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Research Reports

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WORKING WITH LEARNERS' MATHEMATICS: EXPLORING A KEY ELEMENT OF MATHEMATICAL KNOWLEDGE FOR TEACHING

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This paper explores an element of mathematics for teaching (MfT), specifically 'interpreting and judging students' mathematical productions'. The research reported draws from a wider study that includes an examination of MfT produced across teacher education sites in South Africa. We show that this element of MfT is privileged across sites, evidence that it is valued in teacher education practice. Its production varies, however, enabling elaboration of this element of MfT.

INTRODUCTION

A distinguishing feature of mathematics teacher education is its dual, yet deeply interwoven, objects: *teaching* (i.e. learning to *teach* mathematics) and *mathematics* (i.e. learning *mathematics* for teaching *(MfT*)) – the subject-method tension. These dual objects, and their inter-relation are writ large in *in-service* teacher education (INSET) programs where new and/or different ways of knowing and doing school mathematics combine with new and/or different contexts for teaching. Such are the conditions of continuing professional development in South Africa. Post apartheid South Africa has seen a proliferation of formal (i.e., linked to accreditation) and informal INSET programs. Debate continues as to whether and how these programs should integrate or separate out opportunities for teachers to (re)learn *mathematics* and *teaching*. Programs range across this spectrum, varying in degree to which opportunities for teachers to learn are embedded in problems of (mathematics teaching) practice, and so opportunities for learning more of their specialized knowledge, MfT.

In the QUANTUM research project, we are currently studying mathematics and mathematics education courses in three mathematics teacher education sites where the programs differ in relation to their integration of mathematics and teaching. The goal is not to measure impact of these different approaches, but rather, through in-depth investigation of practices within these courses, to understand *what* and *how* *mathematics* and *teaching* come to be (co)produced across and within these settings. We are thus examining practices *inside* teacher education. Specifically, and this is discussed further below, we are investigating how and what knowledge(s) are appealed to as elements of MfT come to be legitimated in pedagogic discourse**.**

Our focus in this paper is on one privileged MfT practice evident across all three sites: working with learners' mathematical productions. Learners' mathematical productions, and teachers' engagement with these, have been a prevalent theme in mathematics education research and widely reported in PME. In this paper we assume the importance of teachers being able to do this work. The concern, rather, is with the mathematical entailments of this work and its elaboration in teacher education practice. Our examination of the practices across three sites reveals that while the notion of working with (interpreting, analyzing, judging) student mathematical thinking is common, it emerges and is approached in quite different ways, illuminating this element of MfT in interesting ways. Our observations are a function of a particular analytic tool, and its underlying theoretical orientation both of which are elaborated below. We begin with a brief discussion of QUANTUM – the wider research project.

THE QUANTUM PROJECT

The overarching 'problem' under scrutiny in QUANTUM[1] is *mathematics for teaching (MfT)*, its principled description and related opportunities for teachers' learning. We regard the mathematical work of teaching as a particular kind of mathematical problem-solving[2] - a situated knowledge, shaping and being shaped by the practice of teaching. More specifically we are concerned with the mathematics middle and senior school teachers need to know and know how to use (i.e. the mathematical work they do) in order to teach mathematics well in diverse classroom contexts in South Africa; and with how, and in what ways, programs that prepare and/or support mathematics teachers provide opportunities for learning MfT. Elsewhere (Adler, Davis & Kazima, 2005), we have problematised the renewed focus on subject knowledge for teaching in mathematics education, its development from Shulman's seminal work on pedagogic content knowledge, how it remains underdescribed, and how mathematics teacher education practice, as well as school teaching practice, is a productive empirical site in the project.

In our earlier work (Adler & Davis, 2004) we exemplified a pedagogic practice where learners are expected to engage with novel mathematics problems, and showed that meanings can and do proliferate. The teacher has considerable mathematical work to do as s/he navigates between varying learner responses, and what would constitute a robust mathematical solution. S/he needs to figure out how to mediate between these interpretations, and the mathematical notion(s) and dispositions she would like all learners in the class to consolidate. S/he needs to figure out suitable questions to ask learners, or comments to make. Both have mathematical entailments.

Ball, Bass and Hill (2004, p.59) describe these mathematical practices as elements of the specialised mathematical problems teachers solve as they teach. These elements

[1] For more detail on QUANTUM see Adler & Davis (2004)

[2] We thank Deborah Ball for this description – personal communication, Adler and Ball.

include the ability to "design mathematically accurate explanations that are comprehensible and useful for students" and "interpret and make mathematical and pedagogical judgements about students' questions, solutions, problems, and insights (both predictable and unusual)". They posit a more general feature, "unpacking", as an essential and distinctive feature of "knowing mathematics <u>for teaching"</u>.[3] We have already noted the extensive work in the field of mathematics education on learners' constructions of mathematical ideas and related work on misconceptions (e.g. Smith, DiSessa, & Roschell, 1993). There has been far less attention, in our view, to the kinds of mathematical and pedagogical judgements teachers make as they go about their work on student productions[4], hence our methodology and focus.

Our overarching theoretical orientation is elaborated in Davis, Adler, Long & Parker (2003) and Adler & Davis (2004). Briefly, the tool emerges from our use of Basil Bernstein's sociological theory of pedagogy. We recruit Bernstein's (1996) proposition that the whole of the pedagogic device (distribution of knowledge; rules for the transformation of knowledge into pedagogic communication) is condensed in evaluation. In other words, any pedagogy transmits evaluation rules. Additionally, evaluation is activated by the operation of pedagogic judgement by both teacher and student.

In QUANTUM we are looking at evaluative events across teacher education programs, on the assumption that these would reveal the kind of mathematical and teaching knowledge that comes to be privileged. Figure 1 presents a network of part of the tool[5] we are using, and includes the codings we refer to in the next section. We have highlight categories of the network pertinent to our focus in this paper. The network reflects our dual and simultaneous focus on mathematics and teaching as specialised activities, and how they emerge as objects of study over time in each of the courses. Each course, all its contact sessions and related materials, were analysed, and chunked into what we have called *evaluative events*. These are marked by punctuations in pedagogic discourse, when meanings are set through pedagogic judgement. Space limitations prevent description of the full network, and the systematic chunking done.

[3] In Adler & Davis (2004) we report QUANTUM: Phase 1. We focused on formal assessment tasks across math and math education courses in 11 institutions in South Africa. A key 'finding' is that across courses, formal assessments of unpacked mathematics in relation to teaching were very limited.

[4] A very recent study by Karin Brodie (Brodie 2005) has explored teacher moves as they engage learner thinking. Her analysis provides an important description of this work of mathematics teaching.

[5] Missing here is an additional set of columns on *subject positions*. These are significant in their relation to particular notions and how they unfold over time, and are the focus of a different paper. See Adler, Davis, Kazima, Parker & Webb, forthcoming.

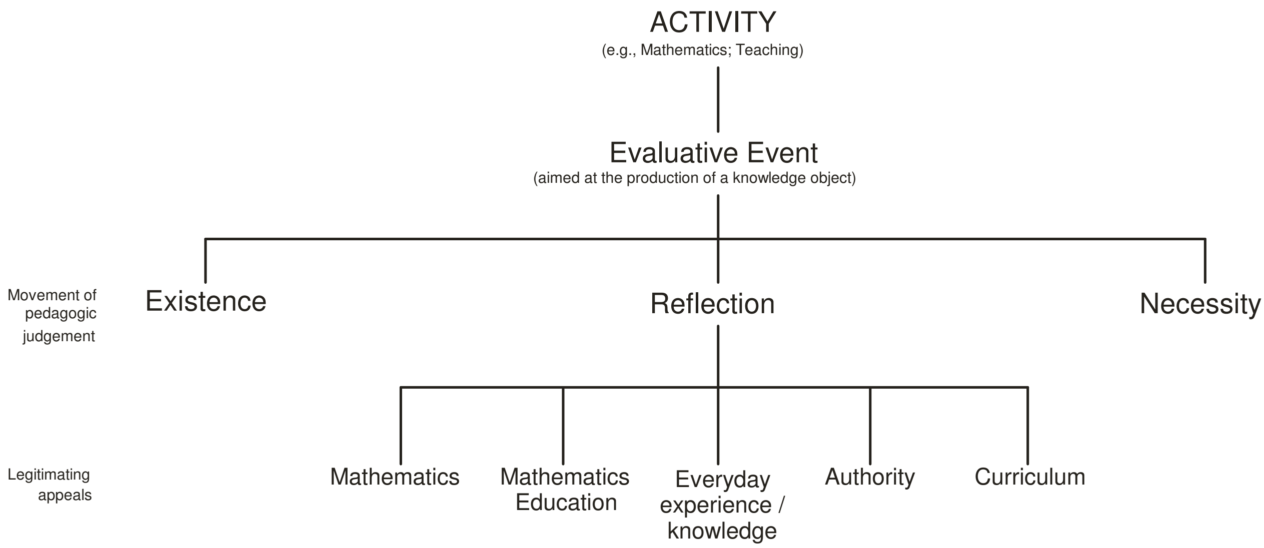

Figure 1: Network describing the movement of pedagogic judgement

Suffice it to say that for each event, we coded first whether the object was a mathematical (M) and/or teaching (T) one, or both, and then whether elements of the object(s) were assumed known, rather than being focus of study (and were then coded either m or t). The additional branches in the network emerge through a recontextualisation of Hegel's theory of judgement (1969). We recruit from Hegel the proposition that judgement in general, and hence pedagogic judgement in particular, is itself constituted by a series of dialectically entailed judgements (of Existence, Reflection, Necessity, and the Notion). Here we are working with the idea that in pedagogic practice, in order for something to be learned, to become known, it has to be represented. Initial orientation to the object, then, is one of immediacy – it exists in some initial (re)presented form, and can only be grasped as brute Existence. Pedagogic interaction (Reflection) then produces a field of possibilities for the object, and through related judgements made on what is and is not the object (Legitimating Appeals), so possibilities are generated (or not) for learners to grasp the object (Necessity).[6] In other words, the legitimating appeals can be thought of as qualifying reflection. An examination of what is appealed to and how appeals are made in the teaching of mathematics delivers up insights into how MfT is being constituted in teacher education.

WORK ON LEARNER MATHS ACROSS THREE COURSES

Table 1, p.8 provides summary information about the course on each site. The last three rows provide a description of the analysis of our data set, particularly in relation to where and how legitimating appeals are made. Each course is for in-service teachers, and part of a larger program towards a qualification. Two courses are aimed at Senior Secondary teachers, one at junior secondary; two are level 6

[6] All judgement, hence all evaluation, necessarily appeals to some or other locus of legitimation to ground itself, even if only implicitly.

(undergraduate) and 1 level 7 (post graduate) courses. They share similar goals (to provide learning experiences that will enable and improve mathematics teaching), with the Level 7 course having an additional academically oriented goal.

The Algebra concepts and methods course (Site 1) is concerned with algebraic thinking at the Grades 7 – 9 level. The underlying assumption in this course, guided by the teachers being primary trained, is that the teachers were unlikely to be adept in algebraic thinking, though they would, like their learners have learned algebraic rules as recipes. They thus needed to learn this way of thinking mathematically. They also needed to learn how to teach this in Grade 7 – 9 classes. These dual goals were integrated in a pedagogic practice that provides experiences for teaching/learning algebra that model the pedagogic practice teachers could/should use in their own classrooms. Teachers could then learn the mathematics needed and at the same time experience how it should be taught. In each of the course sessions dealing with patterns, teachers were given three or four possible formulae that could be generated from a given sequence as if these were produced by learners. Teachers were asked to visualize and explain how each different learner was thinking. In sessions dealing with algebraic rules and operations, teachers were informed of typical learner errors (explained as a result of learning 'recipes'), and provided a way of dealing with these errors. For example, in order to clarify and prevent wrong application of laws of indices, learners could be shown how and why the rule worked (i.e. test it) through substitution of appropriately selected (small) numbers. As indicated in Table 1, legitimating appeals are made to mathematics and everyday life. It is interesting, firstly, that there are moments were everyday experience is appealed to for legitimating mathematical knowledge (specifically algebraic thinking); and secondly when the appeal is mathematical, it is restricted to numerical examples appropriate to learners at Grades 7 – 9.

In Site 2, The Professional Practice in Mathematics Education course provides a structured guide to an action research project teachers are to do. One element of the structured guide is what is referred to as a hypothetical learning trajectory (HLT) – a global teaching practice that includes ways of eliciting student knowledge, generating possible student responses, and analysing student work. As preparation for the weekend session where this aspect of their research was in focus, teachers were meant to bring examples from their own practice where they had elicited student thinking and analysed it. The course materials carried reading on misperceptions. Few teachers brought their preparation[7]. With only some having these available for reflection in the session, the lecturer produced an example of an HLT based on decimal fractions so that all teachers had some object to reflect on. In other words, she provided a model or demonstration of an HLT and related learner productions. Hence the coding of the legitimating appeals in relevant events in this course being described in Table 1 as either teachers' own experience, or a demonstration/assertion

[7] In all three courses, there were sessions were lecturers commented on the importance of the teachers doing the preparation work required.

by the lecturer (authority). The interesting issue here is that the practice that emerges is a function of both the assumptions in the course, and how the teachers respond to demands on them[8].

In the course in Site 3 on Mathematical Reasoning, there were 9 events, over three sessions, with one session entirely on misconceptions. Teachers' experience is the initial resource called in in the introductory session – they were given a task (A learner says that $x^2 + 1$ cannot be zero if x is a real number. Is s/he correct?) and asked to reflect on the kinds of misconceptions their learners were likely to make as they did the task. They were also required to read Smith et al's paper on misconceptions. These together begin to generate a wide field of possible meanings. As the session progresses, the notion of misconceptions is evaluated by appeals to research in mathematics education (classification of misconception types, empirical and theoretical arguments), mathematics itself (complex numbers, justification as testing single cases, justification as generalized argument), curriculum levels (at which complex numbers can be engaged), and records of teaching (a videotape of another teacher working with the same task). It is important to remember that this course is a graduate course. Teachers are thus expected to engage teaching and mathematics (indeed are apprenticed into) discursively. It is nevertheless interesting that it is in this course too where advanced mathematical work is drawn on in the production of MfT in relation to school learners' work.

DISCUSSION

As a study set up to explore the (co)production of mathematics and teaching, we expected legitimating appeals to shift between these two domains. We were surprised, however, at the spread of appeal domains both in relation to mathematics, and to teaching. Across the three courses, appeals included mathematics as would be expected. We were interested to see how this was constrained in pedagogical practice when teaching was being modelled. Mathematics here was then restricted to the levels at which learners would be learning. And there were expected appeals to mathematics education as a disciplinary field, though in effect, in only one of the courses. Ideas about misconceptions in the other two remained at the level of examples provided in the course notes or by the lecturer, and recognized by teachers from their own experience. It is also of interest, that in relation to learners' thinking, there was only one instance of an appeal to curriculum knowledge. This was in Site 3 where learners' responses to the task were considered relative to curriculum levels.

As emphasized at the beginning of this paper, our concern here is neither to compare nor judge of the mathematical and teaching practices in these three courses. It is rather to understand how and why they work as they do. Space limitations prohibit

[8] We note here that, as the course progresses, the lecturer is increasingly aware of the difficulties in the approach, and adaptations needed for the teachers to progress with their action research.

further discussion here. In the presentation of this work, we will reflect further on the questions that arise from our progress so far.

References

Adler, J. & Davis, Z. (2003). An analysis of the structuring of evaluative tasks: A focus on level 6 mathematics INSET. In Jaffer, S. and Burgess, L. (Eds.) *Proceedings of the Ninth National Congress of the Association for Mathematics Education of South Africa.* Cape Town: AMESA

Adler, J. & Davis, Z. (2004). Opening another black box: Researching mathematics for teaching in mathematics teacher education. Presentation to AERA panel on teacher education. San Diego. April 2004.

Adler, J., Davis, Z., & Kazima, M. (2005). The re-emergence of subject knowledge for teaching, its significance and an agenda for research. Working paper #6. QUANTUM. University of the Witwatersrand. Johannesburg.

Adler, J., Davis, Z., Kazima, M., Parker, D., & Webb, L. (in press). Competence and performance in mathematics teacher education pedagogies: Implications for equity. Paper submitted to MES Conference. 2005.

Ball, D., Bass, H. & Hill, H. (2004). Knowing and using mathematical knowledge in teaching: Learning what matters. In Buffler, A. and Lausch, R. (Eds.) *Proceedings for the 12th Annual Conference of the South African Association for Research in Mathematics, Science and Technology Education.* Durban. SAARMSTE.

Bernstein, B. (1996). *Pedagogy, Symbolic Control and Identity: Theory, Research, Critique*, London: Taylor & Francis

Brodie, K. (2005). Learner contributions in mathematics classrooms. In *Proceedings of the 13th SAARMSTE Conference.* Unesco Windhoek Office. Windhoek.

Davis, Z., Adler, J., Parker, D. & Long, C. (August, 2003). Elements of the language of description for the production of data. QUANTUM Research Project. Working paper #2. Johannesburg: University of the Witwatersrand.

Hegel, GWF (1969). *Hegel's Science of Logic*, translated by A.V. Miller, New York: Humanity Books

Smith, J.P., DiSessa, A.A. & Roschelle, J. (1993). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *The journal of the learning sciences*, 3,2, 115-163.

Table 1	Site 1	Site 2	Site 3
Course topic	Algebra concepts and methods	Professional practice in mathematics education	Teaching and learning mathematical reasoning
Qualification	Level 6: ACE Advanced Certificate in Education	Level 6: ACE	Level 7: Hons degree in Math Education
Time and texts	7 X 2 hr contact sessions; Course booklet	Distance learning: written materials; bi-weekly w/end sessions; 10 weeks	7 X 3 hr contact sessions; course reader
# students	25	25	30
Comments on teachers	Experienced elementary teachers upgrading initial diploma to degree level, with qualifications to teach through Grade 9	Experienced secondary teachers upgrading from initial 3 year diploma, to degree equivalent qualification.	Experienced secondary teachers extending 4 year qualification to Honours– first level graduate study
Integration of M and T	Mathematics and teach integrated within a course.	Math and math ed courses separated, with maths courses taught in the Maths Department	M and T courses Separated, each with strong 'eye' on other. Most taught by maths ed staff. Geom taught by tertiary math lecturer
Assumptions and relation to practice	Algebra is focus of course. Algebra is taught to teachers as they would be expected to teach it to Grade 7 – 9 learners. Embedding in practice is thus through *modelling the practice.*	Improving knowledge and practice through systematic *reflection on own teaching* experience. Embedding in practice is hypothetical, assuming teachers can generate problems and related records of practice	Mathematics teaching treated as a discursive practice, that can and should be studied. Embedding in practice is studying research in the field, and records of practice generated from outside of teachers themselves.
Events, appeals and mathematical entailments	10 events identified; appeals mainly to math, restricted, however, to the level of learners. MfT algebra restricted to testing rules with appropriate numerical examples, and so a level of mathematical work that remains at the level of the learners.	4 events where appeals are to teachers' own experience at the start, and in the end to the lecturer modelling /demonstrating a particular instance (generated by the lecturer) of an HLT and related learner work.	9 events where appeals are to Mathematics itself, including advanced mathematics (complex numbers) and justifications; to curriculum (what learners are expected to know at what levels); to research in mathematics education; as well as initially to teachers own experience.

A COMPARISON BETWEEN TEACHERS' AND PUPILS' TENDENCY TO USE A REPRESENTATIVENESS HEURISTIC

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This study builds on a previous research on children's probability conceptions and misconceptions due to the representativeness heuristic. Rasch measurement methodology was used to analyse fresh data collected when a 10-item instrument (described by Afantiti Lamprianou and Williams, 2002, 2003) was administered to a new sample of 754 pupils and 99 teachers. A hierarchy of responses at three levels is confirmed for the teachers' sample, but a hierarchy of two levels is constructed for the pupils' responses. Each level is characterised by the ability to overcome typical 'representativeness' effects, namely 'recency', 'random-similarity', 'base-rate frequency' and 'sample size'. Less experienced teachers had a better performance on the instrument. The educational implications of our findings are discussed.

INTRODUCTION AND BACKGROUND

This paper builds on previous work on pupils' understandings and use of the representativeness heuristic in their probabilistic thinking (Afantiti Lamprianou and Williams, 2002, 2003). One of the aims of the Afantiti Lamprianou and Williams study was to contribute to teaching by developing assessment tools which could help teachers diagnose inappropriate use of the representativeness heuristic and other modes of reasoning based on the representativeness heuristic. The misconceptions based on the representativeness heuristic are some of the most common errors in probability, i.e. pupils tend to estimate the likelihood of an event by taking into account how well it represents its parent population (how similar is the event to the population it represents) and how it appears to have been generated (whether it appears to be a random mixture).

Williams and Ryan (2000) argue that research knowledge about pupils' misconceptions and learning generally needs to be located within the curriculum and associated with relevant teaching strategies if it is to be made useful for teachers. This involves a significant transformation and development of research knowledge into pedagogical content knowledge (Shulman, 1987). Pedagogical Content Knowledge (PCK) "goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (Shulman, 1986, p.9). Pedagogical Content Knowledge also includes the conceptions and preconceptions that students bring with them to the learning. If those preconceptions are misconceptions, teachers need knowledge of the strategies most likely to be fruitful in reorganizing the understanding of learners. Many studies have found that teachers' subject knowledge and pedagogical content knowledge both affect classroom practice and are modified and influenced by practice (Turner-Bisset, 1999).

Along the same lines, Norman (1993) stresses that "there is little in research literature documenting either what teachers know or the nature of their knowledge" (Norman, 1993, p.180). What is more, Hadjidemetriou and Williams (2002) found that some teachers harbour misconceptions themselves (Hadjidemetriou and Williams, 2002). Godino, Canizares and Diaz (n.d.) conclude in their research that very frequently teachers do not have the necessary preparation and training in probability or statistics in order to teach efficiently; they also concluded that student teachers may have various probabilistic misconceptions themselves and this might affect their teaching.

Bearing that in mind, the instrument that was piloted and calibrated to the pupils in our study mentioned above (Afantiti Lamprianou and Williams, 2002, 2003) was now administered to a new sample of pupils and teachers. The administration of this diagnostic instrument to the teachers aimed to investigate (a) whether teachers' probabilistic thinking was affected by the 'representativeness' heuristic and (b) whether teachers were aware of these common misconceptions or of the significance of the representativeness heuristic. This was achieved by asking the teachers not only to answer the items themselves, but also to predict the common errors and misconceptions their pupils would be likely to make on each item, in the manner of Hadjidemetriou and Williams (2002) for a similar instrument assessing graphicacy. Finally, the results of the analyses of the teachers' and pupils' responses are compared.

METHOD

Ten items were used to construct the instrument (reached at http://lamprianou.no-ip.info/pme29/). The items identify four effects of the representativeness heuristic; the *recency* effect, the *random-similarity* effect, the *base-rate frequency* effect and the *sample size* effect. Most of the items have been adopted with slight modifications of these used in previous research (Green, 1982; Kahneman, Slovic and Tversky, 1982; Shaughnessy, 1992; Konold et al, 1993; Batanero, Serrano and Garfield, 1996; Fischbein and Schnarch, 1997; Amir, Linchevski and Shefet, 1999). Other items were developed based on findings of previous research.

The items were divided into three parts. The first part consisted of multiple-choice answers and the respondents were asked to choose an option. In the second part the respondents were asked to give a brief justification for their choice by answering the open-ended question 'Explain why'. Part three was only available in the Teacher version of the instrument and asked teachers to predict which common errors and misconceptions they would expect pupils to make on each question.

Since all items had both a multiple-choice and an open-ended question, a common item Partial Credit analysis (Wright and Stone, 1979; Wright and Masters, 1982) was run. One mark was given for the correct multiple-choice answer and another one for the correct explanation of the open-ended question for each of the ten items.

The calibrated instrument was administered to 754 pupils and 99 teachers from schools in the NW England. For purposes of comparison, the same analysis (i.e. the Rasch analysis described above) was run for the pupils' and the teachers' datasets.

RESULTS FOR THE TEACHERS' SAMPLE

The results of the Partial Credit analysis for the teachers' sample indicated that the data-model fit was appropriate. For example, Item 6 (Random Similarity Effect) had the largest Infit MNSQR (1.16) which is considered to be appropriate for all practical intents and purposes of this study. The item reliability index was 0.95 with a separation index of 4.57. Less than 5% of the respondents had fit statistics indicating poor model-data fit and this is also acceptable for empirical data. The average ability for the teachers was 0.46 (SD=1.01). The ability measures ranged from -3.12 to 2.45 logits. The average raw score was 8.8 (out of 20 maximum possible marks) with a SD of 4.1 but this is difficult to interpret because of the missing responses.

Figure 1 illustrates the ability distribution of the teachers and the difficulty of the items broken down by sub-item (e.g. 3.1 denotes the multiple choice part of item 3 and 3.2 indicates the 'Explain why' part of the same item). According to Figure 1, the test and sample can be interpreted as falling into a hierarchy of three levels. At level 1, approximately -3.0 to -0.5 logits, teachers can succeed on answering correctly questions that tested for the recency effect items (Q1, Q2 and Q3) and also the multiple-choice parts of two Random Similarity Effect items (Q4.1 and Q5.1). At level 2 (approximately from -0.5 to 1 logits), teachers attain higher performance and they can explain their answers to the Random Similarity question 4.2 and also answer correctly the Base Rate Effect questions (Q7 and Q8). Fewer teachers manage to attain level 3 by answering the hardest Random Similarity questions (Q5.2 and Q6) and the Sample Size effect questions (Q9 and Q10).

Overall, the inexperienced teachers were statistically significantly more able than the more experienced teachers in the sense that they had larger average Rasch measures. The largest difference was between the secondary inexperienced and primary experienced teachers. The secondary inexperienced teachers were, on average, at the borderline between Level 2 and Level 3. However, the primary experienced teachers were on the borderline between Level 1 and Level 2.

By averaging the ability estimates of those teachers who made an error, we are able to plot errors on the same logit scale in the table. No teachers gave responses to the multiple-choice parts of questions 1-6 (Recency and Random Similarity effects). Teachers who gave responses indicating the Base Rate (questions Q7 and Q8) misconceptions had a rather low ability. Answers indicating misconceptions based on the Sample Size effect (questions Q9 and Q10) were given by a more able group of teachers.

The teachers were not very successful in describing the most common errors and misconceptions that their pupils were likely to make (this refers to the third part of the items which asked the teachers to predict the common errors and misconceptions of the pupils on each question). Just above 50% of the teachers mentioned that they expected the answers of their pupils on questions Q1, Q2 and Q3 to be influenced by the negative recency effect (62.2% for Q1, 51.6% for Q2, 58.5% for Q3). Around 85% of the respondents expected their pupils' responses to questions Q7 and Q8 to be influenced by the Base Rate effect (83.8% in Q7 and 86.7% in Q8). Very few respondents, however, acknowledged that their pupils' thinking would be influenced by the Random Similarity effect on questions Q4 (12.1%), Q5 (9.8%) and Q6 (0%). The percentages for the Sample Size effect were a bit larger (18.9% for Q9 and 29.6% for Q10).

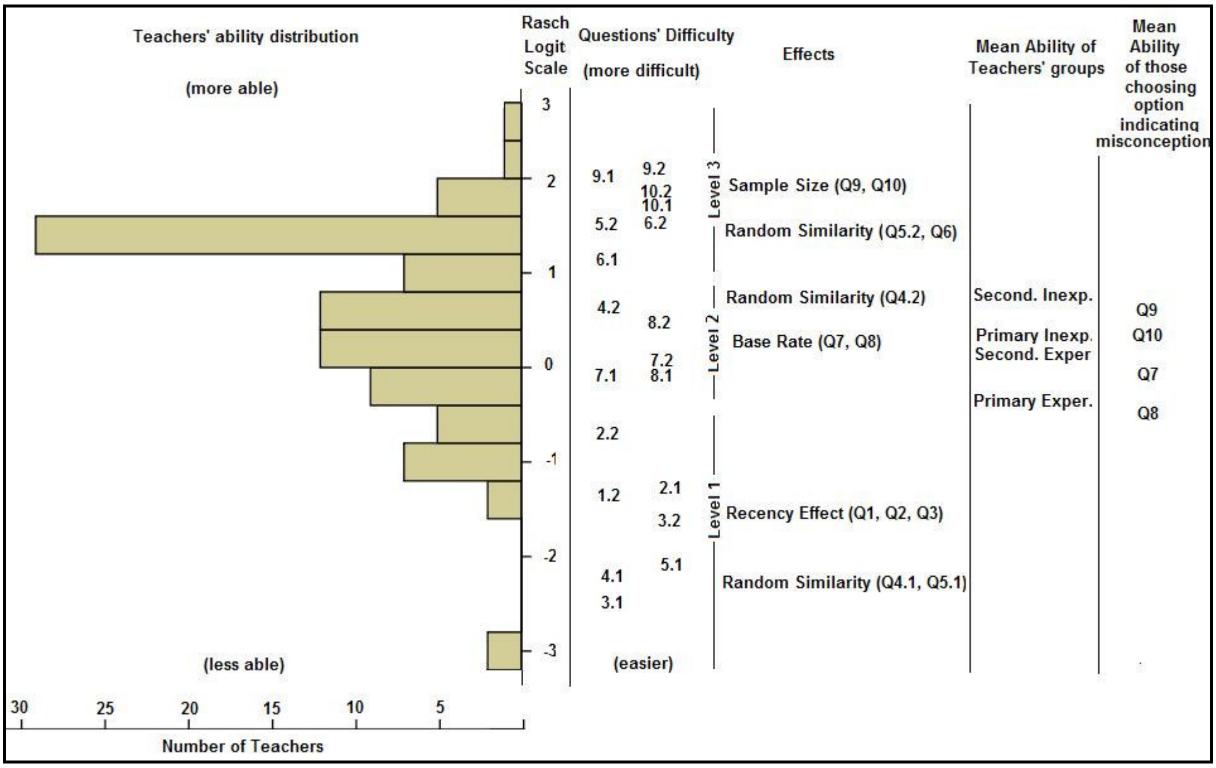

Figure 1: Teachers' ability distribution and item difficulty on the same Rasch scale

When a teacher predicted successfully the common errors and misconceptions of the pupils on a question, he/she was awarded 1 mark. For example, if a teacher predicted successfully the common errors and misconceptions of the pupils on all questions, he/she would receive 10 marks in total (one for each item). However, we could not use the raw score of the teachers across all items as an indicator of their knowledge of pupils' misconceptions because of the large percentage of missing cases. Therefore, we used the Rasch model to convert the raw score of the teachers to a linear scale bypassing the problem of the missing cases. It was found that the 68 inexperienced teachers had an average of 'predictive ability' (to predict the misconceptions of their pupils) of -0.93 logits (SD=1.28). The 31 experienced teachers had an average 'predictive ability' of -0.18 logits (SD=1.40). A t-test showed that the difference was statistically significant (t=-2.643, df=97, p=0.010) and that the experienced teachers were significantly more able to predict the common errors/misconceptions of the pupils.

RESULTS FOR THE PUPILS' SAMPLE

The results of the Partial Credit analysis for the pupils' sample indicated that the data-model fit was appropriate. The fit of Item 6 (Random Similarity Effect) had the largest Infit MNSQR (1.26) which is considered to be appropriate for all practical intents and purposes of this study. All other items had even better Infit MNSQR statistics (between 0.75 and 1.08). The item reliability index was 0.99 with a separation index of 21.65 which is an indication of a very reliable separation of the item difficulties. Just above 5% of the respondents had fit statistics indicating poor model-data fit and this is also acceptable for empirical data. The average ability for the pupils was -0.83 logits (SD=1.12). The ability measures ranged from -3.93 to 3.64 logits. The average raw score was 7.5 (out of 20 maximum possible marks) with a SD of 2.6 but this is difficult to interpret because of the large number of missing responses or not administered items.

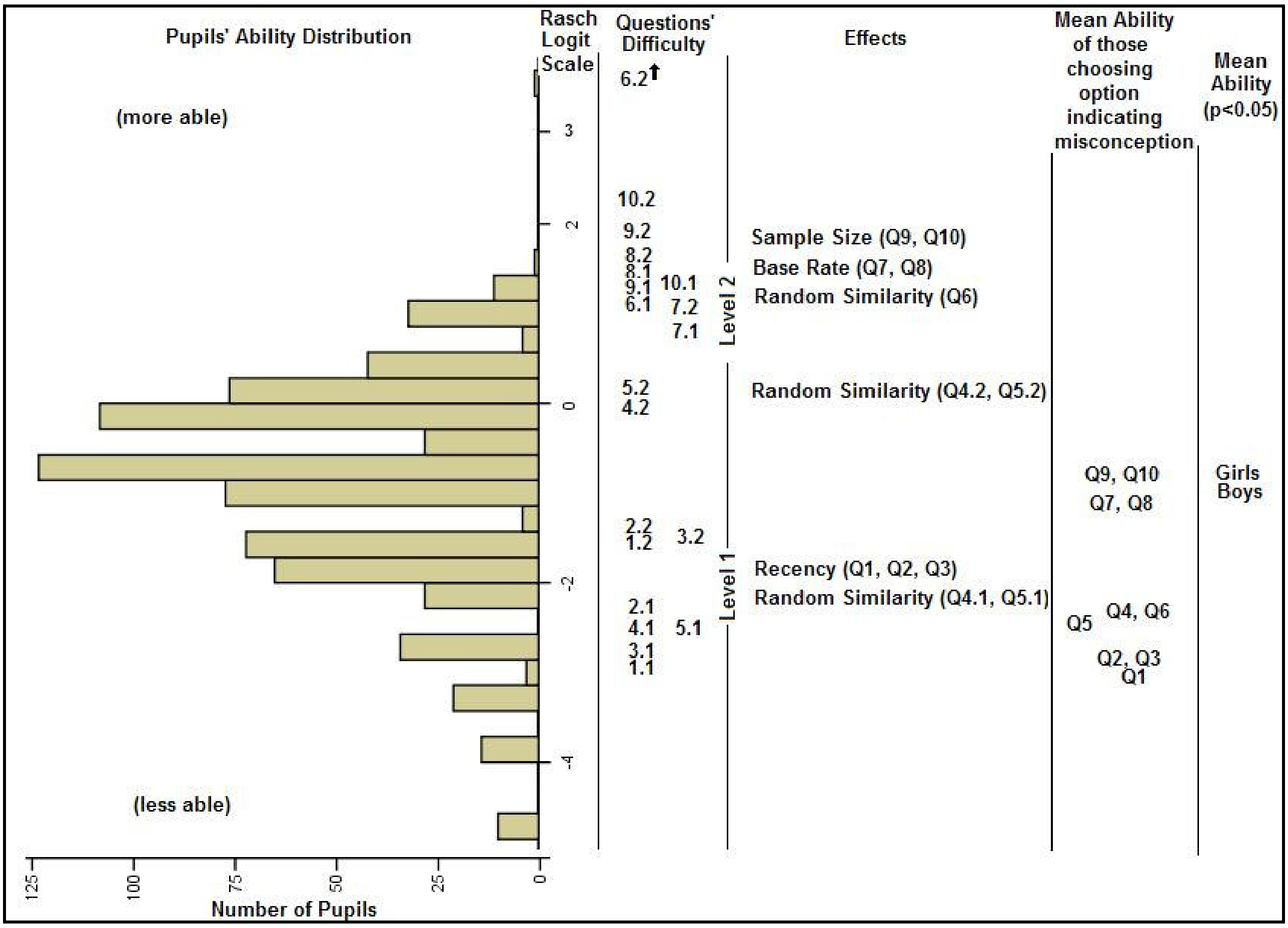

Figure 2: Pupils' ability distribution and item difficulty on the same Rasch scale

According to Figure 2, the test and sample can be interpreted as falling into a hierarchy of two levels. At level 1, approximately -4.0 to -0.5 logits, pupils can succeed on answering correctly questions that tested for the recency effect items (Q1, Q2 and Q3) and also the multiple-choice parts of two Random Similarity Effect items (Q4.1 and Q5.1). At level 2 (approximately from -0.5 to 4 logits), pupils attain higher performance and they can answer the multiple choice of question Q6, explain their answers to the Random Similarity question Q4.2 and Q5.2 and they can also answer correctly the Base Rate Effect questions (Q7 and Q8). Fewer pupils manage to attain the top of level 2 by answering the hardest Sample Size effect questions (Q9 and Q10). Almost nobody managed to give a correct response to question Q6.2.

By averaging the ability estimates of those pupils who made an error, we are able to plot errors on the same logit scale in the figure. Most of the pupils gave responses to the multiple-choice parts of questions 1to 6 (Recency and Random Similarity effects) which indicated that their probabilistic thinking was affected by the representativeness heuristic. The average ability of those pupils for items 1 to 6 was around -2.5 logits (Q1:-3.07 to Q6:-2.36 logits) which is well below the mean ability of the whole sample (-0.83 logits). However, the pupils who gave responses indicating the Base Rate (questions Q7 and Q8) and the Sample Size (questions Q9 and Q10) misconceptions had a mean ability in the area of -1 logit (Q7:-1.10 to Q9:-0.84 logits) which is near the mean ability of the sample.

CONCLUSIONS AND DISCUSSION

Having collected a fresh dataset of responses of pupils and teachers to the instrument which we developed in a previous study (Afantiti Lamprianou and Williams, 2002, 2003), we used Rasch analysis to investigate (a) the degree to which the probabilistic thinking of pupils and teachers suffers from the representativeness heuristic, (b) whether the item hierarchy resulting from the Rasch analysis for pupils and teachers would be similar, and (c) whether the teachers were aware of the common pupils' errors and misconceptions on the items of the instrument.

The analysis of the pupils' data showed that there is a hierarchy of two levels to characterise their probabilistic thinking and this is in agreement with Afantiti Lamprianou and Williams (2002, 2003). Indeed the item hierarchy was found to be the same as the one found by Afantiti Lamprianou and Williams, although the samples were from different schools and were collected two years apart. Pupils' probabilistic thinking was found to be affected by the representativeness heuristic to a great extent in the sense that few pupils managed to reach level 2 (to answer correctly the Base Rate and the Sample Size items). The pupils found the 'Explain why' parts of the Base Rate and the Sample Size items extremely difficult and very few succeeded in answering these correctly.

The analysis of the Teachers' responses showed that the probabilistic thinking of a large number of respondents is influenced by the representativeness heuristic. Few teachers were in a position to answer correctly the most difficult items testing the Sample Size effect (Q9 and Q10).

The item hierarchy resulting from the Rasch analysis of the Teachers' and Pupils' data is not the same. This may be seen by comparing Figures 1 and 2. The rank order of item difficulties does not remain the same when the two figures are compared (although, in absolute numbers, the differences are almost always within the 95% error of measurement). The two hierarchies seem to be qualitatively different in the sense that the Base Rate items were found by the teachers to be substantially easier in comparison to the Sample Size and the Random Similarity items.

One of the most striking findings, however, was the fact that the more experienced teachers were found to have a significantly poorer performance on the instrument compared to the younger and less experienced teachers. One possible explanation could be that the younger and less experienced teachers had the opportunity to receive preparation and training on probabilities and statistics because these topics became more widely available in the relevant teacher training courses in Universities. This finding is in line with the suggestion of Godino, Canizares and Diaz (n.d.) who suggested the need to increase the training opportunities for serving teachers on issues like statistics and probabilities (Godino, Batanero and Roa, 1994; Godino, Canizares and Diaz, n.d.).

This is notably in contrast to the other main result, i.e. that the experienced teachers' pedagogical knowledge was superior (i.e. that the more experienced teachers were in better position to predict the common errors and misconceptions of the pupils): this is in the direction expected, and suggests that the methodology adopted affords the making of nice distinctions between teachers' subject-content and pedagogical-content knowledge. This result reinforces the pilot work in this regard of Hadjidemtetriou and Williams (2004).

References

Afantiti-Lamprianou, T., & Williams, J. (2002). A developmental scale for assessing probabilistic thinking and the tendency to use a representativeness heuristic. In A. D. Cockburn, & E. Nardi (Eds.), *Proc. 26th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 9-16). Norwich, UK: PME.

Afantiti-Lamprianou, T., & Williams, J. (2003). A Scale for Assessing Probabilistic Thinking and the Representativeness Tendency. *Research in Mathematics Education*, 5, 173 - 196.

Amir, G., Linchevski, L., & Shefet, M. (1999). The probabilistic thinking of 11-12 year old children. In O. Zaslavsky (Ed.), *Proc. 23rd Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 25 – 32). Haifa, Israel: *PME.*

Batanero, C., Serrano, L., & Garfield, J. B. (1996). Heuristics and Biases in secondary school students' reasoning about probability. In L. Puig, & A. Gutierrez (Eds.), *Proc. 20th Conf. of the Int. Group for the Psychology of Mathematics Education*. Valencia: *PME.*

Fischbein, E., & Schnarch, D. (1997). The Evolution with age of probabilistic, intuitively based misconceptions. *Journal of Research in Mathematics Education,* 28(1), 96-105.

Godino, J. D., Batanero, C., & Roa, R. (1994). Training Teachers to teach Probability. *Third International Conference on Teaching Statistics.* Retrieved January 2005 from www.stat.auckland.ac.nz.

Godino, J. D., Canizares, M. J., & Diaz, C. (n.d.). *Teaching probability to pre-service primary teachers through simulation.* Retrieved January 2005 from www.stat.auckland.ac.nz.

Green, D. R. (1982). *Probability Concepts in 11-16 Year Old Pupils. Report of Research, CAMET.* Loughborough, England: University of Technology.

Hadjidemetriou, C., & Williams, J. (2002). Teachers' Pedagogical Content Knowledge: Graphs, from a Cognitivist to a Situated Perspective. In A. D. Cockburn, & E. Nardi (Eds.), *Proc. 26th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 57-64). Norwich, UK: *PME.*

Hadjidemetriou, C., & Williams, J. (2004). Using Rasch Models to Reveal Contours of Teachers' Knowledge. *Journal of Applied Measurement,* 5(3), 243 - 257.

Kahneman, D., Slovic, P., & Tversky, A. (Eds.). (1982). *Judgment Under Uncertainty: Heuristics and Biases.* Cambridge: Cambridge University Press.

Konold, C., Pollatsek, A., Well, A., Lohmeier, J., & Lipson, A. (1993). Inconsistencies in Students' reasoning about probability. *Journal for Research in Mathematics Education*, 24(5), 392-414.

Norman, F. A. (1993). Integrating Research on Teacher's Knowledge of Functions and their Graphs. In Romberg, A. T., Fenemma, E., & Carpender, P. T. (Eds.). *Integrating Research on the Graphical Representation of Functions,* Hillsdale, NJ: Erlbaum.

Shaughnessy, J. M. (1992). Research in Probability and Statistics: Reflections and Directions. In D.A. Grouws (Ed.). *Handbook of Research on Mathematics Teaching and Learning*. Reston: NCTM.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform, Harvard. *Educational Review*, 57(1), 1-22.

Turner-Bisset, R. (1999). The Knowledge Bases of the Expert Teacher. *British Educational Research Journal,* 25(1), 39–55.

Williams, J. S., & Ryan, J. T. (2000). National Testing and the improvement of Classroom Teaching: can they coexist? *British Educational Research Journal*, 26(1), 49-73.

Wright, B. D., & Masters, G. N. (1982). *Rating Scale Analysis.* Chicago: MESA Press.

Wright, B. D., & Stone, M. H. (1979). *Best test design*. Chicago: MESA Press.

PURPOSEFUL TASK DESIGN AND THE EMERGENCE OF TRANSPARENCY

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In the Purposeful Algebraic Activity project[1] we have produced a teaching programme of spreadsheet-based tasks, using purpose *and* utility *as the framework for task design. Here we look in detail at the design of one of the tasks, using the notions of visibility and invisibility to examine examples of pupils' activity when working on this task and the role which perceptions of purpose play in the way in which transparency emerges.*

INTRODUCTION

This paper focuses on one aspect of the *Purposeful Algebraic Activity* project. The overall aim of the project has been to study pupils' construction of meaning for algebra in the early part of secondary education. The project takes up the challenge set by Sutherland (1991) to create '*a school algebra culture in which pupils find a need for algebraic symbolism*'. Central to the project is a programme of six tasks, based on the use of spreadsheets. These tasks have been designed to offer purposeful contexts for algebraic activity. In this paper we discuss in detail the design of one task, and use the notion of *transparency* (Lave & Wenger, 1991) to examine potential trajectories through the task, and some specific activity by pupils in response to it.

DESIGNING PURPOSEFUL TASKS

The relative lack of relevance in much of school mathematics, compared to the high levels of engagement with mathematical ideas in out-of-school settings, has been recognised by a number of researchers and curriculum developers. Schliemann (1995) identifies the need for '*school situations that are as challenging and relevant for school children as getting the correct amount of change is for the street seller and his customers*'. However, setting school tasks in the context of 'real world' situations does not provide a simple solution: there is considerable evidence of the problematic nature of pedagogic materials which contextualise mathematics in supposedly real-world settings, but fail to provide purpose (see for example Cooper and Dunne, 2000). Ainley and Pratt (2002) identify the purposeful nature of activity as a key feature which contributes to the challenge and relevance of mathematics in everyday settings, and propose a framework for pedagogic task design in which *purpose for the learner, within the classroom environment* is a key construct.

[1] The Purposeful Algebraic Activity project is funded by the Economic and Social Research Council.

This use of '*purpose'* is quite specifically related to the perceptions of the learner. It may be quite distinct from any objectives identified by the teacher, and does not depend on any apparent connection to a 'real world' context. It may, of course, be true in a trivial sense that learners construct the purpose of any task in ways other than those intended by the teacher. In using purpose as a design principle, we have tried to provide purposeful outcomes through the creation of actual or virtual products, solutions to intriguing questions or explanation and justification of results.

We have also used the notion of the *utility* of mathematical ideas: that is knowing how, when and why such ideas are useful (Ainley and Pratt, 2002). Within a purposeful task, opportunities can be provided for learners to use and learn about particular mathematical ideas in ways that allow them to appreciate their *utility*. In contrast, within much of school mathematics, ideas are learnt in contexts which are divorced from any sense of how or why such mathematical ideas may be useful.

In addition to these two general design principles, we have been concerned to include within the design of our tasks three other features: opportunities to exploit the algebraic potential of the spreadsheet (Ainley, Bills & Wilson, 2004), opportunities for pupils to engage in a balance of *generational*, *transformational* and *meta-level* algebraic activities (Kieran, 1996) and opportunities to build on pupils' fluency with arithmetic to make links to both the spreadsheet notation and standard algebra.

AN EXAMPLE: THE FAIRGROUND GAME TASK

We now describe the design of the sixth and final task in our teaching programme. The task was based on an idea which appears fairly frequently (in the UK at least) in resources for teaching algebra in the early years of secondary school. The example in Figure 1 is taken from the *Framework for Teaching Mathematics* for ages 11-14, which forms the basis of the curriculum which schools in England have to follow (DfES, 2001). It is from the section headed '*Equations, formulae, identities*', for pupils in the first year of secondary school (age 11-12).

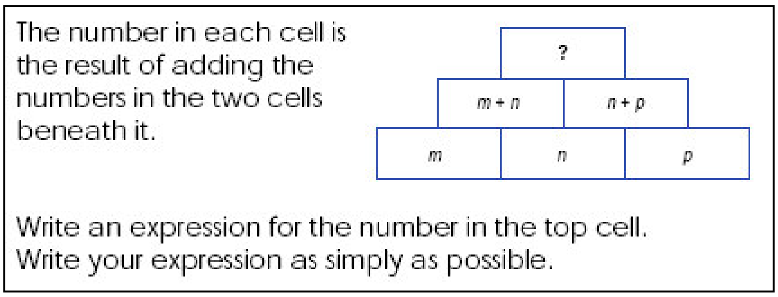

Figure 1: The original example task

The example task given here seems to us to be limited in a number of ways. It is set in a purely algebraic context. Although the text refers to 'numbers' no numbers are given in this example (although further examples based on the same idea appear in the sections for subsequent age groups which ask pupils to find the value of a missing number from a pyramid array). It may be that teachers and pupils would already be familiar with the pyramid array from previous numerical activities, but there is no attempt made in this example to make explicit links to arithmetic experience.

The choice of letters to represent numbers in the array is likely to suggest to some pupils that the numbers in the bottom row are ordered, and indeed consecutive. This task does not give any sense of the letters as variables, representing *any* number, and indeed subsequent tasks based on the pyramid array are concerned with finding the value of particular unknowns.

There is no purpose offered for the task. What is the outcome of adding numbers in this way? And what is the benefit of writing the final expression 'as simply as possible'? For many pupils it would be difficult to see why $m+2n+p$ is a simpler or more usable expression than $m+n+n+p$, because they are offered no context in which the usefulness of simplification might be apparent.

Producing a spreadsheet-based task

Despite these limitations, the pyramid array does seem to offer rich possibilities for algebraic activity, and its cell structure lends itself well to use with a spreadsheet. The spatial arrangement of the cells provides a visual metaphor for the repeating additive structure of the mathematical problem, and thus offers the potential for the array to be *transparent* for users: allowing them to look *at* the visible physical structure so that the content of cells can be manipulated, and to look *through* this (transparent) structure to get a sense of the mathematical structure which underlies it (Lave & Wenger, 1991). However, Meira's study of instructional devices suggests that transparency emerges in the use of tools and symbols, rather than being an inherent characteristic of them (Meira, 1998). Thus the design of tasks may be as significant in the emergence of transparency as the design of tools themselves (Ainley, 2000).

In order to create a task which would offer purposeful activities, we explored the questions which might be asked about the pyramid array, and the challenges which might be set. If the pyramid is used for numerical activities, then one obvious group of questions concerns the effect of changing the numbers used in the bottom row on the subsequent rows, and the final total. Does changing the order of these numbers alter the total? How can the highest or lowest total be achieved from any given set of numbers? Recreating the array on a spreadsheet offers an environment in which it is easy to explore such questions.

In our task we used the structure of the pyramid array as the basis for a game which might be used at a school fair. The game uses a version of the array on a spreadsheet as shown below. The player is given five numbers, which they can enter into the left hand column in any order they like. To win, the player has to make a total (which will appear in the cell on the far right) which is as high as, or higher than, a target set by the stallholder. Pupils are presented with the example shown in Figure 2 on a worksheet, with a description of the game.

	A	B	C	D	E
1	4				
2	2	6			
3	1	3	9		
4	3	4	7	16	
5	5	8	12	19	35

There is a game at the school fair. Players are given five numbers to enter into column A in any order they wish. The stallholder sets a target number. If the number that appears on the right (column E) is the same as or higher than the target number then the player wins!

Figure 2: Extract from the pupils' worksheet

The decisions which we made in transferring the pyramid array to the spreadsheet, and creating the game context, had a number of effects on the potential activity of pupils working on the task.

Because of the 'row and column' structure of the spreadsheet, it was necessary to change the spatial relationship of the cells from that in the original pyramid array[2]. This arrangement may make it less clear which cells were added to produce the next column. In the pupils' worksheet for this task, the whole array of numbers is presented, but how the array is constructed is not made explicit. We chose to rotate the image, partly to make the spreadsheet operations more comfortable, and partly so that pupils would not immediately associate this task with previous experiences they may have had of working with the pyramid array. The array was enlarged to use five starting numbers rather than three to make the challenge more realistic.

The first stage of the task is to recreate the game array on a spreadsheet, and to explore the effects of changing the positions of the numbers in the first column, and in particular to try to make the highest possible total which will become the target number of the game. The next stage of the task concerns what happens when a player wins by making the first target number. The stall holder must then offer a new set of starting numbers, so pupils need to find a method of getting the highest total for any set of five numbers. The final challenge is to find a way for the stallholder to calculate what the target number should be for any set of starting numbers.

TRAJECTORIES THROUGH THIS TASK

We now discuss features of this task and the learning trajectories which we had anticipated in relation to these, and compare these to examples of data from pupils working on this task within our teaching programme. The teaching programme was carried out in five classes in the first year of two secondary schools (i.e. pupils aged 11-12, representing a range of achievement). Four teachers who had been involved in the development of the tasks used them as part of their regular teaching during the year. For each task, pupils' worksheets and detailed teachers' notes were prepared. The teachers were encouraged to introduce the tasks through whole class discussion before pupils began work in pairs, and to bring the class together for further plenary sessions as they felt appropriate. The pupils' worksheets were designed to support the pupils' activity, but not to 'stand alone' in presenting the tasks. The six tasks were

[2] Although a closer approximation to the pyramid structure could have been produced by using alternate cells, this would have added an unnecessary complication.

used as three pairs during the year, with the *Fairground Game* task being the last in the sequence. Classes spent two lessons of about an hour each on each of the tasks. Most of these lessons took place in computer rooms rather than in the normal classrooms. This had the advantage of providing enough access for all pupils to work singly or in pairs at the machines, but had the disadvantages that pupils were generally not very familiar with this environment, and that the layout of the rooms was not well designed for the teacher to be able to circulate and monitor the progress of all pairs of pupils.

During the teaching programme data was collected through fieldnotes and audio recording of the teacher, to give an overall picture of the progress of the lesson, and video and screen recording of one pair of pupils in each class working on each task.

Setting up the game and finding the highest total

In the first stage of the task the intended purpose was to produce a version of the game on the spreadsheet, and then to use this to find how to get the highest total. We anticipated that having to spot the pattern in the array of numbers and generate the formulae to create the array on the spreadsheet would encourage pupils to attend closely to the arithmetic structure of the game. By the time they undertook this task, pupils were reasonably familiar with using the spreadsheet and most could enter formulae confidently. The formulae that are required in the spreadsheet, as shown in Figure 3, make the iterative, column to column structure of the array very clear. However, this view of the spreadsheet was not available to pupils as they worked on the task. The formulae have become (literally) invisible to pupils, and what they see are the numbers in each cell changing.

	A	B	C	D	E
1	1				
2	4	=A1+A2			
3	2	=A2+A3	=B2+B3		
4	3	=A3+A4	=B3+B4	=C3+C4	
5	5	=A4+A5	=B4+B5	=C4+C5	=D4+D5

Figure 3: the completed spreadsheet formulae

Searching for the highest total involves repeatedly changing the values entered in the cells in column A, and seeing the effect of this on the remaining cells in the array. Our intention for pupils' learning was that this would reinforce the notion of the cell reference in a formula representing a variable: any number which may be entered into a particular cell.

Once they had created their own version of the game, most pupils were able to engage with exploring the effects of changing the order of the starting numbers, and many worked systematically to identify a winning strategy. Kayleigh and Christopher, in a low attaining set, did not immediately understand that they needed to produce a spreadsheet made with formulae on which the game could be played. At first they simply reproduced the array they had been shown by typing in the numbers. After an intervention, they were able to put in the formulae, and use their game to explore the effects of changing the order of the starting numbers. However, they seemed to see the purpose at this stage as *getting* the highest total, rather than as *finding how to get* the highest total. Their attention was only on the final total, and they did not see any reason to record how they had used the numbers to get each result. There was no evidence that they were engaging with the notion of variable.

Finding how to get the highest total for any set of numbers

In the next stage of the task the purpose shifts to finding a strategy that will always give the highest total. This not only reinforces the variable nature of the cell reference by increasing the range of possible numbers, but focuses attention on the structure of the array, and how the total is formed. Many pupils had already made a conjecture about a method for placing the numbers to give the highest total, and using a different set of numbers was a way of confirming their ideas.

Pupils' offered a variety explanations for the method they had chosen. In some cases their explanations suggest that the array of numbers on the spreadsheet became a transparent tool which they were able to look through to see features of the underlying arithmetic structure. We conjecture that their experience of entering the formulae supported this as they explored the effects of changing the starting numbers. For example, Hugo, in a middle attaining set, wrote '*you get the highest overall number when the two highest starting numbers are in the middle because they get included in every sum until the overall answer*'. Rupinda, in the same set, said, '*You have to put the largest number in the middle because when you travel through the columns the big number will make a higher total*'. In a high attaining set, a pupil said in a class discussion *' the three middle numbers like carry them on and the other two just get lost somewhere'*.

Kayleigh and Christopher, in a lower attaining set, were initially motivated by a competition to find the highest total with a new set of numbers, but still focussed on the total rather than on a method for getting it. After some further intervention, however, Christopher began to focus on the arrangement of the numbers, and talked about why some gave higher totals in terms of how numbers '*travelled*' across the grid. For him it seemed that opportunities to articulate his exploration were important in allowing him to begin to look through the numbers to gain a sense of the structure and the use of variable inputs.

Explaining the method and calculating the target number

The final stage of the task is designed to introduce purposeful use of standard algebraic notation. The purpose is to give the stallholder a way to quickly calculate the appropriate target number for any new set of starting numbers. Obviously this could be done very easily using the spreadsheet array, or more laboriously by working through the calculations by hand. However, in the context of the Fairground game story the stallholder needs to do this calculation quickly and without his customers seeing the outcome, and so another method is needed. To find such a method, it is necessary to look at the structure of the array in a different way. Using spreadsheet formulae, the cumulative effect of the arithmetic structure is invisible since each formula only refers to the previous column. The teachers' notes for this part of the task suggest that pupils should move away from the computer to find ways of showing why their method will produce the highest total using standard notation.

Many of the higher attaining pupils were able to use letters in place of numbers and work through the array simplifying their answers to give a expression for the total in standard notation. Of these, some made comments in their written work that suggested that they had appreciated the utility of the notation for showing structure. Robin wrote, '*Algebra helped me to find the strategy because it made it easier to see how many letters were used and how often*', and Mandy commented '*Algebra did make it easier because it showed you how the numbers were added up*'.

Other pupils found ways of showing the structure by working through (generic) numerical examples. Amanpreet worked on a paper grid, using the starting numbers 3, 5, 4, 6, 10 (in that order). In each cell he showed the calculation that was to be done, but did not work out any of the results. In the final cell he recorded

$$3+5+5+4+5+4+4+6+5+4+4+6+4+6+6+10$$

He did not feel the need to collect like terms to simplify this result, but was happy that it showed that the number on the middle position (4) was used most, and that 3 and 10 (in the first and last positions) were used least. Other pupils used the grid in similar ways, but showing how to actually get the highest total, and some did simplify their final calculation. Whilst these pupils seemed able to some extent to treat the array as transparent, they had not yet fully appreciated the utility of standard notation to express the generalised structure, or engaged with the purpose of finding a way to calculate the total. In practice the time which teachers allowed for working on this task was not long enough for most pupils to really address this final stage of the task.

Faith also preferred to work with numerical examples to illustrate her general method for finding the highest total for five numbers. However, when challenged to find a strategy for six numbers, she went immediately to algebra, setting out a, b, c, d, e, f in the first column and completing the grid without error to finish with $10c + 10d + a + 5e + 5b + f$ in the final cell. She wrote "***c** and **d** appear most often so that is where the largest numbers should be placed*". The additional challenge of working with six numbers may have helped her to appreciate the utility of using standard notation.

THE ROLE OF PURPOSE

In analysing these examples of pupils' activity we see the role of purpose as significant in shaping the focus of their attention, and thus the ways in which they work with, and look at and through the tools involved, that is, the game array, the spreadsheet formulae and the standard notation. For Kayleigh and Christopher, the challenge of getting the highest total was engaging. Initially the effect of changing the order of the starting numbers on the total was highly visible, but their lack of appreciation of the purpose of the exploration (i.e. finding *how* to get the highest total) prevented them from also attending to the underlying structure of the array. When his attention had been focussed on the purpose of finding a general strategy through the teacher's intervention, Christopher was encouraged to work with the array and articulate his ideas in ways which seemed to make the underlying structure more visible, so that he could describe the numbers 'travelling' through the array.

While some pupils were able to appreciate and articulate the utility of standard notation for clarifying the way in which the total number was calculated, others who were focussing on justifying their strategy for finding the highest total were happy to use generic numerical examples, or generalised descriptions to do this. We conclude that perceptions of the purpose of a task affect the ways in which tools are used within it, and thus the extent to which these tools become transparent for the users.

References

Ainley, J.(2000). Transparency in graphs and graphing tasks: An iterative design process *Journal of Mathematical Behavior*, 19, 365-384

Ainley, J., Bills L. & Wilson, K. (2004). Constructing meanings and utilities within algebraic tasks. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 1-8). Bergen, Norway: PME.

Ainley, J., & Pratt, D. (2002). 'Purpose and Utility in Pedagogic Task Design'. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 17-24) Norwich, UK: PME.

Cooper, C. & Dunne, M. (2000). *Assessing Children's Mathematical Knowledge*. Buckingham: Open University Press.

Department for Education and Skills (DfES). (2001) *Framework for Teaching Mathematics: Years 7, 8 and 9.* Nottingham: DfES Publications.

Kieran, C. (1996). The Changing Face of School Algebra. In C. Alsina, J. M. Alvares, B. Hodgson, C. Laborde & A. Pérez (Eds.), *Proceedings of ICME8: Selected Lectures*. (pp. 271-290) Seville S. A. E. M. 'Thales'.

Lave, J. & Wenger, E. (1991) *Situated Learning: legitimate peripheral participation*. Cambridge: Cambridge University Press.

Meira, L. (1998). Making Sense of Instructional Devices: the emergence of transparency in mathematical activity. *Journal for Research in Mathematics Education* 29(2): 121-142.

Schliemann, A. (1995). Some Concerns about Bringing Everyday Mathematics to Mathematics Education. In L. Meira & D. Carraher (Eds), *Proceedings of the 19th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 45-60). Recife, Brazil,: PME.

Sutherland, R. (1991). Some Unanswered Research Questions on the Teaching and Learning of Algebra. *For the Learning of Mathematics*, 11(3): 40 – 46.

A DEVELOPMENTAL MODEL FOR PROPORTIONAL REASONING IN RATIO COMPARISON TASKS

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The main purpose of this paper is to propose a model that could describe the mode in which people acquire the ability for proportional reasoning. The framework and the analysed data are part of an ongoing research, in which the responses of subjects of different ages and schoolings to different ratio- and rate-comparison tasks are studied. A special accent is placed on the influence of number structure and context upon proportional reasoning; the proposed model is based on a classification of number structure reported in PME-26 and on a classification of contexts in three categories (Rate, Mixture and Probability problems).

This paper reports part of an ongoing research on the strategies used by subjects of different ages and schoolings when faced to different kinds of ratio comparison tasks. In the part conveyed, we are concerned with the following question: Is it possible to describe the way in which the ability for proportional reasoning develops? The results reported and the ensuing proposed model are part of a larger study (Alatorre, 2004).

FRAMEWORK, PROVIDED BY PREVIOUS WORK

In the last three PME's different parts of the research have been put forward. The framework used in the research was presented in Alatorre (2002), an explanation of what are the "different kinds of ratio comparison tasks" as well as a description of the interview protocol used in the experimental part were submitted in Alatorre and Figueras (2003), and in Alatorre and Figueras (2004) the results obtained by six quasi-illiterate adult subjects were described. A succinct summary of these papers will be sketched here; the reader is referred to them for a more complete account.

Among the problems calling for proportional reasoning, those in which the task is a comparison of ratios can be classified according to three issues: context, quantity type, and numerical structure. Figure 1 proposes a classification according to the first one; it blends together the classifications proposed by several authors (Freudenthal, 1983; Tourniaire and Pulos, 1985; Lesh, Post and Behr, 1988; Lamon, 1993).

Rate problems: couples of expositions		*(Two quantities)*
Part-part-whole problems: couples of compositions	*Mixture*	*(One quantity)*
	Probability	*(One quantity)*
Geometrical problems: couples of Σ-constructs		*(Two quantities)*

Figure 1: Taxonomy of ratio comparison tasks according to context

Examples of the first three kinds are shown in Figures 2 to 4 (geometrical problems are not dealt with in this research).

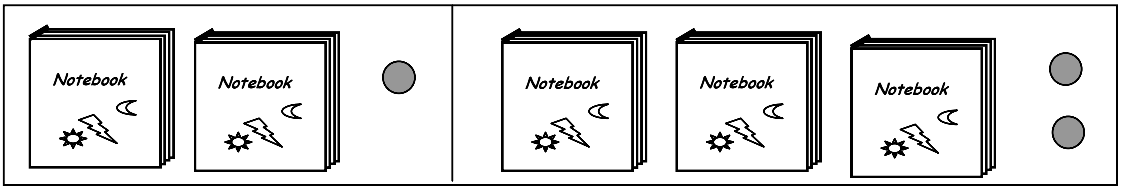


Figure 2. Example of a Rate problem: In which store are the notebooks cheaper? (The round figures stand for coins)

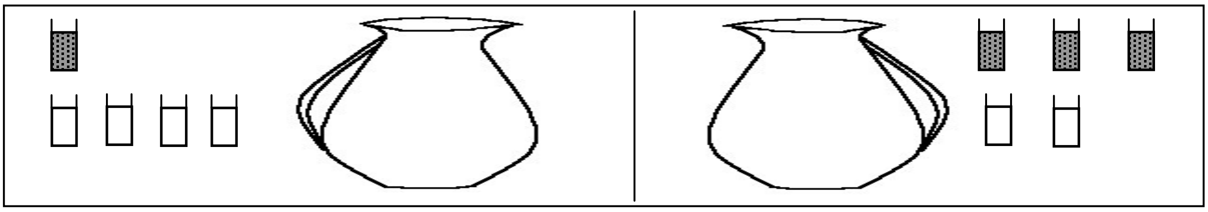


Figure 3. Example of a Mixture problem: In which jar does the mixture taste stronger? (The grey glasses contain concentrate and the clear ones contain water). (Problem taken from Noelting, 1980)

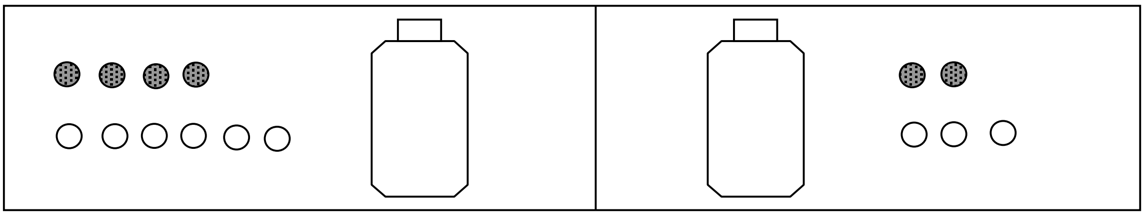


Figure 4. Example of a Probability problem: If bottles are shaken with marbles inside, in which one is a dark marble more likely to come out at the first try?

The second issue is the quantity type. Quantities can be discrete (as the marbles in Figure 4) or continuous (as the amounts of liquids in Figure 3).

The third issue is the numerical structure. In a ratio or rate comparison there is always a foursome: four numbers stemming from two "objects" (1 and 2), in each of which there is an antecedent (e.g. notebooks, concentrate glasses, dark marbles) and a consequent (e.g. coins, water glasses, light marbles). Alatorre's (2002) framework includes a classification of all possible such foursomes in 86 different *situations* that can be grouped in three difficulty levels, labelled L1, L2, and L3; their description will close the section dedicated to the framework.

In the previous paragraphs a description of the classification of ratio-comparison problems was given. Here follows a classification of the strategies used by subjects in their answers to such problems. Alatorre's (2002) framework, as presented in Alatorre and Figueras (2003 and 2004), is to be used. Strategies can be simple or composed; in turn, simple strategies can be centrations or relations. Centrations can be on the totals CT, on the antecedents CA, or on the consequents CC. Relations can be order relations RO (when an order relationship is established among the antecedent and the consequent of each object and the results are compared), or subtractive relations RS (additive strategies), or proportionality relations RP. Composed strategies can be four forms of logical juxtapositions of two strategies.

Strategies may be labelled as correct or incorrect, sometimes depending on the situation (combination and location) in which they are used. The most important correct strategies are:

- RP in all situations (for instance, saying in Figure 2 that in side 1 the notebooks are cheaper because they cost \$0.50, whereas in side 2 they cost \$0.67; or saying in Figure 3 that side 2 has a stronger taste because if three times as much juice was prepared in jar 1 it would need the same three concentrate glasses that are in jar 2, but twelve water glasses, which are more than the two of jar 2; or saying in Figure 4 that in both bottles a dark marble is equally likely, because side 1 is twice as much as side 2, or because in both sides there are three light marbles for every pair of dark ones);
- RO in situations where one of the antecedents equals its consequent, or where one of the antecedents is less than its consequent and the other is more than its consequent (for instance, in Figure 3, saying that jar 2 has a stronger taste because it has more concentrate than water, whereas jar 1 has more water than concentrate);
- In some situations, some composed strategies that can be considered as theorems in action (TA, see e.g., Vergnaud, 1981) (for instance, saying in Figure 3 that jar 2 has a stronger taste because it has more concentrate *and* fewer water glasses than jar 1); there are overall 14 TA’s.

Incorrect strategies are:

- CT in all situations (for instance, saying in Figure 4 that a dark marble is more likely in bottle 1 because it has altogether more marbles than bottle 2);
- CA in most situations (for instance, saying in Figure 2 that side 2 is cheaper than side 1 because it has more notebooks than side 1);
- CC in most situations (for instance, saying in Figure 4 that a dark marble is more likely in bottle 2 because it has fewer light marbles than bottle 1);
- RO in most situations (for instance, saying in Figure 2 that in both sides the notebooks are equally cheap because both have more notebooks than coins);
- RS in all situations (for instance, saying in Figure 4 that a dark marble is more likely in bottle 2 because it only has one more light marble than dark ones, whereas in side 1 there are two more);
- Most composed strategies.

The three difficulty levels mentioned before refer to which correct strategies may be applied. L1 consists of all the situations where, in addition to RP, other correct strategies may be used. In L2 and L3 only RP can be used; the difference among them is that L2 consists of situations of proportionality (both ratios or rates are the same), and L3 consists of situations of non-proportionality. An example of L1 is the array of Figure 3; an example of L2 is Figure 4; and an example of L3 is Figure 2.

METHODOLOGY

A case study was conducted in Mexico City with 23 subjects, aged from 9 to 65 and with schooling from 0 (illiterate adults) to 23 (PhD). Each one was interviewed for a time between 60 and 90 minutes; the sessions were videotaped. Two of the subjects are in fact one, Sofía, who was interviewed twice: when she was aged 10 and 12.

During the interviews, subjects were posed several questions in each of 10 sorts of problems, which were 4 Rate problems (of which the juice problem of Figure 2), 2 Mixture problems (of which the notebook problem of Figure 3), 2 Probability problems (of which the marbles problem of Figure 4), and two forms of partitions problems as controls (one fraction and one pizza problem).

Each of the ten problems was posed in different questions according to numerical structure. Fifteen such questions were designed, five in each of the difficulty levels L1, L2, and L3; all the problems could be posed in each of them. To each subject all of the problems were posed in some of the 15 numerical questions, covering at least a couple of the questions of each level. Each time, the subjects were asked to make a decision (side 1, side 2, or "it is the same") and to justify it.

A total of 2518 answers was thus obtained; 2049 (81%) of them were classified using the strategies system described above, and the rest either consisted of a decision without a justification (9%), or had a justification that was only a description (4%), or consisted of solution mechanisms different from the strategies described before (6%). Two phases of analysis were undertaken: quantitative and qualitative.

QUANTITATIVE ANALYSIS

In order to make a quantitative analysis possible, one point was given to all correct strategies, and 0.5 point was given to answers that could be incomplete expressions of correct theorems in action. Also, 0.5 point was given to all non-classifiable answers that fulfilled the following conditions: correct decision and either no mechanism or a mechanism that could eventually become correct (such as arithmetic or geometric approximations). Then, for each group of answers (e.g., for each subject) a score was obtained, and expressed as a percentage of the answers in that group.

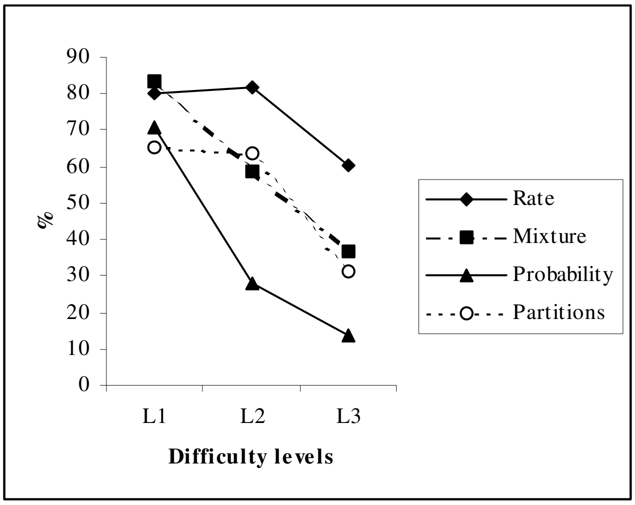
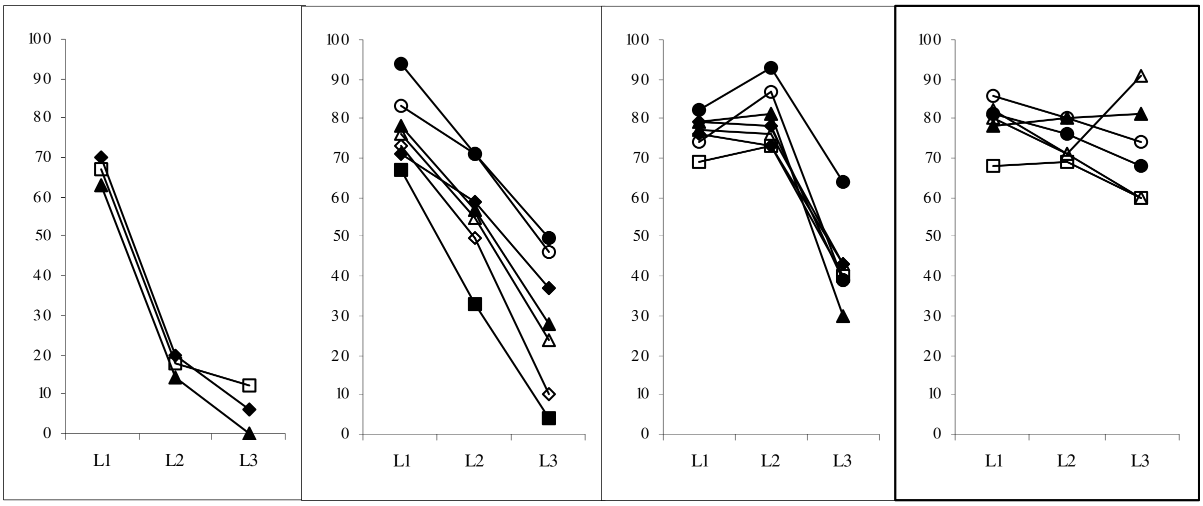


Figure 5. Scores according to context

A first approach consists of verifying that the categories labelled L1, L2, and L3 are indeed difficulty levels. As Figure 5 shows, L3 is, in all of the context types, the most difficult (i.e., the one with lowest scores), and L1 the easiest (highest scores). Except for the Rate and the Partitions (control) problems, L2 has intermediate scores. Figure 5 also allows a comparison of the different context types. The Rate problems are the easiest ones, and the Probability problems are the most difficult ones. Mixture problems are as easy as Rate problems only in level L1, and in levels L2 and L3 lay between Rate and Probability problems.

In a second approach the behaviour of the 23 subjects in the three levels (across all contexts) is studied. The 23 subjects can be classified in four groups, as shown in Figure 6, where the age (child = younger than 15, adult = older than 15) and the schooling of the subjects within each group are also described.



	GROUP A			GROUP B			GROUP C			GROUP D		
	P	S	MS	P	S	MS	P	S	MS	P	S	MS
Child	1	0		1	1		1	2		0	0	
Adult	2	0	0	1	1	3	1	2	1	2	1	3

Figure 6. Four groups of subjects
(P=Primary school or less, S=Secondary school, MS=More than Secondary School)

Group A consists of three subjects in primary school, a child and two adults. For them Level L1 was fairly easy, level L3 was very difficult, and level L2 was almost as difficult as L3. Groups B and C consist of assortments of young and adult subjects of all schooling levels; they all find level L1 rather easy and level L3 rather difficult; the difference between Group B and Group C is that in the former the difficulty of level L2 lies midway between those of L1 and L3, whereas in the latter it is equal or even smaller than that of L1. Finally, group D consists of six adult subjects in the three schooling stages, who had good results in all three levels L1, L2, and L3. Among these subjects a necessary (but not sufficient) condition for belonging to Group A was very little schooling, and a necessary (but not sufficient) condition for belonging to Group D was some age (the youngest of these subjects was aged 16).

An interesting case is that of Sofía, interviewed at ages 10 and 12, because a small longitudinal study can be carried out with her data. Figure 7 shows that when aged 10 Sofía belonged to Group B, and her development over two years took her to Group C. She had an increase in her scores in all three levels, which was small for L1 and much greater in L2 and L3.

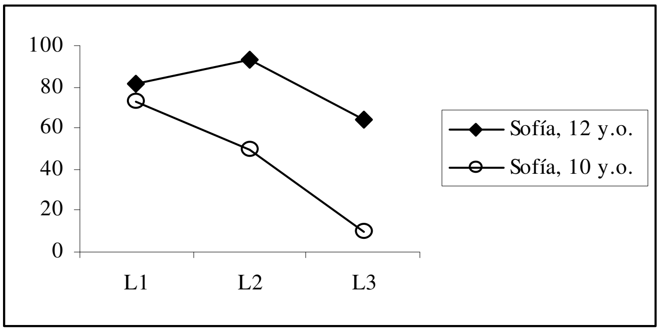


Figure 7. Sofía's results

QUALITATIVE ANALYSIS

The strategies used by the subjects of the four Groups described before differ in some ways.

- In Group A subjects use the correct order relations RO and Theorems in Action (TA) that are applicable in L1, but almost never use the proportionality relations RP. This explains their failure at levels L2 and L3, where they use mainly incorrect centrations.
- Subjects in Group B also use correct RO and TA in L1. They use RP almost only in the proportionality situations of L2, and then again only in some cases of L2 (mainly in Rate problems); they seldom use RP in the non-proportionality situations of L3. The strategies that account for the incorrect answers are mainly centrations.
- In Group C subjects use widely RP in L2, and they even use RP in some L1 questions (although still using correct RO and TA). They still use mostly incorrect strategies in L3, mainly centrations and the additive strategies RS, especially in the most difficult Probability and Mixture problems.
- Subjects of Group D can use RP in all kinds of situations. Some of the subjects go so far as to use exclusively RP, even in L1. The scarce incorrect answers are due to centrations, RS and arithmetically mistaken attempts at RP.

A DEVELOPMENTAL MODEL

The quantitative and qualitative analyses conducted permit the construction of a model that describes how the subjects grow in their ability to respond correctly to ratio comparison tasks. Subjects in groups A, B, C, and D (in that order) have increasingly higher global scores (respectively 40%, 60%, 68%, and 76%); they also use increasingly correct and sophisticated strategies. If one adds the fact that Sofía evolved from Group B to Group C, it can be postulated that within a given context, these groups correspond to stages or moments that occur in that order, as shown in Figure 8.

It may be hypothesized that before responding like Group A, which is the first moment, subjects could go through a moment zero where all three levels are equally difficult (i.e., very young children). Then the first ability to develop is the use of non-proportionality correct strategies RO and TA, which are only useful in level L1 (first moment). After that the ability to use the proportionality relations RP in proportionality situations (L2) would slowly grow, first almost without any change in the non-proportionality situations (second moment), and only when the ability to use RP in L2 equals the ability to use RP or RO or TA in L1 would the ability to use proportionality relations in the non-proportionality situations L3 start to develop (third moment). In the last stage this last ability equals that of the other two levels (fourth moment).

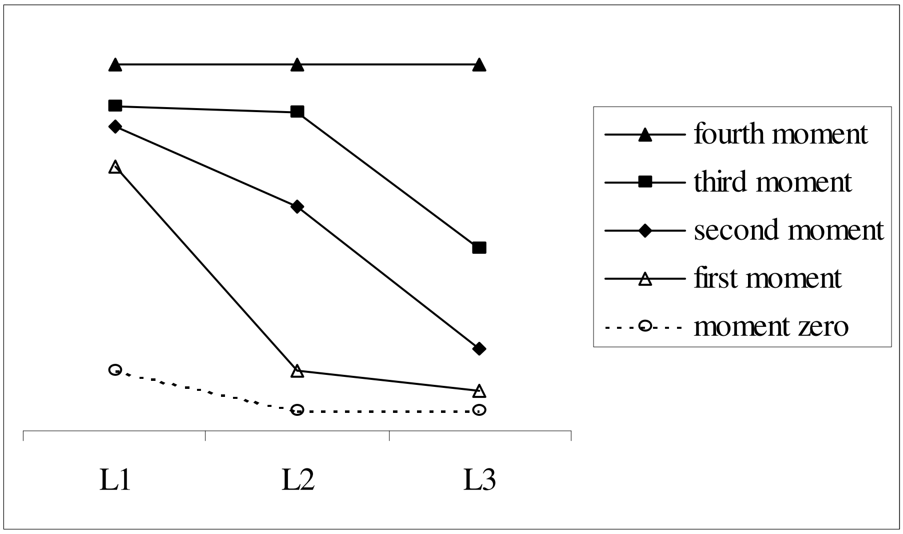


Figure 8. Developmental model

This development, however, is only within a certain kind of context. The whole process would start first with the Rate problems, which are the easiest ones, then with the Mixture problems and finally with the Probability problems, which are the most difficult ones. Thus, at a given instant a person is in different stages or moments regarding his/her response to different kinds of problems. For instance Flor, who is one of the subjects in Group D, is in the first moment in the Probability problems, in the third moment in the Mixture problems and in the fourth moment in the Rate problems (see Figure 9).

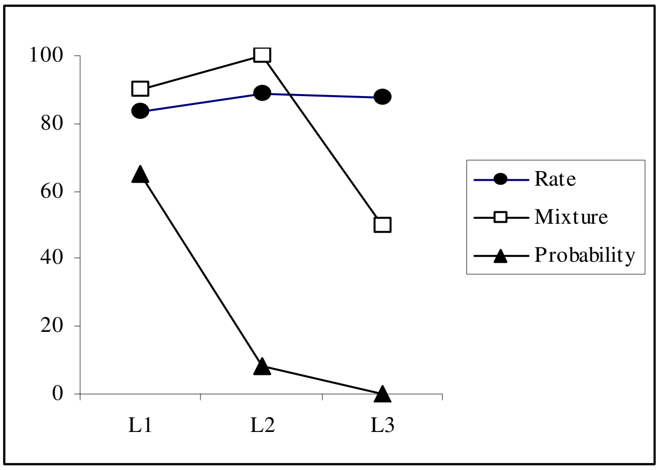


Figure 9. Flor’s results

CONCLUSIONS

It has been said before that it is easier for children to solve ratio comparison tasks in proportionality situations than in non-proportionality ones. However, if one considers that other strategies apart from the proportionality relations can be correct, some non-proportionality situations can be easier (for children as well as for adult subjects) than the proportionality ones. But it is the ability to adequately solve the proportionality situations that can trigger the ability to solve the non-proportionality situations where only the proportionality strategies may be applied.

It has also been said that proportional reasoning is highly context-dependent. This paper has shown that Rate problems are the easiest to solve and Probability problems are the hardest, with Mixture problems between them.

The data obtained from this group of 23 subjects suggest that their ability for proportional reasoning evolves in the form described by the proposed developmental model. It can be conjectured that the model could describe this evolution for other subjects as well. This would in particular entail that only people with very little schooling would be at the first stage of this development, and that only people above a certain age would be at the last stage. In turn, this could imply that neither the school nor life are sufficient conditions for the development of proportional reasoning, but that both can be considered as catalysts for the process.

References

Alatorre, S. (2002). A framework for the study of intuitive answers to ratio-comparison (probability) tasks. In Cockburn & Nardi (Eds), *Proc. 26th of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 33-40). Norwich, GB: PME.

Alatorre, S. (2004). *¿A, B, o da igual? Estudio sobre el razonamiento proporcional* (Tesis doctoral). México: CINVESTAV del IPN.

Alatorre, S. & Figueras, O. (2003). Interview design for ratio comparison tasks. In Pateman, Dougherty, & Zilliox (Eds), *Proc. of the 2003 Joint Meeting of the IGPME (PME27) and PMENA (PMENA25)* (Vol. 2, pp. 17-24). Honolulu, USA: PME.

Alatorre, S. & Figueras, O. (2004). Proportional reasoning of quasi-illiterate adults. In Høines & Fuglestad (Eds.), *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 2-9). Bergen, Norway: PME.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: D. Reidel Publishing Company.

Lamon, S. (1993). Ratio and proportion: Connecting content and children's thinking. *Journal for Research in Mathematics Education, 24(1),* 41-61.

Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In: Hiebert & Behr (Eds), *Number Concepts and Operations in the Middle Grades*. Reston, Virginia: NCTM, Lawrence Erlbaum Associates (pp. 93-118).

Noelting, Gerald (1980). The development of proportional reasoning and the ratio concept. Part I - Differentiation of stages. *Educational Studies in Mathematics, 11,* 217-253.

Tourniaire, F., & Pulos, S. (1985). Proportional reasoning: A review of the literature. *Educational Studies in Mathematics, 16,* 181-204.

Vergnaud, G. (1981). Quelques orientations théoriques et méthodologiques des recherches françaises en didactique des mathématiques. *Proceedings of the PME 5th Conference*, 7-15.

REFERENTIAL AND SYNTACTIC APPROACHES TO PROOF: CASE STUDIES FROM A TRANSITION COURSE

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This paper aims to increase our understanding of different approaches to proving. We present two case studies from an interview-based project in which students were asked to attempt proof-related tasks. The first student consistently took a referential approach, instantiating referents of the mathematical statements and using these to guide his reasoning. The second consistently took a syntactic approach, working with definitions and proof structures without reference to instantiations. Both made good progress on the tasks, but they exhibited different strengths and experienced different difficulties, which we consider in detail.

INTRODUCTION

Writing proofs in advanced mathematics requires the correct use of formal definitions and logical reasoning. However, both mathematicians and mathematics educators have argued that intuitive representations are also necessary for reasoning to be effective (Fischbein, 1982; Thurston, 1994; Weber & Alcock, 2004). This paper highlights the fact that definitions and formal statements can be treated as strings of symbols that may be manipulated according to well-defined rules, or as formal characterizations of meaningful objects and relationships between these, and that either treatment can be the basis for productive reasoning. It is related to the work of Pinto and Tall (1999), who argue that one can extract meaning from a definition by logical deduction, or give meaning to it by refining existing mental images. We say a proof attempt is *referential* if the prover uses (particular or generic) instantiation(s) of the referent object(s) of the statement to guide his or her formal inferences. We will speak of a proof attempt as *syntactic* if it is written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way.

We report two case studies from a project designed to investigate whether students think about the referents of mathematical statements while attempting proofs. In one case the student produces proofs referentially and in the other, syntactically. The specific purposes of examining the case studies are: 1) to show that students in transition-to-proof courses can take two qualitatively different approaches to proof writing, 2) to demonstrate that students taking each approach can be at least somewhat successful in writing proofs, and 3) to highlight what particular difficulties students have when using each approach.

RESEARCH CONTEXT

In this exploratory study, eleven students were interviewed individually at the end of a course entitled “Introduction to Mathematical Reasoning”, the aim of which is to facilitate students' transition from calculation-oriented mathematics to more abstract, proof-based mathematics. It is designed to provide exposure to techniques of mathematical proof, as well as to content on logic, sets, relations, functions, and some elementary group, number and graph theory. The study aimed to 1) investigate the degree to which students at this level tended to instantiate mathematical objects while working on proof-oriented tasks, 2) discern any possible correlation between such a tendency and success at this level, and 3) identify purposes for which students used their instantiations. The participants were asked to complete three tasks, two of which involved producing proofs and one of which involved explaining and illustrating a provided proof. They were then asked to reflect upon their usual practices when trying to produce and read proofs.

This paper will exhibit data from the proof production tasks. These were presented to the students in written form, and are reproduced below.

Relation task

Let D be a set. Define a relation ~ on functions with domain D as follows.

$f \sim g$ if and only if there exists x in D such that $f(x) = g(x)$.

Function task

Definitions: A function f:$\mathbf{R} \rightarrow \mathbf{R}$ is said to be **increasing** if and only if for all $x, y \in \mathbf{R}$, ($x>y$ implies $f(x)>f(y)$). A function f:$\mathbf{R} \rightarrow \mathbf{R}$ is said to **have a global maximum at a real number c** if and only if, for all $x \in \mathbf{R}$($x \neq c$ implies $f(x) < f(c)$.)

Suppose f is an increasing function. Prove that there is no real number c that is a global maximum for f.

The participants were presented with these tasks one at a time on separate sheets of paper, and were asked to describe what they were thinking about as they attempted to answer. They worked without assistance from the interviewer until they either completed the task to their own satisfaction or became stuck. At this point the interviewer asked them about why they had taken specific actions and/or about why they now found it difficult to proceed. These questions focused on the student's choice of actions and conceptions of their own difficulties rather than on conceptual understanding or logical reasoning *per se*.

The interviews were transcribed, and the authors independently identified episodes in which the student used an instantiation and characterized the purpose for which this was used. It became clear that some students took a consistently referential approach, always instantiating in response to a question, and other students took a consistently syntactic approach, almost never instantiating. This was particularly evident in the more successful students. This paper will focus on two students, Brad and Carla, both of whom obtained A's on their midterm examination, and made substantial progress on the tasks in these interviews. Brad instantiated in response to all of the interview tasks. In contrast, Carla never did so. Since they were both articulate in reflecting upon their own strategies, they provide good material for us to see how each approach has distinct advantages and disadvantages. It is worth noting that Brad and Carla had attended the same class and so had been exposed to the same lectures, the same homework assignments etc.

REFERENTIAL APPROACH: BRAD

Response to relation task

Brad read the relation task and, after an initial comment that he was "trying to think what the question's asking", he announced,

> B: Alright, I'm just going to like write out some examples. To try and…like, set a D. And then…yes, write out a function or two. I don't know if that's going to help me.

He wrote the following on his paper:

$D=\{1,3,5\}$ $\quad f(x)=x^2$ $\quad g(x)=x$

He then said,

> B: Would this be an example? Like where f of x is equal to 1, and g of x is equal to 1…and since x is 1, like 1 is in the domain, f is related to g?

He went on to recall that an equivalence relation should be reflexive, symmetric and transitive. In reasoning about reflexivity and symmetry he spoke about f and g as though these stood for general functions, but referred back to his instantiation in which $f(1)=g(1)$ as if to confirm his thinking.

> B: So…so okay if it's reflexive, then…f of x should be equal to f of x. Or there should be x in D with, so that f of x is equal to f of x. Okay. That's all I'm going to say! *Laughs.* And…that's true. Because 1 is equal to 1. Symmetric, is um…x – f implies – f is related to g implies g is related to f. So…so this is really the g, there's an x in D such that f of x is equal to g of x. g is related to x – ah, f, when there's an x in D such that g of x is related to f of x. *Pause*. So…*writing*…implies that g of…*writing*…yes. Because if…because x one, f of x is equal to g of x, then the same x in D that g of x must be equal to f of x.

In reasoning about transitivity, he no longer referred to his instantiation, and made an error based on implicitly assuming that the value of x for which $f(x)=g(x)$ is the same as the value for which $g=h$.

> B: And then transitive. f, g, and g is related to some h, then f is related to h. So f is related to g is…x in D such that f of x is equal to g of x. And g related to h is there's an x in D such that g of x is related to…is equal to h of x. So then…x is in D in both cases. And if x is equal to g of x and g of x is equal to h of x, f of x must be equal to h of x.

The interviewer did not attempt to correct Brad's answer, but instead asked him what role his example had played for him. Brad said,

> B: Um, I guess it just…gives you something concrete […] because this is really general. And you can't really put your hands on this. You know I can't like, get a grasp of it.

It appears important to Brad to feel that he can "grasp" the concepts in the question, and he seemed to achieve this to his own satisfaction. However, he did not maintain explicit links between this example and the general argument, and did not spot his own error in this case.

Response to function task

Brad's response to the function task began in a similar way. He again commented that he was trying to understand the question, and stated:

B: And I'm going to take an example to make sure I'm doing it right.

He wrote the following, along with a small sketch graph of $f(x)=x$:

$f(x)=x$ $x=2$ $f(2)=2$ $y=3$ $f(3)=3$

After overcoming some confusion caused by the fact that the notation was not used in the standard $y=f(x)$ format, Brad suggested a proof tactic.

B: …I think we can do this by contradiction. Assume that…assume that um…if *f* is an increasing function then *c*…ah…then there is…a *c*? For which there is a max. And then prove that that can't happen. And then, so that'll prove it.

He began to work on this idea, but without a very good command of how the variables could be set up to make an argument on this basis.

B: Alright so, if there is…a global max…writing, mumbling…*f* of *c* is greater than both *f* of *x* and *f* of *y*.

After some struggle, he considered a graphical instantiation:

B: I'm just trying to see it by looking at the graph. How I can relate it. Like, the two terms interrelate. Why…because I can't even see – I want to know why, there can't be one […] like know why it can't be and then try to prove.

When the interviewer asked him to talk through his thinking, he said,

B: Alright. I'm thinking that in the definition of increasing, there's never going to be one number that's the greatest. There's always going to be like, a number greater than *x*. Because it's, because it's increasing. So there's always going to be some number greater than the last. So if *x* is greater than – that's what I assumed here. *x* is greater than *y*, then there's going to be some *x* plus 1, that is going to be greater than *y* plus 1, so that *f* of *x* plus 1 is going to be greater than *f* of *y* plus 1. Or something like that. Where like, it's just going to change…. So then, there can't be some number, you know that…if it's increasing there can't be some number that's greater than all of them. Or, some *f* of *c*.

In our view Brad seemed to have a reasonable idea that for any number in the domain, one can always take a greater number, whose image under the function will be greater than that of the original. However, he did not have good control over the way in which the definitions, and in particular the variables x,y and c, could be used to express this argument.

In reflecting upon the function task, the interviewer commented that Brad had spent quite a lot of time thinking at the beginning before writing anything, and asked him what he was thinking in that time. Brad once again indicated that he was using examples to grasp the concepts.

> B: …I didn't, I never heard of a global maximum. I don't think we learned about increasing, but I'm not sure. I don't remember learning about it. So I wanted to teach it to myself first. And, I want to teach myself by examples, you know. And I was kind of starting to understand a bit more when I was trying to, in trying to grasp – I grasped increasing, it seemed like, okay. But then I was trying to grasp the global max.

The interviewer then asked what happened when Brad stopped thinking about examples and wrote "if f is an increasing function". Brad replied,

> B: …it doesn't tell you, proof by induction or proof by contradiction, and so…I'm just trying to think of a way that I *can* prove it. Like, take what's here and then prove it. So then, and then I was just going to write down what, a claim or like what we knew.

Summary

It seems that Brad used examples at the following junctures in his work:

1. To initially understand or grasp the concepts in a given question.
2. To decide on a type of proof to use.
3. To fall back to for more ideas when stuck.

This referential approach served him reasonably well in these respects, affording him a sense of understanding and an ability to decide how to proceed. What it did not seem to afford him was the ability to use this insight to write a full and correct general argument. He did not seem to use his examples to effectively guide his manipulation of the symbolic notation at the detailed level. In fact, his reflective comments on his proof-writing strategies suggest that he was not trying to do this, relying on his knowledge of standard types of proof to provide this structure:

> B: …I start out by forming an example to, you know, get a strong grasp of what they're asking me. And then, ah, probably play around with like, maybe do a few examples, so I can see what it's – actually maybe how I could prove it, which method of proof I should use. And then once I find a method, proceed from there […] because it seems like in all the different types of proofs we've done, there's always some kind of structure. […] Then you can structure it the way you've normally done it before.

SYNTACTIC APPROACH: CARLA

Response to relation task

Carla responded quickly to the relation task. She listed the properties of an equivalence relation, and went on to draw a conclusion.

> C: Oh…okay. It's transitive, symmetric, and reflexive. *Writing*. So to prove that it's transitive…um…*pause*…if x is in D, f of x is…equal to g of x. f of x is equal to f of x, so f is related to g. So it's reflexive…um…symmetric is…if f is related to g, then…f of x is equal to g of x, so g is related to f as well…so…symmetric. And transitive is…f is related to g, that means f of x is equal to g of x, and g is related to…a I guess…so g of x is equal to a of x. So it's transitive as well. So…yes. It's an equivalence relation.

She made an error similar to Brad's by not giving due consideration to the existential quantifier. The interviewer then asked whether she would write anything else if she were going to hand this in for homework. Carla said yes and elected to provide an answer for symmetric. She wrote:

<u>Symmetric</u> YES if f~g, then $f(x)=g(x)$

if $f(x)=g(x)$, then $g(x)=f(x)$

thus g~f, so if f~g, then g~f thus it is symmetric.

As in Brad's case, Carla did not spot her own error.

Response to function task

Carla's response to the function task began in a similar way, with reading of the question followed by immediate writing.

> C: So…I'm thinking the way to prove this is using contradiction. So, I would start out by assuming…there exists…a c…for which…f of x is less than f of c, when x is not equal to c. Okay. *Pause*. So now I'm trying to use the definition of increasing function to prove that, this cannot be. Um…so there exists a real number for which f of x is less than f of c, for all x…and there's…f…is an increasing function…for…all x…y in R, x greater than y implies f of x greater than f of y. Mm…*pause*…I guess what I'm trying to show is if x is in reals, and they are infinite…for all x…there will be…some function f of c greater than f of x. *Long pause*. So…there exists…an element…in R…greater than c. Um…for x…because…f is an increasing function…f of x will be greater than f of c. Um…a contradiction…so that…there is no c for which f of c is greater than f of x…for all x.

Despite successfully producing a proof, she commented that "it seems a bit flaky". When asked why, it seemed she lacked a sense of meaning.

> C: I don't know, it just doesn't make sense for me. It, it feels like, I just, it's just proved systematically, without being able to imagine what's going on. So that's why it feels flaky.

When asked what made her decide to prove by contradiction, Carla answered that she had used the form of statement to decide upon an appropriate proof structure.

> C: Because, in class, whenever we have some statement which says… "there is…no such number", or "there exists no such number", then we assume there is, such number. And then we go on to prove that that would cause a contradiction, thus, it doesn't exist. So it was just, something…automatically ingrained, when I see those couple of words, I think contradiction.

She found it rather difficult to describe how she moved from this point toward finding a link between what was given and what should be proven. What is interesting is that she was *not* referring to instantiations as she did this, as revealed by her later definite negative answer to a leading question from the interviewer.

I: Did you have any sort of picture in your head for this one?

C: No, no…not really. I mean I know what a global maximum is from calculus…I mean I've done these sort of things so many times. But I didn't imagine any, any sort of function. Something that would have a maximum. […] Really…I guess I did it very systematically and theoretically, because I just stepped – this is the rule, and do it through.

Summary

Overall it seems that Carla takes a syntactic approach to proving, beginning by writing down assumptions and using knowledge about standard forms of words to decide upon a structure for the proof. This is confirmed by her later reflective comments. When asked about any general strategies she had for writing proofs, she said,

C: Um, I just start with a claim…I usually don't have anything in my head beforehand. I start off with what I know, and then I assume, what they're talking about, that I should use, in that case. And then I just try to work off of there. And I try to imagine what my goal is, and kind of work from both sides, to the center.

When asked more specifically about the first things she does, she stated that she "thinks of a method to use" and went on to explain how she identifies an appropriate one:

C: If it's something that has to be proven for all…numbers in such a set, then I use induction. And…for instance, if uniqueness is supposed to be proven, I always assume there's two different numbers that produce the same result. Or something to that extent. And use contradiction. Or, for there exists no number such that, I say yes, assume there is and then use contradiction.

