STUDENTS' DIFFICULTIES WITH APPLYING A FAMILIAR FORMULA IN AN UNFAMILIAR CONTEXT

Maureen Hoch and Tommy Dreyfus

Tel Aviv University, Israel

This paper discusses problems many tenth grade students have when asked to apply a familiar formula in an unfamiliar context: specifically factoring a compound expression. Some of their attempts at solution are presented and discussed in terms of structure sense.

Students who have previously displayed proficiency at using algebraic techniques often have difficulty in applying these techniques in unfamiliar contexts. Specifically we have observed that many students are unable to use the formula $a^2 - b^2 = (a-b)(a+b)$, which is familiar to them, to factor the expression $(x-3)^4 - (x+3)^4$, which they have probably never met before.

Rauff (1994) has classified students' beliefs about factoring and shows how these beliefs affect how students approach a factoring exercise. Rauff's categories of students' beliefs about factoring are: reversal – the process of undoing a multiplication; deconstructive - the process of breaking down or simplifying; evaluative – a process that can be checked by multiplying out; formal – a factored expression is in the form of a product; numeric – decomposing numbers into products of primes. We will return to this classification when presenting and analysing our data.

Most of the literature on algebraic techniques is about learning to use them, not about how students use these techniques later. Most of the current research is on the use of computer algebra systems. The emphasis in the research concerning CAS is on conceptual understanding rather than on skill development. However, Pierce and Stacey (2001, 2002) have developed a framework for algebraic insight which is described as two groups of skills needed to use computer algebra systems: algebraic expectation which consists of recognition of conventions and basic properties, and identification of structure and of key features; and ability to link representations. Pierce and Stacey might explain the above difficulty in factoring as a failure of algebraic expectation.

Skemp (1976) might have called this difficulty in factoring a use of instrumental rather than relational understanding. Zazkis and Gadowsky (2001) might attribute it to opaque rather than transparent representations. Tall and Thomas (1991) might view it as a lack of versatile thinking. Another possible way of explaining the difficulty is through concept image (see for example Tall and Vinner, 1981). If a student has never seen an example similar to $(x-3)^4 - (x+3)^4$ then this kind of expression does not fit into his concept image of "the difference of squares".

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We attribute the difficulty to a lack of structure sense. In this paper structure sense is defined, and students' attempts at factoring the above expression are discussed in terms of structure sense.

Linchevski and Livneh (1999) first used the term structure sense when describing students' difficulties with using knowledge of arithmetic structures at the early stages of learning algebra. Hoch (2003) suggested that structure sense is a collection of abilities, separate from manipulative ability, which enables students to make better use of previously learned algebraic techniques. Hoch & Dreyfus (2004) described algebraic structure as it applies to high school algebra and gave a tentative definition of structure sense. Here we present a refined definition of structure sense.

STRUCTURE SENSE FOR HIGH SCHOOL ALGEBRA

A student is said to display structure sense (SS) if s/he can:

- Deal with a compound literal term as a single entity. (SS1)
- Recognise equivalence to familiar structures. (SS2)
- Choose appropriate manipulations to make best use of the structure. (SS3)

Examples to illustrate the definition

SS1 Deal with a compound literal term as a single entity.

Example: Solve: $5(3x-14)^2 = 20$. Here the student is said to display SS1 if s/he can relate to the term 3x-14 as a single entity or object.

Example: Factor: $6(5-x)+2x(5-x)$. Here the student is said to display SS1 if s/he can extract the term 5-x as a common factor (which implies relating to it as a single entity).

SS2 Recognise equivalence to familiar structures.

Example: Factor: $64x^6 - 36y^4$. Here the student is said to display SS2 if s/he can recognise the expression as possessing the structure "the difference of two squares".

Example: Solve: $(x+17)(x-12)=0$. Here the student is said to display SS2 if s/he can recognise the equation as possessing the structure ab=0.

Example: Solve: $(x^2 + 2x)^2 + (x^2 + 2x) - 6 = 0$. Here the student is said to display SS2 if s/he can first relate to the term x^2+2x as a single entity (this is in fact ability SS1), and then recognise the equation as being quadratic in that entity.

SS3 Recognise which manipulations it is useful to perform in order to make use of structure.

Example: Simplify: $(4x^2 - 25x + 13x^2)(7x - 4 - 7x)$. Here the student is said to display SS3 if s/he can recognise the advantage of first simplifying each term in the product before opening brackets.

Example: Find the derivative of: $f(x) = \dfrac{\left(x^3 + x\right)^2}{x^2 + 1}$. A student with the ability SS3 will see the advantage of simplifying the function before differentiating. Having recognised that the function possesses the structure of a ratio of two polynomials (SS2), s/he will be able to cancel the term x^2+1 from the numerator where it appears as a factor and from the denominator (SS1).

In some of these examples the partially hierarchical structure of structure sense can be seen.

