Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education

Melbourne, Australia

July 10-15, 2005

Volume 4

Research Reports

Mul - Wu

Editors: Helen L. Chick & Jill L. Vincent

Department of Science and Mathematics Education
University of Melbourne

Proceedings of the 29th Conference of the
International Group for the Psychology of Mathematics Education

Volume 4

Editors

Helen L. Chick

Jill L. Vincent

University of Melbourne

Australia 3010

The Proceedings are also available on CD-ROM and on-line at
http://onlinedb.terc.edu

ISSN 0771-100X

Cover Design and Logo: Helen Chick

Printing: Design and Print Centre, University of Melbourne

TABLE OF CONTENTS

VOLUME 4

Research Reports

CASE STUDIES OF CHILDREN'S DEVELOPMENT OF STRUCTURE IN EARLY MATHEMATICS: A TWO–YEAR LONGITUDINAL STUDY

Joanne Mulligan*, Michael Mitchelmore* & Anne Prescott**
*Macquarie University, Sydney, Australia
**University of Technology, Sydney, Australia

Two-year longitudinal case studies of 16 Sydney children extended a study of 103 first graders' use of structure across a range of mathematical tasks. We describe how individual's representations change through five stages of structural development. Children at the pre-structural stage showed inconsistent development presenting disorganised representations and incoherent mathematical ideas. High achievers progressed to a more advanced stage of structural development depicted by an increased level of abstraction.

INTRODUCTION

In our PME 28 report (Mulligan, Prescott & Mitchelmore, 2004) we described an analysis of structure present in 103 first graders' representations as they solved 30 tasks across a range of mathematical content domains such as counting, partitioning, patterning, measurement and space. We found that:

- Children's perception and representation of mathematical structure generalised across a range of mathematical content domains and contexts.
- Early school mathematics achievement was strongly linked with the child's development and perception of mathematical structure.

Individual profiles of responses were reliably coded as one of four broad stages of structural development:

1. *Pre-structural stage*: representations lacked any evidence of mathematical or spatial structure; most examples showed idiosyncratic features.
2. *Emergent (inventive-semiotic) stage*: representations showed some elements of structure such as use of units; characters or configurations were first given meaning in relation to previously constructed representations.
3. *Partial structural stage:* some aspects of mathematical notation or symbolism and/or spatial features such as grids or arrays were found.
4. *Stage of structural development*: representations clearly integrated mathematical and spatial structural features.

We build further upon previous analyses (De Windt-King & Goldin, 2001; Goldin, 2002; Gray, Pitta & Tall, 2000; Mulligan, 2002; Thomas, Mulligan & Goldin, 2002), by providing longitudinal case study data with the aim of making as explicit as possible the bases for our identification of developmental stages of mathematical structure. We focus particularly on cases representing extremes in mathematical ability.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 1-8. Melbourne: PME.

THEORETICAL BACKGROUND

Our interest in children's development of structure in early mathematical concepts has been highlighted in our studies of number concepts, multiplicative reasoning (Mulligan, 2002; Mulligan & Mitchelmore, 1997) and measurement concepts (Outhred & Mitchelmore, 2000; Outhred & Mitchelmore, 2004). Related studies have identified that mathematically gifted children's representations show recognisable structure and dynamic imagery, whereas low achievers' representations showed no signs of underlying structure, and the use of static imagery (Thomas et al., 2002). Our findings support the hypothesis that the more that a child's internal representational system has developed structurally, the more coherent, well-organised, and stable in its structural aspects will be their external representations, and the more mathematically competent the child will be.

Our theoretical framework is based essentially on Goldin's model of cognitive representational systems (Goldin, 2002) where we examine our data for evidence of structural development of internal cognitive mathematical ideas and representations. Current analyses have also been influenced from two other perspectives: the study of spatial structuring in two and three dimensional situations (Battista, Clements, Arnoff, Battista & Borrow, 1998); and the role of imagery in the cognitive development of elementary arithmetic (Gray, Pitta & Tall, 2000). We consider 'spatial structuring' a critical feature of developing structure because it involves the process of constructing an organization or form. This includes identifying spatial features and establishing relationships between these features. Pitta-Pantizi, Gray & Christou (2004) discuss qualitative differences between high and low achievers' imagery. Children with lower levels of numerical achievement elicit descriptive and idiosyncratic images; they focus on non-mathematical aspects and surface characteristics of visual cues.

Goldin (2002) emphasises that individual representational configurations, whether external or internal, cannot be understood in isolation. Rather they occur within representational systems. Such systems of representation, and sub-systems within them develop in the individual through three broad stages of construction:

1. An inventive/semiotic stage, in which characters or configurations in a new system are first given meaning in relation to previously-constructed representations;
2. An extended stage of structural development, during which the new system is "driven" in its development by a previously existing system (built, as it were on a sort of pre-existing template); and
3. An autonomous stage, where the new system of representation can function flexibly in new contexts, independently of its precursor.

Our analysis of developmental stages of structure was initially framed by Goldin's three broad stages of construction. From our data with young children we have identified an initial pre-structural stage and two sub-stages (partial structure and structure) preceding Goldin's stage 2 (extended stage). We seek to extend Goldin's model based on longitudinal evidence from young children.

Our analyses have not yet tracked our proposed stages of structural development for individuals over time. Thus, we pose further research questions:

- Do young children continue to develop and use structure consistently across different mathematical content domains and contexts over time?
- Do all young children progress through these identified stages similarly?

METHOD

The sample comprised 16 first grade children, 7 girls and 9 boys, ranging from 6.5 to 7.8 years of age, drawn from the initial 103 subjects. Four children representing each stage of structural development were tracked as case studies in the second year. Selection of a representative sub-sample of children of low or high mathematical ability was supported by clinical assessment data such as IQ tests, and system-based assessments. Four low ability children were classified at the pre-structural stage; one low ability child at the emergent stage; and four high ability children at the stage of structural development. The case study sample was drawn from five state schools in Sydney and represents children of diverse cultural, linguistic and socio-economic backgrounds.

Cases representing extremes in mathematical ability were subject to in-depth study and supporting evidence compiled from classroom assessment data. The same researchers conducted videotaped task-based interviews at approximately three intervals: March and October in the first year and August/September in the second year, including a second phase of interviews.

Thirty tasks, developed for the first year of the study were refined and/or extended to explore common elements of children's use of mathematical and spatial structure within number, measurement, space and graphs. Tasks focused on the use of patterning and more advanced fraction concepts were included. Each task required children to use elements of mathematical structure such as equal groups or units, spatial structure such as rows or columns, or numerical and geometrical patterns. Number tasks included subitizing, counting in multiples, fractions and partitioning, combinations and sharing. Space and data tasks included a triangular pattern, visualising and filling a box, and completing a picture graph. Measurement tasks investigated units of length, area, volume, mass and time. Children were required to explain their strategies for solving tasks such as reconstructing from memory a triangular pattern and to visualise, then draw and explain their mental images (see Figure 1). Operational definitions and a refined coding system were formulated from the range of responses elicited in the first year of interviews and compared with analysis of new videotaped data; a high level of inter-rater reliability was obtained (92%).

Analysis focused on the reliable coding of responses for correct/incorrect strategies and the presence of structural features to obtain a developmental sequence. The coding scheme developed for the first stage of interviews was extended to classify strategies for several new tasks. A fifth stage, an *advanced stage of structural development* was identified, where the child's structural 'system' was developed or extended by using features of the previously existing system. We examined whether this structural development was consistent for individuals across tasks and over a two–year period. Responses to all 30 tasks were coded for all 16 children and the matrix examined for patterns. Achievement scores were compared with individuals' types of representations. It was found that the children could be unambiguously classified as operating at one of five stages of structural development at each interview point.

DISCUSSION OF RESULTS

These results support our initial findings indicating consistency in structural features of individual children's representations across tasks at each interview point. Our report at PME 28 (Mulligan et al. 2004) represents Interview 1 data.

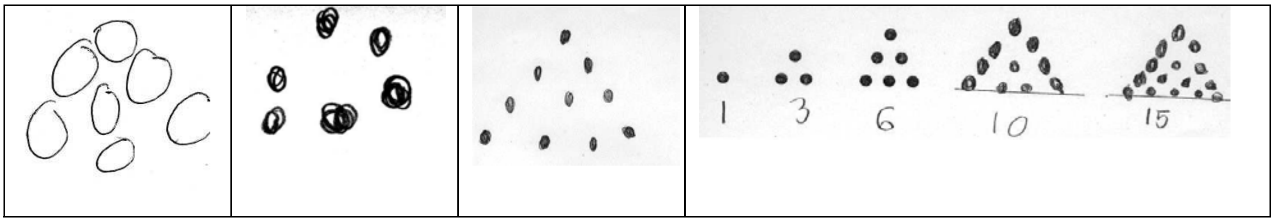
Case Study No.	Interview 1 March 2002	Interview 2 Oct 2002	Interview 3 Sept 2003	Code
1	PRS	PRS	PRS	Pre-structural Stage (PRS)
2	PRS	PRS	ES	Emergent structural stage (ES)
3	PRS	PRS	ES	Stage of partial structural development (PS)
4	PRS	ES	ES	Stage of structural development (S)
5	ES	PRS	PRS	Advanced stage of structural development (AS)
6	ES	ES	PS	
7	ES	PS	S	
8	ES	PS	S	
9	PS	PS	PS	
10	PS	PS	S	
11	PS	PS	S	
12	PS	PS	S	
13	PS	S	AS	
14	S	S	AS	
15	S	S	AS	
16	S	S	AS	

Table 1. Classification of cases by interview by stage of structural development

Table 1 summarises patterns of structural development for the 16 case studies at three interview points across the two-year period. Cases 1 to 5 represent children identified as low ability; cases 12 to 16 as high ability. For most cases there was clearly some developmental progression by at least one stage; cases 7, 8 and 13 progressed by two stages. Cases 1 and 9 showed no observable development of structure in representations or in achievement scores at interviews 2, and 3. For all high ability children there was progression to an advanced stage of structural development encouraged by the inclusion of more advanced tasks. It is not possible to ascertain whether these children may have been operating at this advanced stage at interviews 1 and 2. Cases 1, 4 and 5 showed inconsistencies in their development. Although the low ability children (cases 1 to 5) made some progress, there was more dissimilarity than similarity in their responses, within and between cases.

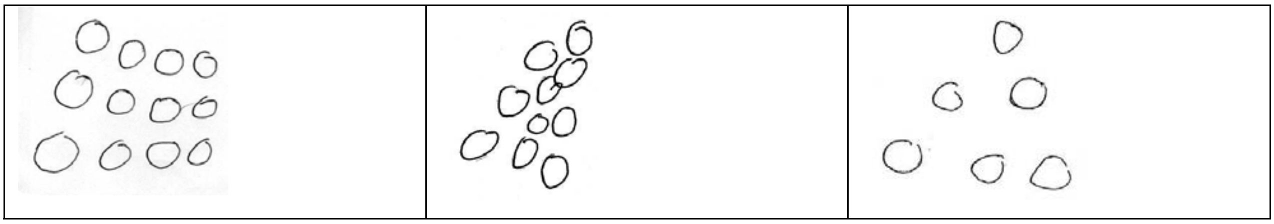
In order to illustrate developmental levels of structure, we discuss representative examples below of children's responses to the triangular pattern task (where the pattern was reconstructed from memory and extended). We selected examples from each stage of structural development identified at the first interview and some exceptions of developmental patterns. The analysis centres on how representations conform to structural features such as numerical quantity, use of formal notation, spatial organization and shape, and construction of pattern.

Figure 1 compares responses given by a high ability child showing the extension to a spatial and numerical pattern of triangular numbers. There is clear development from the stage of partial structure to an advanced stage of structural development. She was able to construct and explain the triangular pattern by repeating the previous row and adding one more circle. Her response indicated that she recognised the pattern, both structurally and numerically, and was therefore, in the early stages of being able to generalise pattern. This ability was also found in her other responses, for example, where she was able to discuss the pattern of digits in a multiple pattern of threes from 3 to 60.



Interview 1 Partial Structure	Interview 2 Structure	Interview 3 Advanced Structure	Interview 3 (second phase) Advanced Structure

Figure 1: Case No. 13. Triangular Pattern Task: Structural Stages

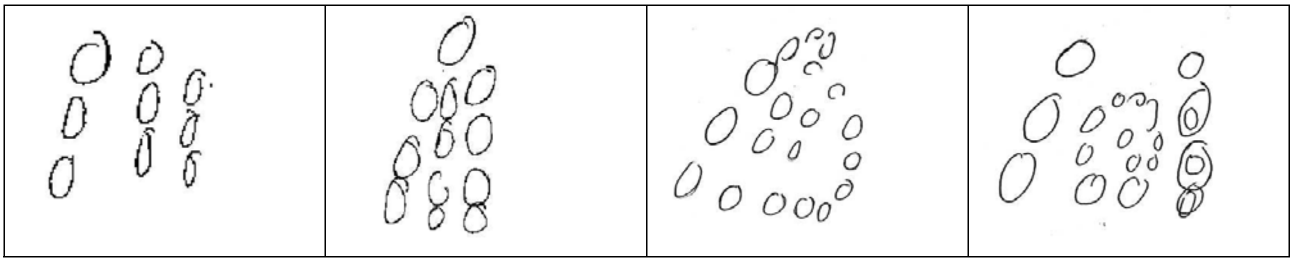


Interview 1	Interview 2	Interview 3
Partial Structure	Partial Structure	Structure

Figure 2: Case No. 10. Triangular Pattern Task: Structural Stages

In Figure 2 the child's first interview shows evidence of some structure in the organization of circles. This becomes more clearly defined as a triangular pattern by interview 3 where superfluous features are excluded.

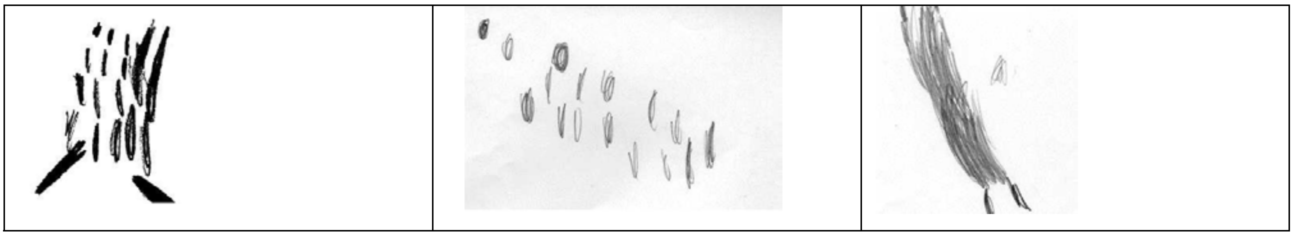
In contrast, Figure 3 shows a child's awareness of a pattern of circles with partial structure. This becomes transformed into triangular form at interview 2, but by interview 3 the image becomes more complex and there is no awareness of the numerical pattern. At a second attempt the image is replicated in a less coherent manner. The images become more disorganised and it can be inferred that the child's internal representational system becomes more 'crowded' with unnecessary icons. It appears that the child loses sight of the initial, clearer numerical and spatial structure that he produced at interview 1. His profile of responses showed no improvement across tasks from interviews 1 to 3.



Interview 1	Interview 2	Interview 3	Interview 3
Partial Structure	Partial Structure	Partial Structure	(2nd phase)
			Partial Structure

Figure 3: Case Study No. 9. Triangular Pattern Task: Structural Stages

Figure 4 shows an initial idiosyncratic image depicting emergent structure; the child draws a triangular form as a 'Christmas tree' and attempts to draw a pattern as vertical rows of five circles. There is little awareness of the structure or number of items in the pattern; there is some indication of spatial structure with equally spaced marks. Interestingly the child produces a completely different image of circles drawn in a diagonal form at interview 2. She could not provide any explanation for an emerging numerical or spatial pattern. At interview 3 the child produced some elements of her initial image but it had fewer structural features. In responses to other tasks she was unable to use multiple counting, partitioning, equal grouping and equal units of measure.



Interview 1	Interview 2	Interview 3
Emergent Structure	Pre-structural	Pre-structural

Figure 4: Case Study No. 5. Triangular Pattern Task: Structural Stages

CONCLUSIONS & IMPLICATIONS

Longitudinal data supported our earlier findings that mathematical structure generalises across a wide variety of mathematical tasks and that mathematics achievement is strongly correlated with the child's development and perception of mathematical structure. This study, however, advances our understanding by showing that stages of structural development can be described for individuals over time. We extend Goldin's model to include two substages of developing structure and an advanced stage of structural development for young children.

There was wide diversity in developmental stages shown for children of the same age range, and some progress shown for most children in their achievement scores across tasks and in their representations. However developmental patterns for low ability cases were inconsistent; the transition from pre-structural to an emergent stage was somewhat haphazard and some children revert to earlier, more primitive images after a year of schooling. There was evidence that some children may not progress because they complicate or 'crowd' their images with superficial aspects. Our data supports the findings of Pitta-Pantazi, Gray & Christou (2004) in that different kinds of mental representations can be identified for low and high achievers. Low achievers focus on superficial characteristics; in our examples they do not attend to the mathematical or spatial structure of the items or situations. High achievers are able to draw out and extend structural features, and demonstrate strong relational understanding in their responses. It was not possible to identify consistently, common features impeding the development of structure in the examples presented by low ability children.

An important new finding gleaned from the cases is the phenomenon of increasingly 'chaotic' responses over time. Representations over time became more complex with configurations and characters of the child's earlier 'system' used inappropriately. In terms of Goldin's theory, we infer that these children fail to perceive structure initially and continue to rely on reformulating superficial and/or idiosyncratic, non-mathematical features in their responses. It appears that these children may benefit from a program that assists them in visual memory and recognising basic mathematical and spatial structure in objects, representations and contexts.

However, our findings are still limited to a sample of 16 cases at three 'snapshots' of development. We plan to undertake longitudinal investigations (using multiple case studies) to track the structural development of low achievers from school entry, and to evaluate effects of an intervention program focused on pattern and structure. In 2003, a school-based numeracy initiative, including 683 students and 27 teachers, was successfully trialled using our research instrument. This initiative implemented a professional development program aimed at developing teachers' pedagogical knowledge and children's use of pattern and structure in key mathematical concepts.

References

Battista, M. T., Clements, D. H., Arnoff, J., Battista, K., & Borrow, C. (1998). Students' spatial structuring of 2D arrays of squares. *Journal for Research in Mathematics Education, 29,* 503-532.

DeWindt-King, A. & Goldin, G. (2001). A study of children's visual imagery in solving problems with fractions. In M. van den Heuvel-Panhuizen (Ed). *Proceedings of the 25th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol 2, pp.345-353). Utrecht, The Netherlands: Freudenthal Institute.

Goldin, G.A. (2002) Connecting understandings from mathematics and mathematics education research. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 161-166). Norwich, England: Program Committee.

Gray, E., Pitta, D., & Tall, D. (2000). Objects, actions, and images: A perspective on early number development. *Journal of Mathematical Behavior, 18,* 401-413.

Mulligan, J. T. (2002). The role of structure in children's development of multiplicative reasoning. In B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.), *Proceedings of the 25th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 497-503). Auckland, New Zealand: MERGA.

Mulligan, J. T., & Mitchelmore, M. C. (1997). Young children's intuitive models of multiplication and division. *Journal for Research in Mathematics Education, 28,* 309-331.

Mulligan, J.T., Prescott, A., & Mitchelmore, M.C. (2004). Children's development of structure in early mathematics. In M. Høines & A. Fuglestad (Eds.) *Proceedings of the 28th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 393-401). Bergen, Norway: Bergen University College.

Outhred, L., & Mitchelmore, M. C. (2000). Young children's intuitive understanding of rectangular area measurement. *Journal for Research in Mathematics Education, 31*, 144-68.

Outhred, L., & Mitchelmore, M.C. (2004). Student's structuring of rectangular arrays. In M. Høines & A. Fuglestad (Eds.) *Proceedings of the 28th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 465-472). Bergen, Norway: Bergen University College.

Pitta-Pantazi, D., Gray, E. & Christou, C. (2004). Elementary school students' mental representations of fractions. In M. Høines & A. Fuglestad (Eds.) *Proceedings of the 28th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 41-48). Bergen, Norway: Bergen University College.

Thomas, N., Mulligan, J. T., & Goldin, G. A. (2002). Children's representations and cognitive structural development of the counting sequence 1-100. *Journal of Mathematical Behavior, 21*, 117-133.

A CASE STUDY OF HOW KINESTHETIC EXPERIENCES CAN PARTICIPATE IN AND TRANSFER TO WORK WITH EQUATIONS

Ricardo Nemirovsky — TERC

Chris Rasmussen — San Diego State University

The broad goal of this report is to describe a form of knowing *and a* way of participating *in mathematics learning that contribute to and further alternative views of transfer of learning. We selected an episode with an undergraduate student engaged in a number of different tasks involving a physical tool called "water wheel". The embodied cognition literature is rich with connections between kinesthetic activity and how people qualitatively understand and interpret graphs of motion. However, studies that examine the interplay between kinesthetic activities and work with equations and other algebraic expressions are mostly absent. We show through this episode that kinesthetic experience can transfer or generalize to the building and interpretation of formal, highly symbolic mathematical expressions.*

INTRODUCTION

How experiences and knowledge from one situation transfer or generalize to another situation has long been a topic of interest (e.g., Thorndike, 1906; Judd, 1908; Wertheimer, 1959). In recent decades researchers have posed alternatives to what now is commonly referred to as a classical or traditional view of transfer (Lobato, 2003; Tuomi-Grohn & Engestrom, 2003). Many of these alternatives are grounded in situated and socioconstructivists perspectives rather than in behaviorist or information processing perspectives. For example, Hatano and Greeno (1999) argue that rather than treating knowledge as a static property of individuals that is correctly or incorrectly applied to new tasks (which is compatible with traditional views of transfer), more emphasis should be placed on the norms, practices, and social and material interactions that afford the dynamic and productive generalization of learning. Hatano and Greeno further argue that alternative views of transfer offer researchers insights into how "students may develop quite different *forms of knowing* when they learn in practices that involve *different ways of participating*" (p. 650, emphasis added).

The broad goal of this report is to bring together a different *form of knowing* with a different *way of participating* in mathematics learning and in so doing contribute to and further alternative views of transfer. Classic forms of knowing include knowing-how and knowing-that (Ryle, 1949). These forms of knowing tend to be static, purely mental, and compatible with traditional views of transfer that look for direct application of knowledge. A different distinction in forms of knowing that is potentially more useful for alternative views of transfer is that of knowing-with and knowing-without. Knowing-with characterizes aspects of meaning making as it

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 9-16. Melbourne: PME.

relates to developing expertise with tools. *Knowing* a mathematical idea *with* a tool, for example, (1) engages multiple and different combinations of dwelling in the tool, (2) invokes the emergence of insights and feelings that are unlikely to be fully experienced in other ways, and (3) is in the moment. The opposite of knowing-with is knowing-without. We all have had experiences of knowing-without embedded in feelings of something being alien, foreign, and belonging to others. The difference between knowing-with and without is not absolute but contextual (Rasmussen & Nemirovsky, 2003; Rasmussen, Nemirovsky, Olszewski, Dost, & Johnson, in press). These characteristics of knowing-with resonate with many of the features of Lobato's (2003) actor-oriented perspective on transfer and Greeno, Smith, and Moore's (1993) situated view of transfer.

In addition to different forms of knowing, Hatano and Greeno (1999) direct our attention to different ways *of participating* in mathematics learning. In this work we draw on recent advances in embodied cognition that highlight the centrality and significance of learners' gestures and other ways of kinesthetically participating in mathematical ideas. Nemirovsky's (2003) review of embodied cognition distills two conjectures regarding the relationship between kinesthetic activity and understanding mathematics that help frame this research report. First, mathematical abstractions grow to a large extent out of bodily activities. Second, understanding and thinking are perceptuo-motor activities that are distributed across different areas of perception and motor action. We also note that the embodied cognition literature is rich with connections between kinesthetic activity and how people qualitatively understand and interpret graphs and motion (e.g., Nemirovsky, Tierney, & Wright, 1998; Ochs, Jacobs, & Gonzales, 1994). It is noteworthy, however, the absence of studies that examine the interplay between kinesthetic activities and work with equations and other symbolic expressions. Thus, the focused goal of this report is to investigate the ways in which kinesthetic activity can participate and transfer to work with conventionally expressed equations.

LITERATURE REVIEW ON TRANSFER

At the beginning of the century Thorndike (Thorndike, 1906; Thorndike & Woodworth, 1901) conducted the first series of "transfer studies." Since then, the overall scheme of these studies became established: subjects who have had experience with a source or learning task are asked to solve a target or transfer task, and their performance is compared to a control group. In looking back at the many studies and debates on the notion of transfer of learning that were developed during the twentieth century, we will describe what we recognize as dominant themes and concerns in the literature.

The aim of most of the transfer research has been to predict and identify the conditions under which transfer does or does not happen. On the one hand we intuitively know that in everyday life we are constantly "transferring" in the broad sense; that is, we are making connections to our past experience, bringing metaphors to life, sensing a stream of thoughts populated by unexpected associations, and so forth. On the other hand, the results of transfer research have led many researchers to conclude that transfer is rare and difficult to achieve unless it is “near” or based on source and target situations that are markedly similar (Singley and Anderson, 1989). This mismatch between common expectations and the results of the transfer studies is, to this day (Anderson, Reder, & Simon, 1996; Lave, 1988), a centerpiece of the debates.

In order to predict the occurrence of transfer and to conduct empirical corroboration, theorists postulated several different types of transfer mechanisms. These mechanisms have centered on the preservation of structures, that is, on the thesis that transfer takes place when certain structures present in the subject dealing with the source task are re-activated when dealing with the target tasks. Thorndike proposed that what one learns in a certain domain transfers to another domain only to the extent that the two domains share "identical elements."

On the other hand, during the period dominated by information-processing approaches, the preservation of mental structures came to be seen as the key for the occurrence of transfer. The idea was that, rather than the features of the tasks themselves, what matters is how people conceptualize the tasks; in other words, the mental structures that subjects bring to bear when they deal with the tasks (Singley & Anderson, 1989).

Transfer studies often cite the literature on “street mathematics” which examined the ways in which people in different cultures solve arithmetic problems from everyday life (e.g., Lave, 1988; Nunes, Schliemann, and Carraher, 1993; Saxe, 1982). We think these studies question the idea that there are some mathematical procedures that are optimal for everyone at all times. This research has repeatedly shown that people compose solutions to the problems they face by combining multiple approaches as well as the resources and demands of the situation at hand. There is nothing exotic about creating idiosyncratic procedures and merging practices, on the contrary it is common and widespread.