This basic strategy still stands when she does not immediately know which technique to use.

C: I would try out just different ones and see which one gets me the farthest. […] We don't really know many methods, so it's not that difficult, to get one right.

This last comment indicates that this syntactic approach affords Carla the ability to answer most of the questions she encounters in this transition course. What it does not appear to afford her is a sense of meaningful understanding of her answers, unlike that which Brad appears to obtain by reference to examples. Indeed, Carla expressed a discomfort with the use examples in proving, both as counterexamples and as a basis for constructing general arguments (the latter at least in graph theory).

C: I could never grasp the, just concept of giving a simple counterexample, any old thing. And those were usually the easiest problems on the exam. And I would always get zeros on them. Because I tried to disprove it in a general manner. And, I guess I'm just not, I don't trust examples, but…

C: …even if I have convinced myself that that proof would be true, and it would happen in certain examples, it wouldn't help me in writing out the proof itself. Because it has to hold for all graphs, and…I don't know how to explain it. I have trouble…generalizing graphs.

It is not clear whether she has over-adopted the maxim "you can't prove by example" or is simply unable to generate a proof based on examining an example.

DISCUSSION

Compared with the majority of the interview participants, both Brad and Carla were doing well in the class, and made good progress on the interview tasks. However, they worked differently: Brad took a consistently referential approach, and Carla a consistently syntactic approach. The referential approach afforded Brad a strong sense of meaningful understanding and a way to decide on an appropriate proof framework, but left him sometimes lacking an ability to coordinate the details of a general argument. The syntactic approach afforded Carla a systematic way of beginning a proof attempt and deciding on an appropriate proof framework, and pursuing this at the detailed level. However, it left her sometimes lacking a sense of meaning as well as confidence in situations in which examples could be useful.

We suggest that these different approaches deserve attention if we wish to help similar students build on their strengths. It would probably be more productive to help Brad describe his examples formally than to reject the examples in favor of a rigid approach to formal work; likewise, to allow Carla to keep using her syntactic strategy as a first approach, but to increasingly recognize situations in which examining examples can be useful. However, we also note that both students seem to have an underdeveloped notion of how to use examples and syntax *together* to construct a proof. Hence we suggest that those taking either approach could benefit from instruction that emphasizes the detail of links between formal statements and proofs and their referent objects and relationships.

References

Fischbein, E., (1982). "Intuition and proof", *For the Learning of Mathematics*, 3(2), 9-24.

Pinto, M. and Tall, D. (1999). Student constructions of formal theory: Giving and extracting meaning. In O. Zaslavsky (Ed.) *Proceedings of the 23rd Conference for the International Group for the Psychology of Mathematics Education*, 3, 281-288. Haifa, Israel.

Thurston, W.P., (1994). "On proof and progress in mathematics", *Bulletin of the American Mathematical Society*, 30, 161-177.

Weber, K. & Alcock, L.J., (2004). "Semantic and syntactic proof productions", *Educational Studies in Mathematics*, 56(3), 209-234.

TEACHERS' BELIEFS ABOUT STUDENTS´ DEVELOPMENT OF THE PRE-ALGEBRAIC CONCEPT OF EQUATION

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Pre-algebraic content has recently been introduced in primary education. One question that is worth examining is to what extend does the teachers' views of the complexity of algebraic tasks match the actual students' difficulty. In this report we focus on teachers` beliefs about the students` difficulty to handle simple equations. Ninety-three 6th grade students completed a test with 14 tasks, while 50 teachers rated the items according to difficulty. It was found that teachers could only partially predict students understanding and reasoning. Contrary to teachers` perceptions, the students could manage word equations and story problems easier than they could handle tasks represented by pictures and diagrams. This mismatch needs to be addressed to help teachers organize productive learning activities.

Introduction

Beliefs constitute one of the three basic components of the affective domain, the other two being emotions and attitudes (McLeod, 1992). Beliefs might be defined as one's personal views, conceptions and theories (Thompson, 1992). The importance of the construct lies in findings that teachers' behavior is primarily determined by their belief system rather than by their own knowledge. Experience and prior knowledge are also important, but beliefs act as the "driving forces" in shaping the structure and content of their practices in the classroom.

The teachers' beliefs shape the type, content and representation format of the activities used in the classroom. As Hersh (1986) put it, "one's conceptions of what mathematics is affects one's conceptions of how it should be presented" (p.13). As Nathan & Koedinger (2000a) mention, "teachers' beliefs about students' ability and learning greatly influence their instructional practices" (p. 168). More specifically, their previous study of teachers' beliefs has revealed that teachers consider students' ability to be the characteristic, which has the greatest influence on their planning decisions. Furthermore, Borko & Shavelson (1990) have found that teachers generally report that information about students is the most important factor in their instructional planning. Raymond (1997) presented a visual model depicting the relationships between mathematics beliefs and practice. She found a direct relationship between mathematics beliefs and mathematics teaching practice.

Recently, the mathematics education community has given more emphasis on studying the teachers' beliefs on specific aspects of mathematics teaching, while little attention has been paid on studying beliefs about the students' ability on developing specific mathematical content. The *Professional Standards for the Teaching of Mathematics* (NCTM, 1991) proposes that teachers must be more proficient in

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 41-48. Melbourne: PME.

selecting mathematical tasks to engage students' interests and intellect. For successful learning outcomes, it is necessary for mathematics teachers to have strong mastery of mathematics content, mathematics pedagogy and knowledge of children's mathematical thinking. Thus, it is important to study how the teachers' beliefs guide them to take into consideration these variables during instructional decision making.

In this study we examine teachers' beliefs about the ways students' develop the concept of equation in the elementary school. This concept has traditionally been taught at middle and high schools. Elementary school teachers' preparation did not include any training on the teaching of pre-algebraic concepts.

The early development of algebra concepts

During the last decade, there has been an effort internationally, to "algebrafy" the mathematics curriculum from as early as the pre-kindergarten years. That is, to introduce algebra content into the elementary school curriculum. According to the National Council of Mathematics (NCTM, 2000):

> by viewing algebra as a strand in the curriculum from pre-kindergarten on, teachers can help students build a solid foundation of understanding and experience as a preparation for more-sophisticated work in algebra in the middle grades and high school (p. 36).

The question though is "What might mean to suggest that algebra should start that early?" Kieran and Chalouh (1993) "consider as pre-algebraic the area of mathematical learning in which students construct their algebra from their arithmetic" (p. 179).

The difference between arithmetic and algebra is in the way questions and problems are expressed. The position of the unknown quantity in a problem statement determines the type of the equation and the required procedure for its calculation. Therefore, we consider the position of the unknown quantity to have a significant effect on the difficulty level of mathematical problems in an early algebra curriculum. For the purposes of this study, we consider as arithmetic equations those that the unknown quantity is the result (at the end), i.e. 32 + 25 = ȋ and as algebraic equations those that the unknown quantity is at the start, i.e. ȋ + 25 = 57.

Additionally, another factor that influences the difficulty level of mathematical problems is the representation format. Specifically, according to the developmental theories of Piaget, for each new concept studied, students use concrete objects to solve problems, next they use pictures, icons or diagrams and finally they use abstract symbols. This sequence of learning must be used for each new major concept that is introduced to elementary school children.

Furthermore, we anticipate that another factor that influences the difficulty level of mathematical tasks is the number of quantities in a problem situation; the difficulty level increases with the number of quantities. Thus, a problem with two quantities (i.e., the number of beads Mary and John have) is less difficult than a problem with three quantities (i.e., the number of beads Mary, John and Peter have).

This study has focused on teachers' beliefs about a specific topic of the mathematics curriculum for the elementary grades. In particular, we studied the teachers' beliefs on the development of an early algebra concept. Since, MacGregor and Stacey (1999) suggest that one of the five aspects of number knowledge that are essential for algebra learning is understanding equality, this study has given emphasis on the ability of 6^{th} graders to solve arithmetic and algebraic equations in different problem contexts and on the teachers' beliefs about the factors that affect the difficulty level of the equations.

The research questions were: (1) Which factors do teachers' of 5^{th} and 6^{th} grades believe that influence the difficulty level of arithmetic and algebraic equations that 6^{th} graders are expected to solve? (2) How do teachers' beliefs compare to students' responses on different types of arithmetic and algebraic equations?

METHODOLOGY

The student questionnaire was made up of 14 mathematical tasks and students were asked to complete it in 40 minutes. The tasks included were designed according to the factors considered to affect the difficulty level of problems, as mentioned in the previous section. Table 1 refers to the specifications of each type of task used with a sample from each type. The first factor considered was the position of unknown quantity. Problems with the result as unknown quantity are considered arithmetic equations, whereas those with start unknown quantity are considered algebraic equations. The second factor considered was the representation format of the equations. Five formats were used: pictures, diagrams, word equations (verbal equations with no context), story problems (verbal equations with context) and symbolic equations where a geometrical shape represented the unknown quantity. The third factor considered was the number of quantities/variables in the equation. Problems were designed with either two or three known quantities and one unknown.

The teacher questionnaire was made up of the same mathematical tasks that were included in the student questionnaire. Teachers were asked to sequence them by giving a value of 1 to 14 (1 for the easiest and 14 for the most difficult) in order to evaluate the level of difficulty of each task. They were given a week's time to complete the questionnaire at their own time.

The student sample consisted of 93 grade 6 students from two urban schools in Nicosia and the teacher sample consisted of 50 grade 5 and 6 teachers in urban and rural elementary schools in Nicosia district. Their teachers administered the student questionnaires. The teachers read aloud the directions to them, supervised the completion of the questionnaires without giving any additional information, collected them and returned them to the researchers the next day.

The data were analyzed using the statistical package SPSS. The 14 mathematical tasks were ordered according to the percentage of students who successfully answered the problem. The same tasks were also ordered according to the average value of level of difficulty teachers had given in the questionnaire.

RESULTS

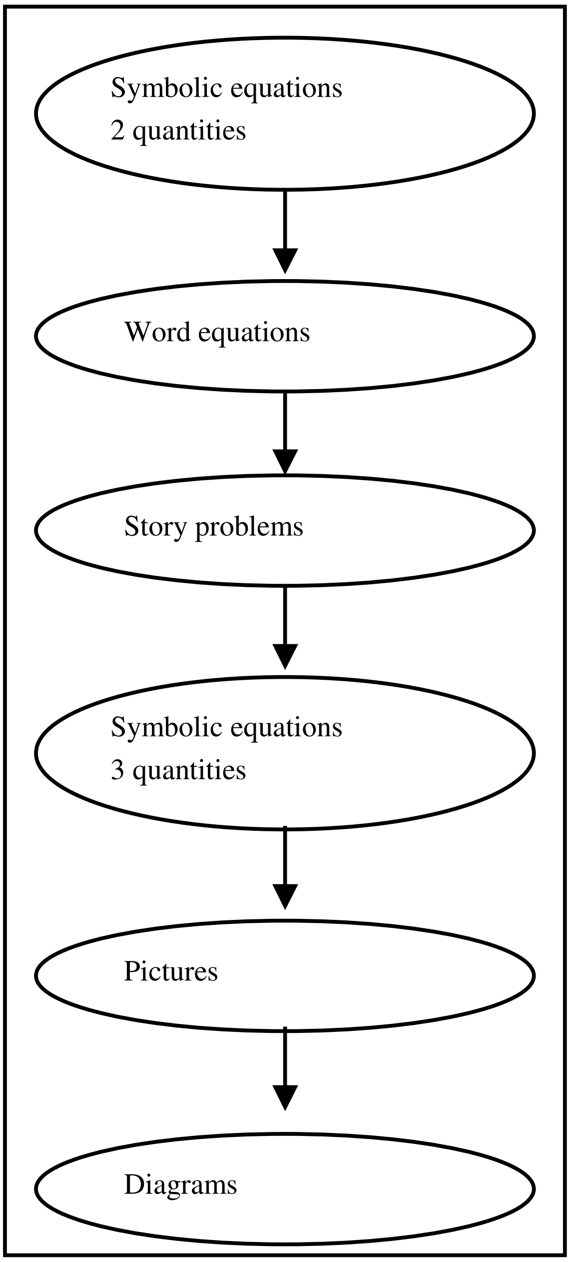
The reliability indices (Cronbach Alpha) for student and teacher questionnaires were 0,67 and 0,83 respectively. Both exploratory factor analyses for each of the student and teacher data confirmed the factors-variables used to design the mathematical tasks included in the two questionnaires.

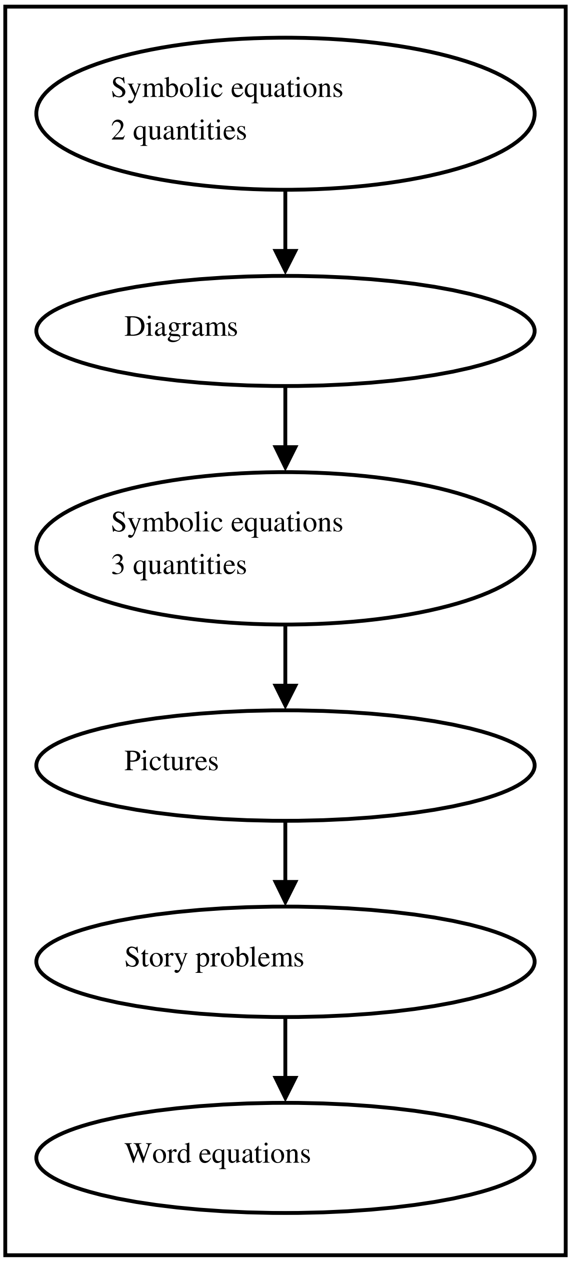
The students' performance in the early algebra problems showed that none of the problems was very difficult for them. The percentages of students' successful responses to the problems were from 98% to 61%. The easiest problems for them were the symbolic equations with 2 quantities, the word equations with 3 quantities, the start unknown story problem with 3 quantities and the result unknown picture with 2 quantities. The problems with medium difficulty for the students were the symbolic equations with 3 quantities, the result unknown story problem with 3 quantities and the start unknown picture with 3 quantities. More difficult tasks were the four result and start unknown diagrams with 2 or 3 quantities.

Overall, teachers' believed that the algebraic equations were more difficult than the arithmetic ones. They systematically ordered problems with result unknown quantity with a smaller value of difficulty level than those with start unknown quantity, for each representation format of the problems. Additionally, they ordered problems with 2 quantities with a smaller value of difficulty level than those with 3 quantities for each representation format. As for the representation format of the problems, teachers believed that the easiest tasks for the students were the symbolic equations. Next, they believed that diagrams with 2 quantities were more difficult, along with the result unknown diagram with 3 quantities, the symbolic equations with 3 quantities and the result unknown word equation and story problem with 3 quantities. Finally, teachers believed that the most difficult problems for 6th graders were the start unknown diagram, picture and word equation with 3 quantities.

Figure 1 presents the way students performed, considering the representation format of the problems, starting from the ones that students found the easiest. They were able to successfully complete symbolic equations with 2 quantities more easily than word equations. Those were easier than story problems and next were the symbolic equations with 3 quantities. Students faced difficulties solving the equations, which were presented pictorially and diagrammatically.

Figure 2 presents the way teachers ordered the problems according to how difficult they believed they were, starting from the easiest ones. They believed that symbolic equations were the easiest tasks for the students. Next were the diagrams and the symbolic equations with 3 quantities and the pictures. Teachers believed that the most difficult tasks were those represented verbally, either in a word equation format or in a story problem format.





Comparing the above two figures, one can understand that there is an agreement on the level of difficulty of the symbolic equations with 2 and 3 quantities, whether they are arithmetic or algebraic ones. Next though one can notice a disagreement between the students' performance and the teachers' beliefs on the level of difficulty for diagrams, word equations and story problems. Although teachers believed that diagrams and pictures were easier than story problems and word equations, the students' performance manifested the opposite direction. Word equations and story problems were less difficult for them than pictures and diagrams. This finding shows that students were able to respond in a better way to equations at the pre-algebraic level, which were represented verbally than pictorially.

CONCLUSIONS

Teachers' beliefs about the difficulty level of early algebra problems indicate their conceptions on the ways that their students are able to respond to them. Teacher decision making about planning and structuring the content of their teaching is greatly influenced by their beliefs on the difficulty of the activities they include in the classroom. The tasks need to be according to the cognitive developmental stage of the students. For these reasons, the accuracy and relevance of teachers' beliefs on a specific topic of the curriculum influence the ways of teaching and, consequently, the learning outcomes.

The results showed that 5th and 6th grade teachers were able to correctly predict the level of difficulty between arithmetic and algebraic equations in different representation formats. This finding is in agreement with previous research outcomes (Carpenter, Fennena & Franke, 1994; De Corte, Greer & Verschaffel, 1996) that problems with result unknown quantities are easier than problems with start unknown quantities.

On the contrary, teachers' beliefs have been found to be discrepant from the students' performance about the level of difficulty of differently represented equations. The representation format is a very important factor to consider when selecting tasks and activities for teaching concepts. As the data showed, students were able to perform better at verbal problems overall, whereas teachers believed that these tasks were harder than pictorial ones. This finding is in line with Nathan & Koedinger (2000a) who mention, "these differences have a significant role on how teachers perceive students' reasoning and learning" (p. 184). Consequently, when tasks are not in accordance with the cognitive level of the students, they are not able to respond successfully to the requirements of the lessons. This may affect their interest, participation, performance and attitude toward mathematics and their mathematical ability.

Teachers' beliefs have been found to follow the ways that this particular mathematical content is presented in the textbooks used in Cypriot schools today. This finding verifies what Nathan and Koedinger (2000b) had concluded. Consequently, it seems essential to include tasks and activities in the mathematics textbooks that are represented in pictorial, diagrammatical and verbal formats. Thus, students will be able to develop the concept of algebraic equation in a natural way as early as the elementary school, in such a way that will help them extend their knowledge later on to the symbolic formats required for further algebra study.

References

Borko, H. & Shavelson, R. (1990). Teacher decision making. In B. F. Jones & L. Idol (Eds.), *Dimensions of thinking and cognitive instruction* (pp. 311–346). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Carpenter, T. P., Fennema, & Franke, M.L. (1994). *Cognitively Guided Instruction: Children's thinking about whole numbers.* Madison: Wisconsin Center for Education Research.

De Corte, E., Greer, B., & Verschaffel, L. (1996). Mathematics learning and teaching. In D. Berliner & R. Calfee (Eds.), *Handbook of educational psychology* (pp. 491–549). New York: Macmillan.

Hersh, R. (1986). Some proposals for revising the philosophy of mathematics. In T. Tymoczko (Ed.) *New Directions in the philosophy of mathematics* (p. 9-28). Boston: Birkhauser.

Kieran, C., & Chalouh, L. (1993). Prealgebra: The transition from Arithmetic to Algebra. In D. T. Owens, & S. Wagner (Eds.), *Research Ideas for the Classroom, Middle Grades Mathematics* (pp. 179-198). New York : Macmillan Pub. Co.

MacGregor, M., & Stacey, K. (1999). A flying start to Algebra. *Teaching Children Mathematics, 6* (2), 78-85.

McLeod, D.B. (1992). Research on affect in mathematics education: A reconceptualisation. In D.A. Grouws (Ed.), *Handbook of Research on Mathematics Learning and Teaching* (pp.575-576). New York: Macmillan.

Nathan, M. J. & Koedinger, K. R. (2000a). Teachers' and researchers' beliefs about the development of algebraic reasoning. *Journal for Research in Mathematics Education, 31*(2), 168–190.

Nathan, M. J. & Koedinger, K. R. (2000b). An Investigation of Teachers' Beliefs of Students' Algebra Development, *Cognition & Instruction, 18*(2), 209–237

National Council of Teachers of Mathematics. (2001). *Navigating through Algebra in Prekindergarten-Grade 2.* Reston, Va.: The NCTM, Inc.

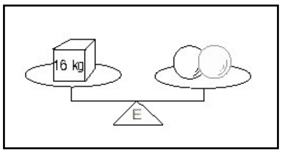
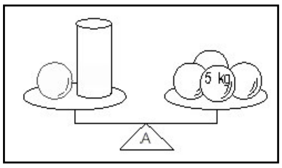
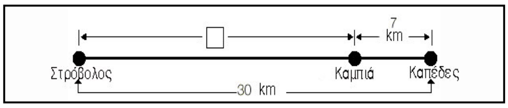
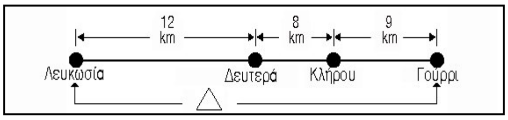
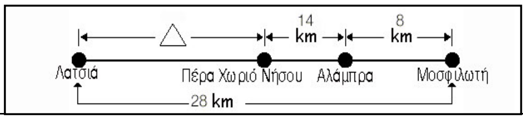
National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics.* Reston, Va.: The NCTM, Inc.

National Council of Teachers of Mathematics. (1991). *Professional Standards for Teaching Mathematics.* Reston, Va.: The NCTM, Inc.

Raymond, A. M. (1997). Inconsistency between a beginning elementary school teacher's mathematics beliefs and teaching practice. *Journal for Research in Mathematics Education, 28*, 550–576.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), *Handbook of research in mathematics teaching and learning* (pp. 390–419). New York: Macmillan.

Appendix 1

Mathematical task	Position of unknown	Representation format	No of quantities
Find the value of . 	Result	Picture	2
Find the value of . 	Start	Picture	3
Find the value of □. 	Result	Diagram	2
Find the value of □. 	Start	Diagram	2
Find the value of Δ. 	Result	Diagram	3
Find the value of Δ. 	Start	Diagram	3
Chris played with his taws. At the beginning, he had 32 taws. At game 1 he lost 12. At game 2 he won 8. How many did he have at the end?	Result	Story problem	3
Steve bought a cheese-pie from the school canteen for 30 cents and an orange juice for 25 cents. He was left with 45 cents in his pocket. How much did he have at the beginning?	Start	Story problem	3
When I multiply 5 by 4 and add 3, what number do I get?	Result	Word equation	3
I think of a number A and multiply it by 3. Next I add 2 and I get 14. What is number A?	Start	Word equation	3
Find the value of □. 24 + 8 = □	Result	Symbolic	2
Find the value of ◊. ◊ + 7 = 30	Start	Symbolic	2
Find the value of Δ. 28 – 16 + 8 = Δ	Result	Symbolic	3
Find the value of ○. ○ + 25 – 12 = 33	Start	Symbolic	3

DEVELOPING STUDENTS' UNDERSTANDING OF THE CONCEPT OF FRACTIONS AS NUMBERS

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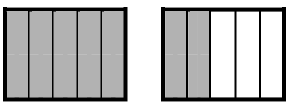
Research has shown that many students have not fully developed an understanding that fractions are numbers. The purpose of this study was to investigate the effects on the understanding of fractions as an extension to the number system of a teaching programme focusing on mixed numbers. Significant differences were found in favour of the programme with greater emphasis on mixed numbers. The study suggests that a programme involving multiple representations for mixed numbers may help students realise that fractions are numbers.

INTRODUCTION

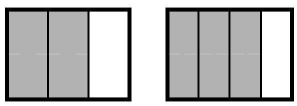
While students may have some facility with fractions, many of them appear not to have fully developed an understanding that fractions are numbers (e.g., Kerslake, 1986, Domoney, 2002 and Hannula, 2003). Kerslake (1986) emphasises the need for students to understand fractions at least as an extension of the number system. Her report presents some of the difficulties 12 to 14 year old students have in connection with fractions. The suggestion is made that many of those difficulties occur because students see fractions as only parts of a shape or quantity and not as numbers. The part-whole model was the only interpretation familiar to all students who took part in her study. Kerslake thinks that the problem starts in primary school when fractions are first introduced merely as parts of geometric pictures. She argues that school practice does not give enough hints to students that fractions are numbers. The work with graphs, algebraic equations and number patterns usually involves only integers.

Research has also shown that students have difficulties in identifying the unit in part-whole diagrams showing more than one unit (e.g., Dickson et al., 1984). When a fraction greater than one is represented in a diagram like the one in Figure 1, many students respond 7/10 rather than 7/5. Similar problems arise when separate part-whole diagrams are used to illustrate addition of two proper fractions (Figure 2) or when the total is greater than one unit (Figure 3).



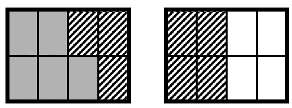
$\frac{7}{10}$

Figure 1



$\frac{2}{3} + \frac{3}{4} = \frac{5}{7}$

Figure 2



$\frac{5}{8} + \frac{7}{8} = \frac{12}{16}$

Figure 3

In the CSMS investigations, Hart (1981) noticed that diagrams sometimes helped in the solution of problems with fractions, or were used to check whether the answer found was feasible. However, the process of interpreting a part-whole diagram often involved: (i) counting the number of pieces which were shaded, (ii) counting the total number of pieces, and (iii) then writing one whole number on top of the other. In the interviews, just after the students answered the fraction shaded in a part whole diagram for 3/5, they were asked to give the fraction not shaded. Hart reports that few subtracted the fraction shaded from one (1 − 3/5) they often used again the counting process just mentioned. It may be here conjectured that those students gave the correct fractions without realising the connection between the fraction 5/5 and the whole number 1. In fact, this counting process of naming a fraction does not require the application of any concept of fractions as parts of a whole. The fraction is interpreted as a pair of whole numbers. Research has also shown that students have difficulties in identifying a proper fraction in a number line showing two units instead of one unit of length (e.g., Kerslake, 1986 and Hannula, 2003). A common misconception is to place the fraction 1/n at (1/n)th of the distance from 0 to 2. So the identification of the unit in number lines seems to be as problematic to some students as in part-whole diagrams.

Although part-whole diagrams are thought to be misleading and a possible inhibitor of the development of other interpretations for fractions (e.g., Kerslake, 1986), Pirie and Kieren (1994) present how 10 year old Katia achieved “a new understanding” (p. 174) of addition of unrelated fractions (halves and thirds) by drawing part-whole diagrams (pizzas) for the fractions and later dividing both into sixths. There is also some agreement that fractions should be introduced as parts of a whole (e.g., English and Halford, 1995). Probably because it is the first aspect of fractions met in a child’s life. So more research needs to be done about how a move from the part-whole aspect to the aspect of fractions as numbers could be achieved (Liebeck, 1985 and Kerslake, 1986). This move was the focus of the present research.

THEORETICAL FRAMEWORK AND RELATED LITERATURE

English and Halford (1995) have developed a psychological theory of mathematics education which combines psychological principles with theories of curriculum development. They discuss the importance of representations and analogical reasoning in helping students construct their mathematical knowledge from prior knowledge. Yet the choice of representation and the actions to be performed upon it can have important consequences for mathematical learning. Some representations can even obscure or distort the concepts they are supposed to help students learn. Certain representations like fictitious stories such as “mating occurs only between fractions, so mixed numbers - 1¾ - become improper fractions - 7/4 ...” may help students remember procedures but do nothing to develop conceptual understanding (Chapin, 1998, p. 611). Some important pedagogical and physical criteria for selecting representations are suggested in the literature (e.g., Skemp, 1986 and English and Halford, 1995), but only the pedagogical versatility criterion will be discussed in this paper.

Skemp (1986) advises teachers to choose versatile representations which can be used to construct long-term schemas. Such schemas are applicable to a great number of mathematical concepts and so make the assimilation of later concepts easier than a short-term schema which will soon require reconstruction. English and Halford (1995) call the criterion of selecting versatile representations “the principle of scope”. They consider the part-whole model to be a representation with scope as it can illustrate many fraction concepts and operations. The idea is to use the same type of representation to communicate several concepts and operations which are related among themselves. It is not just a matter of economy, but of allowing more relationships to become exposed.

Bell et al. (1985) think that some misconceptions may result from new concepts not being strongly connected with the student’s previous concepts. On the other hand, some other misconceptions may result from “the absence of some actually essential detail of the knowledge-scheme which has been overlooked in the design of the teaching material” (p. 2). Therefore, certain misconceptions may also be related to instructional constraints which may result in students’ construction of a schema in a more limited way. Naming improper fractions (Figure 1) or adding the numerators and denominators in addition of fractions (Figures 2 and 3) may be the result of a more limited schema for fractions. The student may see fractions merely as a pair of two whole numbers, one written on top of the other. In order to develop a conceptual knowledge of rational numbers, students should be able to both differentiate and integrate whole numbers and fractions. Yet versatility of a representational model does not imply uni-embodiment. It seems important to use several models for each concept, but two or more related concepts, whenever possible, should be represented together so that their relationship becomes clear. An example which concerns the present study involves using multiple representations to work simultaneously with whole numbers and fractions in order to highlight the relationships between those two sets of numbers.

METHODOLOGY

The purpose of the study was to investigate the effects on the understanding of fractions as an extension to the number system of a teaching sequence for fractions which places emphasis on fractions of the type n/n ($n \neq 0$) and on mixed numbers since from the beginning of the instruction. The study was also concerned with ways of helping students to move from the part-whole aspect to the aspect of fractions as numbers. Each of two teaching sequences was administered to a group of around 60 students of 11 years of age drawn from six schools in England (Amato, 1989). Group X used multiple representations (contexts, concrete materials, pictures and diagrams, spoken languages and written symbols) to represent proper fractions and mixed numbers from the beginning of instruction. Group Y used multiple representations to represent only proper fractions at the beginning of instruction. However, at the end of instruction part-whole diagrams for mixed numbers were also presented.

Some cheap concrete materials which are used to teach place value with whole numbers, like coloured plastic straws, can easily be extended to fractions and decimals through cuts of the unit. For example, the number 135¾ can be represented with straws as in Figure 4. Hundreds, tens and units can be represented together with fractions of those units in both enactive and iconic ways. This type of representation may help students to visualise fractions and decimals as an extension to the right side on a place value system and so as an extension to the number system. The terminology employed in some textbooks does not seem to help students to associate fractions with an extension to the number system. When learning about whole numbers, they read words like units, tens, hundreds, etc. However, when learning about fractions, the word "unit" is substituted by the word "whole". So not many attempts are made to associate fractions with the previously learned numbers by an appropriate use of language.

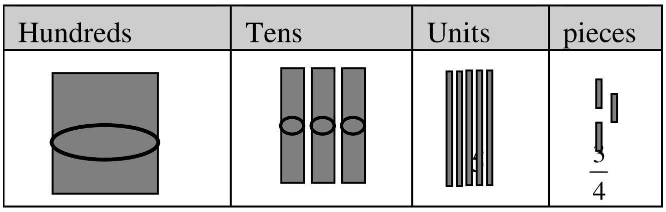


Figure 4

Part-whole diagrams can also be helpful in the development of the concept of fractions as numbers if used in a way that highlights the unit and the connections between fractions and whole numbers. Soon after working with concrete materials and part-whole diagrams for fractions less than one unit (e.g., 1/4, 2/4 and 3/4, Figure 5), diagrams for fractions equal to one unit (e.g., 4/4, Figure 6) and mixed numbers (e.g., 2 units and 3/4, Figure 7) are presented.

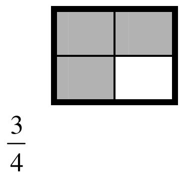


Figure 5

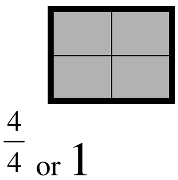


Figure 6

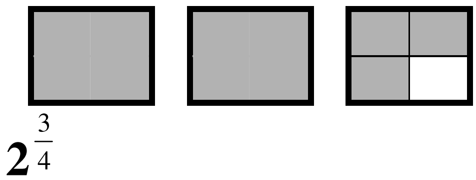
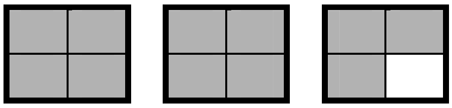


Figure 7

The presence of whole unsliced units in those diagrams may help some students realise that the proper fractions in the mixed number notation are numbers smaller than one. Often mixed numbers are introduced much later in the book or in one of the following books and together with improper fractions. The equivalence between the two notations is usually presented with the help of diagrams where all the "wholes" are cut into equal pieces (Figure 8). This kind of representation does not seem to emphasise the two units as much as when they are not cut (Figure 7).



$\frac{11}{4}$ or $2\frac{3}{4}$

Figure 8

In this study, not only fractions were added in similar manner to that of whole numbers but also the "carrying" process was extended to fractions in a way that reinforces the relation between fractions of the type n/n and the whole number 1. So the study was concerned with ways of helping students to move from the part-whole aspect to the aspect of fractions as numbers and it sought to answer the question: "Does the use of mixed numbers from the beginning of instruction concerning fractions help the development of the concept of fractions as numbers?"

The main activities included in both teaching sequences were:

(1) representing numbers with straws and recording in figures the number being represented with pictures of straws;

(2) counting forward and backwards with fractions: (a) shading diagrams to represent numbers, (b) recording in figures the number being represented with diagrams, and (c) following in figures only a counting number pattern;

(3) using part-whole diagrams for recording the number being represented by the shaded part and the missing number (unshaded part);

(4) using part-whole diagrams to represent three-dimensional divisible units and to help solving sharing problems with whole numbers for both dividend and divisor;

(5) adding fractions: (a) adding fractions in a similar algorithm to the one used for whole numbers (vertical position), and (b) recording resulting fractions of the type n/n as the whole number "1";

(6) multiplying a whole number by a fraction: (a) using part-whole diagrams for changing multiplication into repeated addition and to help combining fractions that together would be equivalent to one unit or a whole, (b) changing multiplication into repeated addition only in figures, and (c) using multiplication tables in a way similar to that which is used when a whole number is being multiplied by another whole number (the sequence of products would form a number pattern); and

(7) Working with number lines associated with the idea of measuring.

The teaching sequences were evaluated by a pre-test, an immediate post-test and a five weeks delayed post-test. The questions on the tests involved the use of fractions in number contexts similar to those in which whole numbers are often used. The questions were extracted from the tests in the projects "Concepts in Secondary Mathematics and Science" (Hart, 1981) and "Strategies and Errors in Secondary Mathematics" (Kerslake, 1986). Covariance analyses were performed on the scores of each post-test, and in both cases, the scores on the pre-test were used as covariate.

SOME RESULTS

The types of activities, the fractions and the quantity of items involving fractions on the worksheets were the same for both groups X (mixed numbers) and Y (no mixed numbers). However, group X spent more time on the worksheets (about 4½ hours) than group Y (about 4 hours). This was expected as group X had at the beginning of instruction three extra worksheets revising place value with whole numbers. Also when group X worked with mixed numbers at the beginning of instruction, they not only had to count pieces and write fractions but also to count units and write whole numbers. A sample of 148 students took the pre-test and started the instructional sequences. Eight of them did not manage to finish 10% of the sequence. On the days when the immediate and delayed post-tests were administered, totals of nine and eleven students were absent respectively. Therefore, the experimental sample was composed of 120 students who had done the three tests and finished 90% of the teaching sequence.

Analysis of covariance with one regression line was used to investigate the effects of using mixed numbers from the beginning of instruction on the acquisition of the concept of fractions as numbers and to allow for initial differences between the experimental groups on the pre-test score. First, it was used to test the operational hypotheses and employed the score on the immediate post-test as the dependent variable. In a second instance, covariance analysis was used for both re-testing the hypotheses and investigating the achievement over time of the two groups. In the latter case the delayed post-test was taken as the dependent variable. The main variable which were thought to relate to the dependent variable in both instances were the pre-test score. The operational hypothesis was tested with differences at the .05 level considered significant.

The majority of students did not perform well on the pre-test. More than 90% of the experimental sample scored less than half of the maximum possible score in the pre-test. It could be noticed that some students had little knowledge about fractions, especially their notation. They could easily talk about halves and quarters but questions like “How do I write one quarter in figures?” were asked in the pre-test and in the initial worksheets. The effect of “Mixed Numbers from the Beginning of Instruction” was significant in both the immediate post-test (scores without covariate adjustment: $F = 13.56$ and $p = .000$ and scores adjusted for pre-test scores: $F = 10.73$ and $p = .001$) and in the delayed post-test (scores without covariate adjustment: $F = 15.01$ and $p = .000$, and scores adjusted for pre-test scores: $F = 12.88$ and $p = .000$).

Student teachers’ understanding of the concept of fractions as numbers has also been found to be limited (Domoney, 2002). More recently, I have been using the idea of focusing on fractions of the type n/n and mixed numbers since the beginning of instruction with student teachers (Amato, 2004a). The idea has proved to be effective in helping them overcome their difficulties in relearning rational numbers conceptually within the short time available in pre-service teacher education (80 hours). I have greatly reduced the number of activities for place value and operations with whole numbers alone. However, through activities involving multiple and versatile representations for concepts and operations with mixed numbers and decimals (e.g., 35¾+26¼ or 24.75-12.53), student teachers have been provided with many opportunities to: (a) revise whole numbers as the representations for mixed numbers and decimals include a whole number part, and (b) make important relationships between rational numbers interpretations and between operations with whole numbers and operations with fractions and decimals. I am also using a similar program to help Brazilian 10 year olds construct rational numbers concepts and the connections among whole numbers, fractions, decimals and percentages.

CONCLUSIONS

Significant differences were found in favour of those students who used mixed numbers from the beginning of instruction. Students' understanding of fractions as an extension to the number system appear to benefit from the use of multiple representations for fractions equal to one unit (n/n) and mixed numbers. It was not difficult to teach the mixed number notation at the beginning of instruction soon after the students had learned the notation for proper fractions. It was interesting to note a student using his fingers to find the solution to "2½ + 2½". He represented 2½ by showing 2 whole fingers and ½ of the third finger. He then covered the other half with his second hand and hide the fourth and fifth fingers behind the palm of his hand and said "2½". After that he showed the 2½ fingers which were hidden and said "plus 2½ makes 5". The process of adding whole numbers with fingers was extended naturally to the addition of mixed numbers with halves. In order to understand fractions as an extension to the number system, students need a variety of experiences with fractions equal to one unit and mixed numbers as well as with numbers between zero and one unit.

Kerlake's suggestion (Kerslake, 1986) that the geometric part-whole interpretation of fractions inhibits the understanding of fractions as numbers and other interpretations of fractions appears to be justified. Part-whole diagrams may be interpreted as a particular way of representing two whole numbers and not as a representation of a single number. The relationship between one whole shape and the whole number 1 may not be recognised by some students. On the other hand, the type of part-whole diagrams used to represent mixed numbers in the activities performed by the students who participated in the present study were seen as beneficial to the understanding of fractions as an extension to the number system. The presence of whole unsliced units in those diagrams may have helped students realise that the proper fractions in the mixed number notation were numbers smaller than 1.

Mixed numbers are often used in everyday life: traffic signs (e.g., 3¼ miles), recipes (e.g., 1½ pints of milk) and ages (e.g., 9½ years). Using decimals in such instances would be more complicated language. To Liebeck (1985) the concept of mixed numbers arises naturally from measuring objects (e.g., 1 metre and 2 tenths of a metre). She thinks that recording a length between 1m and 2m as 1½ m is a strong "hint" that there are numbers between two consecutive whole numbers. Approaches such as "1 + ¼ can be written as 1¼ " and "3½ = 3 + ½ = 6/2 + 1/2 = 7/2" (p. 33) are too formal for the introduction of mixed numbers and improper fractions respectively. Hannula (2003) found that mixed numbers were much easier to locate on a number line than proper fractions. Yet little emphasis appears to be given to mixed numbers. Many textbooks introduce fractions first with pictures of real objects where pieces are missing and then with geometric part-whole diagrams, but normally only fractions "less than one whole" (proper fractions) are presented. Few textbooks work extensively with fractions "equal to one unit" (n/n with $n \neq 0$) and mixed numbers. These fractions may provide the initial link between fractions and whole numbers.

References

Amato, S. (1989). *The Understanding of the Concept of Fractions as Numbers by 11 Year Olds*, Thesis submitted for the degree of Master of Science by Research, Linacre College, University of Oxford.

Amato, S. A. (2004a). Improving Student Teachers' Mathematical Knowledge, *Proceedings of the 10th International Congress on Mathematical Education*, Copenhagen, Denmark, available at http://www.icme-organisers.dk/taA/.

Bell, A., Swan, M., Onslow, B., Pratt, K., Purdy, D. and others (1985). *Diagnostic Teaching: Teaching for Long Term Learning*, Report of ESRC Project, Nottingham: Shell Centre, University of Nottingham.

Chapin, S. (1998). Thinking Deeply about Knowing Mathematics, *The Mathematics Teacher*, 91 (7): 611-614.

Dickson, L., Brown, M. and Gibson, O. (1984). *Children Learning Mathematics*, London: Cassell.

Domoney, B. (2002). Student Teachers' Understanding of Rational Number: Part-whole and Numerical Constructs, in J. Winter and S. Pope (eds.), *Research in Mathematics Education Volume 4, Papers of the British Society for Research into Learning Mathematics (BSRLM)*, 53-67.

English, L. D. & Halford, G.S. (1995). *Mathematics Education Models and Processes*. Mahwah, New Jersey: Laurence Erlbaum.

Hannula, M. S. (2003). Locating Fraction on a Number Line, *Proceedings of the 27th International Conference for the Psychology of Mathematics Education*, 3-17 to 3-24, Honolulu, Hawai'i.

Hart, K. M. (ed.) (1981). *Children's Understanding of Mathematics: 11-16*, London: John Murray.

Kerslake, D. (1986). *Fractions: Children's Strategies and Errors*, London: NFER-Nelson.

Liebeck, P. (1985). Are Fractions Numbers? *Mathematics Teaching*, 111, pp. 32-34.

Pirie, S. and Kieren, T. (1994). Growth in Mathematical Understanding: How Can We Characterise It and How Can We Represent It? *Educational Studies in Mathematics*, 26 (2): 165-190.

Skemp, R. R. (1986). *The Psychology of Learning Mathematics*, Harmondsworth: Penguin Books.

MULTIPLE REPRESENTATIONS IN 8TH GRADE ALGEBRA LESSONS: ARE LEARNERS REALLY GETTING IT?

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The potential benefits to be gained from multiple representations in mathematics education, both where the representations are constructed by learners and where learners use standard representations, have long been recognized. In this paper, qualitative data from 8th grade lessons on linear equations are produced questioning how well this potential, in the case of standard representations, is realized in a real learning environment.

INTRODUCTION

The general case for multiple representations in mathematics education hardly needs defending anymore—most of us have long been persuaded of the central place of multiple representations in problem solving and in the understanding of mathematical ideas (thorough discussions can be found in, for example, Goldin, 2002; Schultz & Waters, 2000; Kaput, 1999; Greeno & Hall, 1997; Janvier, 1987). This paper, therefore, does not aim to adduce further evidence for the importance of multiple representations, nor to challenge it. Rather, we wish to look at the practical question of how ideas about multiple representations are realized in real classrooms. Do teachers succeed in creating learning environments in which they and their students share an understanding of why multiple representations of mathematical ideas and problems ought to be entertained? Are students truly reaping the potential benefits from lessons explicitly designed with multiple representations in mind?

These questions are in fact quite complex for they concern not only students' ability to work with multiple representations as prescribed in documents such as the NCTM *Principles and Standards* (NCTM, 2000), but also their *interpretations* of the meaning and value of what they are doing when they use multiple representations. In the present paper, we can only hope to leave readers with the sense that they ought not be complacent about these practical questions and their complexities, even while they are thoroughly convinced of the correctness of the theory. To this end, based on data from an 8th grade classroom studying systems of linear equations, we shall show that it can happen that in a learning environment where multiple representations have been fully taken into account by a well-informed teacher learners may, nevertheless, fail to grasp the idea of multiple representations and why they are important. Given the allowable length of the paper, we shall give most of our attention to one particular interview, though others could have served as well.

But before we present this data and discuss their significance, we need to circumscribe our treatment of multiple representations. For the word 'representation' itself has multiple meanings in mathematics education, a fact that made discussions

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 57-64. Melbourne: PME.

in the original 1990-1993 PME working sessions on representations at once difficult and rich (see Goldin, 1997). But with respect to classroom practice, it is possible to distinguish two main tendencies concerning multiple representations. One points towards students constructing their own representations, both in pure mathematical contexts and in situations where mathematics is applied to non-mathematical or real-life situations. The other points towards students using or adapting standard representations, particularly, algebraic, graphic, tabular, and verbal representations. Of course, these tendencies are not exclusive. Both tendencies are evident in the NCTM 'representation standard', which stipulates that "Instructional programs from prekindergarten through grade 12 should enable all students to—

- create and use representations to organize, record , and communicate mathematical ideas;
- select, apply, and translate among mathematical representations to solve problems;
- use representations to model and interpret physical, social, and mathematical phenomena" (NCTM, 2000, p.67)

In many classroom situations, especially where standard material is taught, it is the second tendency, that is, towards multiple representations of a standard kind, that naturally dominates (this is true even where the means of presenting the representations are not entirely standard as in Schultz & Waters (2000)). In this case, what the teacher aims towards is chiefly the ability to select, apply, and translate among different representations; this, in turn, demands that learners understand the meaning and value of representations. In this paper, we shall be concerned only with this second tendency.

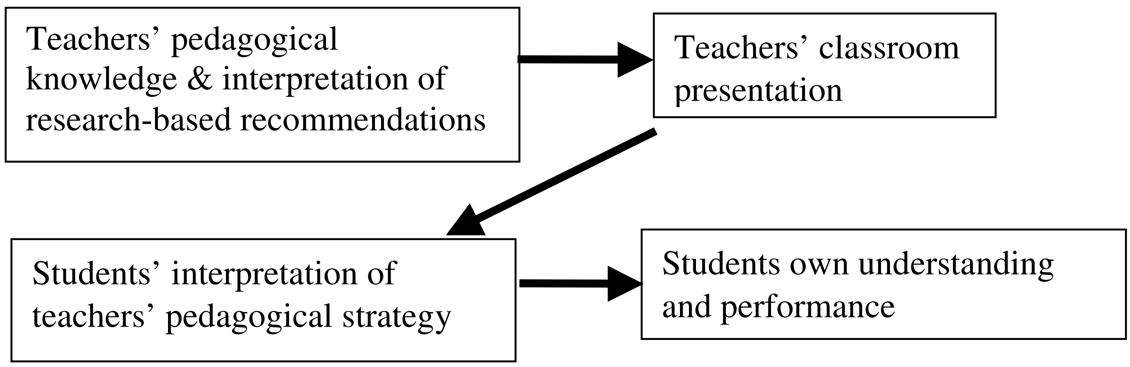
RESEARCH SETTING AND METHODOLOGY

The research setting for the results to be presented here is the *Learners' Perspective Study* (LPS), which is an international effort involving nine countries (Clarke, 1998; Fried & Amit, 2004). The LPS expands on the work done in the TIMSS video study, which exclusively examined teachers and only one lesson per teacher (see Stigler & Hiebert, 1999), by focusing on *student actions* within the context of whole-class mathematics practice and by adopting a methodology whereby student reconstructions and reflections are considered in a substantial number of videotaped mathematics lessons.

As specified in Clark (1998), classroom sessions were videotaped using an integrated system of three video cameras: one viewing the class as a whole, one on the teacher, and one on a "focus group" of two or three students. In general, every lesson over the course of three weeks was videotaped, that is, a period comprising fifteen consecutive lessons. The extended videotaping period allowed every student at one point of another to be a member of a focus group.

The researchers were present in every lesson, took field notes, collected relevant class material, and conducted interviews with each student focus group. Teachers were interviewed once a week. Although a basic set of questions was constructed beforehand, in practice, the interview protocol was kept flexible so that particular classroom events could be pursued. In this respect, our methodology was along the lines of Ginsburg (1997).

This qualitative methodology was chosen in general because the overall goal of LPS is not so much to test hypothesized student practices as it is to discover them in the first place. In this particular instance, however, a qualitative methodology was also necessary because, as we remarked above, our investigation of multiple representations in the classroom involved to a great degree teachers' and students' *interpretations* of the meaning and intent of the classroom activity, as can be seen schematically in the following figure:



The specific case that formed the basis for this paper was a sequence of 15 lessons on systems of linear equations taught by a dedicated and experienced teacher, whom we shall call Danit. Danit teaches in a comprehensive high school whose direction in mathematics education is along the lines of the NCTM standards approach; Danit herself is well-informed about the educational issues involved. The 8th grade class Danit teaches is heterogeneous in ability and multiethnic.

DATA

As mentioned above, Danit is a teacher who, partly through her own interest and partly through the educational framework embodied in the national curriculum, is familiar with new developments in mathematics education. Thus, in constructing her lessons on systems of linear equations, she quite consciously introduces different representations relevant to them. Danit goes back and forth between representations in a way that keeps them always in play and in a way that gives her lessons a flow describable as 'turbulent', albeit carefully controlled turbulence (Fried & Amit, 2004). Her desire that students think about the idea of different representations, that they suggest to the students different approaches to mathematical problems and different ways of conceiving mathematical ideas, that the students do not see them merely "...as though they are ends in themselves" as Greeno & Hall put it (1997, p. 362), is evident in the way Danit makes a transition from the symbolic representation of an equation in two unknowns to a graphical representation. Referring to the equation $x+y=6$ written on the board, she begins as follows:

> [*min 35*]Who is willing to tell me what is written here in Hebrew? I want a translation [with emphasis] into Hebrew, not just “x plus y equals six”!...You’ve seen this [*i.e.* an equation like this] in your book, and you know to do with them [referring to exercises given in the last lesson]—now translate it into Hebrew [*i.e.* into your spoken language].

After some discussion, she finally lets the students know what she is up to:

> [writes: ‘Two numbers whose sum is six’] Find me two numbers whose sum is six. In the language of algebra, we say, ‘x plus y equals six’. [*min. 37*] Today, we’re going to learn to *translate this into another language* [our emphasis]; we’re going to sketch this, that is, what is written here, x+y=6, I don’t have write in the language of algebra, I don’t have to say it in words: I can sketch it.

Thus, besides referring to different kinds of representation, Danit uses words such as ‘language’ and ‘translation’ which refer to the meaning of representation and to moving between representations. She wants the students to know what representations and the act of representing are all about.

In our focus group for that lesson were two boys, Oren and Yuri. By asking them simply “What was the lesson about?” we hoped to find out in the interview whether they grasped Danit’s message as well as her words. Yuri answered “How to solve equations with a number line” [both Yuri and Oren, as well as many of the other students we interviewed, tended to refer to the coordinate system as ‘the number line’—an interesting fact in itself!]. Oren’s answer was somewhat more revealing:

> Oren: We learned [*min 2*] [glances at the whiteboard] about, um, well, equation exercises [*sic*] with two unknowns we started to learn and how to solve them. And, also we learned about the number line and we connect that with equations.

Two observation can be made here. First, Both boys spoke about using the ‘number line’ to *solve* equations. Danit did speak about *solutions* of equations in two unknowns and used yet another representation, a tabular representation, to bring out the pairs of numbers that solve the equation; however, at this point she did not present the graphical representation *as a means* of solving the equation but as a way of seeing the equation in a different light. This tendency was strikingly illustrated in the next lesson, where in the video of the lesson, two girls (that day’s focus group), are seen to continue carrying out only the arithmetical calculations of finding y for a given x, without ever mentioning the graphic representation of the linear relation—and that, just when Danit has been emphasizing aspects of the graphic representation to the whole class!

The second observation is that Oren described the content of the lesson by means of simple conjunctions—this *and* this *and* this *and* this—it is a fragmented account containing the facts of the story but not its theme. Even where he does use the word ‘connect’ (*lechaber*), he is still only reporting factually what Danit has said: indeed, in the previous lesson she said that she would ‘link’ (*lekasher*—which is synonymous to *lechaber*) everything together in the lesson we are looking at now. Here, Oren’s glance at the whiteboard is telling: he has to remind himself what the lesson was about by looking at what *Danit wrote* rather than looking at his own thoughts. We shall return to this point in a moment.

Pushing our original question a little further, we asked what Danit tried to accomplish in her lesson and whether she succeeded in achieving her goals. Oren again emphasized the word ‘connect’, and both Oren and Yuri agreed heartily that Danit did truly achieve her lesson aims:

Interviewer:	What do you think the teacher tried to accomplish in this lesson?
Oren:	She tried to connect [for] us, because before we studied the number line in a separate lesson and equations in the second lesson, so in my opinion she tried to connect [for] us, how the number line is connected to equations.
Yuri:	To equations.
Interviewer:	Do you think she succeeded in her goal?
Yuri:	Yeah, I think so.
Oren:	In my opinion, yes [*min 5*].

They *seem* to have grasped what Danit was trying to do, at least they know the right words to use. But just a couple of minutes later, while discussing one of the exercises they worked on in the class and for which they had asked Danit for help, Yuri described the general procedure, which involved substituting a value for x in the equation, say, x+y=6 (Danit’s example), solving for y, finding the point (x,y) on the ‘number line’, and then repeating the process for another value of x. Yuri describes the procedure in a very disjointed way, and soon afterward, both Yuri and Oren admit that they did *not* understand the point of the lesson:

Yuri:	...Solve it a few times so that the numbers, the unknowns, will be different and afterwards see it on the number line—so it will be a straight line, sort of, that it will be correct—that I didn’t understand—she explained it to me.
Interviewer:	[to Oren] Did you have the same question?
Oren:	Yeah, exactly. I also was a bit mixed up about the teaching, because I understood, but I didn’t understand, it was hard for me to connect with [*sic*] the number line and the equations.
Yuri:	Yeah.
Oren:	The teaching [presumably, “The teaching wasn’t clear to me”].
Interviewer:	[to Yuri] Why didn’t you ask him [Oren]?
Yuri:	Because he asked [Danit] too.
Oren:	I really [with emphasis] didn’t understand either.

It turns out that Oren and Yuri do not truly see the point of the graphic representation. For them, drawing the graphs does not show them equations from a different perspective; for them, drawing the graphs is a redundant exercise. At one point during the video of the lesson, which we watched together with Oren and Yuri, Yuri says ‘Boring!’. We asked what he was referring to. He said drawing the axes. Oren agreed and added, “It takes time.” Asked if the problem was that they had to use a ruler, Yuri expanded and said, “Yeah, drawing the numbers and checking and putting down the points—it’s easy but it takes time. Because of that.” From their point of view, ‘solving’ the equations has weight; drawing the lines was just another task given to them by Danit, a task which could just be easy or hard. This was a typical attitude in Danit’s class. For instance, when on another day we asked Annette and Chanita about why they need the ‘number line’, the ‘axis’, neither could say why. And when we pressed them, and asked why they didn’t ask the teacher, the exchange was as follows:

Interviewer: Chanita, tell me, why didn’t you ask the teacher why? [*min 14*]

Annette: [Answering for Chanita] It wasn’t ( ) interesting.

Interviewer: Sorry?

Annette: Because it’s no so interesting why you need the axis—we just solve, and, that’s it, we go home [everyone chuckles]

Later in the interview, both Chanita and Annette answered that that lesson contained just exercises. And when asked what *they thought* would be in the next lesson, Chanita answered, laughing, that “*She [Danit] said* [our emphasis] in the next lesson we would stop drawing [graphs].” So, like Yuri and Oren, Chanita and Annette see no intrinsic value in pursuing the graphical representation of the linear relations. If Danit decides they should do it or not do it, so be it—but better if she decides not to!

This brings us back to Oren’s glance at the whiteboard to answer what the lesson was about. Although Danit is at pains to make the students *themselves* think about the notion of representation, they take their cues from *her*; her authority is enormous (see Amit & Fried, in press). This could be seen when we asked the students about why the points representing the solutions of a linear equation lie on a straight line. The end of that exchange was as follows:

Interviewer: [Referring to the equation (x-y)/7=(2y-x)/2, which was similar to an equation Danit had written on the board earlier just as an example of an equation in two unknowns] Is it possible, in your opinion, that this won’t be a straight line?

Yuri: I don’t know...to check I need to get [lit. *do*] some results [of calculations] [*min 60*] but I think that it will come out a straight line.

Interviewer: Why?

Yuri: If the results are right then, well, I don’t know exactly, sort of *that’s what the teacher said, so it has to come out a straight line* [our emphasis].

Interviewer: It’s because the teacher said so?

Yuri: I don’t know, no—I can’t explain it—I don’t know.

Interviewer: I see, she said it is a straight line, and you believe her? [the boys laugh]

Oren: Yes.
Yuri: [Sarcastically] No, she's lying.

CONCLUDING DISCUSSION

To summarize, what we see in the case of Yuri and Oren—and, as we suggested in the introduction, they were not atypical—is that despite Danit's conscious attempts to organize her lessons with eye to representations, Yuri and Oren do not appear to understand them as showing different mutually reinforcing views of linear equations; they do not see the line *as* a representation but as a *solution* method, which for them at this stage only means finding the value of y for a given x.

It may be because they expect the graphic representation to be a solution method, rather than a *bona fide* representation, that they think of the graphic representation as redundant. But whatever the reason, in none of our interviews did we find indications that students appreciated the graphic representation as complementary to the algebraic representation of linear relation. On the other hand, they seem to grasp that *Danit* attached importance to the different representations, and, accordingly, they produce statements in line with her approach. These statements, moreover, are sometimes convincing enough to deceive and, therefore, can mask the students' lack of true sympathy with and understanding of what the teacher tries to instil in them.

The division between the teacher's intention of what she was doing and the students' interpretation of what was expected of them (see the figure in the second section) might, then, be one reason why the students in this class did not seem to get the idea that representations are to be selected, applied, and translated. But, of course, this only begs the question. We need to ask *why*, in the first place, there was this gap, why these students seemed able only to give lip service to Danit's emphasis on 'connecting' and 'translating' representations.

One possibility may be the absence of *mediating elements*, that is, not just the presence of different representations said to be connected but 'connectors' as well. Formally, such connectors between representations are isomorphisms, and Powell and Maher (2003) have suggested that students can themselves discover isomorphisms. But in fact what allows learners to connect representations may have much more variety. Thus, Even (1998), speaking about multiple representations of functions, argues that the flexibility and ease with which we hope students will move from representation to representation depends on what general strategy students bring to mathematical situations, what context students place a problem, what previous underlying knowledge students possess, and perhaps other things as well. In other words, the efficacy of multiple representations in the classroom needs more than the multiple representations themselves. Thus, Even writes:

> "...concluding that the subjects who participated in this study had difficulties in working with different representations of these functions is not enough. Much more important is to understand how these subjects think when they work with different representations of functions" (p.119)

It might be then that we need to be more willing to treat multiple representations as a *terminus ad quem* than as a *terminus a quo*, that is, it may be that we have to challenge a multiple representations approach as a framework to begin with in teaching and think of as a distant goal that may not be achieved until the learner has had considerable experience in kinds of thinking that potentially link representations. This conclusion, if it is valid, is sobering for educators who want to promote multiple representations by presenting many representations all at once. But a sobering message such as this may be what is needed for learners to begin and reap truly the potential benefits of multiple representations.

References

Amit, M. & Fried, M. N. (In press). Authority and Authority Relations in Mathematics Education: A View from an 8th Grade Classroom. *Educational Studies in Mathematics*.

Clarke, D. J. (1998) Studying the Classroom Negotiation of Meaning: Complementary Accounts Methodology. In A. Teppo (Ed.) *Qualitative Research Methods in Mathematics Education* (pp. 98-111). Reston, VA: NCTM

Even, Ruhama (1998). Factors Involved in Linking Representations of Functions. *Journal of Mathematical Behavior, 17(1)*, 105-121.

Fried, M. N. & Amit, M. (2004). Time and Flow as parameters in international comparisons: a view from an Eighth Grade Algebra Lesson. In Marit Johnsen Høines, Anne Berit Fuglestad (eds.), *Proceedings of PME 28, 2*, pp. 33-40. Bergen, Norway: Bergen University College.

Ginsburg, H. P. (1997). *Entering the child's mind: the cognitive clinical interview in psychological research and practice*. New York: Cambridge University Press.

Goldin, G. A. (1997). The PME Working Group on Representations. *Journal of Mathematical Behavior, 17*(2), 283-301.

Goldin, G. A. (2002). Representation in Mathematical Learning and Problem Solving. In L. D. English (Ed.), *Handbook of International Research in Mathematics Education* (pp.197-218). Mahwah, New Jersey: Lawrence Erlbaum Associates, Publishers.

Greeno, James G. & Hall, Rogers P. (January 1997). Practicing Representation: Learning with and about Representational Forms. Phi *Delta Kappan, 78*(5), 361-367.

Janvier, C. (Ed.). (1987). *Problems of Representation in the Teaching and Learning of Mathematics*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Kaput, J. (1999). Representations, Inscriptions, Descriptions and Learning: A Kaleidoscope of Windows. *Journal of Mathematical Behavior, 17*(2), 265–281.

NCTM (2000). *Principles and Standards for School Mathematics*. Reston, VA: The National Council of Teachers of Mathematics, Inc.

Powell, A. B. & Maher, C. A. (2003). Heuristics of Twelfth Graders' Building Isomorphism. *Proceedings of the PME 27, 4,* pp. 23-30. Honolulu, Hawaii: University of Hawaii.

Schultz, J., & Waters, M. (2000). Why Representations? *Mathematics Teacher*, 93(6), 448-453.

Stigler, J. W. & Hiebert, J. (1999). *The Teaching Gap*. New York: Free Press.

REFORM-ORIENTED TEACHING PRACTICES: A SURVEY OF PRIMARY SCHOOL TEACHERS

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In line with international recommendations, reform-oriented approaches have been promoted through the Working Mathematically strand of the curriculum for primary school children in New South Wales. Evidence suggests that teachers engage differently with these recommendations depending on their knowledge and beliefs about the role of working mathematically in learning mathematics. Through a self-report survey, this preliminary investigation identified the use of reform-oriented practices. Many teachers reported using such practices and actively plan learning experiences that incorporate a range of processes including reasoning and communicating. However, some respondents appeared to be more informed than others.

INTRODUCTION

Recent curriculum documents typically promote reform-oriented approaches and recognise the importance of engaging students in worthwhile mathematics through a range of processes. For example, the *Standards* of the National Council of Teachers of Mathematics [NCTM] (NCTM, 2000) includes problem solving, reasoning and proof, communication, connections, and representations. Similar processes are included in the latest mathematics syllabus for primary school students in New South Wales [NSW] (Board of Studies NSW [BOSNSW], 2002). The Working Mathematically strand incorporates five interrelated processes – questioning, applying strategies, communicating, reasoning and reflecting.

These processes underpin problem solving; a life skill that is universally considered central to the mathematics curriculum (NCTM, 2000). When such processes are successfully implemented, learning experiences "allow learners to think and create for themselves … discuss their interpretations and develop shared meanings" (Sullivan, 1999, p. 16). The teacher's role is not trivial (Schoen, Cebulla, Finn & Fi, 2003). The teacher needs to choose tasks that engage students in higher order thinking and sustain engagement (Henningsen & Stein, 1997), help students make links between mathematical ideas (Askew, Brown, Rhodes, Johnson & Wiliam, 1997), and meet the needs of the full range of students in classrooms.

Given the centrality of working mathematically in the new mathematics syllabus (BOSNSW, 2002), and the assertion that not all teachers have embraced it (Hollingsworth, Lokan, & McCrae, 2003), it is critical to explore the extent to which it is being adopted and integrated into teachers' practices. It is also essential to identify cases of exemplary practice and to provide advice to teachers about the issues that might constrain their efforts to implement the reform elements of this new syllabus.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 65-72. Melbourne: PME.

SITUATING THE RESEARCH IN AN INTERNATIONAL CONTEXT

It has been argued that the use of non-routine problems and problem-centred activities form the basis of classroom activity in a reformed or inquiry-based classroom (Clarke, 1997; Schoen et al., 2003). There has been substantial advice to teachers to teach problem-solving skills and to use problems as a focus of learning in mathematics (Wilson & Cooney, 2002). Such advice has been accompanied by considerable efforts through preservice and inservice programs to change teaching practices from more traditional approaches to contemporary or reformed methods (e.g., Schifter, 1998).

Investigations into the implementation of reform, or standards-based curriculum (NCTM, 2000), have been undertaken in the United States over recent years. Two studies have particular relevance for this investigation. Schoen et al. (2003) used observation criteria for reform-teaching practices that include open-ended questions, time to learn from investigations, as well as pair and small-group work. Ross, McDougall, Hogaboam-Gray and LeSage (2003) developed a 20-item survey based on nine dimensions of standards-based teaching that include several aspects of the focus of this study (student tasks, discovery, teacher's role, interaction and assessment). While the survey was found to have reliability and validity, the authors advise the use of observations to confirm teacher self-report data.

While teachers may have good intentions and plan to implement reform-oriented approaches, there is evidence that teachers in Australian contexts have not responded to this advice (Hollingsworth et al., 2003), with the suggestion that the culture of schooling and particular teachers' beliefs hinder the implementation of problem-solving approaches in classrooms (McLeod & McLeod, 2002; Stigler & Hiebert, 1999). There is a significant body of research indicating that teacher's knowledge and beliefs about the discipline of mathematics, teaching mathematics, and learning mathematics impact on classroom practice (Wilson & Cooney, 2002). In particular, Stigler and Hiebert (1999) argued that the differences between American and Japanese approaches to teaching mathematics could be explained by differences in teachers' beliefs.

However, it has also been determined that other constraints can impact on teachers' efforts to implement the working mathematically processes. In her investigation of reform in primary schools involved in the *Count Me In Too* professional development program, Bobis (2000) noted teachers' concerns about time, availability of resources and classroom management issues. Similarly, a study into primary school teachers' problem-solving beliefs and practices by Anderson, Sullivan & White (2004) identified several constraints including assessment and reporting practices, parent's expectations, students resistance to new approaches, system requirements of curriculum implementation, and large-scale testing regimes. Jaworkski (2004, p. 18) describes such demands as "sociosystemic factors" suggesting that teachers have to regularly grapple with the tensions and issues that arise in their contexts.

One particular issue for teachers is planning reform-oriented experiences that maintain engagement and cater for the needs of all students (Henningsen & Stein, 1997). There is evidence from the TIMSS 1999 Video Study (Hollingsworth et al., 2003) that teachers plan to use different teaching strategies to teach higher achieving students compared with lower achieving students. However, even with higher achieving students, there was little use of higher-level processes or opportunities for reasoning as emphasised in the Working Mathematically strand. While teachers generally support reform-oriented teaching (Anderson et al., 2004), they appear to have difficulty operationalising it (Ross et al., 2003).

It is possible that teachers may not have an image of what this reform approach looks like in practice, or it may be that particular contextual factors interfere with their intentions. An ongoing concern of the problem-solving research has been the need for descriptions of classrooms where effective practice is occurring with an exemplification of the key role of the teacher (e.g., Clarke, 1997). Identifying successful teachers and providing rich descriptions of their efforts might support implementation for others, particularly if these teachers are able to overcome militating factors.

To investigate the implementation of reform-oriented teaching in NSW classrooms, the research questions for the study include:

1. Which reform-oriented teaching practices do primary school teachers report using?
2. Which particular teaching practices do primary teachers report using for each of the five processes of working mathematically?
3. What knowledge and beliefs distinguish teachers who successfully implement Working Mathematically?
4. How do teachers who successfully implement Working Mathematically cater for the needs of all students in the classroom?

Previous research suggests that teacher self-report surveys provide a relatively accurate picture of classroom practice but that there are some aspects of practice—particularly in the case of working mathematically—that cannot be easily measured in this way (Ross et al., 2003). For this reason, a combination of survey, interview and case study (including classroom observations) approaches were utilised in the study to explore teachers' understandings of working mathematically and their implementation of the various processes of the strand. Only results from the survey component will be discussed in this paper.

METHODOLOGY – SEEKING THE EVIDENCE

A survey was used to determine whether teachers' practices reflect those advocated in reform-oriented curriculum materials produced locally (e.g., BOSNSW, 2002) and internationally (NCTM, 2000). In particular, it focused on specific teaching strategies associated with each of the five processes of the Working Mathematically strand in the *Mathematics K-6 Syllabus* (BOSNSW, 2002).

There were three main parts to the survey. Part A was designed to collect essential background information about respondents and their school contexts. Part B was adapted from the Ross et al. (2003) instrument for measuring the extent to which primary teachers implement reform-oriented teaching practices. It contains 20 Likert items with a 5-point response scale ranging from Strongly Agree to Strongly Disagree. To guard against response bias, seven of the items were worded so that their scoring would be reversed. Ross et al. (2003) provide evidence of the instrument's reliability and validity. Using Cronbach's α, a measure of internal consistency, they obtained a reliability coefficient of $\alpha = 0.81$ in two independent studies. Part C of the survey contained four open-ended questions that explicitly focussed on teaching practices associated with working mathematically.

The aim of the survey was to produce a tentative picture of teacher beliefs and commitment to reform-based teaching practices, and to distinguish teachers—specifically those reporting the incorporation of working mathematically into their teaching—for inclusion in the interview component of the study. Approximately 100 surveys were sent to 12 primary schools located in the Sydney, metropolitan area that had been identified as supporting reform-oriented approaches. Descriptive statistics were used to analyse the items on the survey requiring quantitative responses (Parts A and B). The open-ended items in Part C were analysed according to emergent themes.

RESULTS

Forty surveys were returned. Background information provided by teachers (Part A of the survey) indicated that there was a fairly even representation from each of the grade levels from Kindergarten to Year 6. Similarly, there was an even spread of years of teaching in each of the groups 1-5, 6-10, 11-15, 16-20, and 21 and beyond.

To assist analysis of Part B of the survey, the percentage of teachers indicating that they agreed (including strongly agreed), were unsure, and disagreed (including strongly disagreed) with each statement in the survey was calculated. While there is insufficient space to report the results for each item, we have selected some for discussion to support our analysis and complement the data provided in the open-ended response component of the survey. It must be emphasised, that we intended to use the information gained from the quantitative component as 'tentative', providing starting points for further exploration in the interview and observation components of the project.

As a whole, respondents seemed to be very well aware of what the reform-based movement recommends regarding the teaching and learning of mathematics. For example, 97.5% of respondents indicated that they agreed or strongly agreed with Items 1 and 3 ("I like to use maths problems that can be solved in many different ways", and "when two students solve the same maths problem correctly using two different strategies, I have them share the steps they went through with each other" respectively).

Contrary to the general trend of responses, which were consistent with views expressed by reform-oriented curriculum documents, only 19.7% of respondents disagreed with Item 16 ("I like my students to master basic mathematical operations before they tackle complex problems"). This type of response is contrary to current curriculum documents that advocate the teaching of mathematics through or via problem-solving approaches (e.g., BOSNSW, 2002). Whether teachers are aware of such recommendations or simply disagree with them, it is clear that the majority of our respondents report not implementing such practices. More information on this issue may be gained during the interview component of the study.

Related to this view of mathematics, a quarter of teachers responding to the survey indicated that they considered "A lot of things in maths must simply be accepted as true and remembered" (Item 15). Similarly, 27.5% of respondents indicated their agreement with Item 19: "If students use calculators they don't master the basic maths skills they need to know". Both these responses are indicative of a more traditional view of mathematics. That is, mathematics is seen as little more than a series of facts, rules and procedures that must be learned.

While teachers rarely used the 'unsure' category, two statements attracted high percentages of responses in this category. 30.7% of respondents indicated that they were unsure of Item 12 ("Creating a set of criteria for marking maths questions and problems is a worthwhile assessment strategy") and 27.5% were unsure of Item 18 ("Using computers to solve maths problems distracts students from learning basic maths skills"). The reasons for the higher than expected percentages of 'unsure' responses for each of these items, will be explored in follow-up interviews.

Part C required respondents to list the "specific teaching strategies" they use for each of the five processes of Working Mathematically. Descriptions of three of these processes are presented in Table 1 with samples of teachers' responses.

Process	Description of the Process (BOSNSW, 2002a, p. 19)	Sample Teacher Response
Questioning	Students ask questions in relation to mathematical situations and their mathematical experiences.	Children work together in groups and solve maths problems, which encourage them to ask questions. (23)
Reasoning	Students develop and use processes for exploring relationships, checking solutions and giving reasons to support their conclusions.	Provide opportunities to compare and contrast results of an investigation – expect/encourage explanation of process/product (3)
Reflecting	Students reflect on their experiences and critical understanding to make connections with, and generalisations about, existing knowledge and understanding.	Building upon known concepts, using skills to extend understandings. Applying knowledge to everyday situations (18)

Table 1 Sample responses to three processes in Working Mathematically

The majority of the 31 teachers, who responded to Part C, seemed to be familiar with each process and the associated teaching practices recommended in reform documents. However, some respondents appeared to be more informed than others. To identify those teachers, an adaptation of the Schoen et al. (2003, p. 236) observation criteria for reform teaching practices was used to rate the responses. These criteria (presented below) were used to make holistic judgements about participants' reported level of implementation of reform-oriented practices.

1. The teacher uses open-ended questions to facilitate student thinking and exploration.
2. Students monitor their own work instead of always seeking out the teacher as the authority.
3. Students are given enough time to learn from investigations.
4. Class organisations (i.e., whole-class presentation or discussion, pair or small-group work, and individual work) match expectations for each part of the lesson.
5. Pairs or small groups of students work collaboratively.
6. Manipulative materials are available.
7. The teacher focuses on understanding of the big mathematical ideas by questioning understanding and using problem-solving strategies.

Using these criteria, the response of each participant to the open-ended question was judged as excellent, good, fair, or poor according to the number of criteria that were explicitly addressed. From this, the responses of two participants were rated as excellent, five as good, 17 as fair, and 7 as poor. The responses from those teachers who were rated as fair or poor were either limited in information, repetitive in the practices employed, or suggested that more traditional practices were typically used. For example, an experienced teacher of a Year 3/4 class reported that she uses a "whole class focus first then one to one – needs lots of examples and practise, concrete material or practical applications" for Applying Strategies. This individual focus was repeated for Reflecting with the additional strategy of "sometimes we meet as a group at the board and discuss" for "students who are experiencing difficulties or simply don't understand". These comments were consistent with her responses to the reform-oriented practices in Part B of the survey. Again, this data provides tentative information as respondents may not have given much thought to their responses or they may not have had sufficient time to think deeply about their practice. However, this process helped to identify participants for the interviews and classroom observations.

Ten survey respondents (25%) indicated their willingness to participate in the follow-up interview component of the study. Data from all parts of the survey were considered to develop initial 'profiles' of these teachers so as to determine which teachers we should include. Eight of these teachers had profiles that were very closely aligned with the practices recommended by reform-oriented documents. All eight indicated that they explicitly planned for Working Mathematically either all of the time or at least for approximately 70% of their mathematics lessons. Interestingly, the three teachers with profiles considered to be closest to reform-oriented practices, teach at the same school. The interview component will hopefully reveal if there are any contextual factors operating at the school that may contribute to such a result.

The responses of the other two teachers who volunteered to participate in the interview component of the study were among those respondents who showed least consistency or familiarity with reform-based practices. One teacher indicated that she did not explicitly plan for Working Mathematically, while the other indicated that she planned for approximately 90% of her lessons. Again, what this planning actually entails will be explored further in the interview component of the study.

DISCUSSION AND FURTHER RESEARCH

Considered together, the qualitative and quantitative data gained from the survey provide tentative information (Wilson & Cooney, 2002) and a starting point from which we can now continue to explore aspects of teachers' practices. It would appear that the majority of these teachers support reform-oriented teaching approaches that promote working mathematically in primary classrooms, particularly in a self-report survey. While most responses were consistent for both sections of the survey, a careful reading of the open-ended responses suggests that this may not be what is implemented in practice. Further exploration through interviews and observations is required before in-depth claims can be made.

The next step in our project is to explore particular teacher's practices in detail to form a picture of the successful implementation of working mathematically for *all* students and how teachers confront the sociosystemic factors operating in school contexts. As Wilson and Cooney (2002, p. 131) propose

> in-depth studies of individuals emphasise the value of telling stories about teachers' professional lives and what shapes those lives … good stories are not simply descriptions but are grounded theoretical constructs that have the power to explain what is described.

The knowledge gained from this project has the potential to impact on the implementation of working mathematically in classrooms. It will clarify for teachers what working mathematically actually looks like and provide models of best practice. It will present teachers with evidence that *all* students are able to participate in challenging experiences regardless of their performance on tasks that assess basic skills in mathematics. It will provide teachers with strategies to cope with the tensions and issues that may impede implementation of the Working Mathematically strand.

References

Anderson, J., Sullivan, P., & White, P. (2004). The influence of perceived constraints on teachers' problem-solving beliefs and practices. In I. Putt, R. Faragher, & M. McLean (Eds.), *Mathematics education for the third millennium: Towards 2010*, Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia (pp. 39-46), Townsville: MERGA.

Askew, M., Brown, M., Rhodes, V., Johnson, D., & Wiliam, D. (1997). *Effective teachers of numeracy*. Final Report. King's College: London.

Board of Studies NSW (BOSNSW) (2002). *Mathematics K-6 Syllabus*. Sydney: BOSNSW.

Bobis, J. (2000). *Count Me In Too Report: A Case study of implementation*. Sydney: NSW Department of Education and Training. http//www.currriculumsupport.nsw.edu.au/maths

Clarke, D. M. (1997). The changing role of the mathematics teacher. *Journal for Research in Mathematics Education*, *28*(3), 278-308.

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student congnition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, *28*, 524-549.

Hollingsworth, H., Lokan, J., & McCrae, B. (2003). *Teaching mathematics in Australia: Results from the TIMSS 1999 Video Study*. Camberwell, VIC: Australian Council for Educational Research.

Jaworski, B. (2004). Grappling with complexity: Co-learning in inquiry communities in mathematics teaching development. In M. J. Høines & A. B. Fuglestad (Eds.), *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, pp. 17-36). Bergen, Norway: PME.

McLeod, D. B., & McLeod, S. H. (2002). Synthesis – Beliefs and mathematics education: Implications for learning, teaching and research. In G. C. Leder, E. Pehkonen, & G. Torner (Eds.), *Beliefs: A hidden variable in mathematics education*? (pp. 115-123). Dordrecht: Kluwer Academic.

National Council of Teachers of Mathematics (NCTM) (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM.

Ross, J. A., McDougall, D., Hogaboam-Gray, A., & LeSage, A. (2003). A survey measuring elementary teachers' implementation of standards-based mathematics teaching. *Journal for Research in Mathematics Education*, *34*, 344-363.

Schoen, H. L., Cebulla, K. J., Finn, K. F., & Fi, C. (2003). Teacher variables that relate to student achievement when using a standards-based curriculum. *Journal for Research in Mathematics Education*, *34*, 228-259.

Schifter, (1998). Learning mathematics for teaching: From a teachers' seminar to the classroom. *Journal of Mathematics Teacher Education*, *1*(1), 55-87.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap*. New York: The Free Press.

Sullivan, P. (1999). Seeking a rationale for particular classroom tasks and activity. In J. M. Truran & K. M. Truran (Eds), *Making the difference* (pp.15-28), Proceedings of the 22nd Annual Conference of the Mathematics Education Research Group of Australasia, Adelaide.

Wilson, M., & Cooney, T. (2002). Mathematics teacher change and development. In G. C. Leder, E. Pehkonen, & G. Torner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 127-147). Dordrecht: Kluwer Academic.

THE GENESIS OF SIGNS BY GESTURES. THE CASE OF GUSTAVO

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This paper highlights the role of gestures in communicating and particularly in thinking in mathematics. The research interest is on the relation between the use of gestures and the birth of new perceivable signs. This link is shown through the description of a concrete example, referring to a discussion among 8th grade students around a geometrical problem in 3D, to be solved without the use of devices and paper and pencil. It is interesting to observe the progression in the construction of the solution, obtained with the introduction of new signs from gestures, and at the end even of a common tool used by children, the plasticine.

INTRODUCTION

In the last years psychologists have shown a deep interest on the analysis of gestures and their role in the construction of meanings. More recently, gestures became also relevant in the field of mathematics education, in order to show a strong relation not only with speech, but with the entire environment where the genesis of mathematical meanings takes place: context, artefacts, social interaction, discussion, and so on. Since math is an abstract matter, it often requires signs to be made somehow perceivable by students, the abstract becoming more and more concrete to them. Many times students need to see, to touch, and to manipulate, and as a consequence, the environment plays a crucial role in learning math.

This paper shows the link between the use of gestures and the birth of new perceivable signs through a concrete example. To reach this aim, we use a theoretical framework made of different components, coming from mathematics education, psychology, neuroscience and semiotics, and presented at the Research Forum on gestures at the current PME (Arzarello *et al*., 2005). Three parts constitute the paper. We sketch some ideas of the theoretical framework in a first paragraph. The second paragraph presents an example of a protocol analysed through our theoretical tools. In the last part, conclusions and further problems are introduced.

THE THEORETICAL FRAMEWORK

Some researchers form psychology and mathematics education claim that gestures play an active role in thinking, intending communicating and thinking not as mutually exclusive functions (McNeill, 1992; Goldin-Meadow, 2003). So gesturing is useful to listeners to communicate, and to speakers to think. To exemplify, just think of people speaking on the phone: they are completely conscious of the fact that their

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interlocutor is not in front of them, but all the same they gesture, as well as the listener would be there. Even a subject alone, doing an activity such as studying, could gesture to give expression to her thoughts. Briefly speaking, we can gesture for ourselves or for others, and gesture is both a mean of communication and of thinking. Gesture in fact can contribute to create ideas:

> According to McNeill, thought begins as an image that is idiosyncratic. When we speak, this image is transformed into a linguistic and gestural form. ... The speaker realizes his or her meaning only at the final moment of synthesis, when the linear-segmented and analyzed representations characteristic of speech are joined with the global-synthetic and holistic representations characteristic of gesture. The synthesis does not exist as a single mental representation for the speaker until the two types of representations are joined. The communicative act is consequently itself an act of thought... It is in this sense that gesture shapes thought. (Goldin-Meadow, 2003; p. 178)

Within the perspective of psychology, we refer to the so-called Information Packaging Hypothesis (IPH): Alibali, Kita & Young (2000) describe it as the way gesture is involved in the conceptual planning of the messages. IPH concerns the so called representational gestures (Kita 2000), namely the iconic and the abstract deictic gestures: an iconic gesture represents an entity or a phenomenon, a movement, a shape, and so on; a deictic gesture points to an object (Mc Neill, 1992). This object can be a physical thing or an abstract entity, so deictic gestures can be divided in concrete or abstract (Kita, 2000). According to the IPH, the production of representational gestures seems particularly important, since it helps speakers organise spatio-motoric information into packages suitable for speaking. In such a sense gesture explores alternative ways of encoding and organising spatial and perceptual information. Spatio-motoric thinking (constitutive of representational gestures) provides an alternative informational organisation that is not readily accessible to analytic thinking (constitutive of speaking organisation). Analytic thinking is normally employed when people have to organise information for speech production. On the other side, spatio-motoric thinking is normally employed when people interact with the physical environment, using the body (interactions with an object, locomotion, imitating somebody else's action, etc.). This kind of thinking can be applied even to the virtual environment that is created as imagery.

Gesture can be discussed also from a neuroscientific point of view. In fact some studies from neuroscience argue that there is no inherent distinction between thought and movement at the level of the brain; both can be controlled by identical neural systems (Ito, 1993). Therefore, concepts and ideas can be manipulated just as they were body parts in motion. The 'motor system' is thus a complex computational network that controls and directs the brain's circuitry or internal symbols: counting, timing, sequencing, predicting, planning, correcting, attending, patterning, learning and adapting (Leiner *et al*., 1993). Indeed, gestures that accompany language may facilitate thought itself. With the embodied mind Seitz (2000) introduces a fresh paradigm for thinking about the relation of movements to thoughts, in which the boundaries between perception, action, and cognition are porous. Thought, action, and perception are indissolubly tied. Thinking of embodied activities, although humans may be best characterised as symbol-using organisms, symbol use is structured by action and perceptual systems that occur in both natural environments and artifactual contexts: "[…] body structures thought as much as cognition shapes bodily experiences." (Seitz, 2000).

From another viewpoint, gestures can be seen as signs, as pointed out by Vygotsky (1997: 133):

> A gesture is specifically the initial visual sign in which the future writing of the child is contained as the future oak is contained in the seed. The gesture is a writing in the air and the written sign is very frequently simply a fixed gesture.

As a consequence, semiotics is a useful tool to analyse gestures, but within a wider frame, which involves also their cultural and embodied aspects. Such a wider analysis has been developed in mathematics education with the introduction of the notion of *semiotic means of objectification* (Radford, 2003) that produce the so-called *contextual generalisation*: a generalisation referring still heavily to the subject's actions in time and space, within a precise context, even if he/she is using signs who could have a generalising meaning. In contextual generalisation, signs have a twofold semiotic nature: they are becoming *symbols* but are still *indexes*. We use these terms in the sense of Pierce (1955): an index gives an indication or a hint on the object, like an image of the Golden Gate makes you think of the town of S. Francisco. A symbol is a sign that contains a rule in an abstract way (e.g. an algebraic formula).

In light of the results stated above, our research hypothesis is that meanings construction is supported by a dynamic evolution in the use of gestures and by their role in generating new signs. In this paper, such an evolution is pointed out by the social activity of the students in a geometric context, where the main components are: the hands that shape geometrical figures, and the fingers that point to or trace geometrical entities (sides, angles, faces, vertices, etc.) related to the solution of the problem.

THE CASE OF GUSTAVO: SIGNS ARISING FROM GESTURES

In the following example, some students of the 8th grade in an Italian middle school are working in group to solve a geometric problem. The task is to find the solid figure that fills in the 3D hole obtained if two congruent regular squared-based pyramids are placed (on the same plane) with two sides of the bases touching each other (Fig. 1). The pupils are asked to not use any kind of concrete support, as drawings, paper and pencil, computers, etc. On the contrary, they are required to get the solution 'in their mind', simply imagining and discussing together about it (the tetrahedron ABCD in Fig. 1). The teacher in the classroom observes the work without interfering if not necessary. The students work in a big group

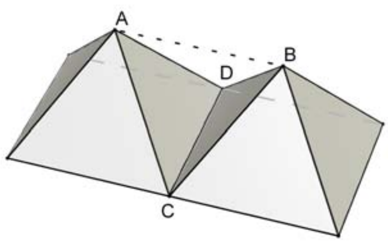

Figure 1

around some desks, in order to pay attention to what each other is doing or saying. During their curriculum they have already met regular tetrahedrons, as belonging to the family of pyramids (a regular pyramid with congruent faces and triangular basis).

The following analysis is based on the video recorded by a moving camera. In the course of the discussion, gestures are used in a massive and social way. Gestures are our data, and they are analysed to figure out the dynamics of the pupils' solution processes and communication acts. In particular, we will show the emergence of signs (arising from gestures) that can be seen, touched and manipulated (not simply imagined) by the students. These observations fit with the hypothesis stated above.

Since the beginning of the work, the actions of the group seem to occur principally around two pupils: Sara and Gustavo. In Fig. 2 Sara is the girl with long hair and eyeglasses on the right of the reader, and Gustavo is the boy on the left, with short hair. It is Gustavo who leads the game: he allows his mates to approach the solution and his gestures guide them and their gestures.

Figure 2

Three phases appear to characterise the dialogue of the students. First, the pupils need to re-construct the geometry of the given configuration. Gustavo performs a lot of gestures to imagine what he has in mind and makes it visible: fingers running along or pointing to Sara's hands open as they were the two close faces of the pyramids (Fig. 3*a*, 3*b*); hands closing the hole in the attempt of figuring out the unknown solid (Fig. 3*c*, 3*d*); fingers tracing sides of the solid itself (Fig. 3*e*, 3*f*).

Figure 3

This is an *exploration* phase in which the students share a space of communication, that over the desks, where they can freely move, gesture, show. We can call it *gesture space*. To Gustavo, such a space represents something more: it is already a space of action and production, other than of communication (APC Space - Action, Production and Communication - Arzarello, 2004). At the moment pupils need an investigation of this kind; spatio-motoric thinking is essential, since gestures are the only available means for them to understand the problem. Imagination alone is not sufficient to figure out the geometric entities, especially the one to be found.

Gustavo's gestures (performed in front of his body over the desks, Fig. 4) start to acquire a symbolic characterisation. Recalling Peirce's terminology, they are still *indexes* of figures (in that his hands in motion represent sides, faces, solids), but start becoming *symbols* when referring to the virtual world where these entities live and being used as really existing objects by Gustavo.

Figure 4

At this point, Gustavo has got the solid in his mind, as highlighted by his words:

Gustavo: It's a triangle, but with a thickness. It's a solid of pyramid.

A new phase begins here: in order to allow his mates to see the shape of the unknown solid, new signs are necessary, with a more concrete nature than the previous gestures. Their emergence is marked by a change of the gesture space, more visible. The (real) desks now represent the new space: in it, Gustavo's index finger traces some virtual segments to explain his solution to the others (Fig. 5).

Gustavo: It is in this way, in this way, and then in this way.

Figure 5

These segments are imaginary inscriptions, but used as if they were real, and so can be deleted (by hand in Fig. 6), just as geometric lines drawn on paper are erased by means of a rubber (the term inscription comes from Sfard & McClain, 2002).

Figure 6

This is a *conjecture* phase, but the explanation is not yet satisfactory to the group. The dialogue goes on among the pupils with an exchange of some sentences:

Gustavo:	It is made of two triangles with the bases below, and two triangles with the bases above.
Sara:	But, it's a solid. It [the problem] tells a solid, one!
Gustavo:	Yeah, it is a solid, made of two triangles placed with the bases below, which are those starting in this way and going up, and two triangles with the bases above that are those going in this way [see Fig. 7]

Figure 7

Lucy:	And how does it call?
Gustavo:	I don't know, but that's it.
Sara:	No, no, because above they have skew sides.
Gustavo:	No, it doesn't matter!

Let us observe that along the discussion, a conflict appears into the whole group: the opposite behaviour of the triangles above and below. Gustavo's metaphor on such triangles (expressed both in his words and gestures) is not useful to accept the shape of the searched solid. Hands are not enough. To overcome the obstacle, something different is to be used: something really existing, that can be effectively seen, touched, manipulated. Again, the emergence of new signs is strong. At this point a last phase of *production* begins: a tool enters the scene, as Gustavo claims:

Gustavo:	Guys, we got a tool! [he takes a piece of plasticine by the hands of a group mate]

Other than gestures, plasticine is stable, motionless and concrete: it is real, visible and manipulable. By plasticine, students are able to check their conjectures, to make the solution apparent, to create it. Figure 8 shows just some moments of this final phase.

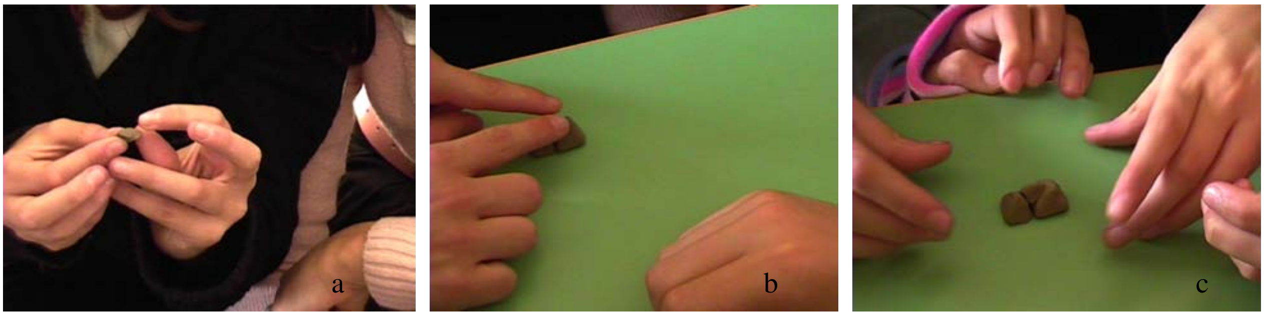

Figure 8

Somebody:	It's a thing made in this way, it's strange [Fig. 8a]
Lucy:	Try to put it [Fig. 8c]
Sara:	But it is a pyramid with triangular basis.

All the geometric entities are visible; the tetrahedron itself is visible. In terms of Radford's words, students have objectified new knowledge: now they know that the solid solving the original task is a "pyramid with triangular basis".

SOME CONCLUSIONS

In the previous example we have described the activity of some students solving a 3D geometry problem. We have pointed out that their understanding grows up around the gestures of Gustavo, who is early able to imagine the solid solving the task, and in the same time has to explain it to the group mates. We have seen the way these gestures mark the birth of new perceivable signs: the virtual segments drawn by Gustavo's forefinger on the desks, and the solids shaped by the use of plasticine.

The relevant point of the activity is the evolution of gestures in generating signs. At first, Gustavo's gestures have an iconic function in that their shape resembles their referents (the geometric solids they express), but they become indexes (in the sense of Pierce) in his communicative attempt of transferring knowledge to the others. The indexical gestures acquire a symbolic function later, when they are used as existing objects of a virtual geometric world and in relation with the genuine geometric objects (e.g., think of the metaphor of the "two triangles with the bases below, and two triangles with the bases above"). This relation consists in a piece of theoretical knowledge. Particularly, when the students identify the unknown solid as a pyramid also their utterances have a two-fold nature: an indexical one and a symbolic one, in encoding information according to the theory at their disposal.

The most significant moment of the activity arises from the use of plasticine, here a tool expressing three different functions. First, plasticine has an iconic function for Gustavo, who wants to show the solution of the task, as well as his mates want to see it. Secondly, it is an index in being manipulated and seen; in these terms, it is a sign useful to make sense of the solution. But it also has the germs of a symbolic function in itself, because it is shaped in a theoretical knowledge, still vague if only supported by the previous gestures and metaphors. Gustavo and the group need to understand those relations. The discovery of the solid as a pyramid happens at this point.

We can interpret this last moment using the theory of instrumentation of Rabardel (1995) within a fresh viewpoint. According to Rabardel, an artefact becomes an instrument in the hands of a subject, by activating different schemes of use; in these terms, the instrument prompts the genesis of meanings. To our students plasticine is already an instrument: they are able to use it, and they have acquired the corresponding instrumentations. As a consequence, the role of plasticine is different from that of an artefact becoming an instrument; in a certain way, the situation is exactly opposite. Our students (especially Gustavo) take an instrument they have and know, with its schemes of use, and, through their perceptuo-motor information about it, they add new schemes of use in order to solve the problem. According to our wider frame developed for the Research Forum (see Arzarello *et al*., 2005), the whole dynamics can be identified as a ***SPO*** (Serial Process of Objectification) in that the situation evolves by the successive production of signs (through gestures and words, and schemes of use).

Acknowledgments

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References

Alibali, M. W., Kita, S., & Young, A. (2000). Gesture and the process of speech production: We think, therefore we gesture. *Language and Cognitive Processes*, 15, 593-613.

Arzarello, F. (in print). Mathematical landscapes and their inhabitants: perceptions, languages, theories, *Proceedings ICME 10*, Plenary Lecture.

Arzarello, F., Ferrara, F., Paola, D., Robutti, O. & Sabena, C. (2005). Shaping a multi-dimensional analysis of signs. In: Arzarello, F. & Edwards, L., *Gesture and the Construction of Mathematical Meaning*, Research Forum. These Proceedings.

Kita, S. 2000. How Representational Gestures Help Speaking. In McNeill, D. (Ed.), *Language and Gesture*, pp. 162-185. Cambridge: Cambridge University Press.

Goldin-Meadow, S. (2003). *How our Hands Help us Think*. Chicago: Chicago University Press.

Ito, M. (1993). Movement and thought: Identical control mechanisms by the cerebellum. *Trends in the Neurosciences*, 16(11), 448-450.

Leiner, H. C., Leiner, A. L., & Dow, R. S. (1993a). Cognitive and language functions of the human cerebellum. *Trends in Neurosciences*, 16(11), 444-447.

McNeill, D. (1992). *Hand and Mind: What gestures reveal about thought*. Chicago: University of Chicago Press.

Peirce, C.S. (1955). *Philosophical Writings* (J. Buchler, Ed.). New York: Dover.

Radford, L. (2003). Gestures, speech, and the sprouting of signs. *Mathematical Thinking and Learning*, 5(1), 37-70.

Seitz, J.A. (2000). The Bodily Basis of Thought, New Ideas in Psychology: *An International Journal of Innovative Theory in Psychology*, 18(1), 23-40.

Sfard, A. & McClain, K. (2002). Analyzing tools: Perspectives on the role of designed artifacts in mathematical learning, *The Journal of the Learning Sciences*, 11(2&3), 153-161.

Vygotsky, L.S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

STUDENTS' EXPERIENCE OF EQUIVALENCE RELATIONS A PHENOMENOGRAPHIC APPROACH

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This paper is based on a doctoral study in which we studied 'lay' students' understanding of equivalence relations through individual task-based interviews. We report a conceptual gap between "the everyday functioning of intelligence and mathematics" as to equivalence relations.

INTRODUCTION

It is "an abstraction, a basic mathematical concept, that includes the way species, phonemes, numbers and many other concepts in many parts of life are best thought of…the name of the concept is "equivalence relation"…it is one of the basic building blocks out of which all mathematical thought is constructed." (Halmos, 1982, pp.245-246)

An equivalence relation is "one of the ideas which helps to form a bridge between the everyday functioning of intelligence and mathematics". (Skemp, 1977, p.173)

In this paper we consider *lay* students' understanding of the notion of equivalence relation. In particular, we report one gap (or two!) between "the everyday functioning of intelligence and mathematics". Despite the fact that the tasks(see below) used do not relate to a *formal educational setting*, we also suggest that it will be useful to pay attention to these gaps in our *standard practice of teaching* the notion of equivalence relation, in which, as Skemp says (ibid, p.137), "we start with everyday examples before defining it mathematically".

LITERATURE

Surprisingly, despite the fact that equivalence relation is one of the most fundamental ideas of mathematics, students' conceptions of it have attracted little attention as a research subject. An exception is a series of papers by Chin & Tall (2000, 2001 and 2002) in which they considered the cognitive growth of "equivalence relation" and "partition" at a time when students have been given the definitions and have been expected to operate in an increasingly "theorem-based" manner (ibid, 2000, p.2). However, as a result of working with students already being exposed to the formal treatment of equivalence relations and partitions the focus of the papers inevitably is on the far end of the bridge, i.e. students' understanding and usage of the formal concepts. Thus, in a sense, we furthered their study by investigating the opposite end of the bridge, i.e., *informal* conception of equivalence relations and partitions. In the discussion of the results we will briefly link these studies together.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 81-88. Melbourne: PME.

METHODOLOGY

The study is based on a detailed phenomenographic analysis of twenty verbatim transcribed audio-taped interviews with students with varied background experience (see also, Asghari, 2004a, 2004b). The participants comprised four middle school students, four high school students, one first year politics students, one first year law student, six first year mathematics students, two second year physics students, one second year computer science student, and one postgraduate student in mathematics. None of them had any formal previous experience neither of equivalence relations nor of the related concepts used to formulate the definition. In a one-to-one phenomenographic interview, each student was introduced to a set of tasks that were designed having the standard definition of equivalence relations in mind (see below). The interviews had a simple structure; the tasks were posed in order, but the timing of the interviews and questions were contingent on students' responses.

Such a varied range of interviewees remind us of a 'pure phenomenography' in which "the concepts under study are mostly phenomena confronted by subjects in everyday life rather than course material in school." as compared to 'developmental phenomenography' in which the concepts under scrutiny are confined to a formal educational setting and the purpose of the study is to help the subjects of the research, or others with the *similar educational background* to learn (Bowden, 2000, p.3). However, in the case of a concept as basic as an equivalence relation, the line between pure and developmental phenomenography fades out.

The Tasks

First, each student was introduced to the definition of a 'visiting law' while they were told that their first task would be giving an example of a visiting law on the prepared grids. (See figure 1.)

> A country has ten cities. A mad dictator of the country has decided that he wants to introduce a strict law about visiting other people. He calls this 'the visiting law'.
>
> A visiting-city of the city, which you are in, is: A city where you are allowed to visit other people/
>
> A visiting law must obey two conditions to satisfy the mad dictator:
>
> 1. When you are in a particular city, you are allowed to visit other people in that city.
> 2. For each pair of cities, either their visiting-cities are identical or they mustn't have any visiting-cities in common.

The dictator asks different officials to come up with valid visiting laws, which obey both these rules. In order to allow the dictator to compare the different laws, the officials are asked to represent their laws on a grid as figure 1.

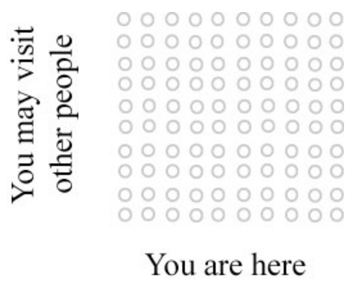

Figure 1: a grid to represent a visiting law

After generating some examples (student-generated, ranging from one example to suggesting a way to generate an example), students were presented with the following three tasks:

Task 1: The mad dictator decides that the officials are using too much ink in drawing up these laws. He decrees that, on each grid, the officials must give the least amount of information possible so that the dictator (who is an intelligent person and who knows the two rules) could deduce the whole of the official's visiting law. Looking at each of the examples you have created, what is the least amount of information you need to give to enable the dictator deduce the whole of your visiting law.

Task 2: One of the officials, for creating an example, uses other officials' examples: he takes two valid examples and put their common points in his own grid. Is the grid that he makes a valid example? [In the discussion following this is termed *the intersection task*]

Task 3: Another official takes two valid examples and puts all of their points in his own grid. Is the grid that he makes a valid example? [Hereafter, this is termed *the union task*]

Our account of equivalence relations when we designed the tasks

Let us use the eloquent, but still informal, account of equivalence relations given by Skemp (1977).He begins by introducing methods of sorting the elements of a parent set into sub-classes in which every object in the parent set belongs to one, and only one, subset (a partition of the parent set). He (ibid, p.174) considers two sorting methods: first, starting "with some characteristic properties, and form our sub-sets according to this"; and second, starting "with a particular matching procedure, and sort our set by putting all objects which match in this way into the same sub-set". The particularity of this matching procedure is in its "exactness", i.e. having an exact measure for the sameness; a necessity that if it is achieved, the matching procedure is called an equivalence relation. The exactness of the matching procedure also accounts for the transitive property. In addition to the transitive property, an equivalence relation has two further properties, reflexivity and symmetry (see below).

In the problem given to our students, two cities are matched together if their visiting-cities are the same, or two columns are matched together if they have the same status in each row. (For a thorough analysis of the task see Asghari, 2004a).

RESULTS

Analysis of the written transcripts led to a categorisation related to the variation in students' focus of attention in this particular situation. It was possible for the same student to experience different things at different times. The categories are: Matching procedure, Single-group experience and Multiple-group experience.

Matching Procedure Experience

In this category, the focus is on the matching procedure between individual elements; what students experienced and described is in terms of the elements involved, without resort to a group and/or groups of elements. Before giving an example, it is worth saying that somehow or other the defining properties of an equivalence relation determine an exact matching. So do the defining properties of a visiting law.

A matching procedure was exhibited by Ali (first year high school student) when he was generating an example.

Ali: I choose the very first things (points) haphazardly, and then I am going to match the things that have not been matched up yet.

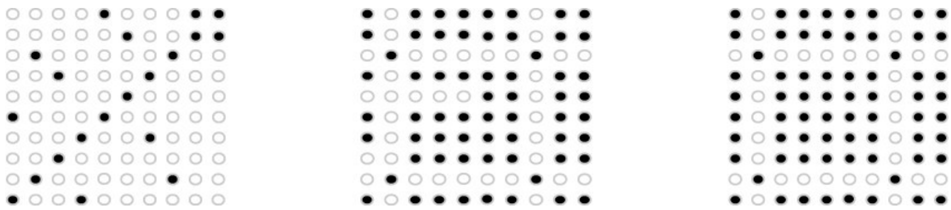

Fig 2: Three stages of Ali's matching procedure

Ali: All right, we start again.

So he paired up city 1 with all the other cities, one-by-one; when two focal columns find something in common, he matched them up, and when they have been already matched or they have nothing in common, he left them as they were. Then he did the same process on city 2 and paired it up and matched it up (if necessary) with all the other city after city 2, and so on. The result of this long process was the middle figure above. Then he continued:

Ali: Now, we are checking from start; it is going to be full (having all points).

And he did so. Eventually the process ended with the right figure above.

Single-Group Experience

In this category, focus is on only one single "group" while all the other elements that do not fall into that group are treated as individuals. The elements in the focal group in one way or another are related to each other while all other elements are in the background as individual elements. Each student in the present study could exemplify this category. However, we have chosen one that at the same time could exemplify different aspects of this category.

Kord (a middle school student) generated the following figures:

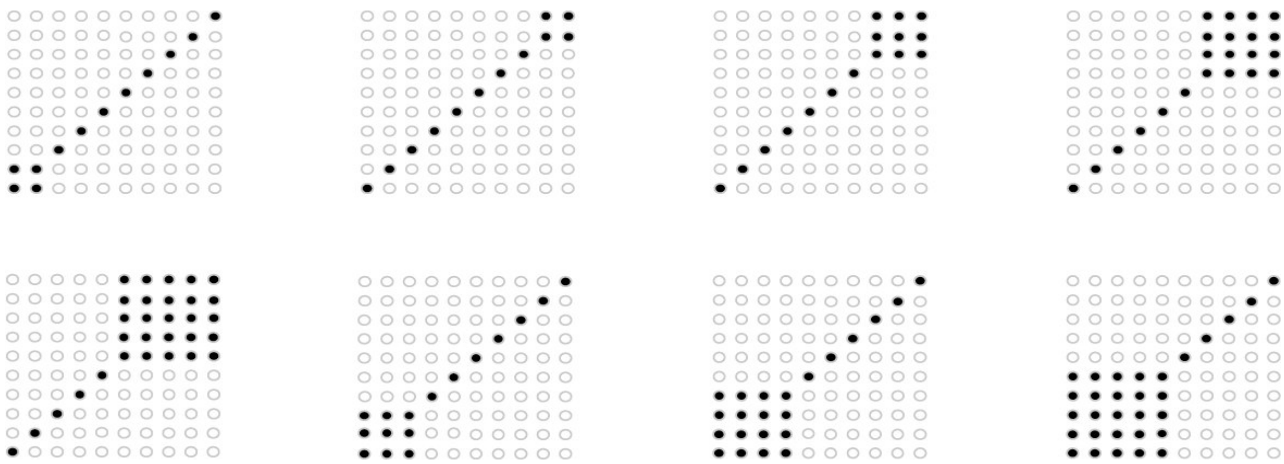

Figure 3: Figures generated by Kord.

Each of these has a square of equivalent points in one corner (lower left or upper right) but in no case did he put together a picture with squares in both corners. Even when confronted with the 'union task', he found it necessary to focus on one square after the other; while he checked whether the square that he has been focusing on has been properly packed, he unpacks the other square and treated its elements on a par with all other individual elements.

Kord: … those that can visit each other are identical and they have no commonality with other cities, so this is correct (this is an example).

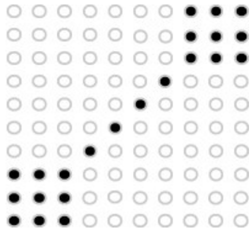

Figure 4: Kord's task putting two examples together.

Since this way of experiencing an equivalence relation has been completely hidden by our formal account (whether formally expressed or informally) we shall give *a few* other examples. Somewhere in his *informal* account of equivalence relation and partition, Skemp[1] (1977, p.174) asks us to imagine that "we are standing on the pavement in London, and in a hurry to get to the station, then we may divide {passing objects} simply into the sub-sets {taxis} and {everything else}". (Let us further his thought experiment) Doing so, we probably could not remember when we went sightseeing in London we divided the very parent set into the sub-sets {double-decker buses designed for tourists} and {everything else}. And still in both situations we do not think of the other passing objects around the world. Given this, it seems in the most practical and/or everyday situation we, ourselves, could exemplify our second category, single-group experience!

Multiple-Group Experience

In this category, "disjoint groups" are experienced; the groups have no elements in common and the elements of each group are related to each other in one way or another. There are only three students who exemplify this category. Let us follow the youngest one (Hess, middle school student) as he dealt with the problem of giving the least amount of information for the following figure on the left, which then was abbreviated to the figure on the right: ("abbreviated" is the way that Hess describes the figure with the least amount of information)

Figure 5: Hess's abbreviation of one of his examples

Hess: For example, one, five and seven make a group (it is the first time that he uses the word "group") with each other, so I only draw five and seven, It doesn't need (to do something) for five and seven, then I see two, nine and ten make a group with each other, I do for two these, it doesn't need for nine and ten; three and six make a group too, four nothing, it make a group for itself, for five, one, no five has been done(suddenly shift to the third category); how many groups are they? It's been finished, eight, it's been finished; that's it.

And then to explain that why this abbreviated figure uniquely determines the original figure he added:

Hess: There is only one case, when we draw the diagonal, the groups are determined; and when the groups were determined there is only one case.

Now, let us enjoy the great extent of the operability of this new idea:

After examining different arguments for the intersection problem he decided to work on the abbreviated figures, since "their abbreviations are themselves" and by using them "our way would be simpler", he suggested.

Hess : Suppose we have an abbreviation, suppose I am deleting certain points, even randomly, it still remain an abbreviation; they have been divided into some groups that have no intersection with each other, certain different groups are created… so if two abbreviations have intersection the intersection is some kind of abbreviation… (In other words) the remained figure is again the abbreviation of another figure.

Reflexivity, Symmetry and Transitivity

Looking at the above categories, we now turn to consider what has happened to the three properties reflexivity, symmetry and transitivity that constitute our normative conception of equivalence relation. In many natural contexts, reflexivity is not made explicit. Family relationships allow A to be a brother of B, but A is not his own brother. Similarly, in some of the earliest formal notions relating to equivalence, the Greek notion of two lines *l, m* being 'parallel' is shown to satisfy the two properties '$a\,\mathsf{P}b$ implies $b\,\mathsf{P}a$' and '$a\,\mathsf{P}c$ and $b\,\mathsf{P}c$ implies $a\,\mathsf{P}b$'. But *a* is not parallel to itself. (How could it be? Two parallel lines have no points in common but *a* has all its points in common with itself). In the case of the example of visiting cities represented on a grid, however, the reflexive law is visible as the main diagonal of the array. (The matter is a little more subtle as the idea of 'matching' usually means matching *two* things. (See Asghari, 2004a for further details.)

Symmetry seems to be the most natural properties of a matching procedure; simply two things are matched together. To see how natural it is, let us recall the example given in matching procedure category where Ali matched up all possible pairs to guarantee examplehood of his figure; however, not quite all possible pairs! Taking symmetry of the matching procedure for granted, he only needed to match forty-five pairs of cities not ninety pairs, as he did so. The ways that our students experienced the geometrical symmetry of each example (see any one of the above examples) or the more algebraic form of symmetry (if (*a, b*) then (*b, a*)) have deeper subtleties.

Our discussion can again start with Skemp who said:

> The importance of the transitive property is that any two elements of the same sub-set in a partition are connected by the equivalence relation. (Skemp, 1977, p. 175)

This suggests that the transitive property is what that makes the vague phrase used in the second (and third) category clear; where we say that "the elements in the focal group in one way or another are related to each other." However, what our students experienced in each single group (of related elements) was the *version of transitivity* formulated above by Euclid and specified by Freudenthal as follows:

> If two objects are equivalent to a third, then they are also mutually equivalent (Freudenthal 1966, p.17).

Let us give an example. Hess is about to explain why the following figure that he has just generated is an example of a visiting law.

Hess: I am going to show that those that have commonality with four are equal to it.

And he did so. And shortly after that, while generalizing his argument he added:

Hess: For each column we check that those that are equal to it, those that must be equal to it, are they equal to it or not.

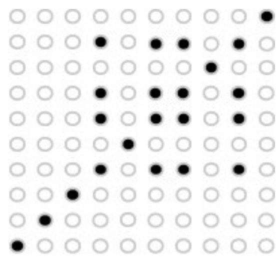

Fig. 6: An example of the visiting law

We will call this version of the property 'F-transitivity' in honour of Freudenthal (following a private communication from Bob Burn). F-transitivity ($a \sim c$ and $b \sim c$ implies $a \sim b$) is equivalent to standard transitivity when dealing with equivalence relations, but it is not satisfied by an *order* relation. The different embodiments of transitivity in order relations and equivalence relations can cause difficulties to students when they are introduced at the same time in a university foundation course (Chin & Tall, 2002).

CONCLUSIONS AND AFTERWORD

Our data suggest that by the standard (and mathematical) treatment of equivalence relation and partition in which we jump from the former to the latter and vice versa, we ignore a gap in everyday experience of the subject, i.e. single-group experience; moreover, If for some purposes we form our focal single-group by a certain matching procedure, it is likely the experience of F-transitivity (not transitivity) that saves us from matching all possible pairs though logically both amount to the same thing.

Being aware of the above deviations from *the standards* could shed some light on our standard practice of teaching equivalence relations and some of its consequences (for example, see the end of the previous section). Furthermore, the above tasks themselves could be used for teaching purposes (though we used them only as a research device).

The first part of The Task of the Mad Dictator (generating an example) was used by a lecturer in one of the top five ranked universities in the UK in a class consisting of fifteen prospective teachers. Following the task he reported:

> The students worked in groups to try to invent new visiting laws. They quickly discovered that just the diagonal and the whole grid were valid laws... one group produced a generic visiting law where each identical equivalence class was coloured the same. They independently 'discovered' the notion of equivalence classes (although they didn't use this terminology of course) and came up with the two main theorems I had on the next seminar's lesson plan.

End note

1- Skemp himself used this example to illustrate that characteristic properties do not necessarily have to have a characteristic property.

References

Asghari, H.A. (2004a). Students' experiences of 'equivalence relations'. In O. McNamara (Ed), *Proceedings of British Society for Research into Learning Mathematics*, 24, 1, 7-12.

Asghari, H.A. (2004b). Organizing with a focus on defining, a phenomenographic approach. In M.J. Hoines & A.B. Fuglestad (Eds), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics*, 2, 63-70.

Bowden, J.A. (2000). The nature of phenomenographic research. In J, A. Bowden& E. Walsh (Eds.), *Phenomenography* (pp.1-18). Melbourne: RMIT University Press.

Chin. E-T. & Tall. D. O. (2000). Making, having and compressing formal mathematical concepts. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education*, 2, 177-184.

Chin. E-T. & Tall. D. O. (2001). Developing formal mathematical concepts over time. In M. van den Heuvel-Pabhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education*, 2, 241-248.

Freudenthal, H. (1966). *The Language of Logic*. Elsevier Publishing Company.

Halmos. P. R. (1982). The Thrills of Abstraction. *The Two-Year College Mathematics Journal*, Vol.13, No. 4, (Sep 1982), 243-251

Skemp. R. R. (1977). *The Psychology of Learning Mathematics*. Penguin Books.

Tall, D. O & Chin. E-T. (2002). University students embodiment of quantifier. In Anne D. Cockburn & Elena Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education*, 4, 273–280.

HOW SERIES PROBLEMS INTEGRATING GEOMETRIC AND ARITHMETIC SCHEMES INFLUENCED PROSPECTIVE SECONDARY TEACHERS PEDAGOGICAL UNDERSTANDING

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In an undergraduate level mathematical problem-solving course, we conducted an experiment with a different methodology in the teaching of mathematical series problems to twenty-eight prospective secondary mathematics teachers. We supplemented the typical series instruction from an arithmetic focus to what we call a geo-arithmetic focus, one that focuses both on visual and analytic skills. What resulted were some inspiring revelations among these future high school teachers. We present the culminating geo-arithmetic series task, describe our interpretative methodology, and report the cases of three case-study students who reported, as a result of these tasks, initial cognitive dissonance, rich discussions in their learning groups, and ramifications for changes in their future teaching practices.

MOTIVATION

Mathematics students in sixth-century B.C. Greece concentrated on four very separate areas of mathematics (called *mathemata*): arithmetica (arithmetic), harmonia (music), geometria (geometry), and astrologia (astronomy). "This fourfold division of knowledge became known in the Middle Ages as the 'quadrivium'" (Burton, 1997, p. 88). To these early Greeks, arithmetic and geometry were as separate as music and astronomy. Mathematicians soon realized that arithmetic and geometry are not separate, and that some intriguing mathematics lies at their intersection. This report attempts to explore the beauty and richness of viewing one problem from a geo-arithmetic perspective.

Studies (e.g., Vinner, 1989) have consistently shown that students' mathematics understanding is typically analytic and not visual. Two possible reasons for this are when the analytic mode, instead of the graphic mode, is pervasively used in instruction, or when students or teachers hold the belief that mathematics is the skillful manipulation of symbols and numbers. It is clear from the literature (e.g., Lesh, Post, & Behr, 1987; Janvier, 1987; NCTM, 2000) that having multiple ways – for example, visual and analytic – to represent mathematical concepts is beneficial.

Our argument is not that one student's representational scheme is superior to another, only that students often construct vastly different personal and idiosyncratic representations that lead to different understandings of a concept. Because student-generated representations provide useful windows into students' thinking, it is productive for teachers to value these personal representations. Moreover, there is a belief among mathematics educators (e.g., Janvier 1987; Lesh, Post, & Behr, 1987) that students benefit from being able to understand a variety of representations for mathematical concepts and to select and apply a representation that is suited to a particular mathematical task. The National Council of Teachers of Mathematics (NCTM) reinforces this belief: “Different representations support different ways of thinking about and manipulating mathematical objects. An object can be better understood when viewed through multiple lenses” (2000, p. 360).

Recently, Aspinwall and Shaw (2002) reported their work with two students with contrasting modes of mathematical thinking – Al, whose mode was primarily visual, and Betty, whose mode was almost entirely symbolic. Their assertion was that students often construct vastly different personal and idiosyncratic representations, which lead to different understandings of concepts. Given problems presented graphically, Betty generally found it nearly impossible to think about the problem in graphical terms; thus, she translated from the graphic representations to symbolic representations, or equations, in order to make sense of the problems. Once she completed analytic operations on the symbols, she translated the problem back to the graphic representations required for the tasks. Al, however, operated directly on the graphic representations without having first to translate to symbolic representations. Betty and Al showcased two very different ways of solving problems, but the study suggested that if students could move freely between the visual (geometria) and the symbolic (arithmetica), their mathematical understanding would be much richer and their problem-solving abilities more robust.

Krutetskii (1976) distinguished among three main types of mathematical processing by individuals: analytic, geometric, and harmonic. A student who has predominance toward the analytic relies strongly on verbal-logical processing and relies little on visual-pictorial processing. Conversely, a student who has predominance toward the geometric relies strongly on visual-pictorial processing predominating over above-average verbal-logical processing. A student who has predominance toward the harmonic relies equally on verbal-logical and visual-pictorial processes. Several aspects of Krutetskii's position are of relevance in our interpretation of the ways that our students, comprising both analytic and geometric, processed mathematical series problems demonstrated geometrically. The use of Krutetskii’s categories permitted us to explore their thinking in the context of their cognitive processing.

The National Council of Teachers of Mathematics (NCTM, 2000) states that problem solving with an array of creative problems is an essential component in students’ construction of meaningful mathematical content. “In high school, students’ repertoires of problem-solving strategies expand significantly because students are capable of employing more-complex methods and their abilities to reflect on their knowledge and act accordingly have grown” (p. 334). The following is one of those creative problems that we developed to generate students’ interests and to engage them in discussing mathematical content as well as geo-arithmetic issues of learning and teaching.

MATHEMATICAL PROBLEM

The teacher stands at the front of the room with a bag and begins to remove four cubes, with side lengths from 1 cm to 4 cm. After ensuring all the students see the four cubes, the teacher returns the cubes to the bag, shakes the bag, then slowly withdraws from the bag … the four cubes? No, she withdraws not four cubes but one single square with side length 10 cm. The students were amazed by this extraordinary feat of conversion of 4 cubes into a square. (For them, it represented a conversion of three-dimensional cubes into a two-dimensional square.)

From an arithmetic perspective, this problem can be represented by the following equation, $1^3 + 2^3 + 3^3 + 4^3 = 10^2$. One student remarked that the conversion was true when using 1, 2, or 3 cubes as well. Another student asked, "Does placing consecutively larger cubes into the magic bag always produce a square with this intriguing property; that is, does this equality always hold: $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$?" A mathematical induction approach is sufficient to show that this relationship is true for any natural number, n. We leave these familiar induction steps for the reader.

From a geo-arithmetic perspective, we can look at this generalized problem in a richer way. First we consider the square, in Figure 1, with size $(1 + 2 + 3 + \cdots + n) \times (1 + 2 + 3 + \cdots + n)$. We divide this large square into smaller squares and rectangles, and calculate the areas of these squares and rectangles based on their dimensions – lengths and widths. But we will add the areas separately based on their placement in groups that we will designate as the Diagonal, Bricked, Vertical-Line, Dotted-Line, and Horizontal-Line regions. Finally, we will demonstrate that the sum of each of these regions is a cube so that the area of the square is the sum of the cubes.

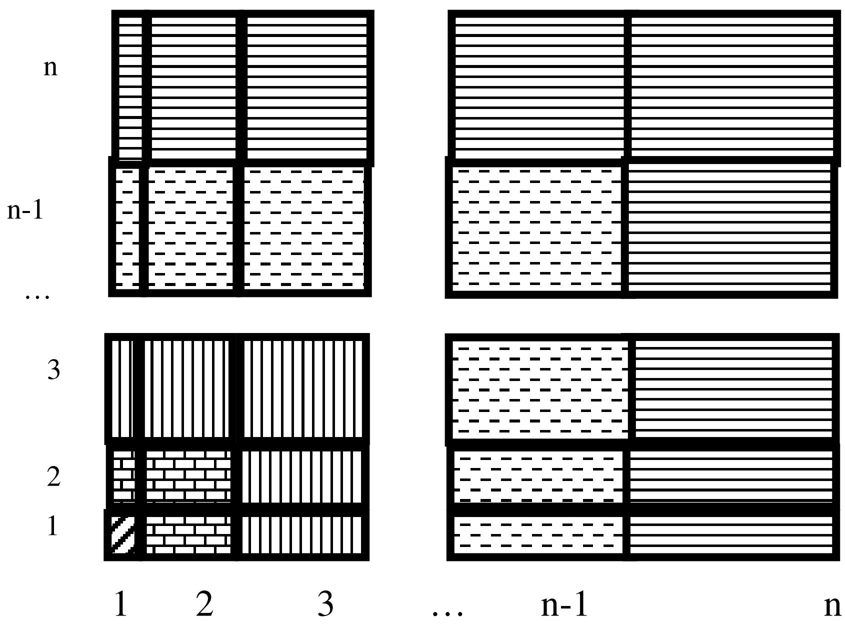

Figure 1: Generalized problem, regions of the square

Sum of the Diagonal Region

$$1 = 1^3$$

Sum of the Bricked Regions

$$1x2 + 2x2 + 2x1 = 2(1+2) + 2x1 = 2((2x3)/2) + 2x1 = 2(3+1) = 2x2^2 = 2^3$$

Sum of the Vertical-Line Regions

$$1x3 + 2x3 + 3x3 + 3x1 + 3x2 = 3(1+2+3) + 3(1+2) = 3[(3x4)/2] + 3[(2x3)/2] =$$
$$3[(3x4+2x3)/2] = 3x3(4+2)/2 = 3^2x3 = 3^3$$

Sum of the Dotted-Line Regions

$$1(n-1) + 2(n-1) + 3(n-1) +\ldots+ (n-1)(n-1) + 1(n-1) + 2(n-1) + 3(n-1) +\ldots+ (n-2)(n-1) =$$
$$(n-1)(1+2+3+\ldots+(n-1)) + (n-1)(1+2+3+\ldots+(n-2)) =$$
$$[(n-1)(n-1)n]/2 + [(n-1)(n-2)(n-1)]/2 = [(n-1)^2n]/2 + [(n-1)^2(n-2)]/2 =$$
$$[(n-1)^2(n+n-2)]/2 = [(n-1)^2(2n-2)]/2 = [(n-1)^2 2(n-1)]/2 = (n-1)^3$$

Sum of the Horizontal-Line Regions

$$1n + 2n + 3n +\ldots+ n(n-1) + nn + 1n + 2n + 3n +\ldots+ n(n-1) =$$
$$n(1+2+3+\ldots+n) + n(1+2+3+\ldots+n-1) = n[(n(n+1)/2] + n[(n-1)(n)/2] =$$
$$n^2(n+1)/2 + n^2(n-1)/2 = n^2[(n+1)+(n-1)]/2 = n^2(2n)/2 = n^3$$

Now, we have as the sum of the areas of the subdivided square:

+	Sum of the Diagonal Region:	1^3
+	Sum of the Bricked Regions:	2^3
+	Sum for the Vertical-Line Regions:	$3^3 + \ldots$
+	Sum for the Dotted-Line Regions:	$(n-1)^3$
+	Sum for the Horizontal-Line Regions:	n^3
=	Area of the square:	$(1+2+3+\ldots+n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3$

A series of other geo-arithmetic problems, similar to this one, was presented to the students over a period of 6 weeks, culminating with the problem above. During the entire semester, students were negotiating these ideas within the small groups of the class, and although many students had valuable insights, we report the thinking of three students as they seemed representative of the students as a whole.

METHODOLOGY

Twenty-eight students (pre-service high school mathematics teachers) from one senior-level mathematical problem solving class participated in the study. Analyzing their responses to Presmeg's (1986) theoretical framework, we determined that some students were non-visual and that others tended to process information visually. Of the three students we chose for interviews, one was visual (Emily) and two non-visual (Ryan and Sara). Students in the class responded to written and oral tasks and questions, and the case studies consisted of students' responses to questions about the classroom activities. In general, the aims of our study were to arrive at a comprehensive understanding of the role of students' personal and idiosyncratic representations in their learning and to develop general theoretical statements about their learning processes.

We explored students' thinking on tasks designed to probe their different ways of understanding and representing series problems. Using multiple sources of qualitative data (e.g., audiotapes of interviews with students, transcripts of those tapes, researchers' fieldnotes, worksheets of case study students, and two researchers' journals), case study analyses were undertaken to identify patterns and changes in students' understanding. In particular, we report how their work on these series problems presented geo-arithmetically influenced the ways they thought about teaching. Analyses of taped sessions included coding of transcripts. We triangulated the data to identify common and distinct strands.

STUDENTS' EXPLORATIONS

As we began investigating these students' geo-arithmetic concepts, assertions in three domains arose from the data: Cognitive Perturbation, Learning Group Dynamics, and Pedagogical Implication. We discuss each of these below with data that support each assertion.

Cognitive Perturbation

Perturbation, although often characterized as negative, is an essential cognitive component of change; to learn and grow, teachers must face cognitive dissonance (Shaw & Jakubowski, 1991). Such dissonance may cause frustration, but can also lead to reflection. We found this task caused students a great deal of reflection as the task was geo-arithmetic and students tended to have a preference toward either the geometric (visual) or the arithmetic (analytic). Thus, non-visual students experienced cognitive dissonance thinking about the visual components, and, similarly, visual students thinking about the analytic (arithmetic) part of the problem saw this as a perturbation.

Ryan, the non-visual thinker above, was initially frustrated by our asking him to solve the series problems geometrically; he said he had always thought "in equations." Ryan said that being confronted with problems presented visually had altered the way he thought about mathematics and his future role as a teacher. But Emily stated that she was

> extremely visual. I have to see things done out; I am sometimes not confident in my mathematical abilities, my algebra skills. I know what I am doing but I am afraid [of mistakes]. If I can do it visually, I know I am on the right track.

She claims that she has a good "3-D mind" and that her "last resort is to write an equation out." She confessed that she looks at problems in creative ways and ways that are "out of the norm." She asserts it "is easier for me to conceptualize it that way." Though Emily was comforted by the blended visual/analytic problem, just because it was partly visual, she found herself mentally challenged as she tied together the visual and analytic aspects of the problem. She said, "I was struggling with the problem algebraically, I did not feel confident in myself."

Ryan said his first approach was to try to write an equation; but Emily's approach was much different. When we asked Emily whether she thought these series problems were algebraic or geometric in nature, she said, "It was a blend for me. You needed to know the algebra behind it, but you had to have that geometry, spatial sense, in order to see the problem." When we asked her how she thought about the problem presented above, she responded, "With the series problems, I had to picture a physical cube, with them lined up next to each other, and figure it out from there."

Sara reported that she found that the inductive proof to be easy, but had "a hard time visualizing it." She said she would "never have thought about the geometric aspect of it." She also stated that it "was confusing to me, and I would still solve them algebraically and then convert it." Recollecting the problem later, after we had given the students cubes for modeling the problems, she said,

> "Once we had the manipulatives, … I can remember working with the actual blocked cubes, colored blocked to build the cubes and then see how they unfolded to make the square. And when I actually had hands-on something to work with, it was a little easier for me to see it, because I wasn't having to depend on my spatial sense."

Here she notes that having physical manipulatives was an aid to her understanding as she had difficulties with mentally picturing the problem. Though the manipulatives were beneficial to her, she still relied on the analytic as her absolute,

> And I still think even though the visual representations were effective, they're not a proof to me. I would still have to do it algebraically for it to verifiably be true in every case.

Learning Group Dynamics

During group activity, Ryan reported he was able to see how some students process information geometrically as he worked through the problems. What was striking was that as a result of the group activities, he felt he would be a better teacher in relating to visual and non-visual learners. "They taught me how to think about a problem so that if you are trying to reach someone who does not think just in numbers, well, you can help the student to see the problem visually."

Sara was also influenced by working within her groups. She said,

> It showed me that there are more visual aspects to math than I ever would have thought…. In the past I tended to rely on algebraic methods to solve problems and now I might be more willing to look at it visually and to think about whether or not my answer makes sense geometrically and visually.

Sara contrasted the way her group partner worked the problem, "She was very visual and I was very non-visual, but together we somehow always seemed to find a solution… we could always find some way to make sense for both of us." She valued working with someone who was a visual thinker,

> I think that if I didn't have someone like that to work with, who looked at it completely different, I would've kept trying the same things over and over and over again, and never have found a solution.

Pedagogical Implication

Our students reported that the activities had altered the way they thought about their future careers in teaching high school. Ryan's experience with the geo-arithmetic problem "opened my eyes to a new way of seeing things that I had never been exposed to before. I consider myself to be not just a better problem solver, but a better teacher seeing how other students are going to see things." Furthermore, he explained,

> Before, I was only thinking of the equations, and I thought everyone else was too. My idea was that everyone was going to learn by my [symbolic] teaching. I wasn't open to visual teaching. Now I'm thinking differently, out of my comfort zone.

Emily reflected on her future teaching practice, "Before these problems, I would have had to just go by the book, teach by breaking the equations down into smaller parts algebraically." As a result of doing these geo-arithmetic problems, she asserted,

> I want to try to incorporate this (visual aspects) into my teaching, into as many lessons as possible. Because I now know I am that kind of thinker (visual), I know there are others like me. Based on this I want to try to accommodate all the different kinds of thinking. I will have to teach it purely algebraically for those who don't think visually. I want to try to incorporate as much visual as I can, and that will help the algebra (analytic) people to see it differently too. Maybe I can create a future engineer. And the people who are visual need to know the numbers, how the equations work and not have to see it visually.

Emily clearly saw a need to provide a balanced approach in teaching students both the analytic and the visual components of problems. Sara stated that,

> In teaching, definitely, I think that I would use more visual aspects, because at least for me as a student it was easier to see why things made sense, because you could visually look at it and tell, as opposed to algebraic methods where you had to think about it and see if it reasoned out.

Since Sara states that she is non-visual, we asked Sara specifically, "What are the ramifications for the non-visual students if she presented something visually?" Sara responded,

> I think you would have to show it the algebraic way, the inductive way, the proof way, and then show it visually to kind of illustrate why it works. And I think that, at least for me, as a non-visual thinker, it still made sense for me to look at it visually.

CONCLUSIONS

We believe students develop mathematical power by learning to recognize an idea embedded in a variety of different representational systems and to translate the idea from one mode of representation to another. A positive result of multiple instructional representations of concepts is that students who are prospective teachers learn to construct and to present representational schemes with which they might not be comfortable.

The geo-arithmetic problems had positive implications for each student in class and in particular, the three students that have been mentioned in this paper. The problems, along with the group interactions caused students to reflect on how they think, whether it be predominantly visual or analytic. They were able to see from their colleagues that not everyone thinks they way they do. The pedagogical discussions were rich in that these prospective teachers began to describe how they might deal with various modes of students' representations in their own classes, especially students who may have a predominance that differs from theirs. The authors intend to continue to investigate how geo-arithmetic problems positively perturb prospective mathematics teachers in their own thinking about mathematics learning and what impact these problems may have on their pedagogical content knowledge.

REFERENCES

Aspinwall, L., and Shaw, K. (2002). Representations in calculus: Two contrasting cases. *Mathematics Teacher*, 95, 434-440.

Burton, D. (1997). The *history of mathematics: An introduction*. New York: McGraw-Hill.

Janvier, C. (1987). Translation processes in mathematics education. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 27-32). Hillsdale, N.J., Lawrence Erlbaum Associates.

Krutetskii, V. (1976). *The psychology of mathematical abilities in school children*. Chicago: The University of Chicago Press.

Lesh, R., Post, T., & Behr, M. (1987). Representations and Translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 33-40). Hillsdale, N.J., Lawrence Erlbaum Associates.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Presmeg, N. (1986). Visualization in high school mathematics. *For the Learning of Mathematics*, 6(3), 42-46.

Shaw, K. L., & Jakubowski, E. H. (1991). Teachers changing for changing times. *Focus on Learning Problems in Mathematics, 13*(4), 13-20.

Vinner, S. (1989). The avoidance of visual considerations in calculus students. *Focus on Learning Problems in Mathematics, 11*(2), 149-156.

DEALING WITH LEARNING IN PRACTICE: TOOLS FOR MANAGING THE COMPLEXITY OF TEACHING AND LEARNING

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Drawing on the so-called CULTIS model of learning theories developed while working with students in the UK and Denmark and insights gained through the experiences of teachers' learning through Networks of Learning developed in Pakistan, we suggest that the complexity of learning can be tackled with the CULTIS model at the conceptual level and can be supplemented while taking insights from the experiences of working through the Networks of Learning. An example of the Network of Learning is the Mathematics Association of Pakistan (MAP). The paper also discusses the implications of how the juxtaposition of CULTIS and Networks for Learning can be used to develop mathematics teachers' understanding for various demands of learning mathematics in an informed manner.

INTRODUCTION

This paper brings in the experiences and ideas developed by each author. Sikunder Ali Baber (SAB) has worked on Networks of Learning and further theorized on this through the creation and continually running of various activities of Mathematics Association of Pakistan (MAP). SAB has chaired MAP the last four years. Bettina Dahl (BD) developed the CULTIS model of learning theories during her Ph.D. study. Below this model is explained. At the end of the paper, we discuss why we think it is necessary to combine both approaches to tackle the complexities of learning theories.

NETWORKS OF TEACHER LEARNING

What are networks? It is difficult to find one suitable definition of a network given the range of purposes for which they are established. However, Clarke (1996) quotes a useful definition proposed by Alter and Hage (1993, p. 46): "Networks constitute the basic social form that permits inter-organizational interactions of exchange, concerted action, and joint production. Networks are unbounded or bounded clusters of organizations that, by definition, are non-hierarchical collectives of legally separate units. Networking is the art of creating and/or maintaining a cluster of organizations for the purpose of exchanging, acting, or producing among the member organizations" (Clarke, 1996, p. 142). Darling-Hammond and McLaughlin (1995) have stressed the importance of networks as a powerful tool in teacher learning for both pre-service and in-service teachers, as cited by the report named Networks@Work (Queensland Board of Teacher Registration, 2002). Networks provide the 'critical friends' or 'peers' that teachers need to be able to reflect on their own teaching experiences associated with developing new practices in their

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 97-104. Melbourne: PME.

classrooms. Teacher networking often provides an opportunity for teachers to visit the various schools of participants and to gain 'practical pedagogical clues' (Moonen and Vooget, 1998, p. 102), from other teachers' classrooms. Also, "Professional relationships forged outside the immediate working environment enable teachers to gain valuable insights into new knowledge and practice beyond that gained from interactions with colleagues in their own schools" (Board of Teacher Registration, 1997, pp. 6-7). Lieberman (1999) says that "Networks are becoming popular, in part, because they encourage and seem to support many of the key ideas that reformers say are needed to produce change and improvement in schools, teaching, and learning".

Networks therefore seem to provide:

- Opportunities for teachers to both consume and generate knowledge;
- A variety of collaborative structures;
- Flexibility and informality;
- Discussion of problems that often have no agreed-upon solutions;
- Ideas that challenge teachers rather than merely prescribing generic solutions;
- An organizational structure that can be independent of, yet attached to, schools or
- universities;
- A chance to work across school and district lines;
- A vision of reform that excites and encourages risk taking in a supportive environment; and
- A community that respects teachers' knowledge as well as knowledge from research and reform (Lieberman and Grolnick, 1997).

Various writers (e.g., Darling Hammond and McLaughlin, 1995; Smith & Wohlstetter, 2001; Lieberman & Wood, 2003) have identified two distinctive features that teacher networks exhibit in their pursuit to better support teachers' learning on a regular basis:

Personal and Social Relationships: improved relationships, flexibility, risk-taking, commitment, openness in interacting with each other and clarifying values and expectations.

Academic and Professional Aspects: innovation, enriching practice, continual development of teachers focused on professional concerns such as student learning, sharing and getting relevant professional information (dissemination), developing healthy and shared norms, enriching curriculum and influencing policy makers.

Networks should also continually get engaged in the process of diversifying their activities and programs so that evolving and changing needs can be accommodated. This requires training of network leaders in managing the complex relationships and meeting the evolving needs in an effective manner. Also networks can get engaged with processes of follow-up of their professional development activities through engaging different individual and institutional members. These follow-up activities can also help participants to develop insights into the issues that the professional networks are supposed to tackle. This continual sharing of professional practice of teachers within the networks can help all the participants to develop the culture of evidence essential to develop teaching practice along professional lines.

Why are networks important in the context of Pakistan?

Recently Aga Khan University Institute for Educational Development (AKU-IED) in Pakistan, a leading Institute mandated to uplift the quality of education through its innovative programs and research initiatives, has supported six professional associations; namely, Mathematics Association of Pakistan (MAP), School Head Teachers Association of Development of Education (SHADE), Science Association of Pakistan (SAP), Pakistan Association of Inclusive Education (PAIE), Association of Primary Teachers (APT) and Association of Social Studies Educators and Teachers (ASSET) to form a network called Professional Teachers Associations Network (PTAN). This network has some funding support from the Canadian International Development Agency (CIDA). The overarching aim of this Network is to promote an enabling environment for the professional growth and development of educators from diverse backgrounds, as a contribution to the improvement of education in Pakistan (PTAN Funding Proposal, unpublished). In the funding proposal of PTAN, an assessment is made about the status of teachers in Pakistan. It states: "Teaching in the context of Pakistan continues to remain as a neglected profession thus leading to poor status for the teachers within society. This status quo also remains prevalent due to the absence of networking amongst Pakistani teachers and an authentic platform to raise genuine issues to broader audiences as well as to support their own professional development. Pakistani teachers today, find themselves as an ignored identity, in most educational reforms and quality improvement initiatives in the country. This despondency has further perpetuated nonchalance and lack of conviction within their profession leading to the educational system working in a dismal situation" (PTAN Proposal, unpublished p. 1.). PTAN, through its constituent members is helping teachers from different sectors (public, private not-for-profit and private for profit) to come together and discuss their professional matters in a more open manner and develop a collaborative strategy to approach their professional matters. For example, the composition of working committees of these professional associations is made up with fair representation of teachers from all the constituencies such as government and private and other non-governmental organizations that they are serving. This coming together of teachers from different sector schools helps members of these networks to understand their particular issues and develop a holistic approach towards creating greater cooperation to deal these issues on a more sustained and focused manner.

MAP was established as a professional association of mathematics teachers to upgrade the quality of mathematics education in Pakistan. Since its inception, July 4, 1997, it has been committed to providing a learning platform for all those related to the field of mathematics education whether directly or indirectly. MAP has adopted a three-pronged approach to address the matter of the continuing development of mathematics teachers. Firstly, it has created and structured focused programs for mathematics teachers both pre-service and in-service to provide opportunities for them to interact freely with each other on professional matters. For example, MAP organizes a regular workshop every month on various topics such as teaching fractions meaningfully or geometry - making connections etc.

Secondly, for children to develop positive attitude towards mathematics, MAP has been very active in organizing separate programs for them. In these programs, the children have opportunities to work in teams to experience mathematics as an interesting and challenging subject. MAP has also organized three Olympiads for children of different grade levels to work on interesting and challenging mathematics in a collaborative fashion.

Thirdly, in order to create a strong support mechanism for teaching and learning worthwhile mathematics, MAP has worked on various projects where important stakeholders are being encouraged to re-learn mathematics so that they can see the broader role of mathematics in daily life situations. In this regard, MAP has been actively engaged into the process of rewriting textbooks with the Provincial bodies such as Sindh Text Book Board, a policy level body to design and produce text books for the province of Sindh in Pakistan. In Pakistan not too distant the government regulates the guidelines of mathematics curriculum to be taught at secondary and high schools in Pakistan. Also the governmental agencies have been significantly involved in the production of the textbooks of mathematics.

MAP is also organizing workshops for parents so they can see what it means to learn mathematics and how they would be able to support children's mathematics understanding. This work with the wider society enables MAP to create greater synergy and networking amongst different stakeholders to achieve quality mathematics education within Pakistan and beyond. Within this scenario the learning of mathematics can be seen as an important subject for making informed decisions in today's fast changing world.

CULTIS AND ITS SIGNIFICANCE FOR TEACHER LEARNING

Dahl (2003, 2004) developed a model combing a number of different widely recognized and classical learning theories. This was done as part of a study on high-achieving Danish and UK high school students' mathematics learning strategies. To have a range of possible analysis, mainly the following theorists were used: von Glasersfeld (1995), Hadamard (1945), Mason (1985), Piaget (1970), Polya (1971), Skemp (1993), and Vygotsky (1962, 1978). These theories express themselves in various categories: Consciousness-Unconsciousness; Language-Tacit; Individual-Social (CULTIS). The categories cut the theories into modules that to some extent interact and overlap but each category has nevertheless its own identity.

Category 1: Consciousness

Polya described four phases for working on a mathematics problem. First: understand the problem; second: device a plan; third: carry out the plan, and the fourth is to examine the solution. The student should also be motivated and "desire its solution" (Polya, 1971, p. 6). Since it is a practical skill to solve problems and since we require all practical skills by imitation and practice, this also applies for solving mathematical problems (Polya, 1971, p. 4-5). Mason writes that practice is important but without reflection it may leave no permanent mark. Time and a questioning, challenging, and reflective atmosphere is also needed (Mason, 1985, p. 153). This reflects many teachers' and students' experience that through practice and repetition, one gets a feeling for the mathematics but also that if one only learns a technique, an algorithm, then soon after, these are forgotten.

Category 2: Unconsciousness

Hadamard (1945, p. 56) states that there are four stages in learning: preparation, incubation, illumination, and verification. Conscious work (preparation) is therefore preparatory to the illuminations. Polya states that "only such problems come back improved whose solution we passionately desire ... conscious effort and tension seem to be necessary to set the subconscious work going" (Polya, 1971, p. 198). This is the experience that after one has worked on a problem, one leaves it, and then later one feels a sudden shed of lighting and everything is clear. The illumination is generally preceded by an incubation phase where the problem solving is completely interrupted (Hadamard, 1945, p. 16). Teachers can organize time for the incubation phase e.g. through repetition and after the illumination spend time on verification, as in Category 1, to reflect consciously on the unconscious inputs.

Category 3: Language as thinking-tool and concept formation

Vygotsky describes language as *the* logical and analytical thinking-tool and that thoughts are not just expressed in words but come into existence through the words (Vygotsky, 1962, p. viii & 125). Mathematics is also itself a language, wherefore the formations of concepts are an essential part of learning mathematics. A basic principle in concept formation is that all concepts, except the primary ones, are derived from other concepts and they take part in the formation of higher order concepts (Skemp, 1993, p. 35). It is therefore important to let new concepts build on old ones and that these old ones are firmly learnt. These concepts form a schema in the student's mind and if a concept is learnt and understood, the student does not need to remember it, he knows it. A change in a schema is always difficult since the existing schema needs to change (accommodate) when it is inadequate to assimilate new knowledge. Assimilation of new knowledge to an existing schema gives however a feeling of mastery (Skemp, 1993, pp. 29-42).

Category 4: Tacit knowledge and obstruction by language

Hadamard argued that thoughts die when they are embodied by word but that signs are nevertheless necessary support of thought (Hadamard, 1945, p. 75 & 96). Piaget (1970, p. 18-19) states that "the roots of logical thought are not to be found in language alone, even though language coordination is important, but are to be found more generally in the coordination of actions, which are the basis of reflective abstraction". Individual actions are thus the root of mathematical thought. In relation to tacit knowledge, one can observe that a person has a certain kind of knowledge but when one asks the person he is not aware that he knows this (Polanyi, 1967, p. 8).

Category 5: Individual

Constructivist epistemology is that "knowledge … is in the heads of persons, and that the thinking subject has no alternative but to construct what he or she knows on the basis of his or her own experience (Glasersfeld, 1995, p. 1). Piaget argues that the basis of abstraction is the action, not the object (Piaget, 1970, p. 16-18). The individual is therefore active and learning comes as the individual manipulates with the objects and reflects on this. These reflective abstractions are based on coordinated actions, not individual actions. Examples of coordinated actions are actions that are joined together or who succeed each other (Piaget, 1970, p. 18). Furthermore: "To know is to assimilate reality into systems of transformations. … knowing an object does not mean copying it - it means acting upon it" (Piaget, 1970, p. 15). Students therefore need to manipulate e.g. with concretization materials, algebraic concepts, or geometrical figures. It is important to leave time for students to do this individually since learning happens as the individual interacts with the surrounding.

Category 6: Social

Social interaction plays a fundamental role in shaping students' internal cognitive structure. This is a gradual process that has two levels: "first between people … and then inside the child" (Vygotsky, 1978, p. 56-57). In the beginning a teacher controls and guides the student's activity but gradually the student takes the initiative and the teacher corrects and guides, and at last the student is in control and the teacher is mainly supportive. The potential for learning is limited to the "zone of proximal development (ZPD)" (Vygotsky, 1978, p. 86), which is the area between the tasks a student can do without assistance and those that require help. The teacher is essential since on his own, the student might not enter his ZPD. Verbal thinking is an example of a social activity since "audible speech brings ideas into consciousness more clearly and fully than does sub-vocal speech" (Skemp, 1993, p. 91-92). Vision is therefore individual and hearing is collective (Skemp, 1993, p. 104). The students should appropriate and internalize. Also discussions among classmates facilitate learning.

CONCLUSIONS

A conclusion in Dahl (2004) is that if a teacher uses teaching methods that are too far away from teaching styles the students are used to, learning becomes difficult. However, the study also confirms that students learn in a variety of ways. Hence balance and eclectism is necessary. This does however not mean that anything is as good/bad as anything else but the teaching style must be targeted towards the specific students. Networks are good at helping teachers establishing new practices in their classroom and CULTIS would be useful to gain input to ensure that the "area" of possible student learning processes is covered.

Networks also respect both teachers' knowledge and knowledge from research. CULTIS could therefore also be a tool from which to discuss the teachers' experience. The teachers might in some of the theories recognize elements of ideas that they have developed from their experience. Kilpatrick argues: "Why is it that so many intelligent, well-trained, well-intentioned teachers put such a premium on developing students' skill in the routines of arithmetic and algebra despite decades of advice to the contrary from so-called experts? What is it that teachers know that others do not?" (Kilpatrick, 1988). CULTIS is a holistic approach and we assume that since CULTIS shows a broad range of different theories, CULTIS might give teachers a language for theories that are not "in" for the moment and give them some arguments and reasons to hold on to their old stuff. We assume that any teacher in CULTIS can find something that "fits" the teacher's own ideas. At the same time CULTIS might give the teachers new insight. It might therefore be a "safe" arena for discussing professional matters in an open manner and hopefully also create some openness for other ideas. Diversity of ideas, trust, and teachers feeling that they are being valued are also essential elements in Networks of Learning.

Networks provide flexibility, informality, and a forum for discussing problems that often do not have an agreed-upon solution. This fits with CULTIS's "neutrality" since it exhibits a wide range of learning theories. These theories are different, opposing, but they have been widely accepted at some point in time. They are thoughts where one might foresee revised versions recurring in the future. This insight is based on Hansen (2004) who argues that there seems to be pendulum swings between child centered/understanding and content centered/skills in the mathematics curriculum reforms. The teachers can disagree with the theories in CULTIS, but they nevertheless need to know the existence of these theories partly since it can provide insight into how to tackle individual student's learning, and partly since it will give the teachers a tool to "recognize" the theoretical roots of future new theories and/or reforms.

In Pakistan the Networks of Learning have up to now not focused on learning theories, but the CULTIS model could be a useful tool for the continual development of teachers focused on professional concerns such as student learning. The implementation of CULTIS into Networks of Learning has not yet happened but based on the experience we anticipate that this will be a useful tool to tackle the complexity of learning.

References

Board of Teacher Registration (April 1997). *Partnership in Teacher Education.* Australia, Queensland: Board of Teacher Registration, Toowong.

Clarke, David (1996). *Schools as Learning Communities.* London: Cassell.

Darling-Hammond, L., & Mc Laughlin, M. W. (1995). *Policies that support professional development in an era of refor*m. New York: National Center for Restructuring Education, Schools, and Teaching.

Dahl, B. (2003). What can we learn about cognitive learning processes by asking the pupils? In N. A. Pateman, B. J. Doherty, & J. Zilliox (Eds.), *Proc. 27th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 277-284). Honolulu, USA: PME.

Dahl, B. (2004). Analysing cognitive learning processes through group interviews of successful high school pupils: Development and use of a model. *Educational Studies in Mathematics, 56*, 129-155.

Hadamard, J. (1945). *An Essay on The Psychology of Invention in the Mathematical Field.* New York: Dover.

Hansen, H.C. (2004). Skolematematikkens formative årtier 1903-1937 [School mathematics 1903-1937]. In *Proc., 2nd Nordic Conference*, (pp. 69-80) NTNU, 17-18 November 2003. NTNU: Norwegian Centre for Mathematics Education.

Kilpatrick, J. (1988). Editorial. *Journal for Research in Mathematics Education, 19* (4).

Lieberman, A. (1999). Network*s. Journal of Staff Development, 20* (3).

Lieberman, A., & Grolnick, M. (1997). Networks, Reform and the Professional Development of Teachers. Chapter 10 in A. Hargreaves (Ed.)*, RethinkingEducational Change with Heart and Mind.* Alexandria VA: Association for Supervision and Curriculum Development, pp. 192-215.

Lieberman, A., & Wood, D. (2003). From Network Learning to Classroom Teaching. *The Journal of Teacher Education and Change*, 3, 3 and 4, pp. 315-337.

Mason, J., Burton, L., & Stacey, K. (1985). *Thinking Mathematically.* Amsterdam: Addison-Wesley.

Moonen, B., & Voogt, J. (1998). Using Networks to support the Professional Development of Teachers. *Journal of In-service Education, 24* (1), 99-110.

Piaget, J. (1970). *Genetic Epistemology.* New York: Columbia University Press.

Polya, G. (1971). *How To Solve It - A New Aspect of Mathematical Method.* Princeton: Princeton University Press.

Polanyi, M. (1967). *The Tacit Dimension.* London: Routledge.

Queensland Board of Teachers Registration (2002). *Networks@Work: A report of Queensland Consortium for Professional Development in Education*. Toowong, Brisbane, Australia: Queensland Board of Teachers Registration.

Skemp R. R. (1993). *The Psychology of Learning Mathematics.* London: Penguin.

Smith, A., & Wohlstetter, P. (2001). Reform Through School Network: A new Kind of Authority and Accountability. *Educational Policy, 15* (4), 499-519.

Von Glasersfeld, E. (1995). *Radical Constructivism: A Way of Knowing and Learning.* London: Falmer.

Vygotsky, L. S. (1962). *Thought and Language.* Cambridge, MA: M.I.T.

Vygotsky, L. S. (1978). *Mind in Society: The Development of Higher Psychological Processes.* Cambridge, MA: Harvard University Press.

SITUATIONS OF PSYCHOLOGICAL COGNITIVE NO-GROWTH

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We present and discuss three classroom situations where failure emerges unexpectedly after initial success and contend that they cannot be sufficiently explained by theories of psychological cognitive growth as surveyed in Tall [2004]. The discussion hinges on the social implication of psychoanalysis as developed by Slavoj Zizek [2002].

INTRODUCTION

Psychoanalysis has made its entrance into Mathematics Education via considerations of affect and cognition. Breen [2004] sought to deal with a case of a student's anxiety through a change in the teacher's attitude. Evans [2004] approaches the relationship of beliefs, emotions and motivations through the study of films that present mathematics as a work of genius. Falcão et al. [2003] discuss affect and cognition approaching the mathematics learner as *possessor of a* subjectivity that is always embedded in culture. Hannula, Maijala and Pehkonen [2004] point out that beliefs related to mathematics (self-confidence) have an influence on students' achievements. Morselli and Furinghetti [2004] consider the connection between cognitive and affective aspects and look for answers in the domain of affect. Walshaw [2004] looks for a conceptualization in Lacan and Foucault that could aid the interpretation of subjectivity. Cabral [2004], Cabral and Baldino [2004], Carvalho and Cabral, [2003] assume a Lacanian perspective and introduce the concept of pedagogical transference. The importance of framing cognition in a wider sociological frame has been demonstrated in PME28 whose main theme was "inclusion and diversity" [Gates, 2004; Johnsen Høines, 2004].

In this paper, we take advantage of another slant of Lacanian Psychoanalysis that has been developed by Slavoj Zizek [2002] and leads to the analysis of social ideological formations. We contend that there is in cognition something more than cognition itself and that, in order to apprehend this surplus, theories of psychological cognitive growth do no suffice. We make an exercise of Hegel's dialectics on Tall's [2004] survey of theories on *psychological cognitive growth* in order to show that these theories have a built-in social exclusion bias. Then we present three episodes of what we call *no-growth situations* that, as such, escape the appreciation of cognitive growth theories. We interpret these situations eliciting their implicit discourse which has the form of present day ideologies: "Yes, I know, but nevertheless…". "I know that school knowledge is important but nevertheless…" Our final discussion relates cognition to three forms of school authority that students, teachers and mathematics educators corroborate in order to disavow (the feeling of) castration: the institutional, the manipulative and the totalitarian forms. It will not be very pleasant to find ourselves as mathematics educators implicated in the support of such forms of authority, but perhaps this is the unbearable dimension of the "P" in "PME".

THE EXCLUSION BIAS OF COGNITIVE THEORIES

Tall [2004] seeks to dress an overall universal picture of PME meetings from the point of view of individual psychological cognitive growth. He makes a comprehensive survey of Piaget's empirical, pseudo-empirical and reflective abstractions, Bruner's enactive, iconic and symbolic representations, Fishbien's intuitions, algorithms and formal aspects of mathematical thinking, Skemp's perception, action and reflection types of activity, Van Hiele's levels, Dubinsky's APOS theory, Sfard's operational operational/structural theory, Lakoff's embodiment of thinking in biological activity. Grounded on the interplay of these theories, Tall attempts a synthesis intended to encompass the developments from conception to mature man and from discalculic children to research mathematicians. He arrives at "three worlds" into which cognitive growth can be categorized: the worlds of perception, of symbols and of properties. "Different individuals take very different journeys through the three worlds" he says [ibid: 285].

The reader is a little deceived since, instead of a synthesis, one could expect a global appreciation of such theories so that they could be sublated (*afhoben)* towards something new. After all, their similarities are much more striking than their differences. Why are there so many theories focusing on the same object, namely, psychological cognitive growth? Besides, they do not stem from an effort to make sense of a large amount of empirical data; on the contrary, they rely more or less heavily on their respective authors' introspection. Experiences and studies tend to confirm, infirm or answer specific questions put by the theory, rather then to discover and tackle new phenomena.

From a philosophical point of view, the general idea of growth implies a change in magnitude while a certain basic entity keeps its identity invariable: the "individual" who transits through the "worlds" remains an invariable seat of magnitude. "A magnitude is usually defined as that which can be increased or diminished" [Hegel, 1998:186]. Hegel shows that this is a circular definition: "magnitude is that of which the magnitude can be altered" [ibid] but instead of discarding the definition as we would do in mathematics, he takes it up as the starting point of the very Notion of magnitude. Indeed, the definition has the merit of pointing out the external agent, the *author*, who first thought of it as a reasonable one. It is the author who provides the invariable background against which growth can be thought.

In so far as theories of psychological cognitive growth refer to mathematics, they rely on a scale of values based on mathematical knowledge itself, a hierarchy rising from numerical pre-linguistic to the axiomatic and formal. Their authors speak from the position of one who has reached the apex of the stages or levels of their scales. They focus on psychological cognitive growth from the perspective of an *autonomous ego* hovering over the changes of magnitude of others, out of reach of any criticism.

Considering the transformation of quantity into quality, Hegel warns us that a field that gets too wet ceases to be a field and becomes a swamp. At what precise amount of humidity did it become a swamp? Which hair thread one has to lose in order to be considered bald? At what precise moment a graduate student becomes a research mathematician? At what precise moment a child surpasses its discalculic condition? These are symbolic determinations and as such they are intrinsically retroactive: once they are verified it is found out that the new situation they constitute existed a little before. Why? Essentially an external agent is responsible for the declaration of the new state of affairs. In order to be able to think changes of levels or states simply as "growth", one has to abstract from the external social agent who attributes different magnitudes to an identical subtract. The identity resumes to the external social agent.

Leaving their authors out, cognitive-growth theories assume the status of scientific subject-free theoretical speeches. This effort leads to an absolute scale of values in which all subjects are positioned, the author occupying the apex. The tendency is almost unavoidable to pass from "growth" to "lack", "deficiency', "shortage", etc. This is the perverse social effect of cognitive theories. We do not claim that a further effort should be made towards a "perfect theory" that would be politically neutral. These theories represent an important logical moment. The contribution of psychoanalysis goes in the opposite direction: simply, the wills and desires of the authors must be brought to the fore. This is what we intend to do below.

THREE NO-GROWTH SITUATIONS

The episodes below were extracted from classes of two freshmen courses, one in Analytic Geometry (AG) the other on calculus (C1) given in August-December 2004 for repeaters in the engineering program of our institution. Ten students enrolled in AG, six concludes the course and four passed; twelve enrolled in C1, seven concluded and two passed. Only one student of each course was not enrolled in the other. Classes met during four consecutive 50-minutes periods on Tuesdays (AG) and Thursdays (C1) totalizing 60 periods for each course. The text book was Stewart [1999] chapter 13 for AG and chapters 1 to 4 for C1. Classes had a tutorial format assisting individuals or couples of students. Each class ended with a 40-minutes hand-in individual exercise, graded and returned to the student's scrutiny in the beginning of the following class. Very seldom students took photocopies of graded exercises. These exercises made 40% of the passing grade the other 60% came from two mid-terms and one final open-book written individual exams. Classes started with a proposition of exercises to be worked out. Students could never do more than two or three exercises per day. Pedagogical remarks stressing important points were inserted at each class as difficulties arose.

Episode 1: Mary

Mary had been our student in a high school course on elementary algebra. The only way she could solve algebraic equations was by trial and error. She entered the university, failed AG and C1 and became our student in the described environment.

In Agu-24, the exercise was: "Given points in the plane *A*, *B*, *C* and *D*, find *x* and *y* such that $AD = x\,AB + y\,AC$". Mary found the system of equations, tried to solve it by substitution but made a mistake:

$(3x-12)/2 = 3x-6$. The class of Sep-9 was dedicated to solving algebraic equations by the "method of transformations": 1) operate simultaneously on both members; 2) replace one member by an equal one. Mary showed some proficiency but made the same mistake again: $(7+14x)/x = 21$. In the class of Sep-21 we made sure that all students could solve systems of two and three equations by Cramer's rule. In Oct-10, one question of the mid-term exam was: Draw the straight lines $\mathbf{r}(t) = (5, 6) + t\langle 1, 3\rangle$ and $y = 8-3x$, write the first one in reduced form and determine their intersection up to three decimal places. Mary solved the system by substitution and this time she got it right. Would we say success?

Mary passed AG but not C1. One of the questions of the second-chance C1 final exam in Dec-21 was: Find the intersection of the tangent line to $f(x) = x^2 + \frac{1}{x}$ at $x = ½$ with the secant line through $x = 1$ and $x = 2$. Mary arrived at the system (with one wrong coefficient) and got stuck.

Mary: Where can I find "intersection of straight lines" in the book?

We showed her the topics of intersection of lines and planes, of two lines in space and the statement of the question in the mid-term exam reminding her that she had got it right. She did not have a copy of the exam with her and her classroom work with a similar question was incomplete. When she finally handed her paper in with the question blank, we checked what sense she made of lines and equations. We drew two lines with their equations $y = 2x-3$, $y = -3x+5$. She indicated the correspondence of *x* and *y* in the equation and points in the plane.

Teacher: (Pointing at the intersection): What happens at this point? What are the values of x and y?

She recognized that the same *x* and the same *y* should fit into both equations. We insisted:

Teacher: How can you find this x and this y? (She remained silent, looking at the picture.)

Teacher: Are you making trials?

Mary: Yes.

Resume: After one semester of intense tutoring work Mary reinforced her confidence in algebraic transformations and was able to solve a system of two equations by substitution. Yet, at the crucial pass/fail moment of the exam, she went back to her old high school strategy of trial and error.

Episode 2: John

In Dec-12 John was able to correctly solve the items below in the final exam.

$d\left(\dfrac{\tan(3x)+4}{\operatorname{sen}(3x^2+4x)}\right)=\ldots$, $\dfrac{d}{dx}\left(\sqrt[5]{32\,e^{3x+4}}\right)=\ldots$, $\sec(10x)\tan(10x)dx=d(\ldots)$, $\lim\limits_{x\to 0}\dfrac{x^2}{e^x-x-1}=\ldots$

He did not get a passing grade and had to take the second-chance final. In the question of finding the minimum distance of a point to a curve, he repeated the mistake that we had pointed out in his exam one week before: the derivative of $\left(x^2+3x-5\right)^2$ was simply $2\left(x^2+3x-5\right)$ and in the question of related rates he differentiated $3\cos\theta = z$ as $3-\sin\theta = dz$.

Resume: After one semester of tutoring John could show proficiency in applying the chain rule to rather involved composition of functions. However, at the final moment he seemed to have forgotten all and scribbled absurd equalities.

Episode 3: Students

During the first two weeks (Aug-19, 26) of C1 we made sure that all students could perform graphical exercises on derivatives and primitives reasonably well. Given the graph of an arbitrary function, draw tangent lines at several points, evaluate the slopes, plot the slopes as the graph of the derivative and conversely, starting from a given graph, interpret the ordinates as the slopes of a primitive and draw its graph through a given initial point. A protractor graduated in tangents was provided. The derivative was introduced as the "name" given to the slope of the tangent line and we made sure that every student could explain the meaning of this definition. Discussions of the relation of increasing/decreasing functions with the signs of derivatives were provided. In the next weeks we worked on algebraic equations (Sep–02), rules of differentiation (Sep-09), derivatives of elementary functions via limits (Sep-16) and graphs of cubics (Sep–23). Finally we came to optimization problems (Sep-30). Students were asked to read the first example in the text book. At a certain point they read: "So the function that we wish to maximize is $A(x)=2400x-2x^2$ $0\le x\le 1200$" [Stewart:278]. They had no problems so far. "The derivative is $A'(x)=2400-4x$, so to find the critical numbers we solve the equation $2400-4x=0$" [ibid]. At this point the six students in class asked "Why?"

Teacher:	Well, if you have a function like this (drawing a graph with a local maximum) how much do you think that the derivative will be at this point?
Students:	I don't know.
Teacher:	The derivative is the name of what?
Students:	(After some help for recollection): It is the slope of the tangent line.

This seemed to suffice for two of the students but the other four still could not make any sense.

Teacher:	(Showing a tangent line just a little to the left of the maximum): Is the slope of this line positive or negative? (The strategy was to move the tangent to the right until it reached the point of maximum.)

Students: I don't know. What do you mean by "slope"?

The exercises of the first two weeks had had to be retaken before they could express any connection between extreme points and derivatives. This took most of the day.

Resume: Everything that they had learned in the first two weeks about slopes and tangents was not available any more.

DISCUSSION

We presented a picture where the natural outcome would point towards growth and in many reports could be held as a bulletin of victory. Mary learned how to solve systems of equations by substitution and abandoned her empirical trial and error strategy; John proficiently learned the chain rule and all students could reasonably perform graphical correspondences between derivatives and primitives. However we went one step further and checked this success in the day after. It had fallen into a black hole! No-growth situations mean *success followed by unexpected failure*.

A new notion such as *no-growth situations* naturally faces criticism. Is it necessary? Do these situations exist at all? Arguments may contend that we did not provide enough data in support of our concept: how was the affective teacher student relation? Were the student's mistakes discussed in class? What sort of extra-class help was provided? Did the students have the necessary requisites to take a calculus course? An endless list of extra data may be required postponing the decision indefinitely or until a point is reached where the reported no-growth situation may be characterized as *failed-growth*: had the teacher behaved more friendly, had the method been adequately applied, had this or that been different, then *growth* could have occurred. True, the reported situations can be considered a peripheral problem in cognitive growth theories; we prefer to take them as a *central problem* in a new way of looking at "growth". Should we call this new look "social cognition"?

We argue that it is important but not sufficient to focus on growth when it occurs. We have to crucially consider *what the student does overnight with what he has learned during the day*, that is, what he does *outside* the school. Every day the students in the reported situations confirmed their will of becoming good professional engineers and behaved accordingly, coming to class and working hard on the exercises. However, from one day to the next they treated their learning in a way as to deny such good intentions. In our interpretation their implicit overnight discourse could be:

Mary: 'I know that my trial and error method to solve equations falls short of the course needs and I have learned other methods; nevertheless trial and error it is *my* method, my deep personal enjoyment and I will stick to it.'

John: 'I know how to operate differentials according to the strict chain rules as I have learned in this course; nevertheless I will do according to my former understanding: squares are replaced by twice the thing and cosine by minus sinus.'

Students: 'We know that what we learn in one class will be necessary for the next one; nevertheless we do not take the trouble of keeping our learning under account.'

According to such interpretations (there may be others) the reported no-growth situations may be referred to one of the three elementary structures of the exercise of authority which function socially as three modes of disavowing castration.

> Traditional authority is based on what we could call the *mystique of the Institution*. Authority bases its charismatic power on symbolic ritual, on the form of the institution as such. (…) Socrates' argument could thus actually be linked to the phrase 'I know, but nevertheless…': 'I know that the verdict that condemned me to death is faulty, but nevertheless we must respect the form of the law as such' . [Zizek,2002:249]

'I know that the value of school knowledge is questionable and that I will have to undergo training in my first job; nevertheless I believe that this knowledge represents the distinctive herald of my social group and I must endeavor to acquire it. The Emperor wears fine clothes because he is the Emperor.' The interpretations we gave of the students' overnight speeches certainly do not support this form of authority.

> The second mode corresponds to what might be called *manipulative authority*: authority which is no longer based on the mystique of the institution – on the performative power of symbolic ritual – but directly on the manipulation of its subjects. This kind of logic corresponds to a late-bourgeois society of 'pathological Narcissism' (…) constituted of individuals who take part in the social game externally, without 'internal identifications' – they 'wear social masks', 'play their roles', not taking them seriously'. (…) The social role of the mask is directly experienced as a manipulative imposture; the whole aim of the mask is to make an impression on the other. [Zizek, 2002:251.

'The social role of the school institution is directly experienced as a manipulative imposture; its whole aim is to make an impression on the other, school knowledge is useless, only the certificate counts.' Would peripheral Third-World countries typify the "late bourgeois societies" mentioned by Zizek? These countries have received the "masks" of neo-liberalism, of globalization, of free trade, of international help and loans as impostures leading to increased exploitation. It is not surprising that such an understanding reflects itself in school, splitting *knowledge* and *belief*: 'yes I know that the Emperor wears fine clothes, nevertheless I believe he is naked and I act accordingly'.

> The third mode, fetishism *stricto sensu*, would be the matrix of *totalitarian authority*. (…) The totalitarian too does not believe in the symbolic fiction in his version of the Emperor's clothes. He knows very well that the Emperor is naked (…).Yet in contrast to the traditional authority, what he adds is not "but nevertheless" but "just because": *just because* the Emperor is naked we must hold together the more, work for the Good, our cause is all the more necessary. [Zizek, 2002:252].

'We know very well that imparting upper class central countries knowledge such as mathematics, to proletarian students of peripheral Third World countries is impossible, that raising the economy of a country through education is a hopeless dream, that all the efforts in favor of Mathematics Education have had a proportionally pale effect. Just because we know, since Freud, that education is one of the four impossible endeavours, Mathematics Education is the more necessary. Commitment to it is our charming mode of disavowing castration.'

References

Araújo, C. R.; Andrade, F.; Hazin, I.; Falcão, J. T. da R.; et al. (2003). Affective aspects on mathematics conceptualization: from dichotomies to an integrated approach. N. A. Pateman, B. J. Dougherty & J. T. Zilliox (Eds), *Proceedings of the 28th PME*, vol. 2, pp. 269-276. Honolulu: CRDG, College of Education, University of Hawai'i.

Breen, C. (2004). In the Serpent's Den: Contrasting Scripts relating to Fear of Mathematics. In M. J. Høines and A. B. Fuglestad (Eds), *Proceedings of the 28th PME*, vol. 2, pp.167-174. Bergen: Bergen University College.

Cabral, T. C. B. (2004). Affect and cognition in pedagogical transference: a Lacanian perspective. In M. Walshaw (Ed.), *Mathematics Education within the postmodern*, pp. 141-158. USA: Information Age Publishing. ISBN 1-59311-130-4.

Cabral, T. C. B. & Baldino, R. R. (2004). Formal inclusion and real diversity in an engineering program of a new public university. In M. J. Høines and A. B. Fuglestad (Eds), *Proceedings of the 28th PME*, vol. 2, pp.175-182. Bergen: Bergen University College.

Carvalho, A. M. F. T. & Cabral, T. C. B. (2003). Teacher and students: setting up the transference. In *For the Learning of Mathematics*, vol. 23, nº 2, pp. 11-15. Kingston: FLM Publishing Association. ISSN 0228-0671.

Evans, J. (2004). Mathematics, popular culture and inclusion: some findings and methodological issues. In M. J. Høines and A. B. Fuglestad (Eds), *Proceedings of the 28th PME*, vol. 1, pp. 298. Bergen: Bergen University College.

Gates, P. (2004). Lives, learning and liberty: The impact and responsibilities of mathematics education. In M. J. Høines and A. B. Fuglestad (Eds), *Proceedings of the 28th PME*, vol. 1, pp.71-80. Bergen: Bergen University College.

Hannula, M. S.; Maijala, H. & Pehkonen, E. (2004). Development of understanding and selfconfidence in mathematics; Grades 5–8. In M. J. Høines and A. B. Fuglestad (Eds), *Proceedings of the 28th PME*, vol 3, pp. 17-24. Bergen: Bergen University College.

Hegel, G. W. F. (1998). *Hegel's Science of Logic*. New York: Humanity Books.

Johnsen Høines, M. (2004). Do we welcome children's mathematics? Who are we? In M. J. Høines and A. B. Fuglestad (Eds), *Proceedings of the 28th PME*, vol. 1, pp.81-86. Bergen: Bergen University College.

Morselli, F. & Furinghetti, F. (2004). Between affect and cognition: proving at university level. In M. J. Høines and A. B. Fuglestad (Eds), *Proceedings of the 28th PME*, vol. 3, pp. 369-376. Bergen: Bergen University College.

Stewart, J. (1999). *Calculus*. Books/Cole Pu. Co.

Tall, D. (2004). Thinking through three worlds of mathematics. In M. J. Høines and A. B. Fuglestad (Eds), *Proceedings of the 28th PME*, vol. 4, pp. 281-288. Bergen: Bergen University College.

Walshaw, M. (2004). The pedagogical relation in postmodern times: learning with Lacan. In M. Walshaw (Ed.), *Mathematics Education within the postmodern*, pp. 121-139. USA: Information Age Publishing. ISBN 1-59311-130-4.

Zizek, S. (2002). *For they know not what they do*. New York: Verso.

GOOD CAS WRITTEN RECORDS: INSIGHT FROM TEACHERS

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Availability of a computer algebra system (CAS) provides a catalyst for teachers to reflect on long-standing practices of teaching mathematics, including how solutions to mathematical problems should be presented. This paper reports on how four teachers implementing CAS for the first time thought about this issue over a school year. The paper analyses their contributions to group discussions about their teaching practices at the beginning and end of the school year. New practice is needed to accommodate lack of intermediate steps available when CAS is used, and a cluster of issues relating to the use of CAS syntax. Their comments show considerable reflection about personal practice, the dominance of the external examination, and concern that new expectations might favour users of some brands of CAS over others.

INTRODUCTION

New availability of a computer algebra system (CAS) in the mathematics classroom and for formal assessment provides a catalyst for teachers to reflect on beliefs and long-accepted practices about teaching mathematics. This paper will report on how a group of teachers reflected on and reconsidered their long-standing practices of how to present written solutions to mathematical problems. They were prompted to reconsider this socio-mathematical norm (Krummheuer, 1995) by the perturbation to normal practice of working with our research team to implement the first mathematics subject permitting the use of CAS in secondary schools in their region. That having a complex calculator in the classroom perturbs normal practice is a common finding in the research literature (see, for example Artigue, 2002; Guin & Trouche, 1999; Stacey, 2003), and many aspects of this have been investigated.

The four teachers, Ken, Lucy, Neil and Meg (not their real names) were participants in the CAS-CAT project (CAS-CAT, n.d.), which researched curriculum, assessment and teaching using CAS in three secondary schools. A new subject, Mathematical Methods (CAS), was accredited for the state examination system for years 11 and 12 mathematics. In MMCAS, CAS could be used for all mathematical work, including in examinations, at the teacher's or student's discretion. Further descriptions and outcomes of the project are described in Stacey (2003), Ball (2003), Flynn and Asp (2002), and VCAA (2002). Previously, only graphics calculators without a symbolic facility had been permitted. The three project schools each used a different brand of CAS, with Lucy's and Ken's classes which were at the same school using the same machines.

The project tracked the progress of the teachers and first cohort of students through the first two years of implementation – year 11 in 2001 and year 12 in 2002. At the end of 2002, the Year 12 students sat for the first externally-set state examinations in

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 113-120. Melbourne: PME.

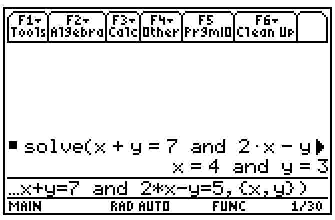
the new subject. Their results contributed to their university entrance scores, and were regarded as very important by the teachers, students and schools. Teachers were always concerned with preparing their students well for examinations - it was a very high priority for them at all times. Ball (2003) and Ball and Stacey (2004) report on the way in which students' recorded their solutions in the 2002 examinations.

During 2001 and 2002, the project team provided extensive teaching material to assist teachers and students, training in the use of CAS and discussions about pedagogy. Consequently, teachers had considered implementation issues prior to the discussions reported in this paper in 2002. The data for this paper is from two meetings. The first was held at the beginning of the school year (February 2002) and involved all teachers and the researchers. The second meeting was at the end of the school year (November 2002) and, at the teachers' request, involved the teachers only. Both meetings were audiotaped and transcribed by the researchers.

HOW AND WHY DOES CAS CHANGE WRITTEN RECORDS?

Early in the planning and implementation of MMCAS, it was evident to the research team, the teachers and also the state-appointed examination setters, that the use of CAS might require changes in the normal way in which students write solutions, and the way in which written solutions are assessed by examiners. The major reason is that, in the phrase of Flynn and Asp (2002), CAS "gobbles up" intermediate working. Figure 1a shows in TI89 syntax, how a CAS can solve simultaneous equations using one input 'solve(x+y=7 and 2x−y=5,{x,y})'. The input line is second from the bottom (above MAIN) and the calculator display above is a restatement of the input followed by the answer $x = 4$ and $y = 3$. Figure 1b shows how multiple CAS steps can often be combined into one 'nested' procedure with one line of CAS syntax and one output. The expression sin(x)cos(x) was differentiated with respect to x using syntax d(sin(x)*cos(x),x) and then x=π substituted into the derivative to give the result 1. Note in particular, that the symbolic derivative is not outputted. In examinations, this intermediate step of finding a correct derivative would often have been awarded a mark, even if the derivative is not explicitly requested.

One step solving (Fig. 1a)



Nested procedure (Fig. 1b)

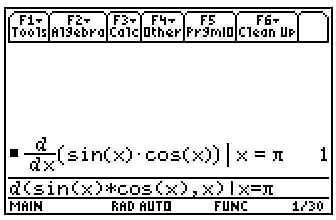


Figure 1. Examples of CAS procedures which do not provide intermediate results.

The state authority, the VCAA, were concerned that partial credit should be able to be awarded for extended response questions and approved the instructions in Figure 2 to appear on the examination papers. The third dot point is relevant here. Throughout

2002, teachers had to decide what 'appropriate working' meant for written records produced in the CAS context.

In order to provide guidelines for the teachers, the research team had devised the RIPA rubric, described with examples in Ball and Stacey (2003), to help establish good practice for the written communication of mathematical solutions. The rubric suggested that students should make sure that their written solutions to problems make the plan of the overall solution (P) clear, specify what was inputted (I) to the calculator although not in calculator syntax, and provide reasons (R). However, they need record only selected answers (A) – there is no reason to transfer to the written record all of the intermediate outputs of the calculator to paper. The February meeting which provided data for this paper, began with researchers initiating a group discussion of how students should be trained to record their solutions in MMCAS classes, during which the RIPA ideas were raised. RIPA promoted much discussion among the researchers and teachers which continued throughout the year. Some teachers found RIPA helpful to share with their class, and others did not.

- Answer **all** questions in the spaces provided.
- A decimal approximation will not be accepted if an exact answer is required to a question.
- Appropriate working must be shown if more than one mark is available.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Figure 2. Instructions for MMCAS examination 2002 (VCAA, 2002).

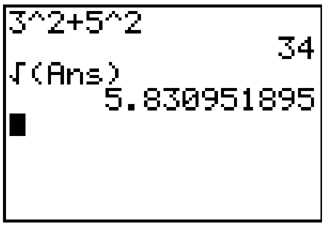
PYTHAGORAS EXAMPLE

The February discussion on teaching students to record solutions in a CAS environment began with examples such as those in Figure 1 but the teachers, planning how to raise these issues with students, wanted to discuss simple examples where students would not find the mathematics challenging. They suggested finding the hypotenuse of a right-angled triangle with sides 3cm and 5cm, and talked about various written solutions such as those in Figure 3, which also shows associated calculator screendumps. Comparing Figures 3a-3c shows that students might be using quite different syntax and calculator methods to solve even basic problems; an illustration of the explosion of methods observed in other studies (e.g. Artigue, 2002).

Figure 3a is a typical solution using a scientific or graphics calculator: the inputs are not symbolic and it is not possible to obtain a surd answer. Ken noted, as a teaching difficulty, that his students (especially the less able students) would often include too many intermediate steps (e.g. $c^2 = 9 + 25$) which were unnecessary to show in senior work, because they could be reasonably taken-as-shared. He attributed this to teachers of more junior classes not adjusting their expectations for written work to the presence of even scientific calculators. *"And some junior teachers actually would make [students] write all of that because they're not used to using technology"*. This, and other comments by the teachers, indicated that the impact of scientific and graphics calculators on written work has not been thoroughly considered in schools.

Figure 3b shows a CAS solution using the TI89. The first input is the equation $c^2 = 3^2 + 5^2$; possible as a symbolic calculator is being used. The next step in the solution is to take the square root of both sides of the equation. The TI89 actually inputs and shows this as an operation on the equation as one entity $\sqrt{(c^2 = 34)}$: a move that is *not* part of standard mathematics. It is important for teachers to make sure that their students are aware of syntax such as this which is used by the CAS but is certainly not standard for mathematical written records. The result of this command (right hand side of line) is $|c| = \sqrt{34}$, a statement which looks unusual to students, who may not immediately deduce that $c = \sqrt{34}$; instead they would expect $c = \pm\sqrt{34}$. Dealing with unexpected output is another issue with which teachers using CAS need to assist students. The solution to this point has been worked with CAS in "exact mode". Obtaining an approximate answer for c is not entirely trivial. Taking the square root of the equation has to be repeated in approximate mode, accessed in this case by pressing the "green diamond" button before ENTER, giving the output shown.

Scientific/graphics (Fig. 3a)



Using Pythagoras

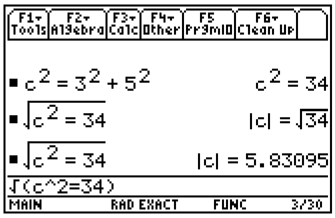
$c^2 = 3^2 + 5^2$

$c^2 = 34$

$c = \sqrt{34}$ since $c > 0$

$c = 5.83$

CAS solution 1 (Fig. 3b)



Using Pythagoras

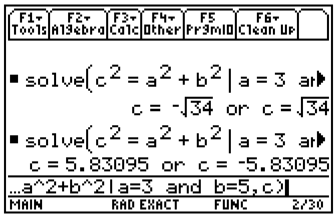
$c^2 = 3^2 + 5^2$

Square root of both sides

$c = \pm\sqrt{34}$

$c = \sqrt{34}$ since c>0

CAS solution 2 (Fig. 3c)



Solve ($c^2=a^2+b^2$| a=3 and b=5, c)

$c = -\sqrt{34}$ or $c = \sqrt{34}$

$c = \sqrt{34}$ since c>0

$c \approx 5.83$

Figure 3. Several written solutions for Pythagoras example, with calculator output.

On the Casio calculator, obtaining an approximate answer is embedded deeper within the menus, requiring the syntax TRNS (F1) then ALPHA then B (or TRNS (F1) then log button). Students became adept at those button sequences which they often used, but this created two issues for teachers. Firstly, students need to commit to memory sequences of button pushes which are not highly visible from the menu structure, and naturally begin to think in these terms. Neil commented "*but my kids use language like TRNS ALPHA B APPROX*" and was concerned that these might appear in their written records. Secondly, different brands of CAS use syntax that familiar users come to regard as intelligible and "standard" but which are very different to other brands. For students who are learning in a CAS classroom, how are they to distinguish between standard and non-standard notation?

Figure 3c shows a different CAS solution to the Pythagoras problem, using the TI89 solve and substitute commands: Solve ($c^2=a^2+b^2$| a=3 and b=5, c). This shows no intermediate working at all, although the plan of the solution seems clear. Should the teacher accept this written record which (a) shows no intermediate steps and (b) uses the calculator syntax directly? If this calculator syntax is permissible, is the illogical Solve ($c^2=a^2+b^2$| a=3 and b=5, x), which produces the same output, also permissible?

The Pythagoras example, simple as it is, shows that in addition to the problem of which steps should be shown in the written record, there are also a second set of issues arising for teachers related to the use of calculator syntax: what syntax can be accepted in written records, how will students know what it special to their learning environment and what is standard. Moreover, there are different problems arising for different brands of calculator. In the rest of this paper, we report on the teachers' thinking on these two sets of issues. As we shall see, there are two aspects to this thinking – what is good mathematical practice and what is necessary to write in order to score marks on the important end-of-year external examination.

CAS "GOBBLES" UP INTERMEDIATE STEPS

When CAS "gobbles" up intermediate steps, what is the key information that should be recorded to show working? This question can be considered from the point of view of good mathematical practice which motivated the RIPA suggestions, or from the point of view of how marks will be allocated in examinations. It was the latter which dominated teachers' comments on this issue in both the documented meetings, since they were always very concerned with maximizing students' performance on the end of year high stakes examinations. Ken, for example, commented on one RIPA example: "*But in an exam you would get maybe credit here and credit here and anything else here you're doing for yourself, not for any marks*". It was probably because RIPA did not directly address the examination question that some teachers did not find it very useful.

In November, Meg commented that her students were still unsure of the validity of a written record that just described the CAS steps, rather than showing intermediate algebraic manipulation. Meg believed that practice in previous years, requiring every step of by-hand working to be shown, made it difficult for her students to accept written records without all the steps that would be necessary in by-hand work.

> "… And my kids had a real problem showing the steps of the working. [They asked]: "If I just write down the process that I have to follow, the mathematical equation and write down that I need to solve for that equation and I need to do this and do that and then just use the calculator, is that enough?"

In February, Ken had also commented that his students, especially the less able, wanted to record too many steps, even when working by-hand. In November Neil believed that it was his more capable students were recording too many steps, inserting algebraic manipulation. By the end of year 12 it might have been expected that students and teachers would feel more confident working in parallel with CAS and pen-and-paper but it remained an issue, more for some students than the teachers. Here we see a struggle to establish a new socio-mathematical norm.

The instruction on the examination paper (Figure 2) "Appropriate working must be shown if more than one mark is available" remained problematic throughout the year. Neil stated "*Well, what does that [mean]? That's meaningless now [with CAS]"*. One teacher responded that a 'bit of work' should be shown, and Neil responded "*Show a bit of work? Show the work that you value? I mean, we don't show, even in year 7, all the work. Even year 7's can skip steps*". There are decisions to be made everyday of what can be taken-as-shared. Lucy suggested that the focus for students might profitably shift from a command to "*show your working*" to a request "*can you let us know what you're doing*", which could be interpreted more broadly.

Lucy clearly supported the idea that CAS might promote more of an overview of solution processes (reflected in a more condensed written record) and she had observed this in her classes.

> "… that's surprised me a bit; just how good some of the kids can get at saying 'Oh, I can see that what I need to do here is [solve] two equations in two unknowns'. They're much clue-ier seeing that [a type of problem exists] inside a problem. [They might say]: 'So now I know that I've got to solve, I've got to define this function that way and I'm going to solve it for this, for this and for that'…"

Lucy believed her students had made progress in that they could focus on solving at a macro level and were content that the details, essentially routine procedures, were to be performed by CAS. If, as Lucy suggests, students see an overview of a solution and they can articulate the processes being used to solve then maybe they are going to be able to produce good written records to describe these solutions, without worrying too much about whether they should include detail within those processes.

WHAT CAS SYNTAX IS ACCEPTABLE IN WRITTEN RECORDS?

The teachers generally discouraged their students from using CAS syntax in their written records, but this was still an issue for them in November. Teachers acknowledged that students had started to use CAS words to communicate mathematical thinking. What represented accepted or CAS specific language was of concern to them in both meetings. Meg explained that she encouraged students to write a description of the process to be used: *"I told them … to write down the procedure, what they were going to [use]; write it down, [such as] solve for x, solve for k, solve for whatever.*" Neil was concerned with the "different feel" of the calculator used at his school. This included the different ways of approaching problems which it encouraged (space precludes examples of this interesting point) and also the different syntax. Neil dealt with this by explicitly instructing his students not to use CAS language:

> "Write down what a mathematician could understand. Write down a logical sequence but don't use the calculator [language]' … So just before the kids went into the exams I just said to them 'Remember don't say CALC-DIFF'…and they [asked] 'what will we say?'

> And someone else in the group would say '[write] differentiate' and then they'd say 'Can we use the word SOLVE?' I said, 'Well, yes, because it's a mathematical word'."

This shows that Neil categorised SOLVE as appropriate to record in contrast to CALC-DIFF (the first and second menu items required to carry out a differentiation) which he saw as brand specific CAS syntax. Here we see that some students were unsure of what was standard mathematical language and what was a specific CAS word right until the examination. Ken helped us to see this issue from another point of view. Ken was new to teaching senior mathematics, and so he was not as firmly enculturated into this world as the other teachers. For example, to him, the notation for solving simultaneous equations in Figure 1 was standard: "*You could even put a and b in [curly] brackets …. That's accepted notation, isn't it.*" Ken made comments such as this throughout the year. When Neil said that his students used the language TRNS ALPHA B APPROX, Ken observed that his brand of CAS "*…doesn't have that* [nonstandard] *calculator language…You don't run into that problem.",* although later he agreed that the symbol | for substitute was an example of calculator language. Ken showed how CAS-specific language had become taken-as-shared in his classroom and highlighted the difficulties that novices may have in distinguishing standard from calculator-oriented practice. Ken also commented: "*I used to say 'Let y equal f(x)', but now you say 'DEFINE f(x) equal..*" and Meg and Lucy both agreed with this, implying that this would be good practice in the examinations. In this case, DEFINE is a command used by the brands of calculators in their two schools. This claim by others that DEFINE should be used, perturbed Neil as he saw it as brand specific language (STORE has a somewhat similar function on his calculator).

Fundamental to this is the question of what is considered 'standard notation' and what is 'syntax'. Comments by teachers suggest that the line between these may be blurring and that some CAS language or syntax had become standard in these classrooms. Student examination scripts in fact showed some use of CAS syntax (Ball, 2003). This suggests that some commands that might be considered syntax by teachers may be standard mathematical practice from the perspective of students. This is not unexpected when CAS is the normal technology in the classroom.

CONCLUDING REMARKS

From February to November, teachers came to moderately comfortable personal positions about the advice they gave to their students. The first main issue was the conflict between expectations that students will show working, and the fact that CAS does not report intermediate results. However, as experience of CAS syntax and the differences between brands grew during the year, they became more aware that students needed explicit guidance about what calculator language was acceptable in written work. The teachers' concern for students' welfare meant that the demands of the examination dominated their actions. To differing extents, they also looked beyond this. Discussion with their students about written records and use of CAS seemed to be a key factor to helping students develop good practices. This involves deciding what can be taken-as-shared and what are acceptable warrants for mathematical explanation, as teachers and students grapple to establish new socio-mathematical norms for the new environment. These teachers raised issues and worked towards agreed understanding about good practice.

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References

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*(3), 245-274.

Ball, L. (2003). Communication of mathematical thinking in examinations: Features of CAS and non-CAS student written records for a common year 12 examination question. The *International Journal of Computer Algebra in Mathematics Education, 10*(3), 183-194.

Ball, L., & Stacey, K. (2004). A new practice evolving in learning mathematics: Differences in students' written records with CAS. In M. Johnsen Høines, & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Bergen: PME. (Vol. 2, pp.87-94).

Ball, L., & Stacey, K. (2003). What should students record when solving problems with CAS? Reasons, information, the plan and some answers. In J. T. Fey, A. Cuoco, C. Kieran, L. Mullin, & R. M. Zbiek (Eds.), *Computer Algebra Systems in Secondary School Mathematics Education* (pp. 289-303). Reston, VA: NCTM

CAS-CAT Research Project. (n.d.). http://extranet.edfac.unimelb.edu.au/DSME/CAS-CAT. Retrieved 10 January 2005.

Flynn. P., & Asp, G. (2002). Assessing the potential suitability of "show that" questions in CAS-permitted examinations. In B. Barton, K. C. Irwin, M. Pfannkuck, & M. O. J. Thomas (Eds.), *Mathematics Education in the South Pacific. Proceedings of the 25th Annual Conference of the MERGA* (pp. 252-259). Sydney: MERGA.

Guin D., & Trouche, L. (1999). The complex process of converting tools into mathematical instruments: the case of calculators. *International Journal of Computers for Mathematical Learning, 3,* 195-227.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: interaction in the classroom culture*. Hillsdale, NJ: Erlbaum.

Stacey, K. (2003). Using computer algebra systems in secondary school mathematics: Issues of curriculum, assessment and teaching.. In W-C. Yang, S-C. Chu, T. de Alwis & M-G. Lee (Eds.), *Proceedings of the 8th Asian Technology Conference in Mathematics* (pp. 40-54). USA: ATCM.

Victorian Curriculum and Assessment Authority (2002). *VCE Mathematical Methods (CAS) Pilot study: Written examination 2*. Melbourne: Author.

DEVELOPING PROCEDURE AND STRUCTURE SENSE OF ARITHMETIC EXPRESSIONS

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This paper describes sixth grade students' performance in tasks related to arithmetic expressions in the context of a design experiment aimed at developing a principled approach to beginning symbolic algebra. This approach, which is centered on the concept of 'term', is described elsewhere. In the paper, students' performance in two kinds of tasks over items that test procedural knowledge and items that test structural understanding is examined. We address questions of consistency in the use of procedures in different task items, and the transfer of procedural knowledge to the more structure-oriented items. The data shows that the relation between procedural knowledge and structural understanding is complex. Developing a structural understanding of expressions requires the consistent use of the procedures and rules in various situations/ contexts and making sense of the relationships between the components of the expression. We cite some preliminary evidence in favour of the effectiveness of the structure-oriented approach both in strengthening procedural knowledge and structural understanding.

BACKGROUND

A sound procedural knowledge in evaluating arithmetic expressions is clearly necessary to build a strong foundation for algebra. Manipulating algebraic expressions requires students to be well aware of the rules, properties and conventions with regard to numbers and operation signs. It has also been recognized that appreciating the structure of arithmetic expressions is useful for understanding algebraic expressions; algebra is at times described as generalized arithmetic exploiting the structure of arithmetic expressions (Bell, 1995). A poor understanding of operational laws might lead to conceptual obstacles and hinder generalizing and recognizing patterns between numbers (e.g. Williams and Cooper, 2001).

Students' experience with arithmetic expressions in traditional classrooms is mainly oriented to procedures but may be ineffective even in inducing sound procedural knowledge. Many studies have reported both the poor procedural knowledge of students and their lack of understanding of the structure of arithmetic expressions (Chaiklin and Lesgold, 1984; Kieran, 1989). Students are seen to use faulty rules of operations and are inconsistent in the way they evaluate an expression (Chaiklin and Lesgold, 1984). Many common and frequent errors are reported, such as doing addition before multiplication and detaching the numeral from the preceding negative sign (Linchevski and Livneh, 1999).

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 121-128. Melbourne: PME.

The larger project, of which this study forms a part, is aimed at developing an instructional sequence for beginning algebra that builds both sound procedural knowledge and understanding of structure of arithmetic and algebraic expressions.

FRAMEWORK ADOPTED IN THE TEACHING APPROACH

The teaching approach adopted in the project explicates the structure of arithmetic and algebraic expressions from the very beginning. It capitalizes on students' prior arithmetic knowledge and is strongly centered around the concept of term. Hence we refer to this approach as the 'terms approach' below. Here we describe briefly the way in which the term concept is used in teaching procedures and concepts. More details of this approach have been described elsewhere (Kalyansundaram and Banerjee, 2004; Subramaniam, 2004).

Students learn at the outset that an arithmetic expression stands for a number, which is the value of the expression. Two numerical expressions are equal if their values are equal. Equality of expressions can also be judged from the relationships between the components or parts of the expressions. This makes it essential for the students to learn to parse the expressions correctly, and explore and identify the relationships between the parts, and of the parts to the whole. We take structural understanding to include this group of skills. This is consistent with Kieran's (1989) definition of structure, which is seen as comprising 'surface' and 'systemic' structure.

The concept of 'term' has proved useful in this context. The concept of 'term' requires students to see the number/numeral together with its sign. Terms may be simple terms (+5) or complex terms. Complex terms can be of various types like product term (e.g. $+3\times2$) and bracket term (e.g. $-(4+2)$). The product term may contain only numerical factor/s or letter factor/s or bracketed factor/s. While simple terms can be combined easily, a product term (or complex term) cannot be combined with a simple term unless the product term (or complex term) is converted into a simple term/s. Identifying the conditions when an expression remains invariant in value leads to the idea of equality of expressions. The meaning of "=" is thereby broadened from the 'do something' instruction to stand for a relation between two expressions which have the same value. The two concepts of terms and equality together give visual and conceptual support to the procedures for evaluating expressions (order of operations) and the rules for opening bracket, as they get reformulated using these two concepts.

METHODOLOGY

A design experiment methodology has been used in developing this instructional approach. The design experiment is conducted with grade 6 students (11 to 12 yr olds) from nearby English and vernacular medium (Marathi) schools. The English medium and the vernacular medium students form separate groups of instruction. The schools cater to low or mixed socio-economic strata. Four teaching intervention cycles have been conducted between summer 2003 and autumn 2004, during vacation periods of the schools.

The four teaching cycles were carried out in summer of 2003, autumn 2003, summer 2004 and autumn 2004 respectively. The first cycle was mainly exploratory in character and is not reported in this paper. There were 3 groups of students in each of the cycles 2, 3 and 4. Each group had 11 to 13 instructional sessions of 90 minutes each. A and B groups in all the cycles were from the English medium, and C groups from the Marathi medium. Subscripts indicate the cycle to which the groups belong. All the nine groups across the three cycles are discussed separately. The students in groups A_4 and C_4 were students who had attended the course in Cycle 3 except a few in C_4 who were first-timers. The students in all the groups in the previous cycles including B_4 attended the course for the first time.

Each group in a particular cycle had one teacher, except for A_2 and A_3, which had separate teachers for the arithmetic and algebra modules, who taught for about equal durations. Three teachers were involved in teaching the English groups across the cycles and one teacher for the vernacular group. Three out of the four teachers, which included the Marathi medium teacher, involved in the project were collaborators in the research project.

The details of the instruction were worked out by the group of teacher-researchers in the course of discussions held both preceding as well as during the cycles. Discussion and reflection by the group on the different teaching cycles has brought out the salient features that are common to and different in the cycles. There is an increasing centrality and coherence to the use of the concept of ‘term’ over the cycles. In the earlier cycles, this concept was used only in the context of judging the equality of expressions, but in the later cycles, increasingly, the procedures for evaluating expressions were brought under this concept. In terms of the evolution and coherence of the approach, cycle 4 represents the most evolved form.

The presence of multiple groups and teachers in and across the cycles helped us trace the development of students as they went through the course of instruction as well as observe the differences among them due to slight variations in the teaching sequence and their prior knowledge. It is therefore difficult to compare the groups directly. The students in Cycle 4 were exposed to the matured ‘terms approach’ and we will focus on their performance looking at the common errors and the extent of structural understanding. The data was collected through daily practice exercises, written tests, video-recordings, teacher’s log book and the pre and the post tests given to the students. Interviews were conducted with 22 students about 6 weeks after cycle 4. The students who showed either very consistent or somewhat inconsistent knowledge of procedures and structure sense during the course were selected from the three groups for the interviews, most of them falling in the average to high category of performance. In the context of the present paper, it is important to note that groups B_2 and B_3 are slightly different in terms of the instruction received. Group B_2 received no instruction in arithmetic, but only in algebra, the extra time being spent on activities in geometry. Group B_3 received instruction mainly on arithmetic expressions that was centered around operations with signed whole numbers.

ANALYSIS OF DATA

Here we discuss the performance of the students in the pre and post tests in tasks dealing with two types of expressions: (a) expressions with a '×' and '+' sign and (b) expressions with '+' and '–' signs only. For each type of expression, we examine a set of tasks: simple tasks and complex tasks requiring essentially procedural knowledge, and tasks that require some structural understanding. The latter tasks call for judging the equality or inequality of expressions based on their structure without recourse to calculation. Since consistent interpretation of conventions used in arithmetic expressions is an essential element in building a structure sense, we examine the consistency of student responses across simple and complex procedural tasks. Specifically we look for the influence of the structure oriented teaching approach using the concept of 'terms', on consistency and on developing a structure sense.

Evaluation of expressions with a '+' and '×' sign

Many children do not absorb the convention of multiplication before addition in evaluating arithmetic expressions even after it has been taught (Linchevski and Livneh, 1999). The most common 'LR' error in evaluating expressions like 7+3×4, is to first add and then multiply, that is, to move from left to right. An earlier study conducted by us (unpublished) showed that the 'LR' error accounted for about 50% of the errors in equivalent contexts made by a group of rural upper primary teachers. Table 1 summarizes the performance of students in the different groups in evaluating an expression with a '+' and '×' sign.

Item		**Cycle 2**		**Cycle 3**		**Cycle 4**	
		Pre	Post	Pre	Post	Pre	Post
e.g., 7+3×4 (one product term)	A	44	88	0	74	68	93
	B	50	62	0	24	15	92
	C	23	89	21	82	74	91

$N(A_2, A_3, A_4)=(25, 23, 28)$; $N(B_2, B_3, B_4)=(21, 29, 26)$; $N(C_2, C_3, C_4)=(34, 38, 42)$

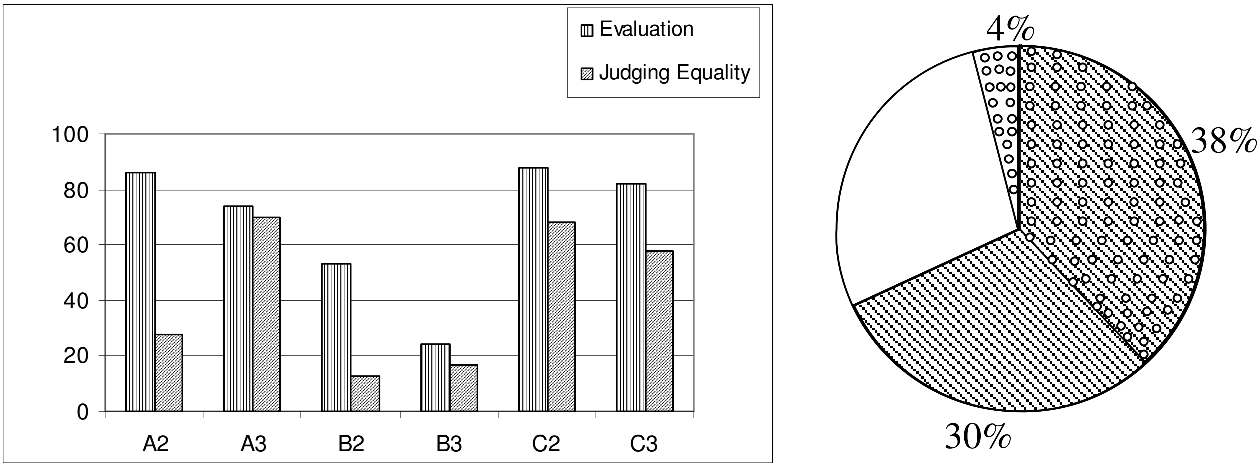
Table 1: Percentage correct in evaluating expressions with '+' and '×'

Students in the present study were not introduced to the rule of operations before class 6, which accounts for the very low rate of correct answers in the pre test of Cycle 3 for all groups in the table. Students in cycles 2 and 4 were briefly exposed to the rules of order of operation during their school instruction before they came for the vacation course. The post test results show a significant improvement in their performance in both the cycles. Also noticeable is the better performance of the students in groups A and C in the pre test of Cycle 4, the students being not only exposed to the rules in the school but also during instruction in Cycle 3. Students in group B_4 were fresh students and had only some idea of evaluating expressions from the school. The post-test scores of groups B_2 and B_3 remain low relative to the pre test and the other groups, because students received very little or no instruction on this aspect during the vacation program. While in Cycle 2, evaluating expressions was taught only as a set of rules, in Cycles 3 and 4, the 'terms approach' with increasing emphasis on the idea of product term was adopted. The incidence of LR error as a fraction of total errors in Cycles 2, 3 and 4 respectively are 6/7, 6/13 and 2/8, the remaining errors being mainly computational errors. (Groups B2 and B3, which did not receive instruction on this topic, have been excluded.)

We now examine the consistency with which students applied the '×' before '+' convention across test items. Some of the tests contained two items of the above type, one with a '+' sign and the other with a '–' sign. Students were consistent in their responses to both questions, with a few (2 to 4) answering one of the questions correctly while making the 'LR' error in the other. However, when the second item was a more complex but similar item (Cycle 2: Evaluate 3×(6+3×5)), around 17% of the students in all the groups made the 'LR' error while evaluating the expression inside the bracket although they had correctly evaluated the corresponding expression in the item without brackets.

In a related item, where a substitution was required to be done prior to evaluation (Cycles 3 and 4: 7+3×x, x=2), the students' performance was low (around 50% or lower, except for C_4 which had around 70%). Although most of the students who performed poorly on this item had a problem with substitution, a significant number of students (12%) in all the groups made the 'LR' error after substituting correctly for the variable, although they had evaluated the corresponding arithmetic expression correctly. This inconsistency on the part of the students shows that although they learnt to parse the expression correctly and had absorbed the convention of multiplication before addition and subtraction in a simpler situation, in a more complex task the 'LR' error may resurface. In Cycle 4, where the 'term' approach was adopted more strongly and the overall occurrence of 'LR' error is low, the inconsistency in the substitution question (that is, responses showing 'LR' error after substitution but not in the evaluation item) is only 7% for all the groups.

Figure 1a shows the performance of students in cycles 2 and 3 on the more structure-oriented task of judging equality for expressions of the above type. These expressions were slightly more complex than the evaluation items and had two '+' signs and one '×' sign each (therefore, two simple terms and one product term, like 28+34+21×19 or 21+34×19+28). The data indicates that knowing how to evaluate expressions of this kind is necessary but not sufficient for judging equality. Nearly all the students who can make the correct judgment about the equality/ inequality of two expressions, can also evaluate the arithmetic expression with '+' and '×' sign (See Figure 1b). The percentage of students, who can succeed in the more complex task of judging the expressions equal to a given expression, is high for the groups C_2, A_3 and C_3. In Cycle 4, the corresponding task was more complex with the options testing their ability to use brackets and splitting terms (like writing –9 as –4 –5) in the expression. We would not discuss the details of these results here.



a b

Figure 1: (a) Percentage of correct responses in evaluation and judging equality tasks for different groups. (b) Overlap of students who perform correctly on the judging equality task (indicated by the region filled with small circles) and those who perform correctly on the evaluation task (indicated by the hatched region) in all groups.

In the interviews conducted after Cycle 4, 19 students out of 22 justified their response by referring to the terms in the pair of expressions. This does not mean however that all were correct in their responses. For example, while comparing the expressions 18-15+13×4 and 4×15+18-13, 6 students identified the terms wrongly as +18, -15, +13 and ×4. This was consistent with their wrongly judging the expressions 4×15+18-13 and 18-13+15×4 as unequal. From the above, it is clear that ability to correctly evaluate simple expressions consistent with the rules of operations does not transfer readily to the more structural task of judging equality. The interview data indicate that the concept of term is readily applied to judging equality and may aid students in forming a structural understanding of expressions.

Evaluation of expressions with only '+' and '-'

An expression like 19–3+6 appears to be easy to evaluate if students know the operations of addition and subtraction. However students frequently evaluate this expression as equal to 19 – 9 = 10, making what has been described the error of detaching the negative sign (Linchevski and Livneh, 1999). In the study with teachers referred to earlier, 'detachment' errors accounted for about 40% of the errors that teachers made in equivalent contexts. One reason for this error could be incorrect perceptual parsing, where students 'detach' the minus sign from the terms to the right of the sign. Another reason, as indicated by the interview responses of some students, is that students mislearn the rule of order of operations, thinking that addition precedes subtraction. (The 'BODMAS' mnemonic actually suggests this misleading rule.) Table 2 shows the performance of students across all the cycles in evaluating this type of expression. The post test results in the even cycles is slightly better than the odd cycle, which could be due to their enhanced exposure to the evaluation task, first in school and then in our project.

Item		Cycle 2		Cycle 3		Cycle 4	
		Pre	Post	Pre	Post	Pre	Post
19-3+6 (only simple terms)	A	64	84	61	74	68	86
	B	48	62	38	69	65	85
	C	74	94	82	74	79	86

Table 2: Percentage correct in evaluating expressions with '+' and '–'

In designing the 'terms' approach, we expected students to avoid making the detachment error as they learnt to parse an expression into terms in the course of evaluating the expression. Although the performance in the even cycles is nearly same, in Cycle 2, the rate of occurrence of the detachment error for all groups in the pre test is 31% and in the post test 17%. In this cycle, it must be recalled, the concept of term was not used in evaluation tasks but only in judging equality tasks. In the post test for Cycle 3, there are only a few cases of detachment error, the rest being mainly calculation errors, and in Cycle 4 there are no detachment errors. This supports our hypothesis concerning the effectiveness of the 'terms' approach in avoiding the detachment error.

Most of the students interviewed after Cycle 4 were confident that 25-10+5 cannot be written as 25-15. Some could not say why they thought so but others said it (i.e., 25-15) can be done only if there is a bracket around 10+5 or that the term –10 has been incorrectly changed to +10 to get 15 and added that it could be -5. These students also evaluated the expressions not in the left to right fashion but combined terms flexibly as it suited them.

The more structure-oriented tasks of judging equality for this type of expressions were specifically designed to test whether students make the detachment error. Only 20%-35% of the students made correct judgments in this type of item in Cycle 2. In the slightly simpler item in Cycle 3 (comparing expressions such as 249+165-328 or 328+165-249), 40%-60% of the students made correct judgments. The item in Cycle 4 was more difficult with a product term included in each expression and was again designed to catch the detachment error (18-27+4×6-15 & 18-20+7+4×6-10+5). Here 40% of the students made correct judgments. The fact that students were splitting the expressions into terms was corroborated in the interviews after Cycle 4. 21 out of 22 students interviewed said that the expression 49-5-37+23-5 is not equal to the expression 49-37+23 because of the extra two '–5's, but readily saw that the latter expression was equal to 49-5-37+23+5, because –5+5 gives 0.

DISCUSSION

The development of the teaching approach during the course of the project, which can be characterized as making the concept of term central to both structural (judging equality) tasks and procedural (evaluation) tasks, has proved fruitful from two points of view. Firstly, it has made the instructional approach internally coherent allowing students to deal more meaningfully with symbolic expressions. Second, it has strengthened students' procedural knowledge and has reduced the occurrence of well-known errors. Subjective assessments of the interviews conducted at the end of Cycle 4 suggest that students feel confident in the justification that they give for their responses. However, the performance in structure-oriented tasks is low even in the later cycles. This is partly due to the increased complexity of the tasks. Classroom discussions indicate that students are more confident in dealing with simpler expressions while judging equality. However, the data indicate to us that the formation of structure sense from a knowledge of procedures and rules is a difficult and long process. It would require abstracting the relationships within and between expressions. Further, it requires consistent use of the rules and procedures in various situations sharing the structural aspects.

One other consequence of our teaching approach needs to be mentioned. Identifying and comparing terms between a pair of expression in order to judge their equality is something of a shortcut in carrying out the task. When this is taught explicitly, for some students it may assume a recipe-like quality, turning what we have called a structure-oriented task to a more procedural one. In the course of the interviews, we noticed that for some students this seems to be the case, while other students develop a more flexible and truly structural understanding. This is an aspect we intend to explore further. However, even for students who interpret the 'terms approach' in recipe-like ways, we hope that the transition to an understanding of structure will be easier than in the traditional approach.

References

Bell, A. (1995). Purpose in school algebra. *Journal of Mathematical Behavior*, 14, 41-73.

Chaiklin, S. & Lesgold, S. (1984). *Prealgebra students' knowledge of algebraic tasks with arithmetic expressions*. Paper presented at the annual meeting of the American Research Association.

Kalyansundaram, S. & Banerjee, R. (2004). Teaching arithmetic and algebraic expressions. In M. J. Hoines and A. B. Fuglestad (Eds.), *Proceedings of the 28th conference of the International group of the Psychology of Mathematics Education*, Bergen, Norway: PME.

Kieran, C. (1989). The early learning of algebra: A structural perspective. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra*, Reston, VA: NCTM.

Linchevski, L. & Livneh, D. (1999). Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics*, 40, 173-196.

Subramaniam, K. (2004). *Naming practices that support reasoning about and with expressions*. Regular lecture at ICME-10, Denmark.

Williams, A. M. & Cooper, T. J. (2001). Moving from arithmetic to algebra under the time pressures of real classroom. In H. Chick, K. Stacey, J. Vincent & J. Vincent (Eds.), *Proceedings of the 12th ICMI study conference: The future of the teaching and learning of algebra*, Melbourne, Victoria: The University of Melbourne.

STRUGGLING WITH VARIABLES, PARAMETERS, AND INDETERMINATE OBJECTS OR HOW TO GO INSANE IN MATHEMATICS

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Pursuing our investigation on students' use and understanding of algebraic notations, this paper examines students' cognitive difficulties related to the designation of an indeterminate but fixed object in the context of the generalization of patterns. Stressing the semiotic affinities and differences between unknowns, variables and parameters, we examine a Grade 11 mathematical activity in which the core of the students' relationship to algebraic formula comes to light. We show how the semiotic problem of indeterminacy reveals the frailty of students' understanding of algebraic formulas and how it puts into evidence the limited scope of the use of formulas as schemas, strongly rooted in student's relationship to algebra.

INTRODUCTION

Making sense of letters is one of the fundamental problems in the learning of algebra. A letter is a sign, something that designates something else. In the generalization of patterns, letters such as 'x' or 'n' appear as designating particular objects –namely, variables. A variable is not a number in the arithmetic sense. A number, e.g. the number 3, does not vary. A variable is an algebraic object. Previous research has provided evidence concerning the meanings that students attribute to variables (e.g. MacGregor & Stacey, 1993; Trigueros & Ursini, 1999; Bednardz, Kieran, & Lee, 1996). One of these meanings consists in conceiving of a variable as an indeterminate number of a specific kind: it is not an indeterminate number *in its own*. For many students, it is merely a *temporally indeterminate number* whose fate is to become determinate at a certain point. Aristotle would have said that for the students, variables are often seen as "*potentially* determinate" numbers, as opposed to the numbers in the elementary arithmetic of our Primary school (e.g. 1, 4, 2/3 and so on), which are "*actual* numbers". Yet, the algebraic object "variable" should not be confounded with another algebraic object –the "unknown" (Schoenfeld & Arcavi, 1988; Radford, 1996). Although both are not-known numbers and, from a symbolic viewpoint, the same syntactic operations can be carried out on them, their meaning is different. In the algebraic equations used in introductory algebra, such as '$x+12 = 2x+3$', the unknown exists only as the designation of a number whose identity will be disclosed at the end. The disclosing of the unknown's identity is, in fact, the aim of solving an equation. In contrast, when 'n' refers to a variable (see e.g. the pattern in Fig 1 below), the focus of attention is not on *finding* actual numbers but on the variable *as such*. The same holds for the expression '$2n+1$', that designates the variable "the number of toothpicks in figure 'number n'" (see Fig 1 below). In other words, in equations, we go from symbols (alphanumeric expressions) to numbers, while in patterns we go the other way around (of course, once established, a formulaic expression of a variable like '$2n+1$' can be used to find out specific values of 'n' or '$2n+1$'). What previous research has suggested is that, for many students, letters (such as 'n' in '$2n+1$') are considered as potential numbers −indeterminate ones waiting in a kind of limbo for their indeterminacy to come to its end. The letter is hence, for the students, an *index* (Radford, 2003), a sign that is indicating the place that an actual number will occupy in a *process* (Sfard, 1991) temporarily in abeyance (we shall come back to this point later).

In this article, we pursue our investigations of students' algebraic thinking about variables. We are interested in understanding the way that students cope with another algebraic object: a parameter, that is, an indeterminate but *fixed* element of the "values taken" by a variable. The paradoxical epistemic nature of this algebraic object rests on its apparent contradiction: it is a fixed, particular number, yet it remains indeterminate in that it is not an actual number. Like the variable from where it emerges, it is indeterminate and is not subjected to an inquisitorial procedure that would reveal (as is the case with unknowns) its hidden numeric identity. From an education viewpoint, the question is: How can such an object become an object of thought for the students? Because of its indeterminate and abstract nature this object cannot be pointed out through a gesture as we can point e.g. to one of the first terms in a given patter (see e.g., the pattern below; Fig 1). The only way that a parameter can become an object of thought is through the interplay of various sorts of signs. The next section provides some details about how we introduced this object in the course of a regular classroom mathematics lesson about patterns. The rest of the article is devoted to the analysis of some of the students' difficulties in making sense of a parameter.

METHODOLOGY

Data collection: The paper reports parts of a five-year longitudinal classroom research program where teaching sequences were elaborated with the teachers. The research involved four northern-Ontarian classes of grade 11, from two different schools. The same methodology was applied in both schools: the classes were divided into small groups of three to encourage students to work together and share their ideas with the others members of the group; then the teacher conducted a general discussion allowing the students to expose, confront and discuss their different solutions. During the implementation of the teaching sequences in the classroom, both the teacher and the researcher were present, willing to answer the students queries as they solved the problem. In each class, three groups were video-taped, the dialogues transcribed and written material was also collected. For the purpose of the present article, we will closely focus on one group we found representative of most students' work. This group was formed by Denise, Daniel and Sam.

About the given task: The teaching sequence included three linked problems concerning the construction of geometric-numerical patterns. The figures that constituted the patterns were described as being composed of toothpicks, triangularly disposed.

In the **first problem** the first figures of the pattern (also called "original" in the subsequent problems) were drawn (see Fig 1). After having been asked to find out the number of toothpicks for specific figures, the students had to write an algebraic formula to calculate the number of toothpicks in figure 'number *n*'.

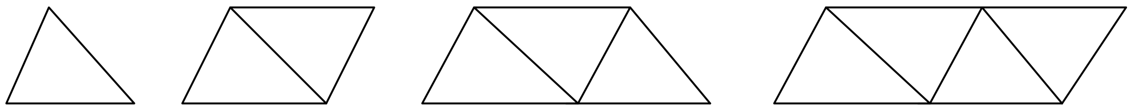

Figure 1 Figure 2 Figure 3 Figure 4

Fig 1: Original pattern.

The pattern in the **second problem** was related to a fictitious character (Mireille) who was said to have begun her pattern at the fourth "spot" of the original pattern (*à la place numéro quatre*, in French). The first figures of the pattern were also provided (see Fig 2) and the questions were similar to those of the first problem.

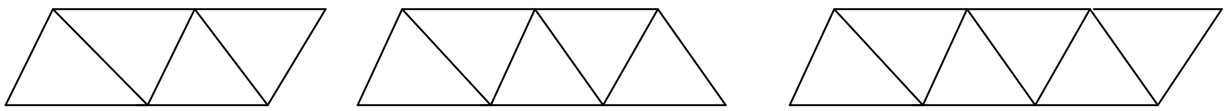

Figure 1 Figure 2 Figure 3

Fig 2: Mireille's pattern.

In order to investigate the students' cognitive difficulties in dealing with parameters, a new pattern (Shawn's) was introduced in the **fourth problem.** The spot where the pattern began was given, yet not specified: students were told that Shawn had begun his pattern at the "spot *m*" of the original pattern. They were then asked to provide an algebraic formula, in terms of *m*, that indicates the number of toothpicks in figure number 1 of Shawn's pattern. In what follows, we will focus on the fourth problem. Special attention will be given, however, to students prior answers, for it provides essential information about the students' relationship to algebraic symbols and, in particular, to their use and understanding of letters.

STUDENTS' RESPONSES

The semiotic problem of multiple referents

Both the first and second problem were easily solved. Thus, in the first problem, right after a quick numerical examination of the link between the number of the first figures and their corresponding amount of toothpicks, the students rapidly worded the description of a sequence of numeric actions: Denise said: "So it's times 2 plus 1, right?" and, to calculate the number of toothpicks in figure number, 25 effected the calculation 25×2+1.

Denise's utterance is the description of a numeric schema (in Piaget's sense) that allowed the students to obtain the formula by translating it into symbols. However, when translating the worded schema into an algebraic formula, students produced a response attesting some lack of precision in the meaning that they gave to symbols:

x = # figure

$n = 2x + 1$

Fig 3: Students' answer to the last question of Problem 1.

Indeed, the translation of the worded schema of the students' response suggests that they do not interpret the letter '*n*' as standing for the number of an unspecified figure (despite the fact this had been suggested in the text). Instead, by writing "$n = 2x+1$" (see line 2 Fig 3), '*n*' designates an amount of toothpicks. Furthermore, a new letter was introduced to designate something that remained *implicit* at the verbal level but which was nonetheless substituted by actual numbers (such as 25, to answer the question of the number of toothpicks in Figure 25). The letter '*x*' used by the students to designate the number of the figure (see line 1 Fig 3) plays the role of *index*, i.e. something indicating a place that will be occupied by a number. The letter '*x*' designates a "temporarily indeterminate" number, suffering from indeterminacy, seen as a kind of sickness that, like a cold, should sooner or later come to its end. The letter '*n*' designates the schema '$2x+1$'. Instead of considering '*n*' as a genuine algebraic variable, the transcripts and video analyses of this and other groups suggest that '*n*' is seen as a "potentially determinate" number, a number that will become "actual" (in the Aristotelian sense) as soon as '*x*' takes on its numerical value.

Bearing these antecedents in mind, let us now turn to the forth problem, where the students encountered the concept of *parameter*. Imagining the letter '*m*' as an indeterminate yet fixed number at the starting point of a new pattern posed many difficulties to them:

1.1 Daniel: OK, but if it begins at spot number *m*, and we want to know figure number 1, isn't this 1? Isn't *m* [equal to] 1?[...]

1.2 Daniel: But isn't *m* the number of the figure?

1.3 Denise: It's... the place where...

1.4 Daniel: OK, it's not... OK, it's a number of figure, but, OK…

The above excerpt illustrates some of the fundamental student difficulties in trying to make sense of the question. In order to understand these difficulties, we need to discuss three different ways of referring to the figures. In the previous problems, indeed, the figures can be seen from different perspectives:

Figure as substance: Each figure can be referred to through the number of toothpicks it is made of. For instance, in the original pattern, there is one 3-toothpick figure, one 9-toothpick figure (namely Figure 1 and Figure 4 respectively). In

Mireille's pattern, there are no figures with 3 toothpicks, but there is one 9-toothpick and one 11-toothpick figures (Figure 1 and Figure 2, respectively).

Names as part of a system: Each figure can also be referred to by a "label". This label is its "name" (Figure 1, Figure 2, etc.). This name corresponds to the *relative* position among the others figures *of the same pattern*. For instance, in the original pattern, as well as in Mireille's pattern, "Figure 1" is the label of the first figure, "Figure 2" of the second figure, etc.

Relativeness of the object's name: Since Mireille's and Shawn's patterns begin at a given place or "spot" in the "original" pattern, the *place* that each figure occupies inside a certain sequence must be distinguished from the place these figures occupy in the original pattern. For instance, the figure called "Figure 2" in Mireille's pattern is called "Figure 5" in the original pattern.

Line 1.1 is representative of the difficulty in seeing the subtle relativeness of the object's name. Indeed, from the point of view of the original pattern, Shawn's Figure 1 is at the spot '*m*'. But each first figure in a pattern starts at spot 1 of its own pattern. By saying that '*m*' is 1, Daniel, probably uncomfortable with the indeterminacy, merges the two referents.

Besides being related to a place in the pattern, '*m*' also corresponds to the number of the figure that occupies this place. In this sense, in Line 1.2 Daniel was right when saying that '*m*' corresponds to the number of a figure: if we consider the original pattern as reference, '*m*' is indeed the number of the figure. In Shawn's pattern, however, this figure is no longer "Figure *m*": it becomes "Figure 1".

The effect of the indeterminate origin on using a schematic formula

As we saw previously, the students rapidly came up with a formulaic schema for the number of toothpicks of a figure located at an *indeterminate* place –namely, '*n*' (see Fig 3). The formulaic schema made sense for the students insofar as it was considered as a process in abeyance. Now, how were they to find an algebraic expression for the number of toothpicks in a figure for which the place ("m") was no longer to be considered temporally indeterminate but *indeterminate as such*? Noticing the students' struggle to make sense of the question and their reaching an impasse, the researcher went to talk to the group:

2.1 Researcher:	They ask you to find an algebraic expression, in terms of 'm', that indicates the amount of toothpicks that there are in the new pattern. It starts at figure 'm' […] How many toothpicks will its first figure have?
2.2 Sam:	Yeah, well we don't know this.
2.3 Daniel:	Well, that's what we have to find out. [...]
2.4 Daniel:	His 1, his 1, where is it located according to this (pointing the "original" pattern) . (...) Where is the 'm' according to this? [...]
2.5 Denise:	So, if you want to find the amount of toothpicks in his pattern (*sic*), if you had the number of the figure you could do it, but we don't have it. That's the only thing I don't know how to do.

The difficulty of conceiving of the indeterminacy of the spot '*m*', not as a temporal indeterminacy but as indeterminacy *as such*, checkmated the students' formulaic schema (see lines 2.3, 2.4 and 2.5). The students needed to understand that a parameter is an indeterminate but fixed element of the "values taken" by the variable and that despite its indeterminacy it makes sense to think about it and of the figure at that place, even if no numerical value can be attributed to them. Understanding this entails understanding that there is a new layer of mathematical generality, a layer where the "existence" of the objects does not depend on numerical determinacy, whether actual or potential. The fact that this indeterminacy directly concerned the first figure of the pattern –its origin– was for the students, to say the least, most disconcerting:

3.1 Daniel: We don't have Shawn's pattern. [...] We don't know where it starts at and where it ends... We can almost not do it [...]

3.2 Sam: I'm going insane.[...] We have nothing...

Acceptance of indeterminacy is a real obstacle to the students. As their dialogue indicates, they seem to feel the need to attribute a numerical value in order to progress in the mathematical activity. This particularity reveals the students' understanding and use of letters in algebraic formulas, suggested elsewhere in their answers for prior problems. For them, even though they are able to produce a formula and manipulate it (e.g. substituting), the formula is still seen as a process and not yet as an object (Sfard, 1991). In other words, we might say that they accept dealing with formulas, dealing with the indeterminacy, but only for a while, for the formulas have to provide a result:

4.1 Daniel: We just don't know how to find '*m*'. [...] What did you say?

4.2 Denise: $x = 2m+1$.

4.3 Researcher: Do you agree with that? [...]

4.4 Sam: Yeah, but it takes you nowhere. It's nice to have a formula, but you have to get a number.

4.5 Researcher: We don't have to have a number!

4.6 Denise: We have nothing.

The students focused on trying to determine *m* and, by analogy with previous problems, they struggled to provide a formula that, at the end, would give an output for *m*. But what exactly is *m* for them? How did the students express it in a formula?

Among all the referents that characterize the figures, there is one to which the students granted a privilege: the number of toothpicks that a figure is made up of (influenced maybe by the questions in the first problems that focused on finding the number of toothpicks in particular figures or on finding a formula that would generalize this amount). As students progressed in solving the problems, their associating of the figure with its number of toothpicks became, indeed, more and more evident:

5.1 Researcher: What spot did Mireille start at?

5.2 Denise and Daniel: At 9.

Not surprisingly, the first attempt in interpreting 'm' was hence to consider it as representing the number of toothpicks and, by analogy with the formula that they provided in the first problem ('$n = 2x+1$'), Denise suggested the formula '$m = 2x+1$' for the first question of Shawn's problem: "(...) It's the same formula as this one (*pointing at the formula '$n = 2x+1$'*) (...) So $m = 2x+1$?". As previously mentioned, in the students' response to the first problem, 'n' designated the number of toothpicks of the figure 'number n'. When Denise proceeded by analogy to solve the problem 4, this would mean that she considered 'm' for the number of toothpicks in figure 'm'. But the question was to provide a formula that would indicate the number of toothpicks in *figure number 1* of Shawn's pattern. Because of their merging of the multiple referents and, more precisely, because of the confusion between the place where Shawn's pattern began (in the original pattern –that is, place m) and the name of the related figure in Shawn's pattern (that is, of its first figure –Figure 1), the formula '$m = 2x+1$' stands for the number of toothpicks in figure 1 of Shawn's pattern. But Denise feels uncomfortable with the formula that she has just provided and says: "That's strange, they say how many toothpicks there will be in figure 1 of Shawn's pattern, but we don't have x." Notice that Denise has transformed the original question into a different one: finding an algebraic expression has been "translated" into finding an amount of toothpicks. What Denise finds strange is that one could ask such a question without providing her with an *actual* number.

It is only after having realized the difference between the multiple referents –and only then− that Denise is able to provide the expected formula: "$x = 2m+1$, because if 'm' is the place where he starts his pattern at, that's still not figure number 1, oh, yeah!". Yet, because the inquisitorial procedure is strongly rooted in students' conceptions of formulas, they do not find this answer acceptable:

6.1 Daniel: Yeah, this would work, yeah, it's just m that we don't know how to find.

6.2 Denise: We don't know how to find it. Yeah, that's the thing.

CONCLUDING REMARKS

The mathematical activity reported in this paper suggests a context in which the core of students' understanding of letters and their conception of formulas comes to light. When considering the ease with which the students solved the first problems, one may be tempted to conclude that the students have successfully conceptualized letters as variables and have been able to meaningfully produce and even manipulate formulas. Indeed, the students were perfectly at ease dealing with the concept of 'figure n' −a concept that posed great difficulties to them and that took time to overcome when first introduced in Grade 8 (see Radford, 2000). Yet, the semiotic problem of indeterminacy brought forward by the concept of *parameter* in problem 4 reveals the frailty of students' understanding of algebraic formulas that the answers provided in first problems hide. In particular, it highlights the frailty of perceiving formulas as schemas, putting into evidence their limited scope. It required a different situation –one demanding the students to deal with a new level of generality– to reveal the students' difficulties. In the context of the generalization of patterns, this means making the students consider the figures not as necessarily characterized by actual or potential numbers but as genuine conceptual objects, objects that can only be referred to through signs. Perhaps the philosopher Immanuel Kant was right in asserting that the possibility of (elementary) geometry resides in our intuition of space and that the same cannot be said of the objects of algebra, whose possibility cannot even be attributed to our intuition of time. Their possibility resides in symbols. From an education viewpoint, our results suggest that a pedagogical effort has to be made in order to make the students understand that there is layer of generality in which mathematical objects can only be referred to symbolically, detached in a significant manner from space and time. The students need to learn to cope with the kind of indeterminacy that constitutes a central element of the concepts of variable and parameter. Although one may very well be asked to begin from "nothing" (see Sam in passage 3.2) there is no reason to go insane: one still can go somewhere else –to symbolic algebra.

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References

Bednarz, N., Kieran, C., & Lee, L. (Eds.) (1996). *Approaches to Algebra, Perspectives for Research and Teaching*. Dordrecht /Boston/London: Kluwer**.**

MacGregor, M. & Stacey, K. (1993). Seeing a pattern and writing a rule. *Proceedings of the 17th International Conference for the Psychology of Mathematics Education* (Vol. 1, 181-188). University of Tsukuba, Japan: PME.

Radford, L. (1996). The roles of Geometry and Arithmetic in the development of Elementary Algebra. In N. Bednarz et al. (Eds.), *Approaches to Algebra: perspectives for research and teaching* (pp. 39-53). Dordrecht: Kluwer.

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis. *Educational Studies in Mathematics* **42**(3), 237-268.

Radford, L. (2003). Gestures, speech and the sprouting of signs. *Mathematical Thinking and Learning* **5**(1), 37-70.

Schoenfeld, A. & Arcavi, A. (1988). On the Meaning of Variable. *Mathematics Teacher* **81**(6), 420-427.

Sfard, A. (1991). On the dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin, *Educational Studies in Mathematics* **22(**1), 1-36.

Trigueros M. & Ursini S. (1999). Does the understanding of variable evolve through schooling? *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, 273-280). Haifa, Israel: PME.

EXPLORING HOW POWER IS ENACTED IN SMALL GROUPS

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This paper presents an analysis of the enactment of power during group discussions in high school mathematics. The class studied was working on introductory calculus using a collaborative learning approach. In analysing a group discussion, I first traced the flow of ideas, looking at when and by whom a new idea was introduced, and how others responded. I next divided the transcript into "negotiative events" and looked at how transitions from one event to the next came about. These analyses made it clear that some students had more power than others to influence the course of the discussion, but that this was not related to their mathematical capabilities.

INTRODUCTION

The research reported here is part of a larger study of student-student interactions during collaborative learning in mathematics (Barnes, 2003) conducted in classrooms where students worked in small groups, with shared goals, on challenging unfamiliar tasks. They were not taught standard solution procedures in advance, but were encouraged to construct new concepts by recalling prior knowledge and combining and applying it in new ways. In whole-class discussions following the group work, students explained solutions, asked questions, and shared insights, and the class tried to reach a consensus. Collaborative learning is encouraged by recent mathematics curriculum documents that emphasize the importance of fostering communication skills and encouraging mathematical dialogue (e.g., AAMT, 2002). Collaborative learning is not always successful, however. This paper explores ways in which social interactions within collaborative groups can interfere with the learning process.

THEORETICAL FRAMEWORK AND LITERATURE REVIEW

The theoretical perspective underlying the study is that of sociocultural theory (see Davydov, 1995; Lerman, 2001). Based on the work of Vygotsky, sociocultural theory asserts that all learning is inherently social, resulting from the internalisation of processes developed in interaction with others. In addition, the theory claims that learning is mediated by signs and cultural tools, including language (both oral and written), symbols, gestures and artefacts. This means that studies of small-group learning need to attend not only to spoken discourse, but also to the participants' body-language, tone of voice, direction of attention, and the artefacts they are using.

Recent research on collaborative learning has studied the interactions within collaborating groups. Most of this has focussed on cognitive and metacognitive aspects of the interactions (e.g., Forster & Taylor, 1999; Goos, Galbraith & Renshaw, 2002), but I believe that social aspects need to be considered also, because poor communication and social relationships within a group can result in failure to engage fully with the task, or can limit the range of solution pathways considered. For collaboration to be effective, appropriate socio-mathematical norms (Yackel & Cobb, 1996) need to be established. These include expectations that everyone will contribute, that others will attend to what is said, and that assertions will be justified.

Cohen (1997) describes status inequalities as a cause of unequal interaction within groups, resulting in unequal learning opportunities. Factors that determine a student's status include perceived ability, popularity with peers, as well as gender, social class and ethnicity. Cohen draws on Expectation States Theory to explain how a student's status sets up performance expectations that can be resistant to change. Cohen and her colleagues used mainly quantitative methods to study inequalities in interactions within groups. My research question was to find ways of using qualitative techniques to investigate how power is enacted, and unequal interaction patterns come about.

THE STUDY

My research was a multi-site case study of classes engaging in collaborative learning, using video to capture classroom interactions. During group work, the camera focussed on one group, and a desk microphone captured their speech. Additional data included interviews with teachers and selected students. This paper focuses on a class of Year 10 students who were following an accelerated mathematics curriculum. The lesson described took place near the end of a sequence on introductory calculus. The class had already investigated gradients of curves, discussed limits, and worked out rules for differentiating polynomials, and how to use calculus in curve sketching. Up to this point, calculus had been presented in an abstract mathematical context, with no discussion of potential applications. The following problem was then presented:

> You have a sheet of cardboard with dimensions 20 cm by 12 cm. You cut equivalent squares out of each corner and fold up the sides to form a box without a lid. What should be the length of the sides of the squares cut out for the box to have maximum volume?

This is a standard problem found in most calculus textbooks, but to these students it was a true investigative task. They had no prior experience of similar problems and no idea of how to proceed. They were not even sure if it was related to their work on calculus, and the teacher gave no hints. There are many possible ways of tackling the problem, with and without calculus. I chose this lesson for detailed analysis, not because it was "typical" in any sense, but because of the contrasting personalities in the group and the complexity of the discussion. This revealed interesting group dynamics which helped to cast light on how power is enacted within small groups.

Introducing the group

During the small-group discussion part of the lesson the camera was focussed on four students, whom I call Vic, Zoe, Charles and Selena. Like everyone in this accelerated class, they were high achievers in mathematics. Vic was a champion athlete, held an elected leadership position within the student body, played in the school band and was popular and confident. He seemed, however, to have a short attention-span and to crave attention. Zoe too was popular and confident, and generally very articulate.

She spoke up frequently in class discussions. In contrast, Charles was awkward, shy and diffident. He appeared to be a loner, with no friends in the class. The teacher commented in an interview on his poor social and communication skills, adding that he was "very bright, a critic". Finally Selena, a new student, and the only class member of Asian background, was shy but eager to be accepted. Other students were unaware of her mathematical thinking capabilities, but did know that some topics which they had studied had not been covered at her previous school, so they may have tended to assume that in general she knew less than they did.

A brief outline of the discussion

During the lesson, the group worked for 35 minutes, following a tortuous solution path that involved many false leads and dead ends. But by the end of the time they had solved the problem by two different methods, one of which used calculus.

They began by trying to make sense of the problem. Although "maximum volume" was stated clearly, Selena and Zoe interpreted it as asking for maximum *base area*, and discussed how small an edge they could turn up and still call the result a box. Selena talked about turning up an edge "as close as possible" to zero, and speculated whether limits were relevant to the problem. Eventually Zoe grasped that the problem was about volume, not area, and claimed that they were now on the right track.

Charles suggested that they let the side of the square cut out be x, and find a formula for the volume in terms of x. Zoe agreed at first, but then abandoned this approach for what she thought was a simpler way and Vic supported her. Selena pointed out a flaw in their reasoning, and the group finally agreed on an expression for the area of the base. After some digressions, Charles prompted them to write the volume as a cubic polynomial, and suggested graphing it (see first transcript below). The others did not think a graph would help, but Selena began to draw the graph on her graphics calculator. Charles explained that a graph would tell them which value of x gave the greatest volume. Zoe ignored this, and proposed asking Miss James if they were on the right track. Miss James first asked them to explain what they had done, followed this with questions like "What are you going to do next?", and then left them.

Zoe invited ideas about what to do, and Selena asked, hesitantly, if they should "do the derivative". Zoe could not see how it would help. Charles supported Selena, and explained why (see second transcript, below). Vic grasped part of what Charles said (about the graph showing where the maximum lay, but not about using the derivative) and acted on it, using a graphics calculator to find the x-coordinate of the maximum turning point. Again, Zoe sidetracked them with the seemingly pointless suggestion of equating the volume to zero, but this eventually led them to conclude that x was between 0 and 6. After an unnecessary substitution to find the greatset volume (not realising that they could read it off the graph) they substituted values of x on either side of their answer to verify that it was indeed a maximum, and announced that they had "done it". The teacher prompted them to explain what they had done, and asked if they could think of another way to solve it, and if they could justify their result.

Selena suggested using the derivative to find the turning point, Charles supported her, (see lines 369-375 below) and Zoe agreed. When they equated the derivative to zero to find the turning points, they struggled for a long time to factorise the resulting quadratic equation. Selena suggested using the quadratic formula, but Zoe and Vic resisted and continued trying to factorise. Eventually Charles concluded that they would have to use the formula, Vic agreed, and he and Selena did the calculation, obtaining the same answer as by the graphical method. As they were explaining to the teacher what they had done, the bell rang bringing the lesson to an end.

ANALYSIS

The complexity of both the range of ideas discussed and the interactions among the students made the transcript difficult to follow and interpret. It was necessary to find methods of data reduction that would help to make visible the phenomena of interest: the interplay between mathematical ideas and the interactions among the students.

Identifying the ideas involved

A first step was to list the different ideas the group discussed, including those that were helpful, and those that proved to be 'red herrings' that led the group astray. I list here the helpful ideas. For reasons of space the 'red herrings' are omitted.

Ideas which helped the group move forward towards a solution:

- Introduce x for the length of the sides of the squares cut out, and find an expression for the area of the base of the box and hence its volume.
- Graph the volume function and, from the graph, find where it is greatest.
- The value of x must be between 0 and 6.
- Substitute the x-value of the maximum point into the volume function to find the greatest volume (only necessary because they did not recognise that the y in their graph represented the volume).
- Check function values on either side of this to verify that it is a maximum.
- Find the derivative and equate it to zero to find turning points.
- Factorise the expression for the derivative to find its zeroes.
- (When factorising proved impossible) Use the quadratic formula.

Tracing the flow of ideas

From the transcript, it was possible to trace the way in which an idea was introduced by one group member, accepted or rejected by others, and perhaps reintroduced later, maybe more than once. To illustrate, I use the idea of *graphing the volume function*. (*Note*: A key to the symbols used in the transcript is given at the end of the paper.)

First mention of idea: Having introduced x to represent the length of the side of the corner square, the group (with some difficulty) found an expression for the volume of the box. They were then unsure what to do. After a short silence, Charles spoke:

298. Chas: Perhaps we should graph it.

299. [8 sec pause. The girls sit back. They seem to be thinking. Vic moves as if stretching his neck. Charles glances a little anxiously in Vic's direction.]

300. Zoe: Wait a minute … Um, okay … Hang on, that was this time … We have to find the limit when X equals zero, maybe /

301. Sel: /How does the graph help it?

302. Zoe: I don't think it does. Oh it might.

303. Sel: Hang on, I'll just see / [begins to draw the graph on her calculator]

304. Vic: /Not in particular, what does it do? It just gives you two points on the axis.

305. [Vic turns round to watch what other groups are doing for 24 seconds.]

306. Zoe: Mm. Well, what we're trying to do is, we're trying to find the value across here. [Points to her diagram] We have to find that.

307. Sel: Um [Uses graphics calculator, murmuring to herself as she presses keys]

308. Chas: Well, that's /

309. Sel: /the graph /

310. Chas: /what value of X gives us the most volume.

311. Chas: [Selena holds out her calculator to Charles.] Is there a turning point there?

312. Sel: Yeah. Two. Um, yeah two.

313. Chas: Yeah, one of them's down there /

314. Zoe: /Shall we ask Miss James if we're on the right track?

315. Sel: Yeah. [Vic has turned back to the group again. He nods.]

Summary 1: Charles' suggestion initially met with no response. Then Zoe expressed doubt and proposed an alternative, based on a misconception. Selena questioned the idea. Vic was dismissive. Then Selena began to draw the graph on her calculator. She and Charles were making progress when Zoe brought the discussion to an end by suggesting that they talk to the teacher, and everyone but Charles agreed.

Second mention of idea: Zoe called the teacher over to them and spoke for the group, but she did not explain everything they had done, and in particular did not mention graphing. As Miss James turned to go, Charles said "We need to graph this". Miss James did not hear, and Zoe interrupted excitedly to propose an unhelpful idea.

Third mention of idea: They discussed a number of suggestions about what to do. Selena asked if they should use the derivative and Charles, in expressing his support, referred to the graph that Selena had drawn on her calculator:

369. Chas: Basically, what I think here is that this turning point [points to the graph on Selena's calculator] um, at the turning point, that's going to be your maximum value for um /

370. Sel: /which is that? [points to something on the table in front of her, possibly on the worksheet, but exactly what is not visible to the camera]

371. Chas: Yeah. Well, maximum value for X, // to get us

372. Vic: //Obviously, so we've to find the value /

373. Chas: /the maximum volume.
374. Vic: [Picks up Selena's calculator] So, trace
375. Chas: So basically you do need to work out the derivative.

Summary 2: Charles was trying to explain why the maximum turning point would give them the answer. This time Vic listened to him, took in part of what he was saying, and acted on it, but gave no sign that he had heard Charles' final statement.

Overview: In this sequence of excerpts, Charles repeatedly made a suggestion without success. Selena was willing to give it a try, but Zoe and Vic repeatedly rejected or ignored what he said. It was not until Vic endorsed part of Charles' final statement that the whole group focused on drawing a graph and used this to find a solution.

I give a second example in less detail. The idea of *differentiating the volume function* and using the derivative to find turning points was first raised by Selena while they were brainstorming what to do (line, 363, just before the start of the second excerpt above). She expressed it tentatively, as a question: "Are we doing, do we do the derivative in that?" Zoe expressed doubt: "Like, what for?" but Charles supported Selena by explaining why it would help (second excerpt). Vic pre-empted him by beginning to use the Trace function on the calculator. The derivative idea seemed to be forgotten until Miss James asked them to think of alternative ways they could use to solve the problem. Selena hesitantly said, "Use the der- deriva-" (line 623). Zoe interrupted to repeat an idea of her own, but Charles spoke in support of Selena. Zoe suddenly seemed to catch on, exclaiming "Yeah, the derivative. It's the turning point." (line 629) and gesturing to show the shape of the graph. The group then used the derivative to find the maximum turning point and hence the maximum volume.

Overview: Again one student, this time Selena, repeatedly tried to make a point, but it was rejected by the group until Zoe gave it her support.

I carried out a similar analysis for each idea discussed. Of eight helpful ideas, Selena initiated three, Charles three, and Zoe and Vic one each, but none were acted upon unless supported by Zoe or Vic or both. This makes it clear that it was not the potential value of an idea that determined its adoption by the group, but whether or not it was supported by at least one of the two students Zoe and Vic. This insight prompted a more detailed look at how the topic of discussion was determined.

Control of the topic of discussion

Clarke (2001) proposed a way of structuring lesson transcripts by dividing them into episodes and further subdividing episodes into *negotiative events*. I adapted his definition slightly to suit the classes I was observing, and defined a negotiative event to be the smallest unit of conversation involving two or more people *with a consistent topic or goal*. A negotiative event may be an entire episode, consisting of many turns or it may be a single utterance followed by tacit assent by another person.

After subdividing the transcript into negotiative events, I set out to investigate how transitions between events came about. Transitions require the complicity of the group: an utterance does not initiate a new negotiative event unless other group members begin to discuss it, or at least assent to it; nor does a declaration such as “That’s done!” necessarily terminate an event, unless other group members agree.

To illustrate, Excerpt 1 is a single negotiative event, initiated when Charles proposed graphing the volume function (line 298) and terminated when Zoe suggested asking Miss James (and Selena and Vic assented). Excerpt 3 shows the end of one negotiative event and the beginning of another. The first (deciding what to do) ended when Vic said “obviously, so we’ve got to find the value” (line 372). The next event (using the graphics calculator to find the maximum) began when Vic said “So, trace” (line 374). Charles’ utterance at line 373 was a continuation of what he had been trying to say in his previous four turns and was ignored by the others.

When the entire discussion had been divided into negotiative events, I analysed who initiated and who terminated each and in what way, and recorded this in a table. These were then counted and the results displayed in another table (see Table 1).

	Zoe	Vic	Selena	Charles
Initiations	16	7	4	3
Terminations	14	9	2	1

Table 1: Negotiative events initiated and terminated by each group member

This clearly shows Zoe’s dominance, and the relative lack of influence of Charles and Selena. Vic spent a lot of time talking to other groups, so had less influence than Zoe.

DISCUSSION AND CONCLUSIONS

The results support the findings by Cohen and her colleagues about the effects of inequalities in status on interactions within groups. To determine a student’s status in the classroom, Cohen (1997) used a combination of peer status (i.e., popularity) and academic status, measured by asking students to nominate who in the class were best at the subject. If such an instrument had been used, it is clear that both Vic and Zoe would have been assigned high status. Both were popular in the class and contributed often to class and small group discussions. In contrast, Charles would have had low status. He was unpopular and inarticulate. The teacher recognised him as “bright” but poor writing skills meant that he did not get high grades in assignments, so it is unlikely that other students would have recognised the quality of his thinking. Selena was new to the class, so had not had enough time become popular, and there was little evidence on which other students could form judgements about her academic ability. Thus, at the time of the study, she too would have had a low status. My analysis has shown that high status students influenced the discussion in the following ways: their ideas (useful or otherwise) were more likely to be accepted by the group; and on most occasions they determined what the group would discuss next. Both of the low status students put forward good ideas, but these were only accepted when endorsed by a high status student. And they had very little opportunity to influence the course of the discussion. By making more transparent the mechanisms by which students establish dominance within a group, this study may help in planning instructional strategies designed to reduce inequities in the classroom and enhance learning for all students.

Cohen and her colleagues identified inequality in participation by counting the number of turns for each student. Looking instead at whose ideas were accepted or rejected, and who determined the topic of discussion, provides a more detailed and more powerful picture of the ways in which power is enacted within small groups.

Finally, a methodological point: tracing the flow of ideas is an innovative approach to analysing complex discussions, as is studying the structure of a discussion to identify how transitions from one topic to another come about. These potentially have wider applications, for example in studying whole-class teaching, or discussions of other kinds, especially in situations where the enactment of power is at issue.

Note 1

Key to symbols used in transcripts:

/	no noticeable pause between turns, along with indications that the first turn was incomplete
//	marks the beginning of overlapping speech
…	a brief pause of 3 seconds or less. (For longer pauses, duration is stated.)
[text]	descriptions of actions, body language facial expressions or tone of voice.

References

Australian Association of Mathematics Teachers (AAMT). (2002). *Standards for excellence in teaching mathematics in Australian Schools*. Adelaide: AAMT. Retrieved 30 August, 2004 from: http://www.aamt.edu.au/standards/.

Barnes, M. S. (2003). *Collaborative learning in senior mathematics classrooms: Issues of gender and power in student-student interactions*. Unpublished doctoral thesis, The University of Melbourne, Melbourne.

Clarke, D. J. (2001). Complementary accounts methodology. In D. J. Clarke (Ed.), *Perspectives on practice and meaning in mathematics and science classrooms* (pp. 13-32). Dordrecht: Kluwer.

Cohen, E. G. (1997). Understanding status problems: Sources and consequences. In E. G. Cohen & R. A. Lotan (Eds.), *Working for equity in heterogeneous classrooms: Sociological theory in practice* (pp. 61-76). New York: Teachers College.

Davydov, V. V. (1995). The influence of L. S. Vygotsky on education theory, research and practice. *Educational Researcher*, 24(3), 12-21.

Forster, P., & Taylor, P. C. (1999, November-December). Learning mathematics through conversation and utilizing technology. Paper presented at the annual conference of the Australian Association for Research in Education, Melbourne.

Goos, M., Galbraith, P., & Renshaw, P. (2002). Socially mediated metacognition: Creating collaborative zones of proximal development in small group problem solving. *Educational Studies in Mathematics*, 49, 193-223.

Lerman, S. (2001). Cultural, discursive psychology: A sociocultural approach to studying the teaching and learning of mathematics. *Educational Studies in Mathematics*, 46, 87-113.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 27, 458-477.

A FRAMEWORK FOR THE COMPARISON OF PME RESEARCH INTO MULTILINGUAL MATHEMATICS EDUCATION IN DIFFERENT SOCIOLINGUISTIC SETTINGS

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The effects of multilingualism have been an explicit focus of a number of PME research reports in recent years. These reports, however, are located in a wide range of socio-linguistic circumstances, making it difficult to compare findings and develop a clearer understanding of the relationship between the teaching, learning or understanding of mathematics. In this paper, I describe a framework that organises the different socio-linguistic settings in which multilingual mathematics classrooms are commonly found. I use this framework to analyse recent PME research reports that focus on multilingualism in mathematics education. My analysis shows that, although the English language has a strong influence in a range of settings, the manifestation of this influence varies.

RESEARCH INTO MULTILINGUAL MATHEMATICS EDUCATION

The prevalence of multilingualism (including bilingualism) in mathematics classrooms around the world is increasingly reflected in research in mathematics education. Research reports at PME meetings in the past 10 years include several concerned with different aspects of the relationship between multilingualism and psychological dimensions of the teaching and learning of mathematics. These papers report research from many parts of the world and with a range of foci, including, for example:

- Clarkson's (1996; Clarkson and Dawe, 1997) research into how multilingual learners from non-English-speaking backgrounds make use of their different languages in solving mathematics problems in Australia;
- Hofmannová et al.'s (2001) research in the Czech Republic into the development and implementation of a curriculum in which mathematics is studied using a language from outside the country;
- Khisty's (2001) ethnographic study of how different languages are used in English/Spanish bilingual classrooms in the United States;
- Setati's (2003) investigation into the relationship between the politics of language and language use in mathematics classrooms in South Africa.

As these examples suggest, PME research in the area of multilingual mathematics education is highly diverse. In this paper, I will focus, in particular, on *sociolinguistic setting*, that is, the constellation of languages available and used within different parts of a society, and the different power and values associated with each of these languages. It is clear that PME research in this area has been conducted in a wide range of sociolinguistic settings. Such settings include, for example, classrooms in

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 145-152. Melbourne: PME.

which many languages are used (e.g., South Africa), and, in contrast, classrooms in which only one language is used, despite the presence of multilingual students (e.g., Australia). This diversity can be both a strength and a weakness. Diversity can be a strength, in that the dangers of generalising from particular situations, or of privileging particular languages or issues are avoided. Research conducted in a range of settings potentially provides a broader picture of the role of multilingualism in the teaching, learning and understanding of mathematics. Diversity can be a weakness, however, if it becomes difficult to build up such a picture, particularly when the number of studies reported remains low. Much of the research, moreover, is concerned with particular issues arising from particular settings. Findings are likely to be highly circumscribed by the particular setting in which the research was conducted. Cummins (2000, pp. 43-44) has argued, for example, that broad social factors, such as sociolinguistic setting, are implicated in patterns of classroom interaction. A current problem for research within mathematics education, however, is that there is no way of comparing, contrasting or otherwise analysing different studies on the basis of sociolinguistic setting. In the rest of this paper, I propose a framework which makes such comparison possible and offer some initial analysis of PME research in this area.

FRAMING SOCIOLINGUISTIC SETTINGS

In applied linguistics, a number of ways of classifying sociolinguistic settings of multilingual education have been proposed (e.g., Skutnab-Kangas, 1988; Baker, 2001, p. 194), many of which are focused on the different institutional approaches to the teaching and learning of second or additional languages (L2), such as second language immersion, for example. This approach does not easily transfer to consideration of classrooms where the focus is on the teaching and learning of mathematics, rather than language. An alternative approach, based on Siegel (2003, p. 179) is to focus on the role of the learner's L2 in the society in which the classroom is situated. Siegel describes 5 different settings using this approach:

Dominant L2: The main classroom language is the dominant or majority language in wider society. Multilingual students are speakers of minority languages, such as many immigrants or indigenous peoples. E.g. Turks learning German in Germany; Native Americans learning Spanish in Peru.

External L2: The main classroom language is a foreign or distant language. Multilingual students are speakers of the dominant language. E.g. Japanese learning English in Japan; English speakers in Western Canada learning French.

Coexisting L2: The main classroom language is a nearby language spoken by a large proportion of the population. Students are from a broadly multilingual environment. E.g. German speakers learning French in Switzerland.

Institutional L2: The main classroom language is an indigenous or imported language with a wide range of official uses. Students speak several local languages and inhabit highly multilingual environments. E.g. learning English in India; Swahili in Tanzania.

Minority L2: The main classroom language is that of a minority group (indigenous or immigrant). Students are speakers of the dominant or majority language. E.g English speakers learning Welsh or Panjabi in the UK.

In Siegel's framework, the five settings describe most situations in which school students may use or learn an L2. The term L2 should be seen as referring to any additional language: the framework does not preclude the use of more than two languages. The framework offers a way of analysing research in mathematics classrooms in different sociolinguistic settings. It is not, however, a precise description of interaction in a classroom. Many classrooms in South Africa, for example, officially use English as the medium of instruction and would be classified as 'Institutional L2' but this does not mean that other languages are not used by students or teachers during mathematics lessons. Finally, different settings may apply within the same geographical area. In the UK, for example, there are examples of mathematics classrooms within the dominant L2 (e.g., with immigrant communities), minority L2 (e.g., English speakers learning Welsh) and external L2 (e.g., French immersion) settings. The framework is, therefore, probably best used at the level of individual classrooms, rather than whole communities or schools.

COMPARING PME RESEARCH ON MULTILINGUAL MATHEMATICS EDUCATION

I have located examples of relevant PME research reports within Siegel's framework (see table, below). I have included all research reports with a clear focus on the role of multilingualism in different aspects of the psychology of mathematics education presented at PME conferences in the past 10 years. I have not included reports in which multilingual issues were tangentially noted or referred to. Nor have I include reports in which the focus was on the relationship between the structure of a language and students' mathematical learning. This survey resulted in the inclusion of 13 research reports.

In applying the framework, I have modified one of the categories. I have divided dominant L2 settings into 'monolingual' and 'bilingual' forms. The former refers to dominant L2 settings in which English is the main language of the curriculum and of classroom interaction, as in the UK, for example. Bilingual dominant L2 settings are those in which both learners' L1 and L2 are legitimately used in the mathematics classroom (a scenario that does not generally occur in language-focused classrooms). Examples include Spanish-English bilingual mathematics classrooms in the USA, where both English and Spanish are used.

Mathematics classroom setting	PME research reports	Location	Focus
Dominant L2 (monolingual)	Barwell (2001; 2003)	UK (mainstream classrooms)	How learners of English make sense of word problems when the classroom language is English
	Czarnocha & Prabhu (2000)	USA (ESL classrooms)	Relationship between learning algebra and learning English as a second language (ESL)
	Clarkson (1996) Clarkson & Dawe (1997)	Australia (mainstream classrooms)	Comparing attainment of bilingual and monolingual students; languages learners use ‘privately’; reasons for ‘switching’ between languages in problem solving
Dominant L2 (bilingual)	Khisty et al. (2003)	USA (Spanish/English bilingual classrooms)	Role of multimodality in a bilingual mathematics lesson
	Khisty (2001)		Features of effective teaching in bilingual mathematics
	Moschkovich, J. (1996)		‘Discontinuity’ and ‘situated’ models of bilingualism in mathematics classrooms
External L2	Hofmannová et al. (2003)	Czech Republic (English-medium classrooms)	Emotional barriers of students training to teach mathematics in English in the Czech Republic
Coexisting L2	NO REPORTS		
Institutional L2	Adler (1995)	South Africa (multilingual classrooms)	Multilingual learners’ ability to express themselves; how a teacher supports them to do this; challenges for teaching
	Prins (1997)		Multilingualism, problem solving and problem readability
	Setati (1999, 2003)		Politics of language and teachers’ use of different languages and language practices in mathematics lessons
Minority L2	NO REPORTS		

Table 1: PME research into multilingual mathematics education and sociolinguistic setting

APPLYING THE FRAMEWORK

My first observation is that two settings are not represented in PME research. The *minority L2* setting involves mathematics classrooms in which the main language used is a minority language within wider society. Whilst there has been research in such settings, such work has not been reported at PME meetings. This absence may be because such research draws more on sociological, anthropological or sociopolitical frameworks, rather than the explicitly psychological perspectives seen to be favoured at PME. This does not mean, however, that there are not important issues relevant to PME research. How, for example, is mathematical understanding influenced by the use of what are generally less widely-used languages? How are learners' motivations to study mathematics related to the use of such languages? The *co-existing L2* setting is also not represented, perhaps reflecting its geographical confinement to one or two locations (e.g. Switzerland, Québec). The research reports I have identified are fairly evenly distributed between the remaining 4 settings. In the rest of this paper, I critically compare the research reported from the three settings represented by more than one report: monolingual dominant L2, bilingual dominant L2 and institutional L2.

The dominance of English in the monolingual dominant L2 setting is reflected in the research reports. Clarkson (1996) compares the performance of bi/multilingual students with monolingual English-speakers, setting the latter as the norm. The students' home languages, such as Vietnamese are portrayed as 'other'. Clarkson seeks to show how these 'other' languages are used by students in solving arithmetic problems. Indeed, his research implies that these languages are largely used covertly. Czarnocha & Prabhu (2000) are interested in how students' mathematical learning can contribute to their learning of English. Similarly, the research reported in my own papers reflects the general absence of languages other than English in the mathematics classrooms reported, despite the students being speakers of one or more languages other than English. It is apparent that both questions and findings in these research reports are closely related to the setting in which they are located.

The three papers from the *bilingual dominant L2* setting are all from the USA, where the use of two languages such as Spanish and English to teach mathematics has been common. Again, the dominance of English is apparent. There is a concern, for example, that students should appropriate mathematical ways of talking, that is, mathematical ways of talking *in English*. Khisty (2001), for example, gives an example of how an effective mathematics teacher introduces the English word 'congruent'. As Moschkovich (1996) discusses in her paper, in most of the research in Spanish-English settings, the relationship between language and learning is described in terms of 'discontinuities'. In particular, the relationship between English in Spanish is seen as a discontinuity. This approach is problematic in several ways, such as its connection to deficit models of bilingual students, who may be penalised for using 'incorrect' mathematical English. Moschkovich does not speculate on the origins or persistence of this approach, but it is arguably related to the nature of the bilingual dominant L2 setting, in which Spanish would be seen as an obstacle that must be overcome on the way to learning to do mathematics in English.

The concerns of the papers in the *institutional L2* setting are recognizably different from those located in dominant L2 settings. All the contributions come from South Africa, a nation of 11 official languages, with English as the main language of education in most schools. Firstly, it is clear that multilingualism is a clear feature of the research. Indeed, in the case of Prins (1997), the research is a comparison between students who have English as L1, L2 or L3, showing that L3 learners were more likely to score badly on written test items, and that this trend is related to the readability of the items, an essentially linguistic issue. On the other hand, Prins's study, like Clarkson's, treats English as the main language. There was no attempt to use test items in Afrikaans or Xhosa, for example, reflecting the institutional importance of English. This institutional position is also apparent in Adler's (1995) paper, in which she explores how a student struggles to explain his thinking due to a lack of familiarity with mathematical language (concerning triangles) *in English*. Thus, the influence of English is apparent in both dominant L2 and institutional L2 settings. This influence is due to the power and opportunities associated with English in both settings. Indeed, Setati (2003) shows how the status of English in society can be related to language use in South African mathematics classrooms. The institutional dominance of English manifests itself in its use for more formal and procedural mathematical talk, such as talking through a standard algorithm. African languages, on the other hand, tend to be used in informal talk, including, for example, conceptual discussions of the mathematics involved in a problem.

The power and opportunities of English arise in different ways in the different settings. In the South African institutional L2 setting, although English has some institutional prestige, other languages are widely used. In the monolingual dominant L2 settings of the UK and Australia, languages other than English are silenced. It is notable, for example, that Clarkson (1997) had to *ask* students if they used languages other than English in working on mathematics; such usage was not generally easily observable. In the bilingual dominant L2 setting represented by Spanish-English classrooms in the USA, Spanish has some institutional recognition and is used in classrooms – a position in between monolingual dominant and Institutional L2 settings. The difference is that Spanish is seen as a stepping stone to English in the US; English is the norm. Dual language mathematics classrooms are part of a system designed to turn students into competent speakers of English.

The differences in the manifestation of the influence of English identified in the above analysis raise questions concerning the teaching and learning of mathematics. What effect does covert L2 use have on students' understanding of mathematics, their relationship with the subject, their motivation and engagement? If a Spanish-speaking students struggles to express their mathematical thinking in English, in a setting in which Spanish is seen as a stepping-stone to English, how do they then value their mathematical understanding?

CONCLUSION

Siegel's framework provides a useful starting point from which to develop a more nuanced understanding of the relationship between multilingualism and the teaching and learning of mathematics. The framework facilitates the comparison of research in different parts of the world. Through such comparisons, it becomes possible, for example, to identify phenomena that are specific to one or more setting and those that arise more widely. Recent research reports at PME have been fairly evenly spread around four different settings, although nothing has been reported from two settings.

Finally, most mathematics classrooms around the world are multilingual, in the sense that most classrooms include teachers or students who are speakers of two or more languages in their day-to-day lives. This multilingualism is rarely acknowledged in PME research reports, perhaps because of the difficulty of concisely describing complex settings when these settings do not form part of the focus of the research. Siegel's framework offers a way in which multilingualism can be acknowledged whenever and wherever it occurs.

References

Adler, J. (1995). Participatory, inquiry pedagogy, communicative competence and mathematical knowledge in a multilingual classroom: a vignette. In Meira, L. & Carraher, D. (Eds.) *PME19,* vol. 3, pp. 208-215.

Baker, C. (2001, 3rd ed.). *Foundations of Bilingual Education and Bilingualism*. Clevedon: Multilingual Matters.

Barwell, R. (2001). Investigating mathematical interaction in a multilingual primary school: finding a way of working. In van den Heuvel-Panhuizen, M. (Ed.): *Proceedings of PME 25*, vol 2, pp. 97-104.

Barwell, R. (2003). *Attention to mathematical structure* during participation in a mathematics classroom task by learners of English as an additional language (EAL). In Pateman, N. A., Dougherty, B. J. & Zilliox, J. (Eds.): *Proceedings of PME 27*, vol. 2, pp. 65-72.

Clarkson, P. C. (1996). 'NESB Migrant Students Studying Mathematics: Vietnamese and Italian Students in Melbourne', in Puig, L. & Gutierrez, A. (Eds.), *Proceedings of the PME20,* vol 2, pp. 225-232.

Clarkson, P. & Dawe, L. (1997). NESB migrant students studying mathematics: Vietnamese students in Melbourne and Sydney. In Pehkonen, E. (Ed.) *Proceedings of PME 21*, vol. 2, pp. 153-160.

Cummins, J. (2000). *Language, Power and Pedagogy: Bilingual Children in the Crossfire*. Clevedon: Multilingual Matters.

Czarnocha, B. & Prabhu, V. (2000). The flow of thought across the zone of proximal development between elementary algebra and intermediate English as a second language. In Nakahara, T. & Koyama, M. (Eds.): *Proceedings of PME 24*, vol. 2, pp. 201-208.

Hofmannová, M., Novotná, J. & Hadj-Moussova, Z. (2003). Attitudes of mathematics and language teachers towards new educational trends. In Pateman, N. A., Dougherty, B. J. & Zilliox, J. (Eds.) *Proceedings of PME27/ PMENA*, vol. 3, pp. 71-78.

Khisty, (2001). Effective teachers of second language learners in mathematics. In van den Heuvel-Panhuizen, M. (Ed.): *Proceedings of PME 25*, vol 3, pp. 225-232.

Khisty, L. L., Morales, H. & Chval, K. (2003) Beyond discourse: a multimodal perspective of learning mathematics in a multilingual context. In Pateman, N. A., Dougherty, B. J. & Zilliox, J. (Eds.): *Proceedings of PME27*, vol. 3, pp. 133 – 140.

Moschkovich, J. (1996). Learning math in two languages. In Puig, L. & Gutierrez, A. (Eds.) *Proceedings PME 20*, vol. 4, pp. 27-34.

Prins, E. D. (1997). Readability of verbal problems in mathematics: some evidence from secondary schools in South Africa. *Proceedings of PME 21* vol. 4, pp. 33-40.

Setati, M. (1999). Ways of talking in a multilingual mathematics classroom. *Proceedings of PME 23* vol. 4, pp. 177-184.

Setati. M. (2003). Language use in a multilingual mathematics classroom in South Africa: a different perspective. In Pateman, N. A., Dougherty, B. J. & Zilliox, J. (Eds.) *Proceedings PME 27/ PMENA*, vol. 4, 151-165.

Siegel, J. (2003). Social context. In Doughty, C. J. & Long, M. H. (Eds.) *The Handbook of Second Language Acquisition*, pp. 178-223. Oxford: Blackwell.

Skutnabb-Kangas, T. (1988). Multilingualism and the education of minority children. In Skutnabb-Kangas, T. & Cummins, J. (Eds.) *Minority Education*. Clevedon: Multilingual Matters.

VYGOTSKY'S THEORY OF CONCEPT FORMATION AND MATHEMATICS EDUCATION

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I argue that Vygotsky's theory of concept formation (1986) is a powerful framework within which to explore how an individual at university level constructs a new mathematical concept. In particular, this theory is able to bridge the divide between an individual's mathematical knowledge and the body of socially sanctioned mathematical knowledge. It can also be used to explain how idiosyncratic usages of mathematical signs by students (particularly when just introduced to a new mathematical object) get transformed into mathematically acceptable usages and it can be used to elucidate the link between usages of mathematical signs and the attainment of meaningful mathematical concepts by an individual.

INTRODUCTION

The issue of how an individual makes personal meaning of a mathematical object presented in the form of a definition is particularly relevant to the study of advanced mathematical thinking. In this domain, the learner is frequently expected to construct the properties of the object from the definition (Tall, 1995). In many instances neither diagrams nor exemplars of the mathematical object are presented alongside the definition; initial access to the mathematical object is through the various signs (such as words and symbols) of the definition.

In this talk, I argue that Vygotsky's theory of concept formation (1986) provides an appropriate framework within which to explore the above issue of concept formation. Specifically I claim that this framework has constructs and notions well–suited to an explication of the links between the individual's concept construction and socially sanctioned mathematical knowledge. Also the framework is apposite to an examination of how the individual relates to and gives meaning to the signs (such as symbols and words) of the mathematical definition.

BACKGROUND

Several mathematics education researchers have considered how an individual, at university level, constructs a mathematics concept and some have developed significant theories in response. The most influential of these theories focus on the transformation of a process into an object (for example, Tall, 1995; Dubinsky, 1991; Czarnocha et al, 1999).

According to Tall et al. (2000), the idea of a process–object duality originated in the 1950's in the work of Piaget who spoke of how "actions and operations become thematized objects of thought or assimilation" (cited in Tall et al, 2000: 1).

In adopting a neo–Piagetian perspective, these researchers and their various followers successfully extend Piaget's work regarding elementary mathematics to advanced mathematical thinking. For example, Czarnocha et al. (1999) theorise that in order to understand a mathematical concept, the learner needs to move between different stages. She has to manipulate previously constructed objects to form actions. "Actions are then interiorised to form processes which are then encapsulated to form objects" (1999: 98). Processes and objects are then organised in schemas.

But much of this process–object theory does not resonate with a great deal of what I see in my mathematics classroom. For example, it does not help me explain or describe what is happening when a learner fumbles around with 'new' mathematical signs making what appear to be arbitrary connections between these new signs and other apparently unrelated signs. Similarly, it does not explain how these incoherent–seeming activities can lead to usages of mathematical signs that are both acceptable to professional members of the mathematical world and that are personally meaningful to the learner.

I suggest that the central drawback of these neo–Piagetian theories is that they are rooted in a framework in which conceptual understanding is regarded as deriving largely from interiorised actions; the crucial role of language (or signs) and the role of social regulation and the social constitution of the body of mathematical knowledge is not integrated into the theoretical framework.

What is required is a framework in which the link between an individual's construction of a concept and social knowledge (existing in the community of mathematicians and in reified form in textbooks) is foregrounded. Furthermore, given that mathematics can be regarded as the "quintessential study of abstract sign systems" (Ernest, 1997) and mathematics education as "the study of how persons come to master and use these systems" (ibid.), a framework which postulates semiotic mediation as the mechanism of learning, seems apposite. I claim that Vygotsky's much–neglected theory of concept formation, allied with his notion of the functional usage of a sign (1986), is such a framework.

VYGOTSKY'S THEORY OF CONCEPT FORMATION

Although Vygotskian theory (but not the theory of concept formation) has been applied extensively in mathematics education, most of the research has focused on the mathematical activities of a group of learners or a dyad rather than the individual (Van der Veer and Valsiner, 1994). Furthermore it has been applied most frequently to primary school or high school learners (for example, van Oers, 1996; Radford, 2001) rather than to individuals at undergraduate level.

Indeed, Van der Veer and Valsiner (1994) claim that the use of Vygotsky in the West has been highly selective. In particular they argue that "*the focus on the individual developing person* which Vygotsky clearly had … has been persistently overlooked" (p. 6; italics in original).

It is important to note that a focus on the individual (possibly with a textbook or in consultation with a lecturer) does not contradict the fundamental Vygotskian notion that "social relations or relations among people genetically underlie all higher functions and their relationships" (Vygotsky, 1981, p. 163). After all, a situation consisting of a learner with a text is necessarily social; the textbook or exercises have been written by an expert (and can be regarded as a reification of the expert's ideas); also the text may have been prescribed by the lecturer with pedagogic intent. Thus a focus on the individual does not undermine the significance of the social.

Functional use of the sign

In order to understand Vygotsky's theory, one needs to understand how Vygotsky used the term 'word'. Vygotsky regarded a word as embodying a generalisation and hence a concept.

As such, Vygotsky postulated that the child uses a word for communication purposes before that child has a fully developed understanding of that word. As a result of this use in communication, the meaning of that word (i.e., the concept) evolves for the child:

> Words take over the function of concepts and may serve as means of communication long before they reach the level of concepts characteristic of fully developed thought (Uznadze, cited in Vygotsky, 1986: 101).

The use of a word or sign to refer to an object (real or virtual) prior to 'full' understanding resonates with my sense of how an undergraduate student makes a new mathematical object meaningful to herself. In practice, the student starts communicating with peers, with lecturers or the potential other (when writing) using the signs of the new mathematical object (symbols and words) before she has full comprehension of the mathematical sign. It is this communication with signs that gives initial access to the new object.

> It is a functional use of the word, or any other sign, as a means of focusing one's attention, selecting distinctive features and analysing and synthesizing them, that plays a central role in concept formation (Vygotsky, 1986: 106).

Secondly but closely linked to the above notion, is Vygotsky's argument that the child does not spontaneously develop concepts independent of their meaning in the social world:

> He does not choose the meaning of his words… The meaning of the words is given to him in his conversations with adults (Vygotsky, 1986: 122).

That is, the meaning of a concept (as expressed by words or a mathematical sign) is 'imposed' upon the child and this meaning is not assimilated in a ready–made form. Rather it undergoes substantial development for the child as she uses the word or sign in her communication with more socialised others.

Thus the social world, with its already established definitions (as given in dictionaries or books) of different words, determines the way in which the child's generalisations need to develop.

Analogously, I argue that in mathematics, a student is expected to construct a concept whose use and meaning is compatible with its use in the mathematics community. To do this, that student needs to use the mathematical signs in communication with more socialised others (including the use of textbooks which embody the knowledge of more learned others). In this way, concept construction becomes socially regulated.

Semiotic mediation

Vygotsky (1978) regarded all higher human mental functions as products of mediated activity. The role of the mediator is played by a psychological tool or sign, such as words, graphs, algebra symbols, or a physical tool. These forms of mediation, which are themselves products of the socio-historical context, do not just facilitate activity; they define and shape inner processes. Thus Vygotsky saw action mediated by signs as the fundamental mechanism which links the external social world to internal human mental processes and he argued that it is

> by mastering semiotically mediated processes and categories in social interaction that human consciousness is formed in the individual (Wertsch and Stone, 1985: 166).

Allied to this, concept formation, as discussed above, is only possible because the word or mathematical object can be expressed and communicated via a word or sign whose meaning is already established in the social world.

In mathematics, the same mathematical signs mediate two processes: the development of a mathematical concept in the individual and that individual's interaction with the already codified and socially sanctioned mathematical world (Radford, 2000). In this way, the individual's mathematical knowledge is both cognitively and socially constituted.

This dual role of a mathematical sign by a learner before 'full' understanding is not well appreciated by the mathematics education community; indeed, its manifestations in the form of activities such as manipulations, imitations and associations are often regarded disparagingly by mathematics educators. That is, they regard such activities as 'meaningless' and without worth. (Conversely, back–to–basics mathematics educators may regard adequate use of a mathematical sign as sufficient evidence of a student's understanding of the relevant mathematical concept. Of course, in terms of Vygotsky's theory, this is not the case).

Vygotsky's theory, that usages of the sign are a necessary part of concept formation, manages to provide a link between certain types of mathematical activities (including those activities regarded pejoratively by many educators) and the formation of concepts.

Different stages

Vygotsky further elaborated his theory by detailing the stages in the formation of a concept. He claimed that the formation of a concept entails different preconceptual stages (heaps, complexes and potential concepts).

During the syncretic heap stage, the child groups together objects or ideas which are objectively unrelated. This grouping takes place according to chance, circumstance or subjective impressions in the child's mind. In the mathematical domain, a student is using heap thinking if she associates one mathematical sign with another because of, say, the layout of the page.

The syncretic heap stage gives way to the complex stage. In this stage, ideas are linked in the child's mind by associations or common attributes which exist objectively between the ideas.

Complex thinking is crucial to the formation of concepts in that it allows the learner to think in coherent terms and to communicate via words and symbols about a mental entity. And, as I have argued above, it is this communication with more knowledgeable others which enables the development of a personally meaningful concept whose use is congruent with its use by the wider mathematical community.

> Complexes corresponding to word meanings are not spontaneously developed by the child: The lines along which a complex develops are predetermined by the meaning a given word already has in the language of adults (Vygotsky, 1986: 120).

Furthermore, in complex thinking the learner begins to abstract or isolate different attributes of the ideas or objects, and the learner starts organizing ideas with particular properties into groups thus creating the basis for later more sophisticated generalisations.

With complex thinking, the learner is not using logic; rather she is using some form of non–logical or experiential association. Thus complex thinking often manifests as bizarre or idiosyncratic usage of mathematical signs.

For example, the learner is using complex thinking when she associates the properties of a 'new' mathematical sign with an 'old' mathematical sign with which she is familiar and which is epistemologically more accessible.

As an illustration, on first encountering the derivative, $f'(x)$, of a function $f(x)$, the learner may associate the properties of $f'(x)$ with the properties of $f(x)$. Accordingly, many learners assume or imply that since $f(x)$ is continuous, so is $f'(x)$. Clearly this is not logical; indeed it is mathematically incorrect.

Another example of activity guided by complex thinking is when the student seems to focus on a particular aspect of the mathematical expression and to associate these symbols or words with a new sign. For instance, when dealing with the greatest integer function $\lceil\lceil x \rceil\rceil$= greatest integer $\leq x$, many students latch onto the word 'greatest' ignoring the condition $\leq x$. They then link the word 'greatest' to the idea of 'greater than' and accordingly state that, say, $\lceil\lceil 4.3 \rceil\rceil = 5$ (whereas of course, the answer should be 4).

My point here is not *how* the student uses the signs but rather *that* she uses the signs. Through this use, the student gains access to the 'new' mathematical object and is able to communicate (to better or worse effect) about it. Through social regulation or reflection (in tandem with the socially constituted definition and for an attenuated or extended time period) the learner will eventually come to use and understand the signs in ways that are congruent with official mathematics.

My observations of undergraduate students over the years ties in very well with the idea that preconceptual thinking is a necessary part of successful mathematics concept construction (this is evidenced by many of these students' apparently confused mathematical assertions prior to mathematical coherence). Of course, the time spent using complex thinking may be very brief or very long, depending on the student, the particular mathematical object, the task, the context and the social interventions.

Vygotsky distinguished between five different types of complexes. For the purposes of this talk it is sufficient to elaborate on the pseudoconcept, which is a construct which effectively bridges the divide between the individual and the social and between complex and concept. (For elaboration and exemplification of the different types of complexes, see Sierpinska (1993), Berger (2004a, 2004b)).

The pseudoconcept: a bridge between the individual and the social

In order to understand the pseudoconcept one needs to know how Vygotsky used the word 'concept': in a concept, the bonds between the parts of an idea and between different ideas are logical and the ideas form part of a socially-accepted system of hierarchical knowledge.

According to Vygotsky, the transition from complexes to concepts is made possible by the use of pseudoconcepts. Hence the pseudoconcept is a very special form of complex.

Pseudoconcepts resemble true concepts in their use, but the thinking behind these pseudoconcepts is still complex in character. This is because the bonds between the different elements of a pseudoconcept are associative and experiential rather than logical and abstract. But the learner is able to use the pseudoconcept in communication and activities *as if* it were a true concept.

The use of pseudoconcepts is ubiquitous in mathematics and is analogous to a child using a word in conversation with an adult before fully understanding the meaning of that word. Pseudoconcepts occur whenever a student uses a particular mathematical object in a way that coincides with the use of a genuine concept, even though the student has not fully constructed that concept for herself. For example, a student may use the definition of the derivative of a function to compute the derivative of the function before she 'understands' the nature of the derivative or its properties.

Vygotsky (1986) argued that the use of pseudoconcepts enables children to communicate effectively with adults and that this communication (the intermental aspect) is necessary for the transformation of the complex into a genuine concept (the intramental aspect) for the learner.

> Verbal communication with adults (…) become a powerful factor in the development of the child's concepts. The transition from thinking in complexes to thinking in concepts passes unnoticed by the child because his pseudoconcepts already coincide in content with adult concepts (Vygotsky, 1986: 123).

Thus the pseudoconcept functions as the bridge between concepts whose meaning is more or less fixed and constant in the social world (such as that body of knowledge we call mathematics) and the learner's need to make and shape these concepts so that they become personally meaningful. This bridging function of the pseudoconcept is the basis for my contention that the pseudoconcept can be regarded as the link between the individual and the social. As such pseudoconcepts are a necessary stage in the child's or student's development of true concepts. Furthermore the notion of the pseudoconcept is entirely consistent with the functional use of a sign.

The pseudoconcept can be used to explain how the student is able to use mathematical signs (in algorithms, definitions, theorems, problem−solving, and so on) in effective ways that are commensurate with that of the mathematical community even though the student may not fully 'understand' the mathematical object. The hope is that through appropriate use and social interventions, the pseudoconcept will get transformed into a concept.

CONCLUSION

In this paper, I have argued that Vygotsky's theory of concept formation provides an apposite framework within which to elaborate how an individual constructs a concept that is personally meaningful and whose usage is commensurate with that of the mathematical community.

In particular, I argued that the notion of functional usage of the sign, together with the construct of the pseudoconcept, can be used to bridge the divide between an individual's concept formation and a socially sanctioned mathematical definition. Related to this, idiosyncratic mathematical activities can be regarded as manifestations of complex thinking. With social regulation, these complexes can be transformed into pseudoconcepts and ultimately concepts can be formed. Finally, I argued that Vygotsky's notion that all knowledge is semiotically mediated is necessary for understanding how students use mathematical signs to gain access to mathematical objects.

What is now required is empirical research which illuminates the bridges between personal and socially sanctified usages of mathematical signs, explicates the transformations from complexes to pseudoconcepts to concepts, and explores the relationships between different usages of signs and meaning−making.

References

Berger, M. (2004a). Heaps, complexes and concepts (part 1), *For the Learning of Mathematics* 24 (2), pp. 2 – 6.

Berger, M. (2004b). Heaps, complexes and concepts (part 2), *For the Learning of Mathematics* 24 (3).11 – 17.

Czarnocha, B., Dubinsky, E., Prabhu, V. & Vidakovic, D. (1999). One theoretical perspective in undergraduate mathematics education research. In 0. Zaslavsky (Ed.), *Proc. 23rd Conf. of the Int. Group for the Psychology of Mathematics Education,* (Vol. 1, pp. 95−110). Haifa, Israel: PME

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.) *Advanced Mathematical Thinking* (pp. 95−123). Dordrecht: Kluwer Academic.

Ernest, P. (1997). Introduction: Semiotics, Mathematics and Mathematics Education, *Philosophy of Mathematics Education,* Journal 10.
http://www.ex.ac.uk/~PErnest/pome10/art1.htm

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis, *Educational Studies in Mathematics* 42: 237–268.

Radford, L. (2001). Factual, Contextual and symbolic generalizations in algebra. In M. van den Heuvel−Panhuizen (Ed.) *Proc. 25th Conf. of the Int. Group for the Psychology of Mathematics Education,* (Vol. 4, pp. 81−88). Utrecht, The Netherlands: PME.

Sierpinska, A. (1993). The development of concepts according to Vygotsky, *Focus on Learning Problems in Mathematics*, Vol. 15, Nos. 2 & 3, 87−107.

Tall, D.O. (1995). *Cognitive Growth in Elementary and Advanced Mathematical Thinking*, In L. Meira and D. Carraher. (Eds.) *Proc. 19th Conf. of the Int. Group for the Psychology of Mathematics Education,* (Vol. 1, pp. 61−75). Recife, Brazil: PME.

Tall, D.O., Thomas, M., Davis, G., Gray, E., & Simpson, A. (2000). What is the object of the encapsulation of a process? *Journal of Mathematical Behavior*, 18, 2, 1−19.

Van der Veer & Valsiner (1994). Reading Vygotsky: from fascination to construction. In R. van der Veer & J Valsiner (Eds. and Trans.), *The Vygotsky Reader* (pp. 1−9). Oxford: Blackwell Publishers.

Van Oers, B. (1996). Learning mathematics as a meaningful activity. In P. Nesher, L.P. Steffe, P. Cobb, G. Goldin, & B. Greer (Eds.) *Theories of mathematical learning* (pp. 91–113). Mahwah, N.J.: Lawrence Erlbaum.

Vygotsky, L.S. (1981). The genesis of higher mental functions. In J.V. Wertsch (ed.) *The Concept of Activity in Soviet Psychology* (pp.144−188). Armonk, NY: M.E. Sharpe.

Vygotsky, L. S. (1986). *Thought and Language*, A. Kozulin, (Ed. and Trans.), Cambridge, MA.: MIT Press.

Wertsch, J.V. & Stone, C.A. (1985). The concept of internalisation in Vygotsky's account of the genesis of higher mental functions. In Wertsch, J.V. (Ed.) *Culture, Communication and Cognition* (pp. 162−179). New York: Cambridge University Press.

PRESERVICE TEACHERS' UNDERSTANDINGS OF RELATIONAL AND INSTRUMENTAL UNDERSTANDING

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This paper reports on the responses of a cohort of preservice primary teachers to a statement about the extent to which helping children achieve relational understanding is a realistic expectation. Although the preservice teachers' course had included teaching about understanding a number of misconceptions about the meanings of relational and instrumental understanding were evident in the responses of a sizeable minority, along with evidence that many held beliefs that were likely to result in them teaching instrumentally. The findings highlight the idiosyncratic nature of preservice teachers' knowledge construction and draw attention to a range of disparate meanings that may be attached to the term 'understanding' even when it is qualified with other words such as 'instrumental' or 'relational'.

BACKGROUND AND THEORETICAL FRAMEWORK

Ongoing calls for reform in education generally and mathematics education in particular have stressed the importance of teaching for understanding (e.g. NCTM, 2000). In several Australian states, including Tasmania where this study was conducted, significant shifts to values-based curricula that place a heavy emphasis on the development deep understanding are underway (Department of Education, Tasmania, DoET, 2002). It thus behoves mathematics teacher educators to prepare preservice teachers to teach for understanding.

This task is by no means simple, with the difficulty due at least in part to the difficulty of defining exactly what is meant by understanding. Madison (1982) sourced the difficulty in the tendency to equate our understandings with reality, and stressed that understandings can really only be described as beliefs. Much that has been written about understanding, including in the two documents cited above, does not attempt to define the concept, but rather a shared 'understanding' of the meaning is assumed. The danger of such an assumption was highlighted by Skemp (1978) in relation to mathematics when he described the existence of two disparate uses of the term that resulted in, in his view, two quite distinct mathematics curricula. Skemp (1978) labelled these types of understanding instrumental and relational and it is the latter which is implied by authors writing from a reform perspective (e.g. Hannula, Maijala, & Pehkonen, 2004). The term relational implies connections and indeed the development of connections is central to advice on teaching for understanding (Mousley, 2004). Mousley (2004) lists three types of connections that are commonly intended. These are connections between: new and existing knowledge; various mathematical ideas and representations; and mathematics learned in school and everyday life. It was the second of these that Skemp (1978) described.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 161-168. Melbourne: PME.

The process by which understanding is achieved (or connections of various kinds are made) has been described by Pirie and Kieren (1989) as recursive in that rather than more sophisticated understandings developing from more primitive ones, there is a need to revisit earlier understandings and view them from a different perspective in order to develop the next level of understanding. Sierpinska (1994) described understanding as emerging in response to difficulties encountered when current knowledge meets new, not readily reconcilable experiences. Wiggins and McTighe (1998), whose work has been influential in the Tasmanian curriculum reforms, provided a framework comprising six not necessarily discrete facets of understanding that they believed could be helpful for teachers in designing learning experiences that fostered the development of understanding. These views have in common that they present the development of understanding as complex, non-linear and unpredictable phenomenon.

All of these perspectives, as well as the underpinning philosophy of calls to reform curricula and specifically mathematics education, are consistent with a constructivist view of learning (Confrey, 1990; Simon, 2000). In describing understandings as beliefs, Madison (1982), is essentially equating understandings with a constructivist view of knowledge in which the distinction between knowledge and beliefs is principally a matter of the degree of consensus attracted by virtue of the amount and quality of information on which they are based, and their powerfulness in terms of explaining and predicting experience (Guba & Lincoln, 1989). Lerman (1997) maintained that researchers should be mindful that theories of learning apply equally to attempts to change the beliefs and practices of teachers. That is, from a constructivist perspective, teachers, including preservice teachers such as those in this study, actively construct knowledge for the purpose of making sense of their experiences (von Glasersfeld, 1990). A further dimension of constructivism derives from the work of Vygotsky (Ernest, 1998) who stressed the critical role of language in social contexts in the development of thinking.

The task of assisting preservice teachers to construct a notion of mathematical understanding as relational (Skemp, 1978) and to value this perspective to the extent that they are likely to teach in ways that foster the development of relational understanding in their students, thus amounts to an effort to change their beliefs about what it means to understand mathematics. Given the established difficulty of influencing beliefs (Lerman, 1997), the strong tendency of teachers to teach in the ways that they were taught (Ball, 1990), the fact that many will have experienced mathematics teaching aimed at achieving instrumental understanding, and the complexities involved in developing understanding of anything, including understanding itself (Pirie & Kieran, 1994; Sierpinska, 1994; Wiggins & McTighe, 1998) this is likely to be a difficult undertaking. In this context it should be remembered that the perception of misunderstanding on the part of a student is also a belief of the teacher. Essentially teachers or educators operating from a constructivist perspective but with particular outcomes for their students in mind are attempting to a greater or lesser extent to replicate their own understandings in their students, with misunderstanding deduced from evidence that students do not share their understanding.

THE STUDY

The study was motivated by a concern that, almost 30 years after Skemp (1978) articulated the problem, teachers including preservice teachers, still attached differing and conflicting meanings to the term ‘understanding’. It was designed to provide evidence in relation to the extent that this was indeed the case for preservice teachers who had notionally ‘been taught’ about understanding in relation to mathematics and was thus part of ongoing course evaluations.

Context of the study

At the University of Tasmania, where this study was conducted, students are required to study mathematics curriculum in three semesters of the B. Ed. program - one in each of their second, third and fourth years. Mathematics curriculum studies are combined with English curriculum studies, and so the students study three half units of mathematics curriculum. Each half-unit is conducted over 13 weeks in a single semester, delivered via a weekly one hour lecture and a one hour tutorial in second and third year, and via a two hour weekly tutorial in the fourth year. Tutorials are conducted in groups of 25-30 students. Instruction in this context is designed to be interactive with students working cooperatively on activities designed to illustrate and explore information presented in the lectures. In the tutorials, the lecturers in the program aimed to model an approach to teaching that was consistent with the principles of constructivism. In both lectures and tutorials the emphasis of teaching was on promoting students’ awareness of broad pedagogical ideas for meaningful learning of mathematics, such as the importance of rich mathematical learning environments for conceptual development, a mathematics curriculum that focuses on problem solving and thinking skills, and appropriate materials for concept representation. In lectures and tutorials, it was the lecturers’ intention to communicate these ideas through modelling best practice, using lecture and particularly tutorial times, to engage students in activities designed for such notions to surface. A further objective of the program in total was to promote students’ beliefs in the importance of mathematics and its teaching, whilst enhancing their confidence in their ability to understand basic mathematics, and fostering positive attitudes to the teaching of mathematics.

Subjects

The subjects were 174 preservice primary teachers enrolled in the first mathematics curriculum half unit, in the second year of the preservice teachers’ study.

Instrument

The statement to which students were asked to respond was contained in question eight of the examination paper for the unit. The two hour examination was comprised

13 questions requiring short answers in the spaces provided (two lines per mark), accounted for 40% of students' result for the unit, and was designed to assess students' understandings of the material covered in the unit rather than simply their ability to recall information. The specific question was:

> Indicate your agreement or otherwise with the following statement, giving reasons for your choice: "Helping children to achieve relational understanding is too time-consuming. There is so much in the curriculum to cover that it is an unrealistic expectation." (4 marks [of a total of 53])

Procedure

Teaching mathematics for understanding was a topic of one lecture. The corresponding tutorial included a discussion of the understanding based on a section of the prescribed text, Van de Walle (2002), and Skemp's (1978) article on instrumental and relational understanding. Incidental references to the importance of teaching mathematics for understanding (relational) were made throughout the course and modelled in tutorials.

At the end of the semester students sat the examination and, after the assessment of the unit had been finalised, their responses to question eight were re-examined specifically for evidence of their understandings of understanding. Those that clearly evidenced misunderstandings were further examined in order to identify categories into which these responses could be divided. Some of the responses that demonstrated misunderstandings were allocated to more than one category on the grounds that they showed evidence of more than one type of misunderstanding.

RESULTS AND DISCUSSION

Of the 174 answers examined 52 (30%) showed evidence of misunderstanding. Table 1 shows the categories of misunderstandings identified, the number of responses falling in each and an example of a response allocated to each category.

Fifteen of the preservice teachers clearly agreed with the statement presented in the question. Given that they responded under examination conditions and that the views of the lecturers who would be marking their papers were likely to have been well known, this figure is likely to be an under-estimation of the numbers who in fact believed that relation understanding was an unrealistic expectation. It seems likely that at least some students in classes taught by these teachers will not be taught with relational understanding of mathematics as the goal.

Categories two to seven all contained responses that presented relational understanding as something additional that should be aimed for, rather than essential, and hence the argument presented in the statement that time is a constraint on teaching for relational understanding is likely to have some merit in the view of these preservice teachers. A likely consequence is that amongst the demands of classroom life the goal of relational understanding will not survive. These preservice teachers may well be among the many who revert to teaching as they were taught (Ball, 1990).

Category of misunderstanding		Example	No. of responses (% of 174)
1. Relational understanding is an unrealistic expectation:	a. For some students	Ideally it would be great to have every student with relational understanding … not every student in the class is going to achieve relational understanding.	10 (5.7)
	b. Under some circumstances	… sometimes there is too much pressure from students, parents and government to allow time for it.	5 (2.9)
2. Relational understanding follows from instrumental understanding		… Children need to move from instrumental understanding so that they can see why …	9 (5.2)
3. Relational/instrumental understanding is a curriculum topic		A well organised teacher can afford to cover such a topic …	13 (7.5)
4. Relational understanding is about relating mathematics to other curriculum areas/real life		… Students should be able to relate mathematics to almost anything as it is ever changing and growing	7 (4.0)
5. Relational understanding is about knowing the purpose/relevance of mathematics topics		… if children only have an instrumental understanding then they are merely memorising concepts and not truly understanding what they're learning and why it is learned …	9 (5.2)
6. Relational understanding is a skill that can be applied to problems in mathematics and other curriculum areas		… it would save time as students would be able to learn to relate the way to understanding one question to another …	9 (5.2)
7. Both relational and instrumental understanding are needed		A child needs to have at least some relational understanding they also need some instrumental understanding …	3 (1.7)
8. Relational understanding is a teaching technique		… Although more time consuming this method is far more beneficial than the instrumental method …	4 (2.3)

Table 1: Types of misunderstandings of understanding

The idea that relational understanding develops from instrumental understanding (Category 2) is perhaps related to the way in which these preservice teachers have experienced coming to understand mathematics. Brown, McNamara, Hanley and

Jones (1999) reported that many primary preservice teachers are pleasantly surprised by their initial experiences of learning mathematics for teaching and in particular enjoy achieving what could be described as relational understanding of various topics for the first time. For them, and arguably for many teachers who have been taught mathematics instrumentally, relational understanding, if it has been achieved at all, has followed instrumental understanding.

The belief that relational understanding is an additional topic in the mathematics curriculum (Category 3) was conveyed in 7.5% of all responses. It would seem that for these preservice teachers the course has been ineffectual in influencing their beliefs in relation to the nature of mathematical understanding.

Categories four and five contained responses that associated relational understanding with versions of the third kind of connections described by Mousley (2004). These students may have been influenced by the word "relational". Their views may also have reflected personal experiences of learning mathematics devoid of context, meaning or applicability to their lives. The importance of connecting school mathematics with the lives of students is emphasised in curriculum documents (NCTM, 2000; DoET, 2002) and born out by research that suggests many students cannot see any use for the mathematics they learn at school beyond passing tests and achieving qualifications (Onion, 2004). While having merit, this view of understanding is neither complete nor that described by Skemp (1978).

Pre-service teachers whose responses fell in Category six saw relational understanding as a skill rather than a quality of understanding. It is possible that at least some of these preservice teachers in fact saw relational understanding in terms of the development of connections between mathematics topics which consequently enhanced students' ability to apply mathematics in a range of contexts. To the extent to which this was the case, and this is not clear, this category is unproblematic and in fact would not represent a misunderstanding.

The view that instrumental and relational understanding are both necessary (Category 7) may be based on the characterisation provided by Skemp (1978) of these types of understanding as respectively knowing 'what' and 'how', and knowing 'why'. As Hannula et al. (2004) pointed out knowledge (what) and skill (how) are inherent in mathematical understanding. The extent to which these preservice teachers regarded instrumental understanding as included within relational understanding is not clear but none articulated this view.

Responses in Category eight conveyed a belief that relational and instrumental understandings are teaching methods. These preservice teachers may have focussed on the descriptions by Skemp (1978) of instrumental and relational teaching. The emphasis on how to teach is consistent with Brown et al.'s (1999) observation that preservice primary teachers wanted to be told how to teach.

CONCLUSION

Up to one third of the 174 preservice teachers in this cohort held some kind of misunderstanding about understanding at the end of a semester in which the topic had been approached in a variety of ways. It is recognised that the use of lectures is neither pedagogically desirable nor effective for many students, as this study attests, but they are sometimes fiscally necessary. There is a need for research on how the effectiveness of courses that are constrained to operate in non ideal modes can be maximised. In the particular context of this study, the findings have lead to the use of an electronic discussion board on which understanding is one of the topics and a variety of questions, similar in nature to that discussed in this paper, are provided to stimulate the discussion. There are also plans to modify the assessment of the unit to facilitate, to the limited extent possible, preservice teachers working with primary school students with a focus on analysing the understandings that students display.

The findings of this study add weight to calls to increase the integration of teacher education in on-campus settings and in schools (Ball & Bass, 2000). Preservice teachers need to experience examples of ‘unlikely’ students achieving relational understanding. They need powerful evidence that their own experience is not the only possible experience of learning mathematics. Mathematics educators approaching their task from a constructivist perspective should not of course be surprised that their students construct idiosyncratic understandings. Findings such as these highlight the inherent difficulty of teaching from such a stance and remind us of the challenging task for which we are preparing preservice teachers. Despite the prominence of the notion of understanding over several decades there clearly remains a need to carefully unpack the meaning attached to it by various users of the term.

References

Ball, D. L. (1990). Breaking with experience in learning to teach mathematics: The role of a preservice methods course. *For the Learning of Mathematics*, 10(2), 10-16.

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 83-104). Westport, CT: Ablex.

Brown, T., McNamara, O., Hanley, U., & Jones, L. (1999). Primary student teachers' understanding of mathematics and its teaching. *British Educational Research Journal*, 25(3), 299-322.

Confrey, J. (1990). What constructivism implies for teaching. In R. B. Davis, C. A. Maher & N. Noddings (Eds.), *Constructivist views on the teaching and learning mathematics* (pp. 107-122). Reston, VA: National Council of Teachers of mathematics.

DoET. (2002). *Essential Learning Framework 1*. Hobart: Department of Education, Tasmania.

Ernest, P. (1998). *Social constructivism as a philosophy of mathematics*. Albany, NY: State University of New York.

Guba, E. G., & Lincoln, Y. S. (1989). *Fourth Generation Evaluation*. Newbury park, California: Sage.

Hannula, M., Maijala, H., & Pehkonen, E. (2004). Development of understanding and self-confidence in mathematics: Grades 5-8. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 17-24). Bergen, Norway: Bergen University College.

Lerman, S. (1997). The psychology of mathematics teachers' learning: in search of theory. In E. Pehkonen (Ed.), *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 200-207). Lahti, Finland: PME.

Madison, G. B. (1982). *Understanding: A phenomenological-pragmatic analysis*. Westport, CT: Greenwood.

Mousley, J. (2004). An aspect of mathematical understanding: The notion of "connected knowing". In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 377-384). Bergen, Norway: Bergen University College.

NCTM. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Onion, A. J. (2004). What use is maths to me? A report on the outcomes from student focus groups. *Teaching Mathematics and it Applications*, 23(4), 189-194.

Pirie, S., & Kieren, T. (1989). A recursive theory of mathematical understanding. For the Learning of Mathematics, 9(3), 7-11.

Sierpinska, A. (1994). *Understanding in mathematics*. London: Falmer.

Simon, M. A. (2000). Constructivism, mathematics teacher education, and research in mathematics teacher development. In L. P. Steffe & P. W. Thompson (Eds.), *Radical constructivism in action: Building on the pioneering work of Ernst von Glasersfeld* (pp. 213-230). London: Routledge Falmer.

Skemp, R. R. (1978). Relational understanding and instrumental understanding. *Arithmetic Teacher*, 26(3), 9-15.

Van de Walle, J. A. (2002). *Elementary and Middle School Mathematics*. New York: Longman.

von Glasersfeld, E. (1990). An exposition of constructivism: Why some like it radical. In R. B. Davis, C. A. Maher & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics* (pp. 19-29). Reston, VA: National Council of Teachers of Mathematics.

Wiggins, G., & McTighe, J. (1998). *Understanding by Design Handbook*. VA: Association for Supervision and Curriculum Development.

THE TRANSFORMATION OF MATHEMATICS IN ON-LINE COURSES[1]

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This paper presents some research findings regarding the changes in the mathematics produced by mathematics teachers in on-line distance courses. Predicated on the belief that knowledge is generated by collectives of humans-with-media, and that different technologies modify the nature of the knowledge generated, we have sought to understand how the Internet modifies interactions and knowledge production in the context of distance courses. The research was conducted over a period of several years, during distance courses proffered annually from the mathematics department at UNESP, São Paulo State University, to teachers throughout Brazil, conducted mainly via weekly chat sessions. Findings presented contrast teachers' knowledge production when using the Internet with production of knowledge when using regular dynamic geometry software or plotters.

INTRODUCTION

In this paper, I will report on partial results and new questions that our practice, as a research group, have raised in the process of engaging in virtual interactions with teachers from different parts of Brazil (and in smaller numbers, from other countries in South America). We have developed Internet-based extension courses for mathematics teachers from different levels as one means of addressing social inequalities in Brazil and, at the same time, to research and learn about Internet-based education. Different research questions are being addressed in this project, some of which are related to the nature of the needs that teachers who participate in on-line courses will have, and others to the different opportunities that teachers and researchers may have with the new possibilities offered by the Internet. In this paper, however we will discuss how mathematics can be transformed by the Internet, which we consider to be an interface. There has been a significant amount of research showing that function or geometry software transform the nature of the mathematics that is produced (Noss & Hoyles, 1997). Our own research (Borba, 2004a; Borba & Gracias, 2004) has strongly suggested that different software lead to different possibilities and different mathematics. The most popular case has been the "click and drag" resource of geometry software which enabled many students and teachers to generate conjectures, test them and connect them to "different levels" of demonstration, depending on the level of the students and the teaching objectives.

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However, it appears no questions have been posed regarding the nature of change that on-line interfaces bring to the production of mathematics. In this paper, we will present one model of on-line course that stresses the use of chat, and how such an interface is changing the nature of the mathematics that is being generated in on-line communities such as the one described. Before we do this, however, we will present theoretical views regarding computers and knowledge production, and methodological issues. No literature review on Internet based courses will be presented as very little has been published in most mathematics education journals in English or PME proceedings (see for instance, Pateman, Dougherty & Zilliox, 2003; Høines & Fuglestad, 2004).

THEORETICAL AND METHODOLOGICAL ISSUES

Our research group, GPIMEM[2], has been developing research on the use of different information and communication technologies (ICT) in mathematics education for eleven years. We have developed the theoretical notion of humans-with-media as a means of stressing that knowledge is always constructed by collectives that involve humans and different technologies of intelligence (Levy, 1993), such as orality, paper-and-pencil, and ICT. Different humans, or different technologies, result in different kinds of knowledge production. There is no knowledge produced without humans nor without media.

This notion has provided important insight as we have analyzed how different interfaces, such as graphing calculators or dynamic geometry software, play an important role in knowledge production (BORBA, 2004). In the last five years, we have also started to conduct research on the possibilities provided by Internet. To this end, some members of our group have been researching how collectives formed of humans-with-Internet have constructed knowledge. In particular, we have offered several on-line courses for mathematics teachers as a means of searching for theories and research methodologies that emerge from engagement in different practices (Lincoln & Guba, 1985; Borba & Araújo, 2004). In this sense, we believe that we need to be involved in on-line courses in order to focus on helpful questions and theories.

One transformation that we soon noticed, in terms of research procedures, is that data collection is much more "natural" than in usual face-to-face educational environments. If we are researching in a regular classroom, on a lab environment, we have to deal with issues about how invasive a video-camera may be, or to struggle with students/teachers to write reports on their findings. In on-line distance education courses, filming, voice recording and/or transcribing are "natural" and non-invasive. For instance, using chat as a means of communication generates transcribed data that can be electronically stored (as the reader will see, the use of chat also has other implications in terms of results). Triangulation of data and "member checks" (Lincoln & Guba, 1985) can be easily done through e-mail, as we can always ask: "what did

[2] www.rc.unesp.br/igce/pgem.gpimem.html

you mean when you wrote such-and-such" in a chat session, or in a forum. On the other hand, an immense amount of data is generated by chats, e-mails and forums and, more recently, by video-conferences.

In theoretical terms, some of us have been emphasizing how theoretical perspectives, research questions and research methodologies shape one another (Borba & Araújo, 2004). I believe that the notion that knowledge is always produced by collectives of human-with-media is consistent with the discussion in the last paragraph, in the sense that research procedures and the nature of the interaction change as different media are being used. Research procedures, results and theoretical frameworks shape each other. In the same way, the research question of this paper interacts with these other components: what is the nature of the change provoked by the Internet, a non-human actor, in the production of mathematics? Next we will describe the context of the study.

The On-line practice developed by GPIMEM

Over the last five years, our research group, GPIMEM, has made efforts to connect teachers and researchers who are interested in fostering change in their classrooms. We have offered on-line courses in "Trends in Mathematics Education", or specific topics such as "Teaching and Learning Geometry Using Software”. These courses have fostered the development of communities that discuss issues related to the topics presented - teaching and learning of functions and geometry using software, ethnomathematics, modeling, adult education in mathematics, critical mathematics education, and so on.

Courses such as these are of paramount importance in Brazil due to the size of the country and the concentration of knowledge production in the southeastern region, where the states of São Paulo and Rio de Janeiro are located. Internet-based courses are one way of connecting research centers such as São Paulo State University (UNESP) with people in remote locations, where the closest university may be more than several hundred kilometers way.

Each course connects about 20 teachers on-line at regularly scheduled times for a period of about four months. They are designed in such a way that interaction is the key word. The model, which has undergone changes over the last four years, is based on synchronous and asynchronous relationships. We have three-hour chat sessions every week for a four months, and also have bulletin boards and e-mail lists. In the last two versions of the course (2003 and 2004), we used a freeware software environment, Teleduc[3], which requires a server in Linux, but can be accessed by computers that use different platforms. Five courses have been offered since 2000.

These extension courses for teachers have become an environment for research. Our research has shown the transformation of the interaction in these courses, when we

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[3] TelEduc é um ambiente de suporte para educação a distância, desenvolvido pelo Nied e Instituto de Computação da Unicamp, sob a coordenação da Profª Drª Heloísa Vieira da Rocha, e disponibilizado no endereço: http://hera.nied.unicamp.br/teleduc.

compare it to our interactions in our regular graduate courses, in which teachers and researchers take part (Gracias, 2003; Borba & Villarreal; *in press*). Based on the assessment made at the end of each course, this model has had a significant impact in terms of bringing members of different communities into the discussion regarding mathematics education, and giving them access to professors from one of the most prestigious mathematics education graduate programs in Brazil, with whom they would otherwise not have an opportunity to interact. The chat has become the principle means of interaction of the course. Forum, an asynchronous tool in on-line environments, has not been used extensively, and the use of e-mail has decreased. A typical course consisted of 11 three-hour synchronous chat sessions. Preparation for a session would be done through asynchronous interactions, mainly e-mail and regular mail. For instance, prior to a session on ethnomathematics, participants would be mailed a book by D'Ambrosio (2001). All the participants were expected to have read it before the session, and two of them (together with myself) would be responsible for raising questions to generate discussion. After the class, a third teacher would generate a summary of the class which would be published in the virtual environment of the course. A different kind of preparation was required when the objective of the class involved doing mathematics; problems regarding the use of function, for instance, would be sent beforehand to the teachers, and they would attempt to solve them before the class. During the chat session, different solutions would be discussed.

The problems were designed to be solved with the use of plotters such as winplot[4]. Since this software is free, teachers could have their own copies installed in their computers. On the other hand, it was not possible to share a figure with the other participants of the course simultaneously. An attached file could be sent to TeLeduc, and everyone could access it minutes later. In this sense, this course was joining together "old" computer interfaces, such as plotters, and "new" ones, such as the Internet. In this paper, we present some of our findings regarding the interaction between teachers and these two types of interfaces.

As a means of explaining this further, I would like to re-emphasize my belief that knowledge is always constructed by collectives of humans-with-media. If the media change, paper and pencil to a plotter, for instance, the manner of teaching the concept of function, for example, will change. For example, a problem that might be particularly provocative and engaging for a collective of students-with-paper-and-pencil could be entirely simple and uninteresting for collectives that include graphing calculators. Might there be analogous changes with intensive use of the Internet? In this paper, we will be presenting excerpts of the interaction of collectives of humans-with-Internet-winplot.

The 20 teachers who took the course each time were, for the most part, high school teachers, but university level teachers, teacher educators and others, such as

[4] http://www.gregosetroianos.mat.br/softwinplot.asp

curriculum developers, have also taken part. It was common in these chat sessions to have simultaneous dialogues, since different teachers would pursue different aspects of a given problem, or would pose a different problem, or talk about something that happened recently in their classroom.

RESULTS

Before we present the main set of data from the 2004 class, we would like to present a short episode that led us to look at the data we have been generating with the on-line courses with different eyes. In the 2003 class, prior to a scheduled chat meeting with all twenty teachers participating in one of the courses, a problem was posed to them regarding Euclidean geometry. Different solutions and questions were raised by all the participants, but one of the student's reflections called our attention. During the discussion, Eliane[5], said: "I confess that, for the first time, I felt the need for a face-to-face meeting right away . . . it lacks eye-to-eye contact". She then followed up, explaining that discussing geometry made her want to see people and to share a common blackboard. In this case, there was no follow-up discussion that to clarify what she meant. While this comment raised some design issues regarding the development of distance education environments, in this paper, we will focus on the conjecture it evoked regarding possible changes in the mathematics practiced in Internet-based environments.

In year that followed, we posed the following problem[6] to the teachers who participated in the course:

> Biology students at UNESP, São Paulo State University, take an introductory course in pre-calculus/calculus. The teacher of this course asks the students to explore, using a graphing calculator, what happens with 'a', 'b' and 'c' in $y=ax^2+bx+c$. Students have to report on their findings. One of them stated: *"When b is greater than zero, the increasing part of the parabola will cross the y-axis . . . When b is less than zero, the decreasing part of the parabola will cross the y-axis."*. What do you think of this statement? Justify your response.

The mathematics involved in the conjecture, and its accuracy according to academic mathematics, is developed in detail in Borba & Villarreal (*in press*). But it is interesting to see how these teachers dealt with it. Some aspects of it were suppressed, since they were seen as irrelevant to the understanding of the dialogue, or because they were part of a different dialogue, as explained in the previous section.

Carlos, a high school teacher, started the debate at 19:49:07 (these numbers indicate the hour, minutes and seconds in when the message reached the on-line course), reporting on what one of his students had said: "When a is negative, or b is positive, the parabola goes more to the right, but when a is negative and b is also negative, the

[5] Eliane Matesco Cristovão, High School teacher, from the 2003 class.

[6] Translation of this problem and of the excerpt from Portuguese into English was done by the author and Anne Kepple.

parabola goes more to the left". He challenged the group to see if the student's sentence could lead them to solve the problem.

Since the debate was not gaining momentum, the professor of the course, the author of this paper, tried to bring the group back to what Carlos had said:

> (19:53:15) **Marcelo Borba**: The solution that Carlos' student presented regarding 'a' and 'b'. Does anyone have an algebraic explanation for it?
>
> (19:54:53) **Taís**: It has something to do with the x coordinate of the vertex of the parabola.
> (19:55:30) **Carlos:** after a few attempts (constructing many graphs changing the value of 'a', 'b, and 'c') the students concluded that what was proposed by Renata is really true.

The issues at stake are distinct. Carlos tried to do what the professor proposed to the group, but Taís raised a new issue, the vertex idea. As can be observed on the excerpt below, the two issues also have intersections:

> (19:57:07) **Taís** : Xv=-b/2a...if 'a' e 'b' have different signs, Xv is positive.
>
> (19:59:16) **Norma**: I constructed many graphs and I checked that it is correct, afterwards I analyzed the coordinates of the parabola vertex Xv= - b / 2a, and developed an analysis of the 'b' sign as a function of 'a' being positive or negative, then I verified the sign of the vertex crossing. . . . with the concavity upwards or downwards, and checked if it was increasing or decreasing. . . .did I make myself clear?

Norma presented her ideas, which according to my analysis, are similar to the one made by Taís, and can be labeled the vertex solution. After further discussion about this, the professor presents another solution based on the derivative of y, y'=2ax+b:

> (20:07:03) **Marcelo Borba**: Sandra, . . I just saw it a little differently. I saw it . I calculated y'(0)=b, . . and therefore when 'b' is positive the parabola will be increasing and analogously. . . .

Since a few people said they did not understand this comment, he went back to explain his solution.

> (20:10:59) **Marcelo Borba**: . . . as I calculate the value of y', y'>0, then the function is increasing, and therefore I consider y'(0), which is equivalent to the point at which y crosses the y-axis, and y'(0)=b, and therefore 'b' decides the whole thing!!!! Got it?
>
> (20:29:24) **Badin**: The parabola always intercepts the y-axis at the point where the x coordinate is zero. In order for this point to belong to the increasing "half" of the parabola (a>0), it should be left of the x_v, this means x_v should be less than zero. Therefore, -b/2a < 0 is equivalent to -b<0 (remember, a>0). But -b<0 is equivalent to b>0. In other words, if b>0, the point where the graph crosses the y-axis is in the increasing part of the parabola. The demonstration por a<0 is analagous.

At this point, some of the teachers had been discussing the problem and both solutions - the vertex and the derivative – for 40 minutes. The large spaces shown by the clock between the different citations from participants of the course, indicate the size and amount of sections which were not transcribed in this paper, as there were about four messages per minute. For ten more minutes, additional refinement and shared understanding of the solutions were presented. More examples of people's writing about their understanding in the chat are available in the naturally recorded data. Educational issues regarding the use of winplot, to explore the problem and generate conjectures, were discussed. But what is new about Internet in this case? This is the topic of the next section.

DISCUSSION

Before going further, the reader should be aware that some sentences were omitted to make it easier to follow the interaction, and that the translation suppressed most of the informality and typos that normally occur in this kind of environment. There were other actors involved in the discussion and refinement of the solutions of the problem, but for the purpose of clarity, only a few were included here. When we compare the solution presented by the teachers, the vertex one, to the original situation that took place in a normal classroom situation in 1997, there are similarities and differences. Students used graphing calculators to generate many conjectures for the problem relating coefficients of parabolas of the type $y=ax^2+bx+c$ to different graphs. Similarly, the teachers used winplot (or other software, in some instances) to investigate the problem just described, and later the problem related to Renata's conjecture. In the face-to-face classroom, the professor/author led the discussion, and eventually presented the vertex solution (as he did not know the answer either, at first). The explanation for the conjecture was never written by the students. In an on-line learning environment based on chat, writing is natural, and everyone involved had to express themselves in writing. Although we know that some aspects of writing in a chat situation are different compared to writing with paper and pencil, there is a fair amount of research showing the benefits of writing for learning (see, for instance, Sterret, 1990). The data presented here is insufficient, and the design of the study is inappropriate, to support arguments about "benefits". However, it can be argued that chats transform mathematics education in a similar way that it changed research procedures. Chats, together with human beings, generate a kind of written mathematics that is different from that developed in the face-to-face classroom, where gestures and looks form part of the communication, as well. I believe that collectives of humans-with-Internet-winplot generate a different kind of knowledge, which does not mean that the mathematical results were different. But if process is considered, I believe that we may be on the way to discovering a qualitatively different medium that, like the "click and drag" tool of the dynamic geometry, offers a new way of doing mathematics that has the potential to change the mathematics produced, because writing in non-mathematical language becomes a part of doing mathematics. At this point, it is too early to confirm this, but I believe that this "working hypothesis" (Lincoln & Guba, 1985) regarding the transformation of mathematics by the Internet is one to be pursued in further research.

References

Borba, M. Dimensões da Educação Matemática a Distância. In: Bicudo, M.A.V.; Borba, M. (Org). *Educação Matemática: pesquisa em movimento*. São Paulo: Cortez, 2004a, p. 296-317.

Borba, M. Humans-with-media and mathematical thinking: Orality, writing and technologies of information and comunication. *Proceedings of 10th International Congress on Mathematical Education - ICME 10*, Copenhagen, Denmark, 2004b.

Borba, M., & Gracias, T. Distance Education in Mathematics. *Proceedings of PME 28*, 1 - 290, Bergen, Norway, 2004.

Borba, M. C.; & Araújo, J. L. (Org). *Pesquisa Qualitativa em Educação Matemática*. Belo Horizonte: Autêntica, 2004.

Borba, M.; Penteado, M. *Informática e Educação Matemática*. Belo Horizonte, Autêntica, 2001.

Borba, M. ; Villarreal, M. *Humans-with-media and the reorganization of mathematical thinking*. Dordrecht: Kluwer Publisher, in press.

D'Ambrosio, U., Etnomatemática – elo entre as tradições e a modernidade. Belo Horizonte: Editora Autêntica, 2001.

Gracias, T. A. S. A. *Reorganização do Pensamento em um Curso a Distância sobre Tendências em Educação Matemática*. 2003. 167 f. Tese (Doutorado em Educação Matemática) - Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Rio Claro, 2003.

Høines, M. J., Fuglestad, A. B. *Proceedings of PME 28*, Bergen University

College, Bergen, Norway, 2004.

Lévy, P. *As Tecnologias da Inteligência: o futuro do pensamento na era da informática*. Rio de Janeiro: Editora 34, 1993.

Noss, R., Healy, L., & Hoyles, C. The construction of mathematical meanings: connecting the visual with the symbolic, *Educational Studies in Mathematics*. 33(2): 203-233, 1997.

Pateman, N. A., Dougherty, B. J., & Zillox, J. *Proceedings of PME 27 and PME-NA 25*, College of Education, University of Hawai'i, Honolulu, HI, 2003.

Sterret, A. (Org.). *Using writing to Teach Mathematics*, Mathematical Association of American (MAA) Notes, nº 16, 1990.

USING COGNITIVE AND SITUATIVE PERSPECTIVES TO UNDERSTAND TEACHER INTERACTIONS WITH LEARNER ERRORS

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Cognitive and situative theories have both proved very useful in furthering our understandings of mathematics learning. An important current area of investigation is to synthesize these perspectives in order to provide more robust theories of learning and to bring theory and practice into better relations with each other. This paper contributes to this endeavour in two ways: 1. by using both theories to understand learner errors, and 2. by focusing on teaching as well as learning.

COGNITIVE AND SITUATIVE[1] PERSPECTIVES

The differences between cognitive and situative perspectives are best captured by Sfard (1998). She argues that cognitive/constructivist[2] perspectives view knowledge as a commodity and the metaphor for learning this knowledge is one of acquisition. We acquire or gain knowledge, through the construction of ever more powerful schemata, concepts or logical structures (Hatano, 1996; Sfard, 1998). Self-regulation is the primary mechanism for learning in this perspective. Contextual and social influences, including teaching, are either ignored, or are seen as means for enabling the acquisition of individual knowledge (Greeno & MMAP, 1998). Social processes are secondary processes, which constrain and influence the primary process of self-regulation (Piaget, 1964).

Situative perspectives view learning as participation in communities of practice (Lave & Wenger, 1991; Wenger, 1998). To learn mathematics is to become a better participant in a mathematical community and its practices, using the physical and discursive tools and resources that the community provides (Forman & Ansell, 2002; Greeno & MMAP, 1998), and adding to them (Wenger, 1998). Situative perspectives argue that a focus on conceptual structures is not sufficient to account for learning. Rather, interaction with others and resources are both the process and the product of learning and so learning cannot be analysed without analysing interactional systems.

Researchers who suggest syntheses of cognitive and situative approaches argue for different possibilities in such syntheses. Schoenfeld (1999) and Sfard (1998) argue

[1] I follow Greeno et al. (1998) in using "situative" rather than "situated" to distinguish a perspective on learning from a particular way of learning. A situative perspective argues that all learning is situated.

[2] Sfard points to differences between information processing views of cognition and neo-Piagetian views, which take a more constructivist and meaning-making approach. I work with neo-Piagetian notions of cognition and constructivism and use these interchangeably in this paper.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 177-184. Melbourne: PME.

that each approach has its strengths and weaknesses, and that we should draw on each in ways that enable coherent progress on particular research projects. For example, Sfard argues that while it may not be helpful to account for learners' thinking only in terms of cognitive structures, it is also not helpful to suggest that we do not impose structure on the world, and that this structuring does not somehow become part of us and help us to make better sense of new situations. Greeno et al. (1998) argue that a situative view is in fact an expanded cognitive view and that we need to develop concepts that will enable us to take features of both into account.

TEACHING MATHEMATICS

Cognitive and situative theories are primarily theories of learning and as such they entail theories of knowledge. It is more difficult to speak about cognitive and situative approaches to *teaching* mathematics because while theories of learning offer implications for pedagogy and general pedagogical principles, they do not directly lead to particular pedagogical approaches. Pedagogical principles do not derive from theories of learning in a one-to-one relationship. The two different theories might suggest very similar approaches, which are distinguished at the level of explanation rather than the level of practice.

One example of this is the common classroom practice of group work. Both cognitive and situative theories suggest that learners talking through their ideas in groups is a useful pedagogical approach. A cognitive perspective suggests that as learners articulate their ideas, they are likely to clarify their thinking, and develop more complex concepts or schemata (Hoyles, 1985; Mercer, 1995). A situative perspective suggests that as learners consider, question and add to each other's thinking, important mathematical ideas and connections can be co-produced. For cognitive perspectives the group is a social influence on the individual; for situative perspectives the group is the important unit, which produces mathematical ideas beyond the individual ideas. Either one, or both, of these purposes for group work might be operating in a classroom at any particular time.

As learners talk through their ideas, either in groups or in whole class situations, they make errors. Teachers' understandings of learner errors and misconceptions are key to reform visions in many countries. In this paper, I begin to develop ways in which we might think about learner errors from both cognitive and situative perspectives. This is an exploratory paper, drawing on an example of classroom interaction where the teacher deals with a number of errors, one of which proves particularly resistant. I argue that this error needs to be seen in both cognitive and situative terms, and in so doing, I begin to expand the notion of misconceptions to take account of situational factors and teaching-learning interactions.

ERRORS AND MISCONCEPTIONS

Research into learners' misconceptions has been a key strand of constructivist research (Smith, DiSessa, & Roschelle, 1993). This research shows that many errors are systematic and consistent across time and place, remarkably resistant to instruction, and extremely reasonable when viewed from the perspective of the learner. To account for these errors, researchers posit the existence of misconceptions, which are underlying conceptual structures that explain why a learner might produce a particular error or set of errors. Misconceptions make sense when understood in relation to the current conceptual system of the learner, which is usually a more limited version of a mature conceptual system3. Misconceptions alert us to the fact that "building" on current knowledge also means transforming it; current conceptual structures must change in order to become more powerful or more applicable to an increased range of situations. At the same time the new structures have their roots in and include earlier limited conceptions. Learners' misconceptions, when appropriately coordinated with other ideas, can and do provide points of continuity for the restructuring of current knowledge into new knowledge (Hatano, 1996; Smith et al., 1993).

Misconceptions can also produce correct contributions (Nesher, 1987). The seminal story of Benny (Erlwanger, 1975) is an example of a learner who constructed many of his own rules for mathematical operations. His rules were partially sensible modifications of appropriate mathematical operations. They were derived from his instructional program and his correct understanding of some mathematical principles. They produced many correct answers and Benny was considered to be a good mathematics student by his teacher. However, many of his underlying understandings of mathematics were incorrect and were never picked up by his teacher.

The notion of misconceptions as part of a cognitive framework suggests that an individual's conceptual structure can account for her productions in the classroom, and that shifts in conceptual structure can account for learning. Situative perspectives argue that a focus on conceptual structures is not sufficient to account for learning and certainly cannot account for teacher-learner interaction in the classroom. Therefore situative perspectives have not focused explicitly on errors or misconceptions. This is of concern for reform visions of teaching, where teachers are asked to focus on learners' thinking, which often exhibits errors or misconceptions. However, situative perspectives can give us additional ways to understand learners' errors. Situative perspectives view learning mathematics as increasingly appropriate participation in mathematical practices using mathematical tools (Forman & Ansell, 2002; Greeno & MMAP, 1998). From this perspective, correct contributions are seen as appropriate uses of tools and resources in a setting. Incorrect productions can be seen as partial or inappropriate uses of the tools and resources in the setting, the use of inappropriate tools and resources, or non-engagement in mathematical practices. Situative perspectives argue that what a learner says and does in the classroom makes sense from the perspective of her current ways of knowing and being, her developing

[3] For this reason, many authors prefer the terms "alternative conceptions" or "naive conceptions" (Smith et al., 1993), preferring to indicate presence rather than absence. I use the term misconceptions to indicate an absence in relation to accepted mathematical knowledge.

identity in relation to mathematics, and to her previous experiences of learning mathematics, both in and out of school. If learners have come to expect particular ways of working in a mathematics classroom, and of what counts as an appropriate contribution in a classroom, they will continue to make use of these expectations. A learner contribution can be an attempt at speaking about an idea in order to grapple with it, to engage someone else in a process of solving a problem or coming to a joint understanding, or to participate in the conversation, either appropriately or innappropriately. It can also be an attempt to resist the classroom conversation or to disrupt it. Researchers can attempt to document these patterns of interaction and show that patterned regularities exist in these kinds of interactions (Greeno & MMAP, 1998). Social and cultural attunements and patterned regularities may be just as widespread, systematic and resistant to instructional intervention as misconceptions are.

AN EXAMPLE

This example comes from a larger research study in which I look at how teachers interact with learners' contributions. In the study, I videotaped and analysed two weeks of lessons of five Grade 10 and 11 teachers in South Africa, and conducted a set of interviews with each teacher. For the classroom analysis, I developed a coding scheme for learner contributions (including errors) and teacher moves in response to learner contributions. My methods have been discussed in detail elsewhere. In this paper, I draw on one example from one of the Grade 10 classrooms to show how cognitive and situative perspectives can help to understand a learner error.

The learners had worked on a task the previous day in pairs and handed their work to the teacher. The task was:

> Consider the following conjecture: "$x^2 + 1$ can never be zero". Prove whether this statement is true or false if $x \varepsilon$ R.

The teacher, Mr. Peters, had read all the responses and chosen three, which he used to structure his lesson the following day. The first response was from Grace and Rethabile, that $x^2 + 1$ cannot be 0 because x^2 and 1 are unlike terms and so cannot be added together. Many other learners had made the same argument. Mr. Peters asked them to explain their reasoning and Rethabile argued:

> what we wrote here, I was going to say that the x^2 is an unknown value and the 1 is a real number, sir, so making it an unknown number and a real number and both unlike terms, they cannot be, you cannot get a 0, sir, you can only get $x^2 + 1$

and

> Yes, sir. There's nothing else that we can get, sir. but the 0, sir

As other learners contributed to this discussion, some made a different error, saying that $x^2 + 1=2x^2$, i.e., they completed the expression, which is a well known algebraic error (Tirosh, Even, & Robinson, 1998). Mr. Peters dealt relatively easily with this error, creating a class discussion and helping many of the students to see their mistake. Grace and Rethabile's error was of a different kind and proved more resistant. They had not completed the expression incorrectly, and they had the correct answer, that $x^2 + 1$ cannot be 0, but for the wrong reasons. Their justification was incorrect and, as the second contribution above shows, their reasoning was confused.

Mr. Peters' interpretation of the girls' error4 was that they saw $x^2 + 1$ as an immutable unit which could not be simplified, rather than as a variable expression that could take different values depending on the values of x. He therefore asked the following question:

> So it will only give you $x^2 + 1$, it won't give you another value. Will it give us the value of 1, will it give us the value of 2?

By asking whether $x^2 + 1$ could take a range of values and suggesting some possibilities other than zero, Mr. Peters was trying to help the learners to see $x^2 + 1$ as a variable expression. Rethabile drew on the first part of Mr. Peters' question to argue:

> It will give us only 1, sir, because x is equal to 1, sir

Mr. Peters followed up this response by asking:

> How do you know x is equal to 1?

which led to Grace making the following, rather confusing contribution:

> Sir, not always sir, because, this time we dealing with a 1, sir, that's why we saying x^2 equals to 1, sir, because, that's how I see my x equals to 1, sir, because, a value of 1, only for this thing, sir

Subsequent contributions by Rethabile and other learners suggested that they thought $x = 1$ because, as they justified it: "there is a 1 in front of the x". Again, this is a common error that many experienced teachers would recognise. Mr. Peters noticed this error and worked on it with the class, as he did with many other common errors. In his interviews Mr. Peters showed a deep understanding of the mathematical thinking and misconceptions that might underlie common errors. However, the crucial error made by Grace and Rethabile proved both more difficult to work with, and more difficult for both Mr. Peters and myself as the researcher to understand. There are many points of contradictory arguments and confused reasoning, for example, if they saw x as 1, why did they not argue that $x^2 + 1$ was 2?

Mr. Peters spend the remainder of the lesson having the class discuss two other solutions, the first where learners substituted different values to show that $x^2 + 1$ could produce a range of positive values, and the second where learners argued that x^2 was always positive or 0, so $x^2 + 1$ would always be positive. Mr Peters spent a lot of time on each of these solutions, emphasising both the testing of the conjecture by substitution and the justification of it by logical argument. In this way, he made available other learners' reasoning as resources for Grace and Rethabile to help them

[4] Mr. Peters discussed this in an interview with me.

think about the parameters of the task. Even after this substantial discussion, Grace was not convinced, and asked the question:

> What about, what's the final, if it's not the zero what is it, sir, if it's not the zero, sir, what's the answer?

Her question suggests that she was still seeing the expression as one that needed to be completed, rather than one that could take on a range of values. Even though others in the class had reached a conclusion that agreed with hers (that $x^2 + 1$ could not be zero), she did not understand the basis for their conclusion, and needed to know what $x^2 + 1$ could be, given that it was not 0. She did not seem to accept that it could take a range of positive values At the end of the lesson, I asked Grace whether $x^2 + 1$ could equal 10 and suggested that she think about it at home and write a response for me for the next day. She wrote two solutions: first that $x^2 + 1$ could not equal 10 or any other number because $x^2 + 1 = x^2 + 1$; and second that $x^2 + 1$ could equal 10, if $x = 3$.

DISCUSSION

How can we best understand this interaction from cognitive and situative perspectives? From a cognitive perspective, the two girls and many other learners in the class were struggling with the idea of $x^2 + 1$ as a variable expression that could take multiple values. Mr. Peters understood this and worked with the learners' ideas, building the lesson around them and using other learners' contributions to do this. He focused on the mathematical reasoning that was required to do the task. However this approach did not help Grace, and possibly other learners, to understand what was faulty in their argument and to restructure their thinking to accept a mathematically correct argument for their conclusion. From a situative perspective, Mr. Peters had provided the learners with a task, which would enable their engagement with the mathematical practices of reasoning and justification. He set up pair work to enable learners' communication and justification processes. In the whole-class discussions he required learners to justify their answers, he probed and pressed their thinking and spoke to them about how they should justify. He also spent much time on other learners' appropriate mathematical reasoning to provide resources for Grace and Rethabile to draw on. Yet Grace and Rethabile still struggled to participate appropriately in the classroom. When asked to justify their thinking, both by the teacher and the researcher, they showed further errors in their thinking. These errors show a number of ways in which the girls were not comfortable in participating in mathematical practices. They did not appreciate the justificatory nature of the task. Having spent many years simplifying expressions, they wanted to continue to do so. They were uncomfortable in reasoning mathematically in the ways in which the task required. They complied with the teacher's requests for justification by trying to say something, even if it was contradictory to their previous position. Grace's written response to the researcher's question shows that she had serious difficulties in reasoning mathematically and could comfortably hold two contradictory positions at the same time. It might also show that she wrote whatever she could think of, hoping that some of it would satisfy me. This might also have been the case in class discussions; that the learners drew on whatever they could think of to be able to comply with the teacher's requests to participate. This relates to the learners' identities as participants in school discourse, rather than mathematical practices. The ways in which they accessed the resources that Mr. Peters provided did not help them to shift their ways of reasoning mathematically nor to participate appropriately in a mathematics discussion. We might even say that through the interaction they co-produced further errors, inappropriate mathematical reasoning and little engagement with important mathematical practices.

In the larger study, I show that Mr. Peters dealt with errors that were relatively familiar to him as an experienced teacher and that he had mathematical and cognitive explanations for them (see also Tirosh et al, 1998). In addition, he understood that his learners were struggling to come to terms with a different way of engaging in mathematics, that of mathematical reasoning and justification, and talked to them about how to do this. He knew that their prior experiences of school mathematics made it difficult for them to engage in the practices that he was trying to teach. As experienced and successful as he was in his teaching, he was still faced with systematic, patterned errors that came out of both the learners' conceptual structures and their ways of participating in mathematics classrooms. How might he go forward with his quest to teach more genuine mathematics to his learners? Taking a situative perspective, some literature suggests teaching the norms of inquiry classrooms (McClain & Cobb, 2001) or the learning practices required to engage in mathematics in this way (Boaler, 2002). These both take account of the patterns of schooling that need to be changed. From a more cognitive perspective, Sasman et al. (1998) have documented how learners easily hold contradictory mathematical positions, or change their positions from one day to the next. In this paper, I have argued that we have to bring these explanations together. We have to understand both the cognitive misconceptions that learners are working with and their difficulties with mathematical reasoning, as well as their issues of participation in class, including identity issues in defending their positions for the teacher and other learners and the ways in which they understand and use the mathematical tasks and resources presented to them.

References

Boaler, J. (2002). Learning from Teaching: Exploring the Relationship between Reform Curriculum and Equity. *Journal for Research in Mathematics Education, 33*(4), 239-258.

Erlwanger, S. H. (1975). Case Studies of Children's Conceptions of Mathematics Part 1. *Journal of Children's mathematical behaviour, 1*(3), 157-283.

Forman, E. A., & Ansell, E. (2002). Orchestrating the multiple voices and inscriptions of a mathematics classroom. *The journal of the learning sciences, 11*, 251-274.

Greeno, J. G., & MMAP. (1998). The situativity of knowing, learning and research. *American Psychologist, 53*(1), 5-26.

Hatano, G. (1996). A conception of knowledge acquisition and its implications for mathematics education. In P. Steffe, P. Nesher, P. Cobb, G. Goldin & B. Greer (Eds.), *Theories of mathematical learning* (pp. 197-217). New Jersey: Lawrence Erlbaum.

Hoyles, C. (1985). What is the point of group discussion in mathematics? *Educational studies in mathematics, 16*(205-214).

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge: Cambridge University Press.

McClain, K., & Cobb, P. (2001). An analysis of development of sociomathematical norms in one first-gade classroom. *Journal for Research in Mathematics Education, 32*(3), 236-266.

Mercer, N. (1995). *The guided construction of knowledge: Talk between teachers and learners.* Clevedon: Multilingual matters.

Nesher, P. (1987). Towards and instructional theory: The role of students' misconceptions. *For the Learning of Mathematics, 7*(3), 33-39.

Piaget, J. (1964). Development and learning. In R. E. Ripple & V. N. Rockcastle (Eds.), *Piaget Rediscovered*. Ithaca: Cornell University Press.

Sasman, M., Linchevski, L., Olivier, A. and Liebenberg, R. (1998). *Probing children's thinking in the process of generalisation.* Paper presented at the Fourth Annual Congress of the Association for Mathematics Education of South Africa (AMESA). Pietersburg.

Schoenfeld, A. H. (1999). Looking towards the 21st century: Challenges of educational theory and practice. *Educational Researcher, 28*(7), 4-14.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher, 27*(2), 4-13.

Smith, J. P., DiSessa, A. A., & Roschelle, J. (1993). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *The journal of the learning sciences, 3*(2), 115-163.

Tirosh, D., Even, R., & Robinson, N. (1998). Simplifying algebraic expressions: Teacher awareness and teaching approaches. *Educational Studies in Mathematics, 35*, 51-64.

Wenger, E. (1998). *Communities of practice: Learning, meaning and identity*. Cambridge: Cambridge University Press.

IDENTIFICATION OF AFFORDANCES OF A TECHNOLOGY-RICH TEACHING AND LEARNING ENVIRONMENT (TRTLE)

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This paper describes how a researcher developed task and four different data collection instruments provide evidence for the identification of various affordances of a technology-rich teaching and learning environment (TRTLE) that were perceived and/or enacted by Year 9 students in their solution of a linear function task utilising a graphing calculator. Each instrument proved valuable in the identification process, with post task interviews being particularly useful in identifying rejected affordances and others that had not been perceived until post task questioning and reflection.

INTRODUCTION

At ICME 5, Pollak (1986) insightfully commented on the power of electronic technologies to transform mathematics, mathematical activity, and mathematics thinking which many proponents of the use of these technologies in secondary school classrooms would have expected to be realised by now. Despite several decades where electronic technologies have played a role in many industralised countries in the mathematics classroom and curriculum documents have advocated their use, the transformational power of a *technology-rich teaching and learning environment (TRTLE)* in secondary classrooms has yet to be universally realised. This transformation includes teaching old things better, new mathematics that was unable to be taught without technology, and increasing the breadth and depth of key concepts such as function (DeMarios & Tall, 1996). Many affordances that would be useful in the teaching and learning of function are offered by classroom environments involving these technologies. However, the realisation of any affordances depends not only on "the technological tool, but [also] on the exploitation of these affordances embedded in the educational context and managed by the teacher" (Drijvers, 2003, p. 78).

The term affordance was first coined in 1966 by the perceptual psychologist J. J. Gibson who used the word for

> something that refers to both the environment and the animal in a way that no existing term does. It implies the complementarity of the animal and the environment … . They are not just abstract physical properties. (1979, p. 127)

Gibson (1977) considered affordances to be relationships between objects and actors involved in interactive *activity*. They are what the environment offers to a particular animal. Following Scarantino (2003), the *affordances of a teaching and learning environment incorporating electronic technologies* will be taken to mean the offerings of such an environment for both facilitating learning (the promises or positive affordances) and impeding learning (the threats or negative affordances). An affordance of a TRTLE is the opportunity for interactivity between the user and the technology for some specific purpose, for example, check-ability.

"Affordances describe how the interaction between perceiver and perceived works - and that is exactly what we need to understand in educational research" (Laurillard, Stratfold, Luckin, Plowman, & Taylor, 2000, p. 3). "Research examining the concept of affordances is crucial if we are to build … a more flexible design orientation to the practices of education" (Pea, 1993, p. 52). Electronic technologies "enhance the individual's capacity to act" (Smitsman & Bonger, 2003, p. 176) but "cannot be considered independently from … the environment" (p. 173).

THE STUDY

The data analysed here are part of a larger study which aims to construct a theory (Strauss & Corbin, 1990) of how teachers and secondary students perceive and enact affordances in a TRTLE so as to maximise the learning of functions. From a research perspective it is necessary to establish what conditions exist in TRTLEs enabling or impeding the realisation of offered affordances for the teaching and learning of functions. The purpose of this paper is to determine which combination of research instruments provide optimal evidence for identifying affordances of a TRTLE perceived by Year 9 students in their solution of a linear function task utilising a graphing calculator.

METHODOLOGY

A case based approach has been adopted for the study proper. In intrinsic case studies "the case itself is of primary, not secondary, interest" (Stake, 1995, p. 171) whereas in instrumental case studies "the case study serves to help us understand the phenomena or relationships within it" (p. 171) as is the situation in this study. The phenomenon being studied is the perception of affordances by secondary teachers and students after the teaching of functions in a TRTLE. The study is an instrumental multiple case study, but only data from one teacher's classes will be analysed in this paper.

A case based approach typically begins with a descriptive phase within which exploratory-descriptive work is undertaken the goal of which is to examine, investigate, and document the phenomenon on its own terms in an open-minded fashion (Edwards, 1998). In the aspects of the case to be reported here, the purpose of data collection has been to undergo this documentation of the affordances perceived by the Year 9 students in a TRTLE and to evaluate the research instruments being used for this purpose.

Forty two students (14-15 year olds), from two Year 9 classes taught by the same teacher (24 from class A and 18 from class B), participated in the data collection. Various electronic technologies, including both laptop computers and graphing calculators owned by the students, have been used extensively by them throughout

Year 9. Both classes had completed a unit of work on linear functions utilising a range of electronic technologies before data collection began.

A graphing calculator program, *Hidden,* developed by the author, was utilised as the basis of the *Hidden Function* task to generate a (linear) function and edit TABLE settings controlling the display of TABLE values by changing from automatic to user initiated display of both. On execution of the program, students were presented with a blank TABLE. The task was attempted in pairs. The pairs were asked to identify several 'hidden functions' and then describe a general method to do so. The program and the methods for accessing numerical data in the TABLE were demonstrated by the author at the beginning of the task. A second task version provided a split-screen view with TABLE and GRAPH side-by-side when the TABLE was accessed.

The task was implemented once in each class during a 50 minute lesson. In both classes two focus pairs were video taped in their regular classroom setting, during the task solution. Six and four pairs of students, from class A and B respectively were interviewed immediately after the task sessions. These included all focus pairs. In Class B the split-screen version of the task was used with four pairs. An application from Freudenthal Institute website that records key strokes of graphing calculator users, *KeyRecorder*, was successfully implemented with five pairs of students in Class B. The application was not functional when class A was administered the task.

Data collected included task record sheets (21 pairs), video transcripts (4 focus pairs), key screens (5 pairs), and post task interviews (10 pairs). The responses of the student pairs to the *Hidden Function* task and subsequent interviews of some pairs were used to document the range of affordances perceived by the students either during the solution process or subsequent discussion of the task in the interview.

AFFORDANCES PERCEIVED BY THE STUDENT PAIRS

All students successfully interacted with the *PROGRAM: Hidden* and the graphing calculator TABLE feature to generate numerical data from which to engage with the task. Eleven additional affordances were perceived by some pairs either during their task solution or the post task interview. These were affordances of the TRTLE to:

1: Access the general equation of the Hidden Function (i.e., $y_0 = \text{m}x\text{+c}$) in the TABLE heading and recognise the function as linear and/or the need to identify m and c.

2: Use the HOMESCREEN to undertake calculations (e.g., to determine required values, or evaluate values of a conjectured function).

3: Generate numerical values in the TABLE and link these to the graphical representation of the function, (e.g., to determine $y(0)$ to find the y intercept).

4: Use the LISTS and LinReg (linear regression feature) to determine the algebraic representation of a function.

5: Make deliberate choices of entering consecutive values for x in the TABLE to simplify the solution path.

6: Use the graphical representation to identify the hidden function as linear (i.e., visually or by displaying $y_0 = mx+c$ in this representation).

7: To access stored values in the graphing calculator (e.g., ALPHA M, ALPHA C) to identify the gradient and y intercept of the function.

8: Use the graphical representation to identify the sign of the gradient, or the y intercept, or to think about the function in some other way.

9: Use the TABLE to perform a local check of conjectured values of the function (e.g., by generating additional pairs of numerical values).

10: Use the function window (*y*=) and TABLE to provide a numerical representation of a given function to verify a conjectured function as the hidden function.

11: Use LIST OPERATIONS to verify a conjectured function (i.e., with data in two lists enter a conjectured function rule in a third list and compare values).

The frequency of identification of these affordances and the sources of evidence for this identification are shown in Table 1. Sources of evidence are record sheets (RS), video transcripts (focus pairs only) (VT), keys screens (KS), or post task interviews (I). Evidence for a pair percieving an affordance may be gained from multiple sources. Affordances that student pairs considered only during the post task interview are indicated by an asterisk (*).

Of the eleven affordances listed in Table 1 the record sheet was evidence for 7 of these, the video transcript for 7, the key screen record for 5, and post task interview for 10. Whilst the greatest number of instances was identified using record sheets (21

Affordance Number	Number of pairs class A, class B	Source of Evidence (number of pairs)			
		RS	VT	KS	I
1	1, 4	2	2	3	1
2	2, 4	0	4	2	2
3	7, 1	6	2	0	4
4	2, 1*	2	0	0	1+1*
5	4, 1	1	1	0	3
6	0, 2	1	1	1	0
7	0, 1*	0	0	0	1*
8	6, 5	1	2	3	5*3
9	4, 7	8	2	2	3
10	0, 1*	0	0	0	1*
11	2*, 1*	0	0	0	3*

Table 1: Frequency and source of perception of various affordances.

in total), this instrument was used to collect data from all pairs unlike the other instruments. Three affordances, 7, 10, and 11, the latter two being opportunities for solution verification, were identified by only the post task interview. A further three affordances, 1, 8, and 9, were identified by all data sources. One of these will be used to illustrate how each instrument contributed evidence for perception of affordances.

EVIDENCE FOR AN AFFORDANCE

One affordance of the TRTLE was to use the graphical representation to identify the sign of the gradient, or the y intercept, or to think about the function in some other way (Affordance 8). The perception of this affordance was identified from the record sheet of Hala and Kay, where in describing their general method they stated, "Also, look at the graph to get an idea of what operations and how large they are". Additional evidence from their post task interview suggested the graph allowed them to identify the sign of the gradient. Whilst other pairs may have perceived this affordance, no other record sheets provided evidence for this.The video transcript of Michelle and Tim provided evidence of Tim using the graphical representation to think about the relationship between the numbers.

Tim: Timesing by 10.1, so 30, the *y* value, is y the across, is it? [Whilst looking at the graphical representation on the split screen view, he runs his finger across screen above and parallel to the *x* axis].

Michelle: The *x* axis is along. The *y* axis is down.

Tim: That's right. So it's hitting the *y* [indicating by running his hand vertically across the screen parallel to the *x* axis], *x* axis at 30 and the y axis at 310.

The key screens of Di and Fiona provided evidence of their observing the graphical representation on three occasions as they unsuccessfully tried to identify a third 'hidden function'. Their view of the function in the standard window did little to assist this identification, however. In contrast Obi and Luke's key screens suggest the graphical representation afforded them the opportunity to think about the 'hidden function'. This is evidenced by the time spent viewing the graph and moving the cursor around the screen. In conjuction with their video transcript, it is apparent that not only did the graphical representation allow this thinking to occur but also it facilitated enactment of the more specific affordance offered by the graphical representation (Affordance 6) as they identified the general equation and *y* intercept.

Obi: [Presses TRACE] $y = mx + c$ according to this.

Luke: I thought it was $y = mx + c$.

Obi: Just write that down. [Passes graphing calculator back to Luke]

Luke: Yep, say, yes ... Look at this [shows screen to Obi]. Zero is three. Y = 3.

Post task interviews provided evidence of extra affordances perceived but rejected during task solution. The conversation between interviewer and students allowed reflection on the task by the pairs and consideration of affordances other than those utilised or rejected during task solution. The instrument, therefore, allowed identification of additional affordances that students could have used. These affordances may well have facilitated solution verification had the students felt there was a need for this. One pair, Rick and Kris, rejected the affordances offered by the graphical representation during their task solution although this was not explicitly evident from their record sheet or the video transcript. The following dialogue shows Kris accessed the graphical representation, and considered the offered affordances. Rick, focussed on an independent solution pathway at the time, was unaware of this.

Interviewer: Could you have used the graph? Could you have solved it graphically?

Rick: No, no graph.

Kris: We did look. We could find out the y intercept using it and then ...

Interviewer: So you did look at the graph in the class then?

Rick: Never!

Kris: Once. We did once but then …

However, the perceived affordance was rejected for other affordances, both students clearly believed other offerings of the TRTLE enabled more efficient task solution.

Kris: Although you can use a, umm, function, the intercept function to find the *y* intercept, but it is easier to just go to table and to do zero.

Rick: Zero.

Interviewer: But you could have done it with the graph as well?

Kris & Rick: Yes

Tim and Michelle used a split-screen version of the task. Whilst TABLE values needed to be deliberately selected, a portion of the graph was visible at all times in the graph window. Whilst saying they ignored the affordances offered (contrary to video tape evidence cited earlier), this pair actually perceived and rejected these as not meeting their current needs.

Interviewer: Are there other ways that you could have found the hidden function?

Michelle: There is usually a graph we could probably have used, which ...

Interviewer: Didn't you have a graph on your screen?

Tim: Yeah.

Michelle: Yeah, but we didn't look at it.

Tim: It didn't have any really [pause] numbers. It just had [pause] the line.

Interview questions asking student pairs not only about the choices they made, but also about other choices that could have been made during their task solution enabled the gathering of evidence of a broad range of perceived affordances of the TRTLE.

THE VALUE OF THE POST TASK INTERVIEWS

Of particular interest was the enactment of affordances for checking or verification of the solution. Checking, where it occurred, was generally at a local level. The post task interviews and in particular questions about other choices and how their equation could be checked allowed the identification of affordances that students may not have considered during task solution, but were possibilities in an environment where solution verification was expected. Evidence of this is presented for Obi and Luke.

After spending much of the allocated task time engaged in off-task behaviour, Obi and Luke's record sheet showed numerical data generated for five hidden functions with only two algebraic representations recorded (both correctly). The post task interview began in a similar fashion as they generated five ordered pairs for a 'hidden function' then appeared to have little idea how to proceed. They started to 'wander around' the various calculator features, reflecting their behaviour at times during task solution. The graphical representation was considered and rejected, as they had forgotten how it proved helpful previously. They continued searching calculator features until Obi appeared to have an idea whilst looking at a statistics menu.

Obi:	Do you want to do that one? ax + b =, umm.
Interviewer:	You went to linear regression? Have you used linear regression in class?
Obi:	No, we haven't.

Their teacher who expected most students would use linear regression on the calculator to solve the task would disagree with this comment. This pair recognised an error message meant they had no data in their LISTS, proceeded to enter their data into the LISTS, and used linear regression to correctly identify the 'hidden function' (Affordance 4), supporting the teacher's comments from a post task interview.

In reponse to questions about checking their equation, Obi immediately suggested using LIST OPERATIONS but then said he preferred to use the function window (y=). Obi worked with Luke to use LISTS to verify their function (Affordance 11).

Interviewer:	Okay. Then $L_3 = 4*L_1 + 6$, yes? [Describing what the pair entered.] And?
Luke:	And the answer is 26, 42, 54, 66.
Interviewer:	So LIST 2 and LIST 3 are the same?
Obi:	Yeah.

Obi was then asked about his alternative idea to check their solution. The pair entered their function as y_1, accessed the table values for y_1 and noted these were identical to the values of the hidden function (Affordance 10). This pair, although relatively unsuccessful during task solution, clearly, when focussed on the task were able to successfully perceive and enact a number of affordances of the TRTLE that allowed both task solution and solution verification.

DISCUSSION OF RESULTS

This study has shown that in the TRTLE's involving these two classes a range of affordances were perceived and rejected or enacted for various purposes. Each of the four instruments proved valuable in identifying these affordances. This identification occurred through use of a single instrument, multiple methods of identifying the same instance of an affordance perceived, and, at times, two or more instruments contributing evidence which when combined identified the perception of an affordance, that evidence from a single source was not enough to substantiate the perception. The post task interview, in particular, provided evidence of affordances that the other instruments did not. Questions relating to alternative choices available to the student pairs with respect to particular features of the graphing calculator, or for the solution pathway, and ways in which they could check or verify their solution proved most informative in this regard. Whilst the use of the post task interview appears to provide the greatest evidence of perceived affordances, each instrument brought valuable information to light. This was sufficient to warrant further use of each in future data collection for this study[1].

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References

DeMarios, P., & Tall, D. (1996). Facets and layers of the function concept. In L. Puig & A. Gutierrrez (Eds.), *Proc. 20th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 297-304). Valencia. Spain: PME.

Drijvers, P. (2003). *Learning algebra in a computer algebra environment: Design research on the understanding of the concept of parameter.* Utrecht, The Netherlands: CD-ß Press.

Edwards, D. (1998). Types of case study work: A conceptual framework for case-based research. *Journal of Humanistic Psychology. 38*(3), 36-69.

Gibson, J. (1977). The theory of affordances. In R. Shaw & J. Bransford (Eds.), *Perceiving, acting, and knowing: Toward an ecological psychology* (pp. 67-82). Hillsdale, NJ: Erlbaum.

Gibson, J. (1979). *An ecological approach to Visual Perception.* Boston: Houghton Mifflin.

Laurillard, D., Stratfold, M., Luckin, R., Plowman, L., & Taylor, J. (2000). Affordances for learning in a non-linear narrative medium. *Journal of Interactive Media in Education, 2*, 1-19.

Pea, R. D. (1993). Practices of distributed intelligence and designs for education. In G. Salomon (Ed.), *Distributed cognitions: Psychological and educational considerations* (pp. 47-87). New York: Cambridge University Press.

Pollak, H. (1986). The effects of technology on the mathematics curriculum. In M. Carss (Ed.), *Proceedings of the Fifth International Congress on Mathematical Education* (pp. 346-351). Boston, MA: Birkhäuser.

Scarantino, A. (Dec, 2003). Affordances explained. *Philosophy of Science, 70*, 949-961.

Smitsman, A., & Bongers, R. (2003). Tool use and tool making: A developmental action perspective. In J. Valsiner & K. Connolly (Eds.), *Handbook of developmental psychology* (pp. 172-193). London, UK: Sage.

Stake, R. E. (1995). *The art of case study research*. Thousand Oaks, CA: Sage.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research*. Thousand Oaks, CA: Sage.

THE "A4-PROJECT" - STATISTICAL WORLD VIEWS EXPRESSED THROUGH PICTURES

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This paper arises from an interest in how students at university level view statistics. We asked 394 students to express their views on statistics by designing an A4 sheet of paper. After providing the theoretical framework, we present the analysis of a random sample of 15 pictures. Based on these works we suggest three categories to describe the expressed statistical world views.

INTRODUCTION

In a world where the handling and interpretation of data is becoming increasingly important, basic statistical skills and statistical literacy build the foundation for many of our decisions (Niederman & Boyum, 2003; Wallmann, 1993; Watson & Callingham, 2003). Indeed, the NCTM standards state that "a knowledge of statistics is necessary if students are to become intelligent consumers who can make critical and informed decisions" (NCTM, 1989, p. 105). Recent dialogues on quantitative literacy or numeracy point in a similar direction (Steen, 1997; 2004).

At the same time, more and more researchers are taking into consideration the role of world views or beliefs as a hidden variable in mathematics education (Leder, Pehkonen & Törner, 2002). The term *statistical world views* is chosen in accordance with the term *mathematical world views* by which we understand subjective beliefs and personal theories related to mathematics (Schoenfeld, 1985; 1998). As many researchers point out, the learning and success in mathematics is influenced by student beliefs about mathematics and about themselves as mathematics learners (Schoenfeld, 1992; Hannula, Maijala & Pehkonen, 2003). While there is substantial research on global beliefs about the nature of mathematics, teaching, and learning (Cooney & Shealy, 1997; Lloyd, 1999), in this study we direct our attention to views about statistics.

This focus on statistical world views seems to us to be fruitful since views on statistics might remarkably differ, for example, from views on algebra. Following Törner (2002), Aguirre (to appear) employs the term *domain-specific beliefs* to describe this phenomenon. Domain-specific beliefs are characterized as beliefs that are associated with a special field or domain of mathematics such as calculus, geometry, or statistics.

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MATHEMATICAL WORLD VIEWS

Dionne (1984) suggests that world views or beliefs are composed of three basic components called the traditional perspective, the formalist perspective and the constructivist perspective. Similarly, Ernest (1991) describes three views on mathematics called instrumentalist, platonist and problem solving, while Törner and Grigutsch (1994) name the three components as toolbox aspect, system aspect and process aspect. All these different notions correspond more or less with each other. According to Dionne (1984), beliefs constitute a mixture of the three components. It is possible that more than one view is expressed by a person, so no clear classification can be made.

In this work, we employ the notions of Törner and Grigutsch (1994) and use this section to briefly explain what is understood by them. In the "toolbox aspect", mathematics is seen as a set of rules, formulae, skills and procedures. In the "system aspect", mathematics is characterized by logic, rigorous proofs, exact definitions and a precise mathematical language. In the "process aspect", mathematics is considered as a constructive process where relations between different notions and sentences, as well as the invention or re-invention of mathematics, play an important role. Besides these standard perspectives, another important component is the usefulness, or utility, of mathematics (Grigutsch, Raatz & Törner, 1997). This is particularly relevant for the students in this study who were undertaking an applied course in statistics.

PICTURES AS A MEANS FOR INVESTIGATING WORLD VIEWS

Traditionally, mathematical world views are investigated with the aid of questionnaires or interviews. In our work we have used pictures for investigating the world views, much less common in the literature. As one research example, restricted to beliefs on learning, Berry and Sahlberg (1996) used four pictures, each showing a real life situation. They asked 13-year old students to choose the picture that – in their opinion – best describes a good learning situation and to give reasons for their choice.

In contrast, we have asked students to express their views on statistics by designing an A4 sheet of paper themselves (see Methodology below). With regard to producing pictures that reveal views on mathematics, there also exist some experiences in Germany. On the occasion of the World Mathematical Year 2000, launched by the International Mathematical Union and supported by UNESCO, students in a competition were invited to draw a picture representing their ideas of what mathematics is. Some of those pictures were awarded with prizes and about 30 pictures were part of a special exhibition (Exhibition, 2000), but as far as we know they were unfortunately never the topic of research in mathematics education.

METHODOLOGY

Sample

The subjects in this study were students undertaking the course "Analysis of Biological Data and Experiments", run at an Australian University in Semester 2, 2004 (from the end of July to the end of October). This course was taken by 433 students who were studying biological sciences, including genuine science students, but also students aiming to study medicine, dentistry, and other professions. A total of 394 students (91%) submitted pictures as part of their assessment for the course.

Study

In the first two weeks of the course, students were asked to use an A4 sheet of paper to describe their views on statistics. Little had been covered in the lectures by this stage, with the pictures aiming to capture the initial understanding that students brought to the course. The wording on the instructions directed students to "take a blank sheet of A4 paper and draw, write, paint, doodle, or whatever" suits them best to express their views on statistics. Since the paper was A4, we referred to this as the "A4-Project".

Analysis

The methodological framework to analyse data about students' statistical world views is informed by grounded theory (Strauss, 1987). The guiding principle of grounded theory is that theorizing grows from the data rather than from a pre-existing theory. We therefore randomly selected 15 out of the 394 pictures, here referred to as pictures (001) to (015). We independently analysed this random sample of 15 pictures and identified some significant themes in the pictures. After discussion between the authors, these significant themes were refined to three key aspects that students appeared to be expressing in their work.

RESULTS

In this section, we first describe the categories by giving a rough definition and an overview of the typical features of each category that were presented in the pictures, providing as foundation some concrete examples that support the category in question. We then summarize these categories and suggest that the statistical world views might be seen as a hierarchy.

Due to the format of the paper, it is unfortunately not possible to include all of the 15 pictures here. Figures 1 and 2 show two examples: pictures (002) and (011), respectively, reproduced with the permission of the students.

Course Aspect

Several pictures do not describe statistics as a discipline. In these the focus is on "statistics" as the course that the students are undertaking. Picture (012) is set at one of the lecture rooms for the course, but goes beyond the course aspect by showing statistical symbols (see Toolbox Aspect) entering the brain of a student. Moreover, the students often express their attitudes towards this course by drawing a person having a certain facial expression, such as concerned (002 - Figure 1), first anxious, later happy and optimistic (008) or smiling (004, 005). In addition to a smiling person, picture (004) consists of a house, a flower, the shining sun; no views on statistics are expressed, but there is a pleasant atmosphere. Only a sheet of paper in the hand of the person with the name of the course on it suggests that the picture has something to do with statistics.

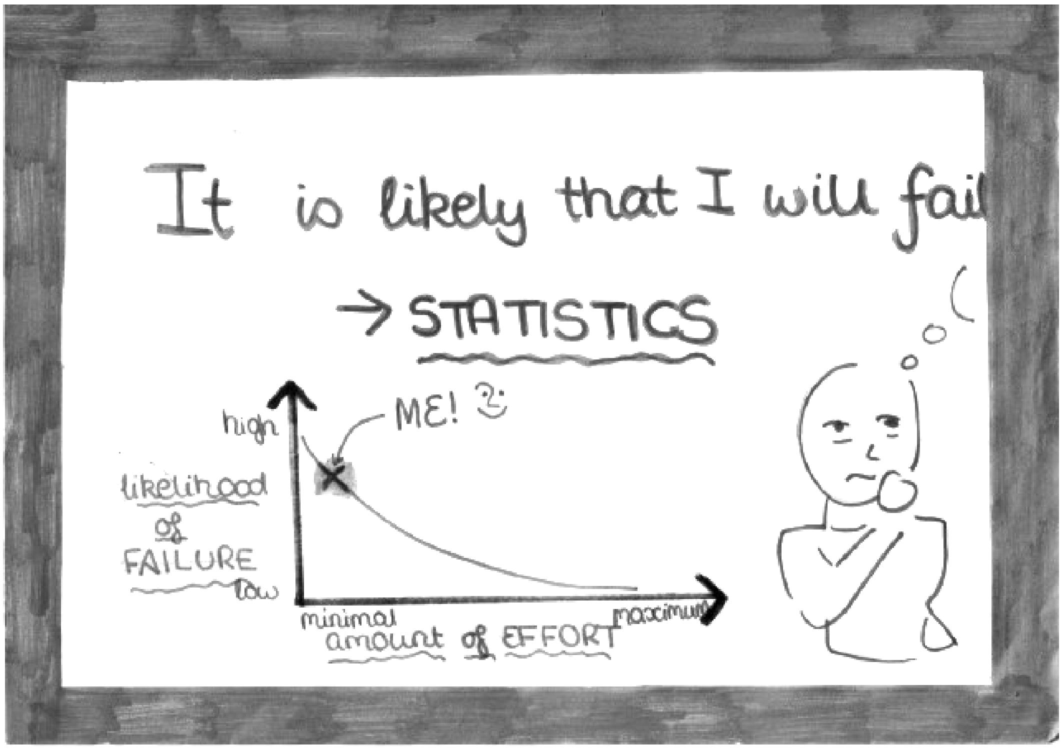

Figure 1. Example picture (002)

Toolbox Aspect

A very common theme is to show statistics as a collection of tools. Typical elements that indicate this view are diagrams, plots and histograms (001, 006, 007, 009, 010, 011 – Figure 2, 013). Another indicator is the inclusion of statistical key words. These are just given, like vocabulary, without explaining the context, such as the mean, null hypothesis, and variation (006), or the mode, median, and mean (011 - Figure 2, 013). Symbols are also used (001, 012), again without explanation of what they mean, as are statistical formulae (009).

Related to statistics as a collection of tools is the view of statistics as numbers and data. This is perhaps expressing lower knowledge of technical statistics, but in the pictures these usually accompany the general toolbox aspect. For example, in two cases (001, 006) numbers and symbols form the frame of the picture, while another picture (015) has a background of numbers.

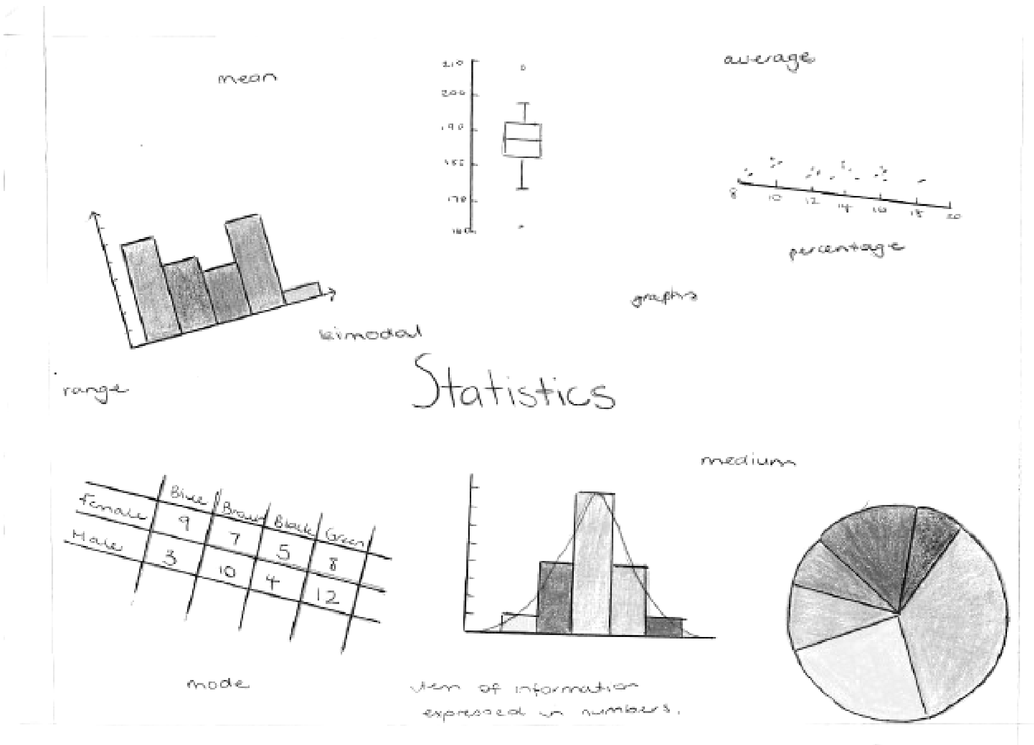

Figure 2. Example picture (011)

Utility Aspect

The final aspect we consider is the expression of the utility of statistics. Picture (001) mentions the role of statistics in “forming the foundations for many reports”. Picture (003) includes a crystal ball with text stating “statistics – a crystal ball to model the future”, while in another corner it hints that statistics helps solving everyday problems.

In addition to the utility of modelling the world, the utility of communication is also identified. Picture (014) asks, “Does there lurk a sunken treasure chest of ideas, investigation, experimentation… Nestled in the bedrock of scientific communication?” Picture (015) features two facing silhouette heads over a field of numbers, suggesting communication.

Hierarchy of World Views

Table 1 presents the three world views as a hierarchy. The description for each attempts to justify the hierarchical nature by specifying what is missing in one aspect that puts it below the next aspect.

Aspect	Description
1. Course	The picture shows "statistics" as a course that the student is studying. Such pictures often describe attitudes, including apprehension about failing the course or the expectation that the course will be boring. No mention of statistics as a discipline or the content of the course is given.
2. Toolbox	The picture shows statistics as numbers and as a collection of tools, such as different ways of plotting data or calculating summary statistics. No indication of where these statistics come from or how they can be interpreted is given.
3. Utility	The picture shows the utility of statistics in describing the world. Rather than just being a tool, the picture shows some relevance of statistics, such as in communication or in scientific experimentation.

Table 1. Summary of world views identified in the sample pictures

Of course the hierarchy in Table 1 is not perfect. Many pictures were seen to express more than one aspect, though it was usually possible to identify a dominant aspect for a picture. It is also likely that a student might draw a picture that expresses the utility of statistics without having a strong understanding of statistics as a toolbox. Since this was not in an interview setting it is difficult to clarify some of these points.

It is also worth noting that this hierarchy of statistical world views is focussing on statistics as a discipline. In studying the pictures we were also aware that many expressed an affective or attitudinal aspect that is not captured in our hierarchy (although some of the affective component has been mentioned in the Course Aspect above). However, most of the pictures do not clearly present an affective aspect and so for this work we have focussed on the actual statistical content of the picture.

CONCLUSION

Almost half of the examined pictures use statistical plots, such as line graphs and histograms. It is not surprising to see such an emphasis on the visual tools of statistics since the nature of the task emphasised the visual. Based on these works, it is difficult to conclude, for example, whether students do normally think visually about statistics. This highlights that what we are measuring here is the *expression* of statistical world views, not the world views themselves. However, the picture task, with the opportunity to write or draw or paint, provides a greater range of possibilities for this expression than does a standard questionnaire, for instance. By starting with a blank sheet of paper we suggest that the students are also less constrained by what they might think are "good answers" to the task. To conclude we discuss two possible areas for leveraging this aspect of the task.

In a broad sense, the pictures drawn by students may give an insight into their level of statistical literacy. However, our results are quite different to those from other research. For example, Watson and Callingham (2003) give a “statistical literacy construct” that has six levels, from “idiosyncratic” to “critical mathematical”. While their research is based on studying how students complete statistical tasks, we have found it more difficult to get fine classifications from the static expression captured by a picture. It would be worthwhile to measure the same students with the tasks given by Watson and Callingham (2003) and see if there are any relationships to our categories.

The timing of the pictures, drawn by students at the start of their tertiary statistics course, was aimed at providing data on the broad beliefs, attitudes, and knowledge of statistics that students brought with them from their prior experience. In itself this has provided academic staff with a richer understanding of the backgrounds of students undertaking this course. However, the wide variety of pictures drawn by students made it difficult to make any clear conclusions from the data. This present study into the categories of statistical world views will make it easier to work with this type of data and provide the needed clarity. For example, we are currently analysing all of the 394 pictures based on the categories given here. Once a larger number of pictures have been categorized it is then possible to look for associations between a student’s initial statistical world views and measures of subsequent performance in this particular course, such as marks on project work or exams. If such associations exist then this will suggest changes to the curriculum in order to address how particular world views need to be incorporated.

References

Aguirre, J. M. (to appear). *“There’s All Kinds of Math”: Teachers’ Domain-specific Beliefs in Response to Mathematics Education Reform*.

Berry, J., & Sahlberg, P. (1996). Investigating pupils’ ideas of learning. *Learning and Instruction*, *6*(1), 19-36.

Cooney, T. J., & Shealy, B. E. (1997). On understanding the structure of teachers’ beliefs and their relationship to change. In E. Fennema & B. S. Nelson (Eds.), *Mathematics teachers in transition* (pp. 87-110). Mahwah, NJ: Lawrence Erlbaum Associates.

Dionne, J. J. (1984). The perception of mathematics among elementary school teachers. In J. M. Moser (Ed.), *Proc. 6th Annual Meeting of the North American Chapter of the Int. Group for the Psychology of Mathematics Education* (pp. 223-228). Madison (WI): University of Wisconsin: PME-NA.

Ernest, P. (1991). *The philosophy of mathematics education.* Hampshire, UK: The Falmer Press.

Exhibition (2000). http://www.math.uni-oldenburg.de/personen/tdm/digitaleAusstellung/Mein-Bild-der-Mathematik-digital.htm, [12.01.2005].

Grigutsch, S., Raatz, U., & Törner, G. (1997). Einstellungen gegenüber Mathematik bei Mathematiklehrern. *Journal für Mathematikdidaktik, 19*(1), 3-45.

Hannula, M. S., Maijala, H., & Pehkonen, E. (2003). Development of understanding and self-confidence in mathematics; Grades 5-8. In M. J. Høines & A. B. Fuglestad (Eds.), *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 17-24). Bergen, Norway: PME.

Leder, G., Pehkonen, E., & Törner, G. (Eds.) (2002). *Beliefs: A Hidden Variable in Mathematics Education?* Dordrecht, The Netherlands: Kluwer Academic Publishers.

Lloyd, G. M. (1999). Two teachers' conceptions of reform curriculum: Implications for mathematics teacher development. *Journal of Mathematics Teacher Education, 2*, 227-252.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

Niederman, D., & Boyum, D. (2003). *What the numbers say: A Field Guide to Mastering Our Numerical World*. New York: Broadway Books.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In A. D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.

Schoenfeld, A. H. (1998). Toward a theory of teaching-in-context. *Issues in Education, 4*(1), 1-94.

Steen, L. A. (Ed.) (1997). *Why numbers count: Quantitative literacy for tomorrow's America*. New York: College Entrance Examination Board.

Steen, L. A. (2004). *Achieving Quantitative Literacy. An Urgent Callenge for Higher Education.* United States of America: The Mathematical Association of America.

Strauss, A. L. (1987). *Qualitative analysis for social scientists*. Cambridge: Cambridge University Press.

Törner, G. (2002). Mathematical beliefs – A search for a common ground: some theoretical considerations on structuring beliefs, some research questions, and some phenomenological observations. In G. C. Leder, E. Pehkonen & G. Törner (Eds.), *Beliefs: A Hidden Variable in Mathematics Education?* (pp. 73-94). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Törner, G., & Grigutsch, S. (1994). Mathematische Weltbilder bei Studienanfängern – eine Erhebung. *Journal für Mathematikdidaktik, 15*(3/4), 211-252.

Wallman, K. K. (1993). Enhancing statistical literacy: Enriching our society. *Journal of the American Statistical Association*, *88*(421), 1-8.

Watson, J., & Callingham, R. (2003). Statistical literacy: A complex hierarchical construct. *Statistics Education Research Journal*, *2*(2), 3-46, http://fehps.une.edu.au/serj.

A WHOLE-SCHOOL APPROACH TO DEVELOPING MENTAL COMPUTATION STRATEGIES

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Curriculum documents in Australia and elsewhere emphasize the importance of mental computation. There has been, however, little advice about developing and implementing mathematics programs that have mental computation strategies as a focus. One primary school's response to this issue was to identify generic mental computation strategies that could be developed across the school in all grades from Kindergarten to Grade 6. The program was implemented within a framework for quality teaching. Initial results suggest that teachers have developed flexible approaches to teaching mental computation. Students are more aware and articulate about the nature of the strategies that they use, and students' learning outcomes have improved.

INTRODUCTION

There is a growing emphasis on the place of mental computation in schools. In Victoria, Australia, for example, the curriculum framework places sufficient weight on mental computation that it is a separate sub-strand. This sub-strand describes expected outcomes in terms of a progression from whole numbers and recall of basic facts, through recognition of decimal and fraction equivalences to the use of a range of strategies to compute mentally with fractions, decimals and percents (Victorian Curriculum and Assessment Authority, 2002). In the UK, mental computation is highlighted in the National Numeracy Strategy as a deliberate process that involves students in developing efficient and effective approaches to calculation (Askew, 2003).

The development of number understanding in the early years is well documented (e.g., Wright & Gould, 2002) and there is continuing research into children's understanding of written computation (e.g., Anghileri, 2004). It is well known that students draw on a range of formal and informal strategies when computing mentally, and strategy development is advocated as an effective approach (McIntosh, 2003).

Jacaranda Public School[1] in central New South Wales (NSW), Australia, took an unusual approach to changing the emphasis of its mathematics programs away from drill and practice of written algorithms to developing mental computation strategies, in the context of applying a Quality Teaching Model (NSW Department of Education and Training (DET), 2003). The program that the staff developed had two major aims: to change pedagogy as a consequence of implementing the Quality Teaching

[1] Names have been changed to preserve confidentiality.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 201-208. Melbourne: PME.

Model, and to transform the mathematics curriculum through an emphasis on mental computation. This paper reports the initial evaluation findings.

The NSW Quality Teaching Model

The model of quality teaching adopted in NSW has three dimensions: Intellectual Quality, Quality Learning Environment, and Significance (DET, 2003). Each dimension is composed of six elements that are used to identify quality teaching in classroom situations. The dimensions and elements are summarised in Table 1.

Intellectual Quality	Quality Learning Environment	Significance
Deep knowledge	Explicit quality criteria	Background knowledge
Deep understanding	Engagement	Cultural knowledge
Problematic knowledge	High expectations	Knowledge integration
Higher-order thinking	Social support	Inclusivity
Metalanguage	Students' self-regulation	Connectedness
Substantive communication	Student direction	Narrative

Table 1: Dimensions and elements of the NSW model of quality teaching

To make the model operational for teachers, each of the elements was first considered from the perspective of mental computation. The promotion of Deep Knowledge, for example, implied a focus on strategy development, linking to key concepts such as place value; Social Support in the classroom indicated that all students would be encouraged to share contributions and accept different approaches to computation; Cultural Knowledge recognized that different groups in the community had various ways of undertaking mental computations and that these would be explicitly discussed. Every element was interpreted in this manner.

Having established how the model could be applied to mental computation, the school staff decided that the program would be based on a whole-school approach. For two weeks at a time, the whole school, from Kindergarten to Year 6, would focus on applying a particular mental computation strategy to content appropriate to the grade level and experience of the students, copying a process that the school had used successfully for writing development. This step proved quite challenging. Although teachers could identify strategies, understanding what this meant across the full range of grades was not simple, and there was considerable discussion at staff meetings about the mathematics program and aspects that were the keys to students' developing skills and understanding. Ten target ideas were identified for the program's focus, including processes for calculation, such as visualising and counting on, and basic concepts, such as place value and pattern recognition. 'Games with a point' was also included to encourage a move away from reliance on text books. Each of the target strategies was then described for every grade with specific links to the NSW syllabus (NSW Board of Studies, 2002). The description of the ‘Using Patterns’ strategy for each grade is shown in Table 2.

<table>
<tr><th>Grade</th><th>Scope</th><th>Examples</th><th>Outcomes*</th></tr>
<tr><td>K</td><td>Copying, and continuing simple patterns</td><td>Continues the pattern ● ■ ♦</td><td>PAES1.1</td></tr>
<tr><td>1</td><td>Finding missing elements in a pattern</td><td>Completes 4, 7, _, 13, _, 19</td><td>PAS1.1</td></tr>
<tr><td>2</td><td>Systematically uses number combinations</td><td>Writes all combinations of 6: 6 + 0, 5 + 1, 4 + 2, 3 + 3, 2 + 4, 1 + 5, 0 + 6</td><td>PAS1.1</td></tr>
<tr><td>3</td><td>Completes number patterns based on tables</td><td>Completes 88, 80, _, 64, 56, _, 40</td><td>PAS2.1</td></tr>
<tr><td>4</td><td>Systematically uses number combinations based on multiplication and division</td><td>Writes all combinations of 24:
6 x 4, 4 x 6, 24 ÷ 6 = 4, 24 ÷ 4 = 6</td><td>PAS2.1</td></tr>
<tr><td>5</td><td>Uses and describes number patterns based on one operation in different ways</td><td>Completes a table of values e.g.,<table><tr><td>1</td><td>2</td><td>3</td><td>4</td></tr><tr><td>3</td><td>6</td><td></td><td>12</td></tr></table>and describes the pattern in words, “It’s the three times tables”, or as a rule: “To get the bottom number, times the top number by 3”.</td><td>PAS3.1a</td></tr>
<tr><td>6</td><td>Uses patterns with fractions and decimals to make calculations easier.</td><td>7 x 11 = 77 so 7 x 1.1 = 7.7
1/3 is the same as 2/6, so
1/3 + 1/6 = 3/6 which is ½.</td><td>PAS3.1b</td></tr>
</table>

Table 2: Description of the ‘Using Patterns’ strategy across grades

* The outcomes refer to the NSW K-6 Mathematics Syllabus (NSW Board of Studies, 2002).

The activities involved were not confined solely to mental computation. There was recognition that recording was an important part of mathematical activity, but the focus of that recording was shifted away from formal written algorithms to purposeful recording of students’ thinking.

PROGRAM EVALUATION

Methodology

During the period from July to December 2004, the program was formally evaluated.

Evaluation was limited to the primary years, Grades 3 to 6, at the request of the school's teachers who wanted to continue to use existing processes in the early years.

A sample of approximately 90 students from four classes, covering Grades 3 to 6, undertook tests of mental computation in early August and late November, using the tests and methodology developed for an earlier study (Callingham & McIntosh, 2001). Students in Grades 3 and 4 took a test of 50 items, and students in Grades 5 and 6 took a longer version of 65 items. The researcher administered all tests. Questions were presented orally using a CD, and students wrote their answers on provided response sheets. Different test forms were used for the pre- and post-tests, but all tests had overlapping items so that they could be linked using Rasch measurement techniques. The responses were entered verbatim into a spreadsheet to provide for further error analysis, and then scored as correct/incorrect. The scored responses were scaled using Rasch measurement techniques using Quest computer software (Adams & Khoo, 1996), anchored to baseline values obtained from the earlier study so that they could be directly compared. Students' performances were estimated in logits (the natural logarithm of the odds of success), the unit of measure used for Rasch measurement. Using a pre- and post-test model provided performance measures at two points in time, and a growth measure over the 15 week period between the tests.

Additional information was collected from lesson observations in each class. Each lesson lasted approximately 40 minutes, and was part of the regular program, not specially prepared. The researcher observed the lesson informally, interacting and talking with the students, making brief notes. Immediately after the lesson ended the notes were written into a coherent account of the lesson, and this was discussed with the teacher concerned. The discussion confirmed the focus of the lesson and allowed teachers and the researcher to agree about the lesson description. Lesson observations were analyzed using the framework provided by the NSW Quality Teaching Model (DET, 2003).

Interviews were conducted with 12 students, three from each grade, chosen by their teachers to cover the range of competence in their class, or because the students had unusual strategies. The students were interviewed twice, first in a group to establish a relationship with the researcher, and then individually. The interview protocol was adapted from one used elsewhere (Caney, 2002) and focused on mental computation strategies used by students.

Initial Results

The results reported include initial analysis of students' performance and growth, lesson observations and interviews, and do not include error analysis. Pre- and post-test results were available for 89 students (18 in Grade 3; 22 in Grade 4; 28 in Grade 5; 21 in Grade 6), although the actual sample was slightly larger. Table 3 shows the mean scores in logits from the two test administrations. The school's mean score improved overall by 0.66 logits. A paired sample t-test indicated that the change was highly significant (t = -4.10, p = 0.000).

	N	Minimum	Maximum	Mean	Std. Deviation
August	95	-4.41	4.19	0.43	2.15
November	99	-3.69	5.67	1.09	2.12

Table 3: Mean performance overall.

Improvement was more marked in the lower grades than in Grades 5 and 6, as shown by the boxplots in Figure 1, where the median and 25th percentile values in Grades 3 and 4 shifted upwards markedly. In Grades 5 and 6 improved performance appeared to be mainly among the upper 25 percent, as shown by the extended 'whisker' at the top of the boxes. Grade 5 also showed an extended lower whisker in the post-test, suggesting that there was considerable variation in performance in this grade.

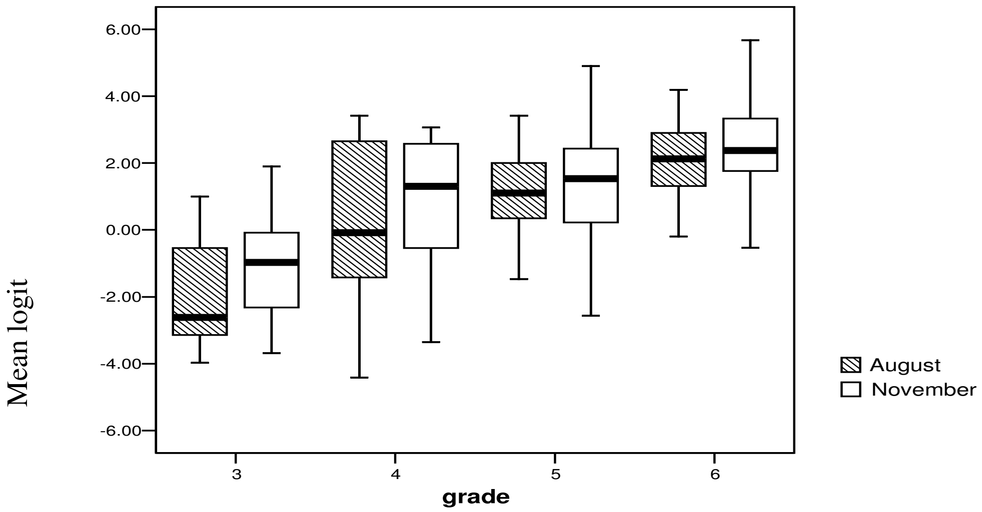

Figure 1: Mean performance by grade.

The four lesson observations were analyzed using the framework provided by the dimensions and elements of the NSW Quality Teaching Model (DET, 2003). All observed lessons were characterized by a very clear and explicit focus – the Deep Knowledge element of the model. Although there were several activities undertaken in each lesson, each addressed the target idea in a different way. For example, a Grade 5 lesson that focused on the doubling strategy included a game in which students had to either double or half the given number and get four correct answers in a row on a grid that they created for themselves. The lesson continued by using the arrangement of desks (4 x 4 grid) as a stimulus to discuss different ways in which number sentences could be made. Finally students were given a number and asked to create two number sentences to make that number. These were shared and discussed, with particular reference to the doubling strategy.

There was considerable discussion and interaction between teacher and students throughout all lessons. Students clearly articulated their strategies, demonstrating the

Deep Understanding and Metalanguage elements of the model. All students were confident about taking risks and suggesting alternative approaches, suggesting that the Quality Learning Environment dimension was well established. In every lesson students were on task throughout the lesson and there was little wasted time, demonstrating the Engagement element. This was probably encouraged by the variety of activities and 'pace' of each lesson observed. Although each activity was completed, in a typical 40-minute lesson there were three or four short activities, each addressing the target idea in a different way.

Teachers also made clear and explicit links to prior experiences of the class. A lesson in a Grade 3 and 4 composite class, for example, started with a discussion about patterns in the nine times table that the class had worked on in the previous week. The rest of the lesson took place in the computer room, where the students practised their previously learnt skills of making a table by creating a grid in which they had to provide number sentences to make given nine times table answers in ten different ways. This teacher skillfully linked the lesson focus of working backwards to prior knowledge, pattern recognition and other learning areas, showing several of the elements of the Significance dimension of the model. This lesson was particularly effective in allowing all students to participate at their own level. Some students took a random approach to finding number sentences, others realised that they could use inverse operations such as making 18 by multiplying it by various numbers and then dividing the answer by the multiplicand. In this way they were able to generate the ten required number sentences quickly.

Teachers were addressing many aspects of the Quality Teaching Model, although most indicated that they did not consciously do this in their planning. It seems likely that the model captures many of the features that contribute to good teaching, and these very competent teachers were drawing on the elements unconsciously. All teachers, however, did indicate that they had changed their teaching to address mental computation more explicitly, with less emphasis on drill and practice of written algorithms. They suggested that their lessons involved students more in discussion, and that they developed a clear focus for every lesson, depending on the target strategy.

Anecdotal evidence from discussions with the school's principal indicated that this changed approach was being implemented across the school. Teachers who had been reluctant to change their mathematics approach, were reporting that their students were more able to talk about their mathematics, and used a wider range of strategies than had been recognized before the program had started.

Students' understanding of strategies and flexibility in strategy use was confirmed by the interviews. In most instances, students were aware of their strategies and could explain them in some detail, even students who were relatively less skilled. Some students were very good at making groups of 10, and could use these ideas flexibly to solve extension problems. Weaker students tended to use counting on strategies, sometimes inappropriately, such as counting on by ones from 24 when asked “How would you work out 24 add 8?” Stronger students would say something like “Take 6 from 8 and add it to 24 to make 30 and then add 2”.

Some of the younger students, not unexpectedly, tended to use repeated addition for some of the multiplication problems, for example counting by 3 seven times to calculate 7 x 3, or, for more complex problems such as 24 x 3, counting on by three 24 times. Although inefficient, these students did have a strategy that would lead to a correct solution and, more importantly, were prepared to try these out. There were very few instances of students not being willing and able to attempt a problem, even when the problem was difficult.

Many students showed high levels of “number sense”, intuitively recognizing an incorrect answer and correcting this. Student Z (Grade 5), for example, in response to 0.5 + 0.5 answered correctly, and then explained “I originally thought zero point ten but it’s lower than 0.5”. His explanation was confused, but he clearly had a feeling for the size of the numbers involved. Some very clearly described visualisation strategies, including “… seeing fingers in my head and counting them” (Student J, Grade 4). In general, younger students seemed more flexible in their thinking. Some of the older students made extensive, and accurate, use of a written strategy, with one girl physically writing the problems on the desk with a finger.

The interviews supported the lesson observations in that students could clearly articulate their strategies, and were confident about using them. Some students had idiosyncratic but effective approaches that they drew on as appropriate.

DISCUSSION

The changed whole school approach to mathematics teaching appeared to have led to improved outcomes for students on standardized tests of Mental Computation Competence. Although some of this improvement may well have been due to increased familiarity with the test format, students also demonstrated clear understanding of many mental computation strategies that they could use effectively, both in a classroom and an interview setting. Teachers reported that their teaching had shifted its focus from written algorithms to strategy and concept development, and this was borne out by the lessons observed. The lessons also addressed many aspects of the NSW Quality Teaching Model adopted by the school. The combined effect of improving teaching quality and a focus on mental computation strategies appears to have been effective.

The whole school focus, across Grades K to 6, on a particular mental computation strategy is unusual. Despite these promising initial results, before any recommendations could be made regarding this approach further work is needed about its efficacy. For example some strategies may be more effective with particular content, such as whole numbers, or strategies may be more appropriately developed in particular grades rather than used across the school. There is also a need to consider further the differential effects shown across the grades, and to establish whether the lower growth observed in the upper primary grades is affected by the strategies used. It may also be possible to establish a hierarchy of strategies that is developmental in nature. These initial results, however, appear to have potential to inform program development in mental computation.

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References

Adams, R.J. & Khoo, S. (1996). *Quest: The interactive test analysis system, Version 2.1*. [Computer software]. Melbourne: ACER.

Anghileri, J. (2004). Disciplined calculators or flexible problem solvers? In M. J. Høines & A. B. Fuglestad (Eds.), *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 41-46). Bergen, Norway: PME.

Askew, M. (2003). Moving from mental arithmetic to mental computation: Lessons from England. In J. Burnett (Ed.), *Making sense of mental computation*. (Proceedings of a one-day mathematics conference for Australian teachers, pp. 6-12). Brisbane: Origo Publications.

Callingham, R. & McIntosh, A. (2001). A developmental model of mental computation. In M. Mitchelmore, B. Perry & J. Bobis (Eds.), *Numeracy and beyond*. (Proceedings of the 24th Annual Conference of the Mathematics Education Research Group of Australasia, pp. 130-138). Sydney: MERGA.

Caney, A. (2002). Exploring mental computation in the middle years. In B. Barton, K.C. Irwin, M. Pfannkuch & M.O.J. Thomas (Eds.), *Mathematics education in the South Pacific*. (Proceedings of the 25th Annual Conference of the Mathematics Education Research Group of Australasia, pp. 163-170). Sydney: MERGA.

McIntosh, A. (2003). Moving to mental: Seven steps along the computation path. In J. Burnett (Ed.), *Making sense of mental computation*. (Proceedings of a one-day mathematics conference for Australian teachers, pp. 13-21). Brisbane: Origo Publications.

NSW Board of Studies, (2002). *Syllabus: Mathematics K–6*. Sydney: Board of Studies.

NSW Department of Education and Training (DET), (2003). *Quality teaching in NSW public schools. Discussion paper*. Sydney: Professional Support and Curriculum Directorate, DET.

Victorian Curriculum and Assessment Authority, (2002) *Curriculum Standards Framework II. Mathematics*. Melbourne: VCAA.

Wright, R. J. & Gould, P. (2002). Mapping overlapping waves of strategies used with arithmetical problems. In A.D. Cockburn & E. Nardi, *Proc. 26th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, 418-425). Norwich, UK: PME.

A COMPARISON OF PERCEIVED PARENTAL INFLUENCE ON MATHEMATICS LEARNING AMONG STUDENTS IN CHINA AND AUSTRALIA

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This paper explores levels of perceived parental influence on mathematics learning among over 700 students in China and Australia. Students in China had stronger perceived parental influences than students in Australia, and while students in China, Chinese speaking students in Australia, and other language speaking students in Australia demonstrated similar levels of perceived parental influence, these three groups of students had higher levels of perceived parental influence than English speaking students in Australia. Possible reasons for the similarities and differences found are discussed.

INTRODUCTION

> My parents come from another country, so my mum makes me do harder work than at school (A Year 5 student's comment on mathematics learning) (Cao, 2004, p.236)

Parents play an important role in influencing students' mathematics learning. Parents from different cultural backgrounds may influence their students learning differently. This paper addressed the issue of parents' role in their children's mathematics learning by comparing parental influence on mathematics learning as perceived by different cultural groups of students in China and Australia.

PARENTAL INFLUENCE IN INTERNATIONAL CONTEXTS

Research on the differences in parental involvement of mathematics learning among different cultural groups has attracted the interest of many researchers. For example, Chen, Lee, and Stevenson (1996) compared students' achievements and their parents' involvement in China and the USA, and found that Chinese parents had higher expectations of their children's performance and spent more time helping their children with school homework than parents in the USA. Mau (1997) investigated differences in parental influence on the academic achievement of Asian immigrants, Asian Americans, and White Americans by using a large representative sample of 10th grade student data in the USA. The findings showed that both Asian immigrant and Asian American parents had higher educational expectations than did White American parents. White American students, however, reported more parental involvement in school activities, such as helping with homework and attending school events, than did Asian immigrant and Asian American students. A recent study conducted by Cai (2003) among over 500 sixth grade students in China and the USA suggested that a larger percentage of Chinese parents reported that they checked

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their children's homework more regularly than did US parents. In contrast, a larger percentage of US parents reported that they often provided their children with reference books and access to libraries. The parents in the two countries did not show significant differences in emotional support for their offspring (i.e., encouraging students to work hard on mathematics).

Even though a rich literature has been produced on parental influences on students' learning of mathematics, in most of the earlier work comparisons were limited to cultural groups within a country, or between people in only two countries. Our literature search did not reveal studies comparing parental influence for the same cultural group in different countries; this study bridges the two approaches and tries to fill the knowledge gap.

RESEARCH QUESTIONS

The purpose of this study was to compare perceived levels of parental influence among different cultural groups of students in China and Australia. The research questions were:

> What are the differences in the levels of perceived parental influence among students from China and students from Australia?
>
> What are the differences in the levels of perceived parental influence among students from different cultural groups?

THE PARTICIPANTS

The participants in this study included 346 primary and secondary school students in China, and 406 primary and secondary school students in Australia. They were distributed at three grade levels: 5, 7, and 9. The students in China were from three primary schools and three secondary schools in Kaifeng, a middle-sized city in Henan Province. The students in Australia were from six primary schools, and seven secondary schools in metropolitan Melbourne. All the participants in China were from Chinese-speaking home backgrounds. Of the students in Australia, 259 were from English-speaking families, 47 were from Chinese-speaking families, and 99 from homes in which other languages were spoken; there were over 30 other languages, with the main ones being Vietnamese, Greek, Italian, Indonesian, Tamil, Arabic, Hindi, and Russian.

THE INSTRUMENT

A Perceived Parental Influence (PPI) scale was developed. Based on the previous work in the parental influence area (Cai, 2003; Poffenberger & Norton,1959), the perceived parental influence measured in this instrument encompassed two aspects: direct involvement, including mother's and father's assistance with homework and difficult problems; and indirect involvement, mother's and father's attitudes towards mathematics, encouragement, and expectations of student learning.

The instrument consisted of 16 items, eight measuring mother's influence on mathematics learning as perceived by students, and eight measuring father's influence.

A four-point Likert scale response format was used. For each statement, the values of 4, 3, 2, and 1 were assigned to the responses: "Strongly Agree", "Agree"(A), "Disagree"(D), and "Strongly Disagree (SD)" respectively.

The Reliability of the Instrument

The reliability analysis of the 16 items of the Perceived Parental Influence (PPI) scale showed that the reliability coefficient (Cronbach Alpha) was 0.876.

Factor analysis was performed to assess the dimensions of the scale. The results are shown in the Appendix. It can be seen from the Appendix that there were four components with Eigenvalues bigger than 1, explaining 66.6% of the total variance. Even though there were items that loaded significantly (>0.3) on Components 2, 3, and 4, all of the items loaded significantly on Component 1. The factor analysis thus indicated that a common construct underpinned the set of items (Hair et al., 1995). Also, since all loadings were in the same direction (all positive), total scores for the scale could be obtained without the need to reverse-score any items.

RESULTS

In this section findings are presented concerning the levels of perceived parental influences on students' learning of mathematics between students from China and Australia, and for students from different cultural backgrounds.

Comparisons between China and Australia

Independent sample t-tests by country were conducted on the mean scores obtained on the *PPI* scale. The results are shown in Table 1. It can be seen from Table 1 that there were significant differences by country in the means on the *PPI* scale for students at each year level and for the whole sample, with students in China having a higher mean score in each case. Effect sizes were medium at grades 7 and 9, and large at grade 5 and overall. The results indicate that there are significant differences in the perceived levels of parental influence between students from the two countries. Overall, and at each grade level, students from China considered that their parents have a stronger influence on their mathematics learning than did the Australian students.

Table 1

Perceived Parental Influence scale: Results of independent samples t-tests for each year level by country

	CHINA			AUSTRALIA					
Grade	N	Mean	SD	N	Mean	SD	t	df	η^2 ♣
5	114	3.50	0.36	120	3.18	0.45	6.00***	226	0.13
7	120	3.23	0.45	137	2.94	0.45	5.11***	255	0.10
9	110	3.00	0.44	114	2.68	0.57	4.88***	222	0.09
All	344	3.25	0.45	371	2.94	0.53	8.33***	713	0.11

*** $p<0.001$

Comparisons by language group

Table 2 shows means and standard deviations for the *PPI* scale among students from the four language groups in China and Australia.

Table 2

Means and Standard Deviations on the Parental Influence scale by language group

Language group	N	Mean	SD
Chinese (CHN[1])	344	3.25	0.45
English (AUS)	235	2.83	0.52
Chinese (AUS)	47	3.11	0.44
Other (AUS)	88	3.13	0.52

[1] CHN = China, AUS = Australia

It can be seen from Table 2 that the students who have the highest mean on the PPI scale are the students from China [Chinese (CHN)], with a mean value of 3.25; next are the other language-speaking students [Other (AUS)] and the Chinese-speaking students in Australia [Chinese (AUS)], with means of 3.13 and 3.11 respectively. The students with the lowest mean on the PPI scale are the English-speaking students in Australia [English (AUS)], with a mean value of only 2.83.

The results suggest that a very strong level of perceived parental influence among the students from China, the Chinese-speaking and the other language-speaking students in Australia; and only slightly strong levels among the English-speaking students in Australia.

♣ Based on Cohen's standard (Cohen, 1988), the effect size is small if η^2 is 0.02, moderate if η^2 is 0.06, large if η^2 is 0.14.]. The effect size is an important index that should be examined to see if a significant test result is of practical significance when sample sizes are large. A small effect size in large samples, even though a significant test result appears, indicates that statistical significant difference is of little practical meaning.

One way ANOVA results (PPI x language group) indicated that there were significant differences in the mean scores on the PPI scale among the students from the four language groups [$F(3, 710) = 34.95$, $p<0.001$, $\eta^2 = 0.127$]. The results suggest that the differences in the levels of perceived parental influence on mathematics learning by language group are quite large.

Post-hoc Scheffe test results are shown in Table 3. The results suggest that the mean scores for students in China, Chinese-speaking students in Australia, and other language-speaking students in Australia are significantly higher than the mean score for English-speaking students in Australia. However, there were no significant differences in the mean scores among three groups of students: students in China, Chinese- speaking students in Australia, and other language-speaking students in Australia.

Table 3

Post hoc tests on mean differences in the Parental Influence scale among the four language groups

Language (I)	Language (J)	Mean Difference (I-J)
Chinese (CHN)	English (AUS)	0.42***
	Chinese (AUS)	0.13
	Other (AUS)	0.11
Chinese (AUS)	English (AUS)	0.28**
	Other (AUS)	-0.02
Other (AUS)	English (AUS)	0.30***

***p<0.001; **p<0.01

The results reveal that students in China, Chinese-speaking students in Australia, and other language-speaking students in Australia have stronger levels of perceived parental influence than English-speaking students in Australia; however, there were no differences in the levels of perceived parental influence among the same three groups of students.

CONCLUSIONS AND DISCUSSION

Two main conclusions emerged from the comparisons among the groups of students in China and Australia, and among the four language background groups.

First, levels of perceived parental influence were stronger among students in China than in Australia. Second, with respect to the levels of perceived parental influence among students from different language backgrounds, students in China had stronger levels of perceived parental influence than English-speaking students in Australia, but there were no significant differences between the students from China, Chinese-speaking and other language-speaking students in Australia. Chinese-speaking students in Australia, and other language-speaking students in Australia demonstrated stronger levels of perceived parental influence than did English-speaking students in Australia.

The explanation for the between-country differences may be due to cultural factors. It has been suggested that education has always been considered the most important path to success in Chinese culture, and parents pay particular attention to their children's education (Hess et al., 1987; Chao, 1996). The consistent findings of this study with other studies indicate that this cultural practice is still strongly held in Chinese society.

As for the similarities and differences reflected among the different cultural groups, it seems to suggest that not only cultural factors, but also societal factors play an important role in shaping perceptions of parental influence. The similarities in the levels of perceived parental influence reflected among students in China and Chinese-speaking students in Australia may be attributed to the Chinese cultural emphasis on education. The similarities in the levels of perceived parental influence among Chinese and other language-speaking students in Australia, and the differences between non-English-speaking and English-speaking students in Australia may reflect an immigrant phenomenon. That is, among immigrant families in Australia for whom English is not their first language, parents recognise that education is vital for success in the new society, therefore they strongly encourage their children and have high expectations of them to fulfil their own dreams.

In conclusion, even though this study revealed that there were differences in the levels of perceived parental influences among different groups of students in China and Australia, and postulated explanations for the differences, more work is needed to identify the underlying causes. Other research techniques, such as interviews with parents and students may assist in this task. On the other hand, as parental influence has been identified as influencing mathematics learning outcomes such as attitudes and achievements, it is necessary for educators to disseminate this knowledge more widely so that more parents realise the important role they can play in their children's schooling, and provide greater support and inspiration for their children.

References

Cai, J. (2003). Investigating parental roles in students' learning of mathematics from a cross-cultural perspective. *Mathematics Education Research Journal, 15*(2), 87-106.

Cao, Z. (2004). *An investigation of students' attitudes towards mathematics in China and Australia*. Unpublished doctoral dissertation, Monash University, Australia.

Chao, R. K., & Sue, S. (1996). Chinese parental influence and their children's school success: A paradox in the literature on parenting styles. In S. Lau (Ed.), *Growing up the Chinese way: Chinese child and adolescent development* (pp.93-120). HongKong: the Chinese University Press.

Chen, C., Lee, S., & Stevenson, H. W. (1996). Academic achievement and motivation of Chinese students: A cross-national perspective. In S. Lau (Ed.), *Growing up the Chinese way: Chinese child and adolescent development* (pp.69-92). Hong Kong: the Chinese University Press.

Cohen, J. (1988). *Statistical power analysis for the behavioural sciences*. Hillsdale, NJ: Lawrence Erlbaum.

Hair, J. F., Anderson, R.E., Tatham, R.L., & Black, W.C. (1995). *Multivariate data analysis with readings* (4th ed.). London: Prentice Hall.

Hess, R.D., Chang, C., & McDevitt, T. M. (1987). Cultural variations in family belief about children's performance in mathematics: Comparisons among People's Republic of China, Chinese American, and Caucasian-American families. *Journal of Educational Psychology, 79,* 179-188.

Mau, W. C. (1997). Parental influences on the high school students' academic achievement: A comparison of Asian immigrants, Asian Americans, and White Americans. *Psychology in the Schools, 34*(3), 267-271.

Poffensberg, T., & Norton, D. A. (1959). Factors in the formation of attitudes towards mathematics. *Journal of Educational Research, 52,* 171-176.

Appendix

The component matrix and the Eigenvalues of the items for the Perceived Parental Influence scale

Item		Component			
		1	2	3	4
1.	My mother is good at maths	0.40	0.43	0.51	
2.	My mother checks my maths homework frequently	0.56	0.36	0.38	
3.	My mother asks me about my assessment results in maths	0.64			
4.	My mother helps me with some difficult maths problems	0.54	0.46	0.43	
5.	My mother makes me feel that I can do well in maths	0.65			-0.36
6.	My mother tells me that a person must do something carefully in order to do it well	0.65	-0.30		-0.33
7.	My mother tells me a person must work hard in order to do something well	0.63	-0.34		-0.35
8.	My mother expects me to be the best student in maths and other subjects in my class	0.48	-0.60		0.43
9.	My father is good at maths	0.47	0.41	-0.38	
10.	My father checks my maths homework frequently	0.64		-0.31	0.35
11.	My father asks me the assessment results in maths	0.69			
12.	My father helps me with some difficult maths problems	0.61	0.35	-0.46	
13.	My father makes me feel that I can do well in maths	0.72		-0.30	
14.	My father tells me that a person must work hard in order to do something well	0.72			
15.	My father tells me that a person must do something carefully in order to do it well	0.73			
16.	My father expects me to be the best student in maths and other subjects in my class	0.45	-0.63		0.48
Eigenvalue		5.90	2.01	1.55	1.19
% Variance explained		36.86	12.65	9.68	7.45

*Extraction Method: Principal Component Analysis; Loadings less than 0.3 omitted.

USING WORD PROBLEMS IN MALAYSIAN MATHEMATICS EDUCATION: LOOKING BENEATH THE SURFACE

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This paper reports on aspects of a project that investigated the influence of Chinese Malaysian students' schooling in a tradition of abstract, technical mathematics and rote learning on ways that they responded to mathematical word problems. Data from an action research project are reported. Supposedly "shallow" and " deep" learning are shown to be interlinked, and assumptions frequently made by Western educators about modelling and practice are questioned.

INTRODUCTION

While most Western pre-university mathematics curricula now incorporate real-life problems and applications, many South-East Asian mathematics curricula remain technical and traditional. Chi's (1999) comparison of Taiwanese and New Zealand curricula revealed that Taiwanese syllabi were comparatively archaic and did not reflect Western developments in mathematics education such as real-life problem solving. Chacko (1999), comparing American and Malaysian students, claimed that the latter learn facts through memorization, so graduates do not think deeply. Chi (1999) described the typical Confucian style of learning in the Taiwan mathematics classroom, where drills, attention to content and not the learning process, emphasis on examinations, technical questions and proofs rather than applications, and learning by memorization are all common features. Lim and Chan (1993) noted similar features in Malaysia. Reports from Japan (Kinoshita, 2000) and Hong Kong (Lucas, 2000) have indicated that students depend on rote learning in mathematics, and concerns have been expressed about the need to implement changes in teaching methods in both of these countries. In Western countries, it is generally believed that such rote learning and memorization do not enhance mathematical understanding.

Biggs and Watkins (1996) also noted that Chinese students use memorization, but concluded that there is a difference between memorizing without proper reflection and "memorization with understanding" (p. 271). Similarly, Marton, Dall'Alba, and Tse (1996) suggested that Chinese students learn repetitively in the belief that memorization could lead to understanding. Confucian tradition emphasizes understanding, reflection, and enquiry as important co-components of learning, and this is achieved by becoming "intimately familiar with the text" (Lee, 1996, p. 35).

> Learning is reciting. If we recite it then think it over, think it over then recite it, naturally it'll become meaningful to us. If we recite it but don't think over, we still won't appreciate its meaning. If we think it over but don't recite it, even though we might understand it, our understanding will be precarious. (Chu, 1990, p. 138)

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 217-224. Melbourne: PME.

Marton, Dall'Alba, and Tse (1996) pointed out that there is a paradox surrounding the Chinese learner because Chinese students have been known to perform admirably in international examinations and competitions, including at higher levels. Suh and Oorjitham (1996) reported that the countries ranked highest in mathematical achievement in a global survey were Singapore, South Korea, Japan, and Hong Kong. However, they claimed that Asian students grind good results out of memorization while their overseas peers are encouraged to be creative. They concluded that curiosity, questioning, and fun were often curtailed at the expense of producing high achievement scores, to the detriment of problem solving ability. In fact, despite the apparent successful performance of Asian students in international competitions and institutions of higher learning in Western countries, academics generally believe that Asian students are more prone to rote learning than their Western peers (Biggs, 1989, 1990). Samuelowicz (1987) surveyed 145 lecturers at one Australian university and over one-third felt that Asian students utilized only a surface approach to learning, characterized by memorization of isolated facts and fragments of arguments.

However, such conclusions need to be balanced with what is valued within countries. As much as it is admirable to produce *thinking students* or to provide *meaningful and deep learning experiences* for the students, both of which are considered desirable learning attributes by Western standards, Alatas (1972) rightly warned of uncritical imitation and unrealistic assumptions when adopting these ideas into an Eastern setting. Similarly, Bishop, Seah, and Chin (2003) cautioned that aiming for "uniformity of practices" (pp. 718–719) results from failing to appreciate the educational differences brought about by different cultural values and practices. After all, it is argued, mathematics and its practices are not culture-free (Bishop, 1988).

Deep and surface learning in the West and East

Marton, Dall'Alba and Tse (1996) identified two approaches that students adopt to learning, namely "deep" and "surface" approaches (p. 69). However, it has been found in several studies that what seem to be surface approaches can be used to develop deep understanding (e.g., Marton & Wenestam, 1987; Marton, Carlsson & Halasz, 1992). Kember and Gow (1990) postulated an "understand-memorize-understand-memorize" sequence (Biggs & Watkins, 1996, p. 271; Hess & Azuma, 1991) where memorizing leads to improved understanding.) It appears that different aspects and perspectives are focused on with each repetition, deepening and widening understanding. Marton, Dall'Alba and Tse (1996) summarized this paradox:

> In the process of repeating and memorizing in this way, the meaning of a text is grasped more fully: "In the process of repetition, it is not a simple repetition. Because each time I repeat, I would have some new idea of understanding, that is to say I can understand better." It is upon this use of memorization to deepen understanding that the solution of the paradox of Chinese learner rests. (p. 81)

Hence, Western educators may equate Chinese learners' memorization to rote learning in error, suggesting that it is necessary to exercise caution when making assumptions about methods from other cultures (Marton, Dall'Alba & Tse, 1996). Biggs and Watkins (1996) explained this phenomenon further:

> … the difference between those who use a lot of repetition in learning for understanding, and those who learn for understanding without much repetition, derives from perceived task demand, which differs between cultures … what differs are the perceived demands of common tasks learners from each culture typically face in the home environment. Chinese learners come to use repetition strategically more often than Westerners do in their attempts to understand their world. (p. 272)

In Malaysia, Chinese schools produce impressive results in mathematics. Their school mathematics is thought to be "superior" to National school mathematics because society perceives mathematics to be only computation and operations. Chinese school students pride themselves on being able to recite their multiplication tables by Year One, and parents spend thousands of *ringgit* to send children to mental arithmetic classes. Fast arithmetic computations by using either the abacus or a finger technique are learnt by rote, and young Chinese children who help their families run small businesses excel in computation. However, it is fair to question whether achieving speedy mastery or arithmetic ensures good mathematical understanding at higher levels—but the college students referred to in this research report attended both primary and secondary schools where such questions were never asked.

In this paper, we report on one aspect of a two-and-a-half year action research project where the main aim was to investigate whether Chinese Malaysian post-secondary students who study mathematics as an enabling science are able to learn mathematics more meaningfully when it is taught not by memorization of procedures but by using word problems. Instead of the usual fare of drills and abstract technical questions, word problems were featured extensively in the curriculum. The specific research question that we focus on in this report is whether the students who apparently prefer "surface learning" in mathematics were able to appreciate deeper concepts and contexts in mathematical word problems. Across a number of action research cycles, the students were encouraged to engage in discussion, peer-group activities and reflection—all of which are Western approaches designed to bring about "deep" learning and not usually adopted in traditional Malaysian education environments.

The introduction of Western teaching methods is increasing in South-East Asian countries, so it is important to learn more about (a) the effects of use of problems and Western classroom methods, and (b) how these new approaches might be adapted to improve teaching processes and hence learning outcomes.

METHOD

This research was undertaken in a Malaysian private college, with a total of 290 students enrolled in the first semester of a computing and information technology diploma course. The majority of the students were 17-18 year-old secondary school leavers from a Chinese school background. Seven 14-week cycles of action research were carried out over two and a half years, using seven cohorts of students.

Action research was appropriate because the overall aim was to explore the possibilities and challenges of instituting change in teaching, over time. Data on students' achievements, interest levels, beliefs and attitudes, and mathematical performance were collected. Instruments used included questionnaires, interview schedules, journal notes of conversations and observations, as well as each cohort's mathematics work and exams. Daily entries were made in a reflective journal. Data from these sources were sorted under headings including the use of word problems, collaborative and reflective learning, learning mathematics in a second language, and the incorporation of values into mathematical concepts and practices (see Chan Kah Yein, 2004). Analysis of data included triangulation by colleagues and a student.

One of the authors, Chan Kah Yein, was the teacher-researcher. Each action research cycle involved making a change in relation to the use of word problems, including using students' interests, consideration of professional needs of the students, small-group discussions, encouraging peer-group reflection, exploration of inculcating values in mathematical concepts and practices, and tackling issues about learning mathematics in a second language. Each new initiative grew from on-going data-analysis and reflection (Kemmis & McTaggart, 1988). Deep and surface learning were investigates in Cycles 4 through 7 as this issue emerged and was problematised towards the later part of the project. In this paper, we report some aspects of what happened in relation to this particular issue.

RESULTS AND DISCUSSION

Since the students had been raised in an environment that privileged mastering procedural skills and ready-made models for solving problems, it was initially assumed by the teacher that the students were using a surface approach to learning, with rote-based and low-level cognitive strategies, as opposed to a deep approach that is characterized by deriving meanings from the learning material. This proved true as they tried tackling unfamiliar word problems for the first time. The following is an example of a word problem used in Cycle 4:

> Compound Interest Formula $$A = A_o\left(1+\frac{r}{k}\right)^{kt}$$
>
> where A = amount after t years, A_o= initial deposit, r = interest rate per annum (in decimals), k = number of times interest is paid in a year, t = number of years invested
>
> Find the amount of money that should be deposited in an account paying 8% interest per year, compounded quarterly to produce a final balance of RM100,000 in 10 years.

First, most students thought they had to find the value of A instead of A_o because that was more predictable and straightforward. They did not bother about the phrase "final balance" in the question, but assumed that the RM100,000 would be the value of A_o. Second, the phrase "compounded quarterly" also did not mean much to them:

CKY: The value of k is how many times the bank pays you interest in a year. For example, if it is annual compounding, the bank pays you interest only once a year, so the value of k would be 1. If it is semi-annual? (Silence) Semi … what does "semi" mean? As in semi-finals in a football match…?

Kah Sui: Half! So k is half!

CKY: You're right that semi means half, but the phrase is semi-annual, so it means interest is paid every half year, every six months. So how many times do you get interest in one year?

Daniel: Half.

CKY: Not quite. They give you interest every six months, every half year, so …

Kah Sui: Oh … two! k is two.

CKY: Yes, very good. Now, what about "quarterly"?

Kah Sui: Three!

CKY: Not quite … quarterly means you divide the year into quarters, so, it IS three months. You get interest every three months, but how many times would you get interest in a year if they give it to you every three months?

Kah Sui: So, k is four? (wrote it down in his notebook) *Cycle 4, Interview notes*

The students were most interested in copying and memorizing the values of k:

Annually : k = 1 ; Semi-annually : k = 2 ; Quarterly : k = 4 ; Monthly : k = 12 ; etc.

It appeared that memorizing the corresponding values of k was more important than understanding what periodic compounding meant or how the values of k could be derived from understanding periods of time. It was thus conjectured that the students were typical surface learners and that exposing them to word problems would prove a good way to engage them in deep learning. Over time, they did learn the common principles underlying such facts, and hence to focus more on general meanings.

In Cycle 5, a different picture emerged:

CKY: I notice you prefer technical questions to word problems …

Eng Li: I like the technical questions, Teacher. They are challenging. I wish you'd us give more difficult ones to do.

Zhi Wei: … we prefer the technical question. It makes us think more, especially the difficult ones. *Cycle 5, Interview notes*

Such remarks changed the teacher's earlier judgmental stance, and raised questions about whether it was fair to assume that the students were surface learners just because they preferred abstract technical questions. Was it right to assume that word problems require a deep approach to learning whereas technical questions do not? While word problems may provide more opportunities for discussion and reflection, and for relating mathematics to other aspects of life, perhaps technical questions could provide an equivalent level of challenge, meaningful learning, and satisfaction.

Eng Li: Maths is logical. … I just need to practise and use my brain. And it is challenging … It helps me build my mental foundation … makes me think

> logically … I think this is very useful for me when I write my computer programs … This training would be useful for me.
>
> *Cycle 5, Interview notes*

Eng Li's comment showed that he viewed mathematics as a tool whose knowledge can be transferred to other subjects. He valued this application aspect of mathematics and how mathematics helped build his mental abilities to think logically. To him, this aspect of abstract mathematics was meaningful in its own sense. Other students also expressed appreciation of working with just numbers and symbols:

> Eugene: I like the problems with lots of numbers … numbers don't lie … Maths is straightforward and accurate. No twisting and turning around … it's like there's just one thing, and no matter how you look at it, from whichever angle … it comes back to that one thing. But with other subjects, it's like you can see it from so many different angles and they're all different!
>
> *Cycle 6, Interview notes*

Eugene's comments demonstrated that he had internalized the universality value inherent in mathematics. To be able to view mathematics in this way was also a form of problematizing and sense making, and hence constitutes one form of reflective learning. Hiebert et al. (1996) suggested that reflective inquiry and problematizing depend more on the students and the culture of the classroom than on the task itself. They explained:

> … tasks such as 63 minus 37 can trigger reflective inquiry because of the shared expectations of the teacher and the students although they may look routine … Whether they become problematic depends on how teachers and students treat them. (p. 16)

Given that the students who practised the drills diligently were the ones who eventually performed well in the range of questions examined, repetition may have helped to develop their mathematical understanding. It would also seem that Kember and Gow's (1990) *understand-memorize-understand-memorize* sequence—a routine that was clearly observable in the classroom—could have lead to improved understanding and performance. The students' beliefs were not identical:

> Suet Yen: The models you gave us (for the word problems) were very useful. It's like you just remember those few models, and then, you identify the question and apply the model, and there's your solution. Actually, there are only a few models. You just have to understand which model fits which type of question.
>
> Yew Loon: Hey, but I don't like to memorize the models, I prefer to approach each question independently and find a method for it. It's better that way. After all, it's mathematical thinking that is required, isn't it? We don't need models, though they are useful. If you have to depend on models, then what happens if you encounter a brand new type of problem which does not fit into any model?
>
> *Cycle 7, Interview notes*

Here, Yew Loon was ready to move on to the level of divergent thinking and explore new ways of solving problems on his own whereas Suet Yen had just gained enough confidence after practising and applying the models given.

CONCLUSION

In this paper we have focused only on those aspects of the research project entitled "Fostering meaningful learning by using word problems in post-secondary mathematics" that pertain to deep and surface learning in the use of word problems with non-English speaking students in a college programme.

We conclude at this point that most of the students in this project felt a need to practise sufficient examples before they developed adequate confidence and curiosity for more independent and diverse ways of solving problems. Hence, it seemed that what could be termed *surface approaches* can be used to build a foundation for the use of deeper learning approaches. Also, it seemed that technical problems were not inferior to word problems in terms of their ability to lead to deeper learning. What appeared to matter was how the students approached the problems, and how their mathematical thinking developed as a result of having tackled numbers of problems.

Last but not least, the research seemed to provide some justification for Alatas' (1972) caution about uncritical imitation, or more accurately in this case, making uncritical *assumptions* about students' ways of learning and perceiving mathematics. Instead, we need to look beneath the surface and recognize the fact that mathematics is not culture-free, and a deeper understanding of how repetitive practice and deeper learning intertwine is important.

References

Alatas, S. H. (1972). The captive mind in development studies. *International Social Science Journal 24*(1), 9–25.

Biggs, J. (1989). Approaches to learning in two cultures. In V. Bickley (Ed.), *Teaching and learning styles within the across cultures: Implications for language pedagogy.* Hong Kong: Institute for Language in Education.

Biggs, J. (1990). *Asian students' approaches to learning: implications for teaching overseas students.* Paper presented at Australasian Tertiary Learning Skills and Language Conference, Brisbane.

Biggs, J. & Watkins, D. (1996). The Chinese learner in retrospect. In D. A. Watkins & J. B. Biggs (Eds.), *The Chinese learner* (pp. 45–68). Hong Kong: Comparative Education Research Centre, University of Hong Kong.

Bishop, A. J. (1988). *Mathematical enculturation: A cultural perspective on mathematics education.* Dordrecht, The Netherlands: Kluwer.

Bishop, A. J., Seah, W. T. & Chin, C. (2003). Values in mathematics teaching: The hidden persuaders? In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick & F. K. Leung (Eds.). *Second international handbook of mathematics education* (pp. 715–763). Dordrecht, The Netherlands: Kluwer.

Chacko, G. K. (1999, Aug 29). Teaching graduate students how to think. *Sunday Star,* p. 28.

Chan, K. Y. (2004). *Fostering meaningful learning by using word problems in post-secondary mathematics*. Thesis submitted in fulfilment of the requirements of the degree Doctor of Philosophy, Deakin University, Australia.

Chi, H. (1999). Mathematics classroom differences: Students' perceptions of Taiwan and New Zealand. *Science and Mathematics Education Papers 1999,* 225–248.

Chu, H. (1990). *Learning to be a sage: Selections from the conversations of Master Chu, arranged topically* (translated by D. K. Gardner). Berkeley, California: University of California Press.

Hess, R. D. & Azuma, M. (1991). Cultural support for schooling: Contrasts between Japan and the United States. *Educational Researcher, 20*(9), 2–8.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., Olivier, A. & Wearne, D. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher, 25*(4), 12–21.

Kember, D., & Gow, L. (1990). Cultural specificity of approaches to study. *British Journal of Educational Psychology, 60,* 356–363.

Kemmis, S., & McTaggart, R. (Eds.) (1988). *The action research planner* (3rd ed.). Geelong: Deakin University.

Kinoshita, A. (2000). Kyoto academia turns to private sector for help [Online]. Available: http://www.yomiuri.co.jp/newse/0605so13.htm (accessed August 7, 2001)

Lee, W. O. (1996). The cultural context for Chinese learners: Conceptions of learning in the Confucian tradition. In D. A. Watkins & J. B. Biggs (Eds.), *The Chinese learner* (pp. 25–41). Hong Kong: Comparative Education Research Centre, UHK.

Lim, S. K., & Chan, T. B. (1993). A case study comparing the learning of mathematics among Malay pupils in primary national schools and primary national schools. *Journal of Science and Mathematics Education in South East Asia, 16*(2), 49–53.

Lucas, L. (2000, May 5). Hong Kong set to abandon rote learning. *The Financial Times.*

Marton, F., & Wenestam. C. (1987). *Qualitative differences in retention when a text is read several times*. Paper presented at the Second International Conference on Practical Aspects of Memory, University College of Swansea, Swansea, UK.

Marton, F., Carlsson, M. A. & Halasz, L. (1992). Differences in understanding and the use of reflective variation in reading. *British Journal of Educational Psychology, 62,* 1–6.

Marton, F., Dall'Alba, G. & Tse, L. K. (1996). Memorizing and understanding: The keys to the paradox. In D. A. Watkins & J. B. Biggs (Eds.), *The Chinese learner* (pp. 69–83). Hong Kong: Comparative Education Research Centre, University of Hong Kong.

Samuelowicz, K. (1987). Learning problems of overseas students: Two sides of a story. *Higher Education Research and Development, 6,* 121–134.

Suh, S. & Oorjitham, S. (1996, December 13). It all adds up to brilliance. *Asiaweek,* p. 30.

CONSTRUCTING PEDAGOGICAL KNOWLEDGE OF PROBLEM SOLVING: PRESERVICE MATHEMATICS TEACHERS

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The paper reports a study of the knowledge preservice secondary school mathematics teachers [PSSMT] hold of problem solving and the role of a reflective-inquiry approach in creating self-awareness of, and in enhancing, this knowledge. The approach included solving problems, narratives, flow charts and observations. The finding shows that the participants were able to construct a deeper understanding of problem solving. It suggests the need for PSSMT to reflect on the learning experiences, not only from the perspective as learner, but also as teacher, in order to be able to construct a meaningful instructional approach for problem solving.

IMPORTANCE OF PROBLEM SOLVING

Problem solving is considered central to school mathematics. NCTM (2000) states,

> Instructional programs should enable all students to build new mathematical knowledge through problem solving; solve problems that arise in mathematics and in other contexts; apply and adapt a variety of appropriate strategies to solve problems; and monitor and reflect on the process of mathematical problem solving. [p. 52]

Similarly, Kilpatrick et al. (2001, p. 420) explained,

> Studies in almost every domain of mathematics have demonstrated that problem solving provides an important context in which students can learn about number and other mathematical topics. Problem-solving ability is enhanced when students have opportunities to solve problems themselves and to see problems being solved. Further, problem solving can provide the site for learning new concepts and for practicing learned skills.

Thus, problem solving is important as a way of doing, learning and teaching mathematics. If problem solving should be taught to students, then it should be taught to preservice teachers who are likely to not have been taught it in an explicit way. If it is to form a basis of teaching mathematics, then preservice teachers should understand it from a pedagogical perspective. This paper is intended to contribute to our understanding of these issues for PSSMT. It reports on an investigation of the knowledge PSSMT hold of problem solving and the role of a reflective-inquiry approach in creating self-awareness of, and in enhancing, this knowledge.

RELATED LITERATURE AND THEORETICAL PERSPECTIVE

Recent studies on PSSMT include investigating their proportional reasoning (Person et al., 2004); pedagogical reasoning on functions (Sánchez & Llinares, 2003); preferred strategies for solving arithmetic and algebra word problems (Van Dooren et al., 2003); reflection on their learning process through collaborative problem solving in geometry (Bjuland, 2004); and deficiencies in specific mathematics concepts, for

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example, division, operations with integers, functions, and exponents (Ball, 1990; Even, 1993; Kinach, 2002; Wilson, 1994). While these studies do not address problem solving in an explicit way, they imply concerns about how PSSMT may conceptualize it. One study that supports this is Leikin (2003). She found that factors that influenced secondary school mathematics teachers' problem-solving preferences were their tendency to apply a stereotypical solution to a problem and act according to their problem-solving beliefs; the way in which they characterized, and their familiarity with a particular, problem-solving strategy; and a mathematical topic to which the problem belongs. A study, then, of PSSMT's thinking of problem solving could provide insights of their sense-making and how to enhance it.

Jaworski & Gellert (2003) explained that when students enter initial mathematics teacher education they already have extensive knowledge about mathematics teaching and have views on the nature of mathematics. But this knowledge is limited because it is based mainly on their experience as students. Jaworski & Gellert added that since this knowledge serves as a basis of their sense-making, an essential part of preservice teacher education is focusing on their initial personal theories and preconceptions. Reflection has been advocated as a necessary process in facilitating this. The study in this paper addresses the use of reflection as a basis of facilitating PSSMT's awareness of their thinking of problem solving. Theoretically, then, the study is framed in reflection and a social perspective of learning.

The reflective process has a long history as a basis of learning (Dewey, 1916). It is widely accepted as a key factor in facilitating teacher education (Sikula, 1996). It can enable teachers to construct the meanings and knowledge that guide their actions in the classroom and gain understanding of themselves as teachers (Schon, 1987). However, achieving effective reflection can be problematic. For example, as Lerman (1997, p.201) noted, "Reflection on one's own actions presumes a dialogical interaction in which a second voice observes and criticizes. In order to lead to learning it would seem that this must be more than the ongoing observation of one's own actions." This suggests that the reflective process could be enhanced through an interactive process with others.

A social/interactive perspective of learning has been discussed by several people including Dewey, 1916; Lave & Wenger, 1991; and Vygotsky, 1978. Lave and Wenger conceive of learning in terms of participation. Dewey emphasized learning through active personal experience and learning as a social process. In his view, purposeful activity in social settings is the key to genuine learning. Similarly, Vygotsky claimed that individual development and learning are influenced by communication with others in social settings. In his view, interacting with peers in cooperative social settings gives the learner ample opportunity to observe, imitate, and subsequently develop higher mental functions. This theoretical perspective, then, emphasizes human interactions as a key factor to facilitate learning. This formed a basis of the reflective-inquiry approach used in this study.

RESEARCH PROCESS

The study was framed in a qualitative, naturalistic research perspective (Creswell, 1998) that focused on capturing and interpreting the participants' thinking about a phenomenon, problem solving in this case.

Participants: The participants were 26 PSSMT in the second semester of their 2-year post-degree education program. This was their first course in mathematics education, so they had no instruction or theory on problem solving prior to this experience. They also were not taking any other mathematics education course in this semester. They had completed all of their mathematics required for the program in their first degrees.

Reflective-Inquiry Approach: Since a goal of the study was to see what the PSSMT knew and what they would learn from this approach, they were not provided with any theory about problem solving before or during it. They worked on problems and in groups without the instructor's intervention. The activities were organized as follows:

Individual reflection: They were required to respond to a list of questions/prompts in sequence that included: What is a problem? Choose a grade and make a mathematics problem that would be a problem for those students. What did you think of to make the problem? Why is it a problem? Is it a 'good' math problem? Why? What process do you go through when you solve a problem? Represent the process with a flowchart

Inquiry activities: This included: (1) They were provided with a list consisting of a non-verbal, algebraic exercise; a simple translation algebraic word problem; a complex translation algebraic word problem; a process [non-routine] word problem; an applied [open] problem, and a puzzle problem. These categories were influenced by Charles &Lester (1982). The categories were not given to the PSSMT. They were asked: Without solving them, how are these problems similar and different? What conclusions can you make about problems? (2) They were required to write narratives of their experiences solving a problem that was assigned to them. The narrative had to be a temporal account not only of the mental and physical activities they engaged in to resolve the problem, but the emotional aspects of the experience. They later analyzed it in terms of 'stuck' and 'aha!' (3) They were required to solve an assigned problem (half got one problem and half a different one) and make notes of their thought processes. They then worked in pairs, with unmatched problems, and took turns to observe each other solve the problem while thinking aloud. They then compared their thought processes. (4) They selected a process problem appropriate for a secondary school student and used it to observe the student solving it while thinking aloud. An example of an assigned problem for item (3)/(4) is:

> Emma was always looking for ways to save money. While in the remnant shop she came across just the material she wanted to make a tablecloth. Unfortunately the piece of material was in the form of a 2m x 5m rectangle and her table was 3m square. She bought it however having decided that the area was more than enough to cover the table. When she got home she decided she had made a mistake because she couldn't see how to cut the material to make a square. But just as she despaired she had a brainwave, and with 3

straight cuts, in no time, she had 5 pieces, which fitted neatly together in a symmetric pattern to form a square using all the material. How did she do it? [Bolt]

Group reflection: This included: (1) Sharing and comparing their individual reflections and their findings from the inquiry activities, preparing summaries of key words of the group thinking to correspond with the questions under individual reflection and a flowchart of the problem solving process. (2) Discussing and summarizing how they would teach problem solving. (3) Whole-class sharing of small-groups' findings.

Data: The reflection and inquiry activities served both research and learning purposes. Thus data consisted of copies of all of the PSSMT's written work for all of the activities. There were also field notes of their groups and whole-class discussions.

Analysis: The analysis began with open-ended coding (Strauss & Corbin, 1998) of the data. The researcher and a research assistant, working independently, coded the data from the pre-intervention (i.e., the self reflection) activities. The researcher and a different research assistant, who did not have access to the pre-intervention data, working independently, coded data from the post-intervention (i.e., inquiry and group reflection) activities. This allowed for cross checks by research team, elimination of initial assumptions/themes based on disconfirming evidence and validation of the findings. Coding involved, for example, identifying significant statements about their thinking of problems and problem solving and problem solving instruction. The coded information was categorized based on common themes and frequency of occurrence. Changes in the PSSMT's thinking resulting from the activities were determined by comparing the pre- and post-intervention coded information.

PEDAGOGICAL KNOWLEDGE OF PROBLEM SOLVING

The findings are presented in terms of the PSSMT's initial knowledge of problems and problem solving, the growth in knowledge resulting from the reflective-inquiry approach and the nature of their instructional knowledge of problem solving.

Initial knowledge: There was consistency in the nature of the PSSMT's initial knowledge about problems and problem solving. However, there were two categories that emerged as their dominant ways of thinking. Category 1 consisted of 83% of the PSSMT and category 2, 17%. 17% of the PSSMT displayed characteristics of both categories, but leaned much more towards category 1 and were thus included there.

Category 1: These PSSMT initially described a problem as something/situation that requires an answer or needs to be solved, or some variation of this, e.g., "Something, which requires an answer which requires a number of steps to find." Their examples of problems were routine or traditional word problems, e.g., for grade 9, "James is twice as old as Laura. The sum of their ages is 24. How old are they today?" and "A building is 9 meters tall and you are standing 12 meters from the base. At what angle do you have to look to see the top of the building?" In order to make the problem, they thought of the topic, mainly, "I thought of grade 8's doing percentages…" "I thought of frequencies/probabilities …" They viewed these as problems because they provided specific/key/some information to arrive at or help guide to answer, or because answer unknown/some unknown to discover/require answer. All but three of them were definite that theirs were good problems. One explained, "I think so because it is pretty straightforward. The numbers aren't too difficult to work with so they are able to focus on the concept rather than the calculation." Others mentioned: appropriate for the students, plenty information available, contributes to learning, deals with answer and process, allows for many solutions. For the exceptions, one said probably not because straightforward, another said not very interesting problem but still useful to practice application, another said okay problem but wording could be enhanced. The process they described to solve problems focused on identifying the known and unknown or identifying relevant and irrelevant information then trying to solve. For example: "I first read the problem carefully then I mark the known clearly. I then look at what is the unknown part of the problem. I then attempt to find the relationship between the known and the unknown." "First, I figure out what is being asked, then I go back through the question to see what is given, then I remember the process I need to take with the given information, and I follow the procedure." Flowcharts of the process included: (1) Read problem carefully → note known → note outcome → relate the known to the outcome if possible → solve; (2) what is the question → what is given → how do I solve → solve.

Category 2: These PSSMT initially described a problem as, e.g., "A question, a challenge, an opportunity for discovery, a search for an unknown." "To be interesting, should be a question of a type that one has not already learned how to answer." "A challenge, something unknown to some parties and possibly knowable, something which can contribute to knowledge about math." Their examples of problems were more process-oriented and included, for grade 8, a diagram of a map with 6 cities in the province, to find the route(s) that allow(s) a salesman to visit each city exactly once, and how might you find the minimum distance. For grade 12, "A box is to be constructed from a piece of cardboard that is square with squares cut from each corner of length x. When the cardboard is folded into the box, x becomes the height of the box. What length of x will give the area of the box a maximum area?" In order to make the problem, they thought of "curriculum for the grade; specific class ability; relate 'local' experience; open to multiple different methods/ techniques to solve." "Something that would be challenging, intriguing, something that would cause the students to think." They viewed these as problems because the answers were not immediately apparent/obvious and they required thought and a process of struggle to find a solution. They viewed them as good problems. One explained: "Because it requires: creativity to develop a solution; multiple methods to arrive at a solution; provides openings to other related problems; disciplined thought process required." The process they described to solve problems focused on, e.g., "First understand the goal(s); examine constraints; 'play' a bit with ideas that might lead to a solution; develop one or more of ideas into a solution." "Read the problem, understand what is being asked; recall what knowledge I have regarding the topic, decide on a strategy, draw picture, equation, etc. try to solve, check to see if answer is reasonable." Flowcharts for the process included: read problem → pick out important information → decide what is being asked → decide on a strategy → try to solve → check.

Growth in Knowledge: The inquiry activities and group reflection served as intervention for the PSSMT to become aware of aspects of problems and problem solving they had either taken for granted, not considered or not been exposed to.

Comparison of problems: This resulted in shift in awareness of problems for most of Category 1. For example: "Problems come in many different forms. They require some thinking on the part of the solver. They can have more than one possible outcome or solution." "The problems are of different types. They require different types of thinking to solve (i.e., logic, spatial, etc.)" "Problems require delving into your thought process and using your skills to sort through information and use that information to find solutions." "They … get students to ponder math in different formats." "Problems can be fun and challenging, but also stressful." From Category 2, "Problems are challenges that require an understanding and application of knowledge. Problems are solved using a variety of strategies and steps. They require thought and often more than one attempt to find a solution."

Group Summaries: The group summaries consisted of their collective thinking resulting from the group-reflection activities. The summaries reflected more depth/ scope in their understanding of problems and the problem solving process. For example, one group's description of a 'good' problem included: "different methods and techniques, focus on problem solving technique - not tedious calculation, students can relate to problem." Another group's: "Should make students think, be challenging." Their description of the problem-solving process was also enhanced, particularly in terms of the flowcharts, which showed the need to move back and forth as opposed to taking a linear path to a solution. The following example of these flowcharts is simplified to fit available space. They actually were drawn with appropriate boxes and arrows. Read the problem → Do you understand the question? [1] → no [arrow to read problem] [1]→ yes → draw a diagram → devise a strategy → does the strategy seem helpful [2]→ no [arrow to devise] [2]→ yes → carry on get an answer → check the answer → is your answer right [3]→ no [arrow to devise] [3] yes → Yay! The whole-class sharing allowed the PSSMT to further extend what their individual groups constructed.

Instructional Knowledge: The groups' responses to what their instructional approach for problem solving would involve focused on what the learner should do. For example, "students should read problem, write down information, determine what is relevant and irrelevant, think of ways to approach problem, write in sentence form." "Kids should learn to: understand the problem, pick out what is important, do not assume there is only one correct solution, relate the problem to what you know, but don't be afraid to try something new, do not worry if you can't see the whole solution at once." However, some groups also noted what the teacher should do. For example, "ask children different ways to do problem and identify wrong ways, demonstrate a couple of ways the children suggested, reflect on which was 'best' way, was there a 'best' way, does it make sense?"

DISCUSSION AND CONCLUSIONS

The initial knowledge of the PSSMT indicated that most of them made sense of problems in terms of the traditional, routine problems they had experienced, directly or indirectly, prior to entering the teacher education program. They also understood these as genuine problems that require thought and logic to arrive at a solution. They understood the problem-solving process in a way consistent with the traditional classroom way of dealing with these problems. This suggests the need for helping them to become aware of, and to expand, their initial views. The series of questions/ prompts were effective in allowing for more depth in their reflection. Their responses to each question revealed another dimension of their thinking of problems. This suggests the importance of providing more than one prompt to facilitate reflection.

The group activities also enhanced their learning. Collectively, they identified a set of characteristics about problems and problem solving with more depth/scope than individually. This was facilitated by each group, although created randomly, having at least one member from Category 2 and/or one who had some characteristics of Category 2. Each group was able to construct knowledge compatible with formal theory of problem solving. This allowed them to relate to theory in a more meaningful way. They were given theory to read following the reflective-inquiry approach, which they seemed to relate to and assimilate more meaningfully than students I worked with in the past who did not engage in the approach. However, they were unable to conceptualize problem-solving instruction from the approach on their own. Their instructional approach implied teaching by telling or being teacher directed. They did not seem to notice/consider the instructional approach they participated in through the reflective-inquiry approach as a basis of constructing/thinking about their own. It required shifting their perspective of the approach from that of a learner to teacher. This was done by allowing them to engage in reflection on the approach and role-play. Details are not provided here given limitation on space.

The paper provides information about PSSMT's initial knowledge and the type of knowledge they could construct on their own from particular self-inquiry activities. It highlights the need to explicitly address pedagogical problem-solving knowledge in teacher education. It suggests that it is essential to provide constructive and reflective opportunities followed by theory to deepen PSSMT's understanding of problem solving. It suggests the need for them to reflect on learning experiences not only from the perspective as learner, but also as teacher to construct meaningful pedagogical knowledge. It provides a structure/model to make sense of PSSMT's knowledge of problem solving and a practical and effective approach to facilitate their self-reflection and construction of meaningful knowledge about problem solving.

References

Ball, D.L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education*, 21, 132-144.

Bjuland, R. (2004). Student teachers' reflections on their learning process through problem solving in geometry, *Educational Studies in Mathematics,* 55(1-3), 199-225.

Charles, R. & Lester, F. (1982). *Teaching problem solving: What why & How*. Palo Alto, CA: Dale Seymour Publications.

Creswell, J.W. (1998). *Qualitative inquiry and research design.* London: Sage.

Dewey, J. (1916). *Democracy and education.* New York: Macmillan.

Even, R. (1993). Perspective secondary teachers and the function of concept. *Journal for Research in Mathematics Education*, 24 (2), 94-116.

Jaworski, B. & Gellert, U. (2003). Educating new mathematics teachers. In A. Bishop, M. Clements, C. Kietel, J. Kilpatrick, & F. Leung (Eds.) *Second International Hand- book of Mathematics Education*. The Netherlands: Kluwer Academic Publishers.

Kilpatrick, J., Swafford, J., and Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

Kinach, B. (2002). Understanding and learning-to-explain by representing mathematics, *Journal of Mathematics Teacher Education*, 5, 153-186.

Lave, J. & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation.* New York: Cambridge University Press.

Leikin, R. (2003). Problem solving preferences of mathematics teachers: Focusing on symmetry. *Journal of Mathematics Teachers Education,* 6(4), 297-329.

Lerman, S. (1997). The psychology of mathematics teachers' learning: In search of theory. In E. Pehkonen (Ed.), *Proceedings of PME* 23 (vol. 3, 200-207). Lahti, Finland: PME.

National Council of Teachers of Mathematics [NCTM]. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Person, A.C., Berenson, S.B., & Greenspan, P.J. (2004). The role of number in proportional reasoning: A prospective teacher's understanding. *Proceedings of PME 28* (vol. 4, 17-24). Bergen, Norway: PME.

Sánchez, V. & Llinares, S. (2003) Four student teachers' pedagogical reasoning on functions. *Journal of Mathematics Teachers Education*, 6(1) 5-25.

Sikula, J. (1996). *Handbook of research on teacher education*. NY: Macmillan.

Strauss, A. & Corbin, J.: (1998) *Basics of qualitative research: Techniques and procedures for developing grounded theory.* Thousand Oaks, California: Sage.

Van Doormen, W., Verschaffel, L., & Onghena, P. (2003). Preservice teachers' preferred strategies for solving Arithmetic and Algebra word problems. *Journal of Mathematics Teacher Education*, 6(1), 27-52.

Vygotsky, L. (1978). *Mind in society*. Cambridge, MA: Harvard University Press.

Wilson, M.R. (1994). One pre-service secondary teacher's understanding of function, *Journal for Research in Mathematics Education*, 25(4), 346-370.

REVISITING A THEORETICAL MODEL ON FRACTIONS: IMPLICATIONS FOR TEACHING AND RESEARCH

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Fractions are among the most complex mathematical concepts that children encounter in their years in primary education. One of the main factors contributing to this complexity is that fractions comprise a multifaceted notion encompassing five interrelated subconstructs (part-whole, ratio, operator, quotient and measure). During the early 1980s a theoretical model linking the five interpretations of fractions to the operations of fractions and problem solving was proposed. Since then no systematic attempt has been undertaken to provide empirical validity to this model. The present paper aimed to address this need, by analysing data of 646 fifth and sixth graders' performance on fractions using structural equation modelling. To a great extent, the analysis provided support to the assumptions of the model. Based on the findings, implications for teaching fractions and further research are drawn.

INTRODUCTION

Teaching and learning fractions has traditionally been problematic. In fact, it is well documented that fractions are among the most complex mathematical concepts that children encounter in their years in primary education (Newstead & Murray, 1998). It has also been asserted that learning fractions is probably one of the most serious obstacles to the mathematical maturation of children (Behr, Harel, Post & Lesh, 1993). During the last three decades researchers and scholars have identified several factors contributing to students' difficulties in learning fractions. In particular, it has been proposed that the obstacles that students encounter in developing deep understanding of fractions are either inherent to the nature of fractions or are due to the instructional approaches employed to teach fractions (Behr et al., 1993; Lamon, 1999). To date there is consensus among researchers that one of the predominant factors contributing to the complexities of teaching and learning fractions lies in the fact that fractions comprise a multifaceted construct (Brousseau, Brousseau & Warfield, 2004; Kieren, 1995; Lamon, 2001).

Kieren (1976) was the first to propose that the concept of fractions consists of several subconstructs and that understanding the general concept depends on gaining an understanding of each of these different meanings of fractions as well as of their confluence. Kieren initially identified four subconstructs of fractions: measure, ratio, quotient, and operator. In his original conceptualization, the notion of the part-whole relationship was considered the seedbed for the development of the other subconstructs; thereby he avoided identifying this concept as a separate, fifth, subconstruct claiming that this notion is embedded within all other subconstructs. In the following years, Behr, Lesh, Post and Silver (1983) further developed Kieren's

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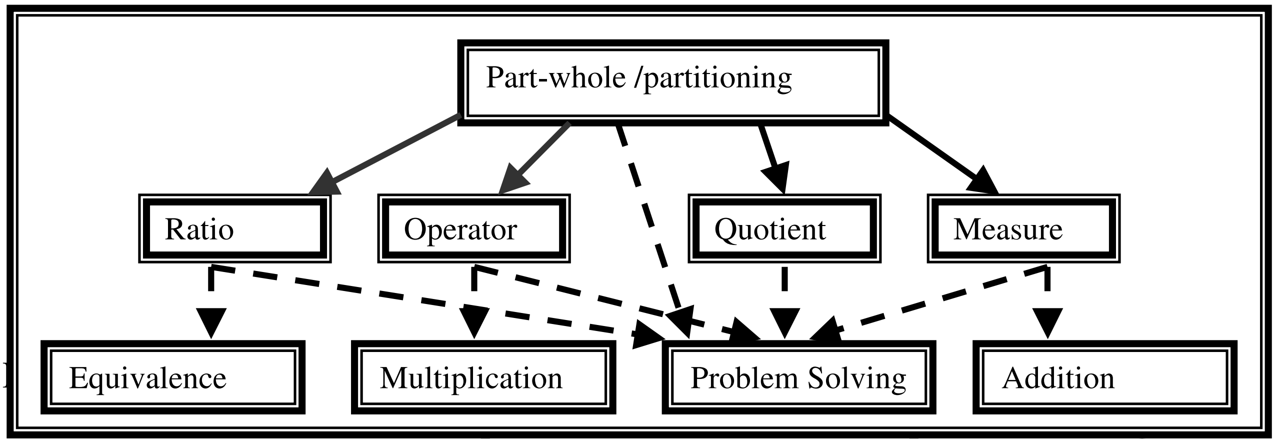
ideas recommending that the part-whole relationship comprise a distinct subconstruct of fractions. They also connected this subconstruct with the process of partitioning. Moving a step forward, they proposed a theoretical model linking the different interpretations of fractions to the basic operations of fractions and to problem solving (Figure 1). The solid arrows presented in this proposed model suggest established relationships among fractional constructs and operations, whereas the dashed arrows depict hypothesized relationships.



Closer examination of the diagram presented in Figure 1 reveals the following. First, the part-whole subconstruct of rational numbers, along with the process of partitioning, is considered fundamental for developing understanding of the four subordinate constructs of fractions. This assumption justifies why the part-whole notion has occupied the lion's share of curricula across different countries and has been the traditional inroad to introduce fractional concepts in primary grades (Baturo, 2004; Lamon, 2001). Second, the diagram suggests that the ratio subconstruct is considered as the most "natural" to promote the concept of equivalence and, subsequently, the process of finding equivalent fractions. Moreover, the operator and measure subconstructs are regarded as helpful for developing understanding of the multiplication and addition of fractions, respectively. Finally, understanding of all five subconstructs of fractions is considered a prerequisite for solving problems in the domain of fractions.

Though the model has been excessively cited in the following years (Carpenter, Fennema & Romberg, 1993), to the best of our knowledge, no systematic attempt has been undertaken since mid 1980s to provide empirical validity to the model. The present study aimed to address this theoretical and research deficiency. Specifically, the purpose of the study was to empirically test the five theoretical assumptions of the model alluded to above and investigate the extent to which any additional associations between the concepts and operations embedded in the model are empirically supported.

THE DEVELOPMENT OF THE TEST

A test on fractions was constructed guided by existing theory and research on rational numbers. An additional requirement in designing the test was its alignment with the curriculum that was operative in Cyprus, where the study was conducted. Table 1 presents the specification table that guided the construction of the test and the items used for examining students' performance on each of the concepts and operations included in the theoretical model. Items in bold letters represent problem-solving tasks related to each of the subconstructs of fractions, since it was decided to use problems related to these subconstructs, rather than general problems on fractions.

CONCEPTS	ITEMS	OPERATIONS	ITEMS
Part-whole /partitioning	1-8, 9	*Equivalence*	34-43
Ratio	10-14, *15*	*Additive operations*	44-46
Operator	16-18, 19	*Multiplicative operations*	47-50
Quotient	20-22, 23-24		
Measure	25-31, 32-33		

Table 1: Specification table of the test used in the study

The first five items of the part-whole subconstruct asked students to identify the fractions depicted in discrete or continuous representations. The remaining three items were associated with unitizing and reunitizing, which are directly related to the partitioning notion of the part-whole relationship (Baturo, 2004). The part-whole problem-solving task (item 9) asked students to reconstruct the whole given a part of it. The subsequent five items requested students to compare ratios, based either on quantitative (10-12) or qualitative information (13-14). Item 15 referred to boys and girls sharing different numbers of pizzas; an item frequently used in studies on ratios and proportions (Marshall, 1993). The following two items asked students to specify the output quantity of an operator machine given the input quantity and the fraction operator. Item 18 was related to the notion of operator as a composite function (Behr et al., 1993), whereas item 19 asked student to indicate the factor by which number 9 should be increased to become equal to 15. In line with previous studies (Lamon, 1999), the three subsequent items, which were related to the concept of quotient, examined students' ability to link a fraction to the division of two numbers; the two problems of this category were related to the partitive and quotitive interpretation of division (items 23-24, correspondingly). In accord with previous studies (Hannula, 2003; Lamon, 1999; Marshall, 1993), the items of the measure subconstruct examined students' performance on identifying fractions as numbers and locate them on number lines. The two problems of this category asked students to find a fraction that was within two given fractions, and identify among a number of fractions the one that was closer to number one. Finally, the remaining 17 items were associated with operations on fractions. Seven of these items (41-45 and 48-49) examined students' procedural fluency in these operations, whereas the remaining ten were related to the conceptual understanding of these operations (e.g., estimating the result of different operations on fractions).

METHODS

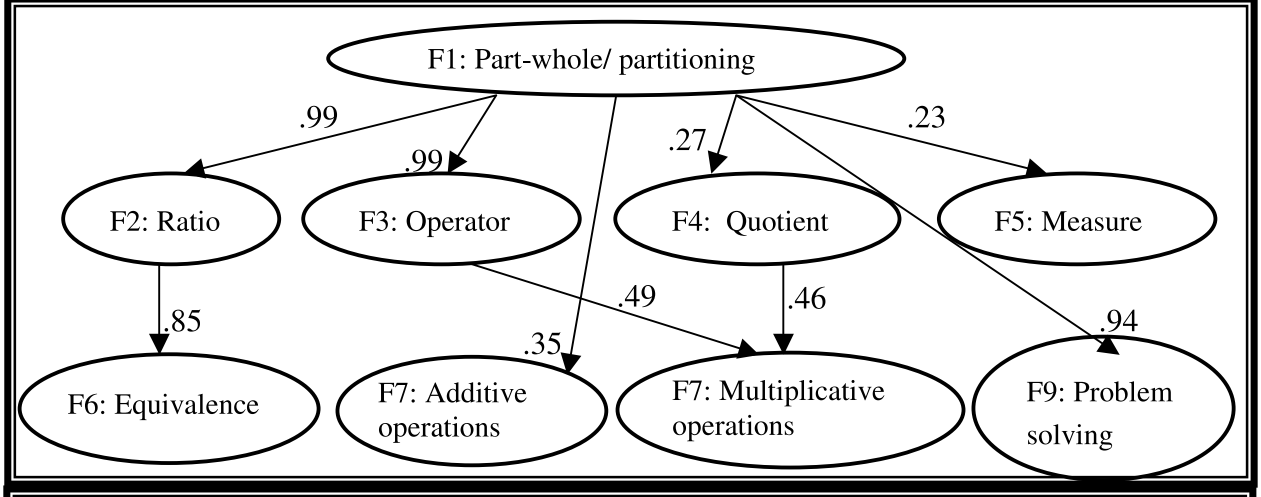
The items of the test were content validated by three experienced primary teachers and two university tutors of Mathematics Education. Based on their comments, minor amendments were made particularly where some terms used were considered as unfamiliar to primary pupils. The final version of the test (available on request) was administered to 340 5^{th} graders and 306 6^{th} graders (301 boys and 345 girls). The test items were split into two subtests which were administered to students during two consecutive schooldays; students had eighty minutes to work on each subtest.

Structural equation modeling and, specifically, maximum likelihood method, was used to test the hypotheses of the theoretical model (Kline, 1998). Goodness of fit of the data to the model was decided by using three fit indices: scaled x^2, Comparative Fit Index (CFI), and Root Mean Square Error of Approximation (RMSEA).

FINDINGS

The theoretical assumptions of the model were tested by using EQS. As reflected by the iterative summary, the goodness of fit statistics showed that the data did not fit the model very well (x^2=9110, df=2129, x^2/df=4.30, CFI=.57, and RMSEA=.07). Subsequent model tests revealed that the model fit indices could be improved by modifying the model in ways that on the whole were consonant with both theory on fractions and the development of the test. Specifically, items 18 and 23 were also linked to multiplicative operations, since both were solved by applying a multiplicative operation. Items 29-31 were also explained by the process of finding equivalent fractions, a relationship that could be attributed to the fact that the foregoing process provided scaffold in solving these tasks. The initial analysis also revealed that the associations of the four subordinate subconstructs of fractions with the problem solving were not significant; nor the association between the measure subconstruct and the additive operation of fractions. All these relationships were eliminated from subsequent analyses. On the contrary, the multiplicative operations of fractions were found to be associated with the quotient subconstruct, which can be explained taking into account that the preceding subconstruct is closely related to the division of fractions. The nine factor model that emerged after these modifications had a very good fit to the data (x^2=1892.55, df=1184, x^2/df=1.598, CFI=.95 and RMSEA=.030). Its goodness of fit was even better compared to a series of other models comprising of one to eight factors, thereby indicating that the emerging model was in alliance with parsimony principle (Kline, 1998). Figure 2 presents the model that emerged from the analysis; the loadings of each variable on the nine factors are presented below the model.

The following observations arise from Figure 2 and Table 1. First, all fifty items were correlated to the factors they were initially supposed to be loaded to, providing support to the construct validity of the test. However, beyond being associated with the measure notion, the three items that concerned locating fractions on number lines (items 29-31) were also related to finding equivalent fractions; their loadings in the latter case were much higher than in the former one. This finding can be partly attributed to the fact that in two of the three items the denominator of the fractions that students were requested to locate on number lines was a sub multiple of the spaces into which the given number lines were divided. Yet, one could also suggest that this finding points to the requirement that students master the equivalence of fractions in order to be able to manipulate number lines efficiently.



<u>EQUATIONS</u>*: *<u>Part-whole</u>*: V1=.25F1, V2=.46F1, V3=.28F1, V4=.26F1, V5=.24F1, V6=.42F1, V7=.52F1, V8=.38F1 *<u>Ratio:</u>* V10=.38F2, V11=.44F2, V12=.25F2, V13=.34F2, V14=.29F2, *<u>Operator</u>*: V16=.49F3, V17=.58F3, V18=.21F3+.37F8, *<u>Quotient</u>*: V20=.51F4, V21=.42F4, V22=.72F4, *<u>Measure</u>*: V25=.46F5, V26=.98F5, V27=.98F5, V28=.93F5, V29=.11F5+.32F6, V30=.11F5+.40F6, V31=.08F5+.42F6, *<u>Equivalence</u>*: V34=.40F6, V35=.76F6, V36=.72F6, V37=.76F6, V38=.81F6, V39=.74F6, V40=.72F6, V41=.52F6, V42=.43F6, V43=.50F6, *<u>Additive operations</u>*: V44=.62F7, V45=.34F7, V46=.61F7, *<u>Multiplicative operations</u>*: V47=.69F8, V48=.43F8, V49=.55F8, V50=.55F8, *<u>Problem solving</u>*: V9=.58F9, V15=.45F9, V19=.41F9, V23=.32F9+.22F8, V24=.59F9, V32=.43F9, V33=.50F9 (*errors were omitted).

Figure 2: Path model linking the five subconstructs of fractions to the operations of fractions and to problem solving

Second, the data provided support to the assumption that the part-whole interpretation of fractions and the partitioning process constitute a foundation for developing an understanding of the four subordinate interpretations of fractions. Specifically, Factor 1 explained about 98% of the variation of the factors related to the ratio and the operator personalities of fractions (given that the percentage of the variation explained is equal to the square of the regression coefficients presented in Figure 2). Yet, one cannot ignore the fact that Factor 1 explained a much smaller percentage of the variation of the quotient and the measure notion of fractions (about 5% and 8%, respectively). Third, the data provided empirical support to the hypothesis that mastering the notion of fractions as ratios contributes predominantly to finding equivalent fractions, since Factor 2 explained a great proportion of the variation of the sixth factor (about 73%). Developing an understanding of the operator subconstruct was also found to explain about a quarter of the variation of students' performance on the items associated with the multiplicative operations on fractions.

Fourth, the model that emerged deviated from the theoretical model in three aspects: (a) the effect of the measure subconstruct on the additive operations of fractions was not statistically significant; however, a statistically significant association between the part-whole subconstruct and the additive operations emerged; (b) the associations of the four subordinate notions of fractions with problem solving were not statistically significant; on the contrary, the part-whole relationship was found to explain a great percentage of the variation of problem solving; and (c) the quotient subconstruct of fractions was found to explain about 20% of the variation of students' performance on the items related to the multiplicative operations of fractions.

In general, the model of Figure 2 verified three of the five examined hypotheses: (a) the part-whole interpretation explained a proportion of the variation of the four subordinate subconstructs of fractions; (b) the ratio notion was associated with equivalence, and (c) the operator concept was linked to the multiplicative operations of fractions. Yet, two hypotheses failed to be empirically validated. In particular, the four subordinate notions of fractions were not statistically related to problem solving, nor was the measure subconstruct related to the additive operations of fractions. Nevertheless, the study supports two additional paths not included in the theoretical model: one linking the quotient subconstruct to the multiplicative operations and the other linking the part-whole relationship to the additive operations of fractions.

DISCUSSION

The findings of the study provide empirical support to the fundamental role of the part-whole subconstruct in building understanding of the remaining constructs of fractions; thereby, they justify the traditional instructional approach in using this notion as the inroad to teaching fractions (Baturo, 2004; Kieren, 1995; Marshall, 1993). However, one cannot overlook the fact that the part-whole interpretation of fractions explains different percentages of the variation of the four subordinate concepts of fractions: almost all the variance of the ratio and the operation subconstructs and only a very small proportion of the variance of the measure and the quotient concepts. Three reasons seem to explain this finding. First, core ideas, such as comparing quantities, are embedded in all three former subconstructs of fractions, whereas they are not required for developing understanding of the latter couple of subconstructs (Lamon, 1999). Second, the measure and the quotient interpretations of fractions could be explained by other concepts not included in the model, such as the notion of the unit fraction, which is considered as contributing predominantly to building meaning for these subconstructs (Behr et al., 1983; Marshall, 1993). And third, students might encounter significant difficulties in gaining an insight into the concepts of measure and quotient, which cannot be surpassed even by developing an understanding of the part-whole "personality" of fractions. Whatever the reason is, this finding validates the claim that, the part-whole interpretation of fractions should be considered as a necessary but not sufficient condition for developing an understanding of the remaining notions of fractions (Baturo & Cooper, 1999; Brousseau et al., 2004). Thereby, though the findings of the study justify the preponderance of the part-whole interpretation of fractions in teaching rational numbers, they also underline the need for emphasizing the other subconstructs of fractions, and especially those that are not so highly related to the foregoing notion.

The study also provides support to the assumption that mastering the five interpretations of fractions contributes towards acquiring proficiency in the operations of fractions. This finding can be partly attributed to the fact that the items used for measuring students' performance on the operations of fractions required both procedural fluency and conceptual understanding of these operations; yet it also spotlights that when teaching fractions, teachers need to scaffold students to develop a profound understanding of the different interpretations of fractions, since such an understanding could also offer to uplift students' performance in tasks related to the operations of fractions. Thereby, instead of rushing to provide students with different algorithms to execute operations on fractions, the findings of the present study, in accordance with previous studies (Lamon, 1999; Brousseau et al., 2004) lend themselves to support that teachers should place more emphasis on the conceptual understanding of fractions. The study also suggests that, the teaching of the different operations of fractions should be directly linked to specific interpretations of fractions. In particular, the findings of study indicate that the teaching of equivalent fractions could be reinforced by learning ratios, whereas the operator and the quotient subconstructs could support developing a conceptual understanding of the multiplicative operations on fractions. Likewise, the associations between the part-whole interpretation and the additive operations of fractions should be highlighted during instruction, in order to promote learning of the latter processes.

Finally, one cannot ignore the fact that only a very small percentage of the variation of items related to number lines was explained by the measure subconstruct of fractions; this finding supports Lamon's (1999) recommendation that researchers employ items beyond locating fractions on number lines to measure students' understanding of fractions. In alignment with previous studies, it also suggests that the number line comprises a difficult model for students to manipulate (Baturo & Cooper, 1999), and that teachers should help students master other notions (such as the equivalence of fractions), before rushing to introduce this model in their teaching.

It goes without saying that further research is needed to cross-validate the model emerged in this study. Specifically, studies could follow at least three different directions. First, the relationships included in the theoretical model and were verified in the present study need to be further examined. Second, further studies need to verify the modifications introduced in the theoretical model, and especially the fact that only the part-whole relationship was found to be associated with problem solving. Finally, provided that the associations among the five subconstructs of fractions were verified, future research could also be directed towards identifying core ideas that permeate the whole domain of fractions and offer significantly to building understanding of all the five subconstructs included in the theoretical model.

References

Baturo, A.R. (2004). Empowering Andrea to help year-5 students construct fraction understanding. In M. J. Høines & A. Fuglestad (Eds.), *Proceedings of the 28th PME Conference, 2* (pp.95-102), Bergen: Bergen University College.

Baturo, A.R., & Cooper, T.J. (1999). Fractions, Reunitisation and the Number-Line Representation. In O. Zaslavsky (Ed.), *Proceedings of the 23rd PME Conference, 2* (pp.81-88). Haifa: Israel Institute of Technology.

Behr, M., Harel, G., Post, T. & Lesh, R. (1993). Rational Numbers: Toward a Semantic Analysis-Emphasis on the Operator Construct. In T. P. Carpenter, E. Fennema, & T.A. Romberg, (Eds.), *Rational Numbers: An Integration of Research* (pp. 13-47). NJ: Lawrence Erlbaum

Behr, M., Lesh, R., Post, T., & Silver E. (1983). Rational Number Concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of Mathematics Concepts and Processes*, (pp. 91-125). New York: Academic Press.

Bentler, P.M. (1995). *EQS-Structural Equations Program Manual.* Encino, CA: Multivariate Software, Inc.

Brousseau, G., Brousseau, N., & Warfield, V. (2004). Rationals and decimals as required in the school curriculum. Part 1: Rationals as measurements. *Journal of Mathematical Behavior, 23,* 1-20.

Carpenter, T.P., Fennema, E., & Romberg, T.A. (1993). Toward a Unified Discipline of Scientific Inquiry. In T. P. Carpenter, E. Fennema, & T. A. Romberg. (Eds.), *Rational Numbers: An Integration of Research* (pp. 1-11). NJ: Lawrence Erlbaum Associates.

Hannula, M. S. (2003). Locating Fraction on a Number Line. In N. A. Pateman, B. J. Dougherty & J. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of the PME and PMENA, 3* (pp. 17-24). Hawaii: CRDG, College of Education, University of Hawaii.

Kieren, T.E. (1995). Creating Spaces for Learning Fractions. In J. T. Sowder and B. P. Schappelle (Eds.), *Providing a Foundation for Teaching Mathematics in the Middle Grades,* (pp. 31-66). Albany: State University of New York Press.

Kline, R.H. (1998). *Principles and Practice of Structural Equation Modeling.* London: Gilford.

Kieren, T.E. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. In R. Lesh (Ed.), *Number and Measurement: Papers from a Research Workshop* (pp. 101-144). Columbus, OH: ERIC/SMEAC.

Lamon, S.L. (2001). Presenting and Representing: From Fractions to Rational Numbers. In A. Cuoco & F. Curcio (Eds.), *The Roles Of Representations in School Mathematics-2001 Yearbook* (pp. 146-165). Reston: NCTM.

Lamon, S.J. (1999). *Teaching Fractions and Ratios for Understanding.* New Jersey: Lawrence Erlbaum Associates.

Marshall, S.P. (1993). Assessment of Rational Number Understanding: A Schema-Based Approach. In T.P. Carpenter, E. Fennema, & T.A. Romberg (Eds.), *Rational Numbers: An Integration of Research*, (pp. 261-288). New Jersey: Lawrence Erlbaum Associates.

STUDENTS' REFLECTION ON THEIR SOCIOMATHEMATICAL SMALL-GROUP INTERACTION: A CASE STUDY

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In this paper we present a case study of a small group of two 11 years old students who participated in a research program whose the purpose was to investigate the way that students can be actively involved in a reform of their own behavior as they cooperate in small-groups to solve mathematical problems. We study the opportunities that were offered for the development of the small-group students' interaction in mathematics in two alternative environments: a) the students' observation and discussion on their videotaped cooperation and b) their participation in dramatic role-play. The results of the research showed that both environments gave the group members the opportunity to reflect on their actions and the consequences of their actions during their cooperation and to achieve the development of new effective social rules.

THEORETICAL BACKGROUND

The investigation on social interaction that takes place in classrooms' microculture continues to be an issue of great interest among the mathematics educators researchers. This is a consequence of the acceptance that in order to make sense of students' learning of mathematics, classroom life has to be interpreted not only from a psychological perspective but from a sociological perspective as well (Cobb & Bauersfeld, 1995; Lerman, 2001). Towards this effort many researchers have developed theoretical constructs for the study of the relation between student's cognitive development and social interactions in the classroom. For example, interpretative constructs for this purpose are the social and sociomathematical norms (Yackel & Cobb, 1996), the thematic patterns of interaction (Voigt, 1995), the meta-discursive rules (Sfard, 2001). It is widely accepted that the way that the members of the classroom develop rules that guide their social behavior determine the evolution of their mathematical discourse. Moreover, there is a reflexive relationship between the sociomathematical interaction and students' beliefs and values about their own role, others' role, the general nature and the goals of mathematical activity (Yackel et al., 2000).

In this tradition the research has mainly focused on teacher's role of initiating and guiding the formation of the rules of sociomathematical interaction (McClain & Cobb, 2001). However, little research has been done on students' role in the development of their social behavior in mathematics classroom (Hershkowitz & Schwarz, 1999). The investigation of the role of different environments that give opportunities for students' reflection on their mathematical discourse is a critical question.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 241-248. Melbourne: PME.

In this paper we present a case study of a small group of two 11-year-old students who participated in a broader research program whose purpose was to investigate the way that students can be actively involved in a reform of their own behavior as they cooperate in small-groups to solve mathematical problems. More specifically, we study the opportunities that were offered for the development of small-group students' interaction in mathematics in two alternative environments: a) the students' observation and discussion on their videotaped cooperation and b) their participation in dramatic role-play.

METHOD

The two students were participated in a research program realized in a fifth grade of a typical public school of Athens in autumn of 2003, which lasted four months. Initially, in order to construct the students' profile, they were interviewed about their beliefs for their own role, others' role, the general nature and the goals of mathematical activity. Furthermore, we recorded their parents' beliefs about the mathematical activity of their children in school as well as in home. During their mathematical activity in the classroom, the two students worked in group and their cooperation was videotaped once a week. The mathematical topic they discussed during the research program concerned the concept and the operations of fractions. After a session of cooperation the members of the group participated in a meeting with the researcher. During this meeting, the students observed and discussed on issues concerning their videotaped cooperation. These discussions were tape-recorded. Moreover, the students of the group were obliged to organize and to present drama role-plays in the classroom based on the experiences of their cooperation. These role-plays were videotaped. At the end of the program, the members of the group were interviewed about their own role and the others' role in mathematics. So, the data consisted of the videotaped recordings of the small-group's work in mathematical lessons, the tape recorded students' discussions about their own videotaped cooperation, the videotaped recordings of the students' role-play and the protocols of tape recorded clinical interviews conducted with each student at the beginning and at the end of the program.

The discourse analysis of the group's engagement in classroom mathematical activities was based on interactivity flowcharts that Kieran and Sfard have developed (Sfard & Kieran, 2001). The mathematical discussion of the group was analyzed according to the way that the members negotiated their mathematical activity (who offered the solution, what kind of solution offered, how explained their thinking, how every member of the group was influenced by the other, etc.). The tape recorded students' discussions about their own videotaped mathematical cooperation were analyzed according to: a) the way that the students assessed their cooperation, b) the critical moments of their interaction and c) the targets they put for their next cooperation. The role-plays were analyzed according to: a) the roles that the students chose to play, b) the relationship between drama text and their cooperation in mathematics and c) their comments for this experience.

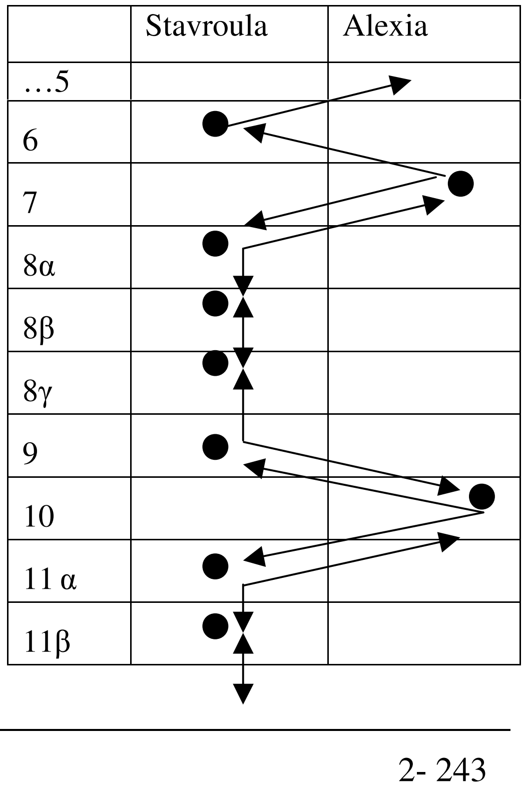
RESULTS

We chose to present the work of this group (Stavroula and Alexia) because these two students had different beliefs about the role of cooperation in mathematics and they had different capacities on this lesson. Firstly, we present the students' profile before their cooperation and then the development of their reflection on their cooperation in mathematics through the two alternative environments.

The children's profile

Both students had developed their beliefs in a traditional context of mathematics teaching. The goals posed by both students for their mathematical activity concerned the result of their effort (right or wrong) and not the process. Nevertheless, they had different conceptions about the role of cooperation in mathematics. Stavroula considered the cooperation to be an obstacle in the understanding of mathematics, because she believed that *"if someone doesn't work on his own, he cannot understand mathematics"*. On contrary, Alexia believed that cooperation could help her to control her thoughts before she announced them in the classroom and so she could *"avoid mistakes"*. Moreover, we should mention that the students' parents attributed to the cooperation in school mathematics a social role and not a cognitive one, that is they conceived the cooperation as a means for students' socialization. As for the two students abilities in mathematics, Stavroula was a student that managed to find solutions on mathematical problems on her own and Alexia was a student that, most of the times, need some help to complete a mathematical activity.

At the beginning of their cooperation Stavroula and Alexia worked individually and they didn't negotiate their ideas. Most of Stavroula's utterances were addressed to herself, revealing, this way, a private discourse and very few utterances indicated a challenge for reaction from her interlocutor's part. The few utterances of Alexia had mostly the form of a challenge for reaction from Stavroula's part and were related to her effort to understand her classmate's solution. Stavroula was guiding the dialogues that were developed by presenting her solution to Alexia without arguments or explanations about it, while Alexia didn't challenge her classmate to explain her solutions. The interactivity flowcharts of their initial cooperation had a form like the next one.



Students' reflection as they observed their videotaped cooperation

Concerning the way that the two students assessed their cooperation, we can notice that the two students experienced it in different ways. Initially, Stavroula assessed the evolution of their cooperation mostly based on the solution (wrong or right) of their mathematical activity. On the contrary, Alexia was based on the type of their interaction, that is if her interlocutor gave her some help. However, both of them had an awareness of the quality of their cooperation (productive or not). For example, after an unproductive cooperation the students commented:

Stavroula: I wanted to write on my own as I was used to, but afterwards I thought that we must cooperate and so sometimes we discussed.

Alexia: We tried to cooperate, I asked Stavroula to discuss the problem, but we sometimes managed it.

The critical moments of the cooperation that they gave the group members the opportunity to reflect on their actions were related to the existence of conflicts. These conflicts were connected to: a) the existence of different ideas and the failure of investigating them and b) the type of explanations that each member offered and the lack of understanding from the partner.

For example, at the beginning of the program, the children discussed about their cooperation in which they had to solve the following problem: In Alexandra's Avenue, public works are being made by 3 different firm of constructors. The works are being made at three different points. The first firm of constructors makes works at a point corresponding to the 1/3 of the avenue, if we count from its beginning. In the ¾ of the avenue there are works of the second firm of constructors and in the 5/6 of the avenue there are works of the third firm of constructors. Note in the following schema where the works are being made. Use red color for the first point, green for the second one and blue for the third one .

beginning ____ end

When they observed their videotaped cooperation, they had the opportunity to reflect on their failure to negotiate Alexia's idea:

Researcher: Did you have different ideas about the solution of the problem?

Stavroula: I said to count with a rule and to put centimeters, but Alexia said to divide it in small pieces.

Researcher: What did you do after your conflict?

Stavroula: I tried to do what I said.

Alexia: Me too, I tried to do what Stavroula said, but I did not manage it.

Researcher: Did you discuss your different ideas, let's say who had right and why?

Alexia: No.

Researcher: Let's observe at the video the solution given in the class… The solution at the blackboard with whose idea does it matche? Stavroula's or Alexia's?

Stavroula: With Alexia's idea…

Observing another video of their cooperation, the students had the opportunity to reflect on the value of explanation for the development of a productive mathematical discussion. For example, the following dialogue took place:

Researcher: How could you understand Alexia's thinking?

Stavroula: She had to explain to me her solution better.

Researcher: Alexia, did you try to solve the problem as Stavroula proposed and you said that you didn't manage it?

Alexia: I had not understand what she said.

Researcher. What could you do then?

Alexia: I could ask her to explain to me her solution again.

About the way by which the two students put goals for the evolution of their next cooperation, we observed the following: At the beginning, the goals posed by the students were common and general (e.g. "*to cooperate more", "to solve the problem together*"). Afterwards, their goals concerned concrete actions that they were addressed to their interlocutor (e.g. *"Stavroula must explain to me her solution*"). Finally, the goals became personal and concerned their own actions about their interaction (e.g. *"I have to think more about what Alexia wants", "I have to listen what Stavroula says").* At this phase, the goals reveal the mutual responsibility that the students managed to develop concerning their cooperation in mathematics.

Students' reflection as they participated in dramatic role-play

Alexia and Stavroula chose to represent a discussion between two students in the classroom, as they tried to solve a problem that was difficult for the one student. The scenario that they designed and played was the following:

[1]Alexia: A fruit-bowl contained 21 apples. George ate 2/3 of the apples. How many apples did they remain?

[2]Stavroula: Ah! It seems difficult!

[3]Alexia: Think again about it. It is easy.

[4]Stavroula: Help me a little.

[5]Alexia: What do you mean "a little"?

[6]Stavroula: Such a little! (She shows with her hands.)

[7]Alexia: If I help you such a little, the problem will be solved by myself and not by yourself !

[8]Stavroula: It doesn't matter at all!

[9]Alexia: It doesn't matter at all? It matters a lot, because you will not learn it.

[10]Stavroula: Oh! You talk like my mother! She told me the same things.

[11]Alexia: She has right! You should solve it alone.

[12]Stavroula: Come here now!

[13] Alexia: What do you want?

[14]Stavroula: Do you want to solve it together, to discuss about it?

[15] Alexia: OK.

[16]Stavroula: How did you solve it?

[17] Alexia: Look here, we can divide the apples in three parts.

[18]Stavroula: Ah! Three times seven …21, every part has 7 apples.

[19] Alexia: Yes, what about the 2/3?

[20] Stavroula: 7 plus 7… Ah! It ill be remained 7.

[21] Alexia: OK.

The above scenario developed in two scenes: at the first scene [1-13] the students chose to represent difficult moments of their cooperation and at the second scene [14-21] represented a productive cooperation. Concerning the choice of the roles, we should mention that Alexia and Stavroula decided to play the opposite roles in relation to those that they experienced at their cooperation in mathematics. Alexia played the student who managed to solve the problem alone and Stavroula chose to play the student who need help. Their experiences during their cooperation were impressed on their play. More specifically, Stavroula's belief that mathematical learning is only an individual process was mentioned by Alexia at the phrases [7, 9, 11]. The continual efforts of Alexia to challenge Stavroula's cooperation printed on Stavroula's phrases [4,8]. The change of Alexia's behavior during the dramatic play prints the evolution of their cooperation in mathematics. Moreover, an interesting fact is the comments made by the students for their parents' beliefs about their mathematical activity [10, 11].

The following discussion took place in the classroom after the dramatic play:

[1] Researcher: Do you want to talk about the roles? How do you feel about your role?

[2] Alexia: I don't think that I am egoist because I finally helped Stavroula. I felt nice because I helped her, I did not solve her the problem, I just helped her.

[3] Stavroula: I felt a little upset at the beginning, when I asked her to help me. When I persuade her to discuss, I felt nice.

[4]Researcher: Very good. Who want to talk about the cooperation that your classmates showed to us?

[5]Student 1: At the point where she told her "solve it alone", she felt upset. Then they began to discuss and they solved the problem together, it was good. I said Kostas the same thing. (Kostas was his partner)

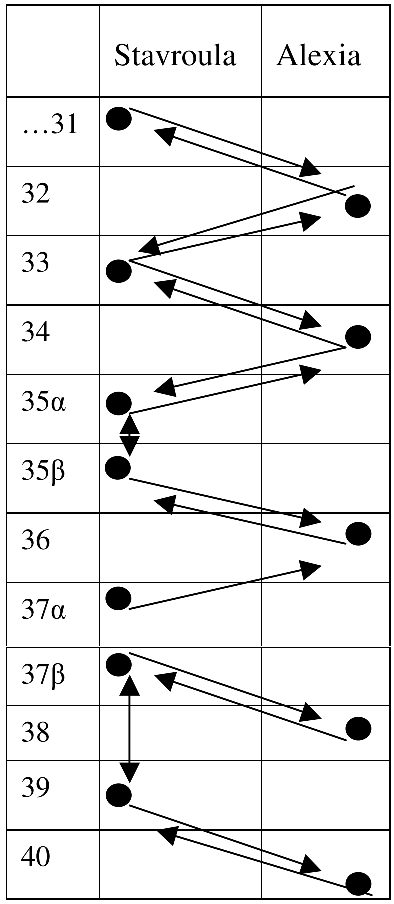
[6] Stavroula: Do you want to explain why I chose this role?

[7] Researcher: Yes.

[8] Stavroula: I chose this role because I usually solve the problems quickly and then I help Alexia, but I did not know how it is if someone does not understand the problem.

This environment gave the students the opportunity to think and to express their feelings about the social actions that take place by themselves as well as by their partners [2,3]. The presentation of the dramatic play in the classroom gave the opportunity to the other members of the classroom to reflect on their own behavior during their cooperation in mathematics.

During the last month of their cooperation, the two students developed productive cooperation as, most of the time, each interlocutor challenged the other's participation. Most of Stavroula's utterances were addressed to her interlocutor. These proactions had mostly the form of questions (request for approval of a suggested mathematical action, request for explanation). Furthermore, she seems to take account of Alexia's reactions in several moments of their discussion. Alexia participated more actively, she did not only tried to understand Stavroula's solutions, but many of her utterances related to the production of a solution and not to the request of an explanation. The interactivity flowcharts of their cooperation had a form like the next one.



Based on the previous analysis, we noticed that the members of this group formed social rules that allowed the development of productive cooperation in mathematics. More specifically, the students explain their thoughts without prompting, they try to make sense to their interlocutor's explanations and justifications, to express their disagreements and to share the responsibility of their actions.

At the final interview about their beliefs for mathematical cooperation, the students said: "It is beautiful to cooperate in mathematics, I can listen other opinions, sometimes better from mine, I don't feel alone when we have to solve a difficult problem…" (Stavroula), " I like to work together because everyone talks about its opinion and we can find a better solution …" (Alexia).

CONCLUSIONS

These environments gave the group members the opportunity to reflect on their cooperation, to evaluate it and to pose goals about the improvement of their mathematical discussion. Although the two students had considerable differences in their beliefs about the role of cooperation in mathematics, they managed to achieve the development of new effective social rules. Both environments (observation of their videotaped cooperation and dramatic role-play) allowed them to reflect on their actions and on the consequences of their actions and to feel the necessity of new rules in their cooperation.

The first environment gave them the opportunity to focus their attention on these moments that were obstacles for the development of a productive cooperation (the consequences of the non exploitation of an effective idea and the lack of understanding of partners' explanation of the mathematical solutions). The second environment offered them the opportunity to experience the role of the other member of the group, to experience the whole history of their cooperation and to express their thinking about it without prompting in front of the other members of the class.

An open question that arises from this research concerns the way that these environments which promote students' reflection on their sociomathematical interaction can be incorporated in teaching practice of mathematics.

References

Cobb, P. & Bauersfeld, H. (1995). Introduction: The coordination of psychological and sociological perspectives in Mathematics Education. In P. Cobb and H. Bauersfeld (eds.), *The emergence of mathematical meaning: Interaction in Classroom Cultures*, pp. 1-16. NJ: LEA.

Hershkowitz, R. & Schwarz, B. (1999). The Emergent Perspective in Rich Learning Environments: Some Roles of Tools and Activities in the Construction of Sociomathematical Norms. *Educational Studies in Mathematics*, 39, 149-166.

Lerman, S. (2001). Cultural, Discursive Psychology: A Sociocultural Approach to Studying the Teaching and Learning of Mathematics. *Educational Studies in Mathematics*, 46, 87-113.

McClain, K. & Cobb, P. (2001). An Analysis of Development of Sociomathematical Norms in One First –Grade Classroom. *Journal for Research in Mathematics Education,* 32(3), 236-266.

Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics*, 46, 13-57.

Sfard, A. & Kieran, C. (2001) Cognition as communication, Rethinking learning by talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture and Activity*, 8(1), 42-76.

Voigt, J. (1995). Thematic Patterns of Interaction and Sociomathematical Norms. In P. Cobb & H. Bauersfeld (eds.), *The emergence of mathematical meaning: Interaction in Classroom Cultures,* pp.163-201. LEA.

Yackel, E. $ Cobb, P. (1996). Sociomathematical norms, argumentation and autonomy in mathematics. *Journal for Research in Mathematics Education,* 27, 458-477.

Yackel, E., Rasmusen.C & King, K. (2000). Social and sociomathematical norms in an advanced undergraduate mathematics course. *Journal of Mathematical Behavior*, 19, 275-287.

INVESTIGATING TEACHERS' RESPONSES TO STUDENT MISCONCEPTIONS

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As part of a study investigating primary/elementary teachers' pedagogical content knowledge teachers were asked to describe how they would respond to some hypothetical situations involving student misconceptions and errors. The nature of the task in which the error arose influenced teachers' emphasis on procedural or conceptual aspects, and teachers' responses revealed aspects of their pedagogical content knowledge. The usefulness of a questionnaire and interview approach for investigating these issues is also discussed.

BACKGROUND TO THE STUDY

Teachers' mathematical pedagogical content knowledge (PCK)—or knowledge for teaching mathematics—has received considerable attention since the mid-1980s (e.g., Shulman, 1986). One of the key components of PCK is knowledge about student misconceptions. As part of a larger study investigating PCK and its influence on instructional practice and student learning, this paper examines the self-reported practices of year 5 and 6 teachers in addressing certain typical misconceptions. It focuses on their strategies and the pedagogical and mathematical knowledge revealed in their responses. It also considers the effectiveness of using a questionnaire and interview to investigate PCK.

Dealing with student misconceptions

An understanding of common student misconceptions, and effective strategies to help students avoid them, is an important aspect of mathematical PCK (Graeber, 1999). In addition to trying to teach in such a way that students avoid misconceptions, teachers must also have approaches for dealing with those that inevitably arise. Once the misconception is recognised teachers must decide what strategies they can use. If reteaching occurs then decisions must be made about what to emphasise and how. Cognitive conflict is another strategy, in which students encounter a situation that contradicts their initial understanding in the hope they will then re-evaluate those beliefs (see Watson, 2002 for a good overview of the literature).

In addressing student misconceptions teachers' approaches may focus on procedural or conceptual aspects. Hiebert and Lefevre (1986) distinguish conceptual knowledge as being rich in relationships, with procedural knowledge having an emphasis on symbolic representation and algorithms. Most curriculum documents place an emphasis on both. One of the challenges in teaching is to address both aspects, a problem exacerbated if teachers lack conceptual fluency in key topics (Chick, 2003).

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Investigating teachers' pedagogical content knowledge

Investigating teachers' PCK and the ways in which it is used by them has proved challenging, because PCK comes into its own in the classroom. Since classrooms vary from moment to moment and are thus different from one another in a myriad ways (as shown by Lampert, 2001) observing two teachers in their own classrooms can show their PCK only for the situations that arise for them, and so comparisons are not easily made. For large scale studies the resources required for an extensive examination of PCK are significant (see, e.g., McDonough & Clarke, 2002, who look at 6 teachers on a single topic). Some researchers have tried to set up a single consistent scenario (e.g., van der Valk & Broekman, 1999), by asking teachers to prepare a lesson plan on a particular topic. This at least makes it possible to investigate many teachers' PCK for that topic, but requires a considerable investment for multiple topics. So, in addition to examining the way teachers address student misconceptions and PCK *per se*, this study also asks if there is a way to investigate teachers' PCK for multiple teachers over multiple topics.

METHODOLOGY

Participants and Procedure

Nine Australian teachers took part in the study. At the time of the study these teachers were teaching Grade 5 or 6 (students 10-12 years old). They had 2 to 22 years of teaching experience, but not all of it in Grade 5 or 6 classrooms.

As part of the larger study on pedagogical content knowledge participants completed a questionnaire and were then interviewed about their written responses. The questionnaire comprised 17 items examining mathematics teaching situations and beliefs, and participants completed it without restrictions on time or resources. The completed questionnaire was returned to the researchers who prepared questions for the interview, partly to clarify ambiguities or omissions in the written responses. Interviews thus varied among teachers, although there were also some common questions. Teachers' responses to four of the items from the questionnaire, together with their follow-up comments from the interview, are the focus of this paper.

The four items were designed to explore how teachers respond to student errors or misconceptions, with a focus on the subtraction algorithm, the division algorithm, fraction addition, and the relationship between area and perimeter. Each item showed work from a hypothetical student, and invited teachers to indicate what they would do in response. The subtraction and the area/perimeter items are shown in Figures 1 and 2. In the division item the hypothetical student disregarded the place value associated with any zeros in the dividend. For the fraction item (7/10 + 2/5) the student correctly obtained an equivalent fraction for 2/5, but the resulting addition of 7/10 + 4/10 was carried out by adding numerators and adding denominators. For this item there were direct questions about what teachers thought the student did and did not understand. The researchers regarded the misconceptions as being archetypal, and expected that most of the teachers would have encountered at least some of them either in teacher training or in actual classroom practice.

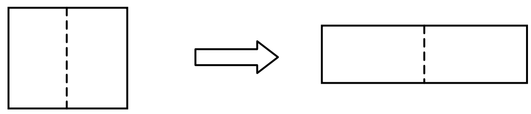
You notice a student working on these subtraction problems:

$$\begin{array}{r} 438 \\ -172 \\ \hline 346 \end{array} \qquad \begin{array}{r} 5819 \\ -2673 \\ \hline 3266 \end{array}$$

What would you do to help this student?

Figure 1. The subtraction item.

A student cuts the following shape in half to make a new shape, saying that the two shapes have the same area and perimeter:



What would you say to this student?

Figure 2. The area/perimeter item.

Analysis

The data were analysed to identify what teachers said they would do in response to the students' work, to identify whether their responses to students were predominantly procedurally or conceptually based, and finally to identify differences in displayed pedagogical content knowledge. The written responses were analysed first, with additional data provided from the interviews.

Strategies used by teachers in their responses to students are shown in Table 1; these categories were refined as the researchers identified common themes from the data.

Category	Definition
Re-explain	Explaining or re-explaining any part of either the concept or procedure.
Cognitive Conflict	Setting up a situation in which the student might identify a fundamental mathematical contradiction between the original response and the new situation, thus encouraging the student to re-evaluate the erroneous approach.
Probes student thinking	Asking the student to explain working or thinking, either to discover what the student is thinking to help the teacher decide what to do next, or to get the student to see the error. [It was not always possible from the data to establish which of these the teacher intended, so no distinction was made.]
Other	Any strategy not clearly in the above categories, e.g., "use simpler examples".

Table 1. Categories of teacher strategies in response to student misconceptions.

Where teachers gave an explanation of how to do the student's problem correctly, this explanation was further categorised as conceptual or procedural or both. To be judged as conceptual the response had to have clear reference to underlying mathematical principles, as opposed to merely giving a recipe for a procedure without justification. Finally, the participants' responses were examined more closely to identify aspects of pedagogical content knowledge evident in their explanations. Teachers' names have been changed for reporting results.

RESULTS

Teachers' approaches to dealing with misconceptions

Table 2 shows the strategies used by teachers in response to the student misconceptions. Re-explain was the most common strategy, although less common for the area/perimeter item. Teachers only suggested probing student thinking in response to the subtraction and division items, usually asking students to say their procedure aloud. Cognitive conflict was a strategy for all items, but the methods teachers used to evoke cognitive conflict varied between the items. For example, using diagrams or materials to compare the sizes of the fractions and demonstrate the unreasonableness of the student's result was common in the fraction addition item, while using inverse operations or a calculator to check answers was common for the subtraction and division items. In the area and perimeter item five teachers said they would ask students to "measure and check" the perimeters of the two shapes to establish the cognitive conflict. A general strategy proposed across items to address misconceptions was to consider simpler examples.

Item	Re-explain*	Cognitive conflict*	Probes student thinking*	Other*
Subtraction	9	2 (2)	1 (3)	1 (1)
Division	8	5 (2)	1 (2)	2 (1)
Fraction addition	7	4 (1)	0	1
Area/perimeter†	4	5 (2)	0	1

*Numbers in brackets refer to responses added in the interview that were different from the original questionnaire response.
† One teacher believed the misconception, so did not offer any strategy for this item.

Table 2. Numbers of teachers using particular strategies in response to misconceptions. Teachers may have used more than one approach.

Nature of teachers' explanations

Table 3 shows the number of teachers whose explanations were categorised as procedural or conceptual in response to student misconceptions. The type of response seemed to depend on the item. Teachers were more likely to respond to the subtraction and division items with a purely procedural explanation, as evident in this response to the subtraction item: "Break this problem down. Treat it as 3 separate or 4 separate subtractions to begin with" [Brian]. In contrast, purely conceptual explanations were only given in response to the fraction addition and, as shown here, the area/perimeter items: "For the first shape the 2 rectangles share a long side, but in the second only a small side and therefore the distance around the outside would be different" [Clare]. Most explanations combined procedural and conceptual aspects, such as the following response to the subtraction item, which emphasised the importance of conceptual understanding in supporting the procedure:

> [After demonstrating "borrow[ing] a ten", and saying that "1 ten and 2 ones … is the same as 12 ones"] We will also look at our MAB equation and note that now we only have three tens. I will ask if we still have the number 42, albeit represented differently … During this initial equation I would not ask the students to write the equation down; the focus would be on making, discussing and solving the equation. [Amy]

Item	Procedural explanation	Conceptual explanation	Procedural and conceptual explanation	Not clear/ no explanation
Subtraction	4	0	4	1
Division	6	0	3	0
Fraction addition	0	2	7	0
Area/perimeter	1	2	4	2

Table 3. Numbers of teachers giving procedural or conceptual explanations in response to misconceptions.

Differences in PCK evident in responses

Teachers' PCK can be examined within the framework described above. Although differences in the approach that teachers took for the different items may have been affected by the nature of the item (as discussed later), a comparison of responses to the same item, by teachers who adopted a similar strategy, is revealing about the PCK held by those teachers.

A closer look at three responses to the fraction item demonstrates this. Brian, Erin and Amy all responded to this item by re-explaining subtraction to the student. Brian and Erin decided to break the first subtraction into three parts (for ones, tens, and hundreds), and all three mentioned using MAB (base 10 arithmetic/Dienes' blocks). However, there are some important differences. Brian suggested breaking the problem into parts, but did not discuss how to bring the parts together again to complete the problem. While he described using MAB to demonstrate that "you can't take 7 from 3", he did not explain what he would do next. Erin's response was similar, but she went on to describe using MAB to help "take from" the hundreds column when there are not enough tens. Amy's response to this problem was noticeably more detailed, and she made an additional decision to begin with a simpler (two digit) problem. She described going through each step in the subtraction algorithm with the MAB, frequently highlighting important aspects for the student, such as the equivalence of the original number, 42, to its new form, 30 + 12. She emphasised the importance of students understanding this process first, and only then "we would look at how we represent our trades on paper".

The differences in the choices made and in the detail and coherence of the three responses reveal differences in knowledge about teaching the subtraction algorithm, and about this misconception. Brian has identified the problem, and actions he might take, but in a limited way, and he has not connected his ideas into a logically sequenced response to the misconception. Erin’s response is more detailed and connected, but still contains many gaps, while Amy’s response is detailed and coherent, and reveals more explicit awareness of the reasons for her decisions.

Teachers’ responses to the division item can be examined similarly. While most responses characterised as “re-explain” mentioned ideas like the importance of place value, zero as a “place holder”, and the use of multiplication in checking answers, a deeper examination of responses reveals more aspects of the PCK of the teacher. The focus of Hilary’s response was to remind students to “put a zero down” in certain cases, without justifying why this is the case. Irene did the same thing, but added that “just because a zero is there doesn’t mean that it’s nothing, it just happens to be no tens.” She also discussed the meaning of the different columns in terms of place value, and was very specific about why she feels students need to understand this.

DISCUSSION AND CONCLUSION

A number of factors are evident in the teachers’ responses. It appears that the nature of the items themselves and the topics covered in each may influence how teachers said they would respond, particularly with reference to the emphasis on procedural and/or conceptual explanations. Although not apparent in the data presented here, there was also evidence that some teachers may be inclined towards one more than the other. Other teacher-based differences are shown in the closer analysis of the teachers’ PCK revealed in their responses. These issues, and the question of the effectiveness of the questionnaire/interview approach, are discussed in detail below.

Differences among teachers’ responses

The data in Tables 2 and 3, and the detailed responses, indicate differences in approach among teachers and items. The subtraction and division items attracted many responses that were purely procedural. This may be because they involved student tasks with an algorithmic calculation, and that while the procedure in the algorithms and the overall concept involved (subtraction or division) are linked, it can be a difficult link to maintain while focussing on the mechanics of the procedure. Indeed, what was noticeable in the teachers’ responses was the lack of conceptual support for teaching the procedure, such as modelling the algorithm using MAB.

In contrast, responses to the area/perimeter item were more often conceptual or both conceptual and procedural, with only one purely procedural response. There is no complex algorithm associated with this item, and the procedure involved in measuring perimeter mimics the concept more clearly than in the algorithm items. Teachers’ responses often indicated that the *meaning* of area and perimeter would be revisited; and where the procedure was stated it was supported by some conceptual explanation. Moreover, five teachers indicated that they would pursue the concepts further through an exploration of shapes with the same area but different perimeter.

The fraction addition item lies somewhere between these two: the link between the procedure and the concept may not be evident to the student, but is readily demonstrated by the teacher. The number of responses that combined procedure and concept offers support to this idea. Three teachers drew a diagram in their questionnaire, and all the teachers at least made mention of drawing a diagram or using physical materials to model the problem. Such a diagram, of two wholes with seven tenths shaded in the first and four tenths in the other, mimics the written problem quite closely.

For the more algorithmically-based items, the prevalence of “re-explain” as a strategy in teachers’ responses is unsurprising. The way in which cognitive conflict was used varied across items. The idea of considering the inverse operations to evoke cognitive conflict for the subtraction and division items was rare. Many teachers sensibly suggested that students estimate their answer for the division item; the cognitive conflict should be dramatic in this case, whereas for the subtraction item the estimated result may not sufficiently differ from the student’s erroneous answer. Finally, for the perimeter item, teachers seemed to believe that the student’s misconception was because the student had generalised some idea of “conservation”, rather than a lack of understanding of the procedure of measuring the perimeter; hence their suggestion that the student should “measure and check”.

Investigating teachers’ pedagogical content knowledge

Some quite significant aspects of PCK were revealed using the questionnaire and interview approach. When two teachers asserted that they would adopt a particular approach in response to the same student misconception, the differences in the details offered by those teachers often revealed evidence of differences in PCK, or at least in the application of their PCK. For some teachers, the careful articulation of a well-developed sequence of ideas seemed to reflect deep understanding of concepts and strategies for making those ideas meaningful for students. In other cases, the concepts were not well-linked, nor were concepts well-supported pedagogically.

It should be noted that the responses in the questionnaire alone were informative, but the follow up interview gave additional insights and allowed teachers to reveal more of their understanding and repertoire of strategies. It certainly seems, perhaps not surprisingly, that investigating PCK is intrinsically labour-intensive. Nevertheless, this approach allows investigation of a wide range of topics, recalling that there were more items on the questionnaire than the four reported here.

Limitations

There are a number of limitations to this study. With such a small number of teachers we cannot quantitatively generalise about the likely responses of other upper primary teachers, although the study has given some qualitative insights. It should also be noted that the questionnaire and interview approach still does not provide legitimate evidence for what teachers may do in a real-life teaching situation, in response to situations they have more control over and with students they know (further insights into this are evident in the work of Lampert, 2001). Furthermore, the absence of a strategy or explanation in a teacher's response does not necessarily imply that it is not part of his or her teacher knowledge.

Conclusion

The study suggests that teachers respond to student misconceptions in a variety of ways. Their responses to the items appeared to be influenced by the nature of the student task, and revealed aspects of their PCK and the emphasis they place on procedural and conceptual aspects of mathematical understanding. Although a framework for investigating teachers' PCK with this method has not yet been developed, it appears from initial explorations that the questionnaire and interview have the capacity to reveal subtle differences between teachers' responses, which may be attributed to differences in knowledge. This will form a background for the larger study, where PCK will be examined in the context of the classroom. It may also highlight aspects of PCK, and conceptual and procedural knowledge, that are important to emphasise for pre-service teachers.

References

Chick, H. L. (2003). Pre-service teachers' explanations of two mathematical concepts. *Proc. 2003 annual conf. of the Australian Association for Research in Education* (Auckland, NZ, 29 Nov. – 3 Dec., 2003). From: http://www.aare.edu.au/03pap/chi03413.pdf

Graeber, A. O. (1999). Forms of knowing mathematics: What preservice teachers should learn. *Educational Studies in Mathematics*, *38*, 189-208.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University Press.

McDonough, A., & Clarke, D. (2002). Describing the practice of effective teachers of mathematics in the early years. In N. A. Pateman, B. J. Doherty, & J. Zilliox (Eds.), *Proc. 27th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 261-268). Honolulu, USA: PME.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

van der Valk, T. A. E., & Broekman, H. H. G. B. (1999). The lesson preparation method: A way of investigating pre-service teachers' pedagogical content knowledge. *European Journal of Teacher Education*, *22*(1), 11-22.

Watson, J. M. (2002). Inferential reasoning and the influence of cognitive conflict. *Educational Studies in Mathematics*, *51*, 225-256.

STUDYING THE DISTRIBUTION OF RESPONSIBILITY FOR THE GENERATION OF KNOWLEDGE IN MATHEMATICS CLASSROOMS IN HONG KONG, MELBOURNE, SAN DIEGO AND SHANGHAI

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In this project, the clichéd 'student-centred' versus 'teacher-centred' dichotomy has been reconceptualized in terms of the distribution of responsibility for knowledge generation in the classroom and applied in the analysis of 'well-taught' mathematics classrooms in Melbourne, Hong Kong, San Diego and Shanghai. This analytical approach enabled the practices of competent teachers in two 'Asian' and two 'Western' settings to be compared in a more meaningful and insightful fashion than previously possible. This analysis was able to distinguish one classroom from another on the basis of the process whereby mathematical ideas are introduced into classroom discussion and subsequently revoiced and accorded authority. In particular, the methodology and analytical technique employed provided the opportunity to track the movement of mathematical ideas in either direction across the public/personal interface. Critical similarities and differences were identified between and within the classroom practices documented in each country with respect to the distribution of responsibility for knowledge generation.

INTRODUCTION: A THEORY OF CLASSROOM PRACTICE

The theory of learning on which this paper is grounded is one that starts from the social situation of the individual in interaction with others, but which accords a significant role to the individual's interpretive activity. Particular significance is attached to social interaction, and learning proceeds by the iterative refinement of intersubjective understandings that include social and content-specific (in this instance, mathematical) meanings, as well as values and modes of collaborative practice. These understandings are enacted as progressively increased participation in valued practice, including the appropriate utilisation of technical language. Essential to an understanding of the nature of social activity in classrooms is the co-constructed nature of the practices of these classrooms, and the role of negotiation not as a subordinate activity *through* which classroom practice is constructed but as an essential activity *of* which classroom practice is constituted (Clarke, 2001).

Teaching and Learning are not simply distinct but interdependent activities that share a common setting, rather they should be conceived as aspects of a common body of situated practice and studied as such. It is ironic that recognition of this fundamental unity is enshrined in several languages other than English and that the

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dichotomisation of Teaching and Learning may be, in part, an artefact of our use of English as the lingua franca of the international Education community (this argument is outlined in greater detail elsewhere (Clarke, 2001)). Classroom Practice as a form of communal collaborative and negotiative activity is constituted as it is constructed through the participation of both teachers and learners and only understood (and consequently optimised) through research that accords value and voice to all participants. It is for this reason that the Learner's Perspective Study, of which this paper is a product, supplements the multi-camera documentation of classroom activity with post-lesson reconstructive interviews of the participants.

THE DISTRIBUTION OF RESPONSIBILITY FOR THE GENERATION OF KNOWLEDGE

Popular in recent educational literature as descriptors of classroom practice are the terms 'teacher-centred' and 'student-centred.' These terms vary in definition and in use, but they represent a key dichotomy driving much of contemporary Western educational (particularly pedagogical) reform. From one perspective they appear to offer mutually exclusive alternatives with regard to the location of agency in the classroom. Western educational reform advocates student-centred classrooms, and research in Western settings confirms the value of practices associated with these classrooms (Chazan & Ball, 1997; Clarke, 2001). Asian classrooms have been typified as teacher-centred by both Western and Asian researchers, yet the students in these classrooms are highly successful in international studies of student achievement ('The Asian Learner Paradox') (Leung, 2001). Recent research in Chinese classrooms suggests that classroom practice is misrepresented by such a dichotomy (Huang, 2002) and that a theoretical framework is needed by which the 'teacher-centred' and 'student-centred' characteristics of classrooms can be more usefully characterised and investigated, without the assumption of an absolute dichotomy.

Clarke and Lobato (2002) (and subsequently Lobato, Clarke & Ellis (in press)) have proposed a theoretical reformulation of teachers' communicative acts in terms of function rather than form. This reformulation is founded on the distinction between "eliciting" and "initiating." By focusing on function (intention, action, and interpretation) rather than form, some of the difficulties experienced in analysing the efficacy of teacher practices from a constructivist perspective are overcome. Such a framework offers a more incisive tool for the analysis of the teacher's contribution to classroom discourse. In particular, it offers a language in which to frame either the devolution of the responsibility for knowledge generation from the teacher to the student, or, alternatively, the concentration of that responsibility in the teacher. For example, teacher acts that take the form of a question but have the function of telling can be identified and the responsibility for the initiation of a new mathematical idea can be correctly located with the teacher rather than the responding student. Equally, the capacity of the student to contribute to the generation of knowledge can be recognized, and classrooms can be compared according to the extent to which the student is accorded the opportunity to make this contribution. The fundamental consideration is the distribution of responsibility for knowledge generation.

There is general assumption in most literature that classroom discourse encompasses any form of interactions that take place in a classroom. Nevertheless, research has seldom studied all the different forms of classroom interactions. Research involving classroom interactions has tended to focus on either the teacher's talk (eg., Wilson, 1999; Young and Nguyen, 2002) or teacher-students' interactions in either whole class (e.g., Klaassen and Lijinse, 1996, and Seah, 2004) or group discussion (e.g., Knuth and Peressini, 2001). There have been however very few studies, if any, that took into account the role of student-student interpersonal interactions in generating knowledge in the classroom. In our study, a more integrated and comprehensive approach was attempted by analysing both public interactions in the form of whole class discussion and interpersonal interactions that took place between teacher and student and between student and student. Interpersonal student-student interactions available for analysis were restricted to a focus group of up to four students. While this approach did not allow all interactions that took place in the classroom to be studied, it provided us with an avenue to track the generation of knowledge that could occur in both the public and interpersonal domains. In this paper, we report the use of an analytical technique applied to data from mathematics classrooms in four distinct cultural settings to explore the nature of the distribution of the responsibility for knowledge generation.

RESEARCH DESIGN

Data collection was undertaken consistent with the 'complementary accounts' approach discussed in detail elsewhere (Clarke, 2001). In the Learner's Perspective Study (LPS), three video cameras documented teacher and learner actions for sequences of at least ten consecutive lessons and this video record was supplemented by post-lesson reconstructive video-stimulated interviews with teacher and students, together with test and questionnaire data and copies of written material produced in class and interview. This data collection procedure was carried out in three mathematics classrooms in each of the participating countries. In each case, the teacher was identified as competent according to local criteria. In each country, the three mathematics classrooms were selected to provide diversity of socioeconomic context within a single major urban setting – Hong Kong, Melbourne, San Diego and Shanghai for the data analysed in this paper.

All teacher classroom utterances and all statements by focus students, together with post-lesson interviews with teacher and students were transcribed and translated into English. The classroom transcript of each lesson was scanned for terms or phrases that expressed, represented, illustrated or explained mathematical concepts or understandings. In this paper, these terms or phrases are referred to as "math-related terms". These might take the form of conventional mathematical terms such as 'gradient' or everyday expressions such as 'slope' or 'steepness'. These math-related terms were classified into three categories:

1. Those 'primary terms' that corresponded to the teacher's stated instructional goal (in lesson plan or interview),
2. Those 'secondary terms' that were subordinate to or supportive of the teacher's main instructional goal (frequently terms that had been introduced in previous lessons or which referred to familiar everyday contexts and which served to explicate the meaning of those terms central to the lesson's intended focus),
3. Those that appeared infrequently and fleetingly in the course of classroom discussion (in either public or interpersonal statements). These were referred to as 'transient terms.'

Once all math-related terms had been identified in the classroom transcript of a particular lesson, the next step involved identifying the speakers from whom each term originated and those by whom the term was then subsequently revoiced (or not). The time at which each math-related term appeared in the transcript was also noted. Each math-related term was annotated on the basis of whether it was mentioned in the personal or public domain. For the purpose of coding, interactions meant for the public domain were defined as those for which the intended audience was the whole class regardless of the number of individuals actually attending to the statement. Similarly, terms were coded as 'personal' if their intended audience were a single individual or a small group, even where the statement may have been audible to the whole class. The purpose of this distinction was to identify the social context in which each math-related term was first introduced and subsequently revoiced. In this way, it was possible to examine the assimilation of terms introduced in the public domain into student interpersonal discourse and, equally, the introduction of terms originating in student interpersonal conversations into the public arena.

The occurrence of each term was then displayed in a tabular form analogous to Barnes' "flow of ideas" display (Barnes, 2004) or to the resource utilization planning charts of engineers (from which both tabular representations derived). If these math-related terms are thought of as resources drawn upon during the collaborative process of classroom knowledge construction, then the analogy is not inappropriate. Table 1 has been significantly abridged for reasons of space: Only the first 6 minutes of the lesson are displayed and only a subset of the lesson's math-related terms are included. The terms are separated within the table by bold lines into the three categories and a brief description is provided of the classroom activity coincident with the occurrence of the various terms. Each vertical column corresponds to one minute and the occurrence of each term is designated by speaker (T = teacher; Andrea, etc = student), by time-code (eg 06:13, seconds and frames within the designated minute) and by "P" if the utterance was an 'interpersonal' rather than a 'public' utterance.

Mathematical Idea/Term	0 – 1 mins	1 - 2 mins	2 - 3 mins	3 - 4 mins	4 - 5 mins	5 - 6 mins
Current Activity	(0:00 to 2:57) T reviewed the things learnt in the previous lesson with the class; drawing *x*- & *y*-axes (coordinate axes), locating the coords. of a pt & features of 2 points having the same abscissa.			(2:57 to 8:19) T discussed the method to find the coords. of a point and marked the 5 points on the blackboard: (1)find the quadrant where the point belongs; (2)draw a line perpendicular to *x*-axis & a line perpendicular to *y*-axis; (3)locate & note coords. of point.		
Coordinate(s)	T (17:15)	T (06:26) Eve (07:15) T (09:15) T (50:01)	T (07:13) T (03:19)			T (27:19)
Area						
Abscissa		Anthea (30:14) T (32:05)		T (03:19) T (34:11)		T (27:19)
x-axis	Sam (43:17) Eve (52:26) T (56:03)	T (32:05)		T (34:11)	T (08:15)	Anthea (29:15)P
Ordinate		Simon (48:17) T (50:01)		T (24:13)	T (08:15)	T (27:19)
y-axis	Sam (43:17) T (52:09) Eve (52:26)	T (09:15)			T (49:29)	Eve (30:12)P
Transient Terms	Eve (51:04): Coordinate axis.	T (50:01): … rectangular coordinate plane.	Anthea (18:22)P: rectangular plane			Eve (30:12)P location

Table 1. The Distribution of Responsibility for Knowledge Generation

The findings that follow are the result of the application of this analytical approach in a preliminary analysis of such tabular representations of the transcribed classroom discourse from between nine and fifteen lessons in each of the four cities. Interpretation of the status and origin of the math-related terms was supported by the teacher and student interview data.

RESULTS: AN OVERVIEW

Our intention in this paper is to illustrate both the viability of the distribution of responsibility for knowledge generation as a conceptual frame through which to compare classroom practice and the utility of an analytical procedure by which it might be applied. Given the constraints of space, we can only summarise some general trends revealed by our analysis in terms of distinguishing characteristics of the classrooms studied.

Shanghai (Lessons SH2 L04-05 and SH3 L01-05)

The style of teaching in both Shanghai schools analysed was such that the teachers generally provided the scaffolding needed for students to reach the solution to the mathematical problems without "telling" them everything. Hence, one could find quite a few math-related terms that were introduced by the students during public discussion which the teacher had not taught. A particularly powerful example of this devolution of responsibility occurred when the teacher in SH2-L04 (Shanghai School 2, Lesson 4) drew the class's attention to an alternative method of solving simultaneous equations being used by a student which the teacher described as more 'elegant' than the standard (textbook) method.

Hong Kong (Lessons HK1 L09-11, HK2 L05-07 and HK3 L05-07)

Students in the Hong Kong classes studied were generally not given the same opportunities to contribute during lessons in comparison with classes in the other three cities studied. The teachers generally stated very explicitly every step for solving the mathematical problems discussed. In other words, students were guided through the steps for each problem type with very little opportunity for original thought or input into class discussion. Where a new math-related term was introduced into whole class public discussion, this was either done by the teacher or by a student in response to very explicit prompting from the teacher. There were, however, math-related terms that occurred for the first time in interpersonal conversation between students, but were not subsequently voiced in the public arena.

Melbourne (Lessons A1 L04-06, A3 L06-08 and A4 L05-08)

In two of the Melbourne classrooms (A1 and A3), math-related terms were frequently first introduced into whole class discussion by the students but this was most commonly in response to specific teacher questions and consisted mainly of terms taught in previous lessons. The teacher's practice of regularly checking student knowledge of previous content provided frequent instances of student introduction of relevant math-related terms. Classroom activities offered frequent structured opportunities for students to contribute math-related terms whose emergence was anticipated and guided by the teacher.

Melbourne classroom A4 provided several examples of math-related terms introduced for the first time in interpersonal conversation between students. Examples of such terms are: obtuse, millimeters, degree, co-interior, and complementary angles. The richness of the student-initiated math-related terms in interpersonal interactions between students reflects the comparatively lower level of overt guidance provided by that teacher.

San Diego (Lessons US1 L03-05, US2 L01-03, US3 L01, L02, & L05)

In classes US1 and US2, student introduction of math-related terms was most frequently elicited by direct teacher questioning. Such terms occurred either as the names of concepts previously taught, in descriptions of patterns identified or in explanations of mathematical procedures employed, typically expressed in terms of specific mathematical rules. In contrast, in US3, the majority of math-related terms introduced by students arose as part of student descriptions of their problem solving attempts. These were typically elementary mathematical terms such as 'equation,' 'squared,' and 'triangle.'

DISCUSSION

As examples of 'Asian' classroom practice, the Hong Kong and Shanghai lessons analysed displayed greater differences in the distribution of responsibility for knowledge generation than those evident from comparison of 'Asian' and 'Western' lessons. Within the sets of lessons analysed for each city, significant variation was evident. The practices of SH2 provided some powerful supporting evidence for the contention by Huang (2002) and Mok and Ko (2000) that the characterization of Confucian-heritage mathematics classrooms as 'teacher-centred' conceals important pedagogical characteristics related to the agency accorded to students; albeit an agency orchestrated and mediated by the teacher. Our preliminary analyses have demonstrated the utility of 'the distribution of responsibility of knowledge generation' as an explanatory framework capable of distinguishing usefully between classroom practices in both 'Asian' and 'Western' settings.

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References

Barnes, M. (2004). *Collaborative learning in senior mathematics classrooms: Issues of gender and power in student-student interaction*, Unpublished doctoral dissertation, University of Melbourne.

Chazan, D. & Ball, D. (1999). Beyond being told not to tell. *For the Learning of Mathematics, 19*(2), 2-10.

Clarke, D.J. (Ed.) (2001). *Perspectives on meaning in mathematics and science classrooms*. Dordrecht, Netherlands: Kluwer Academic Press.

Clarke, D.J. & Lobato, J. (2002). To Tell or Not To Tell: A Reformulation of Telling and the Development of an Intiating/Eliciting Model of Teaching. In Section II of C. Malcolm & C. Lubisi (Ed.) *Proceedings of the tenth annual meeting of the Southern*

African Association for Research in Mathematics, Science and Technology in Education, Durban: University of Natal, pp. 15-22.

Huang R. (2002). *Mathematics teaching in Hong Kong and Shanghai – A classroom analysis from the perspective of variation*. Unpublished Ph.D. thesis. The University of Hong Kong.

Klaassen, C. W. J. M., & Lijnse, P. L. (1996). Interpreting Students' and Teachers' Discourse in Science Classes: An Underestimated Problem? *Journal of Research in Science Teaching, 33*(2), 115-134.

Knuth, E., & Peressini, D. (2001). Unpacking the Nature of Discourse in Mathematics. *Mathematic Teaching in the Middle School, 6*(5), 320-325.

Leung, F.K.S. (2001). In search of an East-Asian identity in mathematics education. *Educational Studies in Mathematics 47*, 35-51.

Lobato, J., Clarke, D.J. & Ellis, A.B. (in press). Initiating and Eliciting in Teaching: A Reformulation of Telling. Accepted for publication March, 2005, in the refereed *Journal for Research in Mathematics Education.*

Mok, I.A.C. & Ko, P.Y. (2000). Beyond labels – Teacher-centred and pupil-centred activities. In B. Adamson, T. Kwan, & K. K. Chan (Eds.), *Changing the curriculum: The impact of reform on primary schooling in Hong Kong*. Hong Kong: Hong Kong University Press, pp. 175-194.

Seah, L. H. (2004). *A Cross-disciplinary Approach to Analysing a Secondary School Science Lesson in Singapore*. Unpublished MEd Thesis, University of Melbourne, Australia.

Wilson, J. M. (1999). Using Words about Thinking: Content Analyses of Chemistry Teachers' Classroom Talk. *International Journal of Science Education, 21*(10), 1067-1084.

Young, R. F., & Nguyen, H. T. (2002). Modes of Meaning in High School Science. *Applied Linguistics, 23*(3), 348-372.

INDIGENOUS AND NON-INDIGENOUS TEACHING RELATIONSHIPS IN THREE MATHEMATICS CLASSROOMS IN REMOTE QUEENSLAND[1]

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In Queensland, Indigenous mathematics performance lags at least two years behind that of non-Indigenous students (Queensland Studies Authority, 2004). This low performance is exacerbated in remote communities where teachers are generally inexperienced, non-Indigenous, usually stay in the school for two years only, and do not know how to work effectively with their Indigenous aides. This paper reports on part of a 3-year study to enhance students' outcomes through improving relationships between teachers, Indigenous teacher-aides, students and community members. It describes three case studies and identifies training, equality in partnerships, communication, and the "westernised" nature of classrooms as issues for effective teacher/aide relationships.

Indigenous students continue to be the most educationally disadvantaged group in Australia with respect to mathematics. With their consistently lower levels of academic performance and higher rates of absenteeism (Bourke, Rigby & Burden 2000; Queensland Studies Authority, 2004), they are poorly prepared to share the benefits of modern society. Adult employment levels are very low necessitating a reliance on welfare.

There is now an expectation that schools must make a difference to Indigenous students' mathematics achievement and should seek strategies to enhance their mathematics learning (Cataldi & Partington, 1998). However, rural and remote schools with Indigenous populations find it difficult to attract experienced teachers. As a consequence, their teachers are nearly always non-Indigenous, young and inexperienced and commonly leave after two years. While ultimately Australia needs more trained Indigenous teachers, an intermediate goal must be the more effective classroom use of Indigenous teacher-aides (who are mostly older, more experienced in dealing with Indigenous students, and have strong commitment and connections to the local community). These aides should be seen as the key to teaching success in a school with indigenous students (Baturo & Cooper, 2004; Clark, 2000).

Indigenous aides in remote community schools are under-utilised in the mathematics teaching/learning process, being more administrative assistants and "crowd controllers" than partners in classroom teaching (Baturo & Cooper, 2004; Baturo, Cooper & Warren, 2004). In many instances, they are not trained in their role,

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provided with sufficient information to assist the teachers, and included in curriculum decisions. However, Baturo and Cooper (2004) found that a small amount of training impacted positively on Indigenous teacher aides' motivation, their ability to assist teachers in mathematics classrooms and students' mathematics learning outcomes.

Furthermore, Indigenous aides have the potential to bridge the gap between culture and western schooling, particularly in contextualising (Matthews, 2003) mathematics learning so that mathematics concepts can have relevance and meaning for Indigenous students. Utilising cultural knowledge in mathematics classrooms is essential with Indigenous learners to offset the current view that Western schooling generally devalues Indigenous culture which it marginalises as primitive, simplistic and insignificant with respect to mathematics (Matthews, Howard & Perry, 2003; Sarra, 2003).

This paper reports on teacher/teacher-aide relationships in three classrooms within a 3-year project in remote North-West Queensland schools to enhance Indigenous mathematics learning through improving relationships between teachers, aides, students and community members.

METHOD

The project's methodology was mixed method. Quantitative data were collected on (a) students' mathematics performance (school- and system-based tests), attendance and attitudes to mathematics and mathematics learning, and (b) teachers' and aides' attitudes and beliefs towards mathematics and mathematics teaching and learning. These data were collected annually across three years. Qualitative data were collected through observations of classrooms, regular interviews with teachers and aides, and artefacts (e.g., examples of teaching units). Each year, the researchers provided professional development in two major mathematics strands from which two units of work were to be developed, taught and shared with other schools. These professional development sessions were undertaken on site with the aides.

The three cases. The three classroom teacher/teacher-aide interactions described in this paper are the result of observations and interviews in three schools representing a range of communities (labeled as *Rural 1*, *Rural 2* and *Regional* respectively). They were undertaken in the second year of the project while the teachers and teacher aides in the three classrooms were completing a unit of work that had to be developed: (1) to cater for Indigenous students; and (2) to form a partnership between the teachers and the teacher aides. All teachers were inexperienced; all teacher aides were long-term members of their communities. However, it should be noted that the schools paid the aides for student contact time only; preparation and reflection time with teachers were not considered part of their aide duties.

Rural 1, a small school of 39 students enrolled in P-7, was situated in a very small Indigenous community (approximately 300 people) where all students and most residents were Indigenous. The community was isolated, being 150 km from the larger regional community. The focus classroom was the Year 4-7 class (14 students) taught by Anne (young, non-Indigenous, newly-graduated) with two Indigenous teacher aides, Betty and Barbara.

Rural 2 was also a small school (47 students enrolled in P-7) in a small country town (approximately 400 people) with a population comprising 50% Indigenous people. It was more isolated than Rural 1 being 300 km from the regional centre. However, the town had more commercial, business, and tourist facilities than Rural 1. The focus classroom was the Year 4-7 class of 16 students taught by Carl (mature, non-Indigenous, 3 years teaching experience) with two aides – Doris (Indigenous) and Deidre (non-Indigenous).

Regional was a larger school (300 students enrolled in P-7) with more than one draft of some Year levels; 60% of its students were Indigenous. The town of approximately 21000 people was the centre for all local, state, and national government agencies and had several large primary and secondary schools run by state, catholic, and independent education sectors. The focus classroom was the Year 1 class of 20 students taught by Eva (young, non-Indigenous, in her second year of teaching) with an Indigenous teacher aide, Fiona.

Procedure and analysis. Each of the classrooms was visited seven times in the year and the interactions between the teacher, aides and students observed (videotape and field notes). Two units of work were collected during the year. The teachers and aides were interviewed at each visit on their perceptions of their teaching and the effectiveness of their units (audiotape). As well, there were three professional development days at the central regional school to which all teachers traveled (the last of which is a conference in which teachers presented their units. There was a pre- and post interview each year at which teachers' and aides' beliefs re learning and teacher-aide partnerships were probed (audiotape).

The students' responses to the tests and surveys were analysed statistically for significant changes. The videotapes and audiotapes were transcribed and combined with attitude and belief survey responses, field notes, artefacts and units to form a profile of the classroom and the actions of the teachers and the teacher aides. This paper comes from analysis of the interviews and the observations.

THREE CASE STUDIES

Rural 1 (Year 4-7 class – 100% Indigenous community). The Indigenous community in which this school was situated held education in high regard. Student attendance at the school was very high (almost 100%), significantly higher than other communities. The reasons for this are uncertain but this excerpt from an interview with Anne (teacher) gives some insight:

Anne — It is cool to go to school … and the kids all know why someone is not here, they know if it is a good reason, or is it not so good reason. … I think the parents push the kids to go to school. … They [parents] get a bit upset when their kids go away and don't do so well, and end up back here, and I know a few parents are very upset that there are kids back here, particularly elders.

Researcher: Why don't parents want their children to stay in (regional city) with relatives?

Anne Because of gambling and drinking, a lot of kids in … they get into paint sniffing and so there …

Researcher And these kids don't have it here? [No] No drugs?

Anne Nope, not with the young kids ….

In this classroom, mathematics was the teacher's least favourite subject and the students' favourite subject and, because of this, Anne taught it in the last teaching period of the day (traditionally a notoriously difficult time for teachers to foster and maintain student interest and learning). The students were well behaved, on task, and appeared to cope with what was predominantly a "westernised" style of classroom. Anne did not include her aides in her mathematics planning. Her lessons were normally structured in whole-class teaching followed by performance-homogenous group investigations and individual work on activities (e.g., computer activities). Her double-spaced classroom had ample room to accommodate individual desk work, group work, quiet reading, and computer work. The teacher aides moved amongst the desks assisting students in the whole-class lessons, worked with a group as they rotated through tasks or undertook an investigation, or helped with an activity as students moved through their work sequence. There was an excellent relationship between students, aides and teacher in the room. They were all very positive about the mathematics lessons and engaged in the classroom activities. The students readily helped each other; the aides encouraged students to stay on task and provided help when they could.

Betty and Barbara often seemed unaware of the particular activities to be taught each day and lacked the training to undertake some of the mathematics being covered. Because of this, they spent the start of each mathematics lesson sitting at their table beside the students and writing detailed notes on what the teacher was saying as she introduced the day's work. This was their way of trying to come to understand what the teacher wanted from the lesson. It was also their way of learning some mathematics. Anne's lack of attention to the aides' knowledge of mathematics and the mathematics that was to be covered in the lesson appeared to have three consequences. First, the aides' notes were often insufficient to enable them to provide appropriate mathematics assistance to the students. Thus, assistance was predominantly affective and behavioural – encouraging the students to keep trying, to stay on task and to not distract others – rather than cognitive (e.g., the mathematics focus of the activity) or even procedural (e.g., the sequence of steps to be followed). Second, the aides were sometimes slow to move to new activities. In one lesson, the students had to take measurements outside after completing some preliminary work in the classroom. These students outside worked unsupervised because the aides appeared not to know that this was to happen. By the time they arrived, the data gathering had degenerated into one person measuring while all the rest watched. Third, the aides sometimes did not know the mathematics being taught. In the same lesson that had the outside measuring, one of the aides was unable to help students having difficulties with multiplying to calculate area. Interestingly, this problem was solved by one of the Year 7 students who came over and with a lovely manner showed both the students and the aide how to do the exercise.

Rural 2 (Year 4-7 class – 50% Indigenous community). The small country town in which the school was situated was 50% Indigenous and appeared to be divided into two communities – Indigenous and non-Indigenous. Like Rural 1, problems with alcohol, substance abuse and violence appeared to be less obvious than in other Indigenous communities (Fitzgerald, 2002).

The classroom in which Carl taught with Doris (Indigenous) and Deidre (non-Indigenous) was a single room. The students sat in three rows in front of a black board with the teacher's desk at the front. In the library next door and in an enclosed verandah beside the classroom, there was extra space in which there were computers and into which students moved for project work. Like Anne, Carl did all the planning and structured his mathematics lessons with a mixture of whole class work (where possible), ability groups and individual activity. The role of the aides appeared to be one of supporting the students with difficulties or giving one-on-one attention to low achievers. Attendance by the approximately 16 students appeared to be good and the students seemed engaged by the lessons. The Indigenous and non-Indigenous students' behaviour, demeanour and presentation was very similar; however, the performance of the Indigenous students was generally lower (they made up the majority of the students receiving special literacy and numeracy support).

To accommodate lack of out-of-school contact with his aides, Carl communicated his daily program to the aides through a notebook that had a section at the start for lessons and sections at the back for each student. Each night, Carl wrote in the front of the book what he intended to cover next day in the mathematics lesson. The aides read this part of the book when they arrived to see what would be covered. As the mathematics lessons occurred between the first and second recess period, the aides had time to ask questions of Carl in the first recess break. During the lessons, Doris and Deidre wrote into the back of the book (in each student's section) anything they noticed about any of the students they worked with, particularly any lack of understanding of topics, and any special efforts and achievements. Each night, Carl read through these notes and used the feedback to modify his teaching, preparing special group lessons for topics for which many students appeared to be having difficulties and finding individual work for students with unique problems.

Regional (Year 1 class – 60% Indigenous community). The Year 1 Indigenous students appeared to be of two types – (1) students whose families had been in the town for a long time and whose attendance, presentation and performance was indistinguishable from non-Indigenous students, and (2) students of families who were new to the city, or who spent only part of the year in the city, and whose attendance was irregular and whose performance was low.

The room in which Eva taught with Fiona (her Indigenous teacher aide) was a double size classroom. Desks were placed in one area with other areas set aside for reading, working with materials, and group discussion. Although she undertook whole-class teaching at times, Eva's predominant modus operandi was to teach via rotating groups. She divided her class into high, middle and low ability groups. For each teaching episode, she developed three types of activities: (1) an initial activity that focused on introducing the idea through manipulating materials, teacher questioning and group discussion; (2) a follow-up activity that related material, language and symbol (if necessary) in a game situation; and (3) an activity that practised the ideas developed (a worksheet). The low achievers started at (1), the middle at (2), and the high at (3). Eva took the initial activities, Fiona supervised the game (after explicit instructions with regard to the mathematical focus of the task, the questions to elicit learning, and the specific mathematics language) whilst the worksheet activities were unsupervised. Eva planned these activities without her aide. Fiona had limited training in mathematics teaching and tutored most effectively in a structured environment such as the game activity where discussion and questions would be restricted by that environment. Therefore, unlike the aides observed in Rural 1 and 2, Fiona was treated as a teaching partner albeit in a limited way.

DISCUSSION AND CONCLUSIONS

The three cases highlighted the need to address the following issues: (1) training for both teachers and aides in mathematics knowledge, pedagogic knowledge, and forming effective partnerships; (2) more equitable partnerships that draw on both teacher knowledge and aide context/community knowledge for planning, delivery and reflection; (3) everyday communication; and (4) the continuing westernised nature of the classrooms.

Training. It was obvious from observations and unit plans that the teachers themselves needed training in mathematics structure and appropriate pedagogy and in how to work with effectively with Indigenous aides. Therefore, with the exception of Eva, the teachers found it difficult to provide an effective teaching framework for their aides. It was also obvious that the aides needed training in basic mathematical concepts, processes and pedagogy. The teachers were aware of their aides' mathematics deficiencies but none of them spent time, or even considered spending time, on training their aides. As strongly argued by Baturo, Warren and Cooper (2004) and RAND Mathematics Study Panel (2003), mathematics learning outcomes are very dependent on teachers' and aides' knowledge of mathematics.

The success of Eva's and Fiona's teaching (Regional) was based on using materials and pictures to build relationships between real world problems, language and symbols and providing Fiona with a role in games with which she could cope. Anne's teacher aides, Betty and Barbara (Rural 1), did not have the mathematics to tutor their students effectively so they had little influence other than to encourage the students. Carl's aides, Doris and Deidre (Rural 2), could share ideas and keep records on misunderstandings but there was no instructional theory for them to follow in supporting the students. Thus, in cognitive terms, Fiona was the most effective in that her games fitted in the activity rotation that developed ideas from materials to symbols.

Equality in partnerships. All teachers were effective to some extent in integrating their aides in the teaching process with differing degrees of effectiveness in terms of student learning. Betty and Barbara (Rural 1) worked with groups or with students with difficulties but did not have the mathematics to do other than encourage and control behaviour; Doris and Deidre (Rural 2) followed the teacher's plan in the notebook but were very instrumental in their assistance; and Fiona (Regional) had a successful role in supervising the game activity but she could do little outside of this. So, although the teachers had found roles in the teaching-learning process for their aides, they had not formed partnerships with some equality between themselves and the aides. In particular, the aides were not involved in planning the programs and their ideas were not sought for other than local knowledge about the children. There was no realisation that the aides could make contributions in other ways (e.g., providing authentic and cultural contexts for learning) to mathematics teaching (see Matthews, Howard & Perry, 2003; Sarra, 2003). Only one classroom (Rural 2) had structured input (reporting on student errors) from the teacher aides – all others were one way, teacher to aide. It was evident that relationships would be more effective if the teachers and aides respect and value each others' culture.

Communication. Everyday communication was crucial for effective teaching and varied across the cases. For example, Anne, who disliked teaching mathematics, provided no prior communication to the aides about the particular mathematics that was to be covered and as such they were often left to "fend" for themselves in the classroom. Carl, on the other hand, communicated with his aides (albeit non face-to-face generally) about what mathematics concepts and processes were to be taught in a lesson and encouraged the aides to provide feedback (albeit written) to him about how the students were achieving. As such, he provided them with an integral function within the teaching and learning process. In Eva's case, restricting Fiona's role to one type of activity (games) enabled Eva to provide information quickly on the purpose, language and questioning required for successful learning from the game. However, there was little communication of how the game fitted into the overall context of the mathematical skills being taught. None of the teachers asked for any contribution from the aides into deciding what should be taught and how it should be taught.

Westernised nature of the classrooms. Finally, all classroom programs were strongly westernised – they could have been used with non-Indigenous students. There was no evidence of contextualising mathematics instruction (i.e., placing it within Indigenous culture) (Matthews, 2003; Matthews, Howard & Perry, 2003;

Sarra, 2003). Neither the Indigenous aides nor other community members were utilised in developing authentic learning contexts to help Indigenous students make sense of mathematics learning. Developing Indigenous contexts for mathematics became the focus of the teacher/aide relationships in the project's third year.

References

Ashbaker, B.Y., & Morgan, J. (2001). Growing roles for teachers' aides. *Education Digest*, 66(7), 1-5.

Baturo, A. R., & Cooper, T. J. (2004). *Training indigenous teacher-aides and parents to support the mathematics learning of educationally-disadvantaged Indigenous students in junior secondary school* (Report to the Australian Department of Education, Science and Training). Unpublished report, QUT, Kelvin Grove, Australia, 4059.

Baturo, A. R., Cooper, T. J., & Warren, E. (2004, April). *Enhancing educationally-disadvantaged Indigenous students' numeracy outcomes through enhancing Indigenous teacher-aides' mathematics and pedagogic knowledge*. Paper presented at the National Council of Teachers of Mathematics research pre-session, Philadelphia, USA.

Bourke, J., Rigby, K. & Burden, J. (2000). *Better practice in school attendance*: *Improving the school attendance of Indigenous students*. Canberra. Department of Education, Training and Youth Affairs.

Cataldi, L. & Partington, G. (1998). Beyond even reasonable doubt: Student assessment. In G. Partington (Ed). *Perspectives on Aboriginal and Torres Strait Islander Education*. (pp.309-332). Katoomba, NSW: Social Science Press.

Clarke, M. (2000). Direction and support for new non-aboriginal teachers in remote aboriginal community schools in the Northern Territory. *Australian Journal of Indigenous Education, 28*(2), 1-8.

Fitzgerald, A. (2002). *Cape York Justice Study Report*. Brisbane: Queensland Govt Dept of the Premier and Cabinet.

Matthews, C. (2003). *Notes on teaching Indigenous students mathematics*. Unpublished paper, QUT, Kelvin Grove, Australia, 4059

Matthews, S., Howard, P. & Perry, B. (2003). *Working together to enhance Australian Aboriginal students' mathematics learning*. Proceedings of the 26th Annual Conference of the Mathematics Education Research Group of Australasia (MERGA 26). Geelong: Deakin University.

Queensland Studies Authority. (2004). *Overview of statewide student performance in Aspects of Literacy and Numeracy: Report to the Minister for Education and Minister for the Arts*. Brisbane, QLD: Queensland Studies Authority.

Roberts, T. (1998) Mathematics registers in Aboriginal languages. *For the Learning of Mathematics, 18*(1), 10-16.

RAND Mathematics Study Panel. (2003). *Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education*. Santa Monica, CA: RAND.

Sarra, C. (2003). Young and black and deadly: Strategies for improving outcomes for Indigenous students. In *Quality Teaching Series: Practitioner Perspectives*. Deakin: Australian College of Educators.

EXPLORING STRATEGIES USED BY GRADE 1 TO 3 CHILDREN THROUGH VISUAL PROMPTS, SYMBOLS AND WORDED PROBLEMS: A CASE FOR A LEARNING PATHWAY FOR NUMBER

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This study investigated how grade 1 to 3 children in South Africa learn early number concepts. A framework was developed and used to assess the children's level of understanding and used to analyse their strategies in solving number problems. Three schools were included and involved 222, 257 and 240 students in grade 1- 3 respectively. The children's level of understanding was assessed through the use of four tests. An analysis of performance, misconceptions and errors made by the learners in each grade was achieved through an in-depth analysis of 48 learners. The results suggested that the majority of learners were unable to solve straight calculations, employed the strategy counting all and counting on, while none engaged in formal or innovative methods. There is no progression in terms of conceptual mathematical development across the Foundation Phase.

INTRODUCTION

This study is located in a three year research and development project where the development of a learning pathway for number is the primary objective, but testing learners and classroom observations are strategies used to provide baseline information to measure the success of the project over three years and to feed into the development of the learning pathway. A detailed quantitative and qualitative analysis of the first round of baseline testing will be presented, a comparative qualitative analysis with the second round of testing, as well as suggestions on how these results motivate for the development of this learning pathway for number.

The development of a learning pathway for number in the early grades of the South African primary school (Foundation Phase) is intended as a mathematical guide for planning instructional sequences. The learning pathway for number is research-based and highlights the main features of children's early number development and describes how number knowledge, number sense, mental and written calculation, estimation and algorithms are developed and relate to each other within and across the Foundation Phase. Concepts and the number range for each grade in this framework are sequenced progressively as the 'stepping stones' that learners will pass on their way to reaching the Mathematics Assessment Standards related to number in the early primary grades. The learning pathway for number highlights the cognitive and didactic continuum of number development across four stages, which give an overview of number development and details the progress that most learners will make on the route to being competent with numbers in different contexts and with different kinds of calculations.

THEORETICAL FRAMEWORK

A survey of international literature on children's numeracy (Kühne, 2004) shows a number of interventions and the research associated with this (in the Netherlands, Australia, the USA and the UK in particular – see Van den Heuvel-Panhuizen (1999), Bobis & Gould (2000), Clarkson (2000), Gould & Wright (2000), Carpenter, Fennema & Franke (1996) and Carpenter et al. (1999). All of these approaches add to our understanding of the development of number in the early years, and all, in some way, point to the need for a learning trajectory for number. This project needs to provide evidence of the efficacy of this pathway on teacher development and learner performance. To do this a baseline study, which measures the performance of foundation phase learners was mooted, hence the basis and the focus for this investigation. The aim of the project is to show a productive, efficient and sustainable way to lift learner's performance in mathematics.

The project design is located within what might broadly be described as "design research" (Brown, 1992). As Cobb et al. (2003) suggests, an important category of design research "seeks to develop an innovative intervention and an underlying theory that constitutes its rationale" (pg 2). The present project builds on a tradition that has already developed in relation to one-on-one teaching sessions, classroom design experiments, teacher development experiments and school restructuring experiments, to elaborate design research in relation to broad instructional tools such as the learning pathway for number described above. The framework draws on Steffe's (1992, 2000) scheme model and on the idea of emergent counting (Wright, 1998). The following framework is used to describe a trajectory for learning and how learners learn, understand and solve number problems.

STAGE 1	STAGE 2	STAGE 3	STAGE 4
Pre-school – Grade R	Grade R – Grade 1	Grade 1 – Grade 2	Grade 3 – Grade 4
Emergent Numeracy	Learning to count-and-calculate Integrated Mental and Written (operations up to 10)	Calculation by structuring Integrated Mental and Written (operations up to 20 and beyond)	Formal Calculation Mental Written (Operations up to 100 and beyond)

Levels within the stages: describe solution strategies

Counting-and-Calculating	Calculation by Structuring	Formal Operations
Strategy Integrated mental and written **Level 1** Count all in ones Count up to Take away Count down **Level 2** <u>Stringing</u> Numbers objects in a counting row – operations along a counting row: beads, lines and tapes Count on forward/ backwards from first then biggest number Groups to count <u>Splitting</u> Breaking down and building up numbers ***Models*** Models ‘of ‘problems Line Group Combination	***Strategy*** Integrated mental and written **Level 3** Contextualise numbers and number relationships in different situations <u>Stringing</u> Leaving first number intact and splitting second number <u>Splitting</u> Breaking down and building up numbers Doubling Halving <u>Varying</u> Range of ‘smart’ strategies ***Models*** Line Group Combination	***Strategy*** Mental Written **Level 4** Transfer knowledge of single digit no’s to 2 and 3-digit numbers Use decades as pivot points <u>Stringing</u> Uses decades as pivot points <u>Splitting</u> Decimal structure – both numbers to be added can be split into tens and ones. Jumps of 10 Jumps via 10 <u>Varying</u> Use of memorised facts, relations between numbers and properties of operations Rounding Off Compensating ‘Easy’ numbers ***Models*** Abstract Representations

TEST DEVELOPMENT

We consulted the relevant exit Assessment Standards (Benchmarked statements) of the Revised National Curriculum Statement (RNCS) for each grade and extracted the skills and knowledge components. A framework (table to be shown at the conference) was developed, which listed the generic skills relating to learning number. These skills remained constant across the grades, the range shows progression. A range of test items from various sources was selected and developed and the skills-framework was used to adapt these items into grade-specific test questions, ensuring that the ranges in the test items corresponded with that of the RNCS, thus ensuring item validity.

Some of the items lent themselves to presentation in three different ways. These were: contextual problems with graphic prompts (referred to as visual in the analysis), number sentences and sequences (referred to as symbols in the analysis), and the contextual question stated in words without graphic prompts (referred to as worded problems in analysis). Only the first six items of these three tests (called test A, B and C) are the same questions posed in different ways and using the same number range. It was intended that these items would give further indication as to possible barriers that might exist for learners when answering test questions. A fourth test (test D) was developed which contained items intended to test number recognition on number grids and in number sequences.

A test administrator’s guide (for the Grade One tests only) was developed as an example to demonstrate the methods of asking the questions. Experienced test administrators received additional training a week before the commencement of testing. Test administration allocation of work was as follows: Grades 1 and 2, three test administrators per class and Grade 3, two test administrators per class.

DATA COLLECTION

The tests were piloted at a Primary School in the city. Three experienced test administrators were trained to present the tests to 20 grade 1 learners, 20 grade 2 learners and 20 grade 3 learners in three different classrooms. In grade 1 the test administrator read each item (question) in each test to the learners. The item (question) was only repeated once. In grade 2 and 3 the test administrator explained briefly the instructions for each test and not per item (question). At the end of the testing process, the Principal, the research supervisor, the test administrators, the grade 1, 2 and 3 teachers, as well as the Foundation Phase remedial teacher at the school discussed the rationale for testing at this level, the test instruments and the testing process. All the learners from two classes in each grade level in the three project schools were tested. The same procedure was used as described in the pilot implementation. The only change in terms of implementation was in grade 2 and grade 3 where each item was read twice to the learners in isiXhosa.

DATA ANALYSIS

A quantitative analysis involved an item analysis in the four tests per grade and a comparative analysis of the first six questions in test A, B and C highlighting scores for visual prompts (pictures), symbol (using +, - and = signs) and worded problems. A qualitative analysis involved the responses (strategies used) of 48 learners. Their methods and strategies, errors and misconceptions and their solutions were captured. However, for this study only the grade 3 results will be presented.

Quantitative Analysis

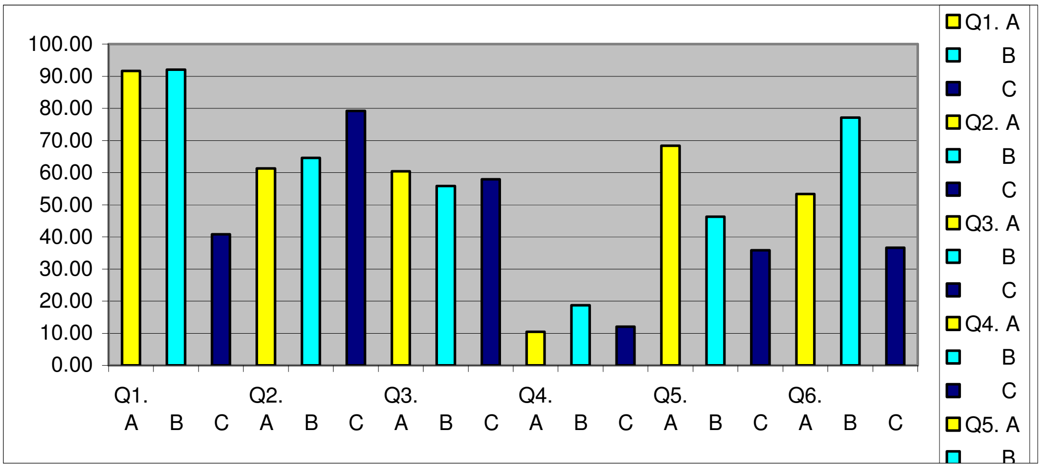
Grade 3

In test A, 91.6% and 68.3% were able to answer the questions on "how many", which involved counting the number of turtles and counting with the use of plus and = signs. The performance on these items was also amongst the best in the four tests. The fact that 9.4% were unable to count the number of turtles is a cause of concern, especially for grade 3 learners. Counting backwards in twenty-five's also proved to be beyond the capabilities of this group since they scored 0.8%. Counting in fives resulted in a score of 53.3% compared with the grade 2's 32.3%. In the division question 62% of the learners were able to solve the problem based on 'sharing'. The learners performed similarly to the grade 2's with scores of 61.2% and 60% in the subtraction problems. The learners were unable to give a mid-number while counting in 50's and scored 10.8% on this item only slightly better than the grade 2 learners.

In Test B only 92% of the learners could add single plus double digit numbers. The subtraction items yielded scores of 64.5% and 55.8% similar to the scores in test A. Adding in fives with plus signs between the numbers proved to be better than counting in fives without the signs. For the place holder type items the performance was only slightly better than the grade 2 learners with scores of 32%, 15% and 11.2% compared to less than 6% attained by the grade 2 learners on similar items.

In test C, the worded problems were also poorly done with the best performance, strangely enough, in a subtraction of double-digit numbers, where 79.1% achieved the correct answer. This was followed by a second best performance of 57.9% also in a subtraction item. In a 'how many' problem the grade 3 learners achieved only slightly better than the grade 2 learners with scores of 40.8% and in a 'spending money' problem they achieved a lower score of 13.7%. A low score of 2% was attained for a repeated addition problem.

In test D only 20% were able to fill in the missing numbers in a full grid of numbers. Again they performed slightly better than the grade 2 learners. The best performance of only 37% and 35.8% was achieved in 'finding a number written in words in a part-grid'. This achievement was similar to grade 2 learners. The learners struggled with this number recognition test with the lowest performance of 4.1% in the item 'find a 3-digit 'between' number written in words in a part grid'. The performance on 'before' and 'after' numbers was only slightly better with scores of 5% and 15% respectively. The graph below shows how grade 3 learners performed on the first six questions in test A, B and C. The graph compares the performance in visual, symbol and context type problems where the numbers were the same for each type.



Scores on first six questions in test A,B and C

Qualitative Analysis

The analysis of the learner responses showed that the strategies used for solving problems from grade 1 to grade 3 did not change, showing very little progression across the grades. Grade 3 learners in particular used counting strategies (count all and count on). There was no evidence of a calculation by structuring, for example grouping or breaking up numbers. The vast majority of learners simply wrote down the answers. Field-staff reported that fingers and in some cases even toes (some learners removed their shoes and some did not have shoes) were used to aid counting. The following represents a few grade 3 learner-responses, from one of the classes at a school, to some of the test items (A, B – indicates the group at the school, while 1,2,3 – indicates the learner number).

Test A

- A2 represented the problem in the answer block with tally marks:
- llllllllll + lllllll = 19 for question 1. (correct response). The learners appeared to need visual prompts in order to count-calculate the answer.
- A28 drew 19 tally marks and cancelled 10 = 9 for question 2 (correct response) to represent subtraction
- A39 and other learners consistently wrote the numbers either upside down or inversely.
- A2 wrote 25 + 25+25+25+25+25 correctly, but did not give an answer. Also used groups of 25 tally marks to assist, but could not get to the answer for question 4.
- A2 counted in 5's by adding the top, middle and bottom rows: 30+20+20 =70 for question 6. A strategy that worked well. Quite a few learners wrote 6 + 20+20, they forgot to multiply the 6 by 5. Others simply added the number of fives and got 14, but forgot to multiply by 5.
- A7 simply counted the bags and not the kg's in the bag in question 7. A18 wrote 30050 instead of 350. This kind of response was quite common, even at this grade level.
- A2 wrote the subtraction number sentence, but gave the wrong answer: 50-12 = 50 for question 8. The majority only counted the visible objects and did not take into account the hidden ones.

Test B

- A2, A14 and others drew tally marks. Others used little circles to represent their counting. It appeared as if they counted all the marks or counted on from one of the given number in question 1.
- A20 drew 19 tally marks – 10 tally marks, then cancelled 10 tally marks to get an answer of 19 in question 2. A number of the learners ignored the minus sign and simply added the two numbers. The majority of learners used tally marks and circles to get the correct answer.
- A2, A8, A14, A20 and others, all used the groups of 25 tally marks or circles, but were unable to give an answer. The majority showed a strategy but failed to give a correct response in question 4.

Learners used tally marks or circles to represent the problem and to assist in their counting-and-calculating. As the numbers range increased, however, errors in counting became common.

Comparative Qualitative Analysis

The strategies used by the 48 randomly selected learners were used to compare their performance in March and November. Table 1 represents a sample of each learner's analysis for test A.

SOME OBSERVATIONS

In general the scores improve in each grade. There is one exception: Grade 3 Question (where the oranges are hidden in a repeated addition problem), where the grade 1's scored 15%, grade 2's scored 24% and the grade 3's only achieved 12%. For most of the questions it is true that the word problems are the most difficult type of questions and problems with visual prompts appeared to be the easiest. Some problems behave "strange". This could be a result of how the problems are designed. Another example shows that the addition word problem in grade 1 (Q1C) is more difficult than the subtraction problem in grade 1 (Q2C), because "crawl in" is more difficult to understand and to see the underlying operation than it is in the case of "eat". None of the learners displayed higher order thinking skills. There was no evidence of formal/flexible operations.

Table 1 - LEARNER CODE: Im2

ITEM	MARCH STRATEGY	SCORE	NOVEMBER STRATEGY	SCORE	COMMENTS
1	Used 8 + 11 tally marks	Correct	No strategy	Correct	Learner able to solve correctly without any strategy
2	Wrote 10 + 19, but 9 in the answer block	Correct	No strategy	Correct	
3	No answer	Wrong	No strategy	Correct	
4	No answer	Wrong	Used 6 groups of 25 tally marks each, counted wrongly	Wrong	Too many tally marks to count
5	No attempt	Wrong	No strategy	Correct	Learner able to solve correctly without any strategy
6	Wrote sum in answer block 30+20+20=70	Correct	No strategy	Correct	
7	Incomplete answer	Wrong	No strategy	Wrong	Learner unable to solve problems. Only counted the visible parts to the question. Learner probably made a slip with the 7.
8	Wrote 50-12=50	Wrong	Counted chocs that were visible	Wrong	
9	Wrote 78-6 but no answer	Wrong	Counted beads that were visible	Wrong	
10	Counted in one's	Wrong	Wrote 735,785,799. did not count backwards in 25's	Wrong	
11	Wrote Γ50 the 7 faced the wrong way	Wrong	No strategy	Wrong	
12	4 groups of 4 circles and wrote 16	Wrong	No strategy	Correct	Learner able to solve correctly without any strategy
13	Wrote 4+8=11	Wrong	No strategy	Wrong	Did not understand the question
14	Shaded 2 x R5 coins and 5 x R1 coins	Correct	Shaded 3 x R5 coins	Correct	Different, but correct responses
		4 correct		7 correct	Improved slightly

CONCLUDING REMARKS

The fact that the tests were conducted very early in the year (only three months into the year) some concepts may not have been taught, revised or consolidated by the teachers at the schools, hence the series of poor responses on a number of concepts or test items. This only improved slightly in November despite a year's teaching. Based on the 4 stages of development, the grade 3 learners are operating at grade 1 and early grade 2 stages. The insights gained from the testing will be used for the development phase in 2005.

References

Bobis, J. (1996). Visualisation and the development of number sense with kindergarten children. In J. Mulligan & M. Mitchelmore (Eds.*) Children's number learning: A research monograph of MERGA/AAMT* Adelaide: Australian Association of Mathematics teachers.

Bobis, J. & Gould, P. (2000). Changing the professional knowledge of teachers. In: J. Bana & A. Chapman Mathematics Education beyond 2000. *Proceedings of the Twenty-third Annual conference of MERGA.*

Brown, A. L. (1992). Design experiments: theoretical and methodological challenges in creating complex interventions in classroom settings. *Journal of the Learning Sciences 2*, (pp 141-178).

Carpenter, T. P. Fennema, E. Franke M. L. (1996). Cognitively guided instruction: a knowledge base for reform in primary mathematics *The Elementary School Journal 97* (1) 3-20

Carpenter, T.P., Fennema, E,, Franke, M. L., Levi, L. & Empson, S. B. (1999). *Children's mathematics: cognitively guided instruction*. Portsmouth: Heinemann

Clarkson, P. (2000). Teachers, students and research: one possibility for teaching early numeracy. In: J. Bana & A. Chapman *Mathematics Education beyond 2000*. Proceedings of the Twenty-third Annual conference of the MERG of Australasia

Cobb, P. Gravemeijer, K., Yackel, E. McClain, K. & Whitenack, J. (1997) Mathematising and symbolising: the emergence of chains of signification in one first-grade classroom. In D. Kirshner & J A Whitson (Eds.) *Situated cognition theory: social, semiotic and neurological perspectives* NJ: Lawrence Erlbaum

Cobb, P., Confrey, J. di Sessa, A, Lehrer, R. & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher32* (1) (pp 9-13).

Gould, P. & Wright, R. (2000). Mapping children's stages of arithmetical solution methods. In: J. Bana & A. Chapman *Mathematics Education beyond 2000*. Proceedings of the Twenty-third Annual conference of the MERG of Australasia

Kühne, C. (2004). *Literature Review of Early Number Development,* Mimeo, University of Cape Town.

Kühne C, Van den Heuvel-Panhuizen M, Lombard A. P. (2004). *A Learning pathway for number for Learning Number: Foundation Phase,* University of Cape Town, South Africa (unpublished).

Revised National Curriculum Statement Grades R-9 (Schools) Policy. (2002), *Mathematics Learning Area,* Gazette No 23406, Vol 443, May, Department of Education, Pretoria, South Africa

Steffe, L. P. (1992). Learning Stages in the Construction of the Number Sequence. *In J Bideaud, C Meljac & J Fischer* (Eds.), *Pathways to number: Children's Developing Numerical Abilities (pp. 83-88),* Hillsdale, NJ: Lawrence Erlbaum.

Steffe, L. P. (2000). Principles and Standards for School Mathematics (PSSM) From a Constructivist Perspective, paper written for the Research Symposium Linking Research and the New Early Childhood Mathematics Standards held at the 2000 National Council of Teachers of Mathematics (NCTM) Research Precession, Chicago, Illinois.

Wright, R. J. (1998). Children's Beginning Knowledge of Numerals and It's Relationship To Their Knowledge of Number Words: An Exploratory, Observational Study, *Psychology of Mathematics Education*: Stellenbosch, South Africa.

Wright, R.J. (1994). A study of the numerical development of 5-year-olds and 6-year-olds, *Educational Studies in Mathematics*, 26, 25-44, Kluwer Academic Publishers, Dordrecht, The Netherlands

Van den Heuvel-Panhuizen M. (1999). *Mathematics Education in the Netherlands: A Guided Tour,* Freudenthal Institute, Utrecht University, Netherlands.

PRIMARY STUDENTS' KNOWLEDGE OF THE PROPERTIES OF SPATIALLY-ORIENTED DIAGRAMS

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The ability to select an appropriate diagram to represent the structure of problem information is a critical step in reasoning. This paper reports on an investigation of Grade 3 and Grade 5 students' knowledge of the properties of spatially-oriented diagrams. The task required the students to select the diagram that corresponded to the structure of a particular problem and to justify their selection. The results revealed that primary students have difficulty in selecting an appropriate diagram and adequately justifying their selections. Although Grade 5 students outperformed Grade 3 students in some aspects, the similarities between Grade 3 and Grade 5 performances on other aspects suggests that it is fallacious to assume that students' knowledge of the properties of diagrams will increase substantially with age.

Diagrams are an important visual-spatial representation in mathematics because they facilitate the representation of problem information (e.g., Diezmann, 2000; Novick, 2001). Diagrams have three key cognitive advantages in problem solving. First, diagrams facilitate the conceptualisation of the problem structure, which is a critical step towards a successful solution (van Essen & Hamaker, 1990). Second, diagrams are an inference-making knowledge representation system (Lindsay, 1995) that has the capacity for knowledge generation (Karmiloff-Smith, 1990). Third, diagrams support visual reasoning, which is complementary to, but differs from, linguistic reasoning (Barwise & Etchemendy, 1991). However, students of all ages are reluctant to employ diagrams, experience difficulty using diagrams or lack the expertise to use diagrams effectively (e.g., Veloo & Lopez-Real, 1994). Thus, students' use of diagrams can inhibit rather than facilitate their mathematical performance.

DIAGRAMMATIC KNOWLEDGE

Three useful diagrams that have broad applicability in mathematics and unique spatial structures are the matrix, network, and hierarchy (e.g., Novick, Hurley, & Francis, 1999) (see Figure 1). For example, the row and column structure of a matrix makes it useful for depicting a combinatorial relationship between two distinct sets.

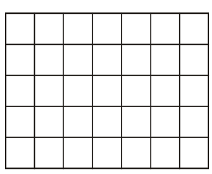
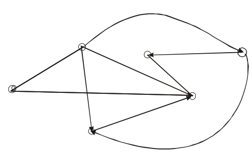
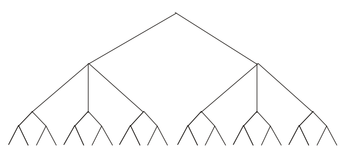
Matrix	Network	Hierarchy
		

Figure 1: Three general purpose diagrams.

While diagrams have been studied intermittently over the past three decades (e.g., Diezmann, 2000), it is only recently that a cohesive framework of ten distinguishing properties of spatially-oriented diagrams has emerged from research with college students (Novick, 2001; Novick & Hurley, 2001). This is a major advance in diagrammatic research because these properties constitute the "building blocks" of diagrammatic knowledge, and are applicable to all spatially-oriented diagrams. Novick and Hurley (2001) confirmed the existence of these properties but found that only six of the ten properties were sufficiently discrete to be readily investigated. These six properties are shown in the first vertical column on Table 1. Each of these properties differs according to the particular spatially-oriented diagram, as shown in the overview on columns two to four on the table.

Properties of Diagrams	Matrix	Network	Hierarchy
1. *Global structure*: the general form	a factorial structure	lacks formal structure	an organisational structure
2. *Number of sets*	ideally 2 sets of information	1 set of information	no limit on sets of information
3. *Item/link constraints:* how items link together	factorial structural constraints	no constraints	organizational structural constraints
4. *Link type:* links between items are best conveyed by a particular diagram	associative non-directional links	flexible links	directional links
5. *Linking relations:* one-to-many links, many-to-one links or both	not salient, but can have both linking relations	both linking relations	either linking relation but not both
6. *Transversal:* the possible paths	paths are not relevant	multiple paths connect item "A" and "B"	only 1 path connects items "A" and "B"

Table 1: Discrete properties of spatially-oriented diagrams.

The ability to identify the properties of diagrams is fundamental to the selection of an appropriate diagram for problem solving (Novick, 2001). This ability involves the recognition of particular representations and knowledge of "where to look and what to look for or look at" (Rogers, 1995, p. 482). Hence, if diagrams are to be useful cognitive tools for problem solving, students of all ages need to know their properties. The focus of this paper is on primary students' knowledge of the properties of diagrams.

METHODOLOGY

This investigation is part of a larger study that aims to increase our understanding of primary students' knowledge about the properties of diagrams and to identify influences on the development of that knowledge. The larger study has an accelerated longitudinal design, in which two differently-aged cohorts are being studied for three years. This paper reports on two aspects of this study which were:

1. To document Grade 3 and Grade 5 students' knowledge about the properties of a matrix, and
2. To determine whether the ability to identify the properties of a matrix increases with age.

The Participants

There are a total of 137 participants in the larger study. The results of eleven students were excluded from this investigation for various reasons (e.g., inconclusive coding). Hence, the results are reported for a total of 126 students comprising 62 Grade 3 students (8- or 9-year-olds) and 64 Grade 5 students (10- or 11-year-olds).

The Tasks

Students' knowledge of the properties of diagrams was investigated in the larger study through a series of 15 scenario-based tasks, which were designed to focus on a range of properties of the matrix, network and hierarchy. Appendix A presents the Matrix task which was the focus of this investigation. The 15 tasks were designed in accordance with the principles used by Novick and Hurley (2001) in the design of scenario-based tasks for college students. The first sentence or two of the scenario tasks sets up a cover story. The same broad scenario of "The Amusement Park" was used for all tasks with primary students to avoid them selecting their responses on the basis of the cover stories rather than the structural information. The next sentence or two focuses on a particular property of a diagram (e.g., the number of sets). The final sentence indicates that someone wants a diagram for a purpose relevant to the cover story. Only two (correct/incorrect) spatially-oriented diagrams were presented for each task. In one of these diagrams, the property was correctly represented, and in the other diagram, the property was not represented (see Appendix A). The scenario-based tasks required students to (1) select a diagram that best suits the given information and to (2) justify their selection and (3) non-selection of particular diagrams. These 15 tasks were presented to students in two individual interviews to avoid undue fatigue. During the first interview, students engaged in a task that emphasised that the diagrams presented were representative of a specific class of diagram rather than the particular problem. This paper reports on one of these tasks.

Students' knowledge of the properties of diagrams was determined by their correct/incorrect selection of a diagram. Categories were developed from the reasons that students gave for selecting and not selecting particular diagrams and frequencies of students' responses calculated.

Results and Discussion

The results reported here are necessarily limited. They focus on the number of students at each grade level who selected correct/incorrect diagrams on the *Sandwich Bar* task (see Appendix A) and the reasons why students correctly selected the matrix to represent the problem information. The reasons why students incorrectly selected the hierarchy and the reasons why students did not select either the matrix or the hierarchy are not discussed here.

The results of the *Sandwich Bar* task for Grades 3 and 5 of 66.1% and 71.9% respectively indicate that many students had difficulty identifying which of the two diagrams (matrix, hierarchy) would best show the information given (chance accuracy = 50%) (see Table 2). The mere 6% difference between the Grade 3 and Grade 5 results suggests that additional two years of schooling have limited impact on students' ability to select the correct diagram.

	Grade 3 (n = 62)		Grade 5 (n = 64)	
Diagram Selection	Number Correct	Percentage Correct	Number Correct	Percentage Correct
Correct	41	66.1%	46	71.9%
Incorrect	21	33.9%	18	28.1%

Table 2: Number and percentage of students selecting a correct/incorrect diagram.

The explanations for students' correct responses are presented on Table 3 together with the frequency of these responses. As shown on Table 3, there was great variation in the type of explanations given by the 87 students who correctly selected the matrix as the best diagram to represent the given information. Of the 16 types of explanations given by students, 11 types of response (indicated by *) were specific to tasks in which the matrix was the correct diagram and five types of response (indicated by $^{\#}$) were more generic and could have referred to any of the spatially-oriented diagrams (see Table 3).

Only three of the total 87 students (3.45%) provided an exemplary or ideal response for their selection of a matrix with a reference to the representation of combinations (CO). However, a further 16 students' explanations showed they had some understanding of the matrix as having a row and/or column structure (LR, LC, RC). Hence, a total of 19 students provided an explanation that was either fully (CO) or partially correct (LR, LC, RC). There was only a 4% difference between the performance of Grade 3 (19.5%, n = 8) and Grade 5 (23.9%, n = 11) students who made fully and partially correct responses.

A total of 32 students (36.8%) based their explanations for selecting the matrix on another visual representation that is used in mathematics, such as a picture graph (PG), (non picture) graph (GF), co-ordinates (UC) or a tally reference (grid) (TR). Fewer Grade 3 students made this type of response (21.9%, n = 9) than Grade 5 students (50%, n = 23). A possible explanation for students' references to other visual representations in mathematics is their attempts to capitalise on prior knowledge of mathematics to make sense of a novel representation. Although some diagrams and other visual representations can be informationally equivalent and content transfer between these is desirable (Baker, Corbett, & Koedinger, 2001), the informational equivalence of representations cannot be assumed. For example, the content of a co-ordinate representation is unlikely to be informationally equivalent to that of a matrix. Of the 28.1% difference between Grade 3 and Grade 5 students who referred to other visual representations used in mathematics, 21% of the variance can be accounted for by the differences between Grade 3 and Grade 5 students' references to a graph format (GF). This response was made by 7.3% (n = 3) Grade 3 students and 28.3% (n = 13) of Grade 5 students. A possible explanation for the greater percentage of Grade 5 than Grade 3 students making reference to a graph could relate to recent instruction about graphs or the use of graphs in the Grades 4 and 5 curricula.

Code	Explanation	Grade 3 (n = 41)	Grade 5 (n = 46)
*BR	Box Reference (not a list); used as a storage space	7	3
#CA	Correct Appearance; "looks right", "would work"	1	2
*CL	Create a List	4	4
*CH	A Checklist; uses ticks	3	1
*CO	Ideal response that described Combinations	1	2
*GF	Graph Format - not a picture graph	3	13
*LR	Create a List using Rows	4	5
*LC	Create a List using Columns	3	1
#NO	Not the Other diagram	3	0
#NS	Response makes No Sense, illogical, vague response, insufficient information supplied	3	1
*PG	Picture Graph	5	2
*RC	Create a list using Rows and Columns	0	3
#SI	Size Issues, could be the right size	0	1
*TR	Tally Reference (number)	1	7
*UC	Used for Co-ordinates	0	1
#VD	Visual/pictorial Description eg., shelves, bread slice	3	0

Table 3: Explanations for why the matrix was selected for the Sandwich Bar task.

A number of students' explanations were inadequate. One type of inadequate explanation focussed on the matrix as a visual representation at a surface level. Students' visually-oriented responses ranged from the broad explanation that the matrix had the correct appearance (CA) to more specific comments about boxes (BR), or a visual/ pictorial description (VD). These responses were made by 10.9% (n = 5) of Grade 5 students and 26.8% (n = 11) of Grade 3 students. It is encouraging that older students made fewer of these types of response than younger students. Another type of inadequate response made by three students was the adoption of the default position that they selected the matrix because the correct response was not the other diagram (NO). All default explanations were made by Grade 3 students (7%).

CONCLUSIONS

Overall, the results suggest that Grade 3 and Grade 5 students have a limited knowledge of the properties of diagrams, in particular the matrix. Although there were some indications of improvement in performance with an increase in age, this was not universally true. There were five key results related to students' performance. First, Grade 3 and Grade 5 students' performed similarly in their ability to correctly select the matrix to represent problem information. Second, students' selections were based on a variety of reasons that may be fully or partially correct, incorrect or inadequate (e.g., default responses). Third, less than 24% of Grade 3 and Grade 5 students made responses that were fully or partially correct. Although Grade 5 students outperformed Grade 3 students, the percentage difference was small. Fourth, over 36% of all students featured another visual representation used in mathematics in their explanations. There was a large difference between Grade 3 and Grade 5 responses with more than double the percentage of Grade 5 students (50%) proposing this type of explanation compared to Grade 3 students (21.9%). Finally, students made a variety of inadequate responses, which included basing their explanations on the surface features of a matrix or providing a default explanation. These visually-oriented responses were made by Grade 3 students (26.8%) substantially more than Grade 5 students (10.9%). Additionally, only Grade 3 students gave default explanations (7%).

If diagrams are to be effective in problem solving, students must be diagram literate (Diezmann & English, 2001). Thus, students need to be able to select the appropriate diagram for a particular problem and adequately justify their selection. The results of this investigation suggest that primary students need considerable teacher support in diagram selection and justification. There are particular concerns with students' performance related to: the scant exemplary responses of students for selecting a matrix; the small differences between Grade 3 and Grade 5 students' performance on correct diagram selection, and the numbers of fully or partially correct responses; and the possible negative effect that an increased familiarity with graphing may have with Grade 5 students making inappropriate transfers between knowledge of graphs and knowledge of the matrix. Limitations of this investigation are that the results are based on the analysis of one task and that task focused on only one of the three spatially-oriented diagrams. However, the generalisability of these results will be informed by other aspects of the larger study, which includes a further 14 tasks, which incorporate the three spatially-oriented diagrams, and the monitoring of students' performance on diagram selection and justification over a 3-year period.

References

Baker, R. S., Corbett, A. T., Koedinger, K. R. (2001). Toward a model of learning data representations. In *Proceedings of the 23rd Annual Conference of the Cognitive Science Society*, (pp. 45-50). Mahwah, NJ: Erlbaum.

Barwise, J., & Etchemendy, J. (1991). Visual information and valid reasoning. In W. Zimmerman & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics* (pp. 9-24). Washington, DC: Math. Assoc. of America.

Diezmann, C. M. (2000). The difficulties students experience in generating diagrams for novel problems. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 25th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 241-248). Hiroshima, Japan: PME.

Diezmann, C. M., & English, L. D. (2001). Promoting the use of diagrams as tools for thinking. In A. A. Cuoco (Ed.), *2001 National Council of Teachers of Mathematics Yearbook: The role of representation in school mathematics* (pp.77-89). Reston, VA: National Council of Teachers of Mathematics.

Karmiloff-Smith, A. (1990). Constraints on representational change: Evidence from children's drawing. *Cognition, 34*, 57-83.

Lindsay, R. K. (1995). Images and Inferences. In J. Glasgow, N. H. Narayanan, & B. C. Karan. *Diagrammatic reasoning* (pp. 111- 135). Menlo Park, CA: AAI Press.

Novick, L. R. (2001). Spatial diagrams: Key instruments in the toolbox for thought. In D. L. Merlin (Ed.), *The psychology of learning and motivation, 40,* 279-325.

Novick, L. R., & Hurley, S. M. (2001). To matrix, network or hierarchy, that is the question. *Cognitive Psychology,42*(2), 158-216.

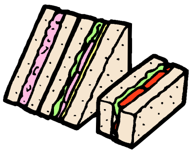
Novick, L. R., Hurley, S. M., & Francis, M. (1999). Evidence of abstract, schematic knowledge of three spatial diagram representations. *Memory & Cognition, 27*, 288-308.

Rogers, E. (1995). A cognitive theory of visual interaction. In J. Glasgow, N. H. Narayanan, & B. C. Karan. *Diagrammatic reasoning* (pp. 481-500). Menlo Park, CA: AAI Press.

van Essen, G., & Hamaker, C. (1990). Using self-generated drawings to solve arithmetic word problems. *Journal of Educational Research*, *83*(6), 301-312.

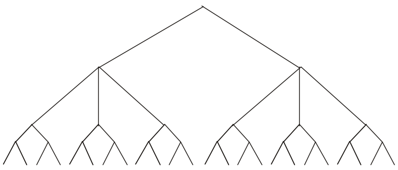
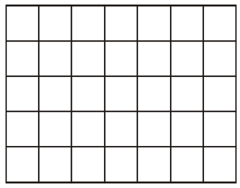
Veloo, P. K., & Lopez-Real, F. (1994). An analysis of diagrams used by secondary school pupils in solving mathematical problems. In J. P. da Ponte & J. F. Matos (Eds.), *Proceedings of the 18th Psychology of Mathematics Education Conference* (Vol. 1, pp. 80). Lisbon, Portugal:PME.

Appendix A. Sandwich Bar Task



The Sandwich Bar sells sandwiches made with different types of bread and different kinds of meat. The Sandwich Bar Manager wants to know which different combinations of bread and meat are ordered the most, so that she can get her workers to prepare the right types of sandwiches for the busy lunch time rush. The Manager would like a diagram to record how many people buy each different combination of bread and meat during one lunch time.

Which type of diagram would best show the information given?

Hierarchy	Matrix
	

1. Tick the box

2. Why?

3. Why not?

A CONCEPTUAL FRAMEWORK FOR STUDYING TEACHER PREPARATION: THE PIRIE-KIEREN MODEL, COLLECTIVE UNDERSTANDING, AND METAPHOR

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This theoretical paper describes a conceptual framework for studying metaphoric mechanisms of the growth of collective understanding among prospective middle and high school mathematics teachers. The framework draws upon a growth of mathematical understanding model, studies of metaphor, and research on collective understanding. Researchers whose studies contributed to this conceptual framework include Pirie and Kieren, English, Lakoff and Nunez, Sfard, Davis, and Simmt. The framework is further defined and illustrated with examples from a teaching experiment in a first mathematics methods class.

FOCUS

This paper develops a conceptual framework to study growth in pedagogical content knowledge (PCK) of prospective teachers. We assume that PCK is grounded in mathematical understanding. Based on this assumption, we use theories on learning *mathematics* as a basis for the framework. These theories include the Pirie-Kieren model, metaphor analysis, and a collective understanding perspective.

THEORETICAL BACKGROUND

We consider three theoretical perspectives as background frameworks to study the growth of understanding among prospective teachers. Shulman (1986), in defining the knowledge base of teaching, initiated the notion of pedagogical content knowledge (PCK) as one of the fundamental categories of teacher knowledge. This type of knowledge is unique within each subject matter domain, and mathematics PCK is accessed by teachers, in concert with their knowledge of mathematics, to communicate mathematical ideas. The Pirie-Kieren model for the growth of mathematical understanding (Pirie & Kieren, 1994b) is fundamental to our studies of prospective teachers. The Pirie-Kieren model is enhanced by coordinating it with metaphoric mechanisms of movement across the layers of the model. Metaphor studies (English, 1997; Lakoff & Nunez, 2000; Sfard, 1997) provide the second theoretical perspective for our framework. Finally, we selected Collective Understanding (Davis & Simmt, 2003; Kieren & Simmt, 2002) to accommodate for the contexts where PCK growth occurs. The Collective Understanding perspective provides the view of learning within a social endeavour, such as a methods class, a school class, or work with others on projects outside of class. Next, we give backgrounds of the three perspectives that provide the theoretical foundation for our research.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 289-296. Melbourne: PME.

Pirie-Kieren Model for the Growth of Mathematical Understanding

The first of the theoretical perspectives we consider is a description of the growth of understanding as a dynamic, levelled but non-linear and recursive process (Pirie & Kieren, 1994b). The Pirie-Kieren model was originally designed as a perspective to study students' changing mathematical ideas. The model provides a framework to map student actions in a variety of contexts, tracing the back and forth movement among eight levels of understanding activities. Within these activities, learners build, search, and collect ideas. The innermost level is *Primitive Knowing*, consisting of one's previous knowledge brought to the learning context. This level serves as a source of materials to build subsequent understanding. Moving outward within the model, *Image Making* and *Image Having* are learner activities for making a new image or revising an existing image, and then for manipulating that image in the mind. These two levels of activity play a prominent role in growth of prospective teachers' understanding (Berenson, Cavey, Clark, & Staley, 2001). The next level, *Property Noticing*, is an action of identifying properties of the constructed image. A method, rule, or property is generalized from the properties in the level of *Formalising*. Beyond are levels of *Observing*, *Structuring*, and *Inventising*. Pirie and Kieren (1994b) describe the process of *folding back* to inner levels of understanding to retrieve primitive knowledge, make or have new images, or notice new properties.

Metaphor

Defined in the past as an embellishment or a figure of speech, metaphor is now seen as a primary mechanism of thinking (Lakoff & Nunez, 2000). The process of metaphoric projection involves a source, consisting of more concrete, better understood images, and a target, which is the new, more formal concept being constructed (English, 1997). This process is a recursive, zig-zag movement between the source and the target, where the target is created and the source is modified (Sfard, 1997).

A metaphor has a certain "life cycle" (Figure 1). When a metaphor is born from its grounding, the target *is* the source. The two parts of the metaphor are inseparable; in other words, the target is not yet constructed. For example, a learner may work with the metaphor of "fair sharing," grounded in sharing actions. In the next stage, the target emerges from the source, which starts to fade, and the metaphor turns into a simile. Now the target is *like* the source; division is *like* fair sharing. Finally, the target disconnects from the source. The metaphor "dies" (Pirie & Kieren, 1994a; Sfard, 1997), and the target lives on as a self-sustained entity. In our example, the learner is able to think about the idea of division without referring to fair sharing.

The paradox of metaphor analysis is that a metaphor can only be studied when it dies, or at least turns into a simile. While the source and the target are inseparable, the metaphor is unnoticeable. Researchers can analyse this first stage in the metaphor's life *retrospectively*, from the vantage point of the future emergence and separation of the target. A participant observer can also influence a metaphor, thus helping metaphor's users learn.

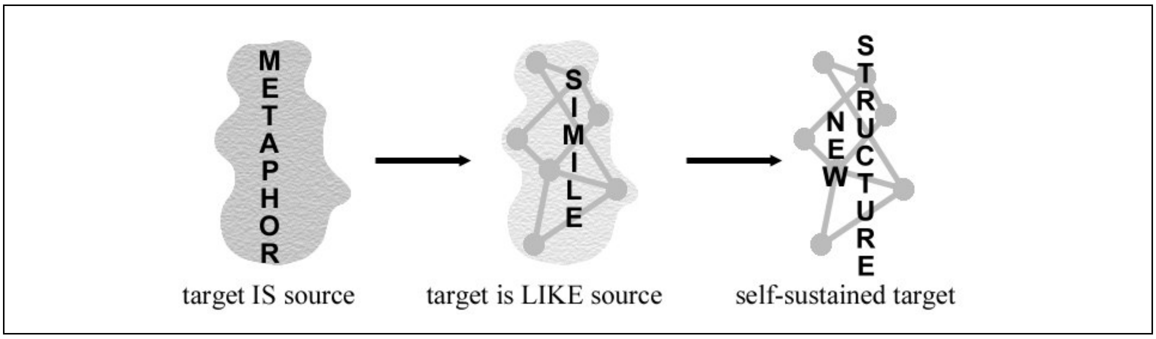

Figure 3: Metaphor life cycle

Collective understanding

The theory being developed by the Collective Understanding Research Collective (Davis & Simmt, 2003; Kieren & Simmt, 2002) focuses on emergent phenomena in groups seen as wholes, rather than simple sums of individuals. A key feature of the complexity of these emergent structures is that they cannot be caused, but may be occasioned, and analysed retrospectively. The authors claim five necessary conditions for such collective understanding: *internal diversity*, *redundancy*, *decentralized control, organized randomness*, and *neighbour interactions. Internal diversity* in response to a task means that members of the collective contribute in different ways. Diversity creates possibilities for new, varied paths toward understanding. *Redundancy* is defined as the occurrence of more actions or ideas than, seen retrospectively, would be necessary to complete the task. Redundancy serves the role of supporting communication and the feeling of "us" within the collective, and of helping to cope with perturbations. *Decentralized control* suggests no single organizing agent within the collective; in the classroom, it means that the teacher is not the only authority on correctness. *Organized randomness* is a set of proscriptions and within it, freedom from prescriptions. "*Neighbours*" interacting in the fifth conditions are not people, but ideas, metaphors, and other representations.

PROBLEM

The research problem this study addresses is constructing a conceptual framework for analysing learning of prospective teachers during an introductory methods class, and for designing ways to help them learn. Each of the three theoretical perspectives described above provided a necessary lens for the framework. The Pirie-Kieren model, initially focused on mathematical understanding and adapted to teacher preparation by Berenson et al. (2001), describes the actions of understanding. The model has been related to metaphor theories (Droujkova, 2004; Pirie & Kieren, 1994a), the lens for examining mechanisms of growth of understanding. The Pirie-Kieren model helps to map *what* is happening with understanding, and metaphor describes *how* it is happening. The Pirie-Kieren model, as well as metaphor theories, was initially developed for describing individuals. However, we looked at a group of prospective teachers interacting, connecting ideas, and building their understanding together. We used the third lens: work on collective understanding (Davis & Simmt, 2003), which arose out of the work of Pirie and Kieren, and was developed, in part, within the context of teacher preparation. The resulting conceptual framework helps to map the growth of collective pedagogical content knowledge of a group of prospective teachers, and to analyse mechanisms of this growth. Applying metaphor analysis to preparation of teachers and tracing collective metaphors are two theory bridges constructed in the teaching experiment that is the source of examples for this work (Figure 2).

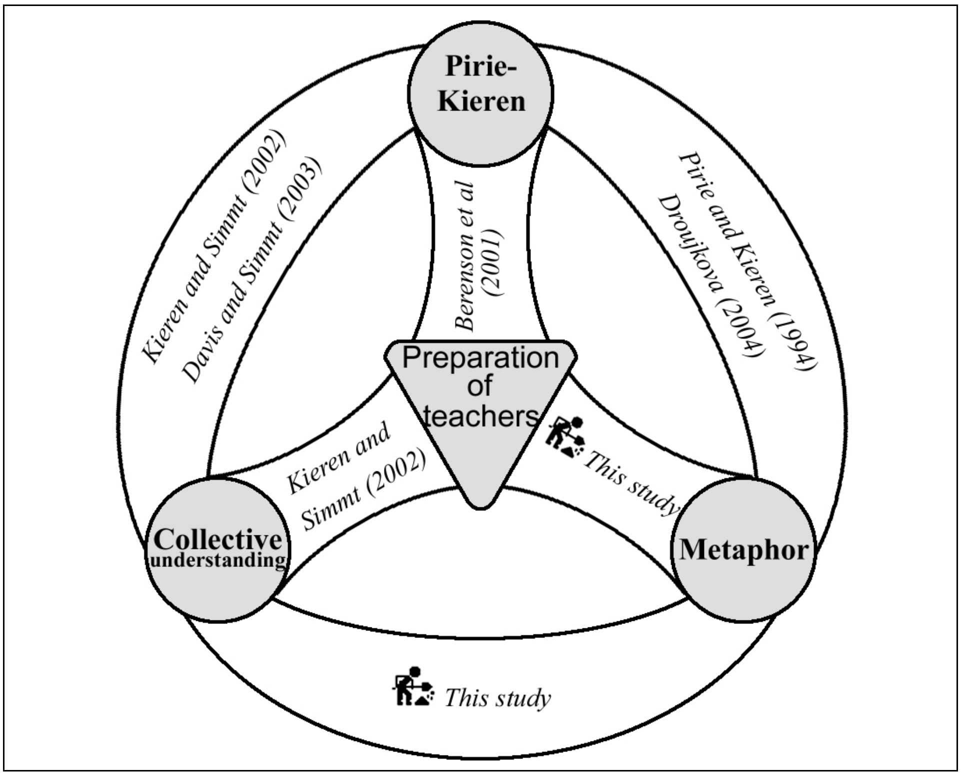

Figure 4: Theories and bridges in our conceptual framework

TEACHING EXPERIMENT METHODOLOGY

Examples in this paper come from a teaching experiment conducted during an Introduction to Teaching class at a large South-Eastern US university. The class met for two hours, one day a week, for fourteen weeks. Paper authors collaborated on planning, teaching and observing during the class. Teaching experiments are defined by the role of researchers as teachers and co-learners, and “environments that are explicitly designed to optimize the changes that relevant developments will occur in forms that can be observed” (Kelly & Lesh, 2000, p. 192). Since the background theories of our conceptual framework use recursive models of learning, it was especially important to allow data from each week to enter the interpretation cycle, and to influence the future data collection. The data include primary artefacts such as home and in-class assignments, teaching portfolios, and videotapes of lessons conducted by prospective teachers; and secondary artefacts such as field notes taken by researchers during class observations and planning meetings.

PEDAGOGICAL CONTENT KNOWLEDGE

In this part, we trace growth of collective PCK in the group of prospective teachers and researchers. Conditions for the emergence of collective understanding help to analyse the features of the class which supported growth.

Image Making: birth of metaphors

“Multiple instructional representations” was a theme included in class activities and homework assignments every day. In first classes and homework assignments, prospective teachers were creating collections of instructional representations of different kinds, such as pictures, definitions and symbols. These collections were based on their Primitive Knowledge of mathematics. The recurring homework of creating four different representations for introducing a mathematical concept, such as slope or ratio, invited the diversity of actions and ideas, and illustrated inter-personal redundancy in actions and ideas – two conditions for emergence of collective understanding. Small group discussions and sharing their results with the whole group supported idea exchange. For example, small groups discussed Pi representations from homework, and sorted them into visual, manipulative, numerical and symbolic and so on to display to the whole class on bulletin boards. An example of a collective image made by the class is “multitude of instructional representations.” This image included the ideas that there are many different ways to learn each concept, and that these choices can be sorted into categories. We, as a part of the group, were folding back to Image Making as well. An example of a change in our image at the time is the focus on overlaps, links and interactions between different types of representations, which we describe in a separate paper (Reference withheld).

Image Having and beyond

Assignments on mathematical and pedagogical connections among representations show that the class was using the idea of multiple representations without the actions of creating them, which indicates Image Having. Class assignments continued to invite individual ideas, either from homework or from a period of contemplation in class, into small group tasks. Each group then reported to the whole class, answered questions, and participated in whole-class discussions. This supported interactions of neighbour ideas within small groups and the whole class, which is another condition for emergence of collective understanding. Such “bumping of ideas” (Davis & Simmt, 2003) looked like animated discussions, and sometimes emotional arguments, within groups, between a group reporter and the rest of the class, and between individuals. In these discussions, we observed Image Having and actions from the further levels.

Property Noticing: metaphor turns into simile

Property Noticing actions are manipulations of images to construct their relevant properties. A transformation of metaphor corresponds to these actions. The newly constructed properties are the metaphor target; they are built on, and separated from, images, which are the source. A plethora of potential properties can be constructed from images. Emphatically, properties are not embedded *in* images, but constructed *from* images, and co-determined by images and actions during Property Noticing. In the case of prospective teachers, mathematical and pedagogical sides intertwine in Property Noticing actions, and metaphor targets are PCK concepts.

An example of Property Noticing comes from the class task of making representations for connections between scaling and ratio, slope, and proportion. In an example of idea redundancy, two groups, working with maps and similar triangles, independently commented that the same representation can be used for scaling-ratio and scaling-proportion connections. The mathematical properties they noticed were connections between ratios and proportions. There were also pedagogical comments: prospective teachers noticed that they can help students learn relationships between ratio and proportion through one activity on scaling, and found two examples of such activities.

During the same task, the organized randomness feature of the collective understanding was expressed as the discussion turned toward the question of what kinds of representations are better as a starting point for students. Prospective teachers used the same image of multiple representations, but were now noticing other pedagogical properties. For example, they determined what representation was more abstract, or what representation was easier for learners to understand. Initially, prospective teachers were talking about particular representations, claiming that maps are easier than similar triangles, or vice versa. When the target of the metaphor began to separate, and the metaphor turned into a simile, prospective teachers started to name noticed properties directly. The sources of metaphors were still present at this point in the learning. A prospective teacher said she would start her lesson from a concrete representation, *like* a map, because it is easier. However, the sources were fading, and the targets became more self-sustained, as the group moved to formalising.

Formalising: death of metaphor, and self-sustained target

During the activities when prospective teachers noticed properties of instructional representations, they also noticed differences in pedagogical uses of representations. Disagreements about uses of representations arose on many counts, such as which representation is better as a starting point, or whether students would have enough prerequisite knowledge to handle a particular representation. The formalising actions were evidenced by abstracting this noticed idea of differences and disagreements about representations as the concept of learner diversity. As evidenced by comments, sometimes emotional, contradictions between ideas were frustrating, and it was imperative for the group to come to an overarching understanding resolving the contradictions. The formalization of the concept of learner diversity was greeted by the group members as a relief of this cognitive tension, and appeared, in many different formalised forms, in the remaining homework assignments, lesson plans, and teaching philosophies of most students.

From the perspective of this formalising we retrospectively see that metaphor started from the images of multiple representations during Image Making and Image Having actions. It turned into a simile while prospective teachers were noticing different, oftentimes contradictory, instructional uses of representations among group members. This property was formalised as the idea of learner diversity. Here are examples of individual expressions of collective formalised understanding, taken from teaching philosophy statements of three prospective teachers:

> Using different types of representations to explain a concept will give students a chance to see the lesson from a different perspective.
>
> Sometimes students lose motivation when they don't understand the material. I plan to use a variety of instructional representations in order to give students who come from different backgrounds and learn differently the opportunity to learn the material.
>
> Students learn best when they are presented with several different ways of looking at a topic. A lecture can only do so much but in combination with hands-on activities and group work students can learn much more effectively.

These examples show variability in individual threads of meaning in the collectively understood idea: one class member focused on learning of a particular concept, another on motivation, and another on class format. Individually and as a collective, prospective teachers started to develop a crucial area of their PCK: understanding of how a particular mathematical concept can be represented in multiple ways, and how differences in learners inform choices, decisions, and development of these instructional representations.

CONCLUSIONS AND FUTURE RESEARCH

Using data from a teaching experiment, we developed a conceptual framework for studying growth of prospective teachers' collective understanding. Our data illustrate metaphoric mechanisms of the growth of understanding, and correspondences between stages in metaphor development, and understanding actions observed at the onset of each stage and mapped by the Pirie-Kieren model (Table 1).

Stage	Metaphor birth from grounding	Inseparably, target *is* source	Metaphor into simile: target is *like* source	Metaphor death: self-sustained target
Actions	Image Making	Image Having	Property Noticing	Formalising
Example	"Multitude of representations"		"Learning differences are like contradicting instructional uses of representations"	"Learner diversity"

Table 1: Co-occurrence of stages in metaphor development, and understanding actions.

Within a metaphor, there is a recursive movement between the emerging target and the source. Growth of understanding, as describe by the Pirie-Kieren model, is also a recursive process where folding back to inner levels is prominent. However, the development of metaphor: birth, turning into simile, and death into the self-sustained target, is unidirectional. This leads us to questions for the future studies: “What is the role of metaphor in folding back? In particular, how does a simile inform folding back from Property Noticing? What is the role of self-sustained targets in folding back from Formalising?” Answers to these questions can help us study ways prospective teachers learn, and ways to prepare teachers.

References

Berenson, S., Cavey, L., Clark, M., & Staley, K. (2001). Adapting Pirie and Kieren's model of mathematical understanding to teacher preparation. Paper presented at the 25th Annual Meeting of the International Group for the Psychology of Mathematics Education, Utrecht, NL.

Davis, B., & Simmt, E. (2003). Understanding learning systems: Mathematics education and complexity science. Journal for Research in Mathematics Education, 34(2), 137-167.

Droujkova, M. (2004). Roles of metaphor in the growth of mathematical understanding. Unpublished Doctoral Dissertation, North Carolina State University, Raleigh.

English, L. (1997). Analogies, metaphors and images: Vehicles for mathematical reasoning. In L. English (Ed.), Mathematical reasoning: Analogies, metaphors and images (pp. 3-20). Mahwah, NJ: Lawrence Erlbaum.

Kelly, A., & Lesh, R. (2000). Handbook of research design in mathematics and science education. Mahwah, NJ: Lawrence Erlbaum.

Kieren, T., & Simmt, E. (2002). Fractal filaments: A simile for observing collective mathematical understanding. Paper presented at the 24th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Athens, GA.

Lakoff, G., & Nunez, R. (2000). Where mathematics comes from: How the embodied mind brings mathematics into being (1st ed.). New York, NY: Basic Books.

Pirie, S., & Kieren, T. (1994a). Beyond metaphor: Formalising in mathematical understanding within constructivist environments. For the Learning of Mathematics, 14(1), 39-43.

Pirie, S., & Kieren, T. (1994b). Growth in mathematical understanding: How can we characterize it and how can we represent it? Educational Studies in Mathematics, 26, 165-190.

Sfard, A. (1997). Commentary: On metaphorical roots of conceptual growth. In L. English (Ed.), Mathematical reasoning: Analogies, metaphors and images (pp. 339-372). Mahwah, NJ: Lawrence Erlbaum.

Shulman, L. (1986). Those who understand: The knowledge growth in teaching. Educational Researcher, 15(2), 4-14.

MATHEMATICAL MODELLING WITH 9-YEAR-OLDS

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This paper reports on the mathematical modelling of four classes of 4th-grade children as they worked on a modelling problem involving the selection of an Australian swimming team for the 2004 Olympics. The problem was implemented during the second year of the children's participation in a 3-year longitudinal program of modelling experiences (i.e., grades 3-5; 2003-2005). During this second year the children completed one preparatory activity and three comprehensive modelling problems. Throughout the two years, regular teacher meetings, workshops, and reflective analysis sessions were conducted. The children displayed several modelling cycles as they worked the Olympics problem and adopted different approaches to model construction. The children's models revealed informal understandings of variation, aggregation and ranking of scores, inverse proportion, and weighting of variables.

INTRODUCTION

With the increased importance of mathematics in our ever-changing global market, there are greater demands for workers who possess more flexible, creative, and future-oriented mathematical and technological capabilities. Powerful mathematical processes such as constructing, describing, explaining, predicting, and representing, together with quantifying, coordinating, and organising data, provide a foundation for the development of these capabilities. Also of increasing importance is the ability to work collaboratively on multi-dimensional projects, in which planning, monitoring, and communicating results are essential to success (Lesh & Doerr, 2003).

Several education systems are thus beginning to rethink the nature of the mathematical experiences they should provide their students, in terms of the scope of the content covered, the approaches to student learning, ways of assessing student learning, and ways of increasing students' access to quality learning. One approach to addressing these concerns is through mathematical modelling (English & Watters, 2004). Indeed, a notable finding across studies of professionals who make heavy use of mathematics is that a facility with mathematical modelling is one of the most consistently needed skills (Gainsburg, 2003; Lesh & Zawojewski, in press).

Traditionally, students are not introduced to mathematical modelling until the secondary school years (e.g., Stillman, 1998). However, the rudiments of mathematical modelling can and should begin much earlier than this, when young children already have the foundational competencies on which modelling can be developed (Diezmann, Watters, & English, 2002; Lehrer & Schauble, 2003). This paper addresses the mathematical modelling processes of children from four classes of nine-year-olds (4th-grade), who are participating in a three-year longitudinal

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program of modelling experiences. The children commenced the program in their third-grade, where they completed preparatory modelling activities prior to working comprehensive modelling problems (English & Watters, in press).

MATHEMATICAL MODELLING FOR THE PRIMARY SCHOOL

The problem-solving experiences that children typically meet in schools are no longer adequate for today's world. Mathematical problem solving involves more than working out how to go from a given situation to an end situation where the "givens," the goal, and the "legal solution steps" are specified clearly. The most challenging aspect of problems encountered in many professions today involve developing useful ways of thinking mathematically about relevant relationships, patterns, and regularities (Lesh & Zawojewski, in press). In other words, problem solvers need to develop more productive ways of interpreting and thinking about a given problematic situation. Interpreting a situation mathematically involves modelling, where the focus is on the structural characteristics of the situation, rather than the surface features (e.g. biological, physical or artistic attributes; English & Lesh, 2003).

The modelling problems of the present study require children to generate mathematical ways of thinking about a new, meaningful situation for a particular purpose (e.g., to determine which set of conditions is more suitable for growing certain types of beans; English & Watters, 2004). In contrast to typical school problems, modelling tasks do not present the key mathematical ideas "up front." Rather, the important mathematical constructs are embedded within the problem context and are elicited by the children as they work the modelling problem. The problems allow for multiple approaches to solution and can be solved at different levels of sophistication, thus enabling all children to have access to the important mathematical content.

The problems are multifaceted in their presentation and include background information on the problem context, "readiness questions" on this information, detailed problem goals, tables of data, and supporting illustrations. In turn, the problems call for multifaceted products (models). The nature of these products is such that they reveal as much as possible about children's ways of thinking in creating them. Importantly, the models that children create should be applicable to other related problem situations; to this end, we have presented children with sets of related problems that facilitate model application (English & Watters, in press).

The problems require the children to explain and justify their models, and present group reports to their class members. Because their models are to be sharable and applicable to classes of related situations, children have to ensure that what they produce is informative, "user-friendly," and clearly and convincingly conveys the intended ideas and ways of operating with these. Because the problems are designed for small group work, each child has a shared responsibility to ensure that their product does meet these criteria.

RESEARCH DESIGN AND APPROACH

Multilevel collaboration, which employs the structure of the multitiered teaching experiments of Lesh and Kelly (2000) and incorporates Simon's (2000) case study approach to teacher development, is being employed in this study. Such collaboration focuses on the developing knowledge of participants at different levels of learning. At the first level, children work on sets of modelling activities where they construct, refine, and apply mathematical models. At the next level, classroom teachers work collaboratively with the researchers in preparing and implementing the child activities. At the final level, the researchers observe, interpret, and document the knowledge development of all participants (English, 2003). Multilevel collaboration is most suitable for this study, as it caters for complex learning environments undergoing change, where the processes of development and the interactions among participants are of primary interest (Salomon, Perkins, & Globerson, 1991).

Participants

Four 4th-grade classes (9 years) participated in the second year of this study, after having also participated in the first year. One of the four class teachers had also been involved in the first year of the study, whereas the remaining three teachers were new to the study. The classes represented the entire cohort of fourth graders in a school situated in a middle-class suburb of Brisbane, Australia. The school principal and assistant principal provided strong support for the project's implementation.

Procedures and activities

At the beginning of the year, a half-day professional development workshop was held with the teachers where we outlined the project and negotiated plans for the year. The four teachers involved in the first year of the study also provided input by sharing their experiences and highlighting what they had learned about implementing modelling activities, as well as describing student learning that had occurred.

An initial preparatory activity (focusing on reading and interpreting data) and three modelling problems were implemented during the year. The first modelling activity was conducted in winter over four weeks and focused on "Skiing for the First Time." The second problem focused on the "Olympics," which was pending at the time of the activity, and the third was conducted during a theme on weather and required the children to decide where to locate a resort in a region subject to Cyclones.

The Olympics problem was undertaken with children working in groups of three or four in four 40-minute lessons conducted over two weeks. Audio-taped meetings were held with teachers to plan the lessons beforehand and to analyse outcomes immediately on conclusion of the activity. The children weer presented with an initial readiness activity containing background information on the history of the Olympics and a table of data displaying the men's world 100 metre freestyle records from 1956 to 2000. The children were to answer a number of questions about the information and data. They were then presented the main modelling problem comprising (a) the data displayed in Table 1; (b) the accompanying information: *We* (Australia's Olympic Swimming Committee) *need to make sure that we have selected our best swimmers. The Olympic Swimming Committee has already selected the women's swim team. However, they are having difficulty in selecting the most suited swimmers for competing in the men's 100m freestyle. The Olympic Swimming Committee has collected data on the top seven (7) male swimmers for the 100m freestyle event. The data collected (see Table 2*; Table 1 in this paper*) show each of the swimmer's times over the last ten (10) competitions. It has been decided by the Olympic Swimming Committee to have you as part of their selection team*; and (c) the problem goal: *Being selectors for the Olympic Swimming Committee, you need to use the data in Table 2 to develop a method for selecting the two (2) most suited swimmers for the Men's 100m Freestyle event. Write a report to the Olympic Swimming Committee telling them who you selected and why. You need to also explain the method you used in selecting these swimmers. The selectors will then be able to use your method in selecting the most suited swimmers for the other swimming events*.

Data Collection and Analysis

Each of the four teachers was fitted with a radio microphone and videotaped during the lesson so that her dialogue with children could be monitored. A second video camera captured critical events as they occurred or was focused on selected groups of students to monitor student interactions. Audio recordings of conversations among children and with teachers complemented video data. Other data sources included classroom field notes, children's artefacts (including their written and oral reports), and the children's responses to their peers' feedback in the oral reports. In our data analysis, we employed ethnomethodological interpretative practices to describe, analyse, and interpret events (Erickson, 1998).

FINDINGS

From our analysis of the children's transcripts as they worked the modelling problems and reported to their peers, we identified a number of different approaches to model development. These included: (a) focusing on personal best times (PBs) only, with some groups also considering the extent of a swimmer's variation from his PB; (b) tallying the number of winning races for each swimmer in each event, and comparing the totals; (c) aggregating the two or three lowest times of each swimmer and comparing the totals; (d) assigning scores (and weighted scores) to the two lowest times of each swimmer and aggregating the scores; (e) in addition to [d], assigning weighted scores to the two lowest PBs, aggregating all the scores, and then ranking the totals (refer Figure 1); and (f) before working with the data, eliminating those swimmers with the most number of DNCs ("Did Not Compete"). Page limit prevents us from providing detailed accounts of the children's developments, however, it is important to note that the groups displayed several cycles of modelling as they worked the problem. That is, they interpreted the problem information, expressed their ideas as to how to meet the problem goal, tested their approach against the given criteria, revisited the problem information, revised their approach, implemented a new version, tested this, and so on. We consider just a couple of the above modelling approaches in this paper.

A Focus on Personal Best Times (PBs)

Lana's group chose to focus on the swimmers' PBs from the outset, but did consider other options in justifying their decision. Initially, the group thought they might "add up the amounts," to which one member responded, "Yeah, and whoever has the smallest…" Later on, when the group revisited this option, Lana felt this was not feasible because "what I'm arguing is, well, I'm not really arguing, what I'm saying is … we can't add up the totals because there are so many Did Not Competes, and they've got uneven amounts, so that wouldn't be fair." Another child responded, "And they would get a lot lower (total)."

In comparing the swimmers' PBs, the group members clarified their interpretation of this notion: "Don't you want the lowest time, whatever? The lowest time is the fastest swimmer." "Because that means they don't take as long to swim." As the group were considering the swimmers' PBs, they also noted an error in the data (Ashley Callus' PB was higher than his score for the 2001 Pan Pacs.) The group spent quite some time arguing about how to resolve this dilemma but decided to accept the error.

In reflecting on their focus on PBs, three of the group members questioned the reliability of these data. In the transcript below, the children are starting to think about trends in the data and swimmers' variation from their PBs.

Kelly continued her argument that "It doesn't just Kelly: Yeah, but Lana they might just one day swim really, really well, like…they might have just had a really, really good day, yeah, or week or whatever.

Lana: Yeah, I know...

Kelly: They might be a really good swimmer and then they sort of you know they might have had an injury and gone back but their not as good, so…it might have changed.

Sam: Yeah it might help to stop swimming, and like start…

Tony: What we would have to do is look at the latest times, compare those, and then we will know.

Kelly: Yeah but see in the Olympics, you don't all get into the Olympics. So, obviously they weren't…

Lana: Oh but he's saying latest times, so everyone's latest times would be in different place really…

depend on their PB, I mean you might be a really good swimmer…your PB might, like, change…because your personal best is your best but changes all the time." The group also spent time considering the PBs in relation to the level of the competition in which these were attained (e.g., a PB earned at the 2000 Olympics was more significant than one at the Telstra Australian Championships). Here the children were displaying an informal understanding of weighted variables, however, they did not pursue this further.

Children in another class were also aware of the limitations of relying solely on PBs, and began to consider possible variations from a given PB. After one group presented their report, a boy asked, "How come you just compared their personal best because they don't do that all the time?" In subsequent class discussion the children explained that the swimmers' most recent times should be considered, "because they could have been slow when they first started, and they could have got stronger…and now they're going faster and faster." The children also commented that the swimmers "could get slower and slower," or "they could just stay on their personal best," or "they could go faster and then slower and then fast."

Assigning Scores and Ranking

We consider here how James' group developed their model. First, they considered each row of Table 1 in turn, and awarded a score of 2 to the swimmer with the lowest time and a score of 1 to the swimmer with the second lowest time. They then added each swimmer's scores and recorded these in their own table (see the first row of Figure 1). As James explained, "Some people got the most amount of points...like, some people both got first…because they need two people, so whoever came first, they would have both got two points." Second, the group considered the PBs of each of the swimmers and awarded 2 points to the lowest PB and one point to the second lowest (see the second row of Figure 1). The group then aggregated each swimmer's scores to find the "total rank." Ian Thorpe and Michael Klim were thus selected.

Other groups only assigned one point to the swimmer with the lowest time in each swimming event and tallied the points to determine the two swimmers to be selected. Ashley's group explained, "Our selection is Ashley Callus and Ian Thorpe because they both had the most winning streaks. We looked at all the competitions and we put a tally on the fastest time and then we counted them up." Ashley's group did not consider the swimmers' PBs.

	Ashley Callus	Michael Klim	Eamon Sullivan	Ian Thorpe	Todd Pearson	Grant Hackett	Adam Pine
Rank	10	10	0	11	4	1	1
PB	0	2	0	1	0	0	0
Total Rank	10	12	0	12	4	1	1

Figure 1: The table created by James' group

CONCLUDING POINTS

The present study is providing young primary school children with opportunities to develop powerful mathematical ideas and processes through mathematical modelling. A modelling problem is a realistically complex situation where students engage in mathematical thinking (beyond that of the traditional school problem) and generate conceptual tools needed for some purpose (Lesh & Zawojewski, in press). Modelling problems foster and reveal children's mathematical thinking thus enabling teachers to capitalise on the insights gained into their children's mathematical developments.

Table 1: Swimming Times Recorded for the Men's 100m Freestyle

Competition	Ashley CALLUS (PB: 48.43 secs)	Michael KLIM (PB: 47.98 secs)	Eamon SULLIVAN (PB 50.06 secs)	Ian THORPE (PB: 48.11 secs)	Todd PEARSON (PB: 48.45 secs)	Grant HACKETT (PB: 48.67 secs)	Adam PINE (PB: 48.68 secs)
2004 Telstra Swimming Grand Prix	51.50 secs	50.44 secs	50.35 secs	49.23 secs	50.19 secs	51.42 secs	51.87 secs
2004 Telstra Olympic Team Swimming Trials	49.31 secs	49.78 secs	50.06 secs	48.83 secs	49.78 secs	50.40 secs	50.24 secs
2003 Telstra FINA World Cup	48.06 secs	50.12 secs	50.24 secs	48.94 secs	48.83 secs	DNC	49.67 secs
2003 Telstra Australian Championships	49.07 secs	DNC	51.86 secs	49.07 secs	49.29 secs	50.32 secs	50.69 secs
2002 Pan Pacs	DNC	48.44 secs	52.43 secs	48.98 secs	49.64 secs	48.67 secs	48.93 secs
2002 Telstra Swimming Grand Prix	51.12 secs	48.58 secs	51.32 secs	48.11 secs	48.45 secs	51.90 secs	51.62 secs
2001 Telstra FINA World Cup	48.43 secs	48.43 secs	51.74 secs	48.81 secs	50.80 secs	DNC	48.90 secs
2001 Pan Pacs	47.81 secs	49.13 secs	53.73 secs	50.79 secs	50.30 secs	51.93 secs	49.46 secs
2001 Telstra Australian Championships	49.46 secs	49.53 secs	55.12 secs	49.05 secs	49.67 secs	51.69 secs	50.27 secs
2000 Telstra FINA World Cup	49.62 secs	47.98 secs	DNC	49.99 secs	48.98 secs	51.42 secs	48.68 secs
2000 Olympic Games	DNC	48.56 secs	DNC	DNC	DNC	DNC	DNC

Note: DNC means did not compete; PB means personal best

Modelling problems are designed to ensure that the product generated embodies the mathematical ideas and processes that children constructed for dealing with the problem situation. In the case of the Olympics problem, the children's creations revealed informal understandings of variation, aggregation and ranking of scores, inverse proportion (the lower the time, the faster the swimmer), and weighting of variables—all of which are not normally part of the 4th-grade curriculum.

REFERENCES

Diezmann, C., Watters, J., & English, L. D. (2002). Teacher behaviours that influence young children's reasoning. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th International PME Conference* (pp. 289-296). Norwich: University of East Anglia.

English, L. D. (2003). Reconciling theory, research, and practice: A models and modelling perspective. *Educational Studies in Mathematics, 54*, 2 & 3, 225-248.

English, L. D., & Lesh. R. A. (2003). Ends-in-view problems. In R. A. Lesh & H. Doerr (Eds.), *Beyond constructivism: A models and modelling perspective on mathematics problem solving, learning, and teaching* (pp.297-316). Mahwah, NJ: Erlbaum.

English, L. D., & Watters, J. (In press.). Mathematical modelling in the early school years. *Mathematics Education Research Journal*.

English, L. D., & Watters, J. J. (2004). Mathematical modelling with young children. In M. Johnsen Hoines & A. Berit Fuglestad (Eds.), *Proceedings of the 28th International PME Conference* (pp. 335-342). Bergen: Bergen University College.

Erickson, F. (1998). Qualitative research methods for science education. In B. J. Fraser & K. G. Tobin (Eds.), *International handbook of science education* (Part 2). (pp. 1155-1173). Dordrecht: Kluwer Academic Publishing.

Gainsburg, J. (2003). *The mathematical behavior of structural engineers.* Doctoral dissertation, Stanford University.

Lesh, R., & Zawojewski, J. (In press). Problem solving and modelling. In F. Lester (Ed.), *Handbook of Research on Mathematics Teaching and Learning*. Information Age Pub.

Lesh, R., & Doerr, H. M. (Eds.) (2003). *Beyond constructivism: A models & modeling perspective on mathematics problem solving, learning, & teaching*. Mahwah, NJ: Erlbaum.

Lehrer, R., & Schauble, L. (2003). Origins and evolutions of model-based reasoning in mathematics and science. In R. A. Lesh & H. Doerr (Eds.), *Beyond constructivism: A models and modelling perspective on teaching, learning, and problem solving in mathematics education* (pp. 59-70). Mahwah, NJ: Lawrence Erlbaum.

Lesh, R. A., & Kelly, A. E. (2000). Multi-tiered teaching experiments. In R. Lesh & A. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 197-23). Erlbaum.

Salomon, G., Perkins, & Globerson, T. (1991). Partners in cognition: Extending human intelligence with intelligent technologies. *Educational Researcher, 22*(3), 2-9.

Simon, M. A. (2000). Research on the development of mathematics teachers: The teacher development experiment. In R. A. Lesh & A. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 335-360). Mahwah, NJ: Lawrence Erlbaum.

Stillman, G. (1998). The emperor's new clothes? Teaching and assessment of mathematical applications at the senior secondary level. In P. L. Galbraith et al. (Eds). *Mathematical modelling: Teaching and assessment in a technology-rich world* (pp.243-254). West Sussex: Horwood Publishing Ltd.

EXPLORING "LESSON STUDY" IN TEACHER PREPARATION

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Prospective secondary mathematics teachers continue to lack images of "reformed" teaching. This investigation studied prospective teachers' development of reform-oriented teaching through Micro-teaching Lesson Study [MLS], an experience based on Japanese lesson study (Stigler and Heibert, 1999). A qualitative analysis of various data sources (video-taped lessons, written lesson plans and reflections, observations, and surveys) for 18 participants working in groups of three was conducted. Findings include growth in understanding and implementing reform-oriented teaching and development of subject matter knowledge. Participants perceived the experience and its components as beneficial in their development as teachers.

INTRODUCTION AND RATIONALE

In many countries, dissatisfaction with teacher education is pressuring toward more field-based programs (Korthagen and Kessels, 1999). More school-based experiences, however, is not the panacea that some expect. Teachers' learning based on their own observations of practices while being students and their individual observations of teachers in placement schools within their teacher education programs leads to teacher preparation that is idiosyncratic and particular (Ball and Cohen, 1999). Zeichner and Tabachnick (1981) and Lortie (1975) have revealed the socialization of teachers into the status quo through the participation in field experiences and personal observations of practice. Prospective secondary mathematics teachers continue to lack images of "reformed" teaching, teaching that engages students in experimenting, analyzing, conjecturing, justifying, and making connections. To better prepare secondary mathematics teachers and reduce the idiosyncratic nature of their development, it seems appropriate to engage them in more collaborative tasks, discourse and environments that will help them more systematically develop images of reformed mathematics teaching and the capabilities to reflect on, reason about and engage in change. As prospective teachers engage in these collaborative experiences, teacher educators should study their development in order to inform the pedagogy of teacher education. The National Academy of Education [NAE] (1999) in a research advisory report underscored as a priority the need for further research on the preparation of teachers.

Mathematics teacher educators need to seek and investigate activities, tasks and contexts that provide prospective teachers with common experiences that will help them more systematically develop images of reformed mathematics teaching. The purpose of this study was to investigate the professional development of prospective teachers engaged in one such experience, named Micro-teaching Lesson Study

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 305-312. Melbourne: PME.

[MLS]. The development of MLS is based on "lesson study," a component of professional development that is highly valued by Japanese teachers and is considered one of the best ideas for improving teaching and education from the world's teachers (Stigler and Heibert, 1999). MLS incorporates key features and phases of Japanese lesson study. The phases of MLS follow those of lesson study: collaborative planning, lesson observation by colleagues and other experts, analytic reflection, and ongoing revision (Curcio, 2002). MLS provides a context for prospective teachers to develop pedagogical content knowledge, knowledge of teaching, content and learning, and images of reform-oriented teaching. MLS may take on different formats. The present investigation studied the use of MLS within an initial course on teaching mathematics. The prospective teachers collaborated in groups of three through three cycles of planning, teaching, analysis and revision of one lesson. As with Japanese lesson study, an important feature of MLS is the focus on learning goals. During this study, the main learning goal for the lessons was to develop students' reasoning and ability to study patterns in discovering relationships or constructing concepts. The following questions were under investigation:

(1) To what extent does MLS help prospective mathematics teachers develop knowledge of teaching aligned with recent reforms in mathematics education?

(2) What were prospective teachers' perceptions of the MLS experience and its components?

RELEVANT LITERATURE

The National Commission on Teaching and America's Future [NCTAF] (1996) identified uninspired methods of teaching prospective teachers as a barrier in the education of qualified teachers that make learning come alive. Prospective teachers are expected to excite and motivate students to learn; yet their own learning experiences are often uninspired and traditional. Methods courses are often taught through lectures and recitation. Prospective teachers complain that methods courses are not intellectually substantive or that discussions of theory and research are not sufficiently oriented toward practice (Commission on Behavioral and Social Sciences and Education [CBSSE], 2000). MLS provides prospective teachers with an intellectually challenging opportunity to explore the use of theory and research based instructional methods.

From a cognitive perspective, knowledge construction is assumed to be a dynamic and active process. Learners construct knowledge while attempting to make sense of their experience. The construction of knowledge involves the interaction of past knowledge with the experience of the moment (Resnick, 1987).

Current information about human learning suggests that learning environments should be learner centered, knowledge centered, assessment centered, and community centered (CBSSE, 2000). The MLS is a collaborative experience encompassing these four components. The MLS is learner centered with prospective teachers working in small groups to develop, implement, and analyze the teaching of mathematics lessons. It is knowledge centered focusing on helping prospective teachers understand and develop images of reform-oriented teaching as envisioned by the National Council of Teachers of Mathematics [NCTM] (1991, 2000) through discussion, exploration, experimentation, and analysis. Knowledge is developed through cycles of planning, implementing, and reflecting on lessons. The opportunity for feedback and revision within the teaching cycles provides for the assessment centeredness of the task. The MLS is community centered in that it provided prospective teachers with opportunities to experiment, make mistakes, discuss, and negotiate among peers and an instructor their emerging understandings of teaching toward goals aligned with reform-oriented teaching.

METHODS, DATA COLLECTION AND ANALYSIS

The research design involved primarily qualitative data collection and analysis. Data collection included observations of planning, implementing, and analysis of lessons, group documentation of three cycles of planning, teaching, analysis and revisions, video-tapes of lessons, and surveys of feedback, analysis, and collaboration. The investigation was conducted in an initial course on learning to teach mathematics that included 18 prospective teachers. Prior to the MLS experience, the prospective teachers were engaged in discussing readings and analyzing videotaped lessons, focusing on reform-oriented teaching of mathematics. Additionally, they were engaged in mathematical tasks modeling the current vision of mathematics teaching proposed by NCTM (1991, 2000). The prospective teachers were given an initial survey to assess their knowledge of potential mathematics topics for lessons. Based on these surveys and instructor observations, the prospective teachers were grouped heterogeneously into six MLS groups each consisting of three prospective teachers.

Each group was involved in three cycles of planning, teaching, analyzing and revising a mathematics lesson with a goal of promoting reasoning and the study of patterns in discovering a relationship or constructing a concept. Mathematics topics for the lessons included fractals, traceable paths, Euler's formula, permutations, prisms and pyramids, and ellipses. The lessons were taught to small groups of peers in the same course; class members lacked understanding of the lesson topics as determined by the initial survey. Each group completed a written assignment consisting of 5 sections that guided them through the phases of the MLS based on Japanese lesson study: Section I included pre-lesson thoughts, materials explored and lesson plan; Section II included video of first teaching, analysis of the lesson (individual and group reflections) and revisions to lesson plan; Section III included video of second teaching analysis (individual and group reflections) and revisions; Section IV included video of the third teaching, analysis (individual and group reflections) and final revisions; Section V included the final revised lesson along with suggestions for teaching the lesson to be distributed to class members. Analysis of the lessons was conducted with respect to a video analysis framework based on the vision of mathematics teaching promoted by NCTM (1991). Groups were observed working together and given feedback on their video-taped lessons by the instructor during the phases of the MLS. Field notes of these observations and interactions were kept. At the end of the four week MLS experience, final surveys assessing prospective teachers' views of lesson feedback, analysis, collaboration and understanding of reform-oriented teaching were gathered.

The multiple data sources described above were used to triangulate the findings. Analysis began with the MLS projects. The videotapes of the lessons and the written lesson plans, submitted as part of the MLS projects, were coded with respect to the pedagogy used and the knowledge of the subject-matter presented. In particular, lessons were coded and compared with respect to the engagement of student peers in discovering, developing or constructing relationships or concepts (student-centeredness), and teacher telling, showing, and stating the relationships or concepts (teacher-centeredness). The MLS projects and observation notes were coded with respect to the prospective teachers' learning about content and pedagogy, and their perceptions of the experience. Findings from the projects and the observations were triangulated with prior findings from the analyses of the videotaped lessons and lesson plans. Finally, responses to the Microteaching Feedback Surveys were tallied and open-responses coded, and then compared and contrasted with the emerging themes. The Microteaching Feedback Surveys were analyzed for participants' perceptions of their learning and views about the MLS experience. As themes developed throughout the data analyses, they were confirmed or disconfirmed through data triangulation.

FINDINGS

From analysis of the video-taped lessons and the written lesson plans, as prospective teachers engaged in the MLS their second lessons became less teacher-centered and incorporated more student experimentation, analysis and reasoning than their first lessons. For example, the group teaching Euler's formula for polyhedrons, had in their first lessons typically provided the formulas or definitions for their topic and focused on application of that information. Later during the second phase of the project, after receiving feedback from the instructor and analyzing the video-tape of their own lesson, they engaged the students in experimenting with a variety of polyhedrons, looking for patterns and using reasoning to generate a relationship. The prospective teachers themselves observed this difference in their own lessons:

> The improvement from our first to second lesson was dramatic. Our first lesson was far more teacher-centered and did not really center around the idea of constructing a concept or discovering a relationship. This time, we kept to the idea of constructing definitions, not stating, and then justifying them. This time it seems as though we had a better understanding of how to do this. (MSL Group 1)

In the MLS group teaching about ellipses, one of the participants was one of the high math content exam scorers. Her group observed that her lessons were more effective in engaging students' thinking and all three improved on their student-centeredness and engagement of students in the discourse during the second lesson:

> As a group we realized how important it was to make each lesson more student led rather than teacher led. Getting away from a teacher-centered classroom is hard, since that is how most of us learned. After seeing how a student-led lesson can improve a student's performance, we see how valuable it is in our teaching. (MSL Group 6)

Participants also reflected on their own changes with respect to student participation. For most of the prospective teachers, the feedback and analysis of their video-taped lessons seemed to be an important aspect of the MLS. This feedback and analysis helped them to think about their own teaching. One prospective teacher reported,

> I feel that the input that I received from the first lesson had an impact on the way I intend to run my classroom. It made me realize that I needed to get the students to participate and play a more active role in the learning process. I think this helped a lot with the success of the [second] lesson" (member MLS Group 2)

Through the MLS experience, in addition to developing their understanding of teaching strategies aligned with recent reforms, the prospective teachers enhanced their mathematics subject matter knowledge. For example, after the first lesson, a few groups struggled with understanding the relationships they were teaching and the need to understand more in-depth arose from questions raised during the lessons. For example, toward the end of the initial lesson on traceable paths, the students being taught asked why a traceable path can have at most two odd vertices (the teacher could not figure it out before time ran out). After the lesson, when the teacher for that lesson got together with her MLS Group members they further explored the relationship in order to understand why this occurred and incorporated this into their next lesson. The next two times the lesson was taught, the teachers did not raise this question and their students did not bring it up as in the first group. The teachers in all of the MLS groups tended to focus less attention on deductive reasoning, justifying the relationships that were discovered or constructed, and more on inductive reasoning, analyzing, looking for patterns and making conjectures. In another example, during the initial lesson on permutations, the group member teaching about permutations had students develop two formulas for determining permutations but did not engage the students in explaining how the formulas were related nor had she thought about it with he group members. The instructor's feedback challenged this aspect of their lesson and the group struggled to understand the relationship and incorporated this connection into their next lessons.

Although most of the lessons improved as the groups progressed through the MLS cycles, the last lesson on traceable paths did not demonstrate as much improvement as the others. This group engaged in limited collaboration as a whole and tended to be overly concerned with others feelings as they discussed the lessons taught. (Their concern for the others feelings was expressed in the Microteaching Feedback Survey.) The member teaching the third lesson did not fully develop her understanding of traceable paths. During her teaching, although she tried to engage the students in reasoning about traceable paths and non-traceable paths, she quickly gave up when the students began to struggle with discovering the relationship (this struggle is a normal aspect of problem solving). She moved on to provide the definition for them and then explained how to use paths to represent maps. Although her MLS group members' assessment of her lesson lacked some depth because of concern for her feelings, through the experience of reflecting on her group members prior lessons on traceable paths, she recognized that she gave up and did not achieve the overarching lesson goal of developing students' reasoning and ability to study patterns in discovering relationships or constructing concepts. She commented, "I believe that I could have been a little more prepared with the definitions and paid a little more attention to the students constructing their ideas of the concepts but I don't think that John and Tami would say that to me."

The Microteaching Feedback Survey provided valuable information about individual and group perspectives on the MLS experience and its components. When asked "What were the two most important things you got out of the Microteaching Project, the participants tended to feel primarily that the feedback they received from their own group members along with their own observations through the videotapes was most beneficial in their learning to teach. Several commented on learning to facilitate student engagement in exploration, analysis and explanation as most important while others felt that the experience help them understand lessons as "works in progress."

On the feedback survey, the participants were asked to provide a rating and explanation of their rating for each statement. The ratings were (1) Strongly Agree, (2) Agree, (3) Nuetral, (4) Disagree, (5) Strongly Disagree. Table 1 summarizes their perceptions by average ratings.

Feedback Item Summary	Ave.
1. Planning in a group broadened my knowledge of teaching ideas	2.0
2. Analyzing others' lessons helped me think more deeply about mine	2.0
3. Analyzing each others' lessons helped me learn to assess lessons	2.3
4. The video analysis framework was helpful in analyzing our lessons	2.6
5. Feedback from group members was helpful	1.8
6. Concern for others' feelings influenced my feedback	3.3
7. I was upset by feedback from my peers	4.4
8. I was upset by feedback from my instructor	4.0
9. Planning together broadened my knowledge of the mathematics	1.9
10. Preparing to teach this topic caused me to engage in mathematical reasoning and problem solving	2.2

Table 1: Summary of MLS Feedback Survey

From the survey, group member feedback and group planning were revealed to be very important to the participants. The participants, however, expressed some concern for other's feelings when providing feedback. This is important to be aware of in experiences where prospective teachers are asked to provide feedback to one another or engage them in thinking critically about their teaching because they may not be helping their partners reach higher levels of thinking about their teaching. The role of the instructor or expert in these cases becomes more important in order to facilitate greater learning opportunities for prospective teachers. The feedback also indicated that students felt they were broadening their mathematics knowledge through the MLS experience.

CONCLUSIONS AND IMPLICATIONS

The MLS experience seems to help prospective teachers understand and begin implementing teaching practices that are consistent with reform-oriented teaching. It also raised the tension between subject matter knowledge, pedagogical knowledge, and practice for prospective teachers to begin experiencing, reflecting on, and working through. This experience provides an opportunity for prospective mathematics teachers to begin linking theory and practice as they engage in cycles of planning, implementing and reflecting on lessons.

In communities where appropriate placements for prospective secondary mathematics teachers are limited, MLS can reduce the burden of school placements. This opportunity can be used in addition to school-based experiences to provide less idiosyncratic teacher preparation experiences.

References

CBSSE (2000). *How people learn: Brain, mind experience, and school*. Washington, DC: National Academy Press.

Ball, D. L. & Cohen, D. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In L. Darling-Hammond and G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice*. San Francisco: Jossey-Bass.

Curcio, F. R. (2002). *A user's guide to Japanese lesson study: Ideas for improving mathematics teaching*. Reston, VA: NCTM.

Korthagen, F. A. & Kessels, J. P. A. M. (1999). Linking theory and practice: Changing the pedagogy of teacher education. *Educational Researcher*, 28(4), 4-17.

Lortie, D. (1975). Schoolteacher: *A sociological study*. Chicago: University of Chicago Press.

National Academy of Education (1999). *Recommendations regarding research priorities: An advisory report to the National Educational Research Policy and Priorities Board*. New York: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author.

National Commission on Teaching and America's Future. (1996). *What matters most: Teaching for America's future*. New York: Teachers College, Columbia University.

Resnick, L. B. (1987). *Education and learning to think*. Washington, DC: National Academy Press.

Stigler, J. W. & Heibert, J. (1999). The Teaching Gap. New York, NY: The Free Press.

Zeichner, K. and Tabachnick, B. R. (1981). Are the effects of university teacher education washed out by school experiences? *Journal of teacher education*, 32, 7-11.

CHILD-INITIATED MATHEMATICAL PATTERNING IN THE PRE-COMPULSORY YEARS

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This paper addresses the nature of child-initiated episodes of mathematical patterning prior to formal schooling. In a multi-site case study, children's engagement in mathematical patterning experiences was investigated as was the teachers' involvement and influence in these experiences. A conceptual framework was used to guide the examination of how children generate, engage in, and direct mathematical patterning activity. The analysis of two child-initiated patterning episodes revealed that they provide rich learning opportunities for both the children who initiate the episodes and their peers who share the episodes. The results also highlight the important role of the teacher in fostering children's patterning development.

BACKGROUND

Educators and mathematicians have emphasized the importance of pattern in mathematics and acknowledge its essential role in the development of mathematical knowledge, concepts and processes. In fact, Steen (1990) argued that "Mathematics is the science and language of pattern" (p. 5). Pattern exploration has been identified as a central construct of mathematical inquiry and as a fundamental element of children's mathematical growth (Burns, 2000; Clemson & Clemson, 1994; Heddens & Speer, 2001; NCTM, 2000). The years prior to formal schooling (pre-compulsory education and care services) are widely recognised as a period of profound developmental change, where many mathematical concepts begin (Clements, 2000; Ginsburg, 1997). The salient role of patterning in the development of mathematical knowledge is evident in its inclusion in various curriculum documents (National Council of Teachers of Mathematics [NCTM], 2000; Queensland School Curriculum Council, 1998; Queensland Studies Authority, 2004; Ministry of Education [N.Z.], 1996).

SIGNIFICANCE OF MATHEMATICAL PATTERNING

Young children's knowledge and skills in mathematics are developed and made meaningful through processes such as comparing, counting, symbolizing, classifying, measuring, representing, estimating and patterning. Within the mathematical domain, patterning can be defined as something that remains constant within a group of numbers, shapes or attributes of mathematical symbols or concepts. The arrangement of the group possesses some kind of clear regularity through the use of repetition. For example, Charlesworth (2000) proposed that patterning is a process of "discovering auditory, visual, and motor regularities" (p. 190). Whilst there are three categories of patterning (repeating, growing, and relationship), repeating patterns are the earliest form of pattern explored (Burns, 2000). Children as young as three and a half years of age display a strong interest in patterns (Ginsburg, Inoue, & Seo, 1999). Ginsburg et al. investigated preschool children's participation in everyday mathematical activities within the educational setting. The aim of their study was to investigate the relative frequency of different types of mathematical activity. Children engaged in pattern and shape activities for 30% of the observed time. This degree of engagement was markedly higher for patterning than for the other mathematically-orientated activities identified, such as dynamics (i.e., the process of change or transformation), relations, classification and enumeration.

Mathematical patterning provides a substructure upon which formal mathematical competencies can be built. Because the study of patterns underpins all mathematical thinking, it has a close connection to mathematical content areas, such as number, geometry, measurement, and data. Although patterning is integral to the mathematics curriculum in the compulsory years of schooling, it is also a feature of other curricula. Patterning opportunities occur across the curricula in science, art, language, music, and physical education. Hence, from a child's earliest years, patterning is foundational within and beyond the mathematics curriculum because it assists children in making sense of their everyday world.

MATHEMATICS LEARNING IN PRE-COMPULSORY SETTINGS

In early childhood education (as with later education), mathematics is not simply "a static network of terms, rules and procedures that are conveyed by teachers and absorbed by students for recall upon demand" (Campbell, 1999, p. 108). Rather, recent curriculum documents describe mathematics as a way of thinking about relationships, quantity, and pattern via the processes of modelling, inference, analysis, symbolism, and abstraction (e.g., NCTM, 2000).

Recent research has provided considerable insight into how children learn mathematics and has influenced current curriculum documents (Ginsburg, 2002). Curriculum documents such as *Early Years Curriculum Guidelines* (Queensland Studies Authority, 2004), *Te Whariki* (Ministry of Education [N.Z.], 1996) and *Principles and Standards for School Mathematics* (NCTM, 2000) currently reflect the constructivist and social constructivist theories of learning. The basic tenet of constructivism, as described by Heddens and Speer (2001) is that "learners construct their own meaning through continuous and active interaction with their environment" (p. 13). Social constructivism, informed by Vygotsky, recognises that learning is a process that occurs within social interactions emphasised by social collaboration and negotiated meanings (Klein, 2000). Social constructivism theory recognises that children's social and material interactions with their environment are the means through which they learn.

Early childhood curricula also recognise the value of play and the use of concrete materials in children's mathematical development (NCTM, 2000; Perry & Dockett, 2002). Young children's play can be elaborate depending on the theme, content, social interaction and the nature of the understandings demonstrated and generated (Perry & Dockett, 2002). Many mathematical experiences occur during children's play. For example, Ginsburg et al. (2000) noted that 42% of observed play activities engaged in by 4 and 5-year-old children featured mathematical experiences. Play is a valuable component of child-initiated curriculum, an approach which recognises children as the source of the curriculum (Perry & Dockett, 2002). Thus, for young children, playtimes provide an opportunity for the exploration of mathematical patterning and other mathematical concepts. Provided that their mathematical experiences are appropriately connected to their world, young children are capable of exploring ideas "in more sophisticated and rich ways than previously believed possible" (NCTM, 2000, p. 103).

RESEARCH DESIGN

Settings and Participants

Two classrooms were chosen for involvement in this study: one preschool classroom (site A) and one preparatory year classroom (site B). The two sites were located in inner city Brisbane and were geographically close (two kilometres apart); they shared similar socio-economic clientele. Both classroom settings were arranged into interest areas such as block corner, home corner, collage table, and sand and water areas. The teachers' daily programs incorporated both teacher-directed times and free play opportunities. Each setting had 13 female and 12 male children and was staffed by a 4-year trained early childhood teacher. Each teacher had in excess of 10 years teaching experience in both informal and formal educational settings.

Data analysis

A case study (Yin, 2003) was undertaken to gain an understanding of the nature and occurrence of mathematical patterning in pre-compulsory settings. Analysis of a total of approximately 80 hours of video observations collected in the two classrooms revealed ten mathematical patterning episodes. Two episodes were initiated by children and the other eight were guided by the teachers. This paper focuses on the observed child-initiated episodes. An episode is defined as an observed occurrence containing some aspect of mathematical patterning behaviour.

CHILD-INITIATED EPISODES

Episodes instigated by the children were explicit (clearly articulated) or implicit (suggested but not clearly expressed). The analysis of these episodes was informed by Stein, Grover and Henningsen's (1996) conceptual framework, which focuses on classroom-based factors that influence student engagement with cognitively demanding mathematics tasks in real classroom settings. The Stein et al. framework was adapted to suit early-childhood settings (see Figure 1) in order to examine episodes that children initiated and that featured mathematical patterning. The framework comprises three phases (as represented by the rectangular boxes). The first phase depicts the *original context* prior to the child-initiated event. A *child-initiated episode* (phase two) is where a child initiates an occurrence that incorporates an aspect of mathematical patterning behaviour. The third phase, *responses to child-initiated task,* considers the responses made to the child's overtures by both the teacher and class members. The child's peers may contribute to the newly initiated task or engage in dialogue with them and extend the task further. This framework also includes factors which influence the initiation and response phases (phases two and three). *Factors influencing the event* include children's knowledge of mathematical patterning, their interests, and prior experiences. The physical environment and availability of resources can also influence the episode. The *factors influencing participation* in the episode include task appeal, the involvement and encouragement of the teacher and peers, and other participants' knowledge of patterning.

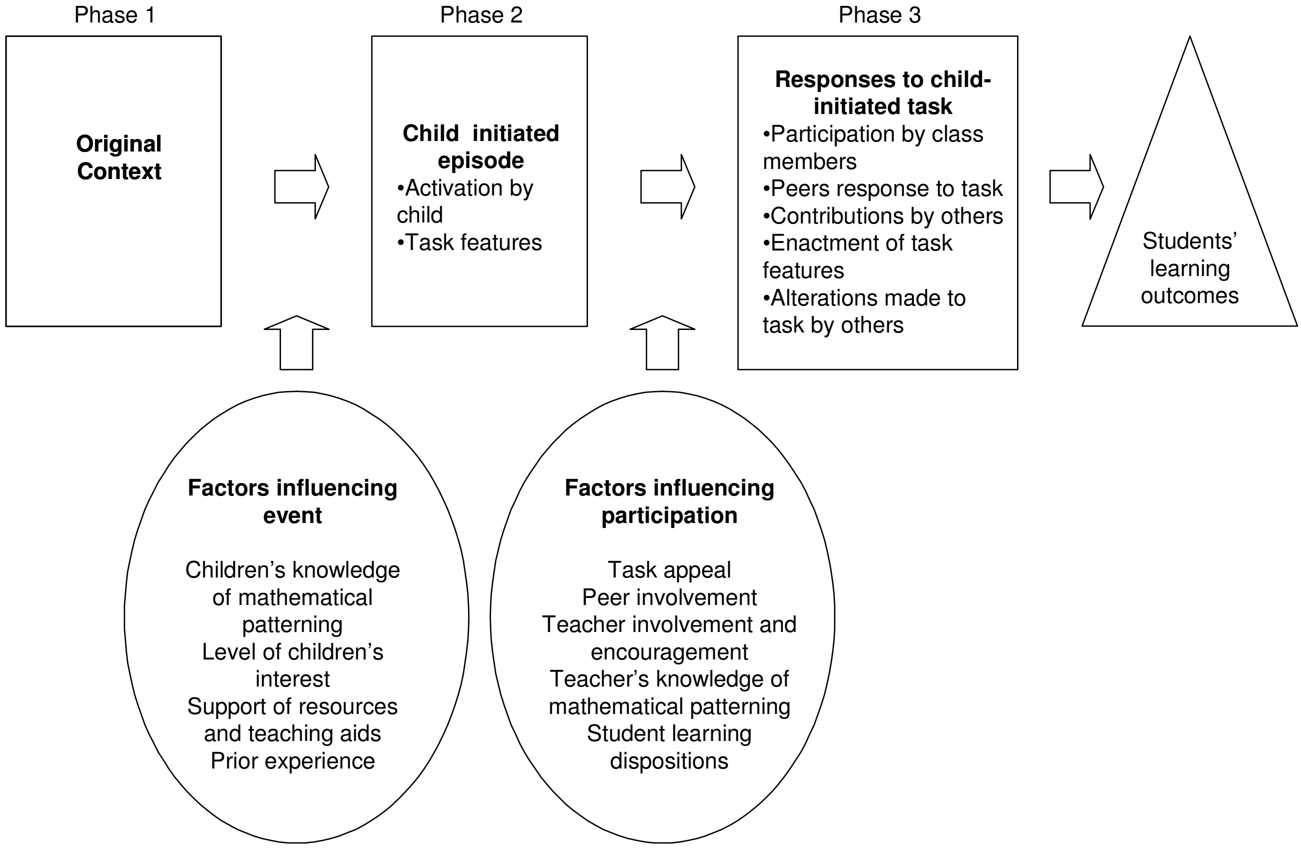

Figure 1: Framework illustrating components contributing to child-initiated activities.

FINDINGS

Two episodes of child-initiated mathematical patterning were observed in the case study. The first episode occurred in the preparatory setting (site B). A child named Ashleigh engaged in an independent activity at the painting easel (*original context*). Paints, paint brushes, paper and toothbrushes were placed in the outdoor area and made available for children to use at their own instigation. Ashleigh was observed using the brushes to paint stripes. She was talking to herself saying "pink, purple, pink, purple." She repeated the set twice before beginning a new set of stripes (*child-initiated episode*-phase 2). Another child, Nicole, observed Ashleigh's painting and said "I'm going to be green, pink, purple." Nicole made four sets of a green, pink and purple pattern. She then painted lots of green stripes and then put orange stripes in between the green lines. Nicole made more AB patterns using orange/green and purple/yellow. Nicole then said aloud (to who-ever was present) "Look at my patterns" (see Figure 2). Nicole's participation in this event constitutes the *response to child-initiated task* (phase 3). From a distance, Mrs Jones (teacher) observed the children discussing their creations. Mrs Jones called out "Looks like you are doing some lovely art work", and continued inside the Centre. Mrs. Jones did not seem to be aware of the opportunity she had failed to capitalise on through not noticing the children's interest in mathematical patterning. Factors influencing this episode included Ashleigh and Nicole's knowledge of repeating linear patterns. The available paint and resources provided them with the opportunity to play and explore. Influencing factors such as the degree of teacher involvement and encouragement may have limited the potential of this activity however the children shared their knowledge and encouraged each other to participate in the experience.

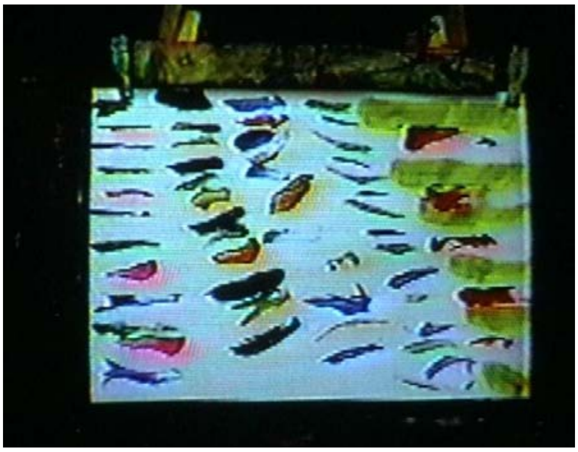

Figure 2: Example of pattern creation using paint colours

Figure 3: Example of pattern creation using tap-tap equipment.

The second episode occurred in the preschool setting (site A). A child named Chelsea was sitting at an inside table independently interacting with manipulative equipment called '*tap tap'* (a hammer and nails construction kit). This construction material had been placed on a table for the children's use. No instructions for its use were provided by the teacher (*original context*). Chelsea initiated an episode (phase 2) by tapping shapes on to the cork board and described it to other children at the table. "It is a necklace with diamonds – diamond, funny shape, diamond, funny shape, diamond, funny shape" (Figure 3). The teacher questioned Chelsea about her creation. After teacher intervention, another child, Harriet, began to use the 'tap tap' equipment to make a repeating pattern (yellow circle–green triangle). A second child, Emma, joined the table and created a necklace utilising an ABBA pattern (*response to child-initiated task*-phase 3). Chelsea's explicit interest in mathematical patterning (*factors influencing event*) seemed to provide the stimulus for other children to join her in creating patterns. The teacher's involvement and intervention also encouraged the children to participate and create patterns. Three children participated in this episode enthusiastically and the episode provided exposure to mathematical patterning concepts in a play-based experience.

The two child-initiated episodes occurred as a result of children's current interests and as the individuals shared their thoughts or creations with peers, their interest grew and developed. As other children became involved and contributed to both activities the learning and knowledge was shared, altered, and extended. Table 1 illustrates that all the components of the learning framework were present with the exception of *factors influencing participation* in the first episode. While this episode involved a successful interaction and exchange of knowledge between two girls, the teacher's acknowledgement and involvement could have also contributed to the episode. As seen in the second episode, when the teacher played a role in the episode more children seemed to engage in mathematical patterning behaviours. During these episodes repeating linear patterns were created by children in unstructured play times. These occurrences were productive exchanges initiated by the children THAT encouraged the exploration of mathematical patterning.

Episode number	Phase 1	Phase 2	Phase 3	Learning outcome	Factors inf. setup	Factors inf. participation
1	✓	✓	✓	✓	✓	×
2	✓	✓	✓	✓	✓	✓

Table 1: Framework elements observed during child-initiated episodes

CONCLUDING POINTS

Mathematical patterning provides an essential foundation for many mathematical concepts and processes. Williams and Shuard (1982) claimed that "the search for order and pattern … is one of the driving forces of all mathematical work with young children" (p. 330). As Piaget (1973) attested, children are natural learners and are motivated to learn – their minds are created to learn. Early childhood curricula support these beliefs in their endorsement of constructivism, social constructivism, and play-based learning (NCTM, 2000; QSA, 2004).

Findings in this case study suggest that child-initiated episodes containing mathematical patterning are productive learning occurrences. During unstructured play times, children initiated activities that explored repeating patterns, pattern language, and the elements of linear patterns. These episodes were rich opportunities where children shared, refined, and developed their knowledge of patterns. The children featured in these episodes manipulated the resources provided in the pre-compulsory settings to explore mathematical patterning. Thus, child-initiated experiences can be powerful learning opportunities with the potential to develop children's knowledge of mathematical patterning in meaningful contexts.

Teachers within the early childhood settings have an essential role in fostering children's mathematical patterning activities. However, teachers need to have knowledge of mathematical patterning and be capable of capitalising on children's interests (Waters, 2004). The role of the teacher in questioning, providing resources, being involved, and offering encouragement has the potential to enrich mathematical patterning experiences and extend the children's existing knowledge. Further studies could provide greater insight into how teachers' knowledge and involvement could influence and enhance similar learning opportunities.

References

Burns, M. (2000). *About teaching mathematics - a K-8 resource*. Sausalito, CA: Math Solutions Publications.

Campbell, P. F. (1999). Fostering each child's understanding of mathematics. In C. Seefeldt et al. (3rd ed.). *The early childhood curriculum: Current findings in theory and practice* (pp. 106-132). New York: Teachers College Press.

Charlesworth, R. (2000). *Experiences in math for young children* (4th ed.). New York: Delmar Thompson Learning.

Clements, D. H., Surama, J. & DiBiase, A. M. (Eds.). (2003). *Engaging young children in mathematics – Standards for early childhood mathematics education*. Mahwah, NJ: Lawrence Erlbaum.

Clements, D. H. (2000). Standards for preschoolers, *Teaching Children Mathematics, 7*(1), 38-41.

Clemson, D. & Clemson, W. (1994). *Mathematics in the early years*. London: Routledge.

Ginsburg, H. P. (1997). *Entering the child's mind*. The clinical interview in psychological research and practice. Cambridge: Cambridge University Press.

Ginsburg, H. P., Inoue, N. & Seo, K. H. (1999). Young children doing mathematics: Observations of everyday activities. In J. Copely (Ed.). *Mathematics in the early years* (pp.88-99). Reston, VA. National Council of Teachers of Mathematics.

Ginsburg, H. P. (2004). Little children, big mathematics: Learning and teaching in the Pre-School. In A. d. Cockburn and E. Nardi (Eds)., *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education* (pp. 3-14). East Anglia: University of East Anglia

Heddens, J. W. & Speer, W. R. (2001). *Today's mathematics concepts and classroom methods*. New York: John Wiley & Sons.

Henningsen, M. & Stein, M. (1997). Mathematical tasks and student cognition: Classroom based factors that support and inhibit high level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, *28*(1), 524-529.

Klein, M. (2000). *Teaching mathematics against the grain – investigation for primary teachers*. Australia: Social Science Press.

Ministry of Education (1996). *Te Whariki early childhood curriculum*. Wellington: Learning Media Ltd.

National Council of Teachers of Mathematics. (2000*). Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Perry, B. & Dockett, S. (2002). Young children's access to powerful mathematical ideas. In L. D. English (Ed.). *Handbook of international research in mathematics education* (pp.81-112). Mahwah, NJ: Lawrence Erlbaum.

Piaget, J. (1973). *To understand is to invent: The future of education*. New York: Grossman.

Queensland School Curriculum Council. (1998). *Preschool curriculum guidelines*. Brisbane: Open Access Unit.

Queensland Studies Authority (2004). *Mathematics Year 1 to 12 syllabus*. Brisbane: Author.

Queensland Studies Authority (2004) *Early years curriculum draft guidelines*. Brisbane: Author.

Steen, L. A. (Ed.). (1990). *On the shoulders of giants: new approaches to numeracy*. Washington DC: National Academy Press.

Stein, M. K., Grover, B. W. & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Education Research Journal, 33*, 455-488.

Waters, J. (2004). A study of mathematical patterning in early childhood settings. In I. Putt, R. Faragher, M. McLean (Eds.). *Mathematics education for the 3rd millennium: Towards 2010* (pp.565-572). Proceedings of the 27th Annual Conference of the Mathematics Education Research Group of Australasia. Sydney: MERGA.

Williams, E. & Shuard, H. (1982). *Primary mathematics today* (3rd Ed.). Harlow: Longman Group.

Yin, R. K. (2003). *Case study research, design and methods*. Thousand Oaks, CA: Sage Publications.

THE TACIT-EXPLICIT NATURE OF STUDENTS' KNOWLEDGE: A CASE STUDY ON AREA MEASUREMENT

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This paper reports on case study that investigated the development of mainly tacit and mainly explicit components of knowledge of area measurement of a student-pair. The research covered two terms or periods of the students' learning of the subject: when they were aged 11 to 12 and when they were aged 12 to 13. The data analysis was based on Ernest's model of mathematical knowledge, with reference to its mainly tacit and mainly explicit components, and Kitcher's ideas about the development of mathematics practice. The results of the research reinforced our hypothesis that students' mathematical knowledge displays a very similar structure to that of the mathematical knowledge of the mathematicians.

INTRODUCTION

The concept of tacit knowledge does not have a single meaning. As discussed in Frade's (2004) work some researchers address what can be called *Polanyi's psychological version of tacit knowledge*: knowledge that functions as subsidiary to the acquisition of other knowledge. Other researchers use the words *tacit* and *explicit* as opposites to refer to different, but complementary ontological dimensions of the same component of a certain practice. Whatever meaning we choose – psychological or ontological – the researchers quoted by Frade (*ibid*) share in some way Polanyi's (1969) epistemological thesis that all knowledge is tacit or constructed from tacit knowledge: put it in another way, language alone is not enough to render knowledge explicit.

We used the two above-mentioned meanings of tacit knowledge in a research to investigate its manifestation in empirical data. Our research was carried out in a mathematics classroom of a Brazilian secondary school and consisted of two sequential studies. In the first study (see Frade, 2004) we analysed an episode related to a class discussion about the difference between plane figures and spatial figures. The aim of this study was to identify how the mainly tacit and mainly explicit components of students' knowledge (see Ernest, 1998) could manifest in learning processes or in a subsidiary way, from Polanyi's (1962, 1969) perspective. The results of the research strongly pointed to a perspective of cognition not necessarily restricted to and coincident with language, but seen as a situated social practice, moving between the poles of the tacit – effective action – and the explicit – intersubjective projection of such an action – dimensions.

In the second study a student-pair (2 boys) in the same class was investigated as they undertook different mathematical tasks on area measurement. Here, the research covered two terms or periods of the students' learning of the subject: when they were aged 11 to 12 and when they were aged 12 to 13. The aim of this study was twofold:

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 321-328. Melbourne: PME.

1) to observe the development of the mainly tacit and mainly explicit components of the student-pair's area measurement knowledge; 2) to provide more information on how the tacit and the explicit interact during tasks involving conversation. The study in question and its results are presented in this paper. In particular, we highlight the first aspect of our analysis, for the later was intensively discussed in the first study.

THEORETICAL FRAMEWORK

Ernest (1998) uses the ontological meaning of tacit knowledge to classify the nature of the components of his model of mathematical knowledge. To the author, mainly explicit mathematical knowledge is related to those types of knowledge that can be communicated through propositional language or other symbolic representation, as for instance: 1) accepted propositions and statements (e.g. definitions, hypotheses, conjectures, axioms, theorems); 2) accepted reasoning and proofs (all types of proofs including the less formal ones, inductive and analogical reasoning, problem solution including all analysis and computing); 3) problems and questions relevant to be solved by the mathematicians (e.g. Hilbert's problems, Last Fermat's theorem). Alternatively, mainly tacit mathematical knowledge is related to the ways in which the mathematicians use their knowledge, as well as how they appropriate mathematical experiences, values, beliefs through their participation in mathematics practice. And this, says Ernest, cannot be fully communicated explicitly. As mainly tacit components of mathematical knowledge he cites: 4) knowledge-use of mathematical language and symbolism; 5) meta-mathematical views, that is, views of proof and definition, scope and structure of mathematics as a whole; 6) knowledge-use of a set of procedures, methods, techniques and strategies; 7) aesthetics and personal values regarding mathematics[1].

From this perspective, we hypothesized that the students' mathematical knowledge could display a similar structure to that of the mathematical knowledge of the mathematicians. Thus, the above-mentioned components were those ones which we search to identify in the case study. To this end, we proposed an adaptation of Ernest's model of mathematical knowledge to the students' knowledge. Such adaptation is illustrated in the next section, and accounted for the fact that the students are learners and part of the learning process consists in a gradual improvement of their understanding and procedures, which in their initial manifestation may seem mistaken from the viewpoint of the discipline. In particular, the component *aesthetics and values* was associated with the students' predisposition, motivation and participation in classroom practices, or else to the students' mathematical identity as, for example, Boaler (2002) and Winbourne (2002) put it. Therefore, this component has a macro character in the sense that it is a necessary

[1] In this presentation of Ernest's model, the first five components were proposed by Kitcher (1984) whereas the last two ones were proposed by Ernest. Due to lack of space we opted not to present the arguments used by Ernest to classify the model's components as mainly explicit or mainly tacit. These arguments are very insightful and can be seen in Ernest (1998).

condition for the development of the remaining components. The component *problems and questions* was not investigated, as it was difficult to adapt it adequately to the students' knowledge.

THE CASE STUDY

This study consisted of a set of short sequential episodes – seventeen in total – constructed to identify the student-pair's stages of development in terms of the mainly tacit and mainly explicit components of area measurement. The data were collected from their work on mathematical tasks (e.g, oral and written exercises, problem solving, individual tests, interviews) and from audio and video recording of the student's class work. In all episodes that involved mathematical conversation we also examined the internal articulations that preceded the students' utterances, applying the categories presented in the first study (see Frade, 2004) : priority of tacit, tacit on the borderline with the explicit, tacit coincides with explicit, explicit separate from tacit, explicit under check. Bellow we provide a description of an episode to exemplify how the data were treated in this study.

Episode 1: Student 1 confuses a counting, and student 2 discovers the multiplication formula length× width.

During the course of classes 1 and 2, student 1 and student 2 were doing some exercises proposed by their textbook. They were trying to calculate how many ceramic tiles covered the floor of a rectangular room. The book displayed the drawing of the room, which facilitated the students in counting the tiles. Calculating area by counting the units of measurement was the only procedure worked at class, until then. While student 1 finds apparent meaningless numbers, student 2 discovers the multiplication formula length×width. Let us see what happened in the protocol below transcribed from audio tapes:

Student 2: What is the result?

Student 1: 71 and 57.

Student 1: 178.

Student 2: 15, 1, 2, 3...15 times 12. 170? Because here, look 1, 2, 3 ... 16. (Student 2 multiplication is incorrect: $15\times12 = 180$)

Student 2: 1, 2, 3... 16. 16 times 12. 192.

Student 2: It's 16. 1, 2...16.

Student 2: The result is 192, isn't it? Because here, look, 16, we have to count the width of the tiles.

Student 1: 178

Student 2: 178?

Student 1: Then write it there: 178. (...) There are 178, 178 tiles on the floor.

Student 2: On the floor, on the floor.

The underlined utterances show that, instead of counting the number of tiles that covered the floor, student 2 chooses to multiply the number of tiles in each row by the number of tiles in each column. However, the utterances in italics suggest that he gave up of that procedure, probably influenced by student 1's insistence to give the final result: 178. Searching for a better understanding of the students' calculations, we have analyzed their written registers of the exercises. Both students wrote: '178 tiles', but they did not record any calculation or reasoning.

Student 2's utterances seemed to be good external representations of what he was thinking while solving the problem, as it was possible to infer about his reasoning. The same cannot be said in relation to student 1's utterances. When he refers to numbers 71 and 57 it is possible that the tacit could be prevailing over the explicit: the clues gave by him were extremely vague. And this would only support an equally vague hypothesis about his reasoning, which could not be publicly checked.

In short, this episode captures a moment of strategy choice by student 2: the above-mentioned multiplication. This strategy choice was identified with a manifestation of the model's component *knowledge-use of a set of procedures, methods, techniques and strategies* by student 2. The component *reasoning and proof*, which includes the students' argumentations and computations, is also identifiable in this episode, for example, in the stated computation '1, 2, 3... 16. 16 times 12. 192' made by student 2. It is interesting to note that this component manifests when student 2's strategy choice looses its 'tacitness', or else when this strategy choice become explicit through the stated computation. The internal articulations identified were: *tacit coincides with explicit* for student 2, and *priority of tacit* for student 1 (see Frade, 2004).

End of episode 1

As exemplified in episode 1, we identified all components of Ernest's adapted model (see table 1) in the episodes with variable intensity and visibility. Further, the analysis showed that, throughout the students' learning, some components of the model predominated over others. The components *statements, proofs and reasoning, language and symbolism, methods, procedures and strategies* appeared more clearly and more often than, for example, the components *propositions* and *aesthetics and values*. It is possible that the component *propositions* was not so evident in the analysis because of the way the study of area measurement was approached during the two stages of the research: propositions was not stressed as an objective of teaching at this level of the course. Yet the identification of the component *aesthetics and values* demanded more effort in terms of reflection and interpretation (probably due to the macro feature attached to it as argued previously). On the other hand, the criteria (motivation, interest, high level of interaction between students and between the students and the teacher) used to select the student-pair for the study directed us towards students who had already shown some identity as participants in mathematics practice in school as well as some taste for mathematics, some sense of aesthetics and values concerning the discipline or some of its aspects.

Component/ Nature	Activity	Example
Propositions and statements/ Mainly explicit	The teacher and the students discussing the following proposition for K=3: if the sides of a rectangle are multiplied by K (K > 1), then its area grows K^2 times.	'It's, its side is 2 and D is 6… Then I saw that the area of the square C equals 4 and that of square D equals 36, which is 9 times bigger. Then after my mother gave me another example that it did not matter, that it was only that the number should be the triple, she put it here side 3 and the area 9. In the other 9, that is, if the side is 9 then it's 81, it's the same thing.'
	Answering a written questionnaire question: what do you mean by 'area' in mathematics?	'I remember that area represents a certain space or place. Based on that we can find that area is used to calculate the size of a space or place. Take the example of a piece of land, how many square meters it has. This is already a way to use area as measure.'
Reasoning and proofs/ Mainly explicit	One of the students of the pair explaining to a classmate how they solved the following problem: a wall with height of 2.30 meters and length of 8.76 meters built with square tiles having sides measuring 2 centimetres. Calculate the number of tiles on the wall.	'We found out that the area of the wall is, we multiply length times width and to obtain the area of the tile we multiply side times side. When we find the result of the two, we divide the area of the wall by the area of the tile. The result was…'
	All the activities in which the students proved a proposition or displayed calculations or any other form of computation.	Area calculation of rectangles by counting the units of measure or using the formula: length × width.
Language, symbolism/ Mainly tacit	All the activities.	Oral and written language, and mathematical symbolism used by the students to communicate their area measurement knowledge in class.
Meta-mathematics views/	Written report in which the students had to reflect and to express the general	Student: …the spatial figures that can have volume, seem to be real. Teacher: Okay, but what does this mean,

Mainly tacit	view they constructed on area measurement and excerpts of conversations where the students made some ontological reference to a mathematical entity.	why did you say that is seems real? Student: This thing [spatial figure that can have a volume] here looks like an egg. Teacher: Oh, yes. Student: This thing [plane figure] here seems to be kind of a drawing. Teacher: Oh, yes, this one here seems to be a concrete object; what about that one? Student: No.
Methods, procedures, techniques and strategies/ Mainly tacit	Problem solving. For example, the students were asked to calculate the perimeter of a rectangular piece of land with an area of 450m2 and 25m in length.	'Now we have to find which number that is. The length is 25, we already know. And what about this one here? 25 times 20. Wait, I understood. 25 times 21. Oh, oh, no God, this is too much.'
Aesthetics and values Mainly tacit	All the activities.	This component was observed in terms of students' curiosity, interest, motivation and participation in classroom practices. In all the episodes that involved conversation we found a high degree of interaction between students of the pair, and, in many cases, between them and the teacher. This was interpreted as indicating the students had some affective components in relation to mathematics.

Table 1 – Components of Ernest's model identified

To explain the development of mainly tacit and mainly explicit components of the student-pair's knowledge of area measurement we found support in Kitcher's ideas (1984) about the development of mathematics practice. First, we could see that many episodes evidenced that the students of the pair had built new statements or rebuilt previously known statements. According to Kitcher, this action results necessarily in the development or in the change of mathematical language and symbolism. For example, a diagnostic questionnaire at the beginning of the first stage of the research showed that student 1 had some previous concept of area as a physical geographic space: 'The word area means a certain location or piece of land, or space.' Another questionnaire at the end of the second stage of research showed that student 1's concept of area had evolved and a specific measure had been incorporated into it: 'I think that area is a place or space…used to calculate the size of a place or space'.

Kitcher says that the component *proofs and reasoning* develops or changes, for example, when new reasoning is added. As this component was identified in various episodes when the students had to work on new concepts and procedures, it seems reasonable to say that both the students developed this component in general. In relation to *proofs*, the analysis demonstrated that both students improved their knowledge: they started their area calculation by counting the units; at a later stage they found, although at different times, that such counting relates to a multiplication (length×width, in case of rectangles); at a third stage, they have this multiplication as a formula. Although the development of the component *methods*, *techniques and procedures* had been identified in many episodes, how the other components affected it was not clear. This could be due to the fact that this component can be said to be the most tacit of all, as we are not given privileged access to mental processes to know when or how a technique or strategy is chosen.

Kitcher argues that the changes in the component *meta-mathematics views* are rooted in the changes of other components. We had evidence that ontological mathematical entities such as plane and spatial figures and area, for example, were created by the students as shown in table 1. The analysis of the component *aesthetics and values* was limited in what concerns how much the other components were linked to it. What we have emphasized is that the component *aesthetics* and *values* involves affective components and thus has an impact on the development of the other components of the model. These affective components certainly depend on factors, which are external to mathematics per se. Boaler and Greeno (2000) show how the way a mathematics class is conducted impacts on the mathematical identity of the students. Once this identity is seen as linked to the component *aesthetics* and *values*, that influence may affect the development of this and of the other components of the model with more or less intensity.

Another result was that the development of the components as a whole was not harmonious. In many episodes one of the students of the pair showed difficulty in expressing his ideas or procedures in mathematical language, producing utterances identified as *explicit separate from tacit*. Despite this difficulty, student 1 was able to develop, for example, the ability to use the rectangle area formula adequately and the ‘know how’ to solve a number of problems. This may indicate that one element of oral mathematical language – the social communication of mathematical knowledge – can be expressed somehow independently from the ‘know how’ factor. And, as in the first study (see Frade, 2004), this independence seemed to be directly related to the manner in which the tacit interacts with the explicit in the process of articulation. Here, once more the teacher can play a crucial role in the student’s development of that component by promoting conversational practices (see Lerman, 2001).

FINAL COMMENTS

The theoretical perspective developed in this paper is innovative in the field of mathematics education in some important ways. Our research offers a contribution to the debate on the theory-practice divide, as it was possible to investigate deep theoretical constructs in practice. The research used Ernest’s model of mathematical knowledge and Kitcher’s ideas about the development of mathematics practice to open up a line of investigation into the nature and development of students’ mathematical knowledge in formal schooling contexts. Although the model does not account for cognitive/sociocultural processes involved in mathematical learning, it helped us understand the types of knowledge – concepts, procedures, attitudes or dispositions – that are presently valued in mathematics curricula. In other words, Ernest's model is a model of scientific mathematical knowledge, and therefore, requires adaptation of the kind suggested in this paper to be applicable to the school context. However, the model is closer to school-acquired mathematical knowledge in the following sense: by the end of a period of learning, and for each level of teaching, learners are expected to have acquired knowledge of a set of statements and propositions; be able to use mathematical reasoning and justify it; use mathematical language and symbolism in individual and social contexts; develop a certain view of the scope and structure of mathematics as a whole; and be able to decide which methods, strategies or procedures are more adequate to the resolution of problems and when to use those methods, strategies or procedures. Moreover, and probably most importantly, learners are expected to have developed a favorable disposition towards mathematical investigation. We believe that such disposition originates mostly from individual experiences with, and values and beliefs about mathematics.

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References

Boaler, J.; Greeno, J. G. (2000). Identity, Agency, and Knowing in Mathematics Worlds, *Multiple Perspectives on Mathematics Teaching and Learning*. In J. Boaler (Ed.), London: Ablex Publishing.

Boaler, J. (2002). Exploring the Nature of Mathematical Activity: Using theory, Research and 'Working Hypotheses' to Broaden Conceptions of Mathematics Knowing. *Educational Studies in Mathematics*, 51(1-2), 3-21.

Ernest, P. (1998b). Mathematical Knowledge and Context, *Situated Cognition and the Learning of Mathematics,* In A. Watson (Ed.), University of Oxford, Department of Educational Studies, 13-29.

Frade, C. (2004). The Tacit-Explicit Dynamic in Learning Processes. In M. J. Høines & A. B. Fuglestad (Eds.), *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 407-414). Bergen, Norway: PME.

Kitcher, P. (1984) *The Nature of Mathematical Knowledge*. Oxford: Oxford University Press.

Lerman, S. (2001) Getting used to mathematics: alternative ways of speaking about becoming mathematical. *Ways of Knowing*, 1(1), 47-52

Polanyi, M. (1962). *Personal Knowledge*. London: Routledge & Kegan Paul.

Polanyi, M. (1969). *Knowing and Being* (M. Grene Ed.). Chicago: Chicago University Press.

Winbourne, P. (2002). Looking for learning in practice: how can this inform teaching? *Ways of Knowing Journal*, 2(2), 3-18.

TEACHERS AS INTERNS IN INFORMAL MATHEMATICS RESEARCH

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This study reports on teacher-interns' observations of research sessions involving urban students participating as subjects in an informal mathematical learning after-school program. The research comes from a project that investigates how minority students from low-income, urban community build mathematical ideas and engage in mathematical reasoning. Videotape data from debriefing sessions of the research team, including teacher interns, show that teacher-interns attend to the behavioural and cognitive aspects of the students' mathematical activity as well as the research interventions.

INTRODUCTION

The purpose of this research is to describe patterns of teacher-interns' observations of research sessions involving sixth-grade students engaged in well-defined, open-ended mathematical tasks in fractions, combinatory and probability, over a 6-month period. The research is an outgrowth of a study, known as the Informal Mathematical Learning project (IML) and funded by the National Science Foundation (NSF award REC-0309062), which investigates how minority students build mathematical ideas, engage in and articulate their mathematical reasoning in the context of an after-school mathematics program. The IML research takes place in Plainfield school district, a low income-urban community whose school population is 98% African American and Latino students. The 9-member team of teacher interns includes elementary and middle-school teachers of the district. During the IML sessions, groups of 2 or 3 teacher interns took ethnographic notes of small group of students working on mathematical tasks. This study reports on such observations. Four research questions guided this study: (1) what behaviours of students did the teacher interns attend to? (2) What evidence, if any, is there that the teacher interns attended to students building of mathematical ideas reasoning? (3) What evidence, if any, is there that the teacher interns attended to the researchers' interventions? (4) What evidence, if any, is there that the teacher-interns' observations varied over time? The general approach was to provide the teacher interns with the opportunity to decide what issues about the IML sessions stood out for them and how they wished to articulate them. However, given the focus of the IML project, there was a special interest in the teacher-interns' observations about the students' mathematical reasoning. It turned out, as described in this study, that the teacher-interns' observations were broader in scope. They included also observations of the students' behavioural and research interventions and varied over time. It may be worth mentioning that, the teacher-interns were more than just observers in the IML project. They also provided important contributions to the o-going research decision-making that helped plan subsequent research sessions. Their observations and teaching experience were useful in designing new seating arrangements for the students, new activities, and providing effective ways of dealing with the students, which helped increased student engagement and productivity.

THEORETICAL FRAMEWORK

According to the discussion document for ICMI study 15, the teachers' role in promoting students' learning is no longer being overlooked or taken for granted. There is a growing awareness that no effort to improve students' learning can succeed without a parallel effort to improve teachers' learning. Research on teacher education has been also expanding rapidly. However, the document recognizes that much more remains to be known about teaching and research on teaching. In particular, it calls for more research on "the knowledge, skills, personal qualities and sensibilities that mathematics teaching entails, and about how such professional resources are acquired (p. 3)." Another area of need is a study of teaching in particular [socio, cultural] contexts. Cohen and Ball (1999) identified two main reasons for the failure of research programs that were based on the "building of instructional capacity" in promoting lasting or even "detectable" improvements in schools. One reason is the use of methodological approaches that are not comprehensive enough to capture the "complex social organizations" that schools often represent. In particular, they claim that, if any teacher development program is to be successful, it must examine the triangle involving students, teachers and class material in an interrelated rather than isolated fashion. The other reason is that most improvement programs do not provide "opportunities for teachers' learning that would be needed to change classroom instruction (p. 1)." In particular, they argue that teachers need to be offered opportunities to learn more about not only the content that they are supposed to teach, but also, and most importantly, about "about how students think about that content (p. 1)." Similarly, Maher (1987), Martino and Maher (1999) and Wood (2004) emphasize the importance of teachers attending to students' mathematical thinking, as they pursue teaching approaches that promote learning in classrooms. Finally, Putnam and Borko (2000) focus particularly on actual professional development programs. They argue that the best models of teacher professional development combine workshops that introduce research-based ideas with on-going support. Such an approach requires the development of teachers as communities of learners who are enculturated into new practices and ways of thinking, which takes time, trust and support to develop (Grossman, Wineburg & Woolworth, 2001; Sherin & Han, 2004). For instance, Sherin and Han studied the effectiveness of video clubs, where groups of teachers come together to discuss video segments of their practice in mathematics teachers professional development programs. They found that through a year of in-depth analysis of videos of their practice, the main focus on teachers' discussions shifted from teacher actions to increasingly in-depth analysis of student thinking.

The design and implementation of this study builds on key theoretical issues discussed above. In particular, given the research its focus, the IML project provided the teacher interns with research-supported learning opportunities on students' mathematical thinking and access to research interventions that could be potentially modelled into classroom pedagogical interventions. The research uses a fairly comprehensive approach, extending the triangle of students, teachers and class material to include also researchers. Finally, similar studies (Beswick, 2004; Siemon, Virgona, Lasso, Parsons & Cathcart, 2004) suggest that examining practicing teachers' observations of teaching experiences, over time, is an important way of influencing teacher practice and a useful mechanism for teachers to reflect on their practice.

METHODOLOGY

This study relies on videotaped observation reports of nine teacher interns who participated regularly in the classroom-based research sessions of the IML project. The reports were collected during 1-hour long debriefing sessions that were held immediately after each research session. Fourteen debriefing sessions constitute the data for this study and correspond to an equal number of research sessions held during the 6 months of IML project. The analysis of the videotapes of the debriefing sessions followed data treatment procedures recommended for videodata (Powell, Francisco & Maher, 2003; Erikson, 1992) and qualitative phenomenological research (Moustakas, 1994; Giorgi, 1985). Seven analytical procedures were used. First, each session was watched entirely to have a sense of its content as a whole. Second, each session was partitioned into segments identified by particular issues being reported. Third, each issue and segment were described in as much detail as possible. Fourth, the issues were scrutinized and significant or critical issues selected on the basis of their significance to the research questions. The fifth and sixth procedure aimed at characterizing the significance of the issues. The fifth procedure involved transcribing entire the segments corresponding to the critical issues. In the sixth procedure, narratives were written that described the significance, or insight, of the issues. Finally, the seventh procedure involved a structural analysis across the fourteen debriefing sessions to identify emerging categories for the entire data set. The characterization of the categories constitutes the teacher-interns' observation described in this study.

RESULTS

The teacher-interns' observations of the IML research sessions can be summarized in three main categories, referring to teacher-interns' observations of the students' behaviour, students' mathematical reasoning and of research interventions, respectively. The categories and the main issues raised by the teacher interns are presented in the table shown below and discussed in the next three sub-sections. Due to limitation in space, supporting excerpts are provided for only some the issues.

Students' behaviour	Students' reasoning	Research interventions
Discipline	Language	Tasks design
Group dynamics	Strategies	Closure
Engagement	Confidence	Withholding answers
Rapport	Meaning	Justification
	Novel ideas	

Students' behaviour

The teacher-interns' observations of the students' behaviour focused on discipline, group dynamic, engagement issues and rapport between students and researchers and among themselves. The rapport issue is discussed in the sections on observations about mathematical reasoning. A few major patterns can be distinguished. First, in a particular debriefing session, the teacher interns almost invariably reported about behavioural issues before making observations about the students' mathematical thinking or research interventions. Second, over time, the teacher interns made fewer and fewer observations about the students' behaviour, as the observations focused more and more on the students' mathematical reasoning. Third, reports about the students' engagement in the tasks were more predominant. As result, the teacher-interns' reports suggested potentially interesting insights regarding students' motivation to do mathematics, In particular, students reportedly tended to enjoy working some problems more than they liked working on others. This suggests a task-dependent type of motivation. The teacher interns also reported observing that some students showed signs of frustration for not being able to express their mathematical ideas and reasoning, particularly in ways that some of the researchers could understand them. The teacher-interns argued that this was particularly the case in the early stages of the program. Interestingly, however, the teacher interns noticed, below, that the language of the students grew increasingly more sophisticated over time. Competition among students was also mentioned as a disruptive aspect group dynamics, which resulted in a diminished engagement in mathematical tasks by some students. Finally, motivation was related to two specific research interventions. In one, some teacher interns reported cases of students whose lack of involvement could be attributed to research sessions dwelling for so long on the same tasks. The students did not necessarily solve the tasks successfully, but reportedly showed sign signs of boredom after working on the same task for too long. The other research intervention related motivational issue follows from the excerpt below:

Teacher Intern 1: Herman was getting frustrated with Robert [research facilitator] because he kept asking "This opposite. I don't understand this opposite." And then after a while, I think he [Herman] really turned off, and then Robert moved on to Dante to see if he could get something out of him because Herman wasn't getting it.

The excerpt suggests that frustration may be the result of intense interviewing that does not provide time for students to think and organize their ideas.

Students' mathematical reasoning

The teacher-interns' observations on the students' mathematical focused on five major issues. Overall, the teacher-interns reported observing that, over time, the students' mathematical reasoning grew increasingly more sophisticated. The teacher interns explained the idea in a number of ways. First, the teacher interns reported that the language of the students became more sophisticated, as the students' vocabulary started to include the names of and the mathematical concepts involved in the tasks they were assigned to work on. Second, the teacher interns reported cases in which they had been surprised or amazed at the strategies used by the students to solve particular problems. Third, the following excerpt suggests that the rapport between teachers and students and among students improved. Most importantly, however, it also suggests that students became intellectually more independent learners:

Teacher Intern 2: I spent the last week looking at a November CD [CD tape with students work in November]. I walked around in today expecting Cuisenaire rods [the students had worked on fraction activities involving cuisenaire rods], and I was really taken aback by the level of sophistication. There is something coming from them that's kind of saying to me, "You know, a problem is worth putting time in. It's worth thinking about." And that's a lot of growth from the beginning, when they kind of looked at us like, "What do we want them to do?" Okay, I see a lot of growing going on, a lot of socializing going on. They're more comfortable, around each other. They are more comfortable with us. I think a lot of genuine personalities are beginning to shine through, and that's okay, because that's part of what they have to learn right now. I really enjoyed it today.

Fourth, the teacher interns reported noticing the students' concern for meaning and, in particular, personally meaningful ideas and ways of expressing them. In the excerpt below, Teacher Intern 3 reports attempts by the students to develop their own idea of what counts as a convincing argument, as they debated whether or not a diagram or picture should also include a text to count as a convincing argument:

Teacher Interns 3: From listening to them, basically posting out, they developed an unspoken rubric to determine what is needed to make this a convincing piece of work. One group stated that the diagrams are fine but there is no text to support the diagram. And then there's text without support for my diagram. So, it seems as if like teachers we use the specific rubric to follow certain things. The students have developed upon themselves the specific rue-brick" to determine what is convincing or not convincing.

Fifth, the mathematical behaviour of the students supposedly became more sophisticated also as a result of the students starting to take up the mathematical behaviours of mathematicians and researchers. This was particularly the case regarding the concern in mathematical for not accepting mathematical statements without a proof:

Teacher Intern 4: Some of the discussion was very detailed. It wasn't frivolous discussion. Like "I don't like this just because I [inaudible]. They were, really, acting, well, they were real mathematicians. Walking around, I was impressed to see the interaction and even the communication among themselves. Within the groups they conversed about what was missing, they asked each other, "Are you convinced?" "Where are you convinced?" or "Why or why not?" And they kept on trying to get more specific. So, I thought the conversation were really good, on point.

Finally, it may be worth mentioning that some teacher interns reported learning new ideas from observing the students working on particular tasks. For instance, Teacher Intern 5 below marvels at and admits never seeing before the strategy a particular student used to solve a mathematics problem, which involved finding all 4-tall towers that could be built when choosing from two colours, yellow and blue:

Teacher Intern 5: I had never seen anyone do it quite that way. Whenever we've done this with a group of teachers or adults, and then he did the other one, and then he noticed the pattern at first, and he might have had it, but I just don't know where to go with it, but he was like, "uh" counting on the top, "yellow, blue, yellow, yellow, blue, two blues" and I think that's the way it went. He knew exactly how it placed them and I thought that was interesting

Research interventions

Some of the issues that the teacher interns attended regarding researcher interventions are already reported above. One is Teacher Intern 3's observation that students seemed to take up researchers' investigative roles. Other insights on the teacher-interns' observations of research roles follow from the four issues mentioned in the table above. Two observations on research interventions are closely related to the claim above by some students seemed to get frustrated with intensive interviewing that did not allow them time to think about the mathematical activity. The two observations concern closure and tasks design. More specifically, regarding the appropriate research intervention to deal with the issue above, some teacher interns took notice of the advantages of avoiding forcing closure and of introducing new problems, which, by design, were extensions of previous problems. However, the teacher interns highlighted the importance of two specific research interventions. First, one teacher-intern reported noticing and reiterated the importance of withholding answers from students and, instead, encouraging them to rely on their thinking, either individually or as a group. In particular, teacher Intern 6 suggests that such an intervention enhances intellectual independence in students:

Teacher Intern 6: One of the things I like about your [the leading researcher's] idea, by the way, is that you are saying that you don't need an authority to figure out if what you've done is right or wrong. It can come from you, and boy, that's a lot of empowerment. That's very special.

The research intervention noticed and supported the teacher interns is the importance of justification of students justifying their answers or mathematical arguments, as part of their normal students' mathematical activity:

Teacher Intern 7: In terms of where, in my opinion, to go next, I do agree that they may feel like "I am finished with it" but the idea of the justification, I think, is really, important for them to see what it looks like. Because we have this issue in the elementary school where I will, you know, give kids an open-ended question and they'll give me an answer and I will write on the paper, "Can you prove it to me?" and fourth-graders will say, "Well, what do you mean by that? They really don't understand that. So, I'm sure kids who are coming into sixth grade have really not had that many experiences. So, if we show them some models of what justification looks like, they'll have a sense of "Oh, this is what you mean." You know, it's not just drawing what I've built; it's also trying to talk about what my thought process is in how I'm determining that I have everything

It may be worth mentioning that the teacher-interns' debriefing reports suggested that that much of the teaching in their schools was based on teachers showing and telling students what to do and did not emphasize justification of mathematical arguments.

CONCLUSIONS

The results of this study provide evidence for the claim that carefully designed professional development programs can help teachers appreciate the importance of attending to students' mathematical thinking. The sophistication of the teacher-interns' observations is evident in the content of their observations and in the comprehensive approach used, which relates observations about mathematical reasoning to observations about behaviour and researcher [pedagogical] interventions. The results are consistent with findings from similar studies discussed above. The teacher interns relate their teaching experiences in schools to their observations (see excerpt from teacher Interns 7). This is consistent with studies (Beswick, 2004; Siemon et al. 2004), which show that teachers' observations of teaching experiences, over time, influences their practice and is a useful mechanism for them to reflect on it. The teacher interns' observations also evolved from a focus on behavioural issues to more detailed accounts of the students' mathematical thinking. This also supports the findings by Sherin and Han (2004) described above. Finally, the results of this study provide evidence of the advantages of research-supported longitudinal teacher professional development programs, particularly those that examine simultaneously teaching and learning.

References

Ball, D. L., & Even, R. (2004). *The fifteenth ICMI study: The professional education and development of teachers of mathematics*. Discussion document for ICMI 15.

Beswick, K. (2004). The impact of teachers' perceptions of student characteristics on the enactment of their beliefs. In M. Hoines & A. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 111-118). Bergen, Norway: Bergen University College.

Cohen, D. K., & Ball, D. L. (1999). *Instruction, capacity, and improvement* (CPRE Research Report No. RR-043). Philadelphia, PA: University of Pennsylvania, Consortium for Policy Research in Education.

Erikson, F. (1992). The interface between ethnography and microanalysis. In M. D. LeCompte & W. L. Millroy & J. Preissle (Eds.), *The handbook of qualitative research in education* (pp. 201-225): San Diego: Academic Press.

Giorgi, A. (Ed.). (1985). *Phenomenology and psychological research*: Pittsburgh, PA: Duquesne University Press.

Grossman, P., Wineburg, S., & Woolworth, S. (2001). Toward a theory of teacher community. *Teachers College Record, 103*, 942-1012.

Maher, C. A. (1987). The teacher as designer, implementer, and evaluator of children's mathematical learning environments. *The Journal of Mathematical Behavior, 6*(3), 295-303.

Martino, A. M., & Maher, C. A. (1999). Teacher questioning to promote justification and generalization in mathematics: What research practice has taught us. *Journal of Mathematical Behavior, 18*(1), 53-78.

Moustakas, C. E. (1994). *Phenomenological research methods*: Sage Publications.

Powell, A. B., Francisco, J. M., & Maher, C. A. (2003). An analytical model for studying the development of mathematical ideas and reasoning using videotape data. *Journal of Mathematical Behavior, 22*(4), 405-435.

Putnam, R. T., & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? *Educational Researcher, 29*, 4-15.

Sherin, M. G., & Han, S. Y. (2004). Teacher learning in the context of a video club. *Teaching and Teacher Education, 20*, 163-183.

Siemon, D., Virgona, J., Lasso, M., Parsons, V., & Cathcart, J. (2004). Elaborating the teacher's role -Towards a professional language. In M. Hoines & A. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 193-200). Bergen, Norway: Bergen University College.

Wood, T. (2004). In mathematics classes what do students' think? *Journal of Mathematics Teacher Education, 7*(3), 173-174.

EXPLORING EXCELLENCE AND EQUITY WITHIN CANADIAN MATHEMATICS CLASSROOMS

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In an attempt to understand the processes that allow all students to successfully learn mathematics this paper conceptualizes a successful mathematics classroom in terms of excellence in mathematics and how equitably achievement is distributed. The study employs multilevel models and the Canadian data from the Third International Mathematics and Science Study to identify the characteristics of successful classrooms. The analysis indicates that the most successful classrooms are those in which students from disadvantaged socioeconomic backgrounds excel in mathematics. Disadvantaged students excel in mathematics classrooms in which instructional practices involve less groupings, the mathematics teachers are specialized, and in schools with lower student-teacher ratio.

INTRODUCTION

One of the major objectives of mathematics education systems around the world is to understand the processes in mathematics education that provide opportunities for all students to successfully learn mathematics. The successful mathematics learning for all requires that schools and school systems function in a way that students' success in learning mathematics is not determined by their background characteristics. That is, in an effective mathematics education system, we would expect the mathematics achievement levels of successful schools/classrooms to be related to their capabilities in helping their disadvantaged students to successfully learn mathematics. In this respect, the relationship between the mathematics achievement level (excellence) of a school and the equitable distribution (equity) of mathematics outcomes within a school is an important indicator of the effectiveness of a school and a mathematics education system.

In a school system where the resources within a school are used in a manner that ensures the successful learning of all students, we would expect equity in high achieving schools. While the relations between excellence and equity are important in defining successful education systems, research on school effect has emphasized exclusively on either achievement levels or equity. As a result, we do not have studies that provide an understanding of how the best schools in an education system function to ensure the successful learning of all students. This understanding is necessary to inform policies on the use of resources and the processes for improving schooling outcomes. The main objective of this paper is to understand how the Canadian mathematics education systems function to include or exclude their disadvantaged students from successfully learning mathematics.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 337-344. Melbourne: PME.

The emphasis will be on school effect on the mathematics outcomes of students from disadvantaged socioeconomic home backgrounds. A substantial body of research points to a consistently strong influence of family background factors, especially their socioeconomic background on mathematics achievement (see Secada, 1992). Unfortunately, many researchers hold the view that these factors are the least amenable to change within an educational policy framework and should therefore be discussed in the context of social policy initiatives rather than from the perspective of school effectiveness. Mathematics education for all makes an understanding of how students from disadvantaged socioeconomic backgrounds come to successfully learn mathematics fundamental to understanding school effect on mathematics outcomes. This paper explores three main issues: the extent to which students' background characteristics affect their mathematics achievement, the extent to which differences in classrooms affect students' mathematics achievement, and the characteristics of mathematics classrooms where students irrespective of their backgrounds succeed in learning mathematics.

Theoretical Perspective (Successful Learning Environment)

When we consider learning as situated in a social and cultural context, the sociocultural perspective provides a useful lens for understanding how schools might function to provide opportunities for all children, especially those from disadvantaged backgrounds, to learn mathematics. The theoretical position of this perspective is motivated largely through the work of Vygotsky, who argues that, in general, learning occurs when an individual internalizes a social experience through interacting with a peer or adult (Vygotsky, 1988). The process of learning occurs through cognitive processes that originate and form through social interaction. Leantev (1981) supports Vygotskys view but stresses the importance of engagement in activity. He maintains that learning occurs through interaction and participation in activity. Other researchers emphasize the importance of locating learning in the co-participation in cultural practices (Lave & Wenger, 1991; Rogoff, 1990). In this model, the student's social engagements through interaction with more experienced others, and through participation in cultural activities are the driving forces for learning.

Bourdieu (1986) argues that often schools operate such that the social and cultural upbringing of students from working class families is not consistent with school norms making it more difficult for these students to engage and participate in learning activities. When school norms and the cultural traditions of children conflict, a school can address the problem from two perspectives. One perspective is to leave students to adapt to the school culture. From this perspective, the success of a disadvantaged student depends on the ability and willingness of the student to function within the two cultures. In cases where a student is unable to function within multiple cultures, success in school leads to losing their cultural traditions. Another perspective is to accommodate all cultural traditions to create a micro-culture that allow all students to participate. In this approach, the success of disadvantaged students depends on school processes and facilities in the school to enhance learning of students from diverse backgrounds. Both perspectives seem to suggest that, for students to succeed in school, they need to acquire certain practices related to understanding a particular subject content.

This suggestion is consistent with an emerging perspective in mathematics education that highlights both the social and mathematical norms in a mathematics classroom. Yackel and Cobb (1996) distinguish between social and sociomathematical norms. Social norms refer to classroom practices that teachers and students engage and that develop gradually over time. They include practices such as learning to participate in group work. Sociomathematical norms are lenses through which teachers and students assess their choices of mathematics teaching and learning activities. To the extent that these norms play important role in learning, we would expect that equitable access to these practices is likely to ensure that students from disadvantaged backgrounds successfully learn mathematics. The question is, whether these practices are the norm in high achieving schools, and to what extent do these practices account for the successful learning of students from disadvantaged backgrounds?

Multilevel models

The concept of "successful schools" defined in terms of excellence and equity poses a considerable methodological challenge as it requires estimates of school achievement levels and inequalities in school outcomes, and most importantly, the processes that account for the variation in these estimates. In the past, researchers assessed school effectiveness through production function models (e.g., Bridge, Judd, & Moock, 1979) from multiple regression statistical techniques, where schooling outcomes are regressed on variables describing students and their schools. However, during the 1980s, there was intense debate as to whether the student or the school was the correct unit of analysis for estimates of school effects (Burstein, 1980). This debate culminated in the development of multilevel statistical models that allow researchers to examine the separate effects stemming from processes at the student, classroom, and school levels (see Goldstein, 1995). These multilevel techniques are now used fairly routinely in analyses of educational data, but rarely by researchers in mathematics education.

This paper employs multilevel statistical analysis techniques that we can describe as regression analyses within and between groups, in this case, schools/classrooms. The analyses provide estimates of regression intercepts (levels of school outcomes) and regression coefficients within schools (measures of, for example, SES achievement gaps). These intercepts and regression coefficients can be regressed on school and classroom characteristics so that the characteristics of schools with high achievement levels and narrow SES achievement gap can be easily identified.

TIMSS

The multilevel models are estimated using the 1995 Third International Mathematics and Science Study (TIMSS) data for Canada. TIMSS is a study of classrooms across

Canada and around the world involving about 41 countries, which makes it the largest and most comprehensive comparative project to assess students= school outcomes in mathematics. TIMSS targeted three populations: population 1 B students in adjacent grades containing a majority of 9-year-olds (grades 3 and 4 in most countries), population 2 B students in adjacent grades containing a majority of 13-year-olds (grades 7 and 8 in most countries), population 3 B students in their final year of secondary schooling (grade 12 in most countries). This research study utilized the Canadian population 2 data describing the mathematics achievement levels of 13-year-old students in Canada. In Canada, these students are in grades 7 and 8 (Secondaire I and II in Quebec). Both grades are part of the secondary school system in all provinces except British Columbia, where grade 7 is part of the elementary program.

The TIMSS Canada population 2 data were collected from a random sample of Canadian schools and classrooms. The random sampling and selection were carried out by Statistics Canada and data were collected in the spring of 1995. Over 16 000 students and their teachers and principals participated in the population 2 component of the study in Canada. Students wrote achievement tests that included both multiple-choice and constructed-response items which covered a broad range of concepts in mathematics. The students also responded to questionnaires about their backgrounds, their attitudes towards mathematics, and instructional practices within their classrooms. Principals completed a school questionnaire describing school inputs and processes, and teachers responded to questionnaires about classroom processes and curriculum coverage.

Instructional Practices and other School Processes

Students responded to a wide range of questions in the questionnaire about instructional activities within their mathematics classroom. In this paper, the classroom instructional practices are classified as grouping, problem solving, traditional, technology, and assessment. Grouping is the extent to which students work in pairs or small groups during mathematics lessons or on projects. Problem solving is a composite score of variables describing the extent and nature of problem-solving activities students are exposed to in a mathematics classroom. The problem-solving activities included giving students problems involving practical and everyday life experiences. Traditional is a composite score of three instructional techniques where students usually copy notes from the board, often work from worksheets, and rely extensively on textbooks. Technology is a description of the extent to which calculators and computers are used in mathematics classrooms. Assessment includes quizzes and homework.

The other school process variables are teacher-specialize, student-teacher-ratio (STR), remedial-tracking, and school disciplinary problems. Teacher-specialize was constructed by dividing the total number of periods a teacher is scheduled to teach mathematics by the total number of periods allocated to that same teacher. This variable served as a proxy for a teacher's specialization in mathematics teaching. Given the challenge mathematics teaching poses to a number of teachers, one will expect that teachers who spend relatively more time teaching mathematics are likely to specialize in this field. There may however, be cases where teachers are assigned to teach mathematics because there are no qualified mathematics teachers. STR is the total number of students per teacher in a school. The STR variable was constructed by dividing the total school enrolment by the full-time teacher equivalent (FTE) of a school. Remedial-Tracking is a dummy variable denoting whether in a particular school, students in remedial classes are removed from regular classes. School disciplinary problems measured the extent of disciplinary problems, such as stealing, in a school.

The dependent variable, students' mathematics scores, is scaled such that the mean score for grade 7 is 7 and the mean score for grade 8 is 8. The scale represents "years of schooling", seven years for an average grade 7 student and 8 years for an average grade 8 student, and is intended to re-express the magnitude of the differences in mathematics scores in a metric based on the mathematics test scores for grade 7 and 8 students in Canada.

Analysis and findings from the multilevel models

	Models			
	1	2	3	4
Within-School Effects				
Socioeconomic status		.52	.50	.50
Female		-.15	-.15	-.16
Immigrant		.36	.36	.34
Variance Components				
%Var. among classrooms	19.2	18.2	17.9	13.6
Corr. Intercept/Gradient		-.14		
Within classrooms (S.D)	2.20	2.13	2.13	2.13
Amongst classrooms (S.D.)	1.07	1.00	.99	.84
Effect on mean achievement				
School mean SES			.47	.38
Students grouped				-.10
Traditional approach				-.12
Use computers				-.15
Use calculators				.26

Regular homework				.15
Disciplinary problems				-.36
Teachers Specialize				.61
Student-Teacher Ratio				-.02
Topic Covered This Year				.31
Topic Covered Last Year				.18
Effects on SES Gradients				
School Mean SES			.14	.09
Students Grouped				.08
Traditional Approach				-.08
Regular Homework				.08
Teachers Specialize				-.14
Student-Teacher Ratio				.02
Note: Bold indicates statistical significance at $p<.05$				

Table 1: Estimates from the multilevel models

Variation among Schools and Classes in Mathematics Scores

The first model tests the hypotheses that schools and classrooms differ in their unadjusted scores. The results indicate that they do indeed differ: 19.2% of the variance is among classrooms (and therefore 80.8% is within schools). The model also yields estimates of the magnitude of the within- and between-school components. Within classrooms, the standard deviation is about 2.20 years. This suggests that in a typical middle school, at each grade level, about two-thirds of all children would have scores within about 2.2 years of the average for their age. But about 16% of all children would fall above or below that range. What this means for most middle school teachers is that in a class of 25 pupils they can expect to have 4 students with scores that are at least *two* year behind those of their peers, and 4 pupils with scores that are at least *two* year above those of their peers. Similarly, the range of classroom means scores vary considerably. The results indicate that about two-thirds of all classrooms have average scores that fall within a year of the national average.

The second model in Table 1 asks whether there is variation among classrooms at these levels after taking account of students= characteristics and family background. The covariates accounted for only about 5.2% of the variance among classes (reducing it from 19.2% to 18.2%). Thus, one cannot claim that schools vary in their mathematics scores mainly because of the types of students they enroll. This model also indicates that the socioeconomic gradients vary significantly among classes in the TIMSS. The intercepts were correlated negatively with the SES gradients providing evidence of converging gradients (magnitude of the correlation, -.14 is small). That is, variation among classes in their mathematics achievement levels tends to narrow at the higher SES level. This seems to suggest that low SES students tend to do well in schools with high mathematics achievement levels.

In the third model, I added the mean socioeconomic status of a class to the second model. The estimate of the effect for mean socioeconomic status is .47, which is comparable to the effect associated with the socioeconomic status of the child. In practical terms, this means that if a child has a socioeconomic status which is one standard deviation below the national average, he or she is likely to have a mathematics score that is the equivalent of about six months below that of his or her peers. But if this child also attends a classroom that has a low average socioeconomic status, say one where the average for the classroom is also one standard deviation below the national average, the child is likely to be a full year (*i.e.*, .50 + .47) below national norms. The classroom mean SES was also positively related to the SES gradient, indicating that the SES gradient is shallower in low mean SES classrooms, and steeper in high mean SES classrooms.

The last model in Table 1 includes several variables describing school and classroom variables. These were modeled on both the intercepts and the gradients, although the model for the gradients was reduced as most of the processes did not have a significant effect. The results indicate that the most successful classrooms are those where: (a) less grouping is practiced, (b) calculators are used but computers are not, (c) there is regular homework, (d) there are few discipline problems, (e) teachers specialize, and (f) there is low student-teacher ratio. Results for the model describing socioeconomic gradients indicate that classrooms have more equitable results (i.e., shallower slopes) when: (a) less grouping is practiced, (b) there is less homework, where teachers are specialized, and (c) there is low student-teacher ratio.

CONCLUSION

This paper conceptualized successful schools in terms of achieving the twin goals of excellence and equity. The analysis indicates that there are schools in Canada that are successful in achieving both excellence and equity. Successful schools and classrooms tend to be those which have relatively high achievement levels for students from lower socioeconomic backgrounds. These schools have low student-teacher ratio, specialized mathematics teachers who rarely employ grouping in their instructional practices. The finding pertaining to small grouping is not consistent with the theory that holds that interaction among students within small groups through discussion, debating, and expressing ideas creates the opportunity for multiple acceptable solutions to mathematics problems. The belief is that, through these interactions, students would experience cognitive conflicts, evaluate their reasoning, and enrich their understanding about mathematical concepts. However, as Springer, Stanne, and Donovan (1999) have noted, without the appropriate structures to make each member of a small group accountable for learning, the expected benefits of small groupings may not be realized, since the interaction would be in most instances merely sharing answers instead of ideas. Effective interactions characterized by high-level deliberations about issues that enhance conceptual understanding occur when teachers clearly define issues, give specific guidelines, and define roles for members in a group (see Johnson and Johnson, 1994). The data from TIMSS do not provide details about small grouping practices in school to allow for further analysis. The finding however calls for a better understanding of how current reform practices should work to provide opportunities for all students to learn mathematics. Current reform in mathematics education in Canada advocates for a more interactive mathematics classroom.

References

Burstein, L. (1980). The analysis of multilevel data in educational research and evaluation.In D. C. Berliner (Ed.), *Review of Research in Education*. (pp.158-231), Washington, DC American Educational Research Association.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education, 23,* 2–33.

Goldstein, H. (1995). *Multilevel statistical models*. London: Arnold.

Johnson, D. W., & Johnson, R. T. (1994). *Learning together and alone: Cooperative, competitive, and individualistic learning*. Boston: Allyn & Bacon.

Lau, C. J. (1979). Educational production function. In D. M. Windham. (Ed.), *Economic dimensions of education*. Washington, D.C: National Academy of Education.

Lave, J. & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge, Cambridge University Press

Leot'ev, A. N. (1981). The problem of activity in psychology. In J. V. Wertsch (Ed.), *The concept of activity in society psychology*. Armonk, NY: Sharp

Springer, L., Stanne, M. E., & Donovan, S. S (1999). Effects of small group learning on undergraduates in science, mathematics, engineering, and technology: A meta-analysis. *Review of Educational Research* 69(1): 21-25.

Secada, W. G. (1992). Race, ethnicity, social class, language, and achievement in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning:a project of the National Council of Teachers of Mathematics* (pp. 623-660). New York, Macmillan Publishing Company.

Vygotsky, L. S. (1978). *Mind and society: The development of higher psychological processes*. Cambridge, Harvard Education Press.