SS1 = see entity

SS2 = SS1 + recognise structure

SS3 = SS2 + useful manipulations

METHODOLOGY

A group of 190 students (from seven tenth grade classes, intermediate/advanced stream), were asked to factor the expression $x^4 - y^4$. A similar group of 160 students (from six tenth grade classes, intermediate/advanced stream), were asked to factor the expression $(x-3)^4 - (x+3)^4$. Both groups were encouraged to use the formula $a^2 - b^2 = (a-b)(a+b)$. In both groups the question was posed within the framework of a larger questionnaire containing several factoring exercises.

Subsequently ten students from the second group were interviewed. Students were chosen on the basis of willingness to be interviewed, and on recommendations from their teachers as to their ability to express themselves verbally.

The students' proficiency in using algebraic techniques is assumed on the basis of the fact that the target population consists of students whose grades in the previous school year were well above average. Their familiarity with use of the formula $a^2 - b^2 = (a-b)(a+b)$ is based on the fact that the majority of the first, similar group succeeded in using it. The assumption that the context is unfamiliar is based on talks with teachers and a survey of the textbooks used.

RESULTS

Whereas 146 out of 190 (77%) students factored $x^4 - y^4$ correctly, only 12 out of 160 (7.5%) students factored $(x-3)^4 - (x+3)^4$ correctly. We decided to study in greater detail the responses of the 148 students who didn't manage to factor $(x-3)^4 - (x+3)^4$ correctly.

Forty-seven students made no (written) attempt whatsoever, leaving a blank. The responses of the remaining 101 students are summarized in Table I.

Table I   Unsuccessful attempts to factor $(x-3)^4-(x+3)^4$

Belief about factoring	Type of response	Number of students	Examples
Deconstructive (the process of breaking down or simplifying)	Opens brackets incorrectly	27	$x^4-12x+81-x^4-12x-81=-24x$
	Obtains difference of squares, does not use formula	6	$(x^2-6x+9)^2-(x^2+6x+9)^2$ $=x^4-36x^2+81-x^4-36x^2-81$
Formal (a factored expression is in the form of a product) Or Reversal (the process of undoing a multiplication)	Manipulates powers incorrectly	29	$(x^2-9)^4$ $(x-3-x-3)(x-3+x+3)$ $(x-3)^2(x+3)^2$ $(x-3)^4(x+3)^4$
	Applies formula incorrectly	16	$(x-3-x-3)^2(x-3+x+3)^2$ $[(x-3)^2-(x+3)^2][(x+3)^2-(x-3)^2]$ $a=x-3 \quad and \quad b=x+3$ but then $[(x-3)(x+3)]^8$ $a=(x-3)^4 \quad and \quad b=(x+3)^4$ and then $[(x-3)^4-(x+3)^4][(x-3)^4+(x+3)^4]$
Numeric (decomposing numbers into products of primes)	Does not use $\left[(x-3)^2\right]^2$	17	$(x-3)^2(x-3)^2-(x+3)^2(x+3)^2$
	Writes 4th power as product	6	$(x-3)(x-3)(x-3)(x-3)-(x+3)(x+3)(x+3)(x+3)$

The responses are arranged according to Rauff's classification of students' beliefs about factoring, as far as it is possible to determine from the written answers.

We were unable to differentiate between a written answer belonging to the formal and to the reversal categories, as both result in a product. Opening brackets clearly belongs to the deconstructive category, and writing the fouth power as a product to the numeric category. We found no example of the evaluative category.

Some of these responses will be examined in greater detail, and analysed in terms of structure sense.

INTERPRETATION IN TERMS OF STRUCTURE SENSE

The data were arranged in the above table according to what they might reveal about students' beliefs about factoring. In this section we will discuss what these students' responses might tell us about their structure sense, and how they use formulas.

The huge difference between results on the two tasks is surprising: 77% success as opposed to 7.5%. The formula was given to the students and they applied it easily in the case of $x^4 - y^4$. What makes $(x-3)^4 - (x+3)^4$ so much more difficult? Students were given the formula (the structure) but were unable to apply it in this instance. It seems that they did not see the connection between the expression and the structure. They were unable to relate to $(x-3)^4 - (x+3)^4$ as being an instance of $a^2 - b^2$.

In the interview some of the students attributed their difficulty to "*We haven't learned this*" or "*I can't remember*". Despite the fact that they succeeded in factoring simple expressions, they were confused when asked what it means to factor. Their answers ranged from "*to get to some formulas*" to "*to break up a number into smaller parts*" to "*to cancel or to simplify as much as possible*". None of the ten students described factoring as writing an expression as a product.

Display of structure sense

The 12 successful students (not shown in the table) displayed structure sense. They were able to relate to x-3 and x+3 as entities (SS1) and recognised the structure $a^2 - b^2$ in the expression $(x-3)^4 - (x+3)^4$ (SS2).

Lack of structure sense

We can say nothing about the 47 students who left a blank, although we may suspect they lack structure sense. Another group who seem to lack structure sense are those 29 students who manipulated the powers incorrectly. They seem to be aiming to simplify the expression, using an "anything goes" strategy.

The 6 students who expressed the fourth power as a product of four factors appeared to be feeling their way, unsure of what to do but attempting to find a meaning for the expression. In the interview Gal was asked why students found this exercise difficult. She said "*when they see the 4th power they think they have to do multiplication between the terms*". 17 students were able to identify $(x-3)^4$ as $(x-3)^2 \cdot (x-3)^2$ but not as $\left[(x-3)^2\right]^2$. These students too are lacking structure sense.

Partial structure sense

We could claim that the 27 students who opened brackets lack structure sense. It is possible that they expect to obtain a simpler expression after opening brackets and collecting like terms. When interviewed Dana said "*I opened brackets to collect like terms, in order to find the common factor*". However, in the process of opening the brackets they are displaying a kind of, if not very useful, structure sense. Some of them "recognised" the expression $(x-3)^4$ as having a structure similar to $(a-b)^2$ and then used a false generalisation of the formula $(a-b)^2 = a^2 - 2ab + b^2$ to get $x^4 - 12x + 81$. As Natalie explained in the interview "*When it's squared you take that squared then that times that times the power, then that to the power. …(In this case) We do x to the fourth, then 3 times 4, that's 12x, then 3 to the fourth. That's how I was taught*".

A very interesting group is the one consisting of 6 students who apparently were able to identify $(x-3)^4$ as $\left[(x-3)^2\right]^2$ since they opened the interior brackets and obtained $(x^2 - 6x + 9)^2 - (x^2 + 6x + 9)^2$. However they quite clearly lacked some structure sense as they did not recognise this as having the structure $a^2 - b^2$. Why is it so difficult for them to see what seems to us so obvious? Remember, the formula (structure) was given to them. Is it the compound expression that hides the connection between the expression and the given formula? Or is it the brackets? Is it a failure of SS1 – seeing the compound expression as an entity – which causes them to fail in SS2 – recognising the structure?

Finally we have the 16 students who valiantly tried to apply the given formula. They are told that the structures are equivalent, but they cannot find the connection. Their structure sense, at least the SS2, is clearly deficient. Let us look at some of their attempts.

The students who wrote $(x-3-x-3)(x-3+x+3)$ seem to have read the fourth power as a second power – perhaps they thought there was a misprint in the question. However they definitely displayed SS1, relating to x-3 and x+3 as entities.

The students who wrote $(x-3-x-3)^2(x-3+x+3)^2$ applied the formula wrongly. They apparently assumed that if $a^2 - b^2 = (a-b)(a+b)$ then $a^4 - b^4$ must be equal to $(a-b)^2(a+b)^2$. These students are also displaying SS1.

It is difficult to interpret the student who wrote $[(x-3)^2 - (x+3)^2][(x+3)^2 - (x-3)^2]$. Perhaps s/he is using a rule wrongly, like Roy who said that $(x-3)^4$ is equal to $[(x-3)(x+3)]^2$ because "*minus times plus gives minus*" and $(x+3)^4$ is equal to $[(x-3)(x-3)]^2$ because "*minus times minus gives plus*".

The student who wrote $a = x-3$ and $b = x+3$ but then $[(x-3)(x+3)]^8$ displays SS1 but an inability to use the formula.

The student who wrote $a = (x-3)^4$ and $b = (x+3)^4$ and then

$[(x-3)^4 - (x+3)^4][(x-3)^4 + (x+3)^4]$ could be seen to be displaying SS1 and SS2, and was probably just being careless in writing the exponent as four instead of two. In the interview Michal gave this identical answer, and then later corrected herself: "*oh, it should be squared*".

The difficulties of those students who seemed to be able to relate to x-3 and x+3 as entities (SS1) but did not see the $a^2 - b^2$ structure may be attributed to their lack of SS2.

CONCLUSION

What do the above data tell us? Many students are unsure what factoring is all about. Some students have difficulties with using a given formula. Very few of the 160 students in this study used structure sense. It should not be inferred that structure sense is solely about using formulas: an ease with using formulas is just one result of a healthy structure sense.

A recurring theme was the inability to relate to *x*-3 and *x*+3 as entities – a lack of SS1. When students were asked to give an example of something "complicated" that *a* could stand for in the formula $a^2 - b^2 = (a-b)(a+b)$ they only gave products (e.g., $2x$ or *xy*), not sums (e.g., *x*+2). As Ben put it "*you have to see that this term represents a variable … you simply have to look at it differently*".

Another common response was "*we didn't learn this*". Since these same students were able to answer standard factoring exercises correctly, this could be interpreted as a lack of SS3 (recognise which manipulations it is useful to perform).

Is structure sense something that can be taught? Should it be taught? We feel that the last two questions should be answered in the affirmative but are not yet ready with an answer for the obvious next question: How should structure sense be taught?

We are convinced of the importance of drawing students' attention to structure. The above definitions should be useful as guidelines for further research and didactic design.

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