As new teaching practices inviting students to invent algorithms are becoming part of schooling, it is increasingly clear that the diversity of approaches and dynamic composition of solutions can be as typical in the school as it is in the street. The old idea that there are some mathematical procedures that are optimal for everyone at all times is an artifact of cultural practices traditionally associated with schooling. The main issue made prominent by research on street mathematics is not, we believe, that school-based algorithms fail to transfer, but that people, rather than using pieces of knowledge as ready-made structures that get applied to new situations, compose solutions by making use of multiple approaches and tuning them to the resources and demands at hand. In this report we examine how prior kinesthetic experiences with a physical tool can offer students resources that can be generalized to work with symbolic equations.

METHOD

We conducted a total of eight, 90- to 120-minute open-ended individual interviews with three students. In the interviews students engaged in a number of different tasks involving a physical tool called the water wheel. As shown in Figure 1, the water wheel consists of a circular plexiglass plate with 32 one-inch diameter plastic tubes around its edge. Each tube has a small hole at the bottom. The plate turns on an axle and is free to rotate. The tilt of the axle can be adjusted between 0 and 45 degrees from vertical. Water showers into the eight uppermost tubes from a curved pipe with holes along its underside.

A submersible pump sends water to the pipe, with a valve to regulate the flow. An oil bath between nested cylinders provides dynamic friction for the axis of rotation. Raising or lowering an oil reservoir varies the oil level in the cylinders. The angular velocity of the water wheel is measured by two photogates that detect the motion of a pattern of black lines on the wheel top.

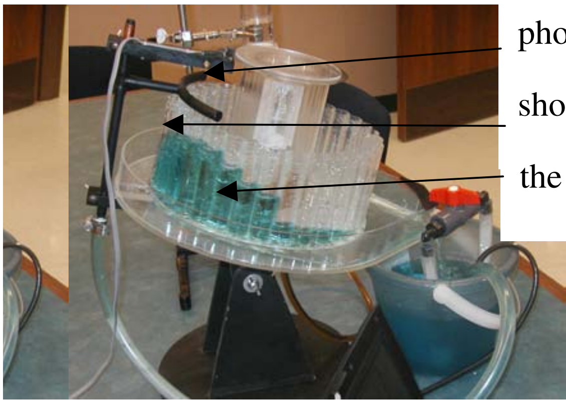


Figure 1. The water wheel

A computer interface permits users to graph angular velocity versus time, angular acceleration versus time, and angular velocity versus angular acceleration while the wheel is turning (Nemirovsky & Tinker 1993). Water showers into the tubes when they are carried underneath the shower pipe. As the wheel turns, the water gathered in each tube provides a torque around the axis of the wheel. Because each tube leaks water from the bottom, the amount of water in each tube decreases over time, until that tube again swings upward to the shower pipe to receive more water. With different choices of tilt angle, flow rate, bearing friction, and initial water distribution, the motion of the wheel exhibits a variety of periodic, almost periodic and chaotic motions, as well as period doubling and transitions into chaos. During periodic motion, water tends to accumulate in a bell-shaped distribution in the tubes, which students often call "the heavy spot" (see Figure 1).

Touching and sensing the heavy spot was a critical and significant experience for students. For example, in the second interview "Jake" predicted that a certain graph of velocity versus acceleration would be circular in shape. Computer generated graphs of actual data, however, indicated the graph to be dimpled on the top and bottom, like an apple. Jake ultimately concluded that the apple shape had to be the case by physically touching and sensing the forces at play in the motion of the wheel (see Rasmussen & Nemirovsky, 2003 for more detail).

Each student we interviewed had completed three semesters of calculus and had taken or was taking differential equations. The interviews used a set of preplanned tasks as a springboard for exploration of mathematical ideas that were of interest to the student, rather than as a strict progression of problems to complete. We also actively worked in the interviews to establish an environment in which the student felt comfortable exploring new ideas and explaining their thinking, however tentative. All interviews were videotaped and transcribed. Summaries of each interview were developed and compared across all interviews. In this report we focus on the learning of one student, Jake, in his third and final interview because it was most helpful in our understanding how kinesthetic activity with a tool can transfer to work with symbolic equations.

MATHEMATICAL IDEAS INVESTIGATED

The first two interviews focused on qualitative and graphical interpretations of motion while the third interview, which is the source of data for this paper, focused on interpretation of the system of differential equations that model the motion of the water wheel.

We planned for students to engage in reasoning about a variety of different phase plane representations. A typical example of a phase plane is the R-F plane for a system of two differential equations dR/dt and dF/dt, which might, for example, model the evolution of two interacting populations of animals such as rabbits R and foxes F. For instance, consider the system of differential equations, $dR/dt = 0.2R - RF$, $dF/dt = -F + 0.8RF$, intended to model the population of rabbits and foxes. Students in modern approaches to differential equations are often required to interpret the meaning of the individual terms in the equations. For example, why is it the case that the first equation has a minus RF term while the second equation has plus RF term? Students in these interviews had engaged in similar analyses in their differential equations course for equations like dR/dt and dF/dt and had developed a number of interpretive strategies. One strategy was to view the RF terms as an indication of what happens to the populations when the two species interact. Another strategy was to interpret the equations when either R or F is zero. An information processing approach would judge successful (or not) transfer in terms of the extent to which these interpretive strategies were employed in the novel task with the water wheel.

The phase plane analyses that we planned to use with the water wheel centered on graphs in the angular velocity-angular acceleration plane, coordinated with time series graphs, and with the motion of wheel. In the third interview we invited students to engage in interpretive analyses of the following system of three differential equations that model the motion of the water wheel: $dX/dt = \sigma(Y - X)$, $dY/dt = -Y + XZ$, $dZ/dt = R - Z - XY$. The variables X, Y, and Z are dimensionless combinations of physical variables, each with a fundamental meaning. X represents angular velocity, Y represents the left-half right-half water imbalance, and Z represents the top-half bottom-half water imbalance. If more water is in the right half of the wheel, then Y is positive. A negative value for Y indicates that more water is in the left half of the wheel (such as the instant in time shown in Figure 1). Similarly, a positive Z value means that more water is located in the top half of the wheel. The parameter R essentially relates to the pump flow rate and tilt while the parameter σ relates to the amount of friction (oil level). All of this was explained to the students in the interview.

The -Y term in the equation dY/dt = -Y+XZ accounts for the fact that water flows out of the tubes in such a way that any differences in their left-right distribution tend to nullify. Both sides tend to have less and similar amounts of water. This happens faster if Y is bigger. From this perspective, dY/dt might be understood as the rate at which the left-right imbalance is evening out. Jake's *knowing* Y *with* the heavy spot cultivated a different perspective on dY/dt. As we elaborate in the next section, Jake's earlier kinesthetic engagement with the water wheel's "heavy spot" afforded him novel and productive ways to make sense of various terms in the differential equations.

ANALYSIS AND DISCUSSION

We often see students designing graphs to produce narratives of perceptuo-motor events, but the use of standard symbolic notations often seems less likely to elicit such direct unfolding of interpretation. An important contribution we make in this report is to clarify and document that kinesthetic experiences can play the role of "bridges" that experientially bring together partial results obtained by symbol manipulation with certain "states of affairs" that students have engaged with physically. In the following example, which is typical of Jake's work with the equations, kinesthetic experiences anchor his interpretations of why the different terms in the differential equations make sense (or not).

His analysis of the differential equation dX/dt = σ(Y–X) began with an attempt to interpret the right hand side of the first equation as a whole. He reasons out what happens to the angular acceleration (since that is what he understands dX/dt to mean) when the amount of damping increases. As Jake worked through this approach, he began to tease out how the individual terms in the right hand side of the first equation might make sense to him. To do so, he returned to the idea of a "heavy spot," which he had introduced in earlier investigations, mainly of periodic motion. In this way, anchored in a special case, he built interpretations that will hold in general. The following excerpt picks up this conversation with Jake reflecting on whether it makes sense for the equation to include a positive Y term rather than a negative one (–Y).

Jake: OK. Now, the positive term of Y, at least, uh, seem to make sense because, if the [holds his hands out, palms up], if it's the more imbalanced [Chris draws a circle on the board next to the equations], the, uh, more [makes a half rotation gesture], uh, the higher the acceleration. Because, if it's much more heavier on this side [cups hand over right side of the circle diagram on the board] than this side [cups hand over left side of the circle], then it seems to make sense that the pull due to this much heavier side [cups right side of circle]. Seems to be, uh, much stronger and, therefore, it [gestures with a grabbing and pulling motion downward] seems to accelerate, uh, much more faster.

Chris: Mmmm. So, that's when Y is positive. [Jake: Right]. How about when Y is negative?

Jake: OK, yes. That's what I was going refer to. Um. Y, Y is negative in a situation where the, uh, uh, the heavier side is, on this side [points to left side of the circle]. And, um, and, if there's. So, the pull is this way [gestures down], therefore, the acceleration is negative [gestures in a counterclockwise swirling motion] instead of positive [gestures in a clockwise swirling motion].

As Jake began his explanation, Chris drew a circle on the board next to the differential equations. Jake's gestures (noted in the transcript) transform this circle into a diagram of the water wheel, with a heavy spot implicitly in evidence. For example, Jake cups a portion of this circle with his hand, as if he were grasping for the heavy spot. Jake's gesture, cupping his hand as if he had taken hold of the heavy spot, suggests a form of being the wheel, in the sense that forces and rotational movement are brought forth through the way he works with the circle diagram of the water wheel drawn on the board. In this way, his physical experience, interpreted through his concept of the "heavy spot," anchors his interpretation of the first equation. In a similar way, his physical experience, combined with key ideas that he has built in order to reflect on that experience, help Jake make sense of the remaining two equations. Other examples will be rendered in the presentation of this paper.

FINAL REMARKS

Representations, such as equations and graphs, are indispensable for mathematical thinking and expression. It is one thing to know, for example, that the slope of the graph of a certain function obeys a certain equation, while it is another thing to sense bodily the need to slow down and the different ways of slowing down. While these aspects can be dissociated, and in fact they often are (e.g., solving an equation without any kinesthetic sense of the motion it describes), they can be related in manifold and complex ways. It is possible that this widespread dissociation leads students to uncritically accept mistaken results obtained through formal calculation, because the latter tends to be performed without the guidance of intuitive expectations. In this report we showed that kinesthetic experience can transfer or generalize to the building and interpretation of formal, highly symbolic mathematical expressions. This existence proof has the potential to open new ground for research on embodied cognition and transfer.

This work has been supported in part by the National Science Foundation (Grants REC-0087573 & REC-9875388). All opinions and analysis expressed herein are those of the authors and do not necessarily represent the position or policies of the funding agency.

References

Anderson, J. R., Reder, L. M. & Simon, H. A. (1996). Situated learning and education. *Educational Researcher*, 25(4), 5-11.

Greeno, J., Smith, D. R., & Moore, J. L. (1993) Transfer of situated learning. In D. K. Detterman & R. J. Sternberg (Eds.), *Transfer on trial: Intelligence, cognition, and instruction* (pp. 99-167). Norwood, NJ: Ablex.

Hatano, G. & Greeno, J. (1999). Commentary: alternative perspectives on transfer and transfer studies. *International Journal of Educational Research*, 31, 645-654.

Judd, C. H. (1908). The relation of special training to general intelligence. *Educational Review*, 36, 28-42.

Lave, J. (1988). *Cognition in practice: Mind, mathematics, and culture in everyday life*. New York: Cambridge University Press.

Lobato, J. (2003). How design experiments can inform a rethinking of transfer and vice versa. *Educational Researcher*, 32(1), 17-20.

Nemirovsky, R. (2003). Three conjectures concerning the relationship between body activity and understanding mathematics. In Pateman, N., Dougherty, B., & Zilliox, J. (Eds.), *Proc. 27th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, pp. 105-109). Honolulu, Hawaii.

Nunes, T., Schliemann, A. D., & Carraher, D. W. (1993). *Street mathematics and school mathematics*. Cambridge, UK: Cambridge University Press.

Nemirovsky, R. Tierney, C. & Wright, T. (1998). Body motion and graphing. *Cognition and Instruction*, 16(2), 119-172.

Nemirovsky, R. & Tinker, R. (1993). Exploring chaos: A case study. *Journal of Computers in Mathematics and Science Teaching*, 12, 47-57.

Ochs, E. Jacobs, S., & Gonzales, P. (1994). Interpretive journeys: How physicists talk and travel through graphic space. *Configurations*, 2(1), 151-171.

Rasmussen, C. & Nemirovsky, R. (2003). Becoming friends with acceleration: The role of tools and bodily activity in mathematical learning. In Pateman, N., Dougherty, B., & Zilliox, J. (Eds.), *Proc. 27th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, pp. 127-132). Honolulu, Hawaii.

Rasmussen, C., Nemirovsky, R., Olszewski, J., Dost, K. & Johnson, J. (in press). On forms of knowing: The role of bodily activity and tools in mathematical learning. *Educational Studies in Mathematics*. [videopaper]

Ryle, G. (1949). *The concept of mind*. London: Hutchinson.

Saxe, G.B. (1982). Developing forms of arithmetical thought among the Oksapmin of Papua New Guinea. *Journal of Educational Psychology*, 18(4), 583-594.

Singley, M. K. & Anderson, J. R. (1989). *The transfer of cognitive skill*. Cambridge, Mass: Harvard University Press.

Thorndike, E. L. (1906). *Principles of teaching*. New York: A. G. Seller.

Thorndike, E. L. & Woodworth, R. S. (1901). *The influence of improvement in one mental function upon the efficiency of other functions*. Psychological Review, 8, 247-261.

Tuomi-Grohn, T. & Engestrom, Y. (2003). Conceptualizing transfer: From standard notions to developmental perspectives. In T. Tuomi-Grohn & Y. Engestrom (Eds.). Between school and work: *New perspectives on transfer and boundary-crossing* (pp. 19-38). Cambridge, UK: Elsevier.

Wertheimer, M. (1959). *Productive Thinking*. New York: Harper & Row (original work published in 1945).

THE CONSTRUCTION OF PROPORTIONAL REASONING

Stephen J. Norton

Queensland University of Technology

The development of proportional reasoning has long been recognised as a central but problematic aspect of mathematics learning. In a Year 6 teaching intervention the part/whole notion of fractions was distinguished from the part:part notion of ratio, and the "between" and "within" relationships in ratio were emphasised. Numerous representations of fractions and ratio including LEGO construction activities were used to develop the multiplicative thinking associated with these concepts. The pre-post results indicated this integrated approach helped students to apply proportional reasoning and to enumerate their responses.

BACKGROUND AND RATIONALE

Ratio and proportional thinking and reasoning abilities are seen as a corner stone of middles school mathematics and this observation is reflected in current syllabus documents (e.g., National Council of Teachers of Mathematics, 2004) and by educators such as (e.g., Nabors, 2002). In this article the term "proportional reasoning" is used to describe the concepts and thinking required to understand rate, ratio and proportionality including scale.

A number of authors (e.g., Ilany, Keret & Ben-Chaim, 2004; Lo & Watanabe, 1997) have noted that the essence of such thinking is essentially multiplicative. Ability in such thinking is needed for and understanding of percentages, gradient, trigonometry and algebra. Lamon (1995) noted that proportional reasoning has typically been taught in "a single chapter of the mathematical text book, in which symbols are introduced before sufficient ground work has been laid for students to understand them" (p. 167). It is hardly surprising then, that many adolescent students who can apply numerical approaches meaningfully in addition context, can not apply such approaches to the multiplicative structures associated with proportional reasoning (e.g., Karplus, Pulos, & Stage, 1983). Indeed many of the error patterns that students demonstrate in relation to proportional reasoning problems illustrate that they apply additive or subtractive thinking processes rather than multiplicative processes (Karplus et al. 1983). Unfortunately, exposing students to routine multiplication and division problems alone, has not been effective in helping students to develop deeper understanding of proportional reasoning. This is in part because students need to understand fractions and decimals as well as multiplicative concepts (Lo & Watanabe, 1997).

The teaching and learning of fractions and decimals is problematic (e.g., Pearn & Stephens, 2004). These authors have noted that many misconceptions that students hold are the result of inappropriate use of whole number thinking, including not understanding the relationship between the numerator and the denominator. Pearn

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 17-24. Melbourne: PME.

and Stephens (2004) found that a major problem for students because they did not understand the part/whole relationships described in fraction notation, and recommended the use of multiple representations of fractions using discrete and continuous quantities and the number line. Given the challenge in learning fractions, it is not surprising that when the multiplicative thinking associated with proportion is added to the learning cycle, many students struggle with cognitive overload, an observation well noted (e.g., Ilany et al. 2004).

The linkage between fractions and ratio is seen in many mathematics texts books. In particular; "students are shown how to represent the information in proportion word-problems as an equivalent fraction equation, and to solve it by cross multiplying and then dividing" (Karplus, et al. 1983, p. 79). The problem with this approach is that in the context of fractions the numerator represents a part and the denominator the whole, while in the case of ratio both the numerator and the denominator represent parts. Thus, while the use of fraction notation in solving some proportion problems may seem expedient in setting out a multiplication and then division algorithm, it is likely to confuse students as to what really is the whole, in fractions this is the denominator, while in ratio it is the sum of the two parts. Since the mathematics text books generally do not teach fractions and proportional reasoning in an integrated way, and usually this distinction is not made explicit, student confusion is understandable.

The particular issues described are set in a wider agenda of curriculum reform. In particular a curriculum shift towards communication of reasoning, problem bases learning and integration based on authentic tasks that include science and technology (e.g., NCTM, 2004). A second level of integration, which is integration between domains within mathematics subject material has also been recommended (Lamon, 1995). By coincidence, the intervention planning model was remarkably similar to that described by Ilany, Keret and Ben-Chaim, (2004., p. 3-83) in which authentic investigative activities for the teaching of ratio and proportion are described. Thus, the aim of this study was to use an integrated approach, across and within the subject domain of mathematics to the teaching of proportional reasoning and assess the cognitive outcomes.

METHOD

The research approach was one of participatory collaborative action research (Kemmis & McTaggart, 2000). The researcher established a working relationship with the teachers and taught one 90 minute lesson in each class, each week over a 10 week period. The researcher and the two teachers involved in the study planned the unit of work during weekly meetings. The collection of data included observations of students' interactions with objects, peers and teachers, students planning and construction of artefacts, their explanations of how things worked, and written pre and post-tests.

Subjects

The subjects were 46 Year 6 students in two classes in a private girls' school in metropolitan Brisbane. The two classroom teachers were also part of the study. Annie (all names are pseudonyms) was a very experienced primary school teacher. Louise was a first Year teacher having recently completed her degree in primary teaching and quickly adapted to the concepts and pedagogy.

Procedure and Instruments.

At the beginning and end of the study were tested for knowledge on proportional reasoning. The pencil and paper test had 18 questions. Some questions had simple and familiar contexts with structures as follows:

> To make drinks for sports day follow the recipe information given. (a) "Mix 1 litre of juice concentrate with 9 litres of water." What is the ratio of juice to water? (b) How many litres of juice concentrate is needed to make a sports drink that is 20 litres in total?

Such a question can be solved with arithmetic thinking, including the construction of tables which can be done with repeated addition. Other questions required a greater abstraction of the notion of proportion, and are not easily solved without a structural understanding of proportion, e.g.:

> My recipe for ANZAC biscuits states that I need two cups of rolled oats to make 35 biscuits. I want to make 140 biscuits, how many cups of rolled oats will I need?
>
> Suppose Challenge College has 800 students and 50 teachers, while Light College has 750 students and 25 teachers. Use mathematics to explain which school is likely to provide better learning opportunities for the students.

The test included questions directly related to the subsequent construction learning contexts such as the inclusion of a diagram, of a bicycle and the following question:

> Explain the effect that turning gear A (attached to the peddles) will have upon gear B which has 16 teeth on it (attached to the rear wheel).
>
> Examine the diagram of the pulleys below. If the circumference of pulley A is 20 cm and the circumference of pulley B is 40 cm and the circumference of pulley C is 10 cm, and pulley B is spun twice, describe how pulleys A and C will spin. Explain your answer.

Scoring was on the basis of correctness and completeness of explanations. Simple items such as the first question above, were allocated 1 mark, while more complex questions requiring symbolic manipulation and justification were allocated 2 marks. Over the life of the study student explanations of their understanding of proportional reasoning was recorded in their written and verbal explanations, which on occasions were trapped on audio or video.

During the intervention fractions were taught emphasising on the sharing division and part/whole relationships. Payne and Rathmall's (1975) principles of constructing relationships between concrete materials, language and symbolism were emphasised through out the study. Various representations were used, including area, line set and volume models. Multiple Attribute Blocks (MAB) was used to link common fractions with fractions with a denominator of 10 or base 10. For example a 10 rod could be viewed as the whole and students were asked to name the fraction shaded if 2 one blocks were shaded (2/10 or 1/5). Similarly the students were asked to view the 100 square as the whole and the 1000 cube as the whole. MAB material was also used to make links to decimal notation, with students having to express parts of the whole as a decimal. While fraction operations were not taught, the multiplicative relationships associated with equivalent fractions were taught using each of the models above (area, set, linear and volume) MAB materials and equivalent fraction strips. Students then represented simple fractions e.g., 2/100 as decimals (0.02) and percentages (2%). That is, percent was seen as a special way of writing decimal fractions with the whole being 1, or 100 hundredths. Linking to the base 10 number system capitalises on students prior experience with the decimal number system.

Ratio or proportion was introduced by emphasising the part/part relationship of ratio as distinct from the part/whole relationship of fractions. The same models used to teach fractions (area, set, line, volume) were used to teach the part:part relationships involved in ratios including equivalent ratios. The linkages between common fractions, decimals and percentage were taught using the models above including the use of MAB and were consolidated in the first one of three critical learning lessons. In this lesson students were taken to the science laboratory and asked to make up solutions 1 part food dye with 9 parts water (volume model), 1 part food dye with 99 parts water, 1 part food dye with 999 parts water. As the students made up the dilutions they recorded the colour, ratio (1:9); fraction 1/10; decimal 0.1 and percent 10%. The students repeated the dilution activity, but rather than making up 1 ml of dye with 99 ml of water, they took 1 drop of 10% solution and mixed it with 9 drops of water to make a 1:99 ratio or 1% solution and compared the colours to the solutions they had made earlier. Students repeated the dilution process to make ratios to 1 part per million. Through such activities students were given multiple opportunities to distinguish between the part/whole relationship of fractions and the part:part relationships of ratio. Contextual links were also made.

The second critical learning activity involved proportional reasoning related to body parts. Students designed their methods to test the hypothesis “Is Barbie a Monster?” They compared their own proportions (e.g. leg length to abdomen length (part:part); waist diameter to bust diameter (part:part), foot length to total height (part/whole) with that of the equivalent ratios and fractions on the Barbie dolls they investigated. Subsequently the students make 2 dimensional scale models of Barbie with cardboard, taking measurements of Barbie and multiplying by 5 to make the scale model the height of the average girl in the class. The students concluded that Barbie was indeed a monster.

The third critical lesson involved the students using their knowledge of proportional reasoning, to construct cars with LEGO materials. These materials included axels, blocks and connecting pieces, motors, gears and pullies, that comprised the Simple and Powered Mechanisms kits (LEGO Educational Division, 2003). The first car was designed to be fast, and the second to win a tug of war competition. Throughout the design and evaluation phases the students were encouraged to make their gearing explanations explicit and formal. In setting out proportion problems, the structural relationships associated with proportion including identifying the "within quantity relationships" and the "between quantity relationships" as described by Lamon (1995. p. 172) was used.

ANALYSIS

The written pre and post-test scores were compared using repeated regression analysis. In assessing the artefacts the associated explanations the description of ratio provided by (Lamon, 1995) and detailed above were used. Emerging assertions were discussed with the teachers and colleagues and tested and refined in the light of further evidence. Triangulation involved the use of multiple data sources identified and this maximised the probability that emergent assertions were consistent with a variety of data.

RESULTS

The results are presented as a number of assertions.

Assertion One: Almost all students improved in their ability to complete questions on the pencil and paper test.

Table 1: Pre and post-test paired results on rate and ratio question, total 24 marks.

Test	N	mean	SD
Maths pre-test	44	10.35	5.24
Maths post-test	44	16.17**	4.62

** significant at $p<0.01$

Not surprisingly on the pre-test many students were able to give correct responses to questions that could be solved with simple additive projections. For example 35% of students correctly answered the drinks question on the pre-test, while only one student failed to get this question correct on the post-test. On questions where simple arithmetic thinking was less able to assist, such as the ANZAC biscuit question the proportion of students who improved was greater for example on the ANZAC question, pre-test 5% correct, while on the post-test 32% of students answered completely correctly and 33% of the total number of students made computational errors rather than errors related to proportional thinking, that is 65% used proportional reasoning. On the questions related to the construction contexts such as the bicycle gearing; 14% of pre-test answered correctly, while on the post-test 55% answered completely correctly and a further 25% used proportional thinking but made computational errors. Likewise on pre-test pulley question 27% students answered correctly while on the post-test 46% answered completely correctly but only two students failed to recognise the application of ratio concepts. The most common misconception of those students who erred in this question was to fail to recognise the inverse relationship, for example stating that the smaller pulley would turn fewer times. The question relating to the ratio of students to teachers was revealing. In the pre-test 9% of students provided a complete solution. However, 38% of students provide an answer that indicated the relationship between student and teachers had accounted for (e.g., "Challenge College because there are more teachers to provide better learning in the classes" or "Challenge College because it is easier for the students to ask questions."). Eight students (18%) provided solutions or reasons which indicated they had added or subtracted. In the post-test 20% provided complete responses, a further 50% provided responses that indicated they had taken account of ratio. While two students provided explanations or solutions indicating additive or subtractive thinking, the responses of the remaining students remained undetermined.

Assertion Two: Most students were able to demonstrate proportional understandings associated with their investigation as to whether Barbie was a monster**.**

All the groups succeeded in constructing scaled cardboard models of Barbie and all students made comments that indicated an appreciation for Barbie's proportionality. For example:

S1 Her thighs are normal, but her legs are too long for her body, her waist is tiny, her chest is larger than normal. If she was a human she would die because her organs will not fit in her body.

S2 Barbie's hands are too small, if she were our height she would have hands the same size as a prep kid, we measured one (5 years old) and that is the age her hands are at.

It is interesting that although students had to use multiplication in the construction of their model, no student provided quantitative explanations in the context of justifying Barbie's proportionality.

Assertion Three: Most students improved in their abilities to construct and explain the proportional concepts associated with the gearing of their cars and tractors.

Early in the teaching phase students were asked to design, construct and explain a fast car using LEGO materials. All students chose to use pulleys to convey the power from the motor to the rear wheels. Only one group (of 14 groups) used the pulley mechanism appropriately (that is a large pulley attached to the motor and a smaller one attached to the wheels). No students provided explanations that indicated an understanding of the nature of the pulley ratios they constructed. At the end of the intervention the students constructed tractors designed to pull loads, seven groups used pulleys and seven groups used cog gearing. Of the 14 groups only one did not attempt to explain their ratios, but while only one of the groups who used pullies tried to quantify it, all of the groups who used gears did so. Three groups used inappropriate gearing or pulley ratios and had incorrect explanations and 10 groups used appropriate ratios and had at least partially correct explanations. Samples of student explanations are as follows:

S3 The gearing is 5:1 which means that the small one goes around 5 times and the one on the wheel, the forty one goes around once.

S4 The most important factor about our tractor is the pulleys because they make the car go slowly so it gets more force and can pull the other cars better.

DISCUSSION AND CONCLUSIONS

Karlpus, Pulos and Stage (1983) found that when middle school students were asked to compare ratios such as 4:6 and 10:15, or only about 18% of students used proportional strategies. The pre-test data presented in assertion one is consistent with Karlpus et al., (1983) findings, that is a low proportion of students apply proportional reasoning when ratio relationships become less obvious in the data, a finding also supported by Ben-Chaim, et al., (1998). The post-test data indicated that between 50% and 75% of students were using proportional thinking. The data associated with the “Is Barbie a monster?” investigation indicates that these Year 6 students tended to use qualitative rather than quantitative explanations of proportionality. That is they had to be encouraged to “symbolise” proportionality, a finding that confirms that qualitative schemas develop before quantitative schemas. Establishing the relationships between representations has long been recognised (e.g., Payne & Rathmall, 1975). The data under assertion three, associated with the LEGO construction indicated that the medium of sense making was important in the process of enumeration. For example, no students who used pulleys in their construction quantified their ratios. This is not surprising since to do so necessitated the use of the intermediate relationship of diameter and circumference. In contrast, all of those who used gears, which afforded a relatively straight forward opportunity to count the gear teeth, quantified their explanations.

The linkage of fraction thinking, decimals and proportional thinking has not been well explored in the research literature, although Nabors (2002) linked fractional reasoning tasks rate, ratio and proportionality and found that this approach helped one case study student to develop proportional schemas. The relatively rare post-trial confusion as to when to use part/whole and part:part relationships and the relative absence of additive strategies suggest that the linkage of multiplicative structures of the base ten number system with part/whole notions of fractions and part:part notions of proportion was helpful. There may well have been important for two reasons. Firstly, the focus on fractions, especially equivalent fractions may well have helped to emphasise the multiplicative relationships underpinning both fractions and proportion. Secondly, it is likely it helped students distinguish those contextual situations that necessitated the use of use additive or subtractive thinking compared to the use of use multiplicative thinking associated with proportion. The emphasis upon “within quantity relationships” and the “between quantity relationships” as described by Lamon (1995) may well have contributed to the high proportion of students who presented correct or partially correct solutions to problems that typically perplex students several years older. In summary, this study has indicated, that affording students the opportunity to make links between the fractions, the decimal number system and proportion through the use of common models and authentic contextual problem situations, has assisted them to develop proportional reasoning. Clearly, further research unpacking how these approaches assist students to develop the multiplicative thinking associated with proportional reasoning is needed.

References

Ben-Chaim, D., Fey, J.T., Fitzgerald, W.M., Benedetto, C. & Miller, J. (1998). Proportional reasoning among 7th grade students with different curricular experiences. *Educational Studies in Mathematics, 36:* 247-273.

Ilany, B-S., Keret, Y., & Ben-Chaim, D. (2004). Implementation of a model using authentic investigative activities for teaching ratio & proportion in pre-service teacher education. In M.J. Høines, & A.B. Fuglestad (Eds.). *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*. Bergen Norway (pp. 3-81-3-88).

Karplus, R., Steven, P. Stage, E. (1983). Proportional Reasoning of Early Adolescents. In R. Lesh & M. Landan (Eds.). *Acquisition of mathematical concepts and processes*. Orlando; Academic Press.

Kemmis, S., & McTaggart, R. (2000). Participatory action research. In N.K. Denzin., & Y.S. Lincoln. (Eds.). *Handbook of qualitative research* (pp. 567-606). Thousand Oaks, Sage.

Lamon, S. (1995). Ratio and proportion: Elementary didactical phenomenology. In J.T. Sowder & B.P. Schappelle (Eds.). *Providing a Foundation for Teaching Mathematics in the Middle Grades.* Albury, NY: State University of NY Press.

LEGO Educational Division. (2003). *Simple and powered mechanisms: Teacher's guide.* UK

Lo, J-J. & Watanabe, T. (1997). Developing ratio and proportion schemes: A story of a fifth grader. *Journal for Research in Mathematics Education 28*, (2) 216-236.

Nabors, W. (2002). On the path to proportional reasoning. In A. D. Cockburn & E. Nardi (Eds.). *Proceedings of the 26th Annual Conference of the 26th Annual Conference of International Group for the Psychology of Mathematics Education*. Norwich, University of East Anglia (pp. 3- 385-3-401).

National Council of Teachers of Mathematics, (2004). *Principles and Standards for School Mathematics*: Introduction. Retrieved March 22, 2004, from http://Standards.nctm.org/documents/chapter1/index/html

Payne, J. & Rathmell, E. (1975). Number and numeration. In J.N. Payne (Ed.). *Mathematics learning in early childhood.* (pp. 125-160). Reston, VA: NCTM.

Pearn, C. & Stephens, M. (2004). Why do you have to probe to discover what Year 8 students really think about fractions. In I. Putt, R. Faragher & M. McLean. (Eds.). *Mathematics education for the third millennium: Towards 2010. Proceedings of the 27th Annual Conference of the Mathematics Education Research Group of Australasia*. Townsville, 27-30 June. (pp. 430-437).

THE TRANSITION OF A SECONDARY MATHEMATICS TEACHER: FROM A REFORM LISTENER TO A BELIEVER

Jo Clay Olson Karmen Kirtley

University of Colorado at Denver & Health Sciences Center

Research on professional development has focused on elementary school teachers. This study is part of a larger study that investigates professional development strategies that support the growth of secondary teachers. Findings indicate that secondary teachers may need experiences that challenge them to re-examine their knowledge and identity before considering implications of reform mathematics.

Mathematics educators (Farmer, Gerretson, & Lassak, 2003; Harel & Lim, 2004; Kazemi & Franke, 2004; Lin, 2004; Patterson & Norwood, 2004) investigated several professional development strategies to deepen teachers' content and pedagogical knowledge. Farmer, Gerretson, and Lassak found that teachers developed their content knowledge when they solved non-routine problems and reflected on how they could be used in their classrooms. By reflecting on the learning environment, these teachers noticed critical aspects in the learning environment that promoted their own learning and were motivated to change facets of their own practice. Kazemi and Franke found that when teachers asked their students to solve the same task and then discussed their students' work during a professional development session, the teachers attended to the details of students' thinking and began to create learning trajectories to develop more sophisticated reasoning. Lin found that helping teachers create situations for students to pose-problems provided student responses for assessment and instructional planning. These studies indicate that helping teachers become more aware of how their knowledge and actions influence students' learning appears to be critical for teachers' professional growth.

Mason (2002) describes the art of noticing as "being awake to situations, being mindful rather than mindless" and can be cultivated through deliberate acts (p. 38). Researchers (Heinz, Kinzel, Simon, & Tzur, 2000; Kazemi & Franke, 2004) suggest that teachers' professional growth is linked to their ability to listen carefully to students' articulation of mathematical ideas, to ask probing questions, and to consider students' responses in light of mathematical concepts. The following five strategies supported elementary teachers to notice details of students' responses: (a) using rich tasks (Stein, Smith, Henningsen, & Silver, 2000), (b) using a research-based framework that maps the development of number sense (Carpenter, Fennema, Peterson, & Carey, 1988), (c) asking questions (Haydar, 2003), (d) examining students' responses (Lin, 2004; Harel & Lim, 2004), and (e) prompting teachers' pedagogical curiosity (Olson, in press).

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 25-32. Melbourne: PME.

This study extends previous research by investigating the complexities of deepening teachers' content knowledge simultaneously with their pedagogical knowledge. It is part of a larger research project which investigates how different professional development approaches influence teachers' beliefs and actions in the classroom. Specifically, this study sought to describe how a course in which secondary teachers created conceptual rubrics to interpret students' responses and plan instruction influenced their beliefs and practices.

THEORETICAL FRAMEWORK

The theoretical framework guiding this study is that of situated learning in which knowledge is co-produced as individuals discuss, adapt, or create models (Boaler, 2000). The situated perspective assumes that learners function in a social context and that learning can not be isolated within a class. Learning is located within classrooms, schools, and communities and the practices in these setting influence how an individual incorporates new ideas into their beliefs and practices.

Research using a perspective of situated learning focuses on how individual develop and use knowledge through their interactions within a social context (Boaler, 2000). To investigate how individuals create new understandings in a social environment, Lave (1993) suggested that researchers collect and interpret data that reflects an individual's learning through his or her actions. This data can then be interpreted to indicate changes in an individual's competence and knowledge, identity, and practices.

METHODS

Six secondary and seven elementary teachers to investigate rational numbers using the textbook *Teaching Fractions and Ratios for Understanding* (Lamon, 1999) during a Master's Degree course taught by Olson. To create collaborative groups, the thirteen teachers answered three opened-ended questions (a) How do students show you that they understand math? (b) Are students born smart in math or do they become smart? (c) What can teachers do to help struggling students become successful? These responses were interpreted as an indication of each teacher's beliefs about teaching and learning. Olson assigned the teachers to one of four groups, comprised of both secondary and elementary teachers who held different views of learning. During each class period, the teachers shared their solution strategies for assigned problems, investigated problems from the textbook, and discussed research that supported constructivist learning theories. Teachers identified underlying mathematical concepts for the grade-level rational number benchmarks. Then, the teachers described student behaviors that indicated a developing, basic, or proficient level of understanding for each underlying concept. These levels of understanding were condensed into a developmental rubric that was used to create a series of lessons that might advance students' reasoning about rational numbers.

A high school teacher (second author) agreed to participate as a case-study to investigate the changes of her beliefs and practices as she collaboratively worked with two elementary and one secondary teacher during the class. Olson made field notes detailing the group's interactions and collected written reflections and solution strategies. Kirtley created a proportional reasoning rubric and designed a series of lessons to enhance students' ability to use proportional reasoning to solve problems.

After completing the course, Kirtley redesigned these lessons using a lesson plan format suggested by Smith (State Mathematics Conference, 2004) that focused her attention on eight aspects: (a) Goals, (b) Previous knowledge, (c) Solution strategies, (d) Engagement, (e) Expectations, (f) Introduction, (g) Questions, and (h) Conceptual understanding. Kirtley recorded her reflections, observations about her students, and conceptions about teaching and learning in a journal. She analyzed these data for her Master's Degree Project (Kirtley, 2004) and described the process of her professional growth.

Olson's field notes, Kirtley's written work, and Kirtley's Master's Degree Project were analysed for changes in Kirtley's attention. These changes were interpreted as evidence that Kirtley was noticing new details as she interacted within a social environment. Matrices were utilized to reduce and synthesize data to describe facets of professional development which influenced Kirtley's competence, identity, and practice.

RESULTS AND DISCUSSION

Before beginning a Master's Degree, Kirtley taught in an urban high school with a history of low achievement, poverty, and high truancy. She described herself as a "savvy high school teacher" who was a strong mathematician and passed the teacher licensure content exam with a high score (Kirtley, 2004, p. 5). As such, she considered herself to be an expert and continually used mathematics books as references while preparing lessons (interview, January 5, 2005). Kirtley wanted students to understand "why" but frequently became frustrated with their low performance. Kirtley explained, "I didn't believe that students could do it [high level math] because they didn't do it after I showed them how. I didn't do all the activities in the reform curriculum because I considered many of them to be fluff." (Kirtley, 2004, p. 13). These comments reveal a core belief that mathematics is best taught through procedures and that students' demonstrated understanding when they arrived at a correct answer.

Kirtley's competence, identify, and practices reflect Stigler and Hiebert's (1999) description of traditional high school teachers. She described hearing the reform message, students need to understand mathematical concepts before procedures can make sense, repeatedly though out the Master's program but they were only words that did not fully resonate with her own experiences (Kirtley, 2004).

Mathematical competence and identity

Kirtley's small group collaboratively solved non-routine problems using manipultives and pictures before linking their strategies to a symbolic representation. By the second session, Kirtly was angry. She wrote, "I wondered why anyone would spend time and effort to go into fractions this way [solve problems using multiple representations] in a way that was quite unfamiliar to me" (Kirtley, 2004, p. 11). Kirtley revealed a core belief about learning when she asked the class to consider the role of procedures, "Isn't it important for students to just know their facts and quickly multiple $^2/_3$ by 2 $^1/_4$ without using pictures?" (class discussion, June 15, 2004) Olson responded, "The question is not the importance of procedural knowledge but how and when it occurs. When students understand a concept first, the procedure makes sense and you don't need to continually repeat instruction to gain mastery. Let's take a look at trying to understand the procedure for dividing fractions."

The class reviewed two representations for 8 ÷ 2. Twelve teachers interpreted division using a measurement model, eight items were placed in groups of two forming four groups. One student used the partitive model, eight items were placed in two groups with four in each group. Olson asked the collaborative groups to represent 1 ½ ÷ $^1/_3$ using the measurement and partitive models of division. Kirtley's group represented 1 ½ with a circle and a semi-circle. An elementary teacher (El Teacher 1) began the discussion.

El Teacher 1: The circle is the unit, so one and a half circles represents 1 ½. To divide by $^1/_3$, we need to find how many groups we can make of $^1/_3$. (The group divided the circle into thirds and the half circle into a third and a left over piece.) We have 4 groups of $^1/_3$ and a sixth.

Kirtley: But when I divided 1 ½ by $^1/_3$, I got 4 ½.

El Teacher 2: Uhm, let's look at the sixth. It is a sixth of the whole circle but only half of a group of $^1/_3$. So, we have 4 groups of $^1/_3$ and half of a group of $^1/_3$, that would be 4 ½ .

Kirtley: I know how to do this with numbers, but I don't get these pictures. I'm starting to buy into the idea that making pictures help you understand the math, but why do kids need to be able to draw pictures?

El Teacher 2: If you draw a picture then you can see what's going on.

Olson: Try to represent the problem by interpreting the problem as, you have 1 ½ and it is in a third of a group. How many would be in one whole group?

Kirtley: I don't know if I can think of it that way. This is really hard.

Olson: In the problem, what is one-third of a group?

El Teacher 1: One and a half?

Olson: What do you think (to the other teachers)?

Sec Teacher: Yes, I have to write down your questions, but if 1 ½ is a third of a group, then we have to have 3 groups of 1 ½, the whole group.

Olson: Can you draw a picture to show it?

El Teacher 1: Yes (draws one and a half circle, labels it $^1/_3$ and repeats it 2 more times.) When I count up the pieces, there are 4 ½ circles. It is right!

Kirtley: And what you drew is what I'd do with the procedure. You invert the $^1/_3$ and multiple. That's what you did with the picture. You made 3 groups of 1 ½. I can't do this by myself. The elementary teachers understand how to make representations with pictures and I need their help.

Olson: So your group is learning from each other. The elementary teachers need help from you [the secondary teachers] to connect the pictures to the algorithms.

In this excerpt, the four teachers worked collaboratively to create two different representations for dividing fractions. One elementary teacher drew a picture using the measurement interpretation but struggled to interpret a symbolic representation for the remaining fractional piece. Kirley used symbols to solve the problem but could not relate the mathematical procedure to the picture. When the group created a representation using a partitive interpretation, Kirtley quickly noticed a connection between this representation and the standard algorithm. She recognized that elementary teachers have a deep understanding of representing mathematical ideas that she lacked and there was more to learn. With this realization, she devised additional problems to solve using pictorial representations. Kirtley later reflected, "I didn't get representing fractions and was upset. Then, it turned me on. I loved seeing multiple representations and how everything connected. I was amazed how deeply you could go into a seemingly simple subject. All students struggle with fractions and I felt empowered to teach. This class was the first time that I experienced math as a student and from that perspective saw the importance of deep understanding." (Kirtley, 2004, p. 12).

Kirtley's mathematical competence was challenged when asked to create a pictorial representation and a new facet of mathematical thinking emerged. In the collaborative problem-solving setting, Kirtley engaged in a mathematical activity in a novel way and her identity shifted as she recognized her reliance on the elementary teachers for creating pictorial representations. Kirtley experienced mathematics in a new way and identified that this experience was critical for her growth.

Teaching practice

After completing the course on rational numbers, Kirtley continued to reflect about the intersection of her beliefs, practice, and students' learning. She described herself as "a changed teacher; I needed to help my students have the same profound and exciting feelings about math that I experienced that summer" (Kirtley, 2004, p. 11). Changing her beliefs about teaching and learning occurred after she struggled to solve problems without using a mathematical procedure which challenged her competence and identity. With this struggle, Kirtley described a fundamental change, "I really believe that my students can learn and that everything I've heard over the past two years [while working on her Master's Degree] fit together and made sense" (reflective notes, September 2004).

Kirtley described two events during her final semester in the Master's Degree Program that cemented her belief in conceptual understanding: attending a State Mathematics Conference and a school district seminar for mathematics teacher leaders. At the conference Kirtley listened to Margaret Smith's presentation on "Thinking Through a Lesson Protocol" (TTLP) and Jeremy Kilpatrick's presentation on the five mathematical proficiencies (September 24, 2004) and these presentations reinforced her new conceptions about teaching and learning.

Kirtley used TTLP to plan 20-activities and noticed that students were more engaged when they created representations to solve problems than when she "skipped the activities" and showed them symbolic procedures (Kirtley, 2004, p. 13). Students used the manipulatives until they "became a liability when the figures became more complex" and created symbolic representation themselves to keep track of data (reflective notes, November 2004, p. 23). For example, Kirtley's students investigated the relationship between the area of triangle and its base and height by constructing all possible triangles with an area of two on a geoboard. Student groups made posters to display all possible triangles but struggled to provide a rationale to prove that their list was complete. Kirtley asked a series of reflective questions like, "Could you make a triangle that was three units long on the base? One unit long? Four units long? Explain." (reflection, November 2004, p. 34). Eventually, her students explained that a triangle had an area of two only if a rectangle could be constructed around it with an area of four. Kirtley noticed a relationship between the cognitive demand of a question, students' mathematical thinking, and wait time. She reflected, "I felt very happy with the students for thinking … I gave a difficult question, and waited for students to answer."

Kirtley was invited to attend the school district's seminar for Mathematics Teacher Leaders. She felt excited about the seminar and the opportunity to learn more. Kirtley wrote, "Some teachers wanted to be leaders in their school but were bitter about being asked to think conceptually. I had great empathy for them, because just one year ago, I was one of them… As I sat in the seminar, a wonderful feeling came over me. I knew I had changed in a way that was exciting and empowering." Kirtley, 2004, p. 14).

Kirtley deepened her conceptual understanding of mathematics, developed a new identity of herself as a learner of mathematics with a unique voice, and changed her teaching practices to build students' conceptual understanding before discussing more efficient strategies. These changes occurred after struggling in a class which provided an opportunity to develop her own mathematical understanding in a collaborative group of teachers and reconsider her beliefs in light of these experiences. Kirtley's professional growth continued as she interacted with students in her classroom and with teachers and experts in seminars and conferences.

CONCLUSIONS

Research (Carpenter et al., 1988; Haydar, 2003; Lin, 2004; Stein et al., 2000) indicates that professional development that deepens elementary teachers' content and pedagogical knowledge by examining tasks, questions, and developmental frameworks can support teachers' growth. Olson initially thought that secondary teachers might demonstrate growth using these same professional development strategies. But, in this study, the professional growth of a highly competent high school teacher was prompted by cognitive dissonance in a social setting.

The collaborative problem-solving environment disrupted Kirtley's identity as a competent, self-confident teacher to confront herself as a student struggling to use mathematical ideas in a new way. She discovered that she needed the help of elementary teachers who were more adept at modeling to be successful. As her competence at solving problems with pictorial representations grew, she recognized how the symbols represented her actions and understood the reform message in a new way. The process of change began when Kirtley's competence and identity were challenged. Kirtley wanted her students to feel the same excitement in the fall and sought follow-up support to cement her new beliefs about teaching and learning into practice.

This study suggests that professional development that supports growth for high school teachers may be different from elementary teachers. The interaction between high school teachers and elementary teachers with their different expertise was critical to help a traditional high school teacher re-examine her own content knowledge and identity. Further research is needed to describe professional development that supports the growth of high school teachers and to investigate whether collaboration between elementary and secondary teachers is a viable professional development model.

References

Boaler, J. (2000). Introduction: Intricacies of knowledge, practice, and theory. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 1-18). Westport, CT: Ablex.

Carpenter, T. P., Fennema, E., Peterson, P. L., & Carey, D. A. (1988). Teachers' pedagogical content knowledge of students' problem solving in elementary airthmetic. *Journal for Research in Mathematics Education, 19*(5), 385 - 401.

Farmer, J., Gerretson, H., & Lassak, M. (2003). What teachers take from professional development: Cases and implications. *Journal of Mathematics Teacher Education, 6*(4), 331-360.

Harel, G., & Lim, K. (2004). Mathematics teachers' knowledge base: Preliminary results. In M. Hoines & A. Fuglestad (Eds.), *Proceeding of the 28th annual meeting of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 25-32). Bergen, Norway: Bergen University College.

Haydar, H. (2003). Daring to ask the hard questions: The effect of clinical interview training upon teachers classroom questioning. In N. Pateman, B. Dougherty & J. Zilliox (Eds.), *Proceedings of the 27th annual meeting of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 33-38). Honolulu, HI: College of Education, University of Hawaii.

Heinz, K., Kinzel, M., Simon, M. A., & Tzur, R. (2000). Moving students through steps of mathematical knowing: An account of the practice of an elementary mathematics teacher in transition. *Journal of Mathematical Review, 19*, 83-107.

Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education, 7*(3), 203-235.

Kirtley, K. (2004). *Teaching for conceptual understanding by creating opportunities for students to engage in high-level tasks* (unpublished Master's Degree Project). Denver, CO: University of Colorado at Denver and Health Sciences Center.

Lamon, S. J. (1999). *Teaching fractions and ratios for understanding*. Mahwah, NJ: Lawrence Erlbaum Associates.

Lin, P.-J. (2004). Supporting teachers on designing problem-posing tasks as a tool of assessment to understand students' mathematical learning. In M. Hoines & A. Fuglestad (Eds.), *Proceeding of the 28th annual meeting of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 257-264). Bergen, Norway: Bergen University College.

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. New York: Routledge Falmer.

Olson, J. C. (in press). Evoking pedagogical curiosity: A coaching approach to support teacher's professional growth. *The Mathematics Educator*.

Patterson, N., & Norwood, K. (2004). A case study of teacher beliefs on students' beliefs about multiple representations. *International Journal of Mathematics Education in Science and Technology, 2*(1), 5-23.

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development*. New York: Teachers College Press.

SUBSTANTIVE COMMUNICATION OF SPACE MATHEMATICS IN UPPER PRIMARY SCHOOL

Kay Owens

Charles Sturt University

A collaborative action research project with preservice teachers built on Wood's (2003) paper on strategy reporting, inquiry and argumentation and the NSW Department of Education and Training documents called substantive communication. There is little research on argumentation about space mathematics in primary schools so this is the focus of the reported study. A qualitative analysis of the data shows that these teachers took account of students' current knowledge and tried to extend it, acted upon their reflections of their teaching, and provided effective challenges and questions. Within space mathematics, we see incidences of strategy reporting and argumentation reminiscent of Wood's (2003) paper.

INTRODUCTION

Classroom Interaction

In 1980, Bauersfeld wrote "the constitutive power of human interaction (is that) interaction constructs the subjects' various realities. Both teacher and students act according to their actual subjective realities" (p. 30). He alerted us to the fact that the teacher and student may be at cross purposes and over the discussion views of purpose and concepts can change. In these circumstances, disagreements are likely to arise. The context of the conversation, for example, what students notice or have experienced previously and interpersonal relations cannot be forgotten in interpreting a classroom. Perret-Clermont (1980) showed that conflict arose, verbal behaviour changed, and the level of reasoning increased when less able students contributed to conversations. This line of reasoning on classroom interaction has developed 25 years later into how change can be brought about in mathematical thinking through argumentation in classrooms (Yackel, 2002).

Substantive Communication

Wood (1999) and her team carefully analysed a number of sequential lessons for a Year 2 classroom and showed the pattern of interaction that occurred during challenges involved turn-taking and explaining until there was agreement. Wood (2003) compared this interaction to conventional classrooms where thinking involves mere recall. There is no "substantive communication" in the typical "initiate-respond-evaluate" teacher-centred pattern in which the teacher asks a question, a student responds, and the teacher makes an evaluative comment (NSW DET, 2003).

Substantive communication is sustained with logical extension or synthesis where the flow of communication carries a line of reasoning and the dialogue builds on statements or questions of another participant. The communication "is focused on the

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 33-40. Melbourne: PME.

substance of the lesson" (NSW DET, 2003, p. 23). In this framework, the degree of quality is mainly in the proportion of the lesson involving substantive communication.

By contrast, Wood's model gives two contexts that are qualitatively different. First, mathematical thinking was revealed in students' "strategy reporting" as recognising, comprehending, applying, and building with analysing. Explainers told different strategies and clarified solutions while teachers accepted or elaborated these and other students listened to decide if their own strategies were different. Second, inquiry and argument showed students building with "synthetic-analyzing" and "evaluative-analyzing" and by agreeing and constructing through synthesising and evaluating. Explainers were giving reasons and justified or defended solutions while teachers asked questions and made challenges, provided reasons or asked for justification. Listening students asked questions for understanding or clarification or disagreed and gave reasons for their challenges. During strategy reporting teachers might prompt with a variety of statements like "I'm confused. Would you tell us what you thought? How did you decide this? … Are there patterns? Is there a different way you can do this?" Inquiry and argument showed the teacher prompting by questions such as "How are the two things the same? Does this make sense? … Does it always work? Why does this happen?" (Wood, 2003, Vol. 4, p. 440). Questions might be structuring, opening-up, or checking questions (Ainley,1988). Questioning, no matter what type, can be carefully linked with the mathematical thinking and level of responsibility in a classroom (Wood, 2003).

Like Wood (2003), Hufferd-Ackles, Fuson and Sherin (2004) presented a framework of improved classroom interaction which outlines shifts from the teacher to students in questioning, explaining mathematical thinking, source of mathematical ideas, and responsibility for learning. At the higher level, teachers expect students to initiate and question. They may ask "why" questions and persist until satisfied with the answer. Teachers will follow students' descriptions of their thinking carefully, encouraging more complete explanations and deeper thinking. Students can defend and justify their answers and are more thorough in explaining. Teachers allow for interruptions from students when explaining in order for students to explain or to own new strategies. While still deciding what is important, the teacher uses students' ideas and methods as the basis of the lesson. Students will spontaneously compare and contrast and build on ideas. Teachers expect students to be responsible for co-evaluation of everyone's work and thinking. They support students helping one another sort out misconceptions and they help when needed. Students may initiate clarifying other students' work and ideas.

Questioning and Teaching Strategies

A broader set of effective teaching strategies than those pivoting around questioning has been identified by the Researching Numeracy Project Team (2004) in Victoria, Australia. The twelve practices are excavating, modelling, collaborating, guiding, convince me, noticing, focussing, probing, orienting, reflecting/reviewing, extending, and apprenticing.

RESEARCH OBJECTIVE

The objective was to describe how preservice teachers were using substantive communication in space mathematics. There has been little action research on argumentation in space mathematics. For this reason, this qualitative study provides data and analysis in teaching and learning space mathematics.

METHODOLOGY

Procedure

Twenty primary school or early childhood pre-service teachers in their third year participated in tutorials for six hours on space mathematics education and substantive communication. During this time, they watched videotapes prepared for the *Count Me Into Space* (CMIS) project in NSW and discussed how students learn space mathematics. They evaluated two videotaped lessons, one on measurement and one space mathematics according to the Quality Teaching framework's section on intellectual quality concentrating on deep knowledge, deep understanding and substantive communication. Their readings included the quality teaching documentation (NSW DET, 2003), Wood's (2003) paper, and excerpts from the paper by Hufferd-Ackles, Fuson and Sherin (2004).

Teachers (cooperating class teachers and preservice teachers) were given a large number of example lessons based on the CMIS project. The lessons covered both two- and three-dimensional space. The strength of these lessons was that they emphasised investigating and visualising as well as describing and classifying. After negotiating with the class teacher, the preservice teachers taught six to ten lessons including their pre- and post-assessment lessons.

Three classes will be discussed in this paper. Class K (Years 5/6) was taught by a primary teacher and an early childhood teacher, Class M (Year 6) by an early childhood teacher and Class P (Year 6) by a primary teacher. All were mature-aged and had received high academic grades but were not particularly confident with mathematics for this Stage, especially the two early childhood teachers.

Data Collection and Analysis

Each preservice teacher kept a journal with anecdotal records and student work samples. They evaluated each lesson using the readings especially the QT in NSW document and prepared a final report. Class K's teachers videotaped and transcribed their lessons. Class M's teacher audiotaped and transcribed her lessons. I observed a lesson in each classroom and viewed the videotapes of Class K. Much of the audible dialogue was from whole class discussions. Other data came from teachers during focus group discussions (preservice and class teachers separately).

This qualitative data was analysed first by marking any recording of interest. Each was annotated with a comment. Many of these comments linked directly back to one of the categories mentioned in the literature review above. From the taped material, I specifically noted how the teachers attempted to extend students' conversations. From these annotations, some perspectives were summarised in order to better understand how beginning teachers can achieve substantive communication in their classrooms.

RESULTS AND DISCUSSION

General Comments

Teachers tended to focus on language and concepts rather than other aspects of mathematical thinking even though they did ask students to report on their thinking or challenges. Teachers noted students' understanding was "uneven" (NSW DET, 2003) meaning it varied between class members or over episodes in the lesson.

Teachers' Analysis of Student Knowledge

Throughout the lessons, preservice teachers realised that students were struggling with three specific concepts – irregular polygons, diagonals and adjacent sides. They were aware of these difficulties in class conversations and from quick quizzes that gave them work samples to look at after the lesson. In Class K, a quiz at the start of the second lesson had most students drawing a triangle when asked to draw a shape with three diagonals indicating that students confused the word diagonal with sloping sides or it was too hard a question. Class P were given a quick quiz of general knowledge of 2D and 3D shapes at the end of the first lesson. The teacher commented

> The students were asked to note if they learned anything today. Surprisingly a large amount [sic] claimed that they knew it all! But their sheets had items which indicated they did not…they said they had forgotten…Only two said something (about properties) … and these were only the number of sides and corners.

Later assessments showed angle size was next to be considered in properties but it seems that the absence of diagonals on diagrams continued to discourage students to mention properties about diagonals in open-ended questions. During lessons, students could use the clues on diagonals when playing the game, what shape am I? In Class P, students made up interesting questions for their peers and some involved angles and diagonals.

Several students in Class P said they had never heard of a polygon with an infinite number of sides, or that a triangle or quadrilateral was a polygon. This was a dissonance that brought about changes in their concepts. Many thought an irregular shape was one that was not common or did not have a name. In Class P, "several students said an irregular shape was not a "real shape" but after a heated argument the class decided irregular shapes with many sides are polygons" (preservice teacher's report). Each class had on the walls the names of shapes and an example for each but they were the stereotypical examples. For example the hexagon and pentagon were regular, the trapezium was isosceles and all had a horizontal side. These diagrams might have helped students with the names but they did not help students to develop a full understanding of different shapes and to know what are essential properties of shapes.

Students did not seem to struggle with enlarging shapes. They knew that the lengths were enlarged but not the angles. However, the majority did not attempt to measure the angles. Deciding on the order of what to measure and draw was not an issue if estimates of angles only were used and just lengths of lines were measured. Hence students were not engaged in reporting different strategies. Lack of time and the need for class control meant the teachers did not persist with deepening understanding. In Class K, most students only doubled the sides even though they were asked to make the shape three times bigger. When reminded to double and measure angles, some students needed assistance with reading the protractor while others had to skip count to work out 6 x 3 or to use repeated addition for 3.5 x 3. Perhaps the multifacets of the problem encouraged some students to take the short cut of estimating angles. The teachers reflected:

> Students were investigating and questioning … engaged with measuring, drawing and discussing how to transfer the information onto another piece of paper. Students were also looking at their peers' work and comparing.

Post-lesson assessment indicated that students had moved in their use of strategies from doing and drawing to static pictorial imagery and some to imagery that contained patterns or required dynamic changes. Class P teacher noted that students could not describe actions of rotation, reflection or slide and felt she may have neglected this area in the lessons.

Challenges

All teachers set students some challenges. In Class M, the teacher gave each group two equal lengths of wool tied together in the centre with cotton. They were to make rectangles. (The wool formed diagonals of the rectangles). In the transcripts, T stands for teacher and other initials for students.

T: What happens when we change the angle in the middle where the cotton is tied? If it's a bigger angle what happens to our rectangle.

M: It gets bigger.

T: The sides get bigger, smaller. Just quickly have a look at your wool and try to make the angle bigger and see whether as the angle gets bigger the rectangle gets bigger or smaller or wider or narrower.

E: Smaller

T: Why do you think its smaller, E.

E: Because the angle makes the ends bigger

T: Will everyone listen to E., when somebody is speaking, everybody else needs to be quiet.

E: When the angles are really really skinny the sides are really, really long and when they move out the sides get shorter and shorter. The angle gets bigger and bigger.

T: Now if we have a piece of wool and either a pencil or piece of chalk, how could we make a circle?

K: Hold one end and draw around (other ideas by students not captured on tape).

Clearly the students were considering different angles and sides of the rectangle but there seemed to be no confusion as they were using the concrete materials (wool) to model their comments. The teacher moved on as the students were unsettled. This was one of the key problems for these beginning teachers in maintaining substantive communication.

Other challenges presented to the classes were:

- how to check whether a piece of paper was square
- making pentomino shapes without repeats,
- deciding why one pentomino shape had a different perimeter than others
- designing and making boxes of different shapes after making a square box
- making cylinders and cones when given a paper-towel roll and a funnel
- deciding on lines of symmetry for the pentomino shapes

In Class M the Z-like pentomino lead to some discussion regarding whether it had one or no lines of symmetry. They grappled with rotational symmetry. The discussion was continued the next week. Students initially did not agree on how many lines of symmetry each shape had but they were able to convince each other by using the concrete cut-outs which the teacher prepared realising this would be an important aid for the discussion. For the same activity, Class P's teacher commented:

> The "magic moment" in this lesson is when a not so bright student argues about the line of symmetry in one of the pentominoes and set about to prove his point or me wrong. He discovered that this shape was symmetrical by rotating the two halves. This student was satisfied because he proved it by himself. Great stuff! (Figure 1 shows) the scrap of paper that the student worked with when he then cut it in half (on the line he thought was a line of symmetry) and placed the two pieces on top of each other and presto they matched.

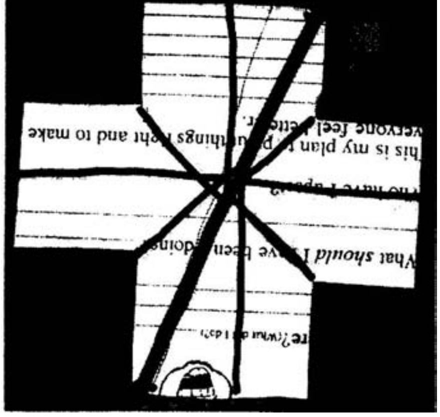


Figure 1: Trying to prove a line of symmetry by cutting and overlapping.

This is a typical example of concrete proof reminiscent of that found by Wood (2003) in strategy reporting in a younger class on number concepts. However, we see that it establishes a new concept. Students in general were reliant on concrete proofs for justifying and explaining although diagrams were used and later some were able to argue verbally and more abstractly. Teachers deliberately provided large examples for whole class discussion to facilitate this kind of proof.

Different Ways of Questioning

The teacher in Class P asked questions in numerous game formats. She noted how these games were both enjoyable but challenging. They made students' question their conceptual knowledge. The games included (a) finding a fellow student for a match of picture and properties, (b) "Celebrity Shapes" in which students ask questions in order to guess the shape drawn or written on the board above their heads, (c) questions provided as chance cards on different self-designed game boards. The best question was one in which the group had to decide if the answer was correct or not. Class K had clues to decide on a shape as a group. These questions were challenging as they focussed on properties other than the number of sides and corners.

Class P's teacher commented that during marking of revision quiz questions there was considerable discussion between neighbouring students and students did not all mark their papers correctly. Were there still areas of disagreement that students needed to discuss rather than quickly marking a quiz? The class teacher noted that questioning by the preservice teacher "is drawing the discovery/information from them rather than giving the answer … the response from students has been great. … The students were beginning to use lesson specific language to describe objects, position etc. It's working!"

Much of the discussion in each class was resulting from direct teacher questioning with some occurring between students during the activities, e.g. from which position the drawing of a 3D model was made or when pentominoes were in different orientations. The videotapes of the whole classroom discussion in the last two lessons of Class K illustrated students' remaining on task throughout the lesson. They may not have talked much but they were thinking and several students were confident to disagree with the teacher or other students' suggestions or to ask their own questions. The students attempted explanations and justifications. The discussion soon moved to students' questions and interests rather than the teacher's initial question.

Teacher Reflection

As a result of reflection, teachers either recognised the effect of having too many students in a small room for group work or re-organised tables to allow closer communication for small groups. They also recognised missed opportunities for group work or sustaining communication and then deliberately allowed themselves more time for discussion and asking more questions to try to encourage discussion.

Classroom teachers pointed out specific lessons in which the students were thinking more mathematically. For example, the lessons on "nets of cubes and making boxes and taking different perspectives for shapes and models made them start to think about nets and shapes and discuss why more than before." "Before, students just gave shape names but now they are discussing them with the correct terminology." There was "far more on tangrams and using more concepts of space than before. Before they made shapes but now they are thinking deeper about concepts of space." "(The lessons were) fun, productive, discovery." "(The preservice teachers) had clear ideas of what outcomes they wanted without restricting it." "They modelled how they wanted them (the students) to ask questions. They talked about lesson expectation and students knew they will be expected to explain and if they were not confident then they got better." "The teachers allowed for incidentals and allowed that deviation to take place. They encouraged and set an atmosphere for taking risk."

CONCLUSION

This paper highlights the importance of the challenge in inquiry/argumentation as Wood (1999) had shown but this paper illustrates the nature of some of the challenges in space mathematics in upper primary school rather than with number in lower primary school. The questioning utilised by the teachers varied. There were (a) quick quizzes with a few challenge questions which the teachers wanted mainly for accessing students' knowledge, (b) planned questions that were strengthened by their reading, reflection, and practice, (c) spontaneous questions as they listened to the students, and (d) questions in game formats. While some of the quizzes and games were followed by whole class discussions, others were left for individuals perhaps talking with their neighbours to resolve conflicts and develop concepts.

References

Ainley, J. (1988). Perceptions of teachers' questioning styles. *Proc. 12th Conf. of the Int. Group for the Psychology of Mathematics Education* (pp. 92-99). Hungary: PME.

Bauersfeld, H. (1980). Hidden dimensions in the so-called reality of a mathematics classroom. *Educational Studies in Mathematics, 11*, 23-41.

Hufferd-Ackles, K., Fuson, K., & Sherin, M. (2004). Describing levels and components of a math-talk learning community. *Journal for Research in Mathematics Education, 25*(2), 81-116.

NSW Department of Education and Training (2003). *Quality teaching in NSW public schools: A classroom practice guide.* Sydney: NSW DET.

Perret-Clermont, A. (1980). *Social interaction and cognitive development in children*. London: European Association of Experimental Social Psychology and Academic Press.

Researching numeracy project team (2004). Researching numeracy teaching approaches in primary schools. *Australian Primary Mathematics Classroom, 9* (4), 27-29.

Wood, T. (1999). Creating a context for argument in mathematics class. *Journal for Research in Mathematics Education, 30*(2), 171-191.

Wood, T. (2003). Complexity in teaching and children's mathematical thinking. In N. Pateman, B. J. Doherty, & J. Zilliox (Eds.) *Proc. 27th Conf. of the Int. Group for the Psychology in Mathematics Education* (Vol. 4, pp. 435-442). Hawaii: PME.

Yackel, E. (2002). A framework for analysing teaching. *Journal for Research in Mathematics Education, 33*(1), 64-69.

TRANSFORMING KOREAN ELEMENTARY MATHEMATICS CLASSROOMS TO STUDENT-CENTERED INSTRUCTION

JeongSuk Pang

Korea National University of Education

Educational leaders have sought to change the prevailing teacher-centered pedagogy to a student-centered approach. Despite the widespread awareness of the reform agenda, there is an increasing concern of whether a real instructional change happens in a way to promote students' mathematical development. This paper deals with successes and difficulties a teacher goes through as she moves on to student-centered instruction. The analysis draws on classroom observation and interviews to illustrate how the teacher and students establish social and sociomathematical norms that emphasize mathematical sense-making and justification of ideas. As such, this paper paves a way by which teachers and reformers open toward possibly subtle but crucial issues with regard to implementing reform agenda.

BACKGROUND

The current mathematics education reform requires substantial changes towards student-centered instruction wherein students' contributions and participations, rather than a teacher's explanations and ideas, constitute the focus of classroom practice (NCTM, 2000). The teacher's role in a reform mathematics classroom is to implement new social norms that foster all students' mathematical learning. For instance, the teacher manages classroom discourse in ways that probe various mathematical ideas and deepen students' conceptual understanding.

In contrast to the widespread awareness of the reform agenda and teachers' positive self-evaluation to their teaching practice, there has been a growing concern that many teachers do not quite grasp the vision of the reform (Research Advisory Committee, 1997; Ross, McDougall, & Hogaboam-Gray, 2003). Teachers too easily adopt new teaching strategies such as the use of manipulative materials or cooperative learning. However, this does not guarantee that students are engaging in worthwhile mathematical activities. Teachers then need to re-conceive their new teaching processes with respect to students' learning processes. What kinds of mathematical and social exchanges occur and in what ways such changes promote students' mathematical development?

In recent international comparisons Korean students have consistently demonstrated superior mathematics achievement not only in mathematical skills but also in problem solving (Beaton et al, 1996; Mullis et al, 2000; OECD, 2004). Despite the high performance, a teacher-centered instruction in Korea has been critiqued as a main factor resulting in learning without deep understanding, negative mathematical disposition, lack of creative mathematical thinking, etc. Broad-scale efforts have been launched to influence the ways mathematics is taught.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 41-48. Melbourne: PME.

Korean reform centers around revision of the national mathematics curriculum and concomitant textbooks and teachers' guidebooks (Ministry of Education, 1997). Main characteristics of the recent reform documents include relating mathematical concepts or principles to real-life contexts, encouraging students to participate in concrete mathematical activities, proposing key questions of stimulating mathematical reasoning, emphasizing problem solving processes, and assessing students' performance in a play or game format (Pang, 2004). These characteristics for enriching learning environment are intended to support student-centered teaching methods.

Whereas typical teaching practices in other countries have been extensively studied through microanalysis of video-recordings of mathematics instructions, those of Korean mathematics classrooms have been little studied in the international contexts. An exceptional study conducted by Grow-Maienza, Hahn, and Joo (1999) reports:

> Teacher behaviors are dominated by question/answer patterns and demonstration of operations in many modes and patterns which lead students through the procedures and conceptual development of the problem, at the same time facilitating student thinking. Student behavior is characterized by choral responses and choral evaluation of individual responses which keep students on task. (p. 6)

Although focusing on typical classrooms makes a valuable contribution to understanding the dynamics of teaching in Korea, it may not always contribute directly to attempts to implement teaching reform. By looking closely at a reform-oriented classroom, this study attempts to understand better what constitutes the process of implementing reform ideals into actual classroom contexts. The study provides a detailed description to explore how the teacher and students establish a reform-oriented mathematics microculture. Given the challenges of substantial implementation of student-centered instruction, in particular, this study probes successes and obstacles a reform-oriented teacher goes through.

THEORETICAL FRAMEWORK

A general guideline to the understanding of mathematics teaching practices is an "emergent" theoretical framework Cobb and his colleagues developed that fits well with the reform agenda (Cobb & Bauersfeld, 1995). In this perspective, mathematical meanings are neither decided by the teacher in advance, nor discovered by students. Rather, they *emerge* in a continuous process of negotiation through social interaction.

Along with the emergent perspective, two constructs of *social norms* and *sociomathematical norms* are mainly used to characterize each mathematics classroom (Yackel & Cobb, 1996). General social norms are the characteristics that constitute the classroom participation structure. They include expectations, obligations, and roles adapted by classroom participants as well as gross patterns of classroom activity.

Sociomathematical norms are the more fine-grained aspects of these general social norms that relate specifically to mathematical discourse and activity. The differentiation of sociomathematical norms from general social norms is of great significant because interest is given to the ways of explicating and acting in mathematical practices that are embedded in classroom social structure. The examples of sociomathematical norms have included the norms of what count as an acceptable, a justifiable, an easy, a clear, a different, an efficient, an elegant, and a sophisticated explanation (Yackel & Cobb, 1996).

METHOD

The data used in this paper are from a one-year project of understanding the culture of elementary mathematics classrooms in transition. The project is an exploratory, qualitative, comparative case study (Yin, 2002) using *constant comparative analysis* (Glaser & Strauss, 1967) for which the primary data sources are classroom video recordings and transcripts. As a kind of purposeful sampling, the classroom teaching practices of 15 elementary school teachers eager to align their teaching practices to reform were preliminary observed and analyzed. An open-ended interview with each teacher was conducted to investigate his or her beliefs on mathematics and its teaching. This extensive search was needed, given the recency of the reform recommendations, and the infrequently of teachers' explicitly advocating reform allegiances.

Five classes from different schools were selected. Two mathematics lessons per month in each of these classes were videotaped and transcribed. Individual interviews with the teachers were taken three times to trace their construction of teaching approaches. These interviews were audiotaped and transcribed.

Additional data included videotaped inquiry group meetings in which the participant teachers met once per month and discussed mathematics, curriculum, and pedagogy. Through the group meetings, the teachers had lots of opportunities to analyze their own teaching practices as well as others, which might help them develop a keen sense of what student-centered teaching practices look like at each classroom level. The interview and inquiry group data were to understand the successes and difficulties that might occur in the process of changing the culture of primary mathematics classrooms, as well as the recursive relationship among the teachers' learning, beliefs, and classroom teaching.

Classroom data were analyzed individually and then comparatively in terms of general social norms and sociomathematical norms. Interview data were included in the analyses whenever they provided useful background information in relation to classroom teaching practices. Because case study should be based on the understanding of the case itself before addressing an issue or developing a theory (Stake, 1998), teaching practices are very carefully scrutinized in a bottom-up fashion. The central feature of these analyses is to compare and to contrast preliminary inferences with new incidents in subsequent data in order to determine if the initial conjectures are sustained throughout the data set. The successes and difficulties that occur in the process of making substantial movement toward reform teaching in classrooms were highlighted in order to explore the issues and obstacles that might point to generic problems of reform.

RESULTS

For the purpose of this paper, one reform-oriented classroom is analyzed in order to examine the extent to which the teacher implements mathematics education reform and to explore challenges of transforming traditional teaching practice toward student-centered instruction. The teacher, Ms. Y, was selected for close examination of her teaching practice because she demonstrated gradual but dramatic changes during the project period.

Initial Classroom Participation Characteristics

To be clear, the preliminary observation of Ms. Y's instruction illustrated that her general teaching approaches would be consistent with the current reform ideas, evidenced by general social norms as follows. The teacher and the students established permissive and open atmosphere so that students' engagement and even their mistakes were welcomed. The teacher introduced mathematical contents in relation to real-life situations, and emphasized the process of problem solving. She also supported students' contributions by providing praise and encouragement. Students presented their ideas to the whole class and usually listened carefully to their friends' explanations.

However, Ms. Y was very concerned about going through all the activities and problems in the textbook. At first, she faithfully followed the sequence of activities in the textbook, not necessarily recognizing the interrelations among them. Ms. Y attempted to induce students' participation by asking questions such as "What shall we do to solve this problem?", "Who will explain?", and "Do you all agree?" However, in most cases, students' answers were limited to short or rather fixed responses. In this way, students were engaged in classroom activities but had little opportunity to develop their own thinking.

Change: Eliciting Students' Ideas

A noticeable change in Ms. Y's teaching practice occurred after she had an opportunity to see other teachers' instruction in the inquiry group meeting. In particular, Ms. Y was influenced by another teacher who was teaching the same grade. Ms. Y could see more directly how a teacher's different approach even with the same mathematical contents and materials would result in different learning environment on the part of students' mathematical development. In an interview, Ms. Y expressed her excitement about the variety and the depth of students' mathematical ideas, and was eager to change her teaching methods:

> I was very impressed by the fact that students could approach a given problem in many ways, something interesting and creative, depending on how a teacher consistently

> pursued to do so. As comparing my teaching practice with others, I realized how much my teaching approach and personality traits influence students' learning and engagement as well as classroom atmosphere. … I also would like to establish a classroom culture in which my students think and discuss actively for themselves.

Instead of relying heavily on the textbook, Ms. Y started to develop a worksheet intended for students to explore important mathematical ideas for themselves. Students were expected to solve a given problem in various methods and to explain their ideas irrespective of the correctness of the answers. Ms. Y also formulated the structure of her lesson as follows: Brief review of the previous lesson or related mathematical topics, her introduction of mathematical contents, students' activities with a worksheet, and whole-class discussion building on students' ideas. In this way, Ms. Y allowed students more time to develop their own sense-making and to explain their thinking to the class.

The following episode shows how Ms. Y orchestrated classroom discourse in a way to elicit students' various ideas. Students in pairs were involved in an activity of choosing 3 number cards, making the biggest and the smallest number, and then figuring out the difference between the two numbers. In the whole class discussion, Ms. Y encouraged students to analyze the results of subtraction.

Teacher:	What do you see? Look at your worksheet and discover something interesting.
Sohee:	Whenever you take three numbers there are two regroupings.
Teacher:	Right! There are always two regroupings. Is there anything else? Why don't you look at the numbers?
Sohee:	As I made the biggest and the smallest number, the number in the middle was always the same.
Teacher:	Yes, it is the same. That's right. Anything else?
Sohee:	When I see hundreds place and ones place, for example, if I had 641 and 146, the 6 and the 1 are the same except their places.
Teacher:	Yes, only the places are switched and the middle number is the same. What happens to the middle number after you subtract?
Giwon:	It always turns out 9 no matter how I subtract.
Teacher:	That's right! The middle number is always 9. Is it really true? Do you have 9 all the time? (Students check their worksheet and agree.) Why do you have 9 in the middle? Because of what?
Students:	Regrouping.

According to the teacher's lesson plan, Ms. Y expected students to discover the fact that the tens place in difference between the biggest and the smallest number is always 9. Students were able to discover the fact and, more importantly, to figure out why. Thanks to the teacher's consistent attempt to soliciting students' ideas, they had an opportunity to analyze mathematical ideas, merely beyond practicing a standard algorithm of subtraction with regrouping.

Change: Focusing on Mathematically Significant Ideas

Another important change in Ms. Y's teaching practice is related to sociomathematical norms. As noted above, Ms. Y indeed asked for different solution methods to a given problem or activity. She then frequently facilitated students to compare and contrast similarities and differences among the various methods. Meanwhile she differentiated mathematical differences from physical or visual differences.

The following episode is an example illustrating how the participants established a norm of mathematical difference. Students were studying the relationship between a part and the whole in the unit of fraction. A rectangle consisting of 3 cells in each of two rows was drawn on a worksheet and students were supposed to color 4 out of the 6 cells. Ms. Y asked students to present their methods in front. Jihoon first showed his method in which he colored 4 cells making a figure of 2x2 square.

Teacher: Is there someone who colored differently?

Sohee: (volunteers and shows her method in which she colored the 3 cells in the first row and then 1 more cell in the second.)

Teacher: What do you see? Jihoon and Sohee colored differently. Who is wrong? (Students express their disagreement in a loud voice.) Who can explain?

Juhyun: The 4 [cells] is the same both in the square figure and in the Giyeok [A Korean alphabet similar to Sohee's figure].

Seungjun: Their shapes are different but we can say that they are the same, because it [the given rectangle] is divided by the same 4 cells.

Teacher: That's right! Because the number of colored cells is the same, we can say that there are the same.

Seonghyun: Although the colored figures are different, the numbers are the same.

Teacher: Yes, that's right. (She shows her drawing in which the first and the third column were colored.) Look at mine! For fun I colored one cell, skipped the next cell, and then colored next cell again. What do you think of this?

Students: (agree that the three figures – Jihoon's, Sohee's, and the teacher's – are the same.)

In the episode, the teacher asked students to compare Jihoon's method with Sohee's. On the one hand, their methods were different (i.e., which cells were colored?). On the other hand, their methods were the same (i.e., how many cells were colored?). Building on this idea, students learned that the fraction 4/6 is the same regardless of the different shapes. In this way, students were able to contrast difference of mathematical ideas or principles applied to solve a problem with difference of physical materials or representation used.

Difficulties in the Change

Despite the promising transition toward successful student-centered teaching practices, Ms. Y experienced some difficulties when students reacted anxiously to the uncertainty associated with the given activity or they did not come up with a specific idea that she thought was important. In those cases, Ms. Y provided a crucial hint that might change the nature of the given task or introduced her own solution strategies, instead of letting students invent them.

Ms. Y was also frequently struggled with how to balance the encouragement of students' conceptual development and the teaching of efficient procedures such as a standard algorithm. After listening to students' various solution methods, she often ended her lessons by summarizing or formulating the most efficient one and implied the students to use it in solving problems for practice. To be clear, students need to compare and contrast different solution methods, for example, in terms of mathematical efficiency. However, this often happened by the teacher's summary at the end of the lesson. In this way, students might think that there was "one" efficient method in solving a problem and their main activity was to find out the method the teacher ultimately waited for.

In addition, Ms. Y was not sure of how to react to students' novel ideas except praise. Although she listened carefully to students' mathematical ideas, Ms. Y had a difficulty in posing questions that further challenge and extend students' thinking after eliciting it. She usually turned out to follow the sequence of the activities prepared in advance rather than students' emergent ideas.

DISCUSSSION

Implementing student-centered teaching practices is fundamentally about significant change, and the teacher remains the key to change. The extent to which substantial change occurs depends a great deal on how the teacher comes to make sense of reform and respond to it. As moving on to student-centered instruction, Ms. Y elicited students' participation and ideas in many ways and then attempted to orchestrate the path of discourse towards conceptual understanding, leading students to be continually exposed to mathematically significant distinctions. In line with many commonalities in the challenges of reforming mathematics classroom culture, this study addresses the need for a clear distinction between attending to the social practices of the classroom and attending to students' mathematical development within those social practices.

On the one hand, Ms. Y was successful in focusing on mathematically significant ideas, in particular, with regard to a norm of mathematical difference. On the other hand, she was limited to be sensitive to students' engagement in order to develop increasingly sophisticated ways of mathematical knowing, communicating, and valuing. The difficulties that Ms. Y had in the process of transforming her teaching practice channel our attention toward the degree by which students' ideas become the center of mathematical discussion and activity.

Another issue to be discussed is a role of a collaborative community where groups of teachers are committed to raise questions on their current instruction, search for alternatives, try on new approaches, and weigh their methods against others' pedagogical alternatives for the common purpose of improving their teaching practices. In fact, many recent studies of teachers' attempts toward reformed mathematics teaching suggest the importance of an inquiry community that provides shared goals and collaboration (Fennema & Nelson, 1997). The message is that participants need to establish new norms for discourse concerning their instructional changes, obstacles and dilemmas of change, as well as the more general nature of mathematics teaching and learning.

References

Beaton, A. E., Mullis, V.S., Martin, M. O., Gonzalez, E. J., Kelly, D. L., & Smith, T. A. (1996). *Mathematics achievement in the middle school years: IEA's third international mathematics and science study*. Chestnut Hill, MA: Boston College.

Cobb, P., & Bauersfeld, H. (Eds.). (1995). *The emergence of mathematical meaning: Interaction in classroom cultures*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Fennema, E., & Nelson, B. S. (Eds.). (1997). *Mathematics teachers in transition*. Mahwah, NJ: Lawrence Erlbaum Associates.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago, IL: Aldine.

Grow-Maienza, J., Hahn, D-D., & Joo, C-A. (1999, April). *Mathematics instruction in Korean primary schools: A linguistic analysis of questioning*. Paper presented at the Annual meeting of the American Educational Research Association. Montreal, Canada.

Ministry of Education (1997). *The 7th elementary school curriculum* (In Korean). Seoul, Korea: the Author.

Mullis, I. V.S., Martin, M.O., Beaton, A. E., Gonzalez, E. J., Kelly, D. L., & Smith, T. A. (2000). *TIMSS 1999 international mathematics report*. Boston, MA: Boston College.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: The Author.

Organisation for Economic Co-operation and Development (2004). *Learning for tomorrow's world: First results from PISA 2003*. Paris: OECE Publications.

Pang, J.S. (2004, July). *Development and characteristics of Korean elementary mathematics textbooks*. Paper presented at the 10th conference of International Congress on Mathematical Education.

Research Advisory Committee (1997). Clarifying the contributions of research with NCTM. *Journal for Research in Mathematics Education, 28*(4), 396-397.

Ross, J. A., McDougall, D., & Hogaboam-Gray, A. (2003). A survey measuring elementary teachers' implementation of standards-based mathematics teaching. *Journal for Research in Mathematics Education, 34* (4), 344-363.

Stake, R. E. (1998). Case studies. In N. K. Denzin & Y. S. Lincoln (Eds.), *Strategies of qualitative inquiry* (pp. 86-109). Thousand Oaks, CA: SAGE.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458-477.

Yin, R. K. (2002). *Case study ressearch: Design and methods* (3rd ed.). Thousand Oaks, CA: SAGE.

THE EFFECT OF IMPROVED AUTOMATICITY OF BASIC NUMBER SKILLS ON PERSISTENTLY LOW-ACHIEVING PUPILS

John Pegg, Lorraine Graham, and Anne Bellert

SiMERR National Centre

University of New England, Australia

This research report summarises the results of an exploratory teaching program in a primary and secondary school in rural New South Wales, Australia, focused on improving basic mathematics skills. Pupils, aged 11 to 13 years, identified as consistently low-achieving in Mathematics were targeted. The program ran for approximately twenty-five weeks with pairs of pupils involved in five thirty-minute sessions per fortnight. Results of the program indicate that these pupils were able to decrease significantly their average response times needed to recall number facts. The results also showed that by the end of the program these pupils exhibited important gains on standardised test scores as well as improvements on State-wide testing measures that were not the focus of instruction. Significantly, pupils maintained performance gains 12 months after the intervention was completed.

INTRODUCTION

Pupils who have problems with learning face a myriad of difficulties in accessing the curriculum. Those who exhibit consistent weaknesses in basic skills such as the recall of number facts are particularly vulnerable. Consequently, there is a critical need for educational researchers to investigate interventions designed to support pupils who experience such difficulties with basic academic skills.

The intervention program described in this report is referred to by the generic title *QuickSmart* because it aimed to teach pupils how to become *quick* (and accurate) in response speed and *smart* in strategy use. This teaching program sought to improve automaticity, operationalised as pupils' fluency and facility with basic academic facts in Mathematics. In terms of research, the study explored the effect of improved automaticity on more demanding mathematics tasks. The fundamental research question addressed was: Does a carefully targeted teaching program aimed at improving automaticity in basic skills free up working memory processing, thereby enabling pupils to undertake more advanced age-relevant tasks that were not part of the intervention program?

THEORETICAL UNDERPINNINGS OF *QUICKSMART*

The *QuickSmart* program brings together research conducted at the Laboratory for the Assessment and Training of Academic Skills (LATAS) at the University of Massachusetts (e.g., Royer & Tronsky, 1998) and related work from the National

Centre for Science, ICT, and Mathematics Education for Rural and Regional Australia (SiMERR) at the University of New England in Armidale, Australia. Part of the theoretical background of the project relates to the work of researchers from LATAS who developed procedures for obtaining reliable assessments of pupil performance using a computer-based academic assessment system (CAAS). Importantly, the assessment tasks used are designed and sequenced in order to help identify particular obstacles that may impede pupil learning. The techniques developed by LATAS have been used successfully as a means of diagnosing the academic problems of pupils who have specific reading and/or mathematics learning difficulties. The *QuickSmart* program has situated CAAS within a teaching approach that incorporates a focus on systematic instruction with the consistent monitoring of pupil performance. This instructional focus is particularly valuable for those pupils who meet the criteria of being ‘treatment resistant’ to usual instructional and remedial efforts/methods.

Based on analysis of the diagnostic information obtained from CAAS assessments, discussions with teachers, and other available test results, *QuickSmart* instructional interventions are tailored to strengthen each pupil’s problematic skills. The interventions are also based on a substantial body of research related to the importance of particular basic academic skills in the development of understanding of the four operations on simple and extended tasks (e.g., Ashcraft, Donely, Halas, & Vakali, 1992; Zbrodoff & Logan, 1996).

Theoretical and pragmatic considerations that point to the importance of developing automatic low-level skills in basic Mathematics underpin the *QuickSmart* intervention. First, it is generally accepted that the cognitive capacity of humans is limited and that working memory has specific constraints on the amount of information that can be processed (Zbrodoff & Logan, 1996). As such, there is good reason to expect that improving the processing speed of basic skills will free up working memory capacity that then becomes available to address more difficult mathematical tasks. Research has already indicated that the ability to recall information quickly uses minimal cognitive capacity (e.g., McNamara & Scott, 2001). Another reason why the automatic performance of low-level academic skills is of prime importance is that it allows for small decreases in response time to accrue across subtasks, again freeing up working memory (Royer, Tronsky, & Chan, 1999). There is evidence that in basic Mathematics, a pupil’s lack of automaticity can result in a reduced ability to solve problems and understand mathematical concepts (Gersten & Chard, 1999).

In summary, *QuickSmart* is a theory-based intervention that supports basic skill development for chronic low-achievers in Mathematics. Specifically, this research implemented instructional program aimed to increase pupils’ understanding and speed of recall of basic number facts by freeing up working memory capacity within the context of a personalised learning environment where pupils are withdrawn in pairs from their normal class.

METHODOLOGY

The *QuickSmart* intervention delivered instruction five times per fortnight for approximately twenty-five weeks to pupils with consistent and long-term difficulties in basic Mathematics. Important to the development and implementation of the *QuickSmart* program was close collaboration with the parents, teachers, support teachers, and principals of the participating schools.

Design The study was designed as a quasi-experiment to measure the effect of increased accuracy and automaticity in basic Mathematics on more difficult mathematics questions for middle-school pupils (11-to-14 year olds) who exhibit long-term poor performance in Mathematics. Measures of improved mathematical ability were operationalised by pupil's performances on more difficult mathematics questions as provided by Australian designed standardized tests. These data were gathered before and after the intervention for the target pupils, as well as for comparison groups of same-age peers. In addition, qualitative data from sources such as interviews and field notes were collected throughout the research.

Participants A total of 12 pupils, six boys and six girls, enrolled in Years 5 or 7 from two schools in a regional district of New South Wales, were selected to participate in the *QuickSmart* Mathematics program. Within this group, three primary school pupils, and one high school pupil, were identified as Indigenous Australians.

Year 5 participants (11 year olds) All pupils in a mixed-ability class were individually assessed on basic academic skills. Based on this assessment information, and in consultation with the class teacher, six low-achieving pupils were selected. The remainder of the pupils in the class became the comparison/control group.

Year 7 participants (13 year olds) In this case the pupils in the secondary school were selected by the Head Mathematics Teacher using the criteria (i) the pupils experienced learning difficulties in basic Mathematics, (ii) performed within the lowest two bands on the State-wide Year 7 screening tests; (iii) had not shown improvement as a result of other school-based intervention or remedial programs, and (iv) attended school regularly. As a means of having a control/comparison group, four Year 7 pupils who were either average or high achieving were also identified. These comparison pupils were assessed using the same materials as the intervention group at the beginning and the end of the *QuickSmart* program.

Procedures The project plan consisted of three phases – an initial assessment, the *QuickSmart* intervention program, and a final assessment phase. The *QuickSmart* program ran for twenty-six weeks for Year 5 pupils and twenty-four weeks with the Year 7 pupils. All pupils participating in the *QuickSmart* intervention were withdrawn from their classes in pairs for five half-hour lessons spread across each fortnight with the same instructor. Where possible, the pairings of pupils matched individuals with similar instructional needs in basic Mathematics. The *QuickSmart* intervention focused on a variety of practice and recall strategies to develop understanding and fluency with basic numeric skills. Each lesson involved at least four components, namely:

- revision of the previous session,
- a number of guided practice activities featuring overt self talk and the modelling of strategy use,
- discussion, clarification and practice of memory and retrieval strategies,
- games and worksheet activities that focused on timed independent practice activities.

Observations and information gained from questioning pupils about their strategy use formed the basis of instructional decision-making and individualization. Information was also derived from lesson activities.

Additionally, CAAS assessments were completed at the end of most lessons. These provided on-going data related to pupils' levels of accuracy and automaticity in basic skills. Pupils evaluated their own learning through recording information obtained during each instructional session and using this information to identify progress and to help set realistic future goals for their achievement. Of importance was that the CAAS assessments represented a random selection of 20 items within different categories drawn from an extensive database of questions.

In order to develop transfer of learning, the *QuickSmart* intervention emphasized knowledge that could be used in classroom and other real-life settings. As well, there were attempts to link *QuickSmart* content to current classroom curriculum whenever possible.

Instruction in the *QuickSmart* program was organised into units of work of three-or-four-weeks duration with a focus on a specific set of mathematics facts. These focus facts were sets of related number facts ranging in difficulty from combinations of numbers that equal 10, to 12 times tables. It is important to note that focus facts for each unit also contained related facts such as 3 + 7 = 10, 30 + 70 = 100; 2 x 12 = 24, and ½ x 12 = 24. This approach helped to facilitate pupils' observations and understandings about the reciprocity of relationships between numbers.

Typically, the lessons began with a review of focus facts starting with those already known, and then moving on to those facts that the pupils still needed to understand and remember. Teacher-led discussion and questioning about the relationships between number facts, and ways to recall them merged into simple mathematics fact practice activities often revolving about highly focused games. These games were developed to complement each set of focus facts and allowed pupils to review and consolidate their learning in a motivating way. Timed performance activities were also used to assist pupils in developing automatic recall. In the last phase of the lesson, pupils practised on carefully selected worksheets that were closely related to the lesson content, before concluding with a brief CAAS assessment.

A feature of the lessons throughout the program was both structured and incidental strategy instruction. The aim of this strategy approach was to move pupils from relying on slow and error prone strategies, especially count-by-one strategies, to using more sophisticated and efficient strategies, including automatic recall.

Dependent Measures Data on dependent measures were collected before, during and after the *QuickSmart* intervention. Results came from four sources: CAAS, standardized tests, qualitative data, and comparison data. The results presented in this brief report focus on CAAS assessment data, and standardised test results, as well as opportunistic data available from the State-wide Year 5 Basic Skills Tests. Detailed analysis and discussion of the qualitative data is currently under preparation.

Assessments using the CAAS provided data on accuracy and automaticity of basic Mathematics. Five sub-tests of CAAS were used in this phase of the research. These were number naming of two digit numerals; addition (single plus single digit, and single plus double digit); subtraction (single and double digit numerals less than 20); triple addition (three numerals less than 20, appearing as 4 + 8 + 3); multiplication facts (to times 12); and related division facts.

Standardised Tests were used to help assess pupils' abilities to engage in more difficult mathematics activities. These tests were administered before and after the intervention. The Progressive Achievement Tests (ACER) were selected to measure this important variable. Specifically, parallel forms of the Progressive Achievement Tests in Mathematics (PATMaths) (ACER, 1997) were administered to Year 5 (Test 1A) and Year 7 (Test 2A) pupils before and after the *QuickSmart* intervention. These tests measure mathematics performance across the range of National Profile strands – number, space, measurement, and chance and data.

RESULTS

The data from pupils' information retrieval times on CAAS tasks, their standardised test scores, and opportunistic data from State-wide Year 5 Basic Skills Tests were all supported by rich observational and field notes. Although not discussed here, these qualitative insights were important in developing profiles of pupils as learners and descriptions of the cognitive obstacles that prevented their success with basic Mathematics.

Data from the Computer-Based Academic Assessment System The CAAS system recorded data relating to retrieval times and accuracy levels on all tasks for all pupils on all occasions. The analyses presented in this section are based on the graphical representation of pupils' information retrieval times similar to Figure 1.

The graph in Figure 1 shows that the average information retrieval times of pupils decreased over time. For example, the Year 5 pupils were able to answer accurately addition sums in an average time of 1.7 seconds by the end of the *QuickSmart* program. At the beginning of the intervention, these same pupils took up to an average of 5.2 seconds to calculate each addition task.

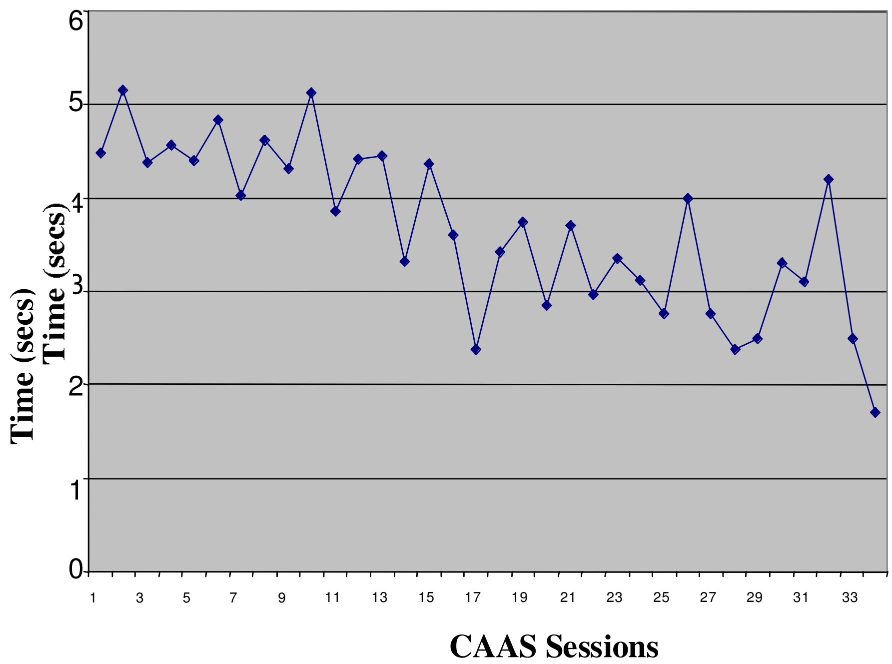


Figure 1: Year 5 Addition Averages

The improvement in retrieval times for Year 7 Mathematics pupils who completed the CAAS multiplication tasks was also dramatic. At the beginning of the program pupils took an average time of approximately 2.6 seconds to respond to each multiplication example. By the end of *QuickSmart,* the average time was more than halved to 1.15 seconds.

A further filter through which to view the results of the intervention program is provided by comparing groups of pupils' response times before and after the intervention. Pupil's *t*-tests (two-tailed with unequal variance) were applied to detect statistical differences between intervention and comparison groups, and paired *t*-tests (two-tailed) were used to detect differences within groups (before versus after). These analyses indicate that the *QuickSmart* intervention was effective in assisting pupils to achieve results comparable to those of their same-age peers. In two out of three mathematics sub-tests of the CAAS there were significant differences between the participants and comparison pupils before the intervention. After the intervention no significant differences were found between the groups' response times. This finding supports the claim that *QuickSmart* can bring pupils 'up to speed' in comparison to their peers on basic mathematics tasks.

Standardised Test Scores Although it is accepted that improvement on standardised measures is hard to achieve through intervention research, all of the Year 5 pupils and five-of-the-six Year 7 pupils increased their post-test percentile rank scores. Individual improvements of up to 63 percentile points were noted.

T-test results indicate that the Year 5 and 7 *QuickSmart* pupils' post-test scores were uniformly higher, at the .05 level of significance, than their pre-test scores ($t = 2.49$, $p < .05$). These results can be interpreted as support for the hypothesis that increased accuracy and automaticity in basic academic skills results in improvements in undertaking more difficult mathematics tasks.

Opportunistic data were also available from the State-wide Year 5 Basic Skills Test. Results indicate that for the first time since this State-wide program of testing began, no pupils in this particular primary school were placed in the lowest band for Mathematics. In fact, only one Year 5 pupil was in the second lowest achievement band (Band 2) while two pupils achieved in the second highest band (Band 5).

Of the six pupils participating in the *QuickSmart* program, three had also been pupils at the same school during Year 3. Consequently, these pupils' State-wide Year 3 Basic Skills Test results were available to the researchers. This information is summarised in Table 1. All these pupils showed improvement in Mathematics greater than Literacy and the state average of 6.5 growth points. The *QuickSmart* Mathematics group scored an average of 9.4 growth points on the Basic Skills Test for Mathematics, compared to an average of 6.5 points for their Literacy scores.

QuickSmart Mathematics PUPILS				
	Year 3	Year 5	Band Year 5	Growth Score
QuickSmart Mathematics Pupil 1				
BST Literacy Results	47.4	54.2	4	6.8
BST Mathematics Results	40.2	49.9	3	9.7
QuickSmart Mathematics Pupil 2				
BST Literacy Results	51.8	55.5	4	3.7
BST Mathematics Results	43.6	51.6	3	8.0
QuickSmart Mathematics Pupil 3				
BST Literacy Results	48.8	57.7	5	8.9
BST Mathematics Results	53.4	63.9	5	10.5

Table 1: Basic Skills Results (Growth Average for the State is 6.5 pts)

CONCLUSION

The *QuickSmart* intervention made a marked difference to the mathematics performance of those pupils who participated in the program. The most marked differences occurred for the Year 5 pupils although the Year 7 pupils also showed statistical significant improvements.

A follow-up study with the same pupils found they did not regress over a period of one year after the intervention program was completed. Hence, these pupils were able to maintain the gains they made. Importantly, this maintenance of performance was sustained across all the mathematics tasks tested by the CAAS system.

Because the *QuickSmart* intervention has a strategy orientation to improving pupils' basic academic skill performance, it moves away from addressing academic problems through 'busy' unsequenced worksheet practice. Instead, it offers an alternative based on supporting pupils to learn to "trust their heads" by encouraging pupils to discard effortful strategies hence freeing up the demands basic Mathematics has on their working memory. As such, the *QuickSmart* program represents a fourth-phase intervention model for offering a new hope for supporting persistent low achievers in Mathematics. This fourth phase is appropriate after initial teacher instruction (Phase 1), teacher remediation in class (Phase 2) and typical in-class remediation by a support teacher (Phase 3) have proven unsuccessful.

In the *QuickSmart* program there are four main themes:

- there is an emphasis on self-regulation, metacognition and self-esteem, with the goal of increasing independence in learning;
- there is extended practice in the application of understanding and strategy use;
- pupil progress is regularly monitored and feedback given; and
- positive reinforcement is provided and initially this needs to be extrinsic, but intrinsic motivation is the long-term goal.

Future research will explore how these key themes relate in helping pupils confront their learning obstacles and whether any one of these points is most significant in leading to improved learning outcomes. Also needed from research is information on whether there are optimal years of schooling in which to offer *QuickSmart* to pupils with mathematics learning difficulties, and to explore, more deeply, relationships between automaticity of basic mathematics skills and working memory capacity.

References

Ashcraft, M. H., Donely, R. D., Halas, M. A., & Vakali, M. (1992). Working memory, automaticity, and problem difficulty. In J. I. D. Campbell (Ed.), *Advances in psychology 1991: The nature and origins of mathematical skills* (pp. 301–329), Amsterdam, The Netherlands: Elsevier.

Gersten R., & Chard, D. (1999). Number sense: Rethinking arithmetic instruction for students with mathematical disabilities. *Journal of Special Education*, *33* (1), 18-28.

McNamara, D.S. & Scott, J.L. (2001). Working memory capacity and strategy use. *Memory and Cognition*, *29* (1), 10-17.

Progressive Achievement Tests for Mathematics. (1997). Hawthorn, Victoria: ACER.

Royer, J.M., & Tronsky, L.N. (1998). Addition practice with math disabled students improves subtraction and multiplication performance. *Advances in Learning and Behavioural Disabilities*, *12*, 185-217.

Royer, J.M., Tronsky, L.N., & Chan, Y. (1999). Math-fact retrieval as the cognitive mechanism underlying gender differences in math test performance. *Contemporary Educational Psychology*, *24*, 181-266.

Zbrodoff, N. J., & Logan, G. D. (1986). On the autonomy of mental processes: A case study of arithmetic. *Journal of Experimental Psychology: General*, *115*, 118–130.

DEGREES OF FREEDOM IN MODELING: TAKING CERTAINTY OUT OF PROPORTION

Irit Peled and Ronit Bassan-Cincinatus

University of Haifa

In its empirical part this paper establishes a general weak understanding of the process of applying a mathematical model. This is also evident in the way teachers regard the application of alternative sharing in their own problem solving and in relating to children's answers. The theoretical part analyses problems that are considered as applications of proportional reasoning. It suggests that the rationale for applying a proportion model varies and includes, for example, cases with a scientific rationale and others with a social one. In some problems there are no degrees of freedom in applying proportion, but in other cases this model should not be taken as "engraved in stone". This analysis is supported by examples of alternative sharing in Talmudic laws or bankruptcy interpreted by game theoretic models.

THEORETICAL BACKGROUND

Modeling

This research expands the existing knowledge of the nature of mathematical modeling by offering an analysis of application rationale. The term modeling refers here to applying a mathematical model in a problem solving situation. As a less "automatic" act, modeling can be defined as the process of organizing and describing a situation or a phenomenon by using a mathematical model (or models) or "mathematizing" the situation by perceiving it through mathematical lenses (Greer, 1993).

Following the re-thinking of math education goals the interest in modeling processes increased, recognizing the importance of modeling expertise as a goal, and noticing that a good modeling activity, in turn, adds meaning to the applied mathematical model, increasing its power and enriching children's mathematical concepts. This meant that the nature of school problems and classroom practice had to change.

Some researchers showed that classroom norms were responsible for the fact that children do not use realistic considerations in problem solving (Reusser & Stebler, 1997; Greer, 1997; Verschaffel, De Corte & Borghart, 1997). Children and pre-service teachers in several different countries were given problems that called for use of everyday knowledge, such as the fact that even a very fast runner cannot keep up his hundred meter speed when running a whole kilometer. In conventional classroom conditions, and even when children were given some hints on the special nature of these problems, children did not use realistic considerations. However, when the didactical contract (Brousseau, 1997) was changed, children changed their problem solving habits (Verschaffel & De Corte, 1997; Verschaffel, Greer, & De Corte, 2002).

Modeling standard problems

Our earlier research (Peled & Hershkovitz, 2004) suggests that a more inquisitive modeling attitude should be used not only in specially designed problems of the type composed by Verschaffel et al. (2002), but become a common practice even in standard problems. Peled and Hershkovitz (ibid) asked teachers and students to solve a conventional proportional reasoning problem. Most of the teachers applied proportional reasoning. A few of them made some drawings and gave a different answer. Class discussion revealed that teachers who solved the problem (correctly) using proportional reasoning engaged in an almost automatic application of proportion without deliberation on the reason why this model fit the given situation. The discussion and comparison of alternative teachers' and children's solutions made teachers analyze the situation and enrich their understanding of ratio and proportion.

This research triggered our further investigation of modeling in standard school problems. We noted that teachers and textbook writers create sets of problem types they consider should be solved by using a certain mathematical model leaving no option for alternative solutions and leading students to view the connection between mathematical models and situations as an undisputed truth.

Doubting certainty in model application

Our concern in this study goes beyond the gap that exists between school mathematics and everyday mathematics. We focus on the assumptions behind applying a certain mathematical structure and analyse their nature. The following example will demonstrate what we mean by that.

The Lottery Problem:

> Two friends, Anne and John, bought a \$5 lottery ticket together. Anne paid \$3 and John paid \$2. Their ticket won \$40. How should they share the money?

A problem of this type appears in textbooks in the ratio and proportion chapter as an example of a situation in which an amount should be shared using a given ratio. In this case, the \$40 sum is expected be split into two amounts using a 3:2 ratio.

Ron, a seventh grader, suggested 3 different solutions to the Lottery Problem (converted here from the original IS to \$):

Solution 1: 40:2=20 Each child gets \$20.

Solution 2: One child (the one who paid \$2) gets \$19 and the other (the one who paid \$3) will get \$21 (although the difference is \$2 and not \$1).

Solution 3: One will get \$16 and the other 24 because $\overset{2\ \ 3}{\overset{\vee}{40:5}}=8$

3x8=24, 24 to the child who paid \$3

2x8=16, 16 to the child who paid \$2

Ron: In my opinion, the first solution is the most fair, but the third is most right because of the ratio.

Ron, aware of classroom norms, knows that the teacher expects him to give the third solution, even if it doesn't feel so right to him. But is proportional sharing really the "right" (and unique) model? Why?

It should be noted that similar answers were given by pre-service teachers in a more complex money-sharing situation described by Koirala (1999) in a problem involving the purchase of shoes in a "3 for 2" sale. Rather than figuring the cost for each of the two friends who are making one purchase by taking care of giving each of them the same percent off, some pre-service teachers suggest different kinds of splits. For example, some (the author identifies them as students with good mathematical understanding) think that dividing the saving evenly is fair. Koirala (ibid) seems to think that there is one correct answer, using the same-percent-off split. In fact, as the title of his article implies, he is worried that academic mathematics might be lost by legitimizing alternative solutions. We do not agree with his point of view.

What is the basis for using proportional sharing? Is it inherent in the situation? Can we use another mathematical model? In the theoretical analysis we will contrast this situation with other situations in showing that the fitting of proportion in this problem is done on a relatively weak basis. Our purpose is to develop and then encourage a meta-analysis of the modeling process that deals with the modeling assumptions, their nature, and the degree of certainty with which we apply a mathematical model.

FINDINGS

Although we split the findings report into a theoretical part and an empirical part, the two parts were conducted simultaneously. The empirical findings start with data that establishes alternative answers in a money sharing situation, and presents attitudes towards these different ways of mathematizing the situation. It continues with a short description of class discussion that was conducted after an initial theoretical analysis was done. Our analysis of this discussion and additional workshop discussions resulted in a refinement of the theoretical analysis.

Empirical findings

We detail here a part of the data collected including children's answers and teachers' reactions, and summarize one of the discussions we conducted.

A group of 24 seventh graders and a group of 43 elementary school teachers were given the original version of the Lottery Problem. They were asked to solve the problem and then react to the following children's answers:

Aviv's answer: 40:2=20 Each one should get 20 IS.

Danit's answer: Anne should get 21½ IS and John should get 19½ IS, because Anne invested 3 IS and John invested 2 IS, the difference is 1 IS therefore the difference in their winning shares should also be 1 IS.

Yaron's answer: Anne should get 24 IS and John should get 16 IS, because 40:5=8 and 3x8=24 and 2x8=16.

The reaction distribution for each of the two groups is depicted in Table 1.

	Aviv (equal)			Danit (diff.)			Yaron (prop)	
*	+	-	+/-	+	-	+/-	+	-
Teachers n=43	7	23	13	1	36	6	43	0
Students n=24	12	12		14	10		16	8

* + regard answer as correct – as incorrect +/- correct and incorrect

Table 1: Teacher and student reactions to ways of money sharing.

As can be seen in Table 1, some of the teachers said that Aviv's answer or Danit's answer were both correct and incorrect. In their explanation they argued that the given solution might be correct socially or morally but incorrect from a mathematical point of view. Some typical answers: "It is their right to share the money anyway they choose, but in principle they should share their winnings using the 3:2 ratio" or "From a moral point of view equal sharing is great, but from a mathematical perspective the sharing ratio should be equal to the investment ratio". There were also some comments such as: "On a second thought, nowhere in the problem does it say that they will receive [money] according to their investment ratio, so it is possible to accept the equal share option".

These (and additional) findings motivated our efforts to develop an analytical tool for modeling. Following an initial theoretical analysis, we conducted several student and teacher workshops where we brought up the issue of modeling rationale.

In one of our first discussions with mathematics education graduate students the Lottery Problem was presented in 3 different versions (the effect of different story conditions is discussed in the theoretical analysis):

1. The original version with a $40 win. 2. A million dollars (actually IS) win.

3. As in version #2 with the additional information that Anne says: *I only have $3,* implying that she could not have bought a ticket if it were not for John's $2 contribution.

Each student was given one of the three versions and then asked to react to a variety of children's solutions. Then the Mixture Problem (shown in the theoretical analysis) was presented and the students were asked to compare the problems. Although students exhibited different reactions as a result of getting the 3 problem versions, and although they expressed positive attitude towards children's non-proportional distributions, the instructor (the first author) felt that the students did not fully understand the differences between the Lottery Problem and the Mixture Problem. As a result, she introduced a third situation that does not involve any chemical reaction.

The new situation involved a car assembly line, where each single car needs N parts type A and K parts type B. In this situation a constant ratio, N:K, exists between the number of parts independently of any given quantity of cars. As a result of this discussion, the Assembly Problem (detailed in the following section) was composed.

Theoretical analysis

One way to highlight and identify the nature of a process is by comparing it in different cases. We did that by composing a problem, the Mixture Problem, that is different in context from the Lottery Problem and yet supposedly (we will refer to the use of this word later) has the same structure. At a later point in the study, following class discussions, we composed a third problem, the Assembly Problem.

The Mixture Problem:

> Ron started painting his garden fence in green that he got by mixing 3 cans of yellow paint with 2 cans of blue paint. When he ran out of paint, he calculated that he needed 40 more cans to finish the fence. He also decided that he would like to mix yellow and blue and get the same shade of green that he had had before. How many of the 40 cans should be yellow and how many should be blue?

The Assembly Problem:

> In a certain car assembly line each car has to be equipped with large cushions for the 2 front seats and smaller ones for the 3 back seats. A load of parts arrived for a certain amount of cars. It included a total of 40 cushions which were indeed used in assembling the cars. How many of them were large cushions?

Although we took care of composing the problems to be analogical in structure (the need to revise the definition of analogical structure will be raised in our work), we claim that they are very different. In the Lottery Problem proportional sharing is a result of the assumption that it is fair to have the same profit for each dollar invested. However, in the Mixture Problem proportion is used because this is how colors behave chemically when they are mixed. The Assembly Problem does not require a moral or a scientific excuse and as will be further analysed, the use of a mathematical model in this case is not only straightforward but also very stable.

Resistance to change in problem conditions

The differences between the three problems become apparent by looking at the solution’s "resistance" to variations in story details. If, for example, John and Anne win a million dollars, will we still expect them to use a 3:2 sharing ratio? And what if Anne only had $3 and would not have been able to buy the $5 ticket without John's contribution?

Several questions are elicited by these problem variations: Who decides how money should be shared? Is there some normative social agreement that it would be fair to distribute the money proportionally? What if John wants more than his proportional share and goes to court, what does the law say about such cases?

The resistance to change criterion applies to the mixture problem in a different way: Sometimes in mixing very large amounts the chemical behavior does not follow the same pattern as in smaller amounts. Greer (1997) refers this phenomenon in cooking, where doubling the amount to be cooked does not necessarily mean that all receipt elements preserve the original ratio. The mathematical model for mixing different amounts depends on the chemical and physical mechanisms that are involved in the process. Some questions may prove pertinent here as well: How was a certain mixture formulae created in the first place? Was it an outcome of experimental observations resulting in a phenomenological connection? Or was it perhaps the result of a theoretical analysis of some chemical relations?

While the mathematical model for the Lottery Problem and the Mixture Problem might depend on problem conditions, this is not the case for the Assembly Problem. The ratio between the total number of large cushions and small ones that are used in the process is constant and independent of the amount of assembled cars. (As a matter of fact even this situation is not completely "clean"… To avoid a quality control issue that would involve unfit parts, the number of which does depend on sample size, the problem refers to the used parts only).

Thus, at this point, we have three different cases (mixture, sharing and assembly) at different locations on the "strength of application" axis (a temporary description): Assembly situations are located on the "very certain" side, moral-social situations on the "less certain" side and scientific situations somewhere in between, not too far from assembly situations.

To use or not to use proportion: Learning from other disciplines

Focusing on the specific mathematical model of proportion, our theoretical analysis takes some of its ideas from other disciplines in several ways: We interviewed specialists such as lawyers and scientist in industrial plants in an effort to understand the actual fitting of a mathematical model.

It is interesting to note that when we asked a lawyer to solve the Lottery Problem, her first reaction was surprisingly similar to Ron's answers and she suggested a proportional distribution of the money. On further prompting she admitted that this is not the way it would work in real practice. She explained that her first answer was based on identifying the problem as a school problem that should be answered as taught in school.

We also looked into ways people solve similar situations in everyday cases, and into solution procedures suggested by disciplines that deal with such problems. One of the cases involves Talmudic laws. As is shown in the following example, it offers a different solution.

Mishnah 3 in chapter 10 of the volume Ketubot (a Ketuba is a document signed by the groom listing what he will pay his bride in case of death or devorce) deals with a case where a man dies leaving 3 widows. In their Ketubot he had promised to give the first woman 100 gold coins, give the second 200 coins, and the third 300 coins. Unfortunately, what he left is smaller than the sum of the promised amounts. Rather than splitting it using a 100:200:300=1:2:3 ratio, the Mishna rules that money distribution (in our terms: the mathematical model that is applied) depends on the given amounts. For example, if the whole inheritance is 100 coins, it is equally distributed. If it is 200 coins, then the second and third wives get 75 coins and the first gets 50 coins.

For years these laws seemed inconsistent and their rationale was not known, until Aumann and Maschler (1985) developed an explanation based on game theory and the distribution of gains suggested by Shapley Value (satisfying the properties of efficiency, fairness and consistency) (Castrillo & Wettstein, 2004). This rationale can be applied in different cases (as in bankruptcy) where existing assets are smaller than the total claims.

DISCUSSION

This study followed our earlier realization (Peled & Hershkovitz, 2004) that the application of proportion in a standard problem is done automatically, with hardly any motivation to explore the situation or the reason a specific mathematical model should be applied.

The empirical findings show that in problems that look like conventional proportion problems most teachers apply a proportion model even in cases that would have called for alternative solutions in reality. Some of the teachers reluctantly accept children's alternative solutions saying that they might be morally fair but mathematically wrong.

Following these results we concluded that teachers need an analytical tool that would make them aware of the differences between situations with regard to the reasons for applying a model. A tool that would identify the modeling rational, establish the degree of certainty for applying a mathematical model, and help indicate where alternative solutions can be legitimate even in the eyes of math educators such as Koirala (1999) who do not want to loose academic mathematics.

Our theoretical analysis describes the direction we take in constructing this tool. We show examples of problems that look analogical in structure but use different contexts. These problems stand for different types of application rationale and can be located at different places on a scale that represents the amount of certainty in using the relevant mathematical model. The Assembly Problem represents a straight case of proportion while the Mixture Problem is a case for a scientific investigation and the Lottery Problem is a case for social norms and existing social laws.

We also found that game theory and Talmudic laws support our claim that the status of proportional distribution of assets (as depicted in the Lottery Problem) is different from the status of corresponding modeling in a scientific problem (as in the Mixture Problem). Several Talmudic laws suggest a non-proportional solution in cases that would probably have been solved in textbooks by applying proportional reasoning.

In our continued research we intend to refine the theoretical analysis, apply and validate it in our work with teachers in an effort to improve their understanding (and subsequently their students' understanding) of the modeling process.

References

Aumann, R., & Maschler, M. (1985). Game theoretic analysis of a bankruptcy problem from the Talmud. *Journal of Economic Theory, 36,* 195-213.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. Dordrecht: Kluwer Academic Publishers.

Castrillo, D. P., & Wettstein, D. (2004). *An ordinal Shapley Value for economic environments.* A working paper.

Greer, B. (1993). The mathematical modeling perspective on wor(l)d problems. *Journal of Mathematical Behavior, 12*, 239-250.

Greer, B. (1997). Modeling reality in mathematics classrooms: The case of word problems. *Learning and Instruction*, *7*(4), 293-307.

Koirala, H. P. (1999). Teaching mathematics using everyday context: What if academic mathematics is lost? In O. Zaslavsky (Ed.) *Proceedings of the 23rd International Conference for the Psychology of Mathematics Education*, 3, 161-168.

Peled, I., & Hershkovitz, S. (2004). Evolving research of mathematics teacher educators: The case of non-standard issues in solving standard problems. *Journal of Mathematics Teacher Education. 7*(4), 299-327.

Reusser, K., & Stebler, R. (1997). Every word problem has a solution – The social rationality of mathematical modeling in schools. *Learning and Instruction*, *7*(4), 309-327.

Verschaffel, L., & De Corte, E. (1997). Teaching realistic mathematical modeling and problem solving in the elementary school: A teaching experiment with fifth graders. *Journal for Research in Mathematics Education*, 28, 577-601.

Verschaffel, L., De Corte, E., & Borghart, L. (1997). Pre-service teachers' conceptions and beliefs about the role of real-world knowledge in mathematical modeling of school word problems. *Learning and Instruction*, 7(4), 339-359.

Verschaffel, L., Greer, B., & De Corte, E. (2002). Everyday knowledge and mathematical modeling of school word problems. In K. Gravemeijer, R. Lehrer, B. van Oers, & L. Verschaffel (Eds.) *Symbolizing, modeling and tool use in mathematics education* (pp. 257-276). The Netherlands: Kluwer Academic Publishers.

"I KNOW THAT YOU DON'T HAVE TO WORK HARD": MATHEMATICS LEARNING IN THE FIRST YEAR OF PRIMARY SCHOOL

Bob Perry — University of Western Sydney
Sue Dockett — University of Western Sydney

Harry completed his first year of primary school (Kindergarten) during 2004 in New South Wales, Australia. He enjoyed school; made great friends; played lots of sport; continued to read quite successfully; was well-liked by his teachers; participated in many activities; and, on reportedly rare occasions, did some mathematics.

In this paper, comparisons are made between the mathematics Harry was capable of doing before he started school and what his parents were told he actually did during his first year of school. The paper was stimulated by Harry's response to his parents when asked, near the end of his first year, what he had learned in mathematics at school: "I know that you don't have to work hard".

INTRODUCTION

Children in New South Wales (NSW), Australia start school in Kindergarten in late January each year. The children must start school by the time they are 6 years old but they may start in the year that they turn 5, provided their fifth birthday is before July 31 of that year. Hence, it is possible for a new Kindergarten class to contain children aged between 4 years 6 months and 6 years.

In NSW primary (Kindergarten to Year 6) schools, there are six syllabuses related to separate key learning areas, one of which is mathematics. The *Mathematics K-6 Syllabus* (Board of Studies NSW, 2002), is based on current research and practice both in Australia and overseas and is organised to match the stages of learning through which students are expected to move. The four stages: Early Stage 1, Stage 1, Stage 2, Stage 3 represent the learning of a typical student across the Kindergarten to Year 6 continuum. While stages of learning and stages of schooling only rarely match for individual children, this organisation does provide teachers with some guidance as to what might be expected of students who are completing a particular stage of schooling. However, the *Mathematics K-6 Syllabus* is clear that

> students learn at different rates and in different ways, so … there will be students who achieve the outcomes for their Stage [of learning] before the end of their stage of schooling. These students will need learning experiences that develop understanding of concepts in the next Stage. In this way, students can move through the continuum at a faster rate. (Board of Studies NSW, 2002, p. 5)

There is clear recognition that children start school with mathematical knowledge and skills that should be considered when developing Kindergarten experiences.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 65-72. Melbourne: PME.

> Early Stage 1 outcomes may not be the most appropriate starting point for all students. For some students, it will be appropriate to focus on these outcomes whereas others will benefit from a focus on more basic mathematical concepts. Still others may demonstrate understanding beyond Early Stage 1 … teachers need to base their planning on the evaluation of current understanding related to all of the strands. (Board of Studies NSW, 2002, p. 14)

Many researchers (Aubrey, 1997; Doig, McCrae, & Rowe, 2003; Ginsburg, Inoue, & Seo, 1999; Perry & Dockett, 2002) have investigated the mathematical power that young children can bring with them when they start formal schooling. The conclusions reached by these authors suggest that teachers in the first year of school need to take into consideration their students' past mathematical experiences and achievements when planning their mathematics programs.

When children start school, there is a lot more going on for them than just their mathematics learning (Dockett & Perry, 2004; Dunlop & Fabian, 2003). For example, compared to the less formal approaches typically found in prior-to-school settings such as pre-schools, day care centres and homes, there is a greater emphasis on whole class approaches to learning, less choice for children as to the activities in which they might involve themselves, less control over these activities and their outcomes and less support from adults. In short, demands go up and support goes down. In mathematics learning and teaching, these changes are typically manifested in terms of a more formal, less play-based, less individual-based and more teacher-centred approach to the development of mathematical ideas (Perry & Dockett, 2004; Tymms, Merrill, & Henderson, 1997).

BACKGROUND TO THE STUDY

This paper reports on a comparison between what one student, Harry, showed he was able to do in mathematics, particularly in number, immediately before he started school in Kindergarten and what he did in number during this first year of school.

The school

Harry started school in 2004. Brightview Heights Public School is located in an upper middle-class suburb of Sydney. The school is a relatively small (almost 150 students) K-6 school with a very stable staff profile. There is very strong parental and community support for the school. Brightview Heights Public School is well endowed with buildings and other resources. In short, Brightview Heights is a school with great potential for its students' learning.

The Kindergarten class

In 2004, Brightview Heights enrolled 18 children (9 boys and 9 girls) into one Kindergarten class. The children's ages ranged from 4 years 7 months to 6 years and 1 month. All but one of the children were of English-speaking background and all had attended pre-school or day care in 2003. All the children lived with at least one of their natural parents and most lived with both parents.

The teacher

The Kindergarten teacher, Mrs Jones, had taught for more than 20 years in NSW primary schools, the last 10 at Brightview Heights. This was her third consecutive Kindergarten class, although she had taught all of the first years of school—Kindergarten to Year 2—throughout her career.

The curriculum

The mathematics curriculum for the 2004 Kindergarten class was determined by the mandatory *Mathematics K-6 Syllabus* but was also influenced by a systemic numeracy program *Count Me In Too* that has been adopted by most NSW government primary schools, and a textbook *New Maths Plus K* (O'Brien & Purcell, 2003). The text

> provides a sound foundation for the teaching and learning of mathematics through the use of comprehensive, student-friendly activities based on the *Mathematics K-6 Syllabus*. (O'Brien & Purcell, 2003, p. v)

Count Me In Too (NSW Department of Education and Training, 2001; Wright, Martland, Stafford, & Stanger, 2002) provides a systematic approach to the assessment and development of students' knowledge in early number.

Harry

When Harry commenced Kindergarten, he was 5 years and 6 months old. He knew none of the other children in his Kindergarten class although he had met some of them during the orientation sessions at the end of 2003. Harry was a quiet child, often shy when meeting new people and sometimes reluctant to seek assistance. At the commencement of school, he was reading at a Year 2 level. He enjoyed music, painting and writing, and he and his mother had constructed a number of books to celebrate special events in his life. He did not like "colouring in" or public performances. He was keen to start school, although, along with many other children starting school, he was concerned about what was going to happen and who his friends might be (Dockett & Perry, 2004).

DATA

The data on which this paper relies consist of the following:

- the Schedule of Early Numeracy Assessment (SENA) from Count Me In Too, administered to Harry four days before he commenced Kindergarten;
- written records of Harry's school number experiences as presented to his parents through his learning portfolio and completed textbook pages;
- written reports and brief discussions between Mrs Jones and Harry's parents; and
- written records of discussions between Harry and his parents about school.

These data provide a snapshot of what Harry was able to do, particularly in number, before he started school and then what he did do during his first year of school.

RESULTS

Schedule of Early Numeracy Assessment (SENA)

The SENA consists of an individual interview in which the student is asked a total of 55 questions across the topics of numeral identification, forward and backward number word sequences, subitising, and early arithmetical strategies (counting objects, addition, subtraction and beginning multiplication and division). The SENA is intended for Early Stage 1 and Stage 1 students. Harry's SENA profile, determined just before he started school, is presented in Table 1.

Number Topic	Level (of highest level) and description
Numeral identification	Level 3 (of 3): Recognises numerals to 100
Forward number word sequences	Level 4 (of 5): Can count to 30 from any number less than 30 and state the number after a given number (In fact he could count to 100 from any number less than 100 except that he consistently counted "69, 50")
Backward number word sequences	Level 4 (of 5): Can count backward from any number up to 30 and state the number before a given number (Again, was on the verge of moving to the next level: counting backwards from numbers up to 100)
Subitising	Level 2 (of 3): Can instantly recognise dice and domino patterns for numbers up to 6
Early arithmetic strategies	Level 3 (of 4): Uses larger numbers and counts on and back to find the answer

Table 1: Harry's SENA number profile before starting school

Learning portfolio and completed textbook pages

Samples of Harry's work were sent home to his parents at the end of each of the four terms in 2004. Mrs Jones explained the source of these samples in the following way:

> Each sample is selected from the work undertaken in class as part of our teaching/learning program and shows the work done by your child. It does not show work specially undertaken in class for the portfolio.

In Term 1, the worksamples for number consisted of two photocopied worksheets. The first asked the students to "Colour the group that has the most in each row and circle the group that has the least". From two to five objects were depicted. The second worksheet depicted two rows of from two to six objects and asked the students to "Circle the number that shows how many more are on the top row than on the bottom row". The Term 2 worksample consisted of two questions: a) count the rows of from four to six counters and write the corresponding numeral; b) two sets of pictures are presented—four and three in the first part and five and three in the second—and students were asked to compete sentences of the form "□ and □ makes

□". For Term 3, the number worksamples consisted of worksheets on subtraction of single-digit numbers, supported by drawings of objects that could be counted plus another where students were asked to colour or circle the objects divided in half. In Term 4, the only number worksample included in the portfolio was a page on which Harry had written the numerals 1 to 31 in the correct positions on a blank December calendar. Highlighted was December 21, the last day of the school year.

The textbook used in Kindergarten covers the learning outcomes required by the Early Stage 1 of the *Mathematics K-6 Syllabus*. These outcomes are:

Whole number:	Counts to 30, and orders, reads and represents numbers in the range 0 to 20
Addition/subtraction	Combines, separates and compares collections of objects, describes using everyday language and records using informal methods
Multiplication/division	Groups, shares and counts collections of objects, describes using everyday language and records using informal methods
Fractions/decimals	Describes halves, encountered in everyday contexts, as two equal parts of an object (from Board of Studies NSW, 2002, pp. 156-162)

Two things are clear from the textbook pages. Firstly, almost all of the pages in the textbook are completed and have been meticulously marked with red ticks and "well done" stamps. Secondly, all of Harry's work has been marked correct. That is, Harry has completed an entire year's work in his textbook without making an error or, even, it appears after close scrutiny, an erasure.

Teacher comments

While Mrs Jones certainly made herself available at the request of parents, she tended not to take the initiative in discussions with Harry's parents. When these discussions did take place, they were, for the most part to tell the parents about how well Harry was going in his school work, especially his reading. At the first formal parents' evening, held about three weeks into Term 1, the only mention of mathematics was when Harry's father enquired about when the individual SENA assessments might be undertaken for Kindergarten. Mrs Jones' reply was that she was not sure because "It has been too hot to think about or do any mathematics". (It should be noted that Sydney had experienced some very hot days in February, 2004.) At the parent/teacher interviews held in Term 2, mathematics was not mentioned either to or by Harry's parents, except that Mrs Jones suggested that Harry was "doing very well".

There were two formal written reports from the school to Kindergarten parents during 2004. The first, at the end of Term 1 was a brief note outlining how Harry was settling into Kindergarten. The following excerpts are illustrative:

> Harry reads fluently with excellent comprehension and is making very pleasing progress. He also shows a great understanding of concepts in Mathematics. …

> Harry has become increasingly more comfortable and happy in the school environment throughout the term. I feel that he has adjusted to Kindergarten very well.

The second written report was received at the very end of the year. It consists of a brief comment from the teacher:

> Harry has made excellent progress academically and in his own level of confidence at school. Harry is always happy to share his ideas and knowledge with his peers and has been a pleasure to have in the class.

and then ticks under “Demonstrating competence” (the most successful category) for all of the 28 learning outcomes listed, including the three listed under “Mathematics – Number”:

> Recognises/compares the size of groups through estimating, matching and counting
>
> Manipulates objects into equal groups (multiplication)
>
> Manipulates groups of objects by combining (addition) or separating (subtraction).

Comments from Harry

Like many children, Harry was never been openly forthcoming in his discussions about his first year of school. He would talk about his friends, playing games during recess and lunchtime, who gained rewards or was “naughty” in class, or what happened in library, dance and drama. Hardly ever, did he mention mathematics.

Harry did continue to do mathematics at home, both in terms of everyday life experiences and with specific activities devised for him by his parents. For example, he became quite adept at some computer games, many of which were either openly mathematical or had a mathematical bent. Generally, he did not talk about the mathematics he did at school, although on one occasion he did say that in mathematics at school “I know that you don’t have to work hard”.

In a book that Harry prepared at the end of 2004, reflecting on what he learned in his first year of school, the only mention of mathematics is:

> Now I know
>
> 220 + 220 = 440
>
> 310 + 220 = 530

DISCUSSION

Harry is clearly not a typical Kindergarten mathematics student. He has had many experiences before starting school that have given him a flying start in terms of reaching the Early Stage 1 mathematics outcomes typically expected of children in Kindergarten. In fact, it would seem that Harry had reached almost all of these outcomes before he started school. However, little seems to have been done to harness the tremendous potential that he brought to the Kindergarten classroom, not only for his own benefit but also for the benefit of all of the children in the class.

Harry was obviously capable of being extended a long way beyond Early Stage 1 outcomes in mathematics but, at least from the evidence of what happened in school mathematics lessons, this was not done at school. There is no doubt that Harry learned many things during his first year of school. In his reflection on this year of his life, he lists the following:

> the importance of friends; school rules; what to do at school; writing; how to play soccer; how to spell; how to play handball; and how to learn by "listening and paying attention".

No doubt, all of these are important but none of them builds on Harry's evident strengths in mathematics.

Harry undertook a SENA early in his Kindergarten year and would have completed it in a competent manner, with results similar to those achieved in the SENA completed before he started school. Such results should have suggested to the teacher that Harry needed special attention in mathematics but it appears that Harry was required to work through the same activities as the rest of the class. To his credit, he has been able to maintain his pride in his learning, at least to the extent of maintaining a perfect record in his much-used textbook. However, given that he has been able to get everything correct, then one must question how much challenge he has experienced and what learning has occurred. Where is the opportunity for Harry to develop to his full potential in mathematics when he apparently spent his time doing things that he already knew? Children faced with such lack of opportunity react in different ways. So far, Harry has continued to maintain the outward semblance of being interested in the work and, at least, being willing to complete it. However, it seems that the strongest lesson he has learned in his Kindergarten mathematics experience is that you do not have to work hard at it.

CONCLUSION

When children start school, they bring much mathematical power with them. This power has grown over the prior-to-school years and is ready to be nurtured, celebrated and extended through a purposeful and meaningful program of learning in Kindergarten. In Harry's case, his capabilities were recognised but not extended. Continued programs that "teach" him what he already knows may eventually turn him away from mathematics and possibly from learning per se.

The solution to this issue is to implement the rhetoric of the syllabus documents, systemic number program and textbook. In the Kindergarten class at Brightview Heights, the restrictions of these programs and written materials, along with an approach that did not actively extend children in mathematics, have conspired to constrain the mathematical learning of a child who has shown great potential.

Harry's experiences in mathematics in Kindergarten have produced dilemmas for all of the stakeholders. For the teacher, choices need to be made to allow the provision of appropriate individual as well as whole class experiences so that all children can be challenged and extended in their mathematics. Teachers need to be able to move beyond the constraints of the syllabus, textbooks and systemic programs so that the full potential of these resources can be realised. For parents, there is a choice to be made between being "pushy" parents at school or being content—if they are able—to extend their children at home, perhaps increasing the disparity between school and home. For the Kindergarten child, the choice is the very difficult one of being like the rest of the class or being different. Should the child be compliant and accept what is handed out or be prepared to generate opportunities for challenge within the classroom? For a child in the first year of school, this is a dilemma that is undoubtedly best avoided but, as we have seen with Harry, it is one that occurs.

References

Aubrey, C. (1997). Children's early learning of number in school and out. In I. Thompson (Ed.), *Teaching and learning early number* (pp. 20-29). Buckingham: Open University Press.

Board of Studies NSW. (2002). *Mathematics K-6 syllabus.* Sydney: Author.

Dockett, S., & Perry, B. (2004). Starting school: Perspectives of Australian children, parents and educators. *Journal of Early Childhood Research*, *2*(2), 171-189.

Doig, B., McCrae, B., & Rowe, K. (2003). *A good start to numeracy: Effective numeracy strategies from research and practice in early childhood.* Retrieved June 6, 2004 from: http://www.dest.gov.au/schools/publications/2003/GoodStart.pdf

Dunlop, A.-W., & Fabian, H. (2003). *Editorial: Transitions.* European Early Childhood Education Research Monograph Series, No. 1, 2-4.

Ginsburg, H. P., Inoue, N., & Seo, K.-H. (1999). Young children doing mathematics: Observations of everyday activities. In J. V. Copley (Ed.), *Mathematics in the early years* (pp. 88-99). Reston, VA: National Council of Teachers of Mathematics.

O'Brien, H., & Purcell, G. (2003). *New maths plus K.* Sydney: Horwitz Education.

NSW Department of Education and Training. (2001). *Count Me In Too: Professional development package.* Sydney: Author.

Perry, B., & Dockett, S. (2002). Young children's access to powerful mathematical ideas. In L. D. English (Ed.). *Handbook of international research in mathematics education: Directions for the 21st century* (pp.81-111). Mahwah, NJ: Lawrence Erlbaum.

Perry, B., & Dockett, S. (2004). Mathematics in early childhood education. In B. Perry, G. Anthony, & C. Diezmann (Eds.), *Review of mathematics education research in Australasia: 2000 – 2003* (pp. 103-125). Brisbane: Post Pressed.

Tymms, P., Merrell, C., & Henderson, B. (1997). The first year at school: A quantitative investigation of the attainment and progress of pupils. *Educational Research and Evaluation, 3*(2), 101-118.

Wright, R. J., Martland, J., Stafford, A. K., & Stanger, G. (2002). *Teaching number: Advancing children's skills and strategies.* London: Paul Chapman.

DISENTANGLING MENTORS' ROLE IN THE DEVELOPMENT OF PROSPECTIVE TEACHERS' EFFICACY BELIEFS IN TEACHING MATHEMATICS

George Philippou[1] and Charalambos Y. Charalambous[2]

[1]University of Cyprus, [2]University of Michigan

The study reported in this paper builds on a broader work aiming to identify factors that affect the development of prospective primary teachers' efficacy beliefs (TEB) with respect to teaching mathematics. Eight student teachers were interviewed in the beginning, in the middle and in the end of a fieldwork course; they were encouraged to reflect on their experiences throughout the course. The analysis of the data revealed that mentors could influence student teachers' beliefs through their own teaching style, the feedback they provide to students, and the latent messages they implicitly convey to students. Yet, mentors' influence on prospective teachers' TEB should be regarded as a function of variable factors including the student-mentor personality matching. Implications for developing practicum courses and for further research are drawn.

INTRODUCTION

Understanding prospective teachers' mathematical beliefs and the circumstances under which these beliefs might develop is considered crucial to educating teachers. Kagan (1992) observed that pre-service teachers tend to leave the university programs holding primarily the same beliefs with which they join it. She thereby recommended that teaching programs afford novice teachers extended opportunities to examine and integrate new information into their existing belief systems. Fieldwork courses have the potential to provide student teachers (hereafter referred as students) with such experiences.

Fieldwork is the most relevant part of teacher education: it aims to help students integrate theory and practice, and learn to view their future role as learning facilitators (Pajares, 1993). On the one hand, fieldwork provides students with hands on experiences, leading them to "practice theory", while, on the other hand, it encourages students to "theorize practice", especially through critical observation and analysis of lessons taught by experienced teachers (Zanting, Verloop, & Vermunt, 2003). Commenting on the catalytic role of fieldwork in changing students' beliefs, Tilemma (2000) pointed out that, though some students hold quite stable beliefs that are well articulated and embedded in their personal belief systems, these beliefs might severely be challenged and become overruled by the conditions set by teaching practice. In fact, fieldwork may evoke negative feelings and attitudes, especially when it turns to be a "sink or swim experience", viz. when students are left alone to fight against the obstacles emerging of school realities (Fullan, 1991; Tschannen-

Moran, Woolfolk Hoy & Hoy, 1998). Hence, it goes without saying that examining the development of students' beliefs during fieldwork, particularly in demanding domains such as mathematics, could inform our understanding of student teachers' professional development.

During the last decades much emphasis was placed on a subset of teachers' beliefs, namely to teachers' efficacy beliefs (TEB). It has been well documented that this construct influences predominantly teachers' professional behavior and pupils' conceptions and performance (Bandura, 1997). In particular, teachers with high TEB were found to exhibit stronger commitment and lower drop out tendency, have higher expectations for their pupils, set high goals and strive to realize them, show enthusiasm, have the view that even non motivated pupils can learn, accept pupils' ideas and be less critical to wrong answers (Gordon, Lim, McKinnon & Nkala, 1998). Though beliefs in general are considered relatively stable, it is supported that fieldwork comprises a critical period, which provides fertile soil for the development of TEB (Pajares, 1993; Tschannen-Moran et al., 1998). It is, therefore, worth examining the effect of factors embedded in training programs on the development of student teachers' TEB.

The present study elaborates on the potential role of mentors in the development of students' TEB. Research has so far indicated that mentoring constitutes a central element in teacher training (Jones, 2001; Templeton, 2003). Zanting et al. (2003) regarded mentors as potential conveyors of their practical knowledge to student teachers, whereas Athanases and Achinstein (2003) found that mentors could help novices focus on pupils' learning, rather than worrying about their own competencies or about issues related to managing the class. Additionally, in Hobson's (2002) study students perceived school-based mentoring as one of the key elements of their initial teacher education experiences. Indeed, students regarded mentors as more effective individuals than other persons involved in their teacher preparation program, in facilitating the development of their class management abilities and their skills in maintaining discipline and for helping them apply different teaching methods, as well.

However, there is growing consensus that the quality of mentoring varies. Examining mentoring in England and Germany, Jones (2001) reported that mentors' role captures a wide spectrum, from mentors being considered as teaching models and critical friends who assist newcomers with planning, teaching and evaluating students to simply being there to provide assistance to student teachers only when requested. Some of the students who participated in Hobson's (2002) study also reported several communication problems with at least one of their mentors. Finally, drawing on the results of their study, Edwards and Protheroe (2003) concluded that, though mentors are in a position to guide student teachers' participation in the practices of teaching and in flexible pedagogic responses to local classroom events, they rarely do so.

Pointing to the necessity for further research in this domain, Feiman-Namser (2001) asserted that, "we still know very little about what thoughtful mentor teachers do, how they think about their work, and what novices learn from their interactions with them" (p. 17). To the best of our knowledge, no previous study has elaborated so far on the role of mentors in the development of student teachers' TEB. Therefore, the present study geared towards unraveling mentors' role in the development of perspective student teachers' efficacy beliefs in teaching mathematics. Specifically, the purpose of the study was twofold. First, to cast some light on the ways through which mentors can influence the development of student teachers' TEB, and second, to investigate factors that affect the magnitude of their impact on students' efficacy.

METHODS

The study took place at the University of Cyprus during the spring semester of 2002. The subjects were 89 four-year prospective teachers, 6 males and 83 females (mean age=22.14 years, SD=1.90) registered in the final teaching practice course. The course was structured in two parts of six weeks each with a week break in the middle for group reflection on practice. In each part, students were assigned interchangeably to a class in either a low primary cycle (1^{st} to 3^{rd} grade) or a high primary cycle (4^{th} to 6^{th} grade), where they attended the daily program, participate in all in-class or in-school events and taught nearly 120 lessons, 30 of which are mathematics lessons. Classroom teachers operated as students' mentors: they were supposed to assign students what to teach, and discuss with students possible ways to present the content; they were also expected to be critical friends, providing students with feedback. Students also had the opportunity to observe their mentors while teaching mathematics and exchange ideas with them on issues related to teaching and managing the class.

The eight interviewees that participated in the present study were selected on the basis of gender, performance in mathematics and level of TEB in mathematics. Specifically, two students (S_5 and S_7) were males; two students (S_1 and S_5) scored below the average in the mathematics courses they had attended at the university, four (S_3, S_4, S_6 and S_8) about the average, and two (S_2 and S_7) scored above the average. The students' teaching efficacy beliefs were measured three times using an existing scale translated in Greek and reworded to reflect efficacy in teaching mathematics (more information is provided in Philippou et al., 2003). The interviewees represented the four different patterns of the development of students' TEB that emerged from the analysis. Namely, S_1 represented those students who entered the program with somewhat higher than the overall mean TEB; these beliefs were improved mainly during the first part of program. S_2 and S_3 started with slightly lower TEB than the average beliefs and got the most out of the program compared to other students. S_4 to S_7 were representative of students who entered the program with the highest TEB and continued to be above the level of the average students' efficacy. Finally, S_8 had extremely low TEB throughout the program.

Students were interviewed three times, one at the commencement of the course, one in the middle and one in the end. The interviews were quasi-structured and were conducted with the aid of a specially prepared interview plan; each interview lasted about 45'. Subjects were asked open-ended questions aiming to clarify their experiences as mathematics students and mathematics teachers, and their interaction with pupils, other preservice teachers, their mentors and the university mathematics tutors during fieldwork. The constant comparative method (Denzin & Lincoln, 1998) was used to analyze the data that emerged from the interviews. Specifically, transcripts were read intending to identify frequently used concepts and integrating themes. In this study we elaborate on the themes that were pertinent to the role of mentors in the development of students' efficacy beliefs.

FINDINGS

The analysis of transcripts indicated that mentors influenced the development of the student' TEB in diverse ways. However, nearly all transcripts suggested three overarching paths of this effect. In particular, mentors were found to affect students' teaching-image by their **teaching style**, the **feedback** they provided to students and the **latent messages** that their behavior conveyed to students, as analyzed below.

Teaching style. The mentors were more influential if they appealed to students as models in organizing and executing teaching tasks. The pattern of this influence, however, was not consistent across all students. A good mentor could be a positive model for a student, as appears in the reflections of S_5:

> Her [the mentors'] teaching was a challenge to me. She was using various teaching approaches and manipulatives, and I was trying to imitate her… I believe that the mentor affects the way the student works and thinks. A good mentor motivates the student to try harder… My mentor, no doubt, exerted a positive effect both on my beliefs and my performance: she made me work harder and believe that I can do well.

Nevertheless, good mentor teaching could also scare some students. S_4's commented, "I would never be able to teach as well as my mentor did. I tried to keep up with her as if there was a competition. I certainly could not override her; she had 12 years of experience". Likewise, referring to the efficiency of her mentor's teaching approach, S_8 pointed out that "She was leading the process as a master. I used to compare my lessons to hers and I was often disillusioned". For the same student, a less competent teacher seemed to be more influential for the development of her efficacy beliefs, as seen from her comments on the problems that her second mentor had in managing the class: "I was relieved to realize that I was not the only one who had problems in managing the class; even more experienced teachers had the same problems. That reinforced my confidence".

The difference between mentors' teaching style and students' beliefs about teaching and learning mathematics constituted another factor influencing students' beliefs. For instance, S_1, S_5, and S_7 described their mentors teaching as traditional. In most cases, that produced a rather "avoidance model", as evident in the following excerpts:

S_5: My mentor used to keep pupils at a distance; his teaching was cool, and showed no interest in learning. I felt sorry for the children and tried to offer them something new and innovative. Being an old teacher, my mentor belonged to a different school of teaching and learning. And that incited me to perform even better in teaching mathematics.

Along the same lines, S_1 noted:

S_1: My mentor was a counterexample for me. She was obsessed with covering the content; I highly disagreed with her teaching approach. I objected to rushing to present the content, since pupils seemed to fail to develop understanding. I was, thereby, incited to try to come closer to pupils and teach mathematics in more pedagogically consonant ways.

Feedback and latent messages: Students' efficacy was also sustained and further developed by the feedback they received from their mentors. Being confronted by the complexities of the teaching practice for such a long period, all eight students seemed to seek for this kind of support, since it reinforced their beliefs that they were doing well. For example, S_2 noted, "On finishing my lessons, she [the mentor] was very supportive. She was telling me that I was doing very well and that pupils were learning", whereas S_8 was indebted to her mentor, for "she persuaded me that our mistakes should be considered as opportunities for learning rather than indications of inefficiencies. That really helped me a lot".

Beyond the verbal channel of communication and persuasion, mentors were additionally affecting students' self-conceptions by the latent messages that their behavior conveyed to students. S_7 confessed that:

Every time I was deviating from the conventional way she was teaching, I felt that she was ready to tell me "you are creating troubles for my pupils". Her behavior was often degrading; she would interfere in my teaching saying, "let's explain this better".

In accord, commenting on one of her mentors' behavior, S_3 pointed out:

His whole attitude instilled doubts about my teaching competence. On seeing him observing my lesson, I often had the impression that he was ready to tell me "My God. Your teaching approach is ineffective!" I felt that I was the worst teacher in the world.

Interaction with a second mentor provided students with two teaching models, sometimes extremely dissimilar. The interviews revealed that students compared the two mentors in terms of their teaching practices and style and the feedback they provided to them. The mentors' impact also depended on the degree of congruence between the students' and mentors' personality, age and sex. For instance, mentors' openness to students' ideas, their teaching style and their avoidance in imposing their own ideas to their neophyte colleagues offered predominantly in the former being considered as accountable advocates, as echoed in S_2's reflections:

My mentor used to assist me in planning the lessons. She was not trying to impose her ideas on me; on the contrary, she was first expecting to listen to my own ideas. Then she was making suggestions. With no exception, I was following her suggestions… I trusted

> her. She was teaching mathematics in a superb way; thereby I considered her recommendations as contributing to improving my teaching approach.

The discrepancy between mentors' and students' age was another issue raised by most participants in the interviews. As S_4 put it:

> My first mentor was the one who had the most determinant effect on the way I envisioned myself as a prospective teacher. She was two years older than me… She made me believe that I could be improved. The second mentor was good; but she was much older than me. She was very helpful; yet I could not communicate with her as openly and efficiently as I was doing with my first mentor.

Finally, S_3's comments surfaced the role that the difference in mentors' and students' gender may have in the development of students' efficacy beliefs. In her own words,

> "My second mentor was a male teacher and I did not feel very comfortable with him. I felt that I could not interact with him the way I was communicating with my female mentor".

DISCUSSION

Though the findings of the present study do not allow for overgeneralizations, they point to some possible ways through which mentors can affect the development of students' TEB during fieldwork. In accord with previous studies (e.g., Bandura, 1997; Wolf, 2003), the present study reveals that mentors can inform students' beliefs and teaching practices by their own teaching and by providing feedback to students. Moving a bit further, the study suggests that mentors can also impact students' TEB by the latent messages that their behavior conveys to students. We elaborate on each aspect in turn.

The findings of the present study seem to provide support to Bandura's (1997) claim that vicarious experiences constitute a significant factor informing individuals' efficacy. Specifically, several participants commented on how influential observing their mentors teaching was, both for their practices and their own efficacy beliefs. Yet, various factors appear to have determined the magnitude of this effect. Mentors' competence was one of these factors, since students seemed to gain more from more proficient mentors. However, the findings of the study provide support to the curvilinear effect of mentors' quality of teaching, given that when their teaching was perceived as more than reachable could degrade students' efficacy. Bandura (1997) would explain this finding asserting that an excessive discrepancy between mentors' and students' competencies sets the bar too high for students and leads them to conclude that there is no meaning in trying to keep up with their mentors. The discrepancy between mentors' teaching style and students' beliefs about teaching and learning was also found to determine the influence that mentors had on neophyte teachers. In fact, students seemed to be less influenced by mentors whose teaching practices were discordant with what they believed about teaching mathematics. Though in most of these cases students were trying to employ teaching approaches that were consonant with current reform ideas in teaching mathematics, mentors could act as stumbling blocks in this endeavor.

The feedback that mentors provided to students was also found to inform students' efficacy beliefs, verifying Templeton's (2003) claim that feedback is more influential if it is provided after students' teaching. Yet, findings also suggest that were a mentor accepted as more knowledgeable and expert, his or her suggestions and feedback would be taken more seriously into account. Verbal interaction was not, though, the sole channel of providing feedback to students, since students could also receive feedback about their performance by the latent messages that the mentors' behavior conveyed to them. Though this factor has not been reported in previous studies, the interviewees' comments are indicative of the predominant effect that this path of "interaction" may have on the way students envision themselves as potential teachers.

In general, the findings of the present study provide support to the argument that there is no single way of informing individuals' efficacy beliefs (Bandura, 1997; Tschannen-Moran et al., 1998). What seems to support the development of a student's TEB can have a neutral or even negative effect on the development of another student's efficacy. It, thereby, seems more legitimate to claim that the development of students' TEB comprises a complex construct. Hence, when examining the mentor-student interaction as a potential contributor to the development of student's efficacy, we should better elaborate on the confluence of several factors, including the mentor-student personality congruence, in terms of age, gender, teaching style, and pedagogical beliefs.

With all this said, it goes without saying that teaching training programs need to place more emphasis on the potential impact that mentors have on the professional development of neophyte teachers. Given that some mentors were found to have reserved or even eliminated students' opportunities to experiment with current teaching ideas and appeared to degrade students' TEB, their role should not be underestimated. On the contrary, mentors' professionalism, their pedagogical ideas, and their competence in establishing channels of communication with preservice teachers should be seriously taken into account when considering the factors contributing to the success of fieldwork. It could even be supported that interacting for so many hours with students during their first teaching steps, mentors are more in place to scaffold the development of students' TEB than any other personnel engaged in teaching training programs.

Though the present study seems to have shed some light on mentors' potential role in the development of student teachers' TEB, several questions remain unaddressed. For instance longitudinal studies could reveal how lasting mentors' influence on students' efficacy beliefs is. Of equal importance is to examine what mentors are thinking of students. Future research could also investigate whether the latent messages that mentors convey to students are a function of students' deficiencies and self-doubts. In sum, the present study seems to have raised more questions than those it answered, verifying Wolf's (2003) assertion that we need to gain more insight into the roles that mentors play to support student teachers' professional development.

References

Athanases, S. Z. & Achinstein, B. (2003). Focusing New Teachers on Individual and Low Performing Students: The Centrality of Formative Assessment in the Mentor's Repertoire of Practice. *Teachers College Record, 105* (8), 1486-1520.

Bandura, A. (1997). *Self-Efficacy: The Exercise of Control*. NY: W.H. Freeman and Company.

Denzin, N. K., & Linkoln, Y. S. (Eds.) (1998). *Collecting and interpreting qualitative materials*. Thousand Oaks. California: SAGE.

Edwards, A., & Protheroe, L. (2003). Learning to See in Classrooms: What are Student Teachers Learning about Teaching and Learning to Teach in Schools? *British Educational Research Journal, 29* (2), 227-242.

Feiman-Namser, S (2001). Helping Novices Learn to Teach: Lessons from an Exemplary Teacher. *Journal of Teacher Education, 52* (1), 17-30.

Fullan, G. M. (1991). *The New Meaning of Educational Change.* New York: Cassell.

Gordon, C., Lim, L., McKinnon, D., & Nkala, F. (1998). Learning Approach, Control Orientation and Self-Efficacy of Beginning Teacher Education Students. *Asia-Pacific Journal of Teacher Education & Development, 1* (1), 53-63.

Hobson, A. J. (2002). Student Teachers' Perceptions of School-Based Mentoring in Initial Teacher Training. *Mentoring and Tutoring, 10* (1), 5-20.

Jones, M. (2001). Mentors' Perceptions of Their Roles in School-based Teacher Training in England and Germany. *Journal of Education for Teaching, 27*(1), 75-94.

Kagan, D.M. (1992). Professional growth among preservice and beginning teachers. *Review of Educational Research, 62,* 129-169.

Pajares, F. (1993). Preservice teachers' Beliefs: A Focus for Teacher Education. *Action in Teacher Education*, 15 (2), 45-54.

Philippou, G., Charalambous, C., & Kyriakides, L. (2003). The Development of Student Teachers' Efficacy Beliefs in Mathematics during Practicum. In N A. Pateman, B. J. Dougherty and J. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of the PME and PMENA, 4* (pp. 7-14). Hawaii: CRDG, College of Education, University of Hawaii.

Tillema, H.H. (2000). Belief Change towards Self-Directed Learning in Student Teachers: Immersion in practice or reflection on action. *Teaching and Teacher Education, 16*, 575-591.

Templeton, L. (2003). Into the Fray on the Very First Day: Lessons From the Unconventional Mentor. *Mentoring and Tutoring, 11* (2), 163-175.

Tschannen-Moran, M., Woolfolk Hoy, A., Hoy, W.K. (1998). Teacher Efficacy: Its Meaning and Measure. *Review of Educational Research, 68* (2), 202-248.

Wolf, N. (2003). Learning to Teach Mathematics for Understanding in the Company of Mentors. *Teachers and Teaching: theory and practice, 9*(2), 87-106.

Zanting, A., Verloop, N., & Vermunt, J.D. (2003). Student Teachers Eliciting Mentors' Practical Knowledge and Comparing it to Their own Beliefs. *Teachers and Teaching: theory and practice, 9* (3), 197-211.

LINEAR FUNCTIONS AND A TRIPLE INFLUENCE OF TEACHING ON THE DEVELOPMENT OF STUDENTS' ALGEBRAIC EXPECTATION

Robyn Pierce

University of Ballarat

The study of linear functions is important as it provides students with their first experience of identifying and interpreting the relationship between two dependent variables. This paper, which builds on previous research, reports a study undertaken with 64, year 9 students from two Australian schools. Linear functions were introduced to these students through a graphics calculator supported, functional approach to modelling contextual problems. The teaching was generally successful. Scrutiny of pre- and post-tests highlights the triple influence of the teaching on their progress in each element of Algebraic Expectation relevant to this stage.

Linear functions provide many students with their first experience of working with two related variables and so this is a significant point of transition in their mathematical development. The typical approach to this topic in Australian text books is to provide an abstract graphical introduction to a general rule, perhaps in the form y= ax+b, with attention given to the effect of each parameter on the graph of the function. In an earlier paper (Bardini, Pierce & Stacey, 2004) the researchers describe the mathematical development of a class of students at this point whose initial teaching followed a functional-modelling approach using context problems common to the students' everyday world, supported by the use of graphics calculators. Those students' ability to write and apply this level of algebra was closely monitored and examination of the data revealed that

> three features of the program exerted a 'triple influence' on students' use and understanding of algebraic symbols. Students' concern to express features of the context was evident in some responses, as was the influence of particular contexts selected. Use of graphics calculators affected some students' choice of letters. The functional approach was evident in the meanings ascribed to letters and rules" (Bardini et al, in press).

This paper reports a further implementation of this teaching program, this time with three year 9 classes (15 year olds) at two different schools. The pre- and post-test results for these students are scrutinised for evidence of this 'triple influence' and its impact on students developing Algebraic Expectation (Pierce & Stacey, 2001, 2002).

THE 'TRIPLE INFLUENCE': THE TEACHING

Instead of following their usual textbook the teacher and students were guided in their approach by the linear functions chapter from Asp, Dowsey, Stacey and Tynan's

(1998) Graphic Algebra, a book that arose from research conducted during the Technology-Enriched Algebra Project of the University of Melbourne. The class teachers (all partner teachers in the RITEMATHS project, HREF1), were aware of the results of the first study. They were encouraged to have students work through the material at their own pace but punctuate this by teacher intervention and whole class discussion. These would be used to teach calculator skills and emphasise important features of the algebra, for example: the meaning of letters as variables, writing algebraic rules, function notation, and transformation of linear function graphs.

The first influence: Real world context problems

The teaching was almost all set in the context of real world problems which were familiar to the students: an approach supported by Freudenthal (1991) who argues that mathematics starts within commonsense and that students' mathematical ideas develop by starting with such experientially real situations. Students in the earlier study commented that working with context problems helped them to 'relate things and to produce answers which made sense'. We may expect to see the imprint of this 'sense' in the students' choice and ordering of symbols in their written algebra.

The unit, which the teachers were asked to follow, begins with a story about a girl selling homemade lemonade. The profit she makes is set up as a function of how much lemonade she sells, first in a table and then on a graph. Students graph the function and read various information related to the problem setting, from the graph. Later, the story introduces other drink sellers with different prices and ingredient costs. These corresponding functions are graphed and the graphs and functions are compared. Points of intersection, slopes, intercepts and intervals are interpreted in context. The program then introduces other real world problems. For example, a comparison of mobile phone charges drew out the significance of slope and intercepts in terms of both the original problem and related algebraic equations, while an investigation of the relationship between height and arm span, led students to model the relationship by drawing a line of best fit by eye through data, writing the rule and then exploring the consequences of varying the parameters.

The second influence: a functional approach

In addition to the modelling approach as described above using real contexts, the perspective taken in this teaching has elements of a functional approach to algebra. While there was no formal teaching of definition of function as a single valued mapping over a domain, function notation is used and the notion of a relationship between a dependent and an independent variable is emphasised, for example, as described in the lemonade problem. Algebraic letters stand primarily for variables. This approach is expected to influence students' meaning of letters, identification of structure and interpretation of features of functions and graphs.

The third influence: the use of graphics calculators

The teaching emphasised a graphic approach to linear functions. Students solved some equations both graphically and symbolically but graphical solution methods predominated. There is strong evidence (see for example Dreyfus, 1991) that students build strong conceptual schema by moving between representations and we expect that understanding gained in the graphical representation will enhance students' ability to identify the structure of two dependent variables and identify and interpret the key features of linear functions, that is the role of the constant and coefficient.

Students' work in the graphical mode was supported by each having a TI-83+ graphics calculator for all lessons. The graphic facility of these calculators allows flexibility of scaling and, in particular, allows students to move easily between different views of a graph by zooming in and out. Even beginning students find little difficulty in entering data, viewing scatterplots and testing the validity of conjectures for a line of best fit by changing the values of parameters.

The use of this technology forms the third influence of the teaching and while we expected the focus on graphs to be a positive influence there was some concern as to whether the use of graphics calculator technology with its own peculiar symbols might impact on students' by-hand algebra. The influence of these three facets of the teaching on students' Algebraic Expectation will be explored as we consider students' written responses on both the pre- and post-tests.

LINEAR FUNCTIONS AND ALGEBRAIC EXPECTATION

Algebraic Expectation was first defined for undergraduate students studying a functions and calculus course with a Computer Algebra System (CAS) available (Pierce & Stacey, 2001). The context of learning environments where sophisticated technology is increasingly on hand, especially in the form of function graphers or CAS, has challenged mathematics educators to reconsider our focus in teaching algebra. Fey (1990) and Arcavi (1994) put forward notions of 'symbol sense' to parallel 'number sense' (see for example McIntosh, 1993). Pierce & Stacey (2001), after considering the full process of mathematical modelling, suggest that the key impact of such technology is in the process of finding a mathematical solution to a mathematically formulated problem. We summarised the 'symbol sense' thinking needed to exploit and monitor work with mathematical analysis tools within the symbolic representation as 'Algebraic Expectation'. The section below considers the key 'common instances' of Algebraic Expectation which we expect to be observable when students learn to write and use linear functions:

- recognition of conventions and basic processes
- identification of structure
- identification of key features.

Creating, interpreting and working with an algebraic (symbolic) rule for a linear function is not trivial for a novice and the foundations of Algebraic Expectation established in this context will be widely applicable. Consider a general rule for a linear function, say y = mx + c or $f(x)$ = ax + b. A student demonstrates Algebraic Expectation by identifying the structure of two related variables, both of degree 1, and hence recognising the rule for a linear function. To create or interpret such a rule the student must understand that x may vary in the values which it represents and that the value of y or $f(x)$ will depend on the value of x. The student needs also to come to see that the choice of letter to represent a variable is arbitrary but that common mathematical convention suggests the use of either a letter related to the context or a letter from the latter part of the alphabet. In developing Algebraic Expectation related to variables, it is also important that the student learns that the variable x, in our general rule, may stand for a variety of numerical values or in fact another object or expression which can replace x. In the rule that describes a relationship, units of measurement need not be included and multiplication is, conventionally, implicit.

For any linear function students should be able to identify the constant term and interpret this as the value of y when x is zero or an 'initial value'. Similarly it is important to identify the coefficient of the variable and interpret this in terms of 'rate of change': the change in y when x changes by 1. The teaching stressed this interpretation from both graphical and real world viewpoints.

This section has briefly noted instances which would show students' competence in Algebraic Expectation. This paper concentrates on understanding in the symbolic representation: other important understandings not considered here relate to other representations. In the next section we scrutinise pre- and post-tests of the students who participated in this study for evidence of the 'triple influence' of the teaching on this aspect of students' mathematical progress.

THE EVIDENCE

The findings discussed in this paper are based on students' scripts from pre- or post-tests completed by sixty-four students from three year 9 classes (approx 15 year olds) at two co-educational secondary schools. One of these three teachers had previously taught the class described in Bardini et al. (in press). The classes each followed the teaching program outlined above over a period of about 4 weeks.

This paper will consider students' responses to three multiple-part items which had equivalent forms on both tests. For the first problem, on the pre-test, students were provided with a graphical representation of the costs of hiring either Jack or Jill's truck and asked to read information and describe the rule verbally and symbolically. The post-test question paralleled this with a graph showing the alternative costs of hiring a plumber, either Bob or Chris. The second question on the pre-test provided a scenario about the cost of a vacuum cleaners with additional dust bags and the post-test described alternative costs for fun park entrance with varying numbers of rides. The third parallel question pair required finding a rule to match a table of values, complementing the previous questions requiring rules from graphs or descriptions of real situations. Examples in the paper are all given from the first item pair, but the percentage changes report on data from all three item pairs.

First it was clear that the teaching was generally successful. We see significant improvement from the pre-test when 50% of students either made no attempt to write algebraic rules or were incorrect to the post-test when only 2% of the students made no attempt to write rules and 18% did not write at least one correct rule. In addition, when writing a rule from graphical information on the pre-test 23% of students were successful and on the post-test 63%, from verbal information pre-test 14% to post-test 55% and from a table the success rate increased from 17% to 69%. Examples and changes in students' responses will provide evidence of the 'triple influence' of the teaching content on their improvement as demonstrated by Algebraic Expectation.

The triple influence on 'recognition of conventions and basic processes'

The influence of context on recognition of conventions and basic processes is seen in students' verbal responses and writing of algebraic rules. The responses of students 38, 34, and 36, below, highlight the range of level of attachment to the context, shown when students were asked to explain, in words, how to work out the cost of hiring Bob if you knew the number of hours he would be working.

Student 38 \$25 set fee plus \$50 an hour *(this is a correct interpretation of the graph)*

Student 34 Bob starts at a fixed rate of \$25 and for every hour after that there is an extra \$50.

Student 36 Ring Bob

Writing an algebraic rule to work out the cost of hiring a truck or plumber allows us insight into students' recognition of conventions, especially meaning of symbols. Some students like students 25 and 34, below, showed a progression from an inability to write in symbolic algebra to writing essentially correct rules.

Student 25 no response	**Student 25** $c = \$25 + \$50x$
Student 34 no response	**Student 34** Bob=25 + ($x \times$ \$50) = C

In other responses we see the influence of context fading as the student becomes more confident in their use of algebra:

Student 109 cost=\$100+\$50x	**Student 109** c=25+50x

Then finally we observed some students who changed from writing their rule by following arithmetic logic, that is initial cost plus rate times hours gives the cost, to the more detached conventional order commonly adopted in algebra.

Student 102 $100+50D=C$	**Student 102** $C = 50t+25$

Through the experience of the context problems and the functional approach, students clearly came to recognize that a letter may stand for a quantity which varies. Many students, such as student 142, added notes which make this understanding explicit.

Student 142 C=25+50n, n = number of hours

This is a vital step forward in their understanding of algebra from their earlier work on equations where they have met letters as standing for fixed unknowns. The imprint of context teaching is clearly seen in the increased incidence of students choosing letters related to the context (e.g. *C*, *h* (hours), *t* (time) etc) in their rules. As students became 'socialized' to the conventions of algebra we see an increase in the use of implicit multiplication (52% to 81% of responses) although the use of $ symbols within the rule remained at 12% of total responses.

Little other impact of the functional aspect of the approach or the use of graphics calculators was evident in students' choice of symbols. Despite the use of notation such as P(x) in the teaching materials, only two students used this when writing rules. Similarly, graphics calculator syntax was not adopted by the students, again only one student wrote '$\ast$' where multiplication could have been implicit. They then replaced this '$\ast$' with the conventional '$\times$'. Using contextualized problems appears to have positively influenced students' use and understanding of the meaning of symbols.

The triple influence on 'identification of structure'

A fundamental aspect of identification of structure is to see that the situations presented in all three question pairs represent functional relationships between two variable quantities. The examples of students 38 and 106 demonstrate progress.

Student 38 $50 \times x + 100$ **Student 38** $C=50 \times x + 25$, x = hours

Student 106 $50x + 100$ **Student 106** $c=50h+25$ c = cost, h = hours

On the pre- and post-tests, two of the questions requiring students to write algebraic rules were based on context scenarios and the third on a table of values without context. On the pre-test 20% of students who responded wrote only expressions in one variable to generalise context scenarios (e.g. students 38 and 106) but interestingly most of these students wrote rules using two variables to express the relationship between variables in a table of values. In addition, on the pre-test 49% of students correctly wrote rules using two variables regardless of context or table and this proportion increased to 97% on the post-test. On the post-test, the only students writing expressions in one variable, instead of relational rules, were from among those who did not respond to these items on the pre-test.

The use of the graphics calculator supported students' explorations of various relationships, for example, the mobile phone charges and especially in the development of a model to link arm-span and height. The repeated use of the function entry format of $y_1=$, $y_2 =$ etc reinforces the structure of two dependent variables. Some students, like student 108 below, who adopted the use of y rather than a context dependent variable, or student 21 who wrote that $b(x)= y$ may have been influenced by this but no student included subscripts in their rules. There was no evidence that the availability of the graphics calculator encouraged students to solve non-graphically presented questions graphically.

Student 108 $y= 50x+25$

The triple influence on 'identification of key features'

Across all three questions most students, who responded, correctly identified and interpreted the parameters. Teaching from context problems has influenced this. Only in pre-test responses to question 1 do we see some errors like those of Student 138 and Student 143. Fourteen percent of students failed to add the constant term for the initial cost while student 138 was the only student to omit the coefficient for the cost per hour. No students who responded to these items on the post-test made such errors.

Student 138 $x+50$ (correct answer $50x + 25$)

Student 143 $x = n \times 50$ (correct answer $50x + 25$)

That students' interpretation of these key features was strongly influenced by context was shown in their verbal descriptions (for example student 34 above) which translated into rules. The teaching approach reinforced the broader 'rate of change' interpretation of the coefficient rather than just 'gradient of a line' as emphasized by the typical abstract graphical introduction to linear functions. In other words, the coefficient was interpreted not primarily as slope, but as the change in the dependent variable corresponding to a change of 1 in the independent variable. Students apparently understood this well and used it to construct correct algebraic rules.

We expect that the inexact modelling will also have contributed to students' strength in identifying and interpreting these key features, although there is no direct evidence in their solutions to these items. The exploration, using the graphics calculator, for a model to link arm-span and height required students to deal with a situation where the initial value was outside of the graph window which displayed the data. Students trialled various constant and coefficient values in order to find their 'line of best fit'. This experience was intended to draw attention to the importance and meaning of these two parameters.

CONCLUSIONS

'Linear functions' is certainly a basic algebra topic but it is of fundamental, not trivial, importance. It marks the point at which many students decide that mathematics is meaningless and difficult. This study complements and extends the work of Bardini, Pierce & Stacey (in press). In both studies, the majority of students made successful progress in writing conventional algebraic expressions and developing Algebraic Expectation. The previous analysis of teaching identified three important features which impacted on students: working initially with modelling contextual problems, following a functional approach and using a graphics calculator.

In this study we have scrutinised students' responses to pre- and post-test items in order to seek evidence of the 'triple influence' of these three features of the teaching on their developing Algebraic Expectation as demonstrated in their writing and interpretation of algebraic rules. When marking students' test responses a teacher will appropriately mark a variety of answers as incorrect or as correct and allocate a grade by which the student will judge their progress. In this paper it has been shown that the variety of answers is also revealing because within both 'correct' and 'incorrect' responses can be seen a range in the students' understandings and their progress towards working competently with de-contextualised symbols in a conventional way.

While the functional approach is demonstrably appropriate for applying algebraic techniques to real world problems and the strategic use of graphics calculators supports this approach, evidence of their direct influence on students' test responses was limited. In contrast, the influence of modelling contextual problems was clear, especially in the work of those students who did not write correct rules on the pre-test. The link to context assisted students in understanding the meaning of symbols and identifying both the structure and key features of linear functions. There was also evidence that students, who could already link symbols to the context, progressed to writing more conventional de-contextual algebraic rules.

References

Arcarvi, A. (1994). Symbol sense: informal sense-making in formal mathematics. *For the Learning of Mathematics, 14*(3), 24-35.

Asp, G., Dowsey, J., Stacey, K. & Tynan, D. (1998). *Graphic Algebra: Explorations with a graphing calculator.* Berkeley, Ca: Key Curriculum Press.

Bardini, C., Pierce, R. & Stacey, K. (2004). Teaching linear functions in context with graphics calculators: students' responses and the impact of the approach on their use of algebraic symbols. *International Journal of Science and Mathematics Education, 2,* 353-376.

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall, (Ed.), *Advanced mathematical thinking* (pp. 25-41). Dordrecht, Netherlands: Kluwer Academic Publishers.

Freudenthal, H. (1991). *Mathematics as a New Educational Task*. Dordrecht, Netherlands: Kluwer Academic Publishers.

Fey, J. T. (1990). Quantity. In L. A. Steen (Ed.), *On the Shoulders of Giants: New Approaches to Numeracy* (pp. 61-94). Washington: National Academy Press.

HREF1 http//:extranet.edfac.unimelb.edu.au/DSME/RITEMATHS (accessed 12-1-05)

McIntosh, A., Reys, B. J., & Reys, R. E. (1992). A proposed framework for examining basic number sense. *For the Learning of Mathematics*, 12(3), 2-8.

Pierce, R. & Stacey, K. (2001). A framework for Algebraic Insight. In J. Bobis, B. Perry, & M. Mitchelmore (Eds.) *Numeracy and Beyond: Proceedings of the twenty-fourth annual conference of the Mathematics Education Research Group of Australasia (Vol.2.pp 418-425).* Sydney: Mathematics Education Research Group of Australasia.

Pierce, R. & Stacey, K. (2002). Algebraic Insight: the algebra needed to use CAS *The Mathematics Teacher 95*(8) 622-627.

Acknowledgements

This study is part of the RITEMATHS project (HREF1), led by Kaye Stacey, Gloria Stillman and Robyn Pierce. The researchers thank the Australian Research Council, our six partner schools and Texas Instruments for their financial support of this project, and especially the teachers and students who took part in this study.

ENGAGING THE LEARNER'S VOICE? CATECHETICS AND ORAL INVOLVEMENT IN REFORM STRATEGY LESSONS

Adrian J Pinel

University College Chichester, UK

This case study is set within the context of a national reform strategy that strongly espouses oral involvement of learners through 'catechetic' interactions, i.e. the use of question-and-answer as a means of teaching, and its inverse, the students' use of question-and-answer with their teacher and their peers in order to enhance their learning, and other oral contributions. Fourteen lessons spanning learners aged 4-15 years were observed and analysed with respect to the extent and quality of such interactions. Attention was paid to observing the catechetics of each lesson,. The incidence and nature of students' oral contributions and interactions could be described as 'the learner's voice' within lessons. Data on these lessons, coupled with data gathered from surrounding structured conversations with teachers and pupils, indicates that the espousal by teachers of some key tenets of the reform strategy is not as yet being significantly enacted in their practice. Also, implicit altered expectations of their role within lessons have not been communicated effectively to most students. The authoritarian model adopted by the reform strategy is suggested as a key factor.

INTRODUCTION

The schools hosting this case study were all in or around a small town with a relatively stable population base, so that the majority of students moved between these schools within the three phases of their compulsory education: 'first' 4-8yrs, 'middle' 9-13yrs, 'high' 14-16yrs. They had already worked together as a consortium for at least two years in attempting to understand and implement the national reform strategy. All of the teachers met within the schools, including all those whose lessons were observed, overtly espoused the reform strategy, including specifically, within informal interviews, supporting the central importance of classroom discussion, question and answer interactions, and students' oral contributions. The fourteen lessons to be seen were negotiated as part of a commissioned audit and review of the health of the mathematics provision within the consortium, and as such spanned all ages of students within 4-15 years. All teachers observed were clear about this and indicated that their chosen lesson was intended to reveal their enactment of the national reform strategy. In a previous presentation (Pinel, 2002) the catechetic interactions of these lessons were provisionally analysed: this analysis is reviewed and extended here to include all oral contributions and opportunities to contribute that were observed and recorded. Each lesson was preceded by a short semi-structured interview with the teacher, and followed by a longer semi-structured discussion with a small sample of students. A two-sided A4 observation sheet was used consistently

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 89-96. Melbourne: PME.

for recording all lesson observations. Further field-notes were made to accompany the interviews and discussions.

KEY MESSAGES OF THE ENGLISH REFORM STRATEGY

The reform strategy in primary schools is based upon a severely truncated 'five-year' project called the National Numeracy Project. This was originally intended to run from June 1996 - June 2001, and then report in 2002, but in practice was required to report in November 1997, so that a Numeracy Task Force (DFEE 1998a,b) could develop the principles upon which to implement a National Numeracy Strategy (DFEE 1999) from September 1999. This also fed into the revised National Curriculum (QCA/DFEE 1999) that came into force in September 2000, and the National Strategy for Key Stage 3 [NSKS3] (DFES 2001), covering the years 11-14, initiated in 2001.

Among the key messages emphasised throughout these reforms, and specifically reiterated within NSKS3 were:

- **Enquiry skills**: these 'enable pupils to ask questions, define questions for enquiry…' Pupils are to present concise, reasoned arguments, explaining and justifying inferences, deductions and conclusions (DFES 2001, 21-22)
- **Creative thinking skills**: these 'enable pupils… to hypothesise…' Pupils are to conjecture, hypothesise, ask questions – 'What if…?' or 'Why?' (DFES 2001, 21)
- **Teaching approach**: to ensure 'a high proportion of direct, interactive teaching' (DFES 2001, 26) where: 'high quality direct teaching is oral, interactive and lively… pupils are expected to play an active part by answering questions, contributing points to discussions, and explaining and demonstrating their methods and solutions to others in the class' (DFES 2001, 26).

This approach leads to: 'regular opportunities [for pupils] to develop oral… skills' (DFES 2001, 26). It is based on 'questioning and discussing', 'exploring and investigation' and 'reflecting and evaluating': questioning in ways that… ensure that all pupils take part; using open and closed questions, skilfully framed, adjusted and targeted…; asking for explanations; giving time for pupils to think before inviting an answer; listening carefully to pupils' responses and responding constructively…; challenging their assumptions and making them think…; asking pupils to suggest a line of enquiry…; discussing pupils' justifications of the methods or resources they have chosen' (DFES 2001, 27)

RESEARCH QUESTIONS

Present policy, as expressed in these ways, would appear to treat the classroom as a 'black box' (Black & Wiliam 2001), where certain government-sponsored inputs are fed in, making demands upon teachers and pupils by setting raised expectations, while strongly recommending an authoritative-sounding set of strategic guidelines for how the teacher and the pupils should interact within the classroom in order to meet these expectations. The key research questions therefore are:

- To what extent were these teachers espousing the above messages?
- To what extent, and in what qualitative ways, were these teachers enacting the above messages through their practice in these lessons?
- To what extent were those students who engaged in subsequent discussions, aware of the nature of their intended contributions to lessons implicit within the above messages?

In addition, as Black & Wiliam surmise, there is the question as to whether these inputs are "counter-productive – making it harder for teachers to raise standards" (ibid.) It is questionable whether the strategic guidelines are as evidence-based as they may seem. Several studies (e.g., Brown, M. 1998; Thompson, I. 2000) have challenged claims made by the Numeracy Task Force that they "aimed throughout our work to look at the evidence" and "attempted to learn not only from this country but from achievements (and mistakes) in other countries" (DFEE 1998b, 7).

PREVIOUS RESEARCH

There are as yet few studies of how the intended oral-interactive nature of lessons in this reform strategy is playing out in classrooms. Exceptions include Denvir et al. (2001) and Coles (2001). In contrast, there are several studies on questioning and responding: Nicol (1999) and Wiliam (1999, 2000) both offer significant thought-provoking contributions, while Sullivan (2001) crystallises the reform strategy's official line. The broader impact of the reform strategy is becoming well documented (Millett et al, 2004) and other studies question its approaches (Brown, M, 2000). Also of direct relevance are studies about belief systems of teachers (Gates, 2001), comparisons of what is espoused and what is planned (Lim, 1997), and how the quality of teaching can be affected by a reform strategy (Shafer, 2001).

METHODOLOGY

Establishing the lesson observations

An e-mail link was established with each school during a delegate meeting. Lessons were scheduled to be observed using e-mail negotiations with school contacts: school headteachers, in first and middle schools; the head of mathematics department, in the high school. Lesson observation visits occurred within one half-term, all being observed by the same experienced observer-researcher. Observed teachers agreed to take part in this audit - school contacts reported that the option not to take part was exercised by a handful of teachers approached. All teachers had notice of the lesson to be observed, the schedule being established at least a week ahead. Reminders were sent by e-mail a few days before visits. Lessons lasted 45–50 minutes and were observed throughout, significant incidents being noted on a proforma of observable processes, modes and styles, including specific spaces within which to record any verbal interactions (Gardner, 1993, 1999; Nicol, 1999; Wiliam, 1999).

Prior to each lesson, a 5-10 minute semi-structured interview was conducted with each teacher. This focused upon [a] putting them at their ease, while reaffirming the agreed purpose of the observation, [b] establishing what views they espoused (Lim 1997) about the national reform strategy, and how far they saw themselves as having progressed in relation to implementing the strategy, [c] more specifically, what level of importance they currently attributed to the oral-language rich, verbally interactive, mutual questioning aspects of the strategy (Busatto 2004). Notes on this aspect were collected by noting down key points immediately afterwards on a proforma, while the teacher was making final preparations for the lesson.

After each lesson, three students were drawn from the class and involved in a 20-30 minute semi-structured interview. This allowed for more in-depth exploration of their relationship to relevant mathematics ideas, and opportunity to gain insight into the penetration of reform strategy ideas about students' roles in lessons as becoming more active and interactive, as outlined above. Their perspectives and expectations in this area were easier to access against a background conversation that focused upon their current mathematical focus (Wiliam, 2000).

Each lesson observation was summarised into two pages shortly after the visit, and a copy e-mailed to school contacts. This allowed them, in discussion with observed teachers, to correct any aspects regarded as factually inaccurate, and to challenge any interpretations. In the event, no changes to the record were suggested.

Certain improvements to the methodology were suggested by the experience and later reflection. Had it been possible to carry out follow up interviews with the teachers, the methodology may have been strengthened. Had the lessons been audio- or video-taped, some more detail may have been added to the data, though perhaps at a cost – it was notable that almost all of these teachers seemed at their ease with the researcher's presence and many were almost able to forget about it when the lesson was underway. Other than these issues, the approach chosen led to considerable richness of data, while retaining an apparently high level of authenticity.

Catechetic interactions

These are various processes of teaching and learning through question-and-answer. Many forms were observed in at least one lesson. To discuss these, categories of catechetic interaction were established – these are seen as qualitatively different. The categorisation provided is related to other analyses (Bloom, 1956; Sullivan, 2001.) Bloom distinguished between 'higher order' and 'lower order' questions: here a third category of 'middle order' question s is included (Table 1).

In summary, the first key research question: *'To what extent were these teachers espousing the above messages?'* was addressed through the pre-lesson interviews. The second question: *'To what extent, and in what qualitative ways, were these* *teachers enacting the above messages through their practice in these lessons?'* was addressed through the lesson observations. Finally, the third question: *'To what extent were those students who engaged in subsequent discussions, aware of the nature of their intended contributions to lessons implicit within the above messages?'* was addressed through the post-lesson student interviews.

	Higher Order Questions:
1	elicit responses centred upon reasoning and justifications - these may include implied questions such as statements to be discussed (e.g. see Dillon, 1985)
2	incite those questioned to consider underlying structures in the mathematics
3	ask group / individuals to reflect upon the mathematics they have been studying
	Middle Order Questions:
4	request that strategies be devised
5	require that a range of possible answers be sought
6	require that known facts be used to find derived facts
	Lower Order Questions:
7	direct the group's attention to specific key features of the topic
8	require that one of a range of possible answers be found
9	require that known methods be recalled and used
10	simply require a specific answer be found
11	require the recalling of a fact

Table 1: Categories of Question.

RESULTS

The teacher-interviews established their awareness of the importance within lessons of oral interactions, of drawing out students' ideas and methods, and of allowing space for their questions. All the teachers espoused this approach, most with some enthusiasm; in a few this was less evident, or even a touch of cynicism was detected.

Oral elements of lessons

All lessons involved some teacher exposition, but in general there appeared to be insufficient opportunity for students to expose their ideas about their mathematics orally. In only three lessons were there short episodes of student exposition. Overall, there was a 'democratic imbalance're question-answer events, as teachers dominated most of these, orchestrating the 'discussion'. This appeared profoundly different to the few cases where the teacher provoked, promoted and 'chaired' