce other combinations. The way is open to consider all possible equally likely outcomes. *NOTE that in Italian the "digits" of the dice are called "numbers".*

In the reported fragment Giulia re-elaborates the distinction (suggested by the teacher) between the dice and the result in terms of numbers: the addends and the results. The intervention of Roberto not only echoes Giulia's intervention, but expands it and suggests a transformation of the content (putting into evidence, by saying “ *for instance* ”, the fact that the couple 3 and 4 is an example; and the fact that the sum is a *"number formed by smaller numbers"*). Then Giulia is able to see how those *"smaller numbers"* can be different from 3 and 4. We can interpret *"a number formed by smaller numbers"* as the key expression that suggests a linguistic expansion that results in a semantic transformation of the original idea of Giulia.

Episode 3 (cf step C)

Grade IV: students compare throwing of two dice with the morra game in order to better understand what a random event is. The occasion of comparing the two games came from the comparison of the histograms of their outcomes after a relatively high number (120 for each game) of trials. Students start the discussion by considering some exterior features of the histograms; then they realise that the comparison must be made by considering the range 0/10 in the case of the morra game, and 2/12 in the case of the dice. A strange fact captures students' attention: in the case of the dice, the outcomes 2 and 12 are the less frequent, while in the case of the morra game 10 is one of the most frequent outcomes, and the frequencies of 0, 1 and 2 are very low.

Elisa: I think that the last number of the morra became high because you decide what number to cast, and, for instance, if you cast 5, probably also the other casts 5

Emanuele: Yes, probably he also casts 5, because sometimes it happens that it is easy to cast 5

Voices: yes, it is easier!

Giulia: 5 is very easy, because you must cast your number quickly, and so you open your whole hand and then 5 is easy

Matteo: with the dice, 12 is less frequent because it has only one possibility, and the same for 10 in the morra, but 10 is more comfortable, on the contrary with dice it is not easy that 6 and 6 happens frequently

Marco: I think that in the morra game 10 was frequent because when you cast your hand it is difficult that you cast the number you have thought, because you must do it quickly

Some reactions: but you do not know what number was cast by the other!

Danilo: Marco does not mean the sum of the two hands, but the number he is thinking of, he was thinking to cast 2 and indeed he casted 5 because 5 is easy to cast

Pietro: because when you make 5 you do open your whole hand

Evandro: two is difficult as well as 1, because if you play quickly you must close 4 fingers and leave 1.

At the beginning, Elisa explains how to get 10; her utterance offers Emanuele the possibility of an interpretation (i.e. a linguistic expansion that transforms the meaning of what Elisa told) in term of easiness to cast 5 (by the other too). Giulia adds an explanation in terms of body mechanisms (the easiness expands in her words into the representation of a physical action probably suggested by the key word "easy"). Matteo comes back to the comparison between 10 (morra) and 12 (dice) in terms of random/not random events. To say *that "10 is more comfortable"* is an interpretation of preceding interventions that expands into a more complex discourse, yet a transformed situation involving the comparison with the purely random outcome 12. The following part of the discussion brings to a comparison between the outcomes 10, on one side, and 1 and 2, on the other, in the morra game. Marco's intervention, appropriately interpreted by Danilo (in psychological terms) and Pietro (in physical terms), seems to offer Evandro the opportunity to shift to the case of 1 and 2. In this case the transformation realised by the linguistic expansion based on the key expression *"open your whole hand"* occurs as an alternative to the situation represented by Danilo's and Pietro's utterances.

DISCUSSION

The analyses of salient episodes seem to provide us with empirical evidence for the hypothesis that the STF of natural language can play a crucial role in social construction of knowledge through *linguistic expansions* based on *key-expressions*, which result in a *transformation of the content* of the discussion. However we must take some distances from such a kind of conclusion. Indeed, collected verbal data need an interpretation by the researcher, and this interpretation can be subjective. In the second episode Giulia could have continued to think about her intuition independently from Roberto's utterances, and finally she could have realized that *"the two numbers can be other numbers "*. One way to escape subjectivity in interpretation might have been asking Giulia where the idea of *"other numbers"* came from. It would have been necessary to ask this question immediately (in order to avoid arbitrary re-constructions based on the subsequent steps of the discussion); but the effect might have been to disturb the flow of the discussion. The main aim of a teacher in classroom situations is different from the aim of a researcher in laboratory situations! We think that another way to reduce subjectivity might be to collect a lot of episodes of advances in classroom discussions and check the frequency of those cases in which advances can be interpreted in terms of STF. If the frequency is high, the plausibility of the hypothesis is reinforced. For instance, in the case of our teaching experiment we have identified 9 advances that occurred in classroom discussions: 7 of them can be interpreted in terms of the STF of natural language.

Let us suppose now that in the reported episodes things worked as it is plausible that they worked according to the STF hypothesis. An interesting issue related to research questions A) and B) concerns the roles of the others' interventions in the discussion. While in the case of the interpretation of one's own written production the unique role of the written text is that of an external reference for an inner evolution of thought, in the case of social interaction the others' interventions play different functions. The voice of a student can provoke (through specific key expressions) an interpretation by another student related to his/her perception of the evoked situation, that comes back to the first student as an enrichment or a transformation of his/her original intuition (like in Episode 2). In that case the other student plays a role that could be interiorised through a mechanism of inner dialogue supported by a written text (like in the episodes analysed by Consogno, 2005). In other situations a chain development happens: different students can play complementary roles to transform the situation under consideration. It may happen that focus moves from a situation to the opposite situation (like in Episode 3); or that two complementary interventions open the way to the consideration of the whole range of possibilities between the two evoked (like in Episode 1). In the last case social construction of knowledge seem to reveal its highest potential. According to these considerations, focussing on the STF of natural language seems to offer the researcher the possibility of classifying different patterns of social construction of knowledge in terms of different mechanisms of linguistic expansion. This suggests the need of characterising the variety of "linguistic expansions" that are of interest in the perspective of the STF of natural language.

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COLLABORATION WITH TEACHERS TO IMPROVE MATHEMATICS LEARNING: PEDAGOGY AT THREE LEVELS

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Current trends in educational professional development favour large-scale professional development in-practice and on-site, involving collaborative planning between teachers and researchers. However, little research is available into aspects of professional development in-practice which effect change in both teacher practice and student learning outcomes. This paper investigates the characteristics of instructional interactions, namely, technical, domain and generic levels of pedagogy that appeared to lead to positive mathematics learning outcomes within one such project.

Within Australia, the focus of major areas of government research funding applicable to mathematics has moved to projects aiming to improve students' mathematics learning outcomes in clusters of schools through large-scale, professional development (PD) in practice, on-site projects which involve researchers and classroom teachers in collaborative planning of teaching units (Baturo, Warren & Cooper, 2004; Bobis, 2004; Higgins, 2004). If well planned, projects involving researchers and teachers jointly planning instructional sequences and classroom interactions (in some cases modelled by researchers) are able to investigate simultaneously both teacher professional learning and change and student learning and development. However, while large-scale professional development in practice is currently in vogue, little is known about the characteristics of instructional interactions that lead to both positive learning outcomes for students and sustainable change in practice for teachers (Bobis, 2004; Higgins, 2004).

This paper. Utilizing a case study approach, this paper will discuss a study of mathematics instructional interactions within two schools participating in a larger PD in practice project in Queensland, Australia. It will further identify and discuss three pedagogy levels observed in the study sample that appeared to lead to positive learning outcomes for both students and teachers, demonstrating the dependence of the success of PD in practice projects on the effectiveness of planned and modelled classroom interactions.

In the Australian context, the tendency for mathematics to be subsumed into literacy as quantitative or scientific literacy within government policy, curriculum development, and the popular press has resulted in a lack of professional learning programs that focus on how to teach mathematics concepts and processes effectively. Queensland Studies Authority (2004) argues the need for students to be effectively taught more than a range of simple quantitative literacy skills required for everyday living. In order to become effective, competent members of communities, students need to be taught a range of mathematical skills that they can flexibly apply to diverse problem solving contexts. However, not all teachers have the combination of pedagogical skill and structural mathematical knowledge required to teach mathematics effectively (Baturo, 2004).

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Classroom interactions. Effective classroom interactions for mathematics require students to construct their own knowledge through interactions and investigations with materials and language and develop their own meanings through discussion with teachers and peers (English & Halford, 1995). That is, in a social constructivist learning environment in which the teacher provides guidance (i.e., cognitive scaffolding) for students so that they can move from dependency upon experts or peers to being able to solve problems independently (Baturo et al., 2004; RAND, 2003).

Successful mathematics classroom interactions can be separated into the four parameters of tasks, talk, tools, and expectations and norms (Askew, Brown, Denvir, & Rhodes, 2000). The interactions are effective if the separate dimensions of these parameters meet the requirements of best practice, namely: (a) tasks incorporate mathematical challenge, integrity and significance, and engage interest; (b) talk is well managed and encompasses teacher talk, teacher-pupil talk, and pupil talk that facilitates mathematics learning; (c) tools encompass a range of modes (oral, visual, kinaesthetic) and types of models that have been found effective in mathematics instruction; and (d) expectations and norms empathetically create a community of learners (Brown, 2002). It is evident that teacher professional learning should be based on excellent mathematics classroom interactions that model combinations of classroom tasks, tools, talk and expectations and norms that are effective in facilitating students' social construction of knowledge.

Mathematics classroom interactions also reflect the quality of the mathematics content and pedagogy knowledge held by teachers (Baturo et al., 2004; RAND, 2003; Shulman, 1986). If the interactions are a result of collaboration between researchers and teachers or modelling by researchers, they should embody what research shows to be quality mathematics content and pedagogy knowledge. That is, the interactions should be designed so that reflection on them simultaneously strengthens teachers' mathematics knowledge, remediates their mathematics misconceptions and builds their pedagogical knowledge of effective teaching sequences of tasks.

Professional learning. Effective professional learning is dependent on four closely interrelated factors, namely: *domain* – the form and type of external source of information or stimulus used by teachers to develop new ideas and practices; *teacher* – the characteristics of teachers including their knowledge, beliefs and attitudes; *classroom trials* – the support for and extent of any trials or experiments with new ideas; and *consequences* – the outcomes of trials as perceived by the teacher, particularly related to student learning and behaviour (Clarke & Hollingsworth, 2002).

Professional learning is optimised when the professional development: (1) enlists support from administration, students, parents/carers, and the broader school community; (2) allows teachers to identify the issues; (3) solicits teachers' conscious commitment; (4) models desired classroom approaches; (5) addresses impediments to teachers' professional growth; (6) realises changes in teachers' beliefs about teaching and learning are derived largely from classroom practice and observing positive student learning; (7) allows time and opportunities for planning, reflection, and feedback in order to share "wisdom of practice"; (8) enables teachers to gain ownership by their involvement in decision-making and by being regarded as true partners; (9) recognises that change is gradual, difficult, and often painful; and (10) encourages participants to set further goals for their professional growth (Clarke, 1994).

THE WIDER PROJECT

The study of the two schools which is the focus of this paper is part of a wider PD in practice project set up by the Australian Government to study factors of mathematics teaching that will enhance mathematics outcomes for students. It was undertaken in Queensland by assigning experienced mathematics-education lecturers/researchers to elementary schools, one researcher to one school (see Baturo, Warren & Cooper, 2004, for a report on the project). The requirement was for the researchers to collaborate for one year with five teachers at their school to exemplify and document best practice in mathematics teaching and learning. There were 8 schools and 37 classrooms in the project. There were 2 schools and 9 classrooms in the study.

The research methodology for both the study and the PD in practice project was mixed method (Brewer & Hunter, 1989), comprising quantitative and qualitative data collection and analyses to facilitate both investigator and methodological triangulation (Cohen, Manion, & Morrison, 2000). The collaborations combined *case study* (Yin, 1994) and *collaborative action research* (Kemmis & McTaggart, 1988) in order to go beyond typical development in natural environments to focus on induced development in controlled environments (Lesh & Kelly, 2000).

Participants. For pragmatic reasons, the elementary schools in the study and the project were chosen to represent the range of remote, semi-rural and urban areas in Queensland rather than excellence in mathematics teaching and, sadly, rather than any a-priori interest amongst teachers at the school in improving mathematics learning. Therefore, the study and the project became professional learning programs in which researchers attempted to involve principals and teachers, many with little commitment to the project, in collaborative programs to improve existing teaching practices and mathematics learning outcomes. The participants in the project were the administrators of the school, the teachers of the collaborating classes, and the students from these classes.

Because of the way schools and teachers were chosen, the researchers could not assume inherent interest or motivation for participation in the project, and therefore had to accept that the teachers may be cynical pragmatists who only engage in professional learning for immediate and successful applications in their classrooms. Thus, the researchers' starting points became the current interests of the teachers (what they were teaching at that time, what they identified difficulties with, and where they would like help). Once this was negotiated, the researchers and teachers worked collaboratively to plan, implement and evaluate new mathematics teaching programs.

Procedure. Quantitatively, the students' mathematics achievement and attitudes in both study and project were measured before and after the collaboration (by tests and surveys) and compared with control schools. Qualitatively, the collaborations were investigated through observations, interviews, journals and artefacts. The procedure for the collaborations was for the researchers and teachers to work together, planning, implementing and gathering data on trials. Analysis of the collaborations was completed by writing rich descriptions of each classroom from transcripts of observation videotapes and interview audiotapes and comparing descriptions within and across classrooms and within and across schools. For this paper, the focus is on the 9 classrooms from the 2 different elementary schools in the study, with each classroom undertaking collaborations in more than two, and sometimes up to five, different mathematics topics.

Professional development. The professional learning within the study and project was designed to incorporate all ten of Clarke's (1994) criteria for successful professional development. However, because of the selection process, much of criteria (1) and (3) (support of administration and teacher commitment) had to be done after selection of the school and, even though there was government money for schools, criteria (7) and (8) (teacher release time and teacher involvement in decision-making) were dependent on the principal and the availability of relief teachers in small and remote areas. In particular, the extent criteria (7) was implemented depended on the opportunities within the teachers' routines for release from classroom duties to plan, share and reflect on the classroom practices they were trialling.

Therefore, the success of the professional learning was dependent on the quality of the support relationships between the researchers and the teachers, that is, the immediate practicality of the resources and activities, the extent to which the researchers could provide "just-in-time" support, the quality and depth of this support, and the students' responses to the trials (Baturo et al., 2004; Bobis, 2004).

LEVELS OF PEDAGOGY

Within the 9 classrooms in the study, some collaborations were more successful than others in terms of student learning of mathematics and teacher learning of effective mathematics classroom interactions. Comparisons of observations across classrooms showed that the highly successful collaborations contained pedagogies that operated at three levels: technical (practical), domain (relevant to the topic), and generic (across topics). This was validated when some unsuccessful collaborations were enhanced by the addition of a missing level.

Technical pedagogies. Technical pedagogies are the practical "tips" that make lessons function (e.g., how to use the constant function process to allow a simple calculator to count in tens). Two examples of this level of pedagogy were evident in more than one classroom in the study. The first example was to do with teaching the part-whole fraction sub-construct and occurred in collaborations in three classrooms. In the collaborations, the teachers used folding of paper shapes to introduce partitioning and unitising. However, the teachers were unable to involve the students in the folding until they had learnt techniques to quickly fold rectangles into thirds, fifths, sevenths, etc., and circles into thirds and sixths. The second example was with teaching the Euclidean transformations (flips slides and turns) and occurred in collaborations in two classrooms in the study. In the first collaboration, the lessons to have the students construct the transformations failed because the transfer of the transformation from the tracing paper back to the paper was difficult. In the second collaboration, this was overcome by the simple technique of shading the back of the tracing paper with pencil.

Technical pedagogies could be considered to be similar to Sternberg's (1997) *practical* skill. In Sternberg's theory of "successful intelligence" there are three skills in teaching: analytic - compare and contrast, connect, analyse, evaluate, critique, judge, explain, evaluate; creative - create, invent, imagine, design, suppose; and practical - apply, show how, implement, utilise, demonstrate). They were evident in the overall analysis of the 37 classrooms in the wider project which found that lesson planning required knowledge of materials and language, student misconceptions, *technical aspects of using the materials*, and generic teaching strategies (Baturo et al., 2004).

Domain pedagogies. Domain pedagogies are the normal teaching and learning methods appropriate to teaching a particular topic. They are, in Askew et al.'s (2002) and Brown's (2002) terms the tasks, talk, tools, and norms and expectations that will facilitate students social construction of the numeration topic and enable teacher reflection to garner insights into the approaches to pedagogy and the crucial underpinnings of mathematics content required for learning within the topic.

Three specific examples from the study show the importance of this level of pedagogy. First, in the three collaborations on the part-whole fraction subconstruct referred to in the section on technical pedagogies, classroom observations showed the importance of maintaining the whole, illustrating partitioning and unitising, and relating names and symbols to the number of equal parts into which the whole has been divided (in particular, ensuring that the relationship between the parts and the whole is kept visible while students unitise the parts into a whole). Such practices were found to extend to other materials (e.g., paper plates, virtual shapes on computers), models (e.g., length and set), subconstructs (e.g., partitive-quotient), and fraction topics (e.g., equivalent fractions). Second, two collaborations in the study on percent showed the power of using 10x10 grids and double number lines in representing percentage and percent problems. Third, a collaboration on the multiplicative comparison concept of multiplication showed the necessity for understanding multiplication in terms of change and the use of arrow diagrams to compare amounts multiplicatively.

Generic pedagogies. Generic pedagogies are approaches to teaching that can be used with most mathematics topics. Utilised in this project were the three generic pedagogies promoted by Krutetskii (1976) and one identified by Hershkowitz (1989): (a) *being flexible* in all representations (e.g., 61 is 6 tens and 1 one, 11 more than 1/2 a hundred, and one hour and one minute); (b) *reversing* or connecting representations in both directions (e.g., "determine the number of lines of symmetry in this shape" is reversed by "construct a shape with 3 lines of symmetry"); (c) *generalising* or "searching for patterns, structures and relationships which transcend the particulars of the data or symbols" (RAND, 2003, p. 38) (e.g., commutative and distributive laws; part-whole structure of fractions; inverse); and (d) using *non-prototypic* examples (e.g., shade 0.24 in this 5 x 20 grid or this chevron-shaped figure) (Hershkowitz, 1989).

These generic pedagogies were shown to be important in planning effective instructional interactions transforming mathematics instruction and significantly improving learning outcomes in three collaborations on fractions in the study (Baturo, 2004, describes the powerful effect of generic strategies in one of the collaborations). In these collaborations, the generic pedagogy of reversing (e.g., denoting a paper rectangle as a whole and constructing 2/3rds and then denoting the same rectangle as 2/3rds and constructing a whole) was found to be effective in building strong fraction understandings, and in enabling teachers to see the power of the part-whole relationship in percent, ratio and rate problems.

The collaborations also showed that learning from interrelationships between representations is enhanced if connections go both ways (e.g., from concrete to symbols and symbols to concrete) and if there is a multiplicity of representations (including non-prototypic as well as prototypic). The importance of these generic strategies was also shown in the wider project (Baturo et al., 2004).

IMPLICATIONS

This paper is too short to explore all the implications of the three levels of pedagogy but four are particularly interesting. First, classroom interactions in which the three pedagogy levels were present were highly effective for students because: (a) the first level, technical, ensured that the lessons worked in a practical manner; (b) the second level, domain, ensured that the particular topic was covered fully and in an appropriate learning sequence; and (c) the third level, generic, enabled deeper insights to be garnered and tethered the topic to other mathematics areas as an interconnected structure. The presence of the three levels appeared to mean that interactions were immediately applicable and translatable into the classrooms and worked in terms of student learning (i.e., were effective in terms of classroom pedagogy and mathematics outcomes – the talk, tools and expectations and norms were appropriate and enhanced learning).

Second, the presence of the three pedagogy levels was effective for teachers because it enabled reflection to transcend the specific mathematical foci of the actual classroom tasks and to operate on classroom techniques, domain practices and generic pedagogies simultaneously. The technical activities appeared to motivate and engage the teachers because of their practicality, the domain methods supported the teachers interest because they ensured successful student understanding, while the generic pedagogies sustained continued interest by providing a window to other uses.

Third, planning and development of instruction that incorporates all three pedagogy levels requires strong mathematics and pedagogy knowledge. For example, for the collaborations on Euclidean transformations, the teachers had to acquire pedagogic knowledge (including technical knowledge of procedures and techniques with tracing paper) as well as substantive mathematics knowledge of transformations to complete any lessons because they had no prior knowledge of task, talk and tools associated with the transformation (e.g., Mira mirrors, tracing paper). Both mathematics and pedagogy knowledge is necessary but neither is sufficient on its own. However, even limited increases in teacher knowledge can lead to student improvements. Most of the eight teachers only had procedural mathematics knowledge and were therefore restricted to using the instrumental pedagogy of explanation and practice. When provided with semantic and structural mathematics knowledge, their pedagogic knowledge was sufficient, with a little assistance in the use of tools, to improve mathematics outcomes. For example, one teacher who taught decimal numeration as a rote extension of whole-number numeration, improved her students' decimal outcomes when she understood the relation of decimal place values to the part-whole fraction concept and learnt partitioning pedagogic techniques to teach this concept. In another example, a teacher had reasonable knowledge of fraction concepts and processes and excellent pedagogic knowledge. She significantly improved students' mathematics learning outcomes when provided with structural knowledge of fractions and generic strategies of reversing, and non-prototypic examples (Baturo, 2004).

Fourth, and most interestingly, there appears to be a relationship between mathematics structure and the action of pedagogies at all levels. For example, fractions, percent and probability all have a part-whole basis. Thus they are all amenable to pedagogies that relate parts to wholes such as geometric shapes and area models, number lines and length models and set models. The teaching fractions, percent and probability all benefits from knowing partitioning techniques and the generic pedagogies of reversing and using non-prototypic examples.

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“AIM HIGH - BEAT YOURSELF”: EFFECTIVE MATHEMATICS TEACHING IN A REMOTE INDIGENOUS COMMUNITY[1]

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In 2004, a young, non-Indigenous, second-year teacher in a remote Queensland Indigenous community developed a mathematics Unit based on an “Aim High - Beat Yourself” theme he developed to overcome the perceived unwillingness of his students to achieve in both sport and school. This paper investigates the apparent effectiveness of this Unit and draws inferences for mathematics teaching and learning in Indigenous communities. It describes the research and teaching contexts in which the Unit was developed and the students’ responses to the Unit. The paper provides further evidence for the efficacy of integrating mathematics learning with more generic programs that build pride, confidence and self worth in Indigenous students and challenge them to perform (Sarra, 2003).

Motivated by the educational disadvantage of Indigenous students (Bortoli & Creswell, 2004; Long, Frigo, & Batten, 1999), the researchers undertook a research project in outback Queensland with teachers and teacher-aides in schools with Indigenous students to improve mathematics learning outcomes. Indigenous students’ performance in mathematics at the Year 3 level in Queensland varies on average at least two years from non-Indigenous students (Queensland Studies Authority, 2004). This lower performance can reinforce notions concerning racial differences in intelligence and contributes to a perceived lack of self worth (Durodoye & Hildreth, 1995). As well, being innumerate can be profoundly disabling in every sphere of life including home, work and professional pursuits (Orrill, 2001).

This paper describes the activity of a teacher (“Jim”) whose Aim High-Beat Yourself approach to mathematics teaching and learning in a small remote community school appeared to have a profoundly positive effect on his students and on the community. Implications are drawn from this practice for effective teaching of Indigenous students.

Indigenous learning of mathematics. As mathematics reflects white middle-class culture (Walkerdine, 1992), learning mathematics in remote communities can be problematic for many Indigenous students who can perceive mathematics as a subject where they “must become ‘white’ to succeed” (Matthews**,** 2003, p. 1), challenging their Indigenous identity **(**Howard, 1998; Pearce, 2001). Teachers tend to have low mathematics expectations of Indigenous students, blaming low performance on absenteeism, social background and culture (Cooper, Baturo, Warren, & Doig, 2004; Sarra, 2003), and to devalue Indigenous cultures as primitive and simplistic (Matthews, 2003). To ameliorate this cultural clash, there is a move towards

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contextualisation which incorporates Indigenous culture and perspectives into the pedagogical approaches to mathematics education (Cooper, Baturo, & Warren, 2005; Matthews, 2003). Contextualisation is being seen as the way to overcome the systemic issue of Indigenous marginalisation with respect to mathematics learning (Board of Studies, 2000; Cronin, Sarra, & Yelland 2002) and to instil a strong sense of pride in the students' Indigenous identity and culture (Cronin et al, 2002; Sarra, 2003).

There is ambivalence about the capacity of schools to be effective with contextualisation. Non-Indigenous teachers with little understanding of Indigenous culture have difficulties with contextualisation, causing some of them to reject it in favour of familiar approaches (Board of Studies, 2000; Connelly, 2002; Howard, 1998). Some teachers in the region believe that Indigenous culture has no place in mathematics (Cooper, Baturo & Warren, 2005). The modern history of Indigenous cultures is one of subservience to Western culture (Kawagley, 1995) which has two consequences: the first is powerlessness and the social problems of modern Indigenous communities (Fitzgerald, 2002); the second is Western ignorance of Indigenous culture and the development of stereotypes (Matthews, 2003).

While it is difficult to determine the stereotypical basis of research findings, one thing is clear – a teaching program can dramatically improve Indigenous students' mathematics learning outcomes if it reinforces pride in Indigenous identity and culture, encourages attendance, highlights the capacity of Indigenous students to succeed in mathematics, provides a relevant educational context, and challenges and expects students to perform (Sarra, 2003). Indigenous students' low performance in mathematics appears to be because of failures to make mathematics culturally appropriate and systemic beliefs that the gap in educational outcomes between Indigenous and non-Indigenous Australian students is "normal" and that educational equality for Indigenous Australians is either not achievable, or only achievable over a long period of time (MCEETYA, 1999).

RESEARCH CONTEXT

The project in which Jim was a participant explored remote primary Indigenous students' learning of mathematics for three years (2002 to 2004) in order to determine and document effective ways to teach mathematics to these students and to develop professionally teachers and teacher-aides. It was based on the expectation that schools must make a difference to Indigenous students' mathematics achievement and should seek strategies to enhance their mathematics learning (Cataldi & Partington, 1998). Although the schools in the region with Indigenous populations attract teachers who are nearly always non-Indigenous, young, inexperienced and commonly leave after two years (Cooper, Baturo, Warren, & Doig, 2004), the teacher-aides, in contrast, are Indigenous, older, more experienced, have strong commitment and connections to the local community and should therefore be the key to teaching success in a school with Indigenous students (Clarke, 2000). The research project (Cooper et al., 2004; 2005) found that teachers lacked mathematics and pedagogic knowledge for Unit development, and that aides were untrained, under-utilised and not seen as partners in facilitating mathematics learning. To assist the teachers and aides with contextualisation of mathematics instruction in relation to local culture, community involvement was also an imperative. Reshaping interactions between Indigenous aides, non-Indigenous teachers, Indigenous students, and the local community was a major focus of this project.

TEACHING CONTEXT

Learning space. The Indigenous community in which Jim taught was relatively homogenous with community members predominantly belonging to the tribe who lived in the area. Adult employment levels were very low necessitating a reliance on welfare, leading to some of the problems described by Fitzgerald (2002) as endemic to remote Indigenous communities (e.g., alcohol and substance abuse, low health, transient population) but noticeably less than in other communities. A particular strength of the community was its attitude to schooling and to youth behaviour. The Elders and the community in general believed that school was important resulting in student attendance close to 100%. The community believed strongly in sport, and travelled as a community to all school sporting activities.

At the time this paper describes (2004), the school's numbers had declined – as a result there were two classes only. The Years P-3 teacher ("Kay") was an Indigenous member of the community while the Years 4-7 teacher was Jim. The principal was new and young with no experience of leadership, having been a classroom teacher for only a few years before this appointment. There were fewer teacher-aides than the year before, with Jim having one Indigenous teacher aide ("Ruth").

Beliefs and attitudes. Jim's first teaching post was in this community (2003-4). He was trained in Brisbane over 2000 km away so this community was a 'culture shock" for him in terms of its isolation, small population, and cultural make up. Similar to many "white" teachers of Indigenous students, he believed that the best way to teach mathematics was "hands-on" activities with a purpose (so that it is not "pointless or just abstract to the kids") (Cooper, Baturo, Warren, & Doig, 2004) in small group rotations. He believed that consistency in language and activity between the teaching staff was essential: "all the staff need to have the same language to be spoken so it doesn't confuse, and teach the same thing and not rush ahead with extra stuff". He believed that it was important for students to be responsible for their own learning and that it was his role to build confidence and motivation to achieve this.

Jim felt that Indigenous students were motivated and supported by their families, but also believed that "the kids out here look for the easy answer, rather then look for the process" and that "some kids get lost" when there are abstract concepts and problem solving that have several steps. He saw a need to set expectations "you know they can reach". He was concerned that his students seemed to forget previous mathematics instruction so he believed he had to overlap ideas in his sequencing. He believed Indigenous students were "very much about what is in front of them" and saw goal-setting as the major difference between Indigenous and non-Indigenous students – "getting the kids to aim beyond what is in front of them – to beat themselves". He believed that community involvement was important for learning stating, "I think … there is a close connection between the child's background and culture." Rather than perceiving deficits in his students' community (Cooper, Baturo, Warren, & Doig, 2004), Jim used the strength of the community focus on sport to support his thematic unit of work.

The teacher aide, Ruth, had been at the school for 15 years. She had no qualifications but had worked with many teachers. She showed a lot of initiative with the students, always willing to take responsibility with respect to students' behaviour and always encouraging them to work. Jim used her to maintain the lower-performing students' attention in situations where they may become distracted, and for contextualisation in areas with which she was familiar: "I am always talking to [Ruth] about the plants because she knows the plants". He described Ruth as having good rapport with the students and being a good support for him: "she shows a lot of initiative to reach the kids … they are willing to take risks with someone that they are comfortable with". However, he was also aware that her weaknesses in mathematics meant that sometimes she did not maintain consistent language and taught ideas for which the students were not ready.

FINDINGS

For the project, Jim developed and undertook a Unit in Chance and Data which would: (a) be cooperatively planned and taught in partnership with his teacher aide, Ruth; (b) cater for Indigenous students in terms of providing culturally appropriate contexts for the instruction; and (c) involve members of the local Indigenous community in developing and implementing the classroom units. Jim was able to integrate the mathematics Unit into the "Aim High - Beat Yourself" approach to teaching he was developing for all his instruction.

Design of the Unit. Jim's attitude towards integrating Indigenous culture into his teaching was interesting. He had taken it into account in his literacy program: "and looked at some different, some aboriginal words and then looked at their meanings and they wrote the aboriginal words into a sentence which they thought was hilarious". The Indigenous Years P-3 teacher, Kay, took his class every Friday for lessons on local Indigenous culture. The school also celebrated Aboriginal week with a camp in which local Elders taught the students about bush survival skills and food and local animals. Jim felt that this activity meant that he should not explicitly bring culture into his mathematics teaching: "See, I tried to avoid, I honestly try to avoid the culture side of it because there's so many other people who could teach it better".

Jim's position was that it was not the overt aspects of Indigenous culture that had to be the focus of his teaching but rather the covert and tacit aspects. In this, he had identified something akin to the notion of shame which the literature (e.g., Board of Studies, 2000; Nichol, 2002) argues makes Indigenous students act so that they do not stand out from the group by either obviously failing or succeeding. Unlike the literature, Jim's experience of this cultural impact was mainly in terms of failure. Jim's answer to this situation was to get the students to focus on "beating themselves" not each other, to ignore winning and losing and to focus on doing their best. He started by reshaping the students' view of winning:

> they'd always get to the end of the race, … who won? … I said that person won because they're the tiredest. … if you haven't tried your hardest, you haven't won.

Jim used sport to introduce his Aim High-Beat Yourself approach:

> I keep trying to challenge them, and to push themselves … I use Cathy Freeman [an Australian Indigenous Olympic Gold medal winner] as an example … she trains by herself, so how does she win? … she's got to beat herself

In planning meetings, Jim proposed adding graphing from the Chance and Data syllabus strand to the Aim High-Beat Yourself Measurement and Sport Unit that he had started earlier in the year. By getting the students to graph their own sport times and distances, he could combine Sport, Measurement, and Chance and Data; and by restricting the graphs to comparisons of students' measurements against their earlier measurements, he could reinforce the "beat yourself" approach. He believed that this combination would enable the motivation of sport to be integrated with the "hands-on" of measurement and the personal focus of graphing and comparing within each student's performance.

> … I wanted to motivate them, so I started with measurement, then we could go out and do measurement outside and stuff like that … cause if there's any way you can motivate these kids, and build their confidence, it's through sport, …

Jim planned to extend the graphing and the Aim High-Beat Yourself approach to literacy and numeracy, for example, to graphing the number correct in a basic-facts drill activity. He involved estimation and checking in the activities and used drawings to record what was being measured.

Student responses. Jim reported that his students' mathematics performance was low (even though their attendance was high), a situation we observed in the previous year when another teacher had Years 5 to 7. He realised that the Indigenous students' low performance was more a problem of resources available to the school rather than his teaching ability; he remained hopeful:

> ... I know if I try to compare what the kids are achieving compared to kids from Brisbane that have got that really rich environment, both home and at school, you know like it'd just be depressing, you'd think that you're a total failure but then at the same time you know that you're not because the kids are still learning …

He was unsure of the correctness of his Aim High-Beat Yourself approach for mathematics, simply wanting to know "if I'm on the right track." He was unhappy with his students' progress on graphs, particularly with respect to scales:

> … they're used to collecting data, but the graphing … they're good when you're there with them, but … to let them go and say you work out the measurements to go up the side, they're just like, no idea.

However, he had evidence of the success of his approach in sport:

> There's another girl from another school that has been flogging them all of their lives anyway so this 800 metres … they went off with this girl and as usual she got way out in front. And usually what would happen in the past, they'd just give up … This year, with teaching them Aim High-Beat Themselves … our girls just kept running and by the halfway through the second lap, they'd passed her!

This sporting success happened across all the athletics races and led to a change in the community. In previous years, when Jim had tried "to push the kids", parents had complained to him to make him "back off", but this year, as his approach had produced success in sport, parents began to support him and his approach – and not just in sport; parents were stating their support for high expectations in education in general. As well, examples of transfer from sport to academic activities had started:

> … this boy that's running well, I got his confidence up by praising him with his running and I knew he was smart in the class but he wouldn't do anything and then all of a sudden he thought, I'll have a go and he jumped 12 levels in a term …

On top of this, the students' graphing skills had picked up and they became proficient in gathering data and constructing tables and better at drawing graphs for a variety of purposes, including their own mathematics performance. Jim believed that the graphing had been particularly successful in changing the students' attitudes.

FINAL COMMENTS

There is not the space in this paper to draw all the inferences from Jim's teaching, although three are fairly obvious:

First, although his teaching approach was inspirational, "first-year-out" teachers should not be the teaching staff of remote Indigenous schools where four Year levels are in each class. In particular, the incentive used to attract teachers to remote communities, that in two years they would receive enough points to have priority in transfer to any school in Queensland, is not supporting Indigenous learning. Rather than producing a continuous turnover of inexperienced teachers, more support must be given to ensure teachers who start to make a difference can stay in the community. Having gained sufficient points from his two years, Jim left the community at the end of 2004 for a school in Brisbane. A new "first-year-out" teacher has taken his class.

Second, Jim's success reinforces findings elsewhere (e.g., Sarra, 2003) that Indigenous students' mathematics performance can improve if instruction: (1) is integrated into generic programs that focus on building pride, confidence and self worth in Indigenous students and challenge them to perform, (2) involves and impacts on the community in general as well as the school, and (3) takes account of Indigenous culture, particularly its strengths, in a positive manner. Jim's initial focus on sport took advantage of an important component of community life. His Aim High-Beat Yourself approach allowed high expectations to exist with practices that enabled the students to achieve success without shame. His focus on students' own practice and his redefinition of winning as "doing your best" circumvented an aspect of Indigenous culture that has made it difficult for Indigenous students, and people in general, to achieve in the competitive aspects of Western culture. It is particularly important for mathematics as it is a subject where success and failure are very public.

Third, Jim's experience also provides some indication that involvement in a generic program is necessary but not sufficient for improving Indigenous students' mathematics outcomes. Jim had introduced graphs on grid paper as a way of recording data but his students had not explored the meaning of scale through early work with squared paper. Therefore, although they used graphs in many situations, the students remained uncertain with respect to scale. Contextualisation, student identity and community support are undoubtedly important for Indigenous students to achieve success, but proficient teacher mathematics and pedagogic knowledge is still required for effective mathematics learning.

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DEVELOPMENT OF CHILDREN'S UNDERSTANDING OF LENGTH, AREA, AND VOLUME MEASUREMENT PRINCIPLES

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This study investigated the concurrent development of children's understanding of length, area, and volume measurement in Grades 1-4. From a 45-minute interview, an attempt was made to derive three parallel scores assessing understanding of each of five measurement principles: the need for congruent units, the importance of using an appropriate unit, the need to use the same unit when comparing objects, the relationship between the unit and the measure, and the structure of the unit iteration. Valid comparisons could only be made across three of these five principles. On the others, student scores increased with grade, with length preceding area and area preceding volume in most cases. The results have several implications for teachers.

Measurement has long been regarded as an important component of numeracy (Cockcroft, 1982). It constitutes a separate strand in the official syllabuses of all Australian states and territories, and receives a similar emphasis in the US and the UK. The focus is on the measurement of the spatial attributes of length, area, and volume, and most concepts are expected to be mastered in primary school.

School syllabuses frequently make statements about children's measurement learning that are at least partly based on research. For example, the recently revised New South Wales mathematics syllabus outlines a developmental sequence for the early grades that includes the following statement of the basic principles of measurement:

> Students develop the key understandings of the measurement process using repeated informal units. These include:
>
> 1. the need for repeated units that do not change
> 2. the appropriateness of a selected unit
> 3. the need for the same unit to be used to compare two or more objects
> 4. the relationship between the size of the unit and the number required to measure, and
> 5. the structure of the repeated units. (Board of Studies NSW, 2002, p. 91)

Most of the research on which such statements are based has investigated length, area, and volume measurement concepts separately. This paper describes an attempt to validate the development assumed in syllabus documents and in particular to relate children's development of the three attributes.

THE RESEARCH BASIS

Each of the above five principles arises from a strand of empirical research.

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There is strong evidence that, in the early stages of learning about measurement, children do not see the need for congruent units when measuring individual objects (e.g., Bragg & Outhred, 2001, for length and Outhred & Mitchelmore, 2000, for area).

The second principle not only refers to the need for appropriately sized units but also to the need to use a length unit to measure length, and so on. This apparently obvious requirement is not so obvious to young children. For example, Bragg & Outhred (2001) found that young children often use a centimetre cube to measure the perimeter of a rectangle but are unaware that they are using the length of its edge as the unit of measurement and not the area of its base or its volume.

The third principle is an extension of the first and is often referred to as transitivity. Hart (1981, p. 13) found that more than 50% of 12 year old students failed to use this principle on a length comparison task.

The fourth principle has a history of research going back to Carpenter (1975). This relationship is, in fact, not needed to carry out a measurement in practice but is rather an interesting consequence of the other principles.

The fifth principle involves two aspects: Firstly, the object being measured must be covered with no gaps or overlaps. Several studies have revealed students' difficulties with this principle (Bragg & Outhred, 2001; Outhred & Mitchelmore, 2000). Secondly, area and volume units form rectangular arrays. The development of this understanding has been studied extensively (Battista & Clements, 1996; Outhred & Mitchelmore, 1992).

THE PRESENT STUDY

Apart from general surveys and a few studies relating two attributes (Battista, 2003; Lehrer, Jenkins, & Osana, 1998; Outhred & Mitchelmore, 2000), we have found no study that has investigated length, area, and volume measurement simultaneously in order to compare development in the three attributes. Perhaps that is why curriculum documents make few recommendations to teachers on linking understanding across the three areas, although there is good reason to think that highlighting such connections could assist student learning in all of them (Hiebert & Carpenter, 1992).

There is, however, one serious problem with comparing development across different attributes: Understanding must be measured using strictly parallel assessment items. Otherwise, any differences found could be simply due to using different items. As we shall see, it is not easy to design parallel items.

A study was therefore designed to investigate the following research questions:

- Can young students' understandings of length, area, and volume measurement be assessed in such a way as to allow valid comparisons?
- If so, how does the development of students' understandings of length, area, and volume measurement differ?

This paper reports the major findings from this study, focussing on tasks linked to the five principles discussed above. Preliminary results on student's understanding of the fifth principle have been reported previously (Curry & Outhred, 2005).

METHOD

The research instrument

A 45-minute clinical interview was designed and pilot-tested and then administered in the final term of the school year to a sample of 96 students chosen from six schools that were representative of the variety of public schools in Sydney. There were 24 students in each of Grades 1-4 (ages 6 to 10 years), balanced for gender.

The interview consisted of a number of sets of three tasks, each set assessing students' understanding of one of the measurement principles for length, area, and volume. There were tasks assessing both volume by packing and volume by filling (capacity), but only volume by packing will be reported here.

We attempted to achieve parallelism across the three attributes by careful design of the tasks and instructions. Many lengthy discussions between the researchers and repeated pilot testing over several months were required before the tasks were judged ready for administration to the main sample.

For example, to assess the first principle, students were given an object (a ribbon, a rectangular region, or a rectangular box) and a number of possible units (thin sticks, square tiles, or cubical blocks). In each task, there were multiple copies of each unit but not enough of any one size to fill the space completely. The object's length, area, or volume was an integral multiple of each unit provided. Students were asked to "use any of these items to measure the space along/inside this object". This phraseology was used after it became apparent that, in English, there are no parallels to "How long?" for area and volume. Pilot testing showed that students had no difficulty understanding the phrase "space along/inside", but to ensure no ambiguity occurred the interviewer clarified its meaning by moving her hand along the length of the object, over the region, or within the box.

In all tasks, neutral probes were used whenever the interviewer was uncertain as to the student's intent or meaning.

Data analysis

All student responses were video-recorded and transcribed. Initial categories were continuously refined and responses re-scored as further data emerged. Analysis proceeded concurrently with the categorisation, and at the same time checks were made that the same phenomena appeared across all three attributes. The several responses to each task were initially scored separately, but gradually some scores were eliminated as invalid or uninformative or merged with other scores to obtain a more valid measure of understanding. The end result was five scores assessing children's understanding of the five principles listed above.

RESULTS

We shall report the scores in the order in which the corresponding tasks occurred in the interview.

Principle 1: The need for repeated units that do not change

The tasks used to assess this principle have been described previously. A response was scored as correct if the student either (a) iterated a single unit or (b) used several units but treated the larger units as multiples of the smaller units. In most of the incorrect responses, students used different sized units to fill the space along/inside the given object and simply counted the number of units they had used.

These tasks proved unexpectedly difficult and also showed a puzzling anomaly: whereas 25% of the students correctly completed the volume task and 23% the area task, only 6% completed the length task correctly. Eventually the reason was found: Students often rejected the possibility of using different sized units for area and volume on the basis that it was physically impossible to fit the different sized tiles or blocks together to fill the rectangle or box, not because it would produce an invalid measurement. In other words, they gave the correct response for the wrong reason. No such problem arose in the corresponding length measurement task.

It was consequently decided that the three tasks were not parallel and therefore did not allow a valid comparison of students' understanding of Principle 1 across attributes.

Principle 2: The appropriateness of a selected unit

These tasks were intended to assess students' understanding of the need to use a length unit to measure length, and so on. Students were given multiple identical copies of a single unit (stick, tile, or block) and asked to measure an object (ribbon, rectangle, or box). They were then asked to show the interviewer the part of the unit they had used to measure. A response was scored as correct if the student provided a clear indication, either verbally or with a finger movement, that the linear, areal or volumetric nature of the unit was understood. Many students incorrectly indicated the boundary of the unit—the ends of the stick, the perimeter of the tile, or the faces of the block.

Difficulties arose in the interpretation of student responses for the length tasks, in that it was impossible to judge whether students who ran their finger along the ribbon were indicating its length or its area. A similar difficulty occurred with volume, where it was often impossible to judge whether a student's statement indicated the surface or the interior of the block. On the other hand, area provided no problems as it was usually possible to make a clear distinction between the interior and the border of the tile. It was therefore decided that these tasks were also not parallel, and student responses could not be validly used to compare understanding of Principle 2 across the three attributes.

Another length measurement task revealed serious shortcomings in student understanding of this principle. Students were asked if it was possible to measure the length of a ribbon using bottle tops and, if so, which part of the bottle top was used. Responses using an imaginary diameter were scored as correct, but many students stated that it was impossible because the bottle tops left gaps or measured "correctly" but indicated they were using the circumference as the unit. Unfortunately, we were unable to construct parallel tasks for area and volume measurement.

Principle 3: The need for the same unit to be used to compare objects

For each attribute, two objects were fixed to a board in a non-parallel orientation. Both objects were exactly covered with multiple copies of identical units, but the units were different for each object. The students were told that "Mary" had measured each object as shown and had compared the sizes of the two objects by subtracting one measurement from the other. They were then asked if this was a good way to compare the amount of space along/inside the two objects. A response was scored as correct if the student stated that a valid comparison could only be made if the same size units were used for both objects. The results are shown in Figure 1(a).

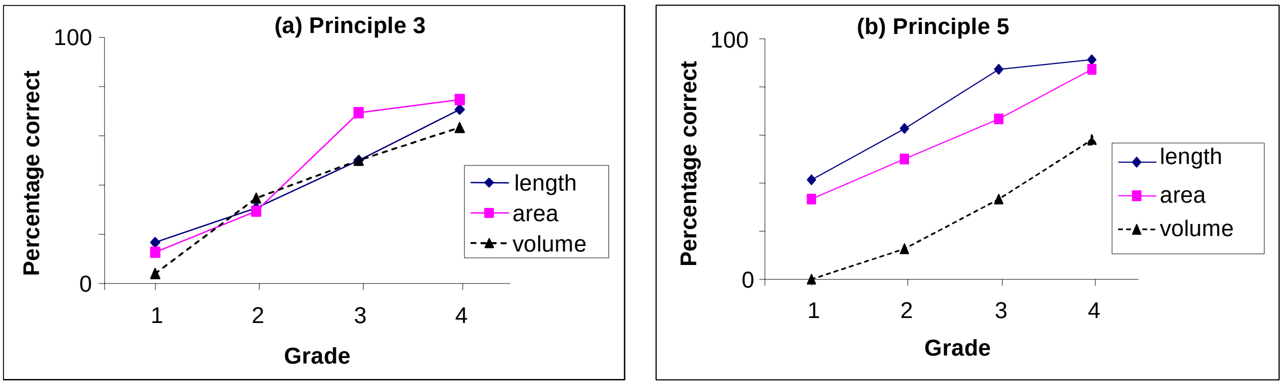

Figure 1. Scores for understanding of Principles 3 and 5, by grade.

Apart from the Grade 3 area score, there was a very similar progression across the three attributes. The percentages of correct responses increased steadily, from 0-20% in Grade 1 to 60-70% in Grade 4.

Some of the younger students carefully checked Mary's measurements and subtraction, and were thereby distracted from the main purpose of the question. However, there was no noticeable difference between the task demands across the three attributes and it was concluded that these tasks did provide a valid comparison of student understanding of Principle 3 across attributes.

Principle 5: The structure of the repeated units

Students were given an object (a line, rectangle, box) and a single unit (a stick, tile, block) and asked to measure the amount of space along/inside the object. A response was scored as correct if the units were iterated reasonably accurately without gaps or overlaps by recording the successive positions of the unit in some way. Many students iterated the unit without appearing to make any attempt to record successive positions. For the length task, several students obtained an incorrect measurement by placing their fingers between successive iterations of the unit. The results are shown in Figure 1(b).

There was a steady and almost parallel progression from Grade 1 to Grade 4, with length slightly ahead of area and area far ahead of volume. By Grade 4, the great majority of students seemed to have mastered Principle 5 for length and area but only about 50% for volume.

On the volume task, many students could only cover the base of the given box; others said they did not know what to do or they needed more blocks. The successful students seemed to have a mental picture of the unit structure, often finding the number of units in the base before multiplying by the number of layers. Two factors seemed to make the iteration of volume units more difficult than the iteration of area units: The unit had to be moved around in empty space instead of on a hard surface, and it was not possible to mark successive positions of the unit as it is moved around.

Another difference between measuring length, area, and volume using a single unit is the way the units are counted. For length, units can be counted one by one as they are laid along the ribbon. For area, students could either multiply the length and width or use repeated addition down columns or across rows. For volume, it is possible to find the number of blocks needed to fill the base of the box and then multiply by the number of layers; or determine the length, width and height of the box and then multiply these three numbers; or visualize the columns of blocks arising from each base block and use skip-counting to find the total number of units. All these methods were used by the students in the present sample.

These differences between length, area, and volume measurement are intrinsic to the attributes and are not artefacts of the tasks used. It was decided that the results give a valid comparison between students' understanding of Principle 5 across attributes.

Principle 4: The inverse relationship between unit size and the number of units

Students were shown a unit which was either half or double the one used for the measurement task described under Principle 5 above and asked to predict the new measurement. Responses were scored as correct if the student attempted to double or halve the original measurement. Responses where students re-measured the given object using the new units were interpreted as showing that the student did not yet understand Principle 4. The results are shown in Figure 2.

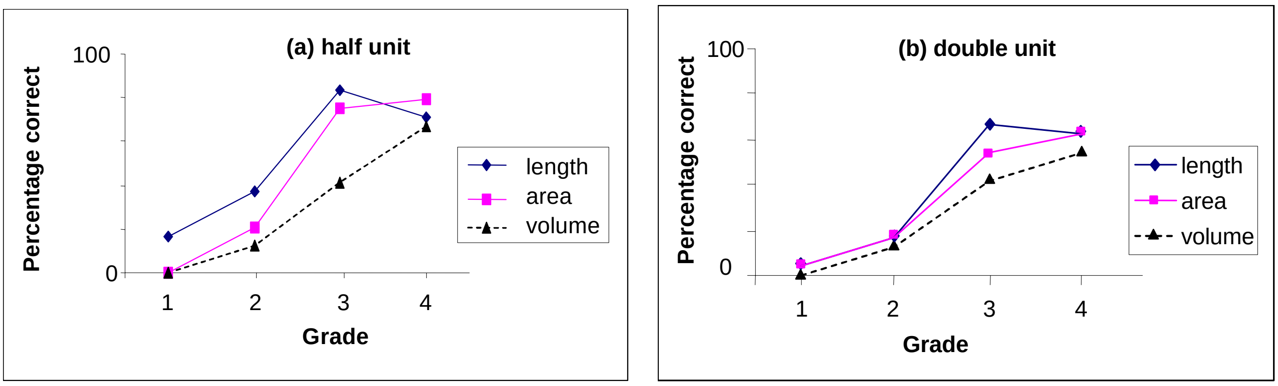

Figure 2. Scores for understanding of Principle 4, by grade.

For both parts of these tasks, there was a clear progression from Grade 1 to Grade 4, with a steep increase in the percentage correct from Grade 2 to Grade 3. All in all, the pattern was very similar for all attributes, with (apart from one or two exceptions) length slightly ahead of area and area slightly ahead of volume. It is very noticeable that students consistently found the half unit task easier than the corresponding double unit task.

No difference was observed in the task demands across the three attributes and it was concluded that they provided a valid comparison of students' understanding.

DISCUSSION

One aim of the present study was to investigate whether it was possible to obtain valid parallel tasks of young students' understanding of five principles of length, area, and volume measurement. This task proved incredibly difficult, and for two of the five principles was not successful.

There were some interesting differences between the three principles for which a valid comparison could be made. The expected length-area-volume order of difficulty appeared most clearly for Principle 5 (understanding the structure of the repeated units), and was readily explained by intrinsic factors related to differences in dimensionality and computational complexity. Most interesting here was the relative similarity between length and area measurement and the much larger gap between area and volume measurement.

On the other hand, Principle 3 (transitivity) showed very little consistent difference between the three attributes. It would appear that transitivity is a general principle that students learn from their experience of measuring several different attributes.

Principle 4 (the inverse relationship) showed only minor differences between length, area, and volume measurement. Again, it would appear that students learn this relationship as a general principle through exposure to measurement in a variety of contexts. Also significant may be experience with doubling in arithmetic, which probably occurs more frequently than halving. This difference could explain why students found the half unit task easier than the double unit task.

Students' responses suggested that they were, in general, not aware of the generality of the principles they were using. However, a few students (mostly from Grade 4) did comment on the similarity of the tasks and of their responses. One student said, *It* [*the interview*] *will be a bit boring on the video because of the same answers.*

The results have definite implications for teaching. Young students appear to have a much poorer understanding of the need for identical units that leave no gaps than teachers often assume, and they may indeed have no clear concept of what they are measuring. Learning activities could profitably include more tasks where errors can occur if students do not understand the basic principles. Such tasks could help focus discussion on the *reasons* for using a fixed unit size, for not leaving gaps, for using multiplication in some contexts, for rejecting certain units and accepting others, and for the inverse principle. The formulation of general principles which could then be applied to the use of formal units of length, area, and volume (and the measurement of other attributes such as angle, mass, and time) could be particularly helpful.

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MATHEMATICS-FOR-TEACHING:
THE CASES OF MULTIPLICATION AND DIVISION

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In this report we offer a theoretical discussion of teachers' mathematics-for-teaching (MfT), using complexity science as a framework for interpretation. We illustrate the discussion with some teachers' interactions around mathematics that arose in the context of a series of in-service sessions in which we, as researchers, focused on some of their tacit/enacted understandings of multiplication and division. We argue that further research into mathematics-for-teaching should take into consideration not just the tacit/enacted dimensions of teachers' knowledge, but the collective, dynamic, and inter-grade aspects of that knowledge.

CONTEXT AND OBJECTIVES OF THE RESEARCH

In this writing, we report on recent developments in an ongoing, longitudinal study of the relationships among teachers' mathematical knowledge, their teaching practices, and their students' understandings. Here we focus in particular on teachers' understandings of multiplication and division, and the work is oriented by recent examinations of the figurative natures and semantic structures of mathematical concepts (Lakoff & Núñez, 2000; Radford, 2002; Rotman, 2000).

As we develop in greater detail below, we are concerned more with teachers' *enacted* (versus *explicit*) knowledge—that is, in the sorts of understandings that they bring to bear when dealing with both novel problems and familiar mathematical topics. We conjecture that this tacit dimension might actually be the most important aspect of their mathematical knowledge, as they draw on images, metaphors, gestures, applications, and other figurative devices to help learners make sense of new concepts. Our premise is that a more explicit understanding of these devices might enable mathematics teachers to be more effective in their roles.

In this report we align ourselves with the assertion that, in endeavouring to understand mathematics-for-teaching (MfT), the common emphasis on *more* mathematics may be inappropriate. It might be that teachers require *more nuanced understandings* of the topics in a conventional curriculum (Freudenthal, 1973). The work is also informed by Ball and Bass's (2003) research into teachers' mathematics knowledge, in which they emphasize "job analysis"—that is, close examinations of the skill requirements of teaching, which include, for example, being adept at interpreting concepts for learners. This competence requires knowledge of how mathematical topics are connected, how ideas anticipate others, what constitutes a valid argument, and so on. The assumption is that the subject matter knowledge needed for teaching is not a watered down version of formal mathematics, but a serious and demanding area of mathematical work.

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We agree with this point, but re-emphasize the need to attend to the tacit/enacted dimensions of teachers' understandings. Our work is thus principally oriented by the question of what mathematics teachers *do* know and how that knowledge is manifested during engagements with mathematical topics, not how well they have mastered pre-specified lists of mathematical competencies, nor how they are able to demonstrate that knowledge on formal assessments. In this report we look specifically at the topics of multiplication and division as they arise and are elaborated through the K–12 curriculum in most North American jurisdictions.

THEORETICAL FRAMEWORK

The work is informed by complexity science, a domain that deals with self-organizing, self-maintaining, adaptive phenomena—in brief, with systems that learn. Examples include anthills, brains, cities, and ecosystems (see, e.g., Johnson, 2001), a diversity makes it difficult to formulate a one-size-fits-all definition of complexity. For this reason, instances of complexity are usually characterized in terms of lists of necessary qualities. A complex unity, for instance, is self-organizing; it arises as the actions of autonomous agents come to be interlinked and co-dependent. This emergence does not depend on central organizers or governing structures. As agents cohere into higher order unities, complex systems manifest transcendent properties and capacities that are not present among the individual agents.

Complex forms are often nested, with many intermediate layers of organization, any of which might be properly identified as complex and all of which influence (both enabling and constraining) the others. Complexity science prompts attentions toward several dynamic, co-implicated, and integrated levels—including, for example, the neurological, the experiential, the contextual/material, the social, the symbolic, the cultural, and the ecological—rather than isolated phenomena. This point is critical to understanding why we refuse a rigid distinction between collective and individual in our research. As we attempt to illustrate in Figure 1, individual understanding might be seen as enfolded in and unfolding from the broader phenomenon of collective dynamics through which, for example, standards of acceptable argument and topics of shared interest might be defined.

In terms of the project of formal schooling, this sort of collectivity is similarly embedded in curriculum structures, which themselves are nested in formal mathematics. Since the boundaries of complex systems are difficult to determine, it is impossible to draw tidy lines between these organizational layers—although, pragmatically speaking, they can be distinguished according to the paces of their evolutions and the sizes of the populations involved. For example, individual understanding tends to be seen as highly volatile (hence able to be readily affected, although not necessarily in predictable ways), whereas formal mathematics is seen as highly stable (and, often, as pre-given or fixed). Whereas in a relatively short span of time, one can observe the transformation of an individual's understanding of a particular piece of mathematics, analogous transformations to the body of mathematics take considerably longer.

Elsewhere (Davis & Simmt, 2003; in press), we have detailed how many of the theoretical discourses that frame current mathematics education research might be understood as focused on one or another of these sorts of nested phenomena—and are thus compatible with complexity science. In particular, radical constructivism, with its interest in the coherence-maintaining activities of individual learners and some versions of social constructionism with their foci on the dynamics of collective knowledge, might both be interpreted as dealing with specific instances of complexity.

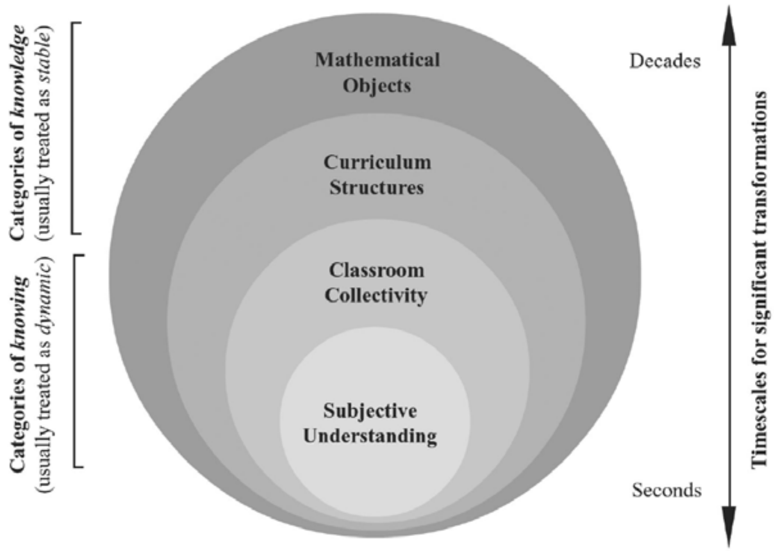

Figure 1: Some of the nested, complex phenomena of immediate relevance to the mathematics teacher.

ORIENTING ASSUMPTIONS

Oriented by complexity science, we frame our ongoing investigations of mathematics-for-teaching in terms of five key principles. First, as already developed, rather than focusing on established processes and conclusions, we are mainly concerned with the substrate of conceptual understandings. We seek to notice and elaborate the images, analogies, metaphors, and applications that teachers use to give shape to concepts.

Second, we interpret mathematical understanding as simultaneously subjective (i.e., rooted in one's physically embodied and socially embedded history) and collective (i.e., dependent on interactions, cultural tools, situated experiences, etc.). Of particular relevance, our studies of MfT are always located in group settings—that is, in spaces that not only invite, but that compel negotiation of meanings and understandings, since teaching always takes place in intersubjective settings.

Third, elaborating the last point and as illustrated in Figure 1, we understand there to be intermediate levels of complex organization between subjective knowing and collective knowledge. These levels of emergent coherence might be distinguished in terms of one's *knowledge of established mathematics* (i.e., the outer layers of Fig. 1) and one's *knowledge of how mathematics is established* (i.e., the inner layers of Fig. 1). Mathematics-for-teaching comprises both these elements.

Fourth, our investigations of teacher knowledge span levels from Kindergarten through Grade 12, oriented by the conviction that mathematical concepts unfold recursively (cf. Pirie & Kieren, 1994). Each moment of learning transforms what has already been learned, contributing to an evermore-elaborate network of associations that is better understood in terms of complex networks that in terms of hierarchies.

Fifth, we recognize that teachers' mathematical understandings are affected by our efforts to study them. We are thus compelled to approach their knowledge not in terms of static deficiency (i.e., looking for gaps in personal knowledge that is assumed to be more-or-less stable), but in terms of dynamic sufficiency (i.e., endeavouring to track collective knowledge and personal understandings as they arise and as they are elaborated).

METHOD OF INQUIRY

This work is anchored in the assumption that experienced mathematics teachers have a wealth of mathematics knowledge that may never have been part of their explicit educations, but that is nevertheless at the centre of their teaching. As such, in our work with a cohort of 24 teachers, the aims are to explicitly represent teachers' mathematics-for-teaching and, in the process, explore the possibilities for knitting articulated understandings into more sophisticated possibilities.

The cohort is a diverse one, with grades from Kindergarten through high school represented. In terms of professional experience, a few of the participants are at the beginning of their careers, several have taught for decades, and most fall somewhere in between. Most of the teachers are generalists, but two are mathematics specialists. Some teach in small urban centres, some teach in rural locations.

We are currently in the fourth year of a six-year study. The cohort meets for daylong sessions every few months. For their own part, the teachers see these meetings as 'in-service sessions'; their principal reasons for taking part revolve around their professional desires to be more effective mathematics teachers. In contrast, for us, these events are 'research sessions'—that is, sites to inform our theorizing. We are explicit in the fact that we are there to try to make sense of their knowledge of mathematics and how that knowledge might play out in their teaching. The common ground arises in the joint production of new insights into mathematics and teaching. Topics have ranged from general issues (e.g., problem-solving) to specific curriculum topics (as in the cases of multiplication and division, developed here).

Informed by our understandings of complexity science, the meetings are organized around extended, group-based engagements with seemingly narrow tasks. These tasks are developed around mathematical topics that are selected by the teachers themselves and they are designed in ways that allow us, as researchers, to map out some of the contours of their mathematical knowledge. For example, the activities that is the focus of the session on multiplication and division were structured around questions about the manners in which the topics are introduced, elaborated, illustrated, and applied across the grades.

The same basic structure was used in the two separate sessions reported here. We began by posing the questions, "What is multiplication?" and "What is division?" Employing a structure for collective interaction presented in Davis & Simmt (2003), teacher-participants undertook to respond first in small groups, and then more collectively by combining and elaborating those initial responses. Through a combination of explanation, discussion, and questioning, in the first session key points were collected into a single summary poster, the contents of which follow:

Multiplication involves

- repeated addition: e.g., $2 \times 3 = 3 + 3$ or $2 + 2 + 2$;
- equal grouping: e.g., 2×3 can mean "2 groups of 3";
- number-line hopping: e.g., 2×3 can mean "make 2 hops of length 3", or "3 hops of length 2";
- sequential folding: e.g., 2×3 can refer to the action of folding a page in two and then folding the result into 3;
- many-layered: e.g., 2×3 means "2 layers, each of which contains 3 layers";
- ratios and rates: e.g., 3 L at \$2/L costs \$ 6;
- array-generating: e.g., 2×3 gives you 2 rows of 3 or 2 columns of 3;
- area-producing: e.g., a 2 unit by 3 unit rectangle has an area of 6 $units^2$;
- dimension-changing;
- number-line stretching or compressing: e.g., $2 \times 3 = 6$ can mean that "3 corresponds to 6 when a number-line is stretched by a factor of 2";
- number-line rotation: e.g., multiplying by -1 can be understood as rotating the number-line by 180°.

By the end of the lengthy discussion of these representations of multiplication, there was consensus that the concept of multiplication was anything but transparent. In particular, it was underscored in the interaction that multiplication was not the sum of these interpretations. It was some sort of complex conceptual blend. One detail that was not lost on the teachers was that new images and analogies were more than variations on a theme. They are not simply 'piled onto' what has already been established, but incorporate and are incorporated into existing ideas. These conceptual blends are recursive elaborations, contributing to much more flexible and powerful notions. As such, we conjecture that access to the web of interconnections that constitute a concept is essential for teaching. More directly, the teachers were occasioned to think about the body of mathematics as a complex system.

This sensibility was brought to bear on a similar exploration of division some months later, during which the following list was generated:

Division involves:

- splitting;
- separating;
- cutting up;
- equal/fair distribution of
    - distances/intervals,
    - contents/measures/volumes/amounts,
    - money/shares;
- a missing dimension of an area/array;
- repeated subtraction (“taking sets of”);
- building up/reassembling;
- subset-making (in “÷ *n*,” *n* can refer to “*n* groups” or “groups of size *n*”);
- factoring (and missing factors);
- fraction-making;
- the opposite of multiplication [i.e., in reference to the previous list; see above].

As with the list of associations for multiplication, this one does not really capture the nuances and depth of the discussion of division. However, it does illustrate the point that, like multiplication, teachers *act* on an embodied understanding that division does not have a singular or consistent definition.

Of particular interest in the discussion of division was a significant shift in the manner of participant engagement. In the first session, most of the teachers indicated some difficulty in seeing the list of interpretations of multiplication as mathematical, even though they readily acknowledged the pedagogical relevance of such a list. Phrased differently, it appeared that many did not accept the suggestion that analogy, metaphor, and image were actually part of the mathematics; rather, as the participants explained, these sorts of devices seemed to serve as illustrations (as opposed to being part of the subject matter). This topic if, of course, hardly a settled issue in discussions of the nature of mathematical knowledge. However, framed in terms of the intertwining dynamics and nested structures of complexity science, it is impossible to separate figurative device from logical inference.

By stark contrast, during the session on division some months after the one on multiplication, participants seemed to have resolved the issue for themselves. Indeed, the debate did not even resurface. Instead, participants readily engaged in discussions of the diversity of interpretations, problems that might arise between certain metaphors, more powerful possibilities that could be created by blending entries on the list, and so on. In other words, there was a not-too-subtle shift in their conceptions of the nature and structure of mathematical knowledge, away from the notion of an idealized logical hierarchy and toward a notion of a linguistically effected, scale-independent network that cannot be dissociated from personal understandings.

DISCUSSION AND CONCLUDING REMARKS

As we consider a teacher's knowledge of mathematics in terms of complex dynamics, we find ourselves compelled to consider a range of topics in ways that have, quite simply, not been well represented in programs intended for teacher preparation and/or ongoing development, nor in much of the current research into mathematics teachers' subject matter knowledge. For us, perhaps the most important conclusion of complexity science is that, in any learning system, complex co-adaptive activity is always happening across several levels simultaneously.

Because it is impossible to affect a part of the complex unity without affecting its global character, and vice versa, it is our conviction that complexity science should prompt very different emphases in teachers' mathematics preparation. For example, we would argue that teachers should be engaged in much more deliberate interrogations of the tacit elements of mathematics knowledge. More directly, we believe that a key (and perhaps *the* key) competence of mathematics teachers is the ability to move among underlying images and metaphors—that is, to translate notions from one symbolic system to another, helping learners to reconcile the diversity of interpretations that arise when dealing with even 'basic' concepts and operations.

We do not mean to suggest that the topics and issues we raised in this writing are not already represented in pre-service teacher education programs. Clearly, topics in formal mathematics, curriculum structures, collective dynamics, and individual understanding are addressed in every comprehensive course of studies. Our main point, rather, is that manners in which these phenomena are framed and integrated are vital. On this matter, we return to the point that courses in mathematics for teachers tend to be framed in terms of either 'beyond' or 'more of' the concepts that are included in grade school curricula. We argue that the mathematics teachers need to know is qualitatively different than the mathematics their students are expected to master—and, further, that the research community has far to go in identifying what these varieties of mathematics might be.

On this point, we are convinced that the places to be looking are in the practices—the embodied and enacted understandings—of experienced teachers. We are further convinced that such a research focus can only serve to underscore the inextricability of teachers' knowledge of established mathematics and their knowledge of how mathematics is established. A fluid mastery of how to organize and mobilize social grouping must be accompanied by a strong background in the subject matter. To that end, we offer the following conclusions based on our research to date.

To begin, an isolated focus on either questions of mathematics or questions of learning is inadequate in efforts to understand teachers' mathematics knowledge. In fact, such emphases may be counterproductive as they place these phenomena in tension rather than foregrounding their nested, self-similar characters.

To get at these qualities, a prominent aspect of teachers' study of mathematics should be the role of metaphor and other figurative devices in the development of formal mathematics concepts and individual mathematics understandings. It appears that a complementary focus on how concepts are presented and elaborated through the course of the K–12 curriculum is also important for these discussions, since the vocabularies, images, and algorithms used in schools clearly play an important role in shaping understandings—and, by consequence, an important role in shaping mathematics.

Unfortunately, we are not yet at a point in the research that we are able to make claims about the relationship between teachers' mathematics-for-teaching and their pedagogical effectiveness. Nevertheless, we are encouraged by the observed transformations in teachers' engagements in the sorts of tasks described, as well as by comments of many of the participants as they remark on their experiences in their classrooms. As one teacher noted, "It's wild to see the lights go on when you help someone realize that things weren't making sense because they were thinking of the right operation but the wrong interpretation."

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GENERATIVE CONCEPT IMAGES

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We report on part of a study of university students' developing understanding of sensitive dependence on initial conditions, one of the essential ingredients of chaos. We discuss how the teaching team's concept images of sensitive dependence on initial conditions changed as a result of their interactions with students. These interactions lead to a substantially simpler, more geometric, way of viewing sensitive dependence on initial conditions. We discuss this simpler way of interpreting sensitive dependence in terms of generative concept images, in which prototypical objects are analysed deeply and then modified by higher level operations.

INTRODUCTION

In dealing with mathematical definitions, students have to bring to mind what the definition is about for them. The process of using a definition in practice is heavily dependent on memory for various parts of mathematics. These memories can take the form of images, feelings, syntactic expressions, and procedures, among other things. Associated with a definition a student will have a host of memories to retrieve. Indeed, for a given definition, Tall and Vinner (1981; see also Vinner, 1983, 1991) associate with each student a concept image that consists of:

> "… the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures."

In this paper we consider the idea of concept image in the context of a somewhat difficult definition – that of sensitive dependence on initial conditions, a basic ingredient of chaos. We argue that in order for students to focus productively on the idea of sensitive dependence it is useful to structure their concept images by assisting them to analyse examples deeply, and by providing them with higher-order operations that allow them to generate further examples. This process works, we argue, because it allows recollections of highly concrete imagery and relieves the strain on working memory.

SENSITIVE DEPENDENCE ON INITIAL CONDITIONS

Over the past 25 - 30 years the idea of sensitive dependence on initial conditions for a continuous function has become a central notion of chaos (Devaney, 1989; see also Holmgren, 1996; Elaydi, 1999). Conceptions of sensitive dependence appear explicitly in Guckenheimer (1979), Ruelle (1979) and Kaplan and Yorke (1979). Prior to these authors May (1976, p. 466) wrote:

> "… it may be observed that in the chaotic regime arbitrarily close initial conditions can lead to trajectories which, after a sufficiently long time, diverge widely."

which he related to Lorenz's (1963) "butterfly effect". Li and Yorke (1975), who first used "chaos" in a technical mathematical sense, did not include sensitive dependence on initial conditions as a criterion for chaos. So explicit mention of sensitive dependence on initial conditions as an ingredient of chaos appeared somewhere between 1975 and 1979. Many popular and technical articles (Gleick, 1987, p. 8; Stewart, 1989, p. 113; Ruelle, 1991, p. 40; Feigenbaum, 1992, p. 6; Peitgen *et a*l, 1992, p. 48 and p. 511; Froyland, 1992) contain concept images, in the sense of Tall and Vinner (1981), but not concept definitions. Devaney (1989) places sensitive dependence on initial conditions as a major ingredient of chaos. Three points make it likely that this definition and associated conceptions will be difficult for students to appropriate:

1. Historically the definition did not come easily to workers in the field.

2. Popular and technical accounts have sometimes simplistic, confusing, or contradictory concept images.

3. The definition, as phrased in Devaney (1989), contains a mixture of five existential and universal quantifiers: a function $f : X \to X$, where X is a metric space with metric *d*, has sensitive dependence on initial conditions if $\exists\delta > 0$ such that $\forall x \in X, \forall\varepsilon > 0, \exists y \in X, \exists$ positive integer n with $d(x, y) < \varepsilon$ and $d(f^n(x), f^n(y)) > \delta$ (here, f^n denotes the nth iterate of *f*, namely, $f^1 = f$ and $f^{n+1} = f \circ f^n$ for $n \geq 1$).

Definitions at this level of mathematics are critically important because they are organizing principles for a variety of phenomena, and because they distinguish subtly different examples. Defined notions constitute the "advanced" in advanced mathematical thinking. However working directly from a definition can be difficult: for example, it is tedious and messy to show directly from the definition that the quadratic function $Q:[0,1] \to [0,1]$ defined by $Q(x) = 4x(1-x)$ has sensitive dependence on initial conditions. Generally, mathematicians work with "higher-level" operations (see Thurston, 1997, p. 118) when trying to settle issues such as this. For example, it is much easier, as we will see below, to establish, that the tent function $T:[0,1] \to [0,1]$ defined by $T(x) = 1 - |1 - 2x|$ has sensitive dependence, via a precise graphical analysis, and then to relate the behaviour of T to that of Q by a change of coordinates.

A major difficulty in working from definitions is that a definition has to relate to examples. However, if a student does not have a useful concept image what, for them, are the examples actually examples of? Both Skemp (1982) and Steffe (1990) cite examples as a necessary, but not sufficient, condition for students to operate successfully with a definition:

> "Concepts of a higher order than those which a person already has cannot be communicated … by a definition, but only by arranging to encounter a suitable collection of examples." (Skemp, 1982, p. 32)

> "Providing a definition can be orienting but using it can be very problematic especially if it has not been the result of experiential abstraction." (Steffe, 1990, p. 100)

Already in 1908 Poincaré observed:

> "What is a good definition? For the philosopher or the scientist, it is a definition which applies to all the objects to be defined, and applies only to them; it is that which satisfies the rules of logic. But in education it is not that; it is one that can be understood by the pupils" (Poincaré, 1908)

We argue that Poincaré, Skemp and Steffe's observations can be considerably strengthened by the process of formation of generative concept images, in which a simple, but typical object is analysed in depth, and appropriate higher level operations are used to generate further examples.

STUDENT DIFFICULTIES WITH SENSITIVE DEPENDENCE

We consider several university students' difficulties with the definition and concept of sensitive dependence on initial conditions. The students were enrolled in a course on one-dimensional dynamical systems at LaTrobe University, Melbourne, Australia. The majority of students were in the final year of their undergraduate degree. The class also included graduate students, and a 1st year student. The class was divided in two, with each group attending a one-hour weekly class for 26 weeks. The course team consisted of two faculty members, two graduate students, and a teaching assistant. All sessions were video-taped for later analysis and to assist the teaching team in real-time development of the course. Students were expected and encouraged to make presentations of their attempts at problem solutions to the rest of the chaos class. They were encouraged to write on an overhead projector and to talk aloud to their solution or attempted solution. Videotapes of class sessions were analysed each week for student difficulties, and exercises and teaching materials were adjusted accordingly.

Student engagement with sensitive dependence on initial conditions began with a lecture on a *Mathematica* experiment. Among the students present, Alice was particularly well qualified in mathematics. She had studied theoretical computer science at doctoral level at Moscow State University, and was, at the time of the chaos course, enrolled in a PhD in mathematics. Nonetheless, Alice experienced considerable difficulty in making sense of the written definition of sensitive dependence on initial conditions. The point that concerned her most about the definition of sensitive dependence was "$\exists$positive integer n". In the following excerpts we see the confusion of Alice, and her classmates Theo and Cherie, in trying to come to terms with the formal definition of sensitive dependence.

Alice: I just can’t understand when you said “there exists an integer n” … why can’t we pick up some very good integer n, for example 0, is it perfectly O.K.? For each x and each y you can pick up n_0 and for each δ.

Greg (teacher): But δ is independent of x and y.

Alice: I don’t understand Veronica’s (a teacher) explanation. Why we pick up two zero points and say this is not expansive - zero points of n iteration? Why we didn’t stop at n -1 iteration? ….. What does it mean “there exists an integer n”?

….

Theo: δ … it can be greater than δ when n-1, but when get to n can be less than, no greater than ... sorry.

Jack (teacher): But it must be true for all x and y - so it must also be true for x and y close together.

Alice: So this n - fixed for all x and y, or for every pair of x and y?

Jack (teacher): Each pair of distinct x and y.

Alice: Now for every pair of x and y we choose n?

Jack (teacher): No!

Alice: It’s fixed for all pairs?

Greg (teacher): The δ is fixed for all n.

Alice: I asked about n, I didn’t ask about δ.

Cherie: You can pick any x and y and you’ve got a δ fixed, and you can iterate however as many times as necessary and there will exist a particular n - you don’t have to specify for every pair - such that this is true.

Alice: I don’t understand Veronica’s (a teacher) explanation.

Veronica (teacher): If you fix the δ, after the iteration, with every pair of x, y in the $[0,1]$ … must be greater than δ.

Greg (teacher): Ask yourself this question: in the tent map could δ be a half?

Alice: No questions!

Students were asked to prepare a statement of their understanding of sensitive dependence for the following class. They took turns to read out their explanations to the rest of the class. Two typical examples are:

Donald: What I wrote sounds a bit bizarre. I wrote that a function has sensitive dependence at a point if, if there is another point near that point so that the functions of the two points are a certain distance apart … after some number of iterations. Like, for example, … there is sensitive dependence at that point of you can find a point close to it so that after some number of iterations the functions of the two points will be some distance apart. Does that make sense?

Michael: A mapping has sensitive dependence at a point if for some number of iterations, if after some, … if for some iterations the function at some nearby point is greater than, than some fixed distance away.

We see, in these two statements, clear examples of what Tall and Vinner (1981) refer to as the students' concept image of sensitive dependence.

WIGGLY ITERATES: REFINING THE CONCEPT IMAGE

As we mentioned in the introduction, checking directly from the definition that a given function has sensitive dependence can be tedious, messy, and difficult. This is not generally how mathematicians work:

"As with most raw definitions, direct use … is rare." (Thurston, 1997, p. 118)

To provide the students in the course with a technique for establishing sensitive dependence the teaching team began with an analysis of the tent function $T:[0,1]\to[0,1]$ defined by $T(x)=1-|1-2x|$. The iterates T^n of T are particularly easy to understand geometrically.

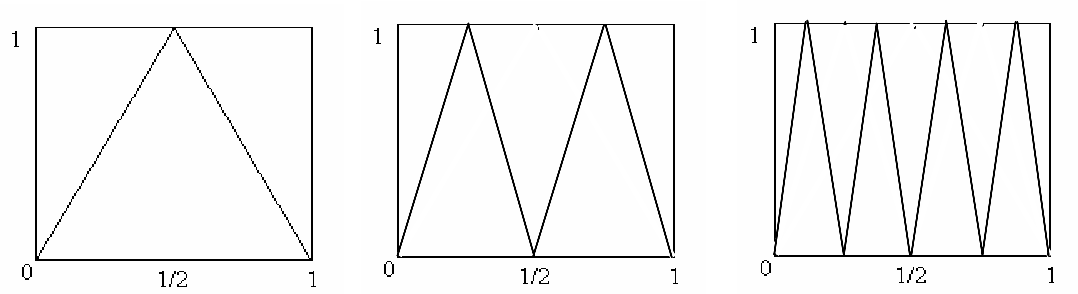

Figure 1: Graphs of the first, second, and third iterates of $T(x)=1-|1-2x|$ on [0, 1]

The n^{th} iterate of T consists of n squashed up copies of T – squashed in the base, but not in height – and the distance between successive zeros of T^n is $1/2^{n-1}$. The slope of the individual linear segments of T^n has absolute value 2^n. These facts are proved easily by induction on n. From this it is clear why T has sensitive dependence on initial conditions: we can choose $\delta=\frac{1}{2}$, for example (any positive number less than 1 will do), and then for any point $x\in[0,1]$ and $\varepsilon>0$, we simply take a $y\neq x$ within ε of x that is not a zero of some iterate of T, and iterate enough times so that the steepness of T^n ensures that $T^n(x)$ and $T^n(y)$ are at least $\delta=\frac{1}{2}$ apart.

The teaching team for the chaos course planned to write a single chapter on sensitive dependence on initial conditions. Chapters were planned to be around 10 pages in length, with no more than one graphic image per page. Because of student difficulties with the idea of sensitive dependence, the proposed chapter was expanded to three chapters. The initial definition, which seemed more or less clear to the course instructors, was found to be difficult for the students to comprehend. Consequently, the teaching team undertook a substantial analysis of the difficulties, and attempted to view them from the students' perspectives. This involved detailed analysis of student responses in class, their attempts at problem solving, and an effort to involve them in the ways in which professional mathematicians attempt to come to grips with the meaning of sensitive dependence.

GENERATIVE CONCEPT IMAGES

A common model for relating concept definitions and concept images involves:

(a) exemplifying the definition with examples (models of the definition), or

(b) organizing different examples by a definition, in the process abstracting or extracting appropriate relevant features,

or both. However, there is no guarantee that a student will:

(a) isolate the relevant parts of the definition as features of any given example, since they can easily focus on what, for the teacher, are irrelevant aspects of a given example, or

(b) extract the appropriate features that the examples are deemed to have in common.

The difficulty for a student in exemplification is to know how to interpret the definition in a particular example. The difficulty in organization is to know why just certain features of the presented examples are picked out as relevant and made into a definition. Tall (1986) refers to the process of extraction from numerous examples as the formation of a generic concept image:

> "… a generic concept being defined as one abstracted as being common to a whole class of previous experiences."

He then suggests that this process of extraction, or abstraction, rather than that of working with a "formal" definition, is more likely to be helpful for a student:

> "Mathematicians analyze concepts in a formal manner, producing a hierarchical development that may be inappropriate for the developing learner. Instead of clean, formal definitions, it may be better for the learner to meet moderately complicated situations which require the abstraction of essential points through handling appropriate examples and non-examples."

Thurston (1997) states however that mathematicians rarely work from raw definitions. From what then do they work? Our answer is that they work from a deep analysis of relatively simple, but prototypical, objects. This is not the same thing as exemplification, since it involves a deep analysis of why the object is an instance of the definition, and what general features of the object can be used as a concept image. Nor is it the same as extraction, because the features of the definition are built into a specific more easily analysed prototypical object, and into the higher order operations that transform that object. In the case of sensitive dependence on initial conditions the tent map is such an object. A deep, but technically straightforward, geometric analysis of its relevant features leads to a visual interpretation of sensitive dependence that carries over to other functions via higher level operations. The quadratic map $Q:[0,1]\to[0,1]$ defined by $Q(x)=4x(1-x)$, for example, can be seen to have sensitive dependence on initial conditions, because a change of coordinates, effected by the transformation $\alpha:[0,1]\to[0,1]$ defined by $\alpha(x)=\sin^2(\frac{\pi x}{2})$, transforms the tent map into the quadratic map – in that $\alpha\circ T=Q\circ\alpha$ – and a change of coordinates preserves sensitive dependence. The two higher level operations – change of coordinates, and preservation of sensitive dependence under change of coordinates – require some technical checking. However, when this relatively straightforward checking is carried out, once and for all, a student's thought processes can operate at a more economical level, and so take less space in working memory. A student then has the possibility of no longer being overwhelmed by a morass of technical details. What formerly seemed difficult now becomes transparent, because of a powerful and generative concept image that involves the action of higher level operations on deeply analysed prototypical objects.

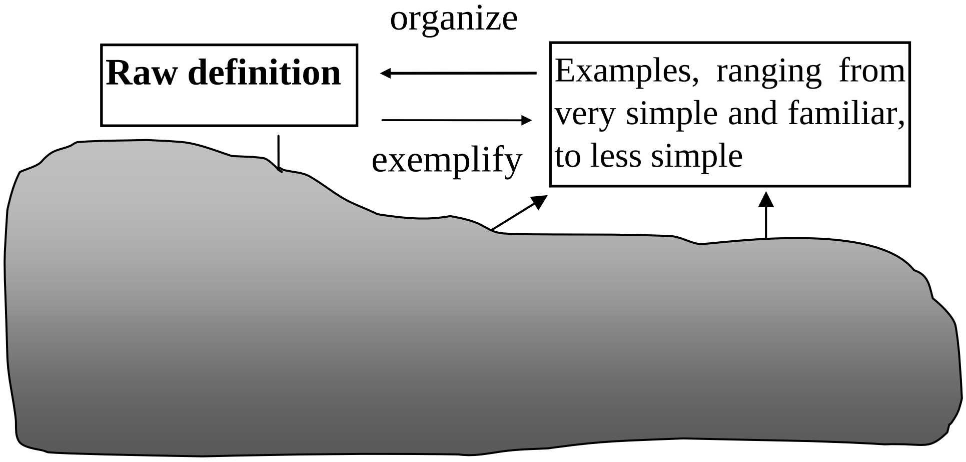

Figure 2. A model for the development of generative concept image and higher level operations: shaded areas contribute to formation of a generative

Tall and Vinner (1981) note that:

> "When a student is given a formal concept definition, the concept definition image that it forms in his (*sic*) cognitive structure may be very weak. … a weak understanding of the concept definition can make … formal proof … very hard ..."

Later, in referring to the connection between mental imagery and concept image they write:

> " [*the students*] are then in the situation where they may have a strong mental picture yet the concept definition image is weak. They understand the statements of theorems as being obvious, but cannot follow the proofs."

We propose that focusing on the relationship between concept definition and examples, through exemplification and organization generally produces a form of concept image that is weak, in the sense of Tall and Vinner, and not usually generative. This form of concept image generally only allows a student to mechanically check the conditions of a definition in particular examples, and possibly draw limited logical conclusions, via simple proofs, from the concept definition. This is in contrast to a form of concept image that we term generative, in which a student has the possibility of higher level operations on deeply analysed prototypical instances of the concept definition. A generative concept image allows a student the possibility of using their imagination in proposing likely general facts that flow from the concept definition. This is because their working memory is not overloaded with tedious checking of minutiae, so that through higher operations they can both generate, and focus on, deeper aspects of more complicated examples.

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DEVELOPMENTAL ASSESSMENT OF DATA HANDLING PERFORMANCE AGE 7-14

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We report a study of 7-14 year old children's performance on data-handling tasks using a diagnostic, age-standardised assessment test developed from the research literature. Rasch measurement methodology was used to construct a difficulty scale for a sample of 8829 children. We interpret this scale as a hierarchy of five levels of data-handling performance. Key errors, prototypes and misconceptions made associated with performance level are discussed. Finally we suggest task difficulty might be accounted for in terms of at least three dimensions: cognitive load, arithmetical demand, and prototypical conceptions. The first two of these tend to explain higher levels of difficulty while the latter are particularly significant to lower performance levels.

INTRODUCTION AND BACKGROUND

The focus of this study is children's performance, conceptions and misconceptions or errors in data handling. Ainley and Pratt (2001) point out that data-handling these days indicates a broader curriculum than just 'probability and statistics'. However, in this paper we restrict our attention to that part of the 'data-handling and statistics' national curriculum in England and Wales for 7 to 14 year olds, which is largely concerned with reading and interpreting information tables, charts and some kinds of graphs. We also draw on research in statistical literacy, e.g. Reading (2002) and Watson and Callingham (2004) and others, who have constructed performance hierarchies similar to the one presented here (see Figure 1).

We report a survey of the data-handling capability of children aged 7 to 14 and is based on a diagnostic formative assessment tool that was developed as part of a new set of age-standardised diagnostic assessment materials for ages 5-14 (Mathematics Assessment for Learning and Teaching with Hodder Murray) (See Williams, Wo and Lewis, 2005). A hierarchy of levels of data-handling capability with associated items is put forward and is explained by examining the typical performance requirements of the items at each level, and associated misconceptions or errors. We conceptualise the hierarchy of levels by means of three constructs which, we tentatively put forward as an early model for difficulty in school-type data-handling items.

It is widely believed that effective teaching should benefit from diagnostic assessment (see also Bell, Swan, Onslow, Pratt and Purdy, 1985; Hadjidemetriou and Williams, 2002). This implies that effective teachers must understand their pupils' ways of thinking, including misconceptions that may underpin their errors. We believe it is useful to classify these in ways which can contribute to a theoretically

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 401-408. Prague: PME.

informed pedagogy: the role of ‘prototypes’ in the formation of concepts being particularly powerful and apt to this study (see eg Tall & Baker, 1991; Hadjidemetriou & Williams, 2002). In addition, theory of cognitive load can be useful in informing pedagogy, (e.g. by idenitifying the need to off-load information during problem-solving, etc., see Miller (1956) and Sweller (1988).

We are careful to draw a distinction between an error, i.e. an erroneous response to a question, and a conception, prototype or *misconception* which may be part of a faulty cognitive structure that causes, lies behind, explains or justifies the error (see Hadjidemetriou and Williams, ibid., p.69). In this paper errors are identified with performances, while misconceptions are a matter of interpretation. Some errors are diagnostic of a conception e.g. a prototypical concept (e.g. the tendency to always read off the tallest bar in a bar chart, see also Perira-Mendosa and Mellow, 1991). These can sometimes lead pupils to construct some well-formulated alternative frameworks of ideas which are not appropriate (Leinhardt, Zaslavsky and Stein, 1990). Other errors may, instead, simply be the result of guessing, of faulty memory or a cognitive overload.

We look to literature on data handling (e.g. Shaughnessy et al., 1996; Jones et al, 2000; Ben-Zvi and Arcavi, 2001, Reading, op cit. and others) but also draw on the graphing literature (e.g Janvier, 1981; Curio, 1987; Perira-Menosa and Mellow, op cit.; Friel et al, 2001 and others). For example, Perira-Mendosa and Mellow undertook research into 9 to 11 year old students’ conceptions in bar graphs. They reported errors in scales, lack of identification of patterns, errors in predictions, and inappropriate use of information. Friel et al (op. cit.) classify graph comprehension into three levels, the first of which (ie classifying and extracting data information graphs) comprises the data-handling in our tests. However, we also attend to arithmetic demand, since, as Curio (op cit.) found, arithmetic competence is required in many data handling tasks and may contribute significantly to difficulty.

Janvier (op cit.) comments that consideration of the role of context may add to the number of elements to which the graph reader must attend: we anticipate that this will often be a factor in data handing too, and influence the cognitive load of a task.

METHOD

A pre-test for this project was carried out in April-May 2004 in order to ensure that the test items were appropriate: a very few items were modified as a result. A nationally representative sample was created, which was stratified by geography, school type/status and pupil attainment for the standardisation procedure conducted in February and March 2005. Thirty two data-handling items were included in the 7 to 14 year old tests, which were taken in total by 14429 children from 120 schools.

The test items (including the data-handling items) were assembled into papers (45 mark points for 7 to 14 year olds) with no common items between papers. Test equating data was achieved by some pupils sitting two standardisation papers (between 31 and 144 per year), and by data from the pre-test (between 209 and 390 per year) in which pupils sat approximately half a paper from the following year's test.

Validation included 'vertically linked' analyses (including subgroup DIF etc.: see Ryan and Williams, 2005 and Bond and Fox, 2001 for details of the procedures followed). While the metric resulting from vertical equation becomes progressively dubious as the curriculum changes, in this paper our interpretation of the data is essentially ordinal, thus allowing consideration of the entire data-set.

Rasch analysis was conducted using QUEST, including the coding and analysis of errors on the same scale as the items. The result is a single difficulty estimate for each item and an ability estimate for each child consistent with the Rasch measurement assumptions. Only one mark point fell outside a model 'infit' tolerance of mean square 0.7 to 1.3 (item 8-17a: to view the items as a whole see www.education.man.ac.uk/lta/pme30/stats).

A developmental hierarchy of the degree of difficulty of data-handling items, along with the associated, significant errors made for the items within any level, was identified. Cut-off points for levels were assigned by identifying groupings of children's ability and items difficulties placed on the same logit scale: children with a given logit score have a 50% chance of providing the correct answer to items at that degree of difficulty (See Figure 1). Clusters of items of similar difficulty may in broad terms be considered as having something in common a purpose of this paper is to identify what this commonality may be. However, we acknowledge that the choice of cut-off between levels is somewhat arbitrary.

We are aware of the debate in mathematics education about the nature of constructing such developmental hierarchies. We accept that there may be serious threats to our understanding about the nature of learning if hierarchies are taken as fixed unidirectional developmental stages that are followed uniformly by all children. On the other hand, hierarchical constructs are useful in describing generalities. In addition, teaching in the UK is dominated by a national curriculum which is so structured, and the engagement of teachers in practice requires us to adapt to this.

RESULTS AND DISCUSSION

A five level hierarchy of item difficulty was constructed based on pupil and item clustering: L0 - non-response or correct response possibly for wrong reasons, L1- simple reading of one piece of information from a table or chart; L2- in addition, 2-step problems involving key conversion or interpretation of context (e.g. simple time context) in a table or chart; L3 - in addition, multi-step problems involving harder arithmetic or time-conversions, and L4 - in addition, one extra conceptual demand, usually involving comprehension of context important in typically multi-step problems using table, charts or graphs e.g. data-handling problems using inequalities & time (See Figure 1).

```
 Item Estimates(Thresholds)
 all on charts (N = 14429
 L = 32 Prob Level=0.50)
-------------------------------------------
        L 4
             5| Bold: items
                  with errors

                  Italics
          x4|   discussed
                items
                  9-11
         XX       14-23b
   XXXXXXXX3|     9-24b 12-07b
     XXXXXX       10-16b
        XXX |     11-12c 12-04
    XXXXXXXX|     14-09
   XXXXXXXX2|     8-17b
           X      9-24a
--------------------------------------
  XXXXXXXXXX
      L 3  X      14-05
        XXX1|     9-12
        XXX       10-13  11-12b
XXXXXXXXXXXXXXX
         XX       8-17a  9-04b
--------------------------------------
       XXXX 0|
      L 2 XXX |   11-08b
     XXXXXX |     9-19
XXXXXXXXXXXX      9-04a   9-15
         -1|      11-08a   11-12a
         XX        10-16a  14-23a
         XX        13-16a
--------------------------------------
         -2|
      L 1
           X      9-02
           X
 XXXXXXXXXXX
         -3
           |
           |       7-06     12-07a
       X -4|
--------------------------------------
      L0
         -5|       8-17c

   XXXXXXX -6|
```

LEVEL 4 CONTEXT / MULTI-STEP

In addition, another concept is added in.

Correct responses are typically to items in which the comprehension of context important to solving multi-step problems using tables, charts or graphs e.g. data handling problems involving use of inequalities *and* time **or** key conversion *and* inequalities

LEVEL 3 MULTI-STEP

Correct: in addition, multi-step problems, typically, involving use of more or less or interpretation of time context from tables, charts or graphs

Errors

Confuses mean and median averages

Ignores or incorrectly uses base in a ratio comparison task

Ignores one condition in solving a multi-condition word problem

Confuses columns when reading a timetable

Treats times as decimal.

LEVEL 2 TWO-STEP/VARIOUS FORMS

Correct: in addition, 2 step problems involving use of a simple data handling operation, key conversion or interpretation e.g. simple time context in a table.

Errors: as above +

Treats time as hours and minutes separated by a point e.g. writes 3:65pm for 4:05pm.

Mistaken magnitude of decimals, Misreads 'less than 11' as 'equal to 11'

Ignores the key in a pictogram

Ignores word problem context and ordered numbers regardless.

Ignores two conditions of a three-part word problem.

LEVEL 1 ONE STEP READING

Correct: simple reading from bar or table eg of a particular bar in a bar chart.

Errors: as above + largest bar identified regardless of question, misinterprets more or less, ig-

LEVEL 0 DEFAULT

Correct: items which elicit correct items despite misconceptions or errors held.

Errors: as above, plus some non-responses.

Figure 1: Data Handling Item Difficulty Hierarchy

Erroneous responses were further used to interpret these levels on the scale as a progression in performance and understanding, by locating the abilities of students performing these errors with the ability levels of the scale (also see Figure 1). As Reading (op.cit.) and Jones et al (op.cit.) found items at higher levels often start to require an appreciation of context. However, although items appeared to cluster into discrete levels using the cut-off points we put forward, as previous literature would anticipate, interpretation of the levels was not wholly straightforward by the type of item e.g. ' bar chart, requiring use of a key ' or ' reading information from a table ' .

For example, a level 1 item may require the height of a bar in a bar-chart; for example, item 9-02 is a bar chart of the number of people on the morning bus from Liverpool to Manchester. For this item, children were asked to circle the departure time of the bus with the smallest number of passengers. A misconception for this item was identifying the time of the highest bar (i.e. the bar depicting the largest number of passengers). Although it is possible that this misconception may be one of reading comprehension, we claim it is perhaps most suggestive of a prototyping error whereby children tap into a incorrect schema, which in their past experience may have held true. Typically, when bar-charts are introduced to children they are asked to identify the most frequent response. Further, another common error elicited from bar-chart and pictogram questions was the ignoring of a key. This may also result of prototyping since typically early teaching begins with one cell/figure representing one object.

However, 11-12c (see Figure 2) is a more difficult item than 11-12b. We explain this by means of another construct, arithmetic demand; when comparing item 11-12b with 11-12c we notice that subtraction (part c) is a more complex operation than addition (part b) and the numbers involved may also entail a greater cognitive load. On the other hand, part (a) only requires identification that the train costs £4.00 per person and the simple arithmetic operation 3*£4.00 = £12.00. Cognitively it is less demanding than either parts b or c, which carry a greater cognitive load in terms of the number of pieces of information that must be processed.

Errors for parts b and c involved: subtracting time using 1hr=100min and writing 3:65 pm for 4:05pm, suggesting time is being conceptualised as decimal, which could also be interpreted as a prototyping error (in the academic context a ‘dot’ prototypically refers to a decimal). Were the same ‘time’ questions asked in a context where the children were looking at a watch in a real situation, we might expect a different pattern of response. Thus, in the ‘school’ situation, context may impose an additional demand rather than a support to problem solving.

We found that the items also varied in their cognitive load (working memory demand) with regard to the number of criteria needing to be considered or the number of steps involved in solving the problem. More difficult items (for the sample) typically involved greater cognitive demand

12

Lisa, George and Ali want to go on a trip.
There are three trips to choose from.

	Lasts for...	Cost per person
Train	30 minutes	£4.00
Boat	50 minutes	£6.25
Coach	70 minutes	£8.50

(a) How much does it cost for all three of them to go on the train trip? £

(b) George goes on the boat trip.
He leaves at 3:15pm.
What time does he come back?

(c) Ali and Lisa went on the coach trip.
They all came back at 5:15pm.
What time did the coach leave?

Figure 2: Items 11-12 a, b &c Train, Boat and Coach Trip

To give another example, in item 12-4 (see below) several pieces of information need to be combined to arrive at the correct answer of £3.00. The most common error elicited from this item was to treat 6 year old Abby as being under six years of age and hence arriving at an answer of 75p. However, apart from a relatively high cognitive demand, this item also involves an arithmetic demand and possibly one of prototyping, with the equation of 6 years with ‘under 6 years’. We acknowledge that errors rarely can be explained simply with reference to just one construct in isolation.

Mr and Mrs Jackson are going to an exhibition with their two children, Abby and Ben. They see this price list.

Ticket type	Cost
Children (under 6)	free
Children (under 18)	£2.25
Adults	£5.25
Family ticket (up to 2 adults and 2 children)	£12

Abby is 6 years old and Ben is 12 years old.
Calculate how much the family saves by buying a family ticket instead of separate tickets. £

Figure 3: Item 12-4: Abby’s Entrance Fee (Calc.)

However, although we can explain some of these errors by reference to prototypes (e.g as with item 9-2 earlier) we found that much of the data-handling difficulty was to be accounted for by arithmetic demand or cognitive load. There may, indeed be other dimensions that could be added, e.g. readability, however, parsimony suggested three ‘dimensions’ (we are of course still open to the suggestion that they are not independent, e.g. that arithmetic demand is also a function of cognitive load and prototyping in arithmetic!).

Thus, qualitatively we interpret item difficulty in relation to three dimensions: consistent with the data and informed by literature. For instance, Curcio (1987) reported that the mathematical content of a graph, (number concepts, relationships, and fundamental operations) was a factor in which prior knowledge seemed necessary for graph comprehension.

Further we argue that, at least for these data-handling problems, prototyping errors were most frequent at the lower end of the difficulty scale than those of cognitive demand. One explanation for this may be that such data-handling questions are introduced early in the school career in England and Wales and, hence, early ‘easy’ problem types are ‘recognised’ and pre-existing schema (Sweller, 1988) latched onto. On the other hand, the prototyping errors might be more prevalent lower down the item difficulty scale because the cognitive and arithmetic demands of such items are relatively less. However, all three kinds of reasons for error were identified at all ability levels.

CONCLUSION

In this paper we have (i) scaled the item-difficulties of data-handling for UK national Curriculum ages 7 to 14, identifying errors on the same scale; (ii)interpreted this scale using five levels of performance L0 –L4 and associated errors, prototypes and misconceptions; and (ii) accounted for difficulty levels using three dimensions: cognitive load, arithmetic demand and prototyping.

Our task is now to to model the data against these three constructs e.g. using multi-dimensional scaling analysis, which we would like to present at the PME-30 conference with this paper.

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THE EFFECT OF DIFFERENT TEACHING TOOLS IN OVERCOMING THE IMPACT OF THE INTUITIVE RULES

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The present study aims at investigating the impact of teaching interventions with the use of various tools in overcoming the impact of the intuitive rules. More specifically the tools that were used in three different teaching interventions were: ruler, concrete geoboard and virtual geoboard. The results reveal that many students' responses in perimeter and area comparison tasks are often in line with the intuitive rules. It also appears that the effect of the intuitive rules decreases after all three teaching interventions. The strongest positive effect in overcoming the impact of the intuitive rules is exerted by the teaching integrating the virtual geoboard.

INTRODUCTION

Stavy's and Tirosh (e.g. 2000) theory suggests that a number of students' erroneous solution are often due to the impact of three intuitive rules: *more A- more B*, *same A- same B* and *everything can be divided*. Various studies (e.g. Stavy & Tirosh, 2000, Zazkis, 1999**)** suggested that the answers in line with the intuitive rules are persistent and deeply rooted. Therefore, the need for designing meaningful interventions in order to limit these strongly embedded intuitive rules effects, arises. Helpful sources for such teaching may be found in theoretical frameworks, as well as in general teaching approaches (Tsamir, 2003). In this paper we present a part of a study that focused on a students' common error in tasks that dealt with the relationship between the perimeter and the area of geometrical figures. This study aims at investigating the impact that different tools may have on limiting the effect of the intuitive rules. Thus, three hypothesis were tested: (a) Some of the students' answers in comparison-of-perimeters-and-area tasks are in line with the intuitive rules *more A-more B* and *same A- same B*, (b) The effect of the intuitive rules decreases after teaching intervention and (c) Different tools used in teaching affect the extent to which students provide answers in line with the intuitive rules.

THEORETICAL BACKGROUND

The intuitive rules theory

Stavy and Tirosh (e.g. 2000) coined the term intuitive rules and formulated the intuitive rules theory for analysing and predicting students' inappropriate responses to a wide range of mathematical and science tasks. In their examinations of students' typical responses to numerous conceptually unrelated problems, which share some external, common features, Stavy and Tirosh found that students tend to react in line with three intuitive rules: *more A-more B* (e.g. Zazkis, 1999), *same A- same B* (e.g. Tirosh & Stavy, 1999) and *everything can be divided* (e.g. Tirosh & Stavy, 1996). Here, we focus on the intuitive rules *more A-more B* and *same A -same B*. Stavy and Tirosh (2000) explained that they regard these types of responses as intuitive since they carry Fischbein's (1987) characteristics of intuitive knowledge. That is, solutions of this type seem self-evident, are used with great confidence and perseverance and have attributes of globality and coerciveness. In particular, the intuitive rule *more A-more B* is reflected in students' responses to comparison tasks in which two objects or systems that differ in a certain silent quantity A (A1>A2) are described. The students are then asked to compare the two objects with respect to another quantity B, where B1 is not greater than B2 (B1=B2 or B1<B2). In several cases, many students judged, in line with the intuitive rule, that B1>B2 because A1>A2 (Stavy & Tirosh, 2000). As for, the intuitive rule *same A -same B* it is identified in comparison tasks in which the students are asked to relate to two systems or entities which are equal with respect to one quantity A (A1= A2), but may differ with regard to another quantity B (B1>B2 or B1<B2). The students are asked to compare the two systems or entities with respect to quantity B. A common incorrect response to these tasks take the form of B1=B2 as A1=A2 (Tsamir & Mandel, 2000).

To overcome the effects of the intuitive rules Tirosh and Stavy (2000) propose the use of three teaching approaches: teaching by analogy, conflict teaching and attention to relevant variables. As for the conflict approach, applied to intuitive rules, a conflict can be generated by first presenting students with a task known to forcefully trigger one of the intuitive rules, leading to incorrect response. Then contradiction may be created in several ways, for instance by presenting students with contradictory concrete evidence, or confronting them with a task that is essentially similar to the initial task but known to elicit a correct response, or describing an extreme condition for which a correct judgement is almost unavoidable.

Manipulatives and mathematical teaching

Concrete manipulative is a hand held object that is designed specifically to be used in mathematics instruction to make abstract ideas more tractable (DeLoache, Uttal, & Pierroutsakos, 1998). According to Hall (1998) concrete materials may be useful because it is easier for the teacher to describe actions on physical objects than to describe operations on symbols, and because it is easier for students to proceduralise correctly such description. Conversely, Hart (1993) summarized evidence that children fail to form bridges between manipulatives and the formalizations they are intended to help develop. As for the virtual manipulative, Moyer, Bolyard, and Spikell (2002) define it as an interactive, web-based virtual representation of a dynamic object that presents opportunities for constructing mathematical knowledge. In the literature review it is clear that the virtual manipulatives are considered to be tools that enhance teaching and learning. Specifically, one feature of virtual manipulatives is their capability to connect dynamic visual images with abstract symbols (Clements, 1999). A web connection makes also these manipulatives free of charge and easily available. It is also worth mentioning that some of them have the potential of alteration and that their interaction supports students' engagement in the problem solving process (Moyer, Bolyard, & Spikell, 2002). Besides, the virtual manipulative technologies allow students to work at their own pace, provide them with immediate and specific feedback, enhance students' enjoyment while learning mathematics (Reimer & Moyer, 2005) and improve their conceptual understanding (Crawford & Brown, 2003).

METHOD

The research was conducted among 56 students (27 boys and 29 girls) aged 10 belonging to three classes of urban elementary schools in Cyprus. In order to examine the hypothesis of this study a test, which was administered in usual classroom conditions before and after the teaching interventions, was developed. Furthermore, worksheets that enhanced teaching were prepared too. The teaching interventions involved tasks known to trigger the intuitive rules *more A-more B* and *same A- same B*, leading to incorrect responses and then creating contradiction by confronting students with contradictory representations. These interventions were undertaken by the first two authors two weeks after the pre-test was given, while the post-test was administered two weeks after the teaching interventions. The teaching procedure and the tasks used in all three teaching interventions were the same. What differed was the tool that each group used in order to carry out the proposed tasks. To be more specific, the first group of students used a ruler to measure the dimensions of the shapes that were constructed on the paper, the second group were asked to construct figures on the concrete geoboard using elastic bands and the group that worked on the computer constructed the figures on the virtual geoboard, while at the same time the corresponding area and perimeter appeared on the screen of the computer. The test included comparison tasks for perimeters and areas. These tasks were multiple choice questions, which also required from students to justify their thinking. These justifications would allow the researchers to see whether the students were affected by the intuitive rules *same A* - *same B* and *more A* - *more B.* The first two tasks which are presented below were adapted from Stavy and Tirosh (2000):

Task 1: George cut a part of a Figure A and then Figure B was formed.

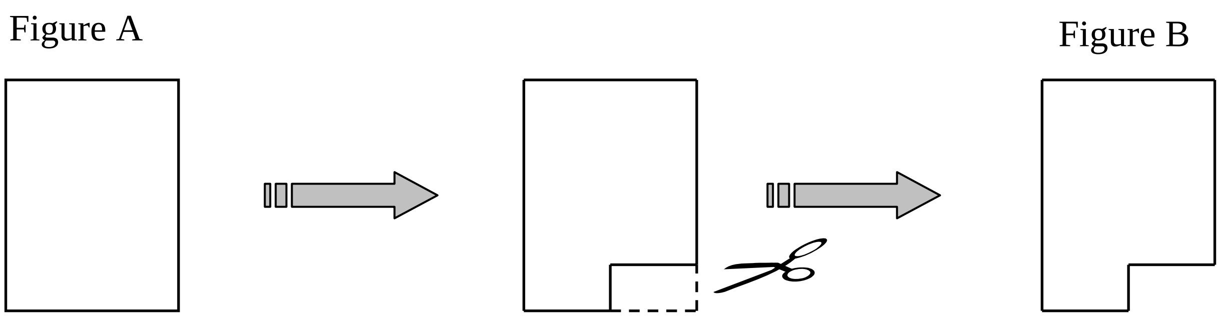
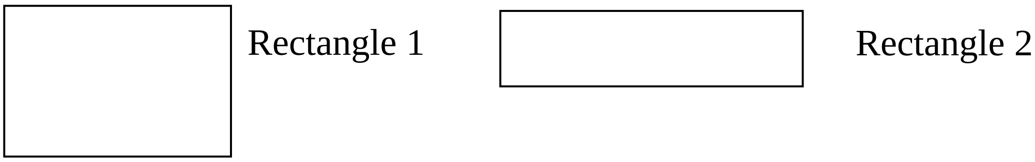


Figure A has a bigger perimeter than Figure B/ smaller perimeter than Figure B/ the same perimeter as Figure B. Explain your answer.

Figure A has a bigger area than Figure B/ smaller area than Figure B/ the same area as Figure B. Explain your answer.

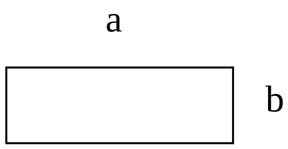
Task 2: Maria has two pieces of rope of the same length. Using these she formed Rectangle 1 and Rectangle 2.



Rectangle 1 has a bigger perimeter than Rectangle 2/ smaller perimeter than Rectangle 2/ the same perimeter as Rectangle 2. Explain your answer.

Rectangle 1 has a bigger area than Rectangle 2/ smaller area than Rectangle 2/ the same area as Rectangle 2. Explain your answer.

Task 3: The rectangle below has length a and width b. What will happen if we double its length and width?



The perimeter of the rectangle will double/ be four times greater/ stay the same. Explain your answer.

The area of the rectangle will double/ be four times greater/ stay the same. Explain your answer.

As it can be noted the application of the intuitive rules to the above tasks leads to an incorrect answer. Specifically, we considered that the students' answer was correct when they answered both the question about the perimeter and the question about the area in a particular task, correctly. The students whose answers were of the kind *bigger perimeter-bigger area* were thought to have answered in line with the intuitive rule *more A-more B*, while those who answered *same perimeter- same area* or *double perimeter-double area* or *four times greater perimeter-four times greater* area were considered to have answered in line with the intuitive rule *same A-same B*. The rests of the other wrong answers were considered to have been caused by other misunderstandings.

RESULTS

Below, the results of data analysis using SPSS statistical package are presented. It is worth mentioning that there is not a statistically significant difference among the three groups of students that took part in the research (F=0.33, p=0.72). In Table 1 the means of the three groups of students' achievement in the two tests in tasks that involve comparison of perimeter and area, are presented. According to the results, the difference between the mean of achievement in pre-test and the mean of achievement in post-test is statistically significant for the group in which the teaching intervention involve the use of concrete geoboard (t = - 2.50, p = 0.00). Besides, the means of achievement in pre-test of the students that made use of the virtual geoboard is significantly higher (t = -3.99, p= 0.00) than in the post-test. The means of success of the students that used the computer doubled in the post-test in comparison to the pre-test due to the potential that the particular tool provides in the learning process.

Tool	Test	$\overline{X}$	SD	t	p
Ruler	Pre-test	4.11	2.54	-2.58	0.02*
	Post-test	5.79	3.39		
Concrete Geoboard	Pre-test	5.65	4.08	-2.50	0.00**
	Post-test	7.41	4.11		
Virtual Geoboard	Pre-test	3.70	3.33	-3.99	0.00**
	Post-test	7.60	3.59		
* $p<0.05$ **$p<0.01$					

Table 1: Means of achievement in comparison tasks in the two tests

In Table 2 the frequencies of correct answers, wrong answers in line with the intuitive rules *same A- same B* and *more A- more B* and other misunderstandings, in perimeter and area multiple choice comparison tasks before and after the teaching intervention, are presented. The results in Table 2 show that before the teaching interventions a big percentage of participants in the comparison tasks were influenced by the accompanying illustration and answered in line with the intuitive rules *same A- same B* and *more A-more B,* even if it possesses to a large extent the formal knowledge of the concepts of area and perimeter. The decrease of answers which are based on the intuitive rules in comparison tasks in post-test is explained taking into account the fact that the teaching interventions focused in overcoming their effects. In addition to this, the answers that were in line with the intuitive rule decreased even more in the case that the teaching intervention involved the use of the virtual geoboard. It may be hypothesised that the prospect of interaction and the feedback of the virtual manipulatives provided this flexibility.

It is also worth mentioning that the related justifications of the answers the students gave in the comparison tasks are categorized in correct justifications, which result from correct reasoning, measurement or application of algorithms and wrong justifications that are in line with the intuitive rules *same A- same B* and *more A- more B*. These are extrapolated by justifications such as *same perimeter - same area* and *bigger perimeter - bigger area,* respectively. Students' responses were also based on visual perception, wrong or correct schematic justifications and other wrong justifications not related to the previous ones.

	Ruler		Concrete Geoboard		Virtual Geoboard	
Answers	Pre-test %	Post-test %	Pre-test %	Post-test %	Pre-test %	Post-test %
Task 1						
Correct	5.3	10.5	17.6	64.7	15.0	60.0
Same A-Same B	21.1	36.8	41.2	5.9	25.0	5.0
More A-More B	42.1	15.8	23.5	5.9	25.0	10.0
Other	31.6	36.8	17.6	23.5	30.0	25.0
Task 2						
Correct	42.1	68.4	47.1	64.7	45.0	70.0
Same A-Same B	10.5	10.5	5.9	5.9	20.0	5.0
More A-More B	15.8	15.8	41.2	11.8	20.0	15.0
Other	26.3	5.3	5.9	17.6	15.0	10.0
Task 3						
Correct	42.1	47.4	17.6	17.6	10.0	55.0
Same A-Same B	36.8	36.8	58.8	41.2	70.0	30.0
Other	15.8	15.8	23.5	41.2	20.0	15.0

Table 2: Frequencies of correct answers, wrong answers in line with the intuitive rules and other misunderstandings in the two tests

CONCLUSIONS

The present study examined the effect of teaching in overcoming the impact of the intuitive rules. According to the findings the students appear to be affected by the intuitive rules *same A- same B* and *more A- more B* in their answers in the comparison-of-perimeter-and-area tasks. This is in accordance with Tirosh and Stavy (2000) who claim that many responses which the literature describes as alternative conceptions could be interpreted as evolving from several common, intuitive rules.

Knowledge about intuitive rules enables teachers, researchers and curriculum planners to foresee students' inappropriate reactions to particular situations so that they can plan appropriate sequences of instruction (Tirosh & Stavy, 1999). The present research undertook and proposes three conflict teaching approaches for overcoming the intuitive rules. These three conflict teaching approaches used three different tools, the ruler, the concrete geoboard and the virtual geoboard, which created different representations. The results of the present study, lead to the conclusion that the use of all these three tools seem to improve the achievement of students in perimeter and area comparison tasks.

Moreover, there appears to be a differentiation in students' achievement after the intervention depending on the tool that was used. A particularly positive effect in overcoming the impact of the intuitive rules was achieved through the incorporation of the virtual geoboard in the teaching intervention. This tool constitutes according to Reimer and Moyer (2005) a combination of multiple representations in a dynamic visual format. Suh and Heo (2005) also point out, that their dynamic nature in relations with various graphics and the interaction they provide, increase the students' interest in solving various mathematical tasks, something that is also confirmed in the present study. It is worth mentioning that in the present study certain students that carried out the tasks in faster pace than the others, using the virtual manipulative had the opportunity to proceed in more tasks and extend them even more than the researchers requested. Furthermore, the students had plenty of time to concentrate on the mathematical relationship which was being studied, since the measurement of the perimeter and the area of geometrical figures appeared automatically on the computer screen, and thus they do not lose time on measuring or calculating. The differentiation between the achievement of students who used the virtual geoboard compared to those who used the corresponding concrete manipulative may be due to the fact that in the case of the latter: (a) The construction of the geometrical figures was time-consuming, since the students were required to have great skill in applying the elastic bands, and (b) The students had to put on paper the calculation of the perimeter and area. These two characteristics were magnified even more in the case of the ruler. In addition to this, in both concrete geoboard and ruler the students had to switch between the tool, measurements and calculations, whereas with the virtual geoboard all these were simultaneously and constantly presented.

Through the teaching intervention the students appeared to realize that their informal, intuitive reactions led them to erroneous solutions. Generally, the students should be encouraged to criticize and test their own responses, relying on scientific formal knowledge (Tirosh & Stavy, 1999). It should be noted that even if the use of tools limited the appearance of answers that were in line with the intuitive rules, their appearance persisted over a small percentage of students. In relation to this, Fischbein (1987) reports that the intuitive perceptions of students, whether they are right or wrong, tend to survive despite the fact that they are contrary to the formal teaching.

However, there is a need for further investigation into the subject with bigger samples than what was used in the present research, with the inclusion of a more extended qualitative and quantitative analysis. In the future, it is interesting to investigate which one of the teaching interventions has the strongest long lasting effect. This can be done through the administration of a delayed test.

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INVESTIGATING SOCIAL AND INDIVIDUAL ASPECTS IN TEACHER'S APPROACHES TO PROBLEM SOLVING

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The study aims at investigating how teachers approach reform-based problem-solving lessons. The study is framed in a socioconstructivist perspective that focuses on both the social and the individual dimension of learning. To grasp the social perspective we developed a coding scheme relating to (1) the focus of the learning environment, (2) the instructional techniques and classroom organization forms, and (3) the set of tasks. For understanding the individual level, we administered three instruments to measure teachers' and students' beliefs about the teaching and learning of mathematical word-problem solving and to assess students' problem solving processes and skills. The paper presents the results with respect to both the social and the individual perspective.

THEORETICAL BACKGROUND

Starting in the 1970s, a worldwide consensus grew that mathematics education should mainly aim at students' mathematical reasoning, problem-solving skills, attitudes, and the ability to use these skills in meaningful, real-life (application) situations, instead of focusing on the acquisition of definitions, formulae and procedures (e.g., Lesh & English, 2005). This reform movement was also influential in Flanders and resulted in the formulation of new standards for primary mathematics education (Ministerie van de Vlaamse Gemeenschap, 1997).

As a contribution to the implementation of these new standards, Verschaffel, De Corte, Lasure, Van Vaerenbergh, Bogaerts, and Ratinckx (1999) designed, implemented and evaluated a learning environment that is powerful in eliciting in upper primary school children the appropriate learning processes for acquiring competence in mathematical problem-solving as well as positive mathematics-related beliefs. This learning environment emphasizes students' acquisition of an overall metacognitive strategy for solving mathematical problems involving five stages: (1) build a mental representation of the problem; (2) decide how to solve the problem; (3) execute the necessary calculations; (4) interpret the outcome and formulate an answer; (5) evaluate the solution. Moreover, it also aims at eliciting positive mathematics-related beliefs. To attain these goals, the learning environment is based on three pillars: (1) the use of a varied set of realistic, complex, and open problems; (2) the use of a varied set of powerful instructional techniques; and (3) the creation of a classroom culture aimed at establishing new norms about teaching and learning mathematical problem-solving. The results revealed that the learning environment

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was indeed powerful in producing significant and stable positive effects on students' beliefs and problem-solving capabilities.

In an attempt to generalize the benefits of this experimental learning environment, some Flemish textbook makers reconceptualized their materials by explicitly appealing to Verschaffel et al.'s (1999) work. However, implementation of this environment on a large scale is problematic for at least two reasons. First, translating research into practice is context sensitive (Burkhardt & Schoenfeld, 2003). Second, successful implementation of the innovative ideas requires not only good materials but also highly motivated and capable teachers who are prepared and able to use the materials appropriately (Remillard, 2005).

In this paper, we describe a study aimed at investigating how textbook problem-solving lessons that explicitly appeal to the above-mentioned experimental learning environment are implemented in regular classrooms. Based on the findings of previous research on curriculum implementation, showing that teachers, as active interpreters of the curriculum, create their own meanings concerning what should be implemented and how this should be done (Remillard, 2005), we first expect difficulties in the implementation of these lessons by the teachers as well as considerable differences in implementation quality between different teachers. Second, we anticipated that these distinct approaches to implement the same problem-solving lessons are related to teachers' beliefs, on the one hand, and students' beliefs and problem-solving capabilities, on the other hand.

SOCIOCONSTRUCTIVISM

The socioconstructivist perspective enables us to meet the intended objectives, i.e., to investigate the way in which teachers approach problem-solving lessons and to unravel the (complex) relation between these approaches and individuals' beliefs and performances. By valuing both the individual and social dimensions of mathematics learning and contending that both dimensions are reflexively related with neither having precedence over the other, the socioconstructivist perspective tries to take into account features from a cognitive/rationalist as well as a situative perspective (Voigt, 1995). Several authors developed theoretical constructs to describe the reflexive relation between individuals' cognitive development and the social context, such as the didactical contract (Brousseau, 1998), thematic patterns of interactions (Voigt, 1995), and norms and beliefs (Cobb, Stephan, McClain, & Gravemeijer, 2001; Yackel, 2001).

Especially this last theoretical account for analysing mathematical classroom processes was useful for our study. Cobb and associates (Cobb et al., 2001) describe mathematics cultures in terms of classroom norms and practices, on the one hand, and teacher's and students' beliefs, conceptions and activities, on the other hand. These clusters of concepts reflect the social and psychological perspective underlying socioconstructivism: the social perspective refers to ways of acting, reasoning, and arguing that are normative in a classroom community, while the psychological perspective is concerned with the nature of individual students' reasoning and beliefs, and their particular ways of participating in communal activities.

METHOD

We selected ten sixth-grade teachers who all used the textbook *Eurobasis* and expressed a willingness to participate in our study. Major criteria for the selection of that textbook series were its representativeness for the Flemish situation in elementary school teaching (i.e., it is the most frequently used textbook in Flanders), and its explicit reference to the principles and materials of the aforementioned experimental learning environment by Verschaffel et al. (1999). For instance, the textbook applies explicitly the proposed overall metacognitive strategy for solving mathematical application problems.

In each classroom we selected and videotaped the same two problem-solving lessons from the textbook that contained, besides some routine tasks, several non-routine problems. In addition, all students' materials (e.g., their workbooks) were collected. Moreover, we administered three instruments to measure teachers' and students' beliefs and attitudes about the teaching and learning of mathematical word-problem solving, and to assess students' problem-solving processes and skills.

In line with Cobb's framework we focused on both social and individual processes to understand how teachers approach problem-solving lessons. To grasp the **social perspective** we developed a coding scheme relating to three important characteristics of the classroom culture that are assumed to enhance students' mathematical beliefs and problem-solving competencies, namely (1) Is the focus of the lessons in line with reform-based ideas? (2) Do the teachers apply powerful instructional techniques and classroom organization forms that foster such a reform-based focus? and (3) Does the set of tasks and the way in which the tasks are addressed reflect these innovative ideas? The focus of our analysis was the teacher, i.e., each speech act[1] of the teacher was coded for the first two characteristics. For the third one, we analysed all tasks that were given to and discussed with the whole class.

Concerning the *focus of the learning environment* we focused on the extent to which the teachers explicitly addressed heuristic and metacognitive skills, on the one hand, and on their establishment of particular classroom norms, on the other hand. The teacher's manual that accompanies the textbook proposes some important heuristics to which teachers should pay attention, such as distinguish relevant from irrelevant data; reword the problem; contextualize the problem; draw a picture; make a scheme/table; guess and check; evaluate the solution; interpret the outcome. Teacher's speech acts were analysed with regard to the classroom norms that were explicitly negotiated between teacher and students during the lessons. We distinguished between ten norms, such as: "Solving problems is enjoyable"; "Solving

[1] In this context, a speech act is defined as the whole set of sentences, which is uninterruptedly spoken by one actor.

problems is time-consuming, also for smarter students"; "A problem can be solved in different ways", "A problem may have different solutions", etc.

The second part of the coding scheme focused on *teachers' instructional techniques and on classroom organization forms*. The instructional techniques distinguished in the cognitive apprenticeship model (Collins, Brown, & Newman, 1989) were used as coding categories. However, we split coaching into two subcategories and added to the list the technique of praising, which relates to the motivational aspect of learning. This resulted in the following instructional techniques: modeling, non-directive coaching, directive coaching, scaffolding, articulation, reflection, exploration, and praising. With respect to classroom organization we distinguished between whole-class instruction, group work, individual work, and combined organizational form.

The last category pertained to the *set of tasks* the students were confronted with. Each task was analysed with regard to its degree of realism and its degree of complexity.

To better understand **individuals'** conceptions and activities, we assessed teachers' and students' beliefs about the teaching and learning of mathematical word-problem solving and students' problem-solving processes and skills.

To measure *teachers' beliefs*, we used a self-made questionnaire containing three parts. The first part asks for some personal information. The second part consists of a series of statements relating to teachers' beliefs about mathematics and problem solving (e.g., "There is only one correct way to solve a problem"). Teachers had to express their level of agreement on a five-point Likert-scale and were also invited to comment on their choices. The third part addresses teachers' evaluations of possible realistic and non-realistic student answers to two realistic mathematical problems: "Jan's best time to run 100m is 17sec. How long will it take him to run 1km?", and "Steven has bought 4 planks of each 2,5m. How many planks of 1m can he saw form these planks?". As in the study of Verschaffel, De Corte, and Borghart (1997), teachers were asked to evaluate four possible student responses to these realistic problems; one of the answers did not take into account any realistic consideration (e.g., 4 x 2,5 = 10 planks for the above planks problem) whereas another answer could be considered as a (more) realistic one (e.g., 4 x 2 = 8 planks)

Students' beliefs were measured by means of a questionnaire developed by Verschaffel et al. (1999), consisting of 21 statements that students had to value on a five-point Likert-scale. These statements represent two reliable factors: students' pleasure and persistence in solving word problems (7 items, e.g., "I like to solve word problems"), and a problem- and process-oriented view on word problem solving (14 items, e.g., "There is always only one solution to a word problem"). The beliefs addressed in this questionnaire corresponded with the norms distinguished in our coding scheme (cf. supra), which enabled us to investigate whether the approaches to implement these problem-solving lessons (and more specifically, the norms that are negotiated) are related to individual's beliefs.

Students' mathematical problem solving was measured by a paper-and-pencil test, containing 10 non-routine problems for which the solution process appeals especially to the application of heuristic/metacognitive skills and to a disposition towards realistic mathematical modeling and problem solving (e.g., "Michael builds little figures with matches. To construct 2 figures, he needs 11 matches. To build a row of 5 figures, he needs 26 matches. How many matches will he need to build a row of 10 figures?"). This test was also developed and used by Verschaffel et al. (1999), but was slightly modified with respect to the current situation of our research.

RESULTS

Social perspective

The analysis of teachers' speech acts from the perspective of the focus of the learning environment showed that some of the *heuristics/metacognitive skills* were frequently emphasized during the problem-solving lessons. In general, most attention was paid to "distinguish relevant from irrelevant data", "reflect critically on the problem", "make a scheme/table", "compare different solution processes", and "evaluate the solution". Other skills were almost never addressed, such as "guess and check", "simplify the numbers", "contextualize a calculation", "make a flowchart", and the "overall metacognitive strategy for solving mathematical application problems". Substantial differences among the teachers were observed in the frequencies with which they stressed the heuristics/metacognitive skills. Remarkably, in only a very few cases the teachers explained *why it is important* to use a particular heuristic or metacognitive skill.

Interestingly, little or no attention was paid to the deliberate and explicit establishment of mathematics-related *norms*. Only to a very small extent did some teachers emphasize that a problem can be solved in different ways and/or may have different solutions. Other norms were almost never articulated by the teachers.

Teachers' speech acts were also analysed with regard to their *instructional techniques*. Apart from modeling, scaffolding, and exploring, most of the instructional techniques described in the model of cognitive apprenticeship were used frequently. Notable is the importance given by the teachers to articulation and reflection, which certainly fosters students' problem-solving competency. Teachers praising students was observed rarely. Again, substantial differences between teachers in using these instructional techniques were observed.

Although most of the total lesson time was devoted to whole-class instruction, the combined *classroom organization form* was also popular. In this format teachers mostly guided a specific group of weaker children, while the others worked individually on tasks at their own pace. Although the mathematics education literature emphasizes the importance of small-group instruction for the development of students' problem-solving skills and dispositions (Good, Mulryan, & McCaslin, 1992) and this organizational form was also principal in Verschaffel et al.'s (1999) learning environment, only two of the ten participating teachers used group work during the video-taped problem-solving lessons. In general, the analysis of the sequence of organization forms showed that most of the teachers frequently alternated between different organization forms.

All *tasks* were analysed with regard to their realism and their complexity. Almost all tasks referred to contexts taken from students' experiential worlds. Conversely, the percentage of complex problems was much lower. Interestingly, some teachers systematically dropped the complex tasks provided in the textbook. However, it should be noted that in all of these classes some of the more able students were given the opportunity to work individually and independently on complex tasks.

Individual perspective

The *teacher's questionnaires* revealed that, in general, the teachers did have positive beliefs towards mathematical problem solving. However, three out of the ten participating teachers believed that a good math problem always has an exact solution. With respect to the runners-item, teachers valued non-realistic answers more than realistic ones, whereas the inverse was found for the planks-item. These results are in line with the findings of Verschaffel et al. (1997).

Students' questionnaires revealed that, compared to the results of the design experiment of Verschaffel et al. (1999), in which students were asked to complete the same questionnaire, our students' beliefs and attitudes about the teaching and learning of mathematical word-problem solving were less in line with the reform ideas reflected in the new standards. This suggests that the attempts to innovate mathematical textbooks do not automatically result in an enhancement of students' mathematics-related beliefs. Although this is in line with the related research literature (Remillard, 2005), the kind of information we collected, as well as the restricted data set, does not allow us to derive in this regard general conclusions.

Students' performance on the *mathematical application test* showed that their problem-solving competency had increased more substantially compared to their peers in the aforementioned study of Verschaffel et al. (1999). This better performance may be explained by a more problem-oriented instruction towards mathematical problem solving in our ten classes, suggesting that the innovated textbook had indeed a positive effect in this respect. Also here remarkable differences between classes were observed, with scores varying from 1.94 to 5.04 out of 10.

Attempt at integrating both perspectives

Although our restricted data set does not allow us to draw any causal relations between the individual and social aspects investigated in our study, our results revealed already some provisional trends concerning the connection between the social and individual dimension of mathematical learning and teaching. First, only one teacher frequently stressed the norm that problems can be solved in different ways (e.g., "All roads lead to Rome, as long as you take the road that is the easiest one for you"). This was the only class wherein all students expressed the belief that problems can be solved in different ways (59% totally agreed and 41% agreed with this statement); in all other classes the overall agreement with this statement was considerably lower. Second, the class that performed worst on the problem-solving test had the least availing beliefs. The teacher of that class coached more directive than non-directive, whereas all other teachers (except one) did the inverse. This directive teacher also never explicitly negotiated an appropriate norm for mathematical problem solving, she never explicitly emphasized the importance of a certain heuristic, and she never used group work or individual work.

CONCLUSION AND DISCUSSION

This study is relevant for at least two reasons. First, it confirms the gap between theory and practice, in this case the tension between the intended and the implemented curriculum. This tension is often neglected in educational research (Remillard, 2005). Second, the study yields a useful instrument for understanding the complex processes going on in the mathematics classroom.

However, our study has also several limitations. We restricted our analyses to low-inference coding, i.e. directly observable classroom activities (Reusser, 2005). As an example we refer to our coding of classroom norms. In line with the theoretical framework of Cobb et al. (2001) we were especially interested in the normative aspects that are negotiated in the different classrooms. However, as we only coded for norms that were *explicitly* negotiated between teacher and students (i.e., low-inference coding), we could only partly map out the ways of acting, reasoning, and arguing that are normative in a classroom community. Indeed, norms are also *implicitly* negotiated between teacher and students. Moreover, we videotaped and analysed only two lessons in each classroom, and, arguably, classroom norms develop and get only installed gradually over time. Although our coding of classroom norms has led to a reliable identification of these norms, it should be possible to achieve a better grasp of the normative aspects of classroom activities by looking for patterns of regularities (i.e., practices that frequently occur in a classroom) that implicitly reflect certain norms, and by conducting more in-depth case studies (involving video-based stimulated recall with teachers and/or students) spread over a longer period of time. Therefore, we are now carrying out a new study with two teachers who represent contrasting approaches to implementing the same problem-solving lessons in an attempt to unpack these approaches more in-depth in view of distilling explicitly and implicitly negotiated norms.

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MATHS AVOIDANCE AND THE CHOICE OF UNIVERSITY[1]

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In the last decade, in Europe, there has been a constant decrease of enrolment in university degrees with scientific orientation. In Italy, those degrees are characterised by at least one compulsory mathematics course. Our hypothesis is that a negative attitude towards mathematics (i.e. a negative emotional disposition towards mathematics) plays a role in the refusal to enrol in university scientific courses. After presenting arguments to maintain the existence of a link between attitude and university choice, it seems crucial from researchers' point of view to understand which kind of experiences with mathematics characterise a negative emotional dispositions towards the discipline. With these aims a study has been carried out within a national Project, using a purposefully designed questionnaire, namely QCM (Questionnaire on the Characterization of Mathematics). This report discusses the main features of the QCM, describes some results of that study and suggests some implications for research on attitude.

INTRODUCTION AND THEORETICAL BACKGROUND

In the last years, in Europe, the matriculation to university degrees with a scientific orientation has been constantly decreasing. This decrease could bring about, in the near future, an insufficient number of both professionals and researchers with a strong scientific background. For this reason, the European Union included the reversal of such a phenomenon of decrease among the main goals of education. Data referring to tha Italian situation are quite alarming: in the last 20 years the number of Science faculties undergraduates decreased from 13,3% of the total number of undergraduates to 10,3%. Furthermore, in the academic year 2003/2004[2] enrolments increased by 1,8% (if compared to year 2002/2003), but new entries to Science degrees decreased by 5,2%. We wonder whether this phenomenon is (at least partially) caused by a sort of dislike for one common feature to all the scientific degrees, i.e. specific mathematics courses. This would mean that the refusal to enter a university degree within Science faculties may be linked to the well known problem of maths avoidance. Hart (1989, p.38) already underlined that:

> It is relatively clear that decisions about how many and which mathematics courses to take in middle school, high school, and college can be influenced by affective characteristics of the students that have developed over a period of many years.

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[1] Research supported by the MIUR (Project FIRB RBAU01S427).

[2] This is the last year for which official data from the Ministry of Education are available at the moment.

Maths avoidance was traditionally studied through the construct of *attitude towards mathematics*. Considerations such as the link between attitude and university choice bring to the fore the importance of a theoretical study of attitude towards mathematics. This construct has been used with different meanings in the literature (Di Martino & Zan, 2001); in particular, two typologies can be identified: what we called the *simple definition*, that identifies attitude with the emotional disposition (like/dislike) towards the object, and the definition linked to a *tripartite model*, according to which attitude has a cognitive, an affective, and a behavioural component (Eagly & Chaiken, 1998). However, as Kulm (1980, p.358) points out:

> It is probably not possible to offer a definition of attitude toward mathematics that would be suitable for all situations, and even if one were agreed on, it would probably be too general to be useful.

In this way, the definition of attitude assumes the role of a '*working definition*' (Daskalogianni & Simpson, 2000). If in situations of choice/refusal, even in social psychology, the simple definition of attitude is considered suitable to highlight a correlation between attitude and behaviour (choice/refusal), on the other hand, the simple definition of attitude does not give any indication for possible interventions on and changes of the maths avoidance phenomenon. For this reason, it is necessary to access those beliefs and intepretations of one's own experiences with mathematics, that are the source of emotional disposition. This is also in line with the studies of many cognitive psychologists (Mandler, 1984; Ortony et al., 1998) that underline the importance of the cognitive dimension of emotion. Hence, it is very important to design appropriate observational tools taking into account the complex nature of the problems to be described through these constructs.

The standard approach for research on attitude in mathematics uses questionnaires made of a list of statements (beliefs, emotional dispositions, interpretation of experiences) that are considered important *by the researcher* and that are a priori scored by the researcher (Munby, 1984; Eagly & Chaiken, 1998). The answerer is asked to express his/her agreement (often, a Likert scale is used) with the statements (e.g. *maths is useful* or *maths is difficult*). In this way, each answer has a score and the sum of the scores is the *result*, which would be the *measure* of the emotional disposition toward mathematics. This approach certainly led to important results, but it is related to a number of theoretical issues. First, the a priori selection of the statements does not take into account the psychologically central role of the topic for the answerer (Green, 1970): the latter may be asked to express his/her agreement on sentences that are meaningless for him/her, or which he/she never thought about. Furthermore, the approach amplifies the problem of the well-known difference (Schoenfeld, 1989) between *beliefs expoused* and *beliefs in action*, because the answer may be influenced by the answerer's idea that there are a "right" and a "wrong" answer. Finally, the score given by the researcher is linked to the arbitrary assumption that single answers are necessarily correlated with a given emotional disposition. (Di Martino & Zan, 2003).

The design of new and suitable observational tools for the study of affective factors (emotions, beliefs and attitudes) towards mathematics is one of the major aims of a three-years Italian Project about the evolution of students' attitude towards mathematics: "*Negative attitude towards mathematics: analysis of an alarming phenomenon for the culture in the new millenium*". The strenght of the project is the possibility of analyzing jointly results obtained with different instruments (one of these observational tools was presented in Di Martino, 2004).

In this report we present an innovative semi-open questionnaire, used in the Project and called QCM[3] (Questionnaire on the Characterization of Mathematics), which was designed in order to characterize the *view of mathematics* associated with an emotional disposition. We also analyze some results obtained through the administration of the QCM to a wide sample of Italian 18-19 years old students.

METHOD

QCM "turns upside down" the usual way of designing questionnaires for research on attitude towards mathematics: the emotional disposition is no more the final focus but the starting point to gather new information. This approach is in line with the previous remarks about the importance of not only describing the phenomenon but also outlining possible interventions. As a matter of fact, emotional disposition towards mathematics is not inferred from the answers but rather asked directly through a Likert scale with four modalitities: VM (very much), E (enough), L (little), NA (not at all). Afterwards, the questionnaire is developed in different directions. On one side, the student is asked whether he/she is going to carry on with his/her studies and, in the affirmative case, whether and how the presence of a mathematics course may influence the choice of the university degree. A cross-cutting analysis of these answers with the expressed emotional disposition towards mathematics, makes it possible to study the influence of attitude towards mathematics on the choice of a university degree. Furthermore, if this is the case, it is interesting to investigate the possible symmetry between the influence of positive/negative attitude towards maths in choice/refusal of scientific courses. On the other side, an open question asks to describe mathematics using three adjectives. A statistical analysis of the answers to this open question gives clusters of adjectives that are linked to a given emotional disposition towards mathematics. Finally, in order to study the correlation (not discussed in this paper) between emotional disposition and success in mathematics, the mathematics mark in the latest school report is asked.

QCM was distributed to more than 70 classes of secondary schools (to students attending 12th and 13th grades of secondary school, i.e. students that are next to the choice of university) nationwide. The students were given half an hour to fill the questionnaire and were told that the questionnaires were anonymous. We collected 1837 questionnaires (805 from students of grade 12, 1032 of students of grade 13).

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[3] See the appendix. In the following, we will refer to the questions in the QCM, that are numbered.

As concerns the characterization of the four groups (VM, E, L, NA) through the adjectives used to describe maths (the analysis of the answers to point (3) of QCM) we have used T-Lab[4], a software for textual analysis. We choose a variable (emotional disposition) with different modalities (four in our case) and the software analyses data (the three adjectives in our case) identifying specific features for each group through the chi square test, with a threshold value of 3.84 (df=1, p. 0.05).

RESULTS AND DISCUSSION

The analysis of QCM suggests many reflections starting from the division of students into four groups according to their declared emotional disposition towards mathematics: VM, E, L, NA. In the following, we analyse the answers focusing on: a) the correlation between declared emotional disposition towards mathematics and choice/refusal of a scientific university course; b) the characterization of the four groups through the adjectives used to describe mathematics.

Table 1 summarises the partition of the sample into the four groups (according to answers to (1)). Almost 40% of the sample has a negative or very negative attitude towards mathematics.

emotional disposition	**VM**	**E**	**L**	**NA**
n.of answers	260 (14%)	869 (47%)	516 (28%)	179 (10%)

Table 1: Emotional disposition towards mathematics

Answers to (4) show that 85% of the sample (1565 students) express their intention to enter university. Crossing this information with the partition into groups according to answers to (1), we see that the percentage of those who express their intention to study at university decreases when moving from group VM to group NA. Probably, many of those who answered “No” have a negative attitude towards all the disciplines; anyway, this result is in line with the hypothesis that a negative attitude towards mathematics is strictly linked to school drop-out (Moscucci et al., 2005):

(4)	**VM**	**E**	**L**	**NA**
Yes	242 (93,5%)	752 (86,6%)	428 (83,1%)	135 (73,4%)
No	17 (6,5%)	117 (13,4%)	87 (16,9%)	44 (24,6%)

Table 2: Intention to study at university

Answers to (4a) give some information on the influence that the presence of a Mathematics course may have on the choice of a university degree. Data are alarming: 4% (55) of those who are going to study at university will never choose a university degree that includes Mathematics courses. And 27% (414) views the

[4] The bibliography related to T-lab is available on-line: http://www.tlab.it/en/presentazione.asp by F.Lancia

presence of a Mathematics course as a negative aspect, whereas only 12% views it as positive. All the others (57%, 870 students) declare that the presence of a Mathematics course does not affect their choice. Once again, it is interesting to cross these data with the partition into four groups:

(4a)	VM	E	L	NA
Negatively decisive	0 (0%)	8 (1%)	20 (5%)	20 (16%)
negative	6 (3%)	100 (13%)	228 (54%)	82 (68%)
I don't care	125 (52%)	556 (75%)	171 (40%)	20 (16%)
positive	109 (45%)	80 (11%)	6 (1%)	0 (0%)

Table 3: Influence of the presence of a Mathematics course

We stress that the presence of a Mathematics course seems to influence more those who belong to groups NA and L. For groups E and VM, the most common answer is "I don't care", while for groups L and NA the most common answer is "negative". We may think that the negative emotional disposition is often linked to emotions and experiences that are stronger than those linked to a positive emotional disposition: for this reason, the negative emotional disposition seems to have a stronger influence on the choice/refusal.

As regards point (b), we try to characterize the four groups (VM, E, L, NA) through the analysis of the adjectives (answers to (3)) used to describe maths. First of all, we underline that sometimes students did not use adjectives to describe mathematics (they rather used sentences or substantives or neologisms very hard to translate from Italian). Anyway we will use the word *adjective* to indicate all answers given to (3). The adjectives used to describe mathematics were 5387, instead of the predicted 5511, due to the fact that some students have written less than three adjectives, as requested. But it is very important that the *different* adjectives used to describe mathematics were 490: on one hand this highlights the richness of the data collected through an open-ended question, on the other hand we had to face the problem of managing different adjectives that have a similar meaning. Using T-Lab there is the possibility to manually merge different adjectives (and analyse them as a single one), but we decided not to use this option because we believe that often two different adjectives with similar meanings may have a different *emotional charge.*

As we mentioned earlier, in order to constitute clusters, we constructed a file including all the collected answers, to be analyses through T-Lab. For each of the four groups (VM, E, L, NA) T-Lab provides the list of adjectives which, within the groups, get a value over the threshold value 3.84 at the chi square test. In this short paper we only point out and analyse the first 5 adjectives in each group (the most meaningful ones): the complete list certainly provides a detailed characterisation of clusters, but also this partial datum suggests important remarks. For each adjective we will orderly indicate the chi square value (chi2), the number of occurrences in the considered group (sub) and the total number of occurrences (tot):

Group VM				Group E			
Adjective	**Chi2**	**Sub**	**Tot**	**Adjective**	**Chi2**	**Sub**	**Tot**
enthralling	44,47	20	39	interesting	72,36	362	560
interesting	39,01	128	560	useless	29,82	19	95
amusing	39,00	25	59	difficult	13,00	128	335
relaxing	29,84	6	7	pleasant	10,14	23	30
beautiful	29,35	19	45	basic	9,31	11	12
Group L				**Group NA**			
Adjective	**Chi2**	**Sub**	**Tot**	**Adjective**	**Chi2**	**Sub**	**Tot**
boring	116,84	168	304	useless	206,57	49	95
difficult	35,58	142	335	boring	139,62	86	304
heavy	30,56	26	38	nasty	55,84	9	13
complex	23,05	154	399	annoying	50,42	7	9
tedious	13,02	11	16	incomprehensible	24,20	11	32

Table 4: the 5 most meaningful adjectives for each group

The first striking aspect in these data is that *interesting* characterises both groups of positive emotional disposition, whereas *boring* (that can almost be considered as the opposite of *interesting*) characterises groups with a negative emotional disposition. Hence**,** the answer to a semantic differential having these two adjectives as endpoints seems to be meaningful in a questionnaire on attitude.

An unusual aspect is the fact that in two cases the same adjective is used to characterise groups with a different emotional disposition. The two adjectives are *difficult* and *useless* and this makes the result even more interesting because often questionnaires ask agreement or disagreement with sentences like "*math is useful*" or "*math is difficult*" in order to observe and measure attitude. This suggests that this agreement is not extremely meaningful if the aim is to attribute a score in order to deduce the emotional disposition as final result. This is an important remark from the theoretical point of view because it stresses the arbitrary nature of the choice to measure attitude and the need for a descriptive approach. According to this approach, attitude is '*a construct of an observer's desire to formulate a story to account for observations*', rather than '*a quality of an individual*' (Ruffell et al., 1998, p. 1).

As concerns *difficult*, it characterises the two central groups: it is not in NA and its opposite (easy) is not in VM either. Besides this, a possible hypothesis is that those who link *difficult* with negative emotions tend to identify two different levels: the level *like/dislike* with that *can/cannot*, whereas often there is a taste for a challenge, increasing as this gets more difficult.

As concerns *useless* it may seem weird to find it as the second adjective characterising group E. In actual fact this is not surprising if also great mathematicians (e.g. Hardy, Hadamard) believe that real mathematics is completely useless. Moreover the fact that mathematics is basic to many objects used in everyday life does not imply that users need to know mathematics. However it is interesting that *useless* strongly characterises group NA (chi2 = 206,07) and that *useful* is generally one of the most widely used adjectives in our questionnaire (376 occurrences) but not specific of any group. What is a possible interpretation of this fact? We believe that while agreement with a commonplace, a myth about mathematics (such as "*maths is useful*") is often a-critical and not psychologically central, disagreement can be the result of a personal reflection as well as a way to rebel against the way of presenting mathematics in the classroom. This remark is important in relation with standard questionnaires which ask for agreement on these sentences: two opposite answers from the content viewpoint may be not *symmetrical* from the emotional viewpoint.

CONCLUSIONS

Analysis of answers to QCM concerning the correlation between choice of university degree and attitude (in the simple meaning of emotional disposition) towards mathematics raises at least two points for reflection. On the one hand, a cross-cutting analysis of data about declared emotional disposition and possible influence of the presence of mathematics courses confirms that the two aspects are strictly related. On the other hand it seems to point out that a negative attitude influences the choice more than a positive attitude.

But the most interesting results obtained using QCM descend from the analysis of answers to open ended questions (only partially discussed in this paper). In actual fact not only the creation of clusters of adjectives characterising the four types of emotional dispositions is important as a theoretical result: the comparison between the different clusters, the analysis of similarities and differences provides elements for a discussion on the standard approach to research on affective factors. In particular it suggests the need for a descriptive approach in which attitude is not a number resulting from a more or less articulated sum of scores but rather is a mosaic made of many pieces that merge and get a meaning only in a global view.

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Appendix: The QCM

School: Town: Grade: ☐12th ☐13th

1. Do you like mathematics? ☐ not at all ☐ little ☐ enough ☐ very much

2. What mark did you get in mathematics in your last school report?

3. Choose 3 adjectives to describe mathematics:____________________

4. Do you think you are going to enter University? ☐ yes ☐ no

4a. How does the fact that there is a compulsory mathematics course in a certain university degree influence your choice?

☐ it is indifferent: what matters is that I am interested in the course

☐ it is a positive fact

☐ it is a negative fact, although not decisive: if I am interested in the course I choose it anyway

☐ it is a decisive fact: even if I am interested in a course I will never choose it if it includes a mathematics course

☐ OTHER (specify)

PRIMARY STUDENTS' REASONING ABOUT DIAGRAMS: THE BUILDING BLOCKS OF MATRIX KNOWLEDGE

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This paper reports on the development of primary students' ability to justify their reasoning for the selection of the matrix as an appropriate representation for specific mathematical scenarios. The results over a 3-year period revealed that less than 50% of students were able to justify their selections adequately and that this percentage dropped substantially with age. Additionally, although 20% of students' provided more adequate explanations over time, 73% of students declined in their explanations or provided erratic or consistently inadequate explanations. These results suggest the need for explicit instruction in diagram use because there is scant evidence that a generic focus in the curriculum on representation, reasoning and communication is sufficient to support students to develop sound diagrammatic knowledge.

INTRODUCTION

The visually-oriented Information Age has sparked a dramatic upsurge in diagrammatic research and theory development across a range of disciplines including applied psychology, linguistics, visual programming, data visualisation, graphics design, history and philosophy of science, architecture, and cartography (e.g., Blackwell & Engelhardt, 2002; Glasgow, Narayanan, & Karan, 1995). However, there is scant research on diagram literacy in mathematics education. Diagram literacy involves knowing about diagram use and being able to use that knowledge appropriately in representation, reasoning and problem solving (Diezmann & English, 2001). This paper focuses on primary students' ability to select an appropriate diagram to represent given information. Being able to select an appropriate diagram is one of three key skills for effective diagram use (Novick, 2001), and fundamental to problem solving (Booth, & Thomas, 2000).

THEORETICAL FRAMEWORK

A useful definition of "a diagram" across disciplines is elusive. However, following a cross-disciplinary review of diagram taxonomies, Blackwell and Engelhardt (2002) proposed a meta-taxonomy for diagram research consisting of nine aspects of diagrams and diagram use. Their taxonomy included *graphic structure*, which is important in the identification of an appropriate diagram for a given mathematical situation. Three useful diagrams that have unique graphic structures are the *matrix*, *network* and *hierarchy*. These diagrams serve particular purposes (Novick, in press):

> The matrix stores static information about the kind of relation that exists between pairs of items in different sets, the network conveys dynamic information by showing the local connections and global routes connecting the items being represented, and the hierarchy depicts a rigid structure of power or precedence relations among items.

Novick and colleagues (e.g., Novick, in press, 2001; Novick & Hurley, 2001) in a series of studies with college students identified the properties of spatially-oriented diagrams and examined the diagnosticity (i.e., relative importance) of each of these properties in diagram selection. Novick and Hurley (2001) proposed that spatially-oriented diagrams are defined by ten properties, namely *global structure, building block, number of sets, item/link constraints, item distinguishability, link type, absence of a relation, linking relations, path,* and *transversal*. However, the profile of the property varies according to the diagram. For example, the *global structure* has a factorial structure in a matrix, lacks formal structure in a network and has an organisational structure in a hierarchy. The variation in diagram properties extends to their diagnosticity in diagram selection. Novick (in press) argues that there is instructional benefit in knowing which properties are strongest diagnostic because these are the most important in diagram selection (See Table 1).

Diagram	Properties
Matrix	Global structure, Link type, Absence of a relation
Network	Item/link constraints, Link type, Linking relations, Transversal
Hierarchy	Global structure, Building blocks, Linking relations

Table 1: Properties with the strongest diagnosticity for college students.

The work of Novick and colleagues (e.g., Novick, 2001, in press; Novick & Hurley, 2001) has informed a study of primary students' knowledge of diagrams. Previous reports of this study revealed three points of interest and relevance. Firstly, Grade 3 and Grade 5 students' ability to select the correct diagram to represent a particular property varied according to the property, the diagram and the age cohort (Diezmann, 2005a). Both cohorts' performances were greater than chance (.50) for: (a) the matrix tasks on the *global structure, number of sets, item/link constraints* and *link type* properties, (b) the network tasks on the *link type* and *transversal* properties, and (c) the hierarchy task on the *transversal* property. Hence, *global structure* and *link type* properties on the matrix and *link type* and *transversal* properties on the network appear to be diagnostic for primary students as well as college students (See Table 1). Secondly, there was a lack of significant differences in the performances of Grade 3 and Grade 5 students on these four diagram selection tasks (Diezmann, 2005a). However, this outcome could have been impacted by cohort or time period effects (Willett, Singer, & Martin, 1998). Thirdly, there was great variation in students' reasons for correctly selecting the matrix on the *link type* task with only 19.5% of Grade 3 and 23.9% of Grade 5 students giving reasons that demonstrated some understanding of the row and column structure of the matrix or its purpose (Diezmann, 2005b). Hence, it is fallacious to assume that students who can correctly select a diagram can adequately justify its selection.

These previously reported results focus on the performance of two cohorts of students in a single year (Diezmann, 2005a, 2005b). This paper reports longitudinal data on one cohort's knowledge of *global structure* and *link type* properties for the matrix.

The matrix can be useful in combinatorial and deductive problems. In a matrix, the *global structure* or general form is factorial and the *link types* are associative and non-directional (Novick & Hurley, 2001). These properties were diagnostic for both primary students (Diezmann, 2005a) and college students (Novick, in press).

DESIGN AND METHODS

This investigation is part of a larger study of the development of primary students' knowledge about spatially-oriented diagrams. Within the larger study, the performance of two cohorts of differently-aged primary students was studied for three years. This paper explores two research questions.

1. *What justifications were given by students who correctly selected a matrix on the global structure or link type tasks and how adequate were these reasons?*
2. *How consistent were students' reasons for selecting a matrix across the global structure and link type tasks?*

The Participants

The participants were 37 students drawn from one of two cohorts in the larger study (N = 137). These students had correctly identified the matrix as the correct diagram for three consecutive years on *global structure* or *link type* property tasks. They commenced in the 3-year-study as Grade 3 students (8- or 9-year-olds) and did not engage in any specific instruction about diagram use over this period.

The Tasks

Students' knowledge of the properties of diagrams was investigated in the larger study through a series of 15 scenario-based tasks, which focussed on a range of properties of the matrix, network and hierarchy. The tasks were designed in accordance with the principles used by Novick and Hurley (2001) and set within the context of an Amusement Park. The first sentence or two of the task sets up a cover story. The next sentence or two focuses on a particular property of a diagram (e.g., global structure). The final sentence indicates that someone wants a diagram for a purpose relevant to the cover story (See Diezmann, 2005a for further details about the tasks). The *global structure* and *link type* property tasks for the matrix are shown on Figure 1. The property-focusing sentence has been underlined for illustrative purposes but it was not be underlined for students. These tasks required students to (1) select the diagram that best suits the given information and to (2) justify their selection and (3) non-selection of particular diagrams. Only two (correct/incorrect) spatially-oriented diagrams were presented for each task. The property was correctly represented on one of these diagrams but not the other. Students were presented with these same tasks in three annual sets of clinical interviews and were encouraged to elaborate on and clarify their responses. Justification codes were developed from all students' reasons for selecting and not selecting particular diagrams. However, only justifications for the correct selection of the matrix on either the *global structure* or *link type* tasks for the matrix are reported here. Students' reasons were coded independently by two coders with a third more experienced coder arbitrating on disagreements.

Lunch Orders (*global structure*): Sam's class is visiting the Amusement Park and will be buying food and drink for lunch. There are many different food items and many different drinks. Each child can only choose one food item and one drink item for lunch, but any of the foods can be combined with any of the drinks. To help the children decide on their lunch orders, Sam's teacher would like a diagram to show them all the possible choices of food and drink for lunch.	**Sandwich Bar** (*link type*): The Sandwich Bar sells sandwiches made with different types of bread and different kinds of meat. The Sandwich Bar Manager wants to know which different combinations of bread and meat are ordered the most, so that she can get her workers to prepare the right types of sandwiches for the busy lunch time rush. The Manager would like a diagram to record how many people buy each different combination of bread and meat during one lunch time.

Figure 1: Matrix tasks.

RESULTS AND DISCUSSION

The findings for the two research questions are discussed in turn.

1. *What justifications were given by students who correctly selected a matrix on the global structure or link type tasks and how adequate were these reasons?*

A total of 13 codes were used to describe students' reasons for selecting the matrix on either the *Lunch Orders* (global structure) or *Sandwich Bar* (link type), tasks over the 3-year period (See Table 2). These reasons comprise four broad categories. (1) *Adequate* reasons refer to information stored in Rows and Columns and its association (RC) or to the Combinations of two sets of information (CO). The other three categories contain inadequate reasons that focus on: (2) *Visual* or surface details of a matrix; (3) *Actions*, such as listing, ticking and tallying, without an explanation of how these are relevant to a matrix; or (4) *General* statements with no explicit or comprehensible explanation for the selection of a matrix. The proportion of justifications in each of these categories over the 3-year period for each task is shown in Figure 2.

Students' justifications revealed three key points related to the adequacy of their reasons over time and across tasks (See Table 2, Figure 2). Firstly, on both tasks, at least 26% of students provided adequate reasons for selecting the matrix from Grade 3 onwards. However, on both tasks, the lowest proportion of adequate responses occurred at Grade 5. Hence, students were less proficient at articulating their reasons for selecting a matrix after two additional years of schooling. On the *Sandwich Bar* (link type), nearly 10% fewer Grade 5 students provided adequate reasons for selecting a matrix than when they were in Grade 3.

Categories and Codes	Lunch Orders (*n* = 23)			Sandwich Bar (*n* = 31)		
	Gr. 3	Gr. 4	Gr. 5	Gr. 3	Gr. 4	Gr. 5
1. ADEQUATE						
CO (Describes Combinations)	4.3	0	0	0	3.2	0
RC (List using Rows and Columns)	26.1	30.4	26.1	45.2	35.5	35.5
TOTAL	**30.4**	**30.4**	**26.1**	**45.2**	**38.7**	**35.5**
2. INADEQUATE-VISUAL						
BR (Box Reference -not a list)	4.3	0	0	3.2	0	0
CA (Correct Appearance)	8.7	8.7	0	3.2	3.2	0
GF (Graph Format – not a picture graph)	8.7	17.4	13.0	3.2	3.2	3.2
PG (Picture Graph)	8.7	0	0	0	0	0
VD (Visual Description)	0	4.3	0	16.1	9.7	3.2
TOTAL	**30.4**	**30.4**	**13**	**25.7**	**16.1**	**6.4**
3. INADEQUATE-ACTIONS						
CH (Checklist; uses ticks)	4.3	8.7	13.0	3.2	3.2	9.7
CL (Create a List)	17.4	13.0	13.0	3.2	38.7	45.2
TR (Tally Reference)	4.3	8.7	34.8	0	3.2	3.2
TOTAL	**26**	**30.4**	**60.8**	**6.4**	**45.1**	**58.1**
4. INADEQUATE-GENERAL						
DK (Don't Know)	4.3	8.7	0	6.5	0	0
NO (Not the Other diagram)	4.3	0	0	3.2	0	0
NS (Makes No Sense)	4.3	0	0	12.9	0	0
TOTAL	**12.9**	**8.7**	**0**	**22.6**	**0**	**0**

Table 2: Percentages of students providing each type of explanation.

Secondly, by Grade 5, no students provided Inadequate-General responses, and compared to Grade 3 there was at least a 20% reduction in Inadequate-Visual responses. Unfortunately, there was an increase in Inadequate-Action responses, which in Grade 5 were at least 58% on both tasks. Thus, while over a 3-year period many students might have learned what constitutes an inadequate General or Visual reason, they did not learn what constitutes an Adequate reason. The large proportion of students providing Inadequate-Action responses by Grade 5 could be explained by a procedural use of the matrix for checking, listing and tallying.

Finally, students' explanations across the 3-year period were similar (≤ 10% variation) for 23 of the 26 explanations on each of the tasks with three exceptions. (1) There was an increase of over 36% in the number of students who incorporated the notion of a list (CL) into their explanations from Grade 3 to Grade 4 on the *Sandwich Bar* (link type). Recognition that a matrix is composed of lists is a step towards understanding its graphic structure. (2) There was a rise of over 26% in the number of students associating the matrix with tally references (TR) on *Lunch Orders* (global structure). This association illustrates confusion between the uses of two grid-like structures in mathematics. Students' confusion between the representation of surface and structural information has long been recognized (e.g., Dufour-Janvier, Bednarz, & Belanger, 1987). (3) There was a 12.9% decrease in the number of students who provided answers that made no sense (NS) from Grade 3 to Grade 4 on the *Sandwich Bar* (link type). Hence, there is some evidence of an improvement in sense making.

Figure 2: Reasons for Matrix Selection on the Lunch Orders and Sandwich Bar tasks.

2. *How consistent were students' reasons for selecting a matrix across the global structure and link type tasks?*

Students were classified into five groups based on the adequacy and consistency of their reasons for diagram selection over time (See Table 3). (1) The Consistently Adequate (CA) group always gave an adequate reason for diagram selection. (2) The Inadequate to Adequate (IA) group provided an inadequate response in Grade 3 and an adequate response by Grade 5. (3) The Adequate to Inadequate (AI) group provided an adequate response in Grade 3 and an inadequate response by Grade 5. (4) The Erratic Explanations (EE) group oscillated between adequate and inadequate responses in successive years. (5) The Consistently Inadequate (CI) group always gave inadequate reasons for diagram selection.

This analysis revealed that only 3.2% of students provided a consistently adequate (CA) response for the selection of a matrix and this was limited to the *Sandwich Bar* (link type) (See Table 3). Over the 3-year-period, more than 20% of students' improved their performance from inadequate to adequate explanations (IA). However, in the same period, more than 73% of students declined in their explanations (AI) or provided erratic (EE) or consistently inadequate explanations (CI). There were similarities (≤ 10% variation) in the percentages of students responding across tasks in the CA, IA, AI and EE groups. The CI group had a slightly higher variance. Hence, the tasks appear to be of similar difficulty across groups of students.

Group	Lunch Order Task (*n=23*)	Sandwich Bar Task (*n=31*)
CA	0%	3.2%
IA	21.8%	22.6%
AI	26.1%	32.3%
EE	17.4%	19.4%
CI	34.8%	22.6%

Table 3: Comparison of groups across tasks.

An analysis of individual responses revealed that 58.8% of the 17 students who correctly identified the matrix for both tasks gave reasons of mixed adequacy (MA) comprising IA, AI, or EE responses. A further 35.2% gave reasons that were a combination of MA for one task and CI for the other. Only 5.9% of students provided CI responses on both tasks. No student provided CA reasons across tasks.

CONCLUSION AND IMPLICATIONS

The results of this investigation indicate that many students are either not developing appropriate knowledge of the properties of diagrams or are unable to communicate their knowledge adequately. Thus, there is a need for explicit instruction about the properties of diagrams, which essentially constitute the “building blocks” of diagram literacy. The results also serve as a caution that over time changes in students’ reasoning about diagrams or their communication of this reasoning are not necessarily positive. Hence, it is essential to monitor students’ knowledge. Moreover, the results indicate the need to be aware of spikes or troughs in reasoning about the matrix or communication of this reasoning at particular grade levels and to capitalise on or address salient issues as appropriate through instruction.

There are two implications of the results for instruction. Firstly, there is a need for strategic instruction to develop mathematical proficiency in diagram use. Key components of this instruction should be reading diagrams with a focus on diagnostic properties of diagrams (Novick, in press) and supporting students to move from the visually-oriented focus of beginners towards the conceptual focus of experts (Lowe, 1993). Instruction should commence early in the primary years to build on students’ existing knowledge. Secondly, instruction at the broadest level should account for what could be termed the “Emperor’s New Clothes” effect, that is the outcomes of a curriculum that emphasizes representation, reasoning, communication and context-based mathematics should be <u>visible</u> in the results of tasks that draw on these skills.

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INTEGRATING ERRORS INTO DEVELOPMENTAL ASSESSMENT: 'TIME' FOR AGES 8-13

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It is widely agreed that measurement is of paramount importance to students' overall development in mathematics. This paper describes a developmental 'map' of students' understanding and skills in measurement, focussed on the topic of Time, that integrates correct and incorrect student ideas. The map is based on a Rasch analysis of data from a large-scale UK national survey for standardising assessment for children from 5 to 14 years of age. It is demonstrated how a partial credit strategy enables a developmental map to be constructed to show students' strengths and weaknesses in a meaningful and useful summative and formative manner. This map provides evidence, of both a summative and a formative nature, which may enable teachers to craft appropriate and successful learning experiences for children.

INTRODUCTION

The application of mathematics to daily life is firmly based on those aspects of mathematics commonly denoted as 'measurement' in primary school curriculum documents, a situation affirmed precisely thirty years ago in the National Council of Teachers of Mathematics annual year-book (Nelson & Reys, 1976). While there are some regional differences in nomenclature and definition, we argue that there is sufficient common ground for a description of the development of measurement understandings and skills in students to be of benefit to teachers and students. We will show that the large-scale collection of student responses to Time items enables a description of a hierarchy of student development, and that the use of a partial credit strategy enhances our understanding of how certain 'errors' may indicate progress towards understanding and eventual success.

The most common description of children's development of measurement understandings and skills are to be found in curriculum documents, and their close relatives, text books. However, other education stakeholders have contributed support for this view of the development of measurement. The Cockcroft Report (Cockcroft, 1982) strongly endorsed the usefulness and necessity of learning to measure (see, for example, § 79 and § 269). In the United States, the late 1980s saw the National Council of Teachers of Mathematics (NCTM) publish its 'Standards' which included standards for measurement (National Council of Teachers of Mathematics, 1989). The measurement standards for grades K to 4 included awareness of the measurable attributes of objects, units of measurement, estimation of measurements, and using measurement in everyday contexts. The standards for

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grades 5 to 8 further developed these topics to rates, indirect measurement, and derived units.

However, some mathematics education researchers have explored alternative ways of describing students' mathematical development. Under the banner of Developmental Assessment, the work of Masters and Forster (Masters & Forster, 1996a, 1996b) has provided a great deal of encouragement to those seeking an alternative approach for thinking about development and assessment, and particularly, how assessment can inform curriculum (Doig & Lindsey, 2002). Similar research has yielded such maps, in such areas as mental computation (Callingham & Watson, 2004) and probability (Gagatsis, Kyriakides, & Panaoura, 2001).

In a recent study, drawing on aspects of Piaget's work on the development of measurement ideas, Bladen and her colleagues (2000) describe a developmental 'map' that showed discrepancies between curriculum and children's development. The authors claim that their 'findings may serve as a warning to organizations that develop content standards to reconsider expectations of students … [and that] These data reveal a wide discrepancy between educational expectations and development of the concept of measurement' (p. 11). Further, the authors raise the question of 'how can these results be used to help teachers and students meet the state and national standards … ?' (p. 11). This question was also raised by Williams and Ryan (2000) who examined the responses of a large number of students to national test items.

Diagnostic, or formative, information in most developmental maps is implied, and is carried by the notion of *more is better*: that is to say, development is shown through descriptions or examples of more achievement (usually as more correct responses) whereas we argue that a richer, more useful map, in O'Connor's (1992) phrase, includes "*accounting for errors*" (p. 20) rather than simply counting errors (italics in the original). Thus, in order to explore students' measurement development better, we set out to describe how student responses from a large-scale assessment program could be analysed, for some aspects of measurement, to produce a 'map' of typical development, one that would include errors and mis-understandings. Such a developmental map would describe increasing ability, and importantly, it would also describe typical mis-understandings and errors that accompany this development (Williams & Ryan, 2000). This 'accompanying' is a critical point, as it illustrates that development includes, for some students at least, new mis-understandings or the persistence of earlier problems. As Williams and Ryan (2000) suggest, the errors illustrated in the developmental map "provide a concrete reference point for teachers to engage with research findings and conceptual frameworks in the literature that would otherwise remain obscure and arcane" (p. 67). Further, they argue that "[the] analysis and interpretation [of mis-understandings and errors] may mediate between the research community and the teaching profession" (p. 68).

INSTRUMENTS AND DATA SOURCES

As part of a continuing program to develop formative assessments that also serve summative purposes (see, for example, Ryan, Williams, & Doig, 1998), a new set of age-standardised diagnostic assessment materials for ages 5-14 were developed by the Mathematics Assessment for Learning and Teaching (MALT) team from the University of Manchester in collaboration with the publishers Hodder Murray (see, Williams, Wo, & Lewis, 2005, for further details). The MALT items draw on content from the UK National Curriculum, and the measurement items assessed aspects of time, length, capacity and weight, measures of measures, conversion of units, problem solving, and estimation and rounding.

A nationally representative sample was drawn from schools in England and Wales, and students' responses to this large set of mathematics items were collected. A total of over 14 000 students took part.

The items were placed into papers (30 mark points for Reception to Year 2 and 45 mark points for Years 3 to 9) with no common items between papers. Vertical test equating was conducted using a sub-set of students who sat two papers, and by using data from a pre-test in which students sat approximately half a paper from the following year. Validation included analyses at both the development and main test stages that suggested that the construct, scale and the vertical equating was safe (Ryan & Williams, 2005) although vertical equating becomes less reliable with changes to curriculum content in subsequent years.

The complete set of MALT measurement items can be accessed at www.education.man.ac.uk/lta/pme30/measurement.

ANALYSIS AND RESULTS

Like most of the research cited above, responses to the MALT items were analysed using Item Response Theory (IRT) (Bond & Fox, 2001; Rasch, 1960). Whilst there are several benefits of using this form of analysis, a particularly useful aspect is the nexus between a student's ability, as measured by the assessment items, and the difficulty of correctly answering these items. In brief, this is that for any given level of student ability, it is possible to identify the likelihood of that student responding correctly to any item. Importantly, this means that the greater the difference between student ability and item difficulty the more likely the student will answer correctly and vice versa. In addition to this formative affordance, we may interpret positive differences between positions on the IRT scale as indications of progress or development. While the basis for connecting the difficulty of assessment items and student achievement is probabilistic, the relationship is of critical importance in diagnostic interpretations of a student's assessment performance.

In a simple Rasch analysis items are taken to have two values: correct or incorrect. While this provides useful summative information about student achievement and understanding, responses that are incorrect are all treated as being equal, that is, they are the same. However, as an examination of the students' responses shows, a range of responses exists, many responses presenting possible insights into the different ways students think about the item to which they respond.

In the eight-year-old MALT test, item 30, shown in Table 1, children had to read an analogue clock and select the equivalent digital time (2:45). This generated three error-responses, of two qualitatively different kinds: the first pair refer to problems in reading a clock-face, while the last describes a difficulty in correctly interpreting the hour hand (before or after the hour). It will emerge that the latter is an error made by much less able students than the former two, which are quite close in developmental terms.

Table 1: Item codes and interpretations for a MALT item 8 – 30

Item	Response	Description of imputed student thinking
Age 8 Q 30 Reading times	2:45 (correct)	Read o'clock time (a quarter to)
	3:45	Confused hour on the clock with the next hour when reading a clock-face
	9:15	Confused hour hand and minute hand when reading a clock-face
	2:15	Interpreted quarter to as quarter past on reading a clock problem

Although a single item may not provide a key to unlocking every student's mis-understandings, collectively, item responses can reveal markers of students' mathematical development. Thus, across a range of items, the patterns of incorrect responses provide formative information that adds richness to the developmental map.

However, these descriptions of responses need to be placed on the developmental map at an appropriate place. In order to do this, an alternative analysis, using a Masters' PCM (Partial Credit Model, Masters, 1982) was conducted with ConQuest (Wu, Adams, & Wilson, 1998). As its name suggests, in a Partial Credit analysis each response is viewed as a step towards a fully correct response to the item. This requires the ordering of the erroneous responses from a 'least developed' group to more highly developed groups, to 'correct'. This was done by ordering the responses by the mean ability (obtained from a dichotomous Rasch analysis) of those students making the error: then ordering the error responses so that errors made by students with higher abilities are given a higher order. In the case of item 8 Q30 discussed above, involving the reading of a clock, there were four responses (three errors and one correct) associated with increasingly 'able' students (meaning simply students of increasing competence in measurement overall). Only in this ordering will the Rasch PCM be well-fitting with acceptable fit statistics, and thresholds correctly ordered as for this item (shown at those points marked with "A" in Fig 1.)

```
MORE ABILITY                           HARDER ITEMS
   2            |
                |
                |
                |
                |
               X|
                |
               X|
              XX|
              XX|C 10Q32b.3 Reads & interprets time-distance graph
               X|
              XX|
   1         XXX|D 10Q32a.2 Able to round time to decimal part
             XXX|B 10Q19.3 Solves problems with decimal hours
             XXX|C 10Q32b.2 Mis-interprets time-distance graph
            XXXX|
          XXXXXX|D 10Q19.2 Rounds down to nearest hour, resists decimal
          XXXXXX|A 8Q30.4 Reads clock time (a quarter to)
          XXXXXX|C 10Q32b.1 Misreads scale on time-distance graph
         XXXXXXX|A 8Q30.3 Confuses hour on clock with next hour
       XXXXXXXXX|A 8Q30.2 Confuses hour and minute hand
        XXXXXXXX|B 8Q16.1 Treats hours and minutes as separate
      XXXXXXXXXX|D 10Q19.1 Rounds up to nearest hour, resists decimal
   0   XXXXXXXXX|D 10Q32a.1 Rounds to nearest hour, resists decimals
         XXXXXXX|
        XXXXXXXX|B 11Q12c.1 Subtracts time using 1hr=100min
        XXXXXXXX|A 8Q30.1 Reads quarter to as quarter past
       XXXXXXXXX|
      XXXXXXXXXX|
         XXXXXXX|
         XXXXXXX|
            XXXX|
            XXXX|
           XXXXX|
             XXX|
  -1          XX|B 12Q13.1 Adds times as if 1hr=100mins
              XX|
              XX|
               X|
               X|
               X|B 13Q14.1 Treats hrs/minutes as decimal
               X|
LESS ABILITY                           EASIER ITEMS
```

Figure 1: Developmental map for some Time item-responses, n=14420
(approximately 1500 per year group), age 8-13

INTERPRETATION

A selection of meaningful responses to eight items focussed on the topic of Time are presented in the developmental map in Figure 1 (four correct responses and twelve errors). The item response descriptions are numbered with their test number, question number, and their step level: thus, 12Q13.3 is the third step (and, in this case, is the correct response) of item 13 in the 12-year-old MALT test. Correct responses are in bold text. An examination of the map shows that there are several sub-texts that provide formative information. In order to see these sub-texts clearly, related items are labelled similarly alphabetically. Four sub-texts are described here:

Sub-text A deals with the ability to read an analogue clock. The conventions of the clock-face are not as simple to master as many adults believe. This difficulty persists across a range of abilities, shown by the responses 8Q30.1, 8Q30.2, 8Q30.3, and is not fully achieved until the response 8Q30.4 is given. Note that the chart on the left of the figure indicates that only a minority of the sample are expected to achieve this.

Sub-text B focuses on the conceptual aspect of the representation of time by numbers. The idea that numbers can be used in other than a decimal form is apparently a difficult concept for many students. As responses 13Q14.1, 12Q13.1, 11Q12c.1, and 8Q16.1 show, this causes trouble for students across a wide age range from 8 to 13 years.

Sub-text C describes student development in reading information from a distance-time graph. These items may reflect also student development in graphicacy as well as in concepts of time and distance. Notwithstanding this, responses 10Q32b.1, 10Q32b.2, and 10Q32b.3 (a correct response) are all at the more difficult end of the developmental scale. This information enables teachers to plan for successful learning by being aware of the likely difficulties students may encounter.

Sub-text D describes the development of student flexibility in dealing with time represented in a decimal format. While responses 10Q19.1 and 10Q19.2 describe difficulties, success is achieved for students of slightly higher ability, shown by the higher position on the scale of response 10Q32a.2. Interestingly, rounding down appears to be a more difficult undertaking than rounding up.

While these interpretations are informative, they are not exhaustive, even for this one topic: they serve, however, to illustrate the point that well-designed summative assessments can be made to provide formative information.

CONCLUSIONS

We have shown how the award of credit for increasingly sophisticated errors in large-scale assessment can enrich diagnostic assessment through a Time map which charts the full range of student responses. While the description offered here provides a picture of students' development in the measurement topic of time, it must be remembered that students have been exposed to the UK curriculum, which, by its very nature, attempts to scaffold a particular developmental pathway. However, we recommend the methodology to those involved in large-scale assessment in other countries, and although our argument has been largely technical, we hope that the general principal of rewarding errors with credit may be a significant part of the work of changing perspectives on teaching and learning.

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VYGOTSKY'S EVERYDAY CONCEPTS/SCIENTIFIC CONCEPTS DIALECTICS IN SCHOOL CONTEXT: A CASE STUDY

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The aim of this paper is to analyse, through a case study, how Vygotsky's everyday concepts/ scientific concepts dialectics, intentionally promoted by the teacher, can result in an evolution of conceptualisation in mathematics. I will present the case at study, then, on a theoretical level, I will characterise conceptualisation in school context. This will result from combining Vygotsky's elaboration about scientific concepts with Vergnaud's definition of concept in a dynamical perspective inspired by the Vygotskyan dialectics. On a pragmatic level, I will specify the conditions that can favour building up this dialectics in school. This will be backed by Boero's "experience fields" didactic theory, and will concern argumentation and student's implication in experience field activities.

VYGOTSKY'S EVERYDAY CONCEPTS / SCIENTIFIC CONCEPTS DIALECTICS

Scientific concepts are characterised by Vygotsky (1985, Chapt. 6) through their use, as they are used in a conscious, intentional and explicit way. They are related to other concepts in systems. They are general, have a theoretical character and can be presented through a definition. Everyday concepts are rich in meaning, but their relation to systems is rather implicit. They are developed spontaneously through subject's experience, within culture and social context. Generally they have a local scope and do not need to be defined. The existence and development of everyday concepts and scientific concepts are closely related: scientific concepts evolve from abstract to particular groundings; everyday concepts can be refined into more precise (and eventually into scientific) concepts; but relating to a system and the conscious use of a concept imply that some everyday concepts have already been developed, which can back its scientific use. I wish to stress the relative and dynamical character of these two poles of the dialectics.

THE EXPERIENCE FIELDS DIDACTICAL SETTING

Our example comes from a second grade class involved in the Genoa Project for primary school (20 children, about 8 years old). The aim of this Project is to teach mathematics, native language, natural sciences, history, etc… through systematic activities concerning "fields of experience" from everyday life (Boero, 1994; Boero et al., 1995). These activities are intended both to develop mastery of the fields of experience themselves, and to approach and develop concepts and skills belonging to those disciplines. For instance, in Grade I the development of numerical knowledge is rooted in the "money" and "class history" fields of experience; moreover they allow to promote students' argumentation as well as the use of specific symbolic representations.

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In the "fields of experience" didactical setting, a fairly common classroom routine consists: of individual production of written hypotheses on a given task; of classroom comparison and discussion of selected students' products, orchestrated by the teacher (cf Bartolini Bussi, 1996); of individual written reports about the discussion; and of a classroom summary, usually constructed under the guide of the teacher and finally written down in the students' copybooks. In most cases, classroom summaries represent students' reached knowledge (with possible ambiguities and hidden mistakes). Final institutionalisation phases (Brousseau, 1986) are attained only in few circumstances. From the researcher's point of view, this style of slowly evolving knowledge offers the opportunity to observe the transformation of students' knowledge in a favourable climate. Argumentation has a strong status in this didactical setting, and most problem situations, particularly in mathematics, are solved through argumentation, avoiding manipulative activities (cf Douek, 2003). Argumentation reflects fairly well each student's level of mastery of knowledge, his/her level of use of references, and the class' common stable references (cf Douek, 1999). Justification and explanation are essential in the didactical contract. As in all Italian primary schools, the student group remains with the same two teachers for the first five years of primary education. Therefore didactical contracts are very stable. I must add that Italian curricula are rather loosely prescriptive around a core of necessary knowledge.

Students deal with the *plant cultivation* experience field for about 4 months in Grade II. Students had measured wheat plants taken in a field, then had to follow the increase in time of the heights of plants of the classroom pot. They described them and drew them on their copybooks. They solved various problems concerning their growing up, which involved mathematical knowledge; our example concerns one of those problem situations. We will consider data deriving from direct observation, students' texts, and videos of classroom discussions concerning this problem situation as well as previous activities.

THE CASE AT STUDY

Here is the problem posed to the class:

> On Friday December 3, the plant in the pot, kept in class and in the light, was 26 cm tall, and the one in the dark was 7 cm tall. Today, Friday December 10, the plant in the pot kept in the light is 28 cm tall and the one in the pot in the dark is 10 cm tall. Which plant has grown the most from December 3 till now?

Students had to deal with two kinds of conceptual difficulties. The first difficulty is related to the *comparison of transformations* (increases) in the additive conceptual field, a complex and new situation for them (cf Vergnaud's categorisation, in Vergnaud, 1990). The second difficulty is inherent in the students' concept of increase, which was at the level of an everyday concept, although students were able to solve problems as "*how much such plant has increased from that date till now*", and to compare heights of two plants. The closeness between the concepts of *height* and *increase* added complexity. However, it will be interesting to see how the evolution of the conceptualisation of *increase* depended on, and simultaneously favoured, the mastery of the additive field and its extension to include the comparison of transformations.

Phase 1: Individual oral production with teacher's mediation

13 students (out of 20) engaged correctly into the problem solving, through calculations and comparison of results. But they were confused when the teacher asked them to *explain their problem solving*. Only 6 of them arrived easily at a stable correct conclusion. Some students answered "*the plant that increased the most is the one which is the tallest*". They deduced increase from height, thus contradicting their own numerical result. Others said "*this plant increased the most, so it is the tallest*". They deduced height from increase, disregarding the visible state of the plants.

Performing the right calculations was not enough to grasp the situation as a whole. To solve problem numerically, most students interpreted increase as a supplementary length in the plant. But the dynamical character of increase was not grasped with its dependence on time and the implicit fact that it may be slow or quick between two given moments: they were not able to use such an argument to interpret the situation. Their everyday conceptualisation of increase was not sufficient. But through argumentation guided (or stimulated, depending on cases) by the teacher, this concept evolved and most students became able, at least momentarily, to handle the problem situation through verbal activity, paying attention to the various elements of the situation, and were able to interpret them according to a hierarchy that organised calculations. Many students used an expression that became meaningful and stabilised: *"pezzo cresciuto" ("the bit that grew").*

The other 7 students had difficulties to master and interpret data, and did not succeed in producing a solution. The teacher encouraged them (as well as 2 students out of the group of 13 who had produced a correct solution) to draw the two plants, mark the drawing with signs corresponding to the different heights at different dates, and follow the increases on the drawing. Hence for all the students the situation gained a spatial character, either through the drawing or through the image of *"the bit that grew"* (which is not correct scientifically, since growth is an evolution of the whole plant). This spatial frame helped most students to "see" and grasp the dynamical aspect of the plant growing as well as to fix for each plant the elements that need to be determined and compared. Given students' mastery of the additive field, I conjecture that those spatial representations allowed the representation of numbers on the number line to be transferred (gradually, depending on students) to this new additive situation. This favoured the mathematisation of this new situation while putting its dynamical character into new evidence.

As a whole, student/teacher interactions developed argumentation concerning three types of references: *reality* of plants cultivation, their "stories" with their dynamical character and time dependence; *schematisation* through drawing and/or its equivalent verbal spatial organisation, which helped to master the hierarchy between dynamical and static elements (either as data or as parts of the sketched plants); and *calculations* that helped finding and fixing intermediate elements, but which could have been mastered abstractly and independently from the particular situation.

Confusion due to the insufficient everyday conceptualisation of *increase* was overcome thanks to a sufficient mastery of the additive situation and related external representations (numerical, verbal, schematic). This made the situation challenging and attainable at the same time. We can conjecture that such conditions made the scientific/everyday concept dialectics effective. Argumentation was the way by which that dialectics enhanced students conceptualisation. It also allowed some students to appropriate elaborate verbal expression reflecting refined understanding of the situation. We heard sentences such as: "*the plant that increased the most is the one which was in the dark although the highest was the one in the light*."

At the end of Phase 1, all students could distinguish and calculate increases and heights; 9 expressed the relation between them as in the excerpt mentioned above.

Phase 2: comparison activity and collective discussion

This text was distributed for individual comparison, before collective discussion:

> JESSICA : the plant in the dark increased the most because it increased 3 cm, on the contrary the plant in the warm place increased 2 cm. I say that the one in the dark increased the most because the centimetres of the part of the plant that increased are the most.
> GIUSEPPE : the plant that increased the most since Friday, December 3 is the one in the dark, because it increased 3 cm, but the one in the warm increased 2 cm and so 3 are more than 2. But in fact the plant which is the tallest is the one that makes 28 cm, and it is the one in the warm.
>
> Questions :
> 1) In which way did Giuseppe and Jessica reason ?
> *2) In Giuseppe's text there is a consideration we cannot find in Jessica's. Which one is it?*

This metacognitive reflection about their problem solving led students to share the reasoning and stabilise it within the class, as well as to deepen, for some students, and to pursue, for the others, the effort of clarification and interpretation of the situation.

At the beginning of the collective discussion, the weak everyday conceptualisation of *increase* appeared again: many students mixed it up with *height* in their reasoning, even though they had been able to discriminate them during their interaction with the teacher. But once some students referred to the *bit that increased,* the evoked spatial situation helped to restore the "right" argument. However, the drawing was not needed anymore. A new example given by a student established another criteria to distinguish increase from height in relation to time: a child's *height* is his/her *increase* since birth (note that this is rigorously true for plants, not for children…); while *increase* is determined in relation to two dates. All along the discussion the arguments to interpret and compare the texts concerned "reality" (the known development of the plants), spatial organisation, and calculations, with their hierarchical organisation.

Phase 3: Individual synthesis

The teacher brought the students to a synthesis by the following question:

> Is it the same to say "which plant increased the most" and to say "which plant is the tallest"? Why?

The generality of this question aims a more *scientific* handling of the involved concepts. 12 students out of 20 produced a general and valid synthesis, most gave examples (and some of them were creative in doing it). 14 students referred to time. 6 out of these 14 had produced a drawing in Phase 1.

A CHARACTERISATION OF SCIENTIFIC CONCEPTUALISATION IN SCHOOL CONTEXT

Let us now describe scientific conceptualisation as we consider it pertinent in school context, and present a characterisation that allows to evaluate its evolution. We will use it to analyse the effect of Vgotsky's everyday concepts/ scientific concepts dialectics on conceptualisation, as well as the conditions that helped this dialectic to take place. *This characterisation captures two entangled aspects*: the epistemological construction, and the development of cognitive activities. Vygotsky (1986; cf Vergnaud, 2000) helps us to recognise the quality of the use of the concept, and its evolution. Vergnaud (1990) helps us to recognise explicit links with experienced "reality" through *reference situations* and *external representation*; and with subject's activity interpreted through *operational invariants*[1]. *External* representation, in coherence with the experience field didactical theory, includes gesture and language in the broadest way (cf Radford, 2003). We consider two characters of conceptualisation and at the same time we try to describe how to detect them:

<u>- The evolution of systemic links of concepts.</u> Linked objects can be reference situations, external representations, other concepts, or components of concepts (according to Vergnaud's definition). Links can be propositions about the objects (e.g. propositions that express properties, characteristics, analogies or differences or hierarchies), metaphors, examples, etc. Operational invariants or generalisations can underlie these links.

<u>- Students' awareness in handling the concept, its components or its systemic links:</u> it consists in the mastery of the concept, and the intentional recourse to it or to some of its components, that can be detected specifically in using and expressing operational invariants and through using its external representations.

AN ANALYSIS OF CONCEPTUALISATION IN OUR EXAMPLE

Students had to discover the crucial elements of the situation and organise their hierarchical relations to back a procedure of calculation. This reflects the

[1] Vergnaud defined a concept as composed by the set of its reference situations, the set of its operational invariants ('schemes', theorems in actions, etc.) and the set of its symbolic representations (see Vergnaud, 1990).

development of an intentional and explicit use of additive relations and a mastery of important systemic links. But they also dealt with the central notion of *increase*. This everyday use of *increase* was crucial to interpret the problem situation in terms of additive relations; at the same time the possibility of interpreting it with the scientific concepts of the additive field made it necessary, and possible as well, to develop the conceptualisation of *increase* (and specially to make explicit its dependence on time). Teacher's stimulation and direction of argumentation made this dialectics alive and productive. As a result of setting this dialectic, the everyday concept of *increase* was made more "scientific" (though remaining in an everyday use).

Evolution of the concept of increase

Development of <u>systemic links</u> of *increase*:

- distinction between *height* and *increase*: It was possible thanks to the implicit relation to the number line, either by drawing or indirectly by mental reference (because of student's mastery of such elements of the additive conceptual field), which contributed also to determine their relation to time. Increase was determined as a variation of length through time;

- linking *increase* to reference situations: The real situation made it possible through its dynamical character, thanks to the interpretation needed to solve the problem; indeed *increase* it is a variation related to time that is not really visible, but has to be elaborated. Also referring *height* to *increase since birth* as different from *increase between two dates* had a positive effect: students had another reference situation and another explanation of the role of time at their disposal.

Concerning <u>students' awareness</u>:

students syntheses revealed that they assimilated this new understanding of the concept (all of them referred to time) and were able to use it in a personal way to answer the question put in general terms (it was not possible to just repeat the newly established knowledge). The ability to determine and distinguish the different components of height and increase was related to the possibility to master and combine intentionally that set of additive situations (variation of length, reference and dependence to time, and comparison of variations).

Extension of the scientific conceptualisation of the additive field: comparison of additive transformations

Comparing transformations in the context of plants' increase (familiar because of the long term work in the experience field of plant cultivation) was meaningful and efficient: the growing of the plants confers to the additive situation a dynamical character and a possibility of mental spatial representation. This allowed to master and organise the various phases of calculus.

Development of <u>systemic links</u> of the additive conceptual field:

they can be identified through the relationship established between *comparison of transformations* and a "real" dynamical situation, as well as through schematic (direct or mental) treatments of the static visible aspects and the dynamical, only thinkable, aspects. Hierarchic organisation between the various additive calculations, in relation to their interpretation as static or dynamical, was an operational invariant, as well as the arithmetic *theorems in action* underlying the numerical treatment.

Concerning students awareness:

at the beginning of the sequence one part of the students mastered the additive situation only through calculations. At the end, though, all of them were able to interpret calculations in relation to the reference situation and give examples of comparable additive situations. They revealed then a conscious mastery of this additive situation and an ability to relate it or compare it to other situations (as comparing increases of children by comparing increases between two dates, and heights as increases since “birth”).

CONDITIONS THAT ALLOWED THE VYGOSTKIAN DIALECTICS TO WORK

The analysis of the didactical situation put into evidence two main conditions that favoured the setting of this dialectics and its functioning.

- The important use of the “reality”, the development of various expression about it and the development of everyday concepts in school context. This comes from the experience fields didactical setting, which implies making sure that students are sufficiently involved in some activities related to this “reality”, and important everyday concepts are explicitly shared, and enriched with verbal and other external representation through observations and discussions. In fact, everyday concepts do not quite correspond to spontaneous ones, though they remain in the everyday stream of conceptualisation (cf Vygotsky, 1985, Chapt. 6).

- Argumentative activity managed by the teacher allowed students to interpret the situation, express and discuss their views, handle and compare various (verbal, schematic, numerical) external representations. It helped students to assimilate the new construction, because they had to express themselves in a variety of ways to answer to a variety of detailed questions. Argumentation played a dynamical role so that the dialectic could evolve.

CONCLUSION

Vygotsky’s scientific concepts/everyday concepts dialectics works as a powerful *model* for conceptualisation in teaching and learning contexts. But through observation of examples such as the one reported here, I found that it can turn out to be also an efficient tool to plan teaching situations in order to achieve some important educational aims in mathematics. The "experience fields" didactical setting does use this dialectics as one of the levers to develop conceptualisation, with some specific favourable conditions:

- the development of everyday concepts in school context up to a certain level of expression and activity in the perspective of a “mastered reality”, in parallel to the building of scientific conceptualisation of related notions;

- the explicit rooting of mathematical concepts within systems that may contain nonmathematical concepts, and by this way can support the students' mathematical activity by providing them with meaningful references.

We can also note how, in the "experience fields" didactical setting, a continuous development of argumentative skills allows to nourish the backing of mathematical reasoning on other (and easier to master) kinds of reasoning. As a result, argumentation can effectively move the everyday concepts /scientific dialectics under the teacher's guide.

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CREATING MATHEMATICAL MODELS WITH STRUCTURE

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This paper reports on a study that traced the development of year 4 students who were taught to structurally organise text through top-level structuring, as they engaged in two textually-based, mathematical-modelling problems. Top-level structuring is an organizational tool used to structure text for recall and comprehension. First, prior to top-level structure instruction, students investigated a modelling problem where they used data to determine and report on the best condition for growing beans. They were then taught the top-level structuring strategy. Finally, the students worked on a modelling problem where they analysed results to report on the best way to choose a winner of a paper-plane race. Results showed evidence that after students were taught to structure the text, they applied the strategy when explaining and justifying their ideas and models.

INTRODUCTION:

Mathematical modelling (MM) provides students with opportunities to explore meaningful problem situations and generate explanatory models that empower them to cope with contemporary global problems. Lesh and Doerr (2003) warn of misinterpretations of the term "model". In MM, the term refers to models that problem solvers develop "to construct, describe or explain mathematically significant systems they encounter" (Lesh & Doerr, 2003, p. 9). Generally speaking, mathematical models are representations of reality constructed by mathematicians in their every day activities. In this paper, a model is a mathematically-based, explanatory system that is designed to represent relationships among variables associated with a complex system.

MM problems allow for multi-interpretations and approaches to problem solving. English and Lesh (2003) have emphasised that it is not just reaching the goal that is important, but also the interpretation of the goal, the information provided and the possible steps to solution. "Children not only have to work out how to reach the goal state, but also have to interpret the goal itself as well as all of the given information, some of which may be displayed in representational form e.g., tables of data" (English & Watters, 2004, p. 336). Students work in small groups to develop a model that is presented to their peer group for constructive criticism and feedback. So, MM is specifically designed as a social experience (Zawojewski, Lesh, & English, 2002). Models can be transferred to similar situations and reused. An example MM problem is described in Appendix A.

MM problems reflect real world contexts in that; the problem is embedded within texts that can be based in any of the other disciplines for example; science or social studies. Students faced with a MM problem must negotiate substantial textual information in order to arrive at the purpose of the problem-solving task. The "key

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mathematical ideas" must be "elicited by the children as they work the modelling problem" (English & Watters, 2005b, p. 298). Hence, to access mathematical knowledge from MM problems, students need high levels of literacy to comprehend the information embedded in text. In addition, students need to be sufficiently mathematically literate to navigate the mathematical texts like tables or graphs.

This paper builds upon the work of English (2002), and English and Watters (2004, 2005a, 2005b) in understanding learning through mathematical modelling. The research is also informed by Bartlett's studies on top-level structuring of texts (2001; 2003) which provide a framework for understanding student literacy development. The study investigated the question: to what extent does top-level structure (TLS), a strategy grounded in text comprehension research, effect children's ability in mathematical modelling? The purpose in this paper is to compare the students' presentations of models after the two MM problems, 'Beans, Glorious Beans' and 'Paper Planes Contest" (Appendices A & B), to determine in what ways TLS instruction impacts on children's learning through MM problems.

Theoretical Framework

MM problems require literacy, that is, an ability to read and process the given information selectively so that the main idea/s can be extracted from the texts. Only then can mathematical knowledge be constructed as the problem is negotiated. Applying the TLS strategy to MM problems is the focus of this paper.

Literacy involves decoding and comprehension (Gough, Hoover & Peterson, 1996). An important component of literacy is memory, short-term and ultimately long-term memory. Inability to remember what one reads as one reads will negatively effect decoding and comprehension, and therefore the cognitive processes needed for full understanding of the text (Helwig, Almond, Rozek-Tedesco, Tindal & Heath, 1999). So the process of reading text and comprehending, that is being literate, follows the continua model demonstrated in Figure 1.

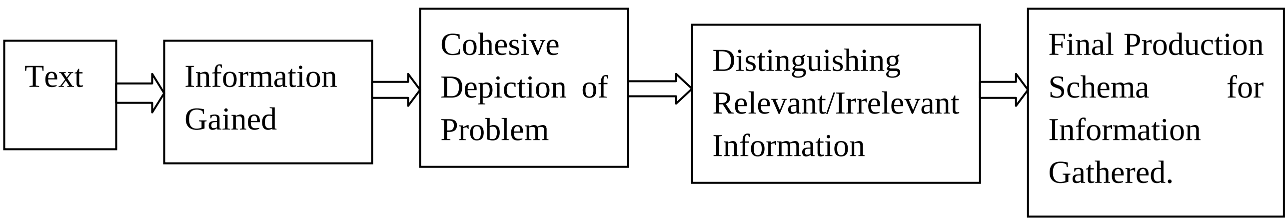

Figure 1: From text to understanding.

In mathematical modelling, students are faced with a maze of textual information. They must negotiate a wealth of background information from a diverse range of subject areas. Making sense of the textual information pre-empts being able to make sense of mathematical texts such as tables and graphs. In order to learn mathematically, students need to be textually literate (Cobb, 2004).

Top-level structure

Top-level structuring of text gives students a tool to organize textual information according to four basic plans for example, the author of the text could have constructed the text using (a) a list, (b) a comparison, (c) a problem and solution, or (d) a cause and effect. Students are taught key words or phrases for example, _so_, _because_, _as a result_, to help identify the author's plan which they will use to organise or 'structure' the text (Bartlett, 2003; Meyer & Poon, 2001). An example of how a text could be structurally organised is given in Appendix C. This text is scientifically based, but could be used as background information leading into a MM problem such as, students could investigate geometrical shapes, find patterns, create quilts, and reuse their model for further quilt patterns using alternative colours or fabrics.

While one overall plan for a text may be better than others, ultimately the important issue is for students to choose a plan, and therefore organise the textual information so that their recall and comprehension of text can be enhanced. In turn, this aids the students to communicate orally and in writing about the text (Bartlett, 2003).

RESEARCH DESIGN AND METHOD

The study used a qualitative, interpretative approach in which students were presented with two MM tasks, 'Beans, Glorious Beans' and 'Paper Planes Contest' (Appendices A & B). The class was taught by the researcher with the classroom teacher an active observer. Data included video recordings of classroom interactions and groups were audio taped. Student work was collected and field notes compiled. The aim of the study was to analyse the presentation phase of both problems and determine the effect the TLS strategy would have on students' ability to explain and justify their models The students were video and audio taped throughout these sessions. Their written work samples were also collected as complementary data to the taping. . Five questions were analysed in this study:

1. Did students use key words in their written and oral presentations?
2. Were key words an aid to students in explaining mathematical reasoning?
3. Did students demonstrate their ability to organize text information?
4. Were questions to presenters from peers enhanced after TLS instruction?
5. Was mathematical reasoning evident in explaining and justifying models?

Participants

The study used a year group of twenty-eight year four students. Ten of these students have significant learning or behavioural disorders. They attend a primary school situated in an outer shire of an Australian capital city.

Procedure

Firstly, students investigated the MM problem "Beans, Glorious Beans" (Appendix A) in small groups over a four-day period. Presentations were made on the fifth day. Secondly, the students were taught the plans and key words for the four organisational structures of top-level structuring over a period of two weeks. Subsequently, the students investigated the MM problem "Paper Planes Contest"

(Appendix B) over three days. As for the first MM problem, students reported to their peers on the fourth day. During the presentations, peers and the researcher challenged students to justify assertions and to explain the basis of their reasoning.

RESULTS AND DISCUSSION

The analysis, based on the five questions listed in 'research design and method' illustrates that students were not structurally organizing text in the 'beans problem but began employing TLS strategies in the 'planes' problem presentations. The individual analysis of each question follows: 1. Students used limited connective words in their language after the 'beans' investigation. The transcript (student 1 outlined below) shows the child telling the farmer what to do with vague justification. The child gave no specifications using data such as, a comparison of the weight of beans with other rows.

Student 1: Farmer Sprout you should use sunlight if you want more beans. Row 3 weighed the most in summer the first week from group 5. If you go diagonally you will find a sum.

Also, another child could not adequately explain his reasoning as to why he went diagonally when adding the amounts on the table, rather than horizontally or vertically.

Jacob: Well, I did that … because I hadn't done the other things yet so I thought it was 30 times better, I mean 30 times worse because it was shade. That's all.

However, as is evident in the 'planes' transcript words such as, _ but _and _comparing_, were in deliberate use. This indicated that students used a structurally organized method of explaining and justifying their ideas.

Josh: Dear Judges, You should pick team E. But Jacob thinks it is team C. Our group found this out by comparing team E with all the others. Josh found this out by seeing that team E had no scratches. Matt B found it out by seeing that team E has most metres going straight.

2. After TLS instruction, the students used key words to organize their responses and appeared confident in explaining their ideas with mathematically sound reasoning. The students' initial letter to the judges, read by Josh and shown above, used the words _but_ and _comparing_ to structure their response. They used mathematical data to justify their responses and provide proof as seen in Jacob's following explanation

Jacob: Metres! …Team E only beat them by one and I thought it was pretty good that team E only beat them by one and they got 31metres in 4 seconds.

However, the 'beans' transcript shows that students had difficulty explaining mathematical reasons for their ideas.

Researcher: Explain to me what you've just said. I didn't quite catch it.

Student 1: If you go diagonally you will find a sum.

Researcher: What does that mean? Diagonally what? Across the table or??

Student 1: Across the table. (Student 2 comes in) Across the table. (They pointed diagonally across the table)

Researcher: Why would you do that? What does that sum tell you?

Student 1: Jacob's the one who found the sum.

Researcher: OK Jacob, would you like to explain about that?

We just want to know why you added up the diagonal.

Jacob: Well, I did that … because I hadn't done the other things yet so I thought it was 30 times better, I mean 30 times worse because it was shade. That's all.

3. The 'beans' transcript reveals that students had not organized the textual information. But, the 'planes' transcript confirms that the students compared the teams both in the letter and in justifications. 4. No peer questioned this group after the 'beans' presentations. It was also noted on the video that students were also reluctant in their questioning of other groups. The planes 'transcript' shows class members asking relevant questions requiring groups to justify their positions with mathematical proof. Again the video revealed that this was reflected in all group presentations for 'planes'.

Question: Where's your proof?

Answer: It's in the letter like we found it out by seeing who had the most metres. Josh found it by seeing who had the most scratches. We just like compared it. We compared it to every single one.

Question: How did you get team C?

Answer: … by comparing the first two (scores) with team E. It beat it the first two times.

5. The 'planes' transcript shows that students referred to some data to explain and justify ideas after TLS instruction.

Student: Matt B found it out by seeing that team E has most metres going straight

Student: … by comparing the first two (scores) with team E. It beat it the first two times

The video tape also revealed that students had drawn a table which they showed to the class to demonstrate how they had worked out their solutions mathematically.

The author acknowledges that the 'planes problem' was the students' second MM investigation, so the process was familiar. But, the evidence remains that they began applying. top-level structuring strategies in the 'planes' problem, and this indicated that the students explanations and justifications were improved as they applied structure to oral and written text.

CONCLUSION

Overall, the results indicate that young students can be taught to structurally organize textual information. Learning to structure text has helped students to comprehend, communicate in writing, and present oral explanations by using the structure of comparison and applying key words in their communication. Students demonstrated thoughtful questioning after TLS instruction and students' explanations and justifications were more mathematically sound because they used some mathematical data to verify their models. This positive response to TLS instruction within the MM learning process is significant. Further research is warranted on the outcomes of teaching children organizational strategies for learning through textually-based problems. Also, teachers need to be made aware of the major part literacy plays in mathematics learning (Cobb, 2004), and inserviced on strategies that provide positive tools to aid students in overcoming textual barriers in mathematical problem solving.

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APPENDIX A: BEANS, GLORIOUS BEANS

This and the following problem were developed by English and Watters (2004, 2005a, 2005b).

Farmer Sprout must decide on the best light conditions for growing his butter beans. To help make his decision, he went to visit the Farmers' Association who test growing climbing butter bean plants using two different light conditions:1. growing beans out in the full sun with no shade at all, and 2. growing beans underneath shadecloth.

The Farmer's Association measured and recorded the weight of butter beans produced after eight weeks. They grew 4 rows of butter bean plants using each type of light condition

Sunlight				**Shade**			
Butter Bean Plants	Week 8	Week 9	Week 10	Butter Bean Plants	Week 8	Week 9	Week 10
Row 1	9 kg	12 kg	13 kg	Row 1	5 kg	9 kg	15 kg
Row 2	8 kg	11 kg	14 kg	Row 2	5 kg	8 kg	14 kg
Row 3	9 kg	14 kg	18 kg	Row 3	6 kg	9 kg	12 kg
Row 4	10 kg	11 kg	17 kg	Row 4	6 kg	10 kg	13 kg

Using the data, determine which of the light conditions is suited to growing Butter beans to produce the greatest crop. In a letter to Farmer Ben Sprout, outline your recommendation of light condition and explain how you arrived at this decision.

Your second investigation:

Predict the weight of butter beans produced on week 12 for each type of light. Explain how you made your prediction so that Farmer Ben Sprout can use it for other similar situations.

APPENDIX B: THE ‘PLANES RESULTS’ TABLE

Team	Attempts	Air-Time (seconds)	Distance (metres)	Team	Attempts	Air Time (seconds)	Distance (metres)
A	1	2	11	D	1	2 ½	12
	2	1 ½	12		2	scratch	scratch
	3	scratch	scratch		3	1	8
B	1	1	12	E	1	1 ½	9
	2	½	7		2	1	10
	3	½	8		3	2	13
C	1	1	9	F	1	1	9
	2	1	11		2	2	11
	3	2	11		3	scratch	scratch

Investigation

Write a letter to the judges of the contest explaining to them how to determine who wins overall award for the contest.

APPENDIX C: TOP-LEVEL STRUCTURING TEXT EXAMPLE

The Giant’s Causeway

In Northern Ireland there is a place called “the Giant’s Causeway”. There are some 40 thousand columns of volcanic basalt rock jutting out to sea. These were formed as a result of volcanic action. The columns were formed because of the slow and even cooling and contraction of molten lava.

<u>Key Words</u> **-** as a result, because   <u>Author’s Plan</u> – cause/effect.

Cause	KeyWord	Effect
Forty thousand years ago molten lava contracted and cooled slowly and evenly after volcanic action	***so***	the giant’s causeway in Northern Ireland was formed.

The sentence you have written forms the <u>main idea</u> of the text.

MECHANISMS FOR CONSOLIDATING KNOWLEDGE CONSTRUCTS

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Processes of abstraction of a group of students working together in a classroom on tasks from a unit on probability are analyzed with the aim of identifying mechanisms for consolidating recent knowledge constructs. Three such mechanisms are identified by means of indicative epistemic actions: Consolidating during building-with the construct, consolidating during reflecting on the construct, and consolidating during processes of constructing further constructs. While the first two mechanisms have been reported in previous research studies, in different settings, the third one is presented here for the first time.

The dynamically nested epistemic actions model of abstraction in context proposed by Hershkowitz, Schwarz and Dreyfus (2001) provides researchers with a tool for the analysis of learners' constructing and consolidating processes. The central elements of that tool are the three epistemic actions of Recognizing, Building-with and Constructing (RBC); therefore, the tool has been given the shorter name *RBC-model*. Previously reported research studies based on the RBC-model have either treated the first emergence of new knowledge constructs (e.g., Dreyfus, & Kidron, in press; Hershkowitz, Schwarz, & Dreyfus, 2001; Kidron, & Dreyfus, 2004; Tsamir, & Dreyfus, 2003), or they have treated consolidating as a process that follows constructing somewhat independently (e.g., Dreyfus, & Tsamir, 2004; Monaghan, & Ozmantar, 2004, in press; Tabach, & Hershkowitz, 2002; Tabach, Hershkowitz, & Schwarz, in press). In this contribution, we argue that constructing processes and consolidating processes are often narrowly intertwined, and we present data in support of our argument.

The study presented in this paper forms part of a long-term research project, in which we observed five classes, and six additional pairs of students, working on sequences of tasks in probability. Their processes of knowledge construction were supported by a ten-lesson learning unit with three stages: (i) calculating probabilities in 1-dimensional sample space (1d SS); (ii) calculating probabilities in 2-dimensional sample space (2d SS) for cases where the possible simple events in each dimension are equi-probable; in such cases, the 2d simple events can be counted and organized in a table; (iii) calculating probabilities in 2d SS for cases, where the simple events in each dimension need not be equi-probable. The data for the present paper stem from the work of three female students working as a group in a classroom on the first tasks of stage ii. More information on the project, including reasons for the choice of probability as a content domain and a discussion of some of the learning goals of the probability unit may be found in parallel papers submitted to this conference

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(Azmon, Schwarz, & Hershkowitz, 2006; Hershkowitz, Hadas, & Dreyfus, 2006; Ron, Hershkowitz, & Dreyfus, 2006). Each of these papers presents data taken from a different group of students in a different classroom, and each paper is related to a different theoretical aspect of the RBC-model.

CONTENT AND SETTING

The learning unit introduces students to 2d SS by means of

Question 1

1a Yossi and Ruthie throw two white dice. Ruthie wins if the numbers of points on the two dice are equal, and Yossi wins if the numbers are different.
Do you think that the game is fair? Explain!

1b The rule of the game is changed: Yossi wins if the dice show consecutive numbers.
Do you think the game is fair now?

1c How many possible outcomes there are when throwing 2 dice?

1d If Yossi and Ruthie play with one red die and one white die, will this change the answers to 1a, 1b and 1c?

A thorough a priori analysis of this and the following questions led us to the formulation several mathematical underlying principles. We list the ones that are relevant for the analysis in the present paper (see Hershkowitz, Hadas, & Dreyfus, 2006, for principle E1, relating to the nature of events in 2d SS):

E2 (All Events) All possible simple events in the given 2d situation are obtained by combining all possible values for first dimension with all possible values for the second one. Thus the number of all possible events is a product. (e.g., there are $6\times6=36$ simple events when throwing 2 dice with six faces each).

E3 (Relevant Events) All simple events, which are relevant to a given problem, have to be identified (e. g., the six 2d events with equal numbers of points on the two dice).

O (Order) The order of the elements in the pair describing a 2d simple event is essential: (a,b) and (b,a) are two different simple events (unless a=b).

We note that questions 1a and 1b can be solved without principle E2 because these questions are comparative. Thus question 1c has been added. On the other hand, principle O is necessary to answer 1b: If O is not applied, Ruthie wins; if O is applied, Yossi wins. Question 1d presents an opportunity for sharpening students' awareness of principle O.

The following question again produces contradictory answers depending on whether principle O is applied or not:

Question 2

We again throw 2 regular dice. This time we observe the difference between the bigger number of dots and the smaller number of dots on the two dice. (If the numbers on the two dice are equal, the difference is 0.)
Make a hypothesis whether all differences have equal probability. Explain!

In analogy to question 1b, students who do not apply principle O, will find that the probability for a difference of 1 is smaller than that for a difference of 0. A subsequent computer simulation of the experiment shows that the opposite is the case: The probability for a difference of 1 is bigger than that for a difference of 0. After the computer simulation, students are asked to reconsider questions 1a and 1b.

The data we are going to present and analyze have been collected in an 8th grade classroom. During the entire unit, a researcher (denoted by R) was present in the classroom with a video camera and an audio-tape recording a focus group of three female students C, S, and Y. The analysis is based on the transcripts of these recordings as well as on observations by the researcher.

CONSTRUCTING AND CONSOLIDATING ASPECTS OF 2D SS

In this main section of the paper, we present the processes of constructing and consolidating of principles E2 and O by the focus group. While working on questions 1a and 1b, the focus group concentrated on identifying the relevant events (principle E3). They only briefly raised the question how many possible events there are altogether (principle E2), and quickly settled on 12:

9	Y	[Equal numbers of points] can happen six times: 1,1; 2,2; 3,3; 4,4; 5,5; and 6,6. So what are the possibilities? Good question! How do we write this?
10	C	I think 2 times 6.
11	Y	Why 2 times 6?
12	S	Because there are 6 possibilities and 2 dice: 6 and 6.
13	Y	So [the probability for Ruthie to win is] 6 out of 12.

For a while, the students worked under the assumption that there are 12 possible events. Several times, they were briefly touching on issues related to principle O. For example, when working on question 1b, and considering five events with consecutive numbers on the two dice:

59	Y	No, no, no, it can be different; each die, that's ten; [Yossi] still remains with the same 10: (1,2) and (2,1).

And after a while

95	C	Its' possible to reverse the dice: 1 and 2, and 2 and 1.
98	Y	I completely change my opinion; it won't change anything. (2,1) or (1,2), that's the same consecutive numbers. This won't change a thing. It's the same consecutive number - the same consecutive number. I was wrong; I completely change my opinion.

And

113	S	[Y] is right; it does not make a difference whether the outcome is 1,2 or 2,1.
114	C	I wonder. Maybe [Yossi] has more chances …
115	Y	He doesn't; he doesn't. He only has the consecutive numbers from 1 to 6.
116	S	5 out of 12, right?

117	C	[to R] What shall I do, 10 out of 12 or 5 out of 12?
118	Y	It's finished; thanks, C! We still haven't read the (next) question.

And her opinion carried the day, at least for the moment.

With respect to question 1b, they thus agreed that Ruthie wins since she has 6 chances (as above), whereas Yossi has only 5, which they list as 1,2; 2,3; 3,4; 4,5; and 5,6. Their considerations were based on E3, without making use of E2.

They return to the principle E2 when dealing with question 1c:

129	S	[Reads] How many possibilities are there when throwing two dice?
130	Y	Ah! I can tell you: (1,1), (2,2), (3,3), …

And after a brief discussion on the meaning of the question:

144	Y	I can tell you, just a second [Writes 1 – 1, 2, 3, 4, 5, 6 2 – 2, 3, 4, 5, 6 3 – 3, 4, 5, 6 4 – 4, 5, 6 5 – 5, 6 6 – 6]
150	Y	Wait, I can tell you the result: 10, 9, 8, 7, 6, … 21.
152	Y	I can also explain why: 1 can appear either with 1, or with 2, or 3 or 4 or 5 or 6, right? 2 can appear with the remaining numbers 2 or 3 or 4 or 5 or 6, …

After the first brief phase of constructing E2 in rows 9-13, rows 130-152 constitute a second phase. Y leads the group to a 21 element sample space. Neither Y nor any of the other students expresses any awareness of the fact that this contradicts and replaces their earlier 12 element sample space; they even occasionally mention the number 12 later on.

While C's attention to Y's explanations about E2 appears to be limited by her preoccupation with principle O (153-156), S is following Y (158-162), and Y is charging ahead to question 1d (163):

153	C	(1,2) and (2,1), are they different results?
155	S	What?
156	C	(1,2) and (2,1), are they different results?
157	Y	Yes – no! I counted this as one result.
158	S	Like, she says if you get 1, then it can be with 1, 2, 3, 4, 5, 6, no? Isn't that what she says? [C writes this explanation.]
159	R	You need all possibilities altogether.
160	S	That's how she did it: 21 possibilities she did.
161	C	How did she do it?
162	S	[based on Y's writings] Look: the first is 1, and then with 1 or 2, 3, 4, 5, and 6; and then 2 and 2, 2 and 3, 2 and 4, got it?
163	Y	I'm a champ at solving this kind of stuff! [Reads 1d] "If there is one red and one white die …" What's the connection? What's the connection

between a red and a white die? What's the connection between a red and a white die, tell me! There is no connection.

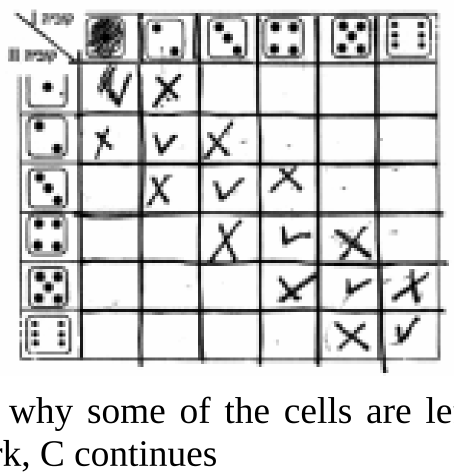
Tasks 1a and 1b were the first tasks in the curriculum that dealt with a 2d SS. While working on these tasks, the students constructed E3 in the sense that they were clearly aware, presumably in analogy to parallel 1d SS tasks, that they had to find all relevant events, and they were fairly successful at doing so, modulo principle O. They related only briefly and unsuccessfully to principle E2 (9-13) until they were explicitly asked to find all possible events (question 1c). Then they succeeded, again modulo principle O, and thus we may interpret that they also constructed E2 (129-162). However, some care is indicated with this interpretation since they were explicitly asked to find all possible events; therefore, we don't know to what extent they are aware of the importance of counting all the possible events in 2d SS for the purpose of calculating probabilities. With respect to principle O, the students' constructing is still in its initial stages and takes the form of a growing awareness, mainly by C, and general uncertainty (98, 153, 177). Indeed, when answering question 1d, they express three different points of view:

177 S There won't be a difference, right? The color of the dice will not make a difference.

178 C Sure, there will be a difference!

179 Y No way!

Toward the end of the lesson, they worked on question 2, a background task intended to prepare them for the next day's computer simulation. In accordance with their 21 event sample space, they expected six possibilities for a difference of 0, five possibilities for a difference of 1, etc. The computer simulation on the next day confronted them with the correct results. After the simulation, the teacher left little time for the students to reflect or discuss; she presented a 6 by 6 table as a tool for organizing all possible events of the 2d SS for two dice, and thus as a basis for calculating probabilities. She then initiated a whole class discussion concerning principle O. For this purpose, she colored one die yellow and the other one green, and asked the students to color yellow (green) all the cells that represent a difference in which the number on the yellow (green) die is bigger. The class concluded that every cell in the table represented a different simple event.



Next, the teacher asked the class to work in groups and reconsider questions 1a and 1b. The focus group students now identified consecutive numbers with the difference of 1 in the table. While Y worked separately, S observed how C filled a table for 1a and wrote: 6/36 same, 30/36 different, Yossi wins. Next she filled a table for 1b, in which Ruthie's wins are marked by × and Yossi's by ∨ (see Figure). S has some difficulty in understanding why some of the cells are left empty in this table but after a quick explanatory remark, C continues

380 C The probability for Yossi to win is 10/36 and for Ruthie 6/36.

Working alone, Y obtains a similar table including the information for both, questions 1a and 1b in the same table, and writes the correct results for question 1a.

The researcher then reminds them of the answers they had given the previous day:

388 R You wrote 21. What's the number of possible results now?

389 C Now 36.

390 R So why did you think 21?

391 S Because we thought (1,1), (1,2), (1,3), …

393 C Because (1,2) and (2,1), that's not the same thing; it's a different event.

394 R [to Y, who joined them] What are the probabilities in the two games?

395 Y Either way Ruthie loses

396 R What are the probabilities in the first game?

397 Y Ruthie 6/36, and Yossi 10/36.

398 R Where does he have a better chance to win, in the first or the second?

399 Y The first. Come let's go!

We observe three differences between the way, in which at least two out of the three focus group students, namely C and Y, go about answering questions 1a and 1b now, as opposed to how they went about them the previous day: They use the event table, they take into account principle O, and they correctly and efficiently apply principle E2. While these three differences are intimately linked, they are not identical.

More specifically, the students confidently and immediately made use of principle E2 to build-with it the answers to questions 1a and 1b. Judging from their written and oral (380, 397) answers, the action of recognizing all events in the 2d SS for this problem appears to be self-evident to them now. Adopting the criteria for consolidation proposed by Dreyfus and Tsamir (2004), immediacy, self-evidence, confidence, flexibility, and awareness, we conclude that the students have consolidated principle E2 within the particular context of using the event table as a tool for a sample space for two dice. Evidence from later activities and from the posttests of the unit shows that all three students confidently and flexibly use the event table and recognize and build-with principle E2 also in other contexts. We thus conclude that their process of consolidating E2 continued beyond the activity presented in this paper.

Students typically don't discuss mathematical issues such as principle O directly, unless they either run into a disagreement, as the focus group did earlier (e.g., 113-118), or they are explicitly asked as in 177-179, and 388-393. Thus much of the evidence about constructing and consolidating must be inferred from the students' epistemic actions when they solve the problems. This evidence is that when answering questions 1a and 1b after the whole class discussion, as opposed to before it, they build-with principle O. Whether the process of constructing principle O has taken place while they were listening to the teacher and the class discussion, or while they were using the event table themselves during their renewed work on questions 1a and 1b, is immaterial. But there is no doubt about their having constructed principle O within the context of using the event table for a sample space of two dice: They now work with a sample space that includes (a,b) and (b,a) as different events.

This leads us to the central observation of this paper: The students have been consolidating their construct of principle E2 during the process of constructing principle O. More generally, an earlier construct has been consolidated during the process of constructing a new one, with the earlier one intervening as an element in constructing the new one. This is one mechanism for consolidating recent knowledge constructs.

Two other mechanisms have been extensively described by Dreyfus and Tsamir (2004), and confirmed by Monaghan and Ozmantar (2004; in press). We therefore limit their discussion here to brief remarks. The second mechanism, consolidating a recent construct during building-with this construct is the most frequent and most easily observed one. In the framework of the present research, evidence for it can, for example, be found in students' answering further questions, which are similar but set in different contexts, progressively more quickly, more flexibly, and with more self-confidence. Such flexibility expressed itself not only in students' adapting to changing contexts but also in their becoming independent in generating their own, sometimes idiosyncratic tables.

The third mechanism, consolidating a recent construct when recognizing it as an object of reflection, depends on students being offered opportunities for reflection. The data presented in the present paper show reflective activity mainly with respect to principle O. And in accord with this, we can observe C's progressively more elaborate language for dealing with O, from 1 and 2, and 2 and 1 (95) via (1,2) and (2,1), are they different results? (153) to (1,2) and (2,1), that's not the same thing; it's a different event (393). This more elaborate language expresses a more acute and fine-tuned awareness of principle O; and as mentioned above, awareness is an important characteristic of consolidation.

CONCLUDING REMARKS

In this paper, we have analyzed the intertwining of processes of constructing and consolidating. In the framework of RBC-model based research, the study of consolidation processes necessitates not only the identification of epistemic actions but also of their "internalization" or "interiorization". These can only be observed by means of their expression in further epistemic actions. We thus look at consolidating as a psychological process expressed through recognizing actions, primarily during reflection, through building-with actions, for example during problem solving, and through further constructing actions.

We conclude with a methodological remark: A part of the learning process discussed in this paper occurred in a whole class discussion led by the teacher. The information we have about the focus group students for this part is very sparse. Thus, some of our interpretations are based on less detailed data than we might have wished. This is a price we willingly pay for the opportunity to do research in classrooms, rather than in laboratories. But as a community of researchers, we need to be aware that there is a limit to the grain size of data that it is possible to obtain in a classroom, and therefore to the detail, with which learning processes in classrooms can be analyzed.

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RECONCILING FACTORIZATIONS MADE WITH CAS AND WITH PAPER-AND-PENCIL: THE POWER OF CONFRONTING TWO MEDIA

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The study presented in this report is part of a larger project on the co-emergence of technique and theory within a CAS-based task environment for learning algebra, which also includes paper-and-pencil activity. In this paper, an example is presented in which 10th graders in Canada use handheld computer algebra to factor expressions of the form $x^n - 1$. The factorizations made by the computer algebra system are compared with the students' by-hand factorizations. This confrontation, and the students' wish to reconcile the two, turned out to be productive for the development of both theoretical understanding and paper-and-pencil and machine techniques. These findings are in line with the anthropological theory of didactics.

THE FOCUS OF THE STUDY

In school algebra, technique and theory used to collide. Both *technique* and *theory* are broader in meaning than *procedures* and *concepts* (Artigue, 2002). The availability of computer algebra systems (CAS) technology in schools, along with the development of theoretical frameworks for interpreting how such technology becomes an instrument of mathematical thought, have both contributed to a recent increase of attention to the interaction between technique and theory.

The research study, of which this report is a part, is an ongoing one. It has as a central objective the shedding of further light on the co-emergence of technique and theory within the CAS-based algebraic activity of secondary school students (Kieran & Saldanha, 2005). Because of space restrictions, this report will highlight the design and findings from one task set on the factorization of expressions of the form $x^n - 1$.

THEORETICAL FRAMEWORK

The instrumental approach to tool use encompasses elements from both cognitive ergonomics (Vérillon & Rabardel, 1995) and the anthropological theory of didactics (Chevallard, 1999). An essential starting point in the instrumental approach is the distinction between an artifact and an instrument. Whereas the artifact is the object that is used as a tool, the instrument involves also the techniques and schemes that the user develops while using it, and that guide both the way the tool is used and the development of the user's thinking. The process of an artifact becoming an instrument in the hands of a user -- in our case the student -- is called instrumental genesis. The instrumental approach was recognized by French mathematics education

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researchers (e.g., Artigue, 2002; Lagrange, 2003; Trouche, 2000) as a potentially powerful framework in the context of using CAS in mathematics education.

Chevallard's anthropological theory of didactics, which incorporates an institutional dimension into the mathematical meaning that students construct, describes four components of practice by which mathematical objects are brought into play within didactic institutions: task, technique, technology, and theory. By *technology*, Chevallard means the discourse that is used to explain and justify techniques; he is not referring to the use of computers or other technological tools. In their adaptation of Chevallard's anthropological theory, Artigue and her colleagues have collapsed *technology* and *theory* into the one term, *theory*, thereby giving the theoretical component a wider interpretation than is usual in the anthropological approach. Furthermore, Artigue notes that *technique* also has to be given a wider meaning than is usual in educational discourse.

Lagrange (2003, p. 271) has elaborated this latter idea further: "Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration. It also serves as an object for a conceptual reflection when compared with other techniques and when discussed with regard to consistency." It is precisely this epistemic role played by techniques that is a focus of our study, that is, the notion that students' mathematical theorizing develops as their techniques evolve. It is noted, as well, that our perspective on the co-emergence of theory and technique is situated within the context of technological tool use, where the nature of the task plays an equally fundamental role. Thus, the triad Task-Technique-Theory (TTT) served as the framework not only for constructing the tasks of this study, but also for gathering data during the teaching sequences and for analyzing the resulting data.

METHODOLOGY

The research involves six intact classes of 10th graders (15-year-olds) in Canada and Mexico, as well as a class of older students in Oregon. Five of the 10th grade classes were observed during the 2004-05 school year; the sixth class, the following year. One of these 10th grade classes from the 2004 study is featured in this report. This class consisted of 7 girls and 10 boys, all of them considered by their teacher to be of upper-middle mathematical ability Their teacher had five years of experience, and, along with encouraging his pupils to talk about their mathematics in class, believes that it is useful for them to struggle a little with mathematical tasks. He elicits students' thinking, rather than quickly giving them answers. The students in this report had already learned a few basic techniques of factoring (for the difference of squares and for factorable trinomials) and had used graphing calculators on a regular basis; however, they had not had any experience with symbol-manipulating calculators (i.e., the TI-92 Plus CAS machines used in this project).

All project classes were observed and videotaped (12-15 class periods for each of the seven project classes). Students were interviewed, alone or in pairs, at several instances -- before, during, and after class. Thus, data sources for the segment of the study presented in this report include the videotapes of all the classroom lessons, videotaped interviews with students, a videotaped interview with the teacher, the activity sheets of all students (these contained their paper-and-pencil responses, a record of CAS displays, and their interpretations of these displays), and researcher field notes.

Guided by the TTT foundations of the study, we developed a priori descriptions of the techniques and theories that we considered might emerge among students participating in the study. These descriptions provided the lens for gathering and analyzing the data drawn from the classrooms, from student work, and from student interactions. The structure of this article in fact makes explicit these a priori descriptions of technique and theory that we generated, as well as the way in which they served as a focus for our analysis.

TASK, TECHNIQUE, AND THEORY

The activity that exemplifies this theme is inspired by a task described by Mounier and Aldon (1996), two teachers who presented, over the course of a few years, to their classes of 16- to 18-year-old students the task of conjecturing and proving the general factorizations of $x^n - 1$, in a first year with paper-and-pencil, and in a second and third year with computer algebra. However, as Lagrange (2000) pointed out, the CAS factorizations and the paper-and-pencil factorizations are not always identical, and reconciling them is a non-trivial task. This latter issue is central in this paper (see also Kieran & Saldanha, in press).

First, students discovered the so-called 'general factorization' of $x^n - 1$:

$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \ldots + x + 1)$$

Then, a next part of the activity focused on the differences between CAS factorizations and by-hand factorizations (see Figure 1). A final part of the activity, not discussed here, concerned conjecturing on the forms of the CAS factorizations and on proving these conjectures.

Which techniques play a role in this activity? Table 1 provides an overview, including both paper-and-pencil techniques and CAS techniques. It should be noted, that the relation between the CAS techniques and the corresponding paper-and-pencil techniques is not as close as it might seem. Furthermore, the task of reconciling paper-and-pencil factors and CAS factors can involve a combination of the techniques mentioned in Table 1.

The students' theoretical development includes, for example, the notion of complete factorization, which comes to the fore when students attempt to factor an expression with a non-prime even exponent, such as $x^4 - 1$, according to the general rule, and are confronted with a CAS factorization that they do not anticipate. Reconciling paper-and-pencil factors with CAS factors is considered by us, indeed, an important part of the process of extending students' theoretical views on factoring. Reflection on these issues is related to techniques 1 to 4.

In this activity each line of the table below must be filled in completely (all three cells), one row at a time. Start from the top row (the cells of the three columns) and work your way down. If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

Factorization using <u>paper and pencil</u>	Result produced by the <u>FACTOR command</u>	Calculation to reconcile the two, if necessary
$x^2 - 1 =$		
$x^3 - 1 =$		
$x^4 - 1 =$		
$x^5 - 1 =$		
$x^6 - 1 =$		

Figure 1. Task in which students confront the CAS factorizations with their paper-and-pencil factorizations (note that the task also included integral values for the exponents up to 13)

Technique	**CAS command (using TI-92 Plus)**	**Paper-and-Pencil variant**
1. Expanding an expression completely	Expand command	Expanding all of the expression by hand, combining like terms, and ordering final terms
2. Expanding a sub-expression	Expand command, using as argument the desired part of the expression	Expanding, by hand, usually two factors of the given expression
3. Factoring completely an expression (if factorable)	Factor command	Factoring by hand, often with a choice of several possible methods
4. Factoring a sub-expression	Factor command, using as argument the desired part of the expression. The CAS may not always succeed in this regard.	Factoring, by hand, a particular factor of a given expression, often with a choice of methods possible

Table 1. CAS and paper-and-pencil techniques for this activity

SOME RESULTS

The students' first surprise while working on the task in Fig. 1 arrived when they entered Factor $(x^4 - 1)$ into their CAS, which yielded $(x-1)(x+1)(x^2+1)$, in contrast with $(x-1)(x^3+x^2+x+1)$, which all of them had written for their paper-and-pencil version. It did not take long before students could be heard commenting: "it can be factored further," "it's not completely factored," "it gives you all the factors".

To reconcile the CAS factors with their own paper-and-pencil factors for $x^4 - 1$, students did the following:

- Multiplied the second and third CAS factors to produce their second paper-and-pencil factor (Figure 2, half the students);
- Factored by 'grouping' the second paper-and-pencil factor to produce the second and third CAS factors (Figure 3, a little fewer than half the students);
- Refactored the given $x^4 - 1$ as a difference of squares (Figure 4, one student).

For some students, the first of the three methods of reconciliation had initially been carried out with the CAS (using Expand) and then transferred to paper. However, most of the reconciliation work was done with paper-and-pencil, as had been requested by the teacher.

Factorization using paper and pencil	Result produced by FACTOR command	Calculation to reconcile the two, if necessary
$x^2-1=(x-1)(x+1)$	$(x-1)(x+1)$	N/A
$x^3-1=(x-1)(x^2+x+1)$	$(x-1)(x^2+x+1)$	N/A
$x^4-1=(x-1)(x^3+x^2+x+1)$	$(x-1)(x+1)(x^2+1)$	$(x-1)(x+1)(x^2+1)$ $(x-1)(x^3+x^2+x+1)$

Figure 2. Reconciling by multiplying the second and third CAS factors of $x^4 - 1$ to produce the second paper-and-pencil factor

Factorization using paper and pencil	Result produced by FACTOR command	Calculation to reconcile the two, if necessary
$x^2-1=(x-1)(x+1)$	$(x-1)(x+1)$	
$x^3-1=(x-1)(x^2+x+1)$	$(x-1)(x^2+x+1)$	
$x^4-1=(x-1)(x^3+x^2+x+1)$	$(x-1)(x+1)(x^2+1)$	$(x-1)(x^2(x+1)+1(x+1))$ $(x-1)(x^2+1)(x+1)$

Figure 3. Reconciling by 'grouping' the second paper-and-pencil factor to produce the second and third CAS factors of $x^4 - 1$

Factorization using paper and pencil	Result produced by FACTOR command	Calculation to reconcile the two, if necessary
$x^2-1=(x-1)(x+1)$	$(x-1)(x+1)$	
$x^3-1=(x-1)(x^2+x+1)$	$(x-1)(x^2+x+1)$	
$x^4-1=(x-1)(x^3+x^2+x+1)$	$(x-1)(x+1)(x^2+1)$	$x^4-1=(x^2-1)(x^2+1)$ $(x-1)(x+1)(x^2+1)$

Figure 4. Reconciling by refactoring the given $x^4 - 1$ as a difference of squares

In the class discussion that followed the completion of the first set of examples for n from 2 to 6 in the factoring of $x^n - 1$, some clarification of the notion of complete factorization took place, which included the teacher's comment that: "Sometimes, they can be factored further. What we did initially is not wrong, it's just not complete." The notion that expressions with even exponents greater than 2 could also be regarded as a difference of squares was not obvious for some students, as suggested by the remark uttered by one student: "I can't get [the factors of] x^4-1". Furthermore, while it was mentioned by a few students that x^6-1 could be treated either as a difference of squares, $(x^3)^2-1$, or as a difference of cubes, $(x^2)^3-1$, the upcoming task involving the factoring of x^9-1 was to provide evidence that *seeing* a difference of cubes was even more difficult for some students than seeing a difference of squares.

Based on their limited set of examples thus far, it was inevitable that most students conjectured that, for odd values of *n*, the general rule seemed to be holding. In other words, they thought that the complete factorization of x^n-1 had exactly two factors for odd ns, while for even values of *n*, it contained more than two factors, of which $(x+1)$ was one of them. The following conversation between two students illustrates how the CAS helped them realize that their conjecture regarding odd ns was incorrect.

Chris	The only time it contains two factors is when it is odd, I think, which means it can be, [pause] like, our pattern can't be broken down anymore. 'Cause it always ends up being all positive. And uh, then, because, it's sort of hard to explain.
Peter	When the exponent is [pause], when the exponent is an even number you'll have more than two factors, but when the exponent is not an even number, you'll have exactly two factors all the time.
Chris	Yeah. [Types Factor(x^7-1) into the CAS]
	Yeah, because any time you plug in an odd number as the exponent power, it's uh, the calculator always stays at the most simplified [pause] and [Types in Factor(x^9-1); the CAS displays: $(x-1)(x^2+x+1)(x^6+x^3+1)$]
	And, no!!! [a look of utter surprise on Chris's face]

In light of the classroom discussion related to confronting paper-and-pencil factors with CAS factors for the expressions from x^2-1 to x^6-1, some students began to adjust their paper-and-pencil factoring techniques with the aim of "playing a game" with the CAS. They tried to anticipate what it would produce as its factored form, for the expressions from x^7-1 to $x^{13}-1$, and to thereby reduce the amount of reconciliation that would need to be done.

Within this part of the task where students were confronting their paper-and-pencil factors with the CAS factors, the CAS played a role that was quite different from that which it had played in other parts of the activity. The CAS technique of Factor, with its accompanying output, disclosed to the students that there were certain factoring techniques that they were missing from their repertoire. As a consequence, they wanted to learn these techniques. This need to understand the factored CAS outputs and to be able to explain them in terms of a certain structure, or by means of paper-and-pencil techniques that would produce the same results, seemed important to the students (and to us!).

In all, the confrontation of students' paper-and-pencil factors with the CAS factors led to the development of new theoretical ideas. In the process of making sense of the CAS factors, the students extended their view of the range of the difference-of-squares technique. They came to see that exponents that have several divisors can generally be factored in more than one way. They began to look at expressions in terms of multiple possible structures. Their understanding of the notion of complete factorization evolved. Finally, some students were even able to detect new patterns, and with the aid of the CAS, developed another general rule.

CONCLUSION

From our analysis we conclude that the emergence of theory among the students would not have been possible without the accompanying technical demands raised by the tasks. In fact, the development of CAS techniques, simple as they were in most cases, was vital to theoretical advances. The confrontation of the CAS factored forms with those that students produced by paper-and-pencil was found to be very productive for most students, but especially so for those who, upon realizing that they could not generate the same factors as had the CAS, insisted on finding out how to do so. These CAS encounters resulted in the evolution of not only students' paper-and-pencil factoring techniques but also their theoretic perception of the structure of expressions (e.g., seeing that x^6-1 could be viewed either as a difference of squares, or as a difference of cubes, or as an example of the general rule that they had earlier generated). The need to make sense of the CAS outputs, and the ability to coordinate these with existing theoretical notions and paper-and-pencil techniques, was fundamental to the students' theoretical and paper-and-pencil-technical progress.

The main finding of the study is that we clearly found evidence for the relation *theory – technique* within the setting of the designed tasks, which confirms the importance and productiveness of the TTT approach. Technique and theory emerge in mutual interaction. The observations show how techniques gave rise to theoretical thinking, and, the other way around, how theoretical reflections led students to develop and use techniques. This interaction proved to be very productive in cases of confrontation, or even that of conflict, between the techniques – particularly the CAS techniques – and the students' theoretical thinking. A tendency to reconcile CAS work and theory was observed; students seemed to strive for consistency, and used the CAS on several occasions as a means of checking their theoretical thinking.

Our research findings lead us to suggest that the epistemic value of CAS techniques by themselves may depend both on the nature of the task and on the limits of students' existing learning. When students cannot explain, in terms of their current theoretical and technical knowledge, that which a CAS technique produces, reliance on additional CAS techniques may not suffice. In such cases, the epistemic value of paper-and-pencil techniques would seem to play a complementary, but essential, role. Recent research that has used the TTT theoretical framework for analyzing the learning of mathematics in technological environments has tended to pay less attention to the role of paper-and-pencil in interaction with CAS techniques in promoting theoretical growth. Our results point to this as a fruitful area for research involving, in particular, young high school algebra learners.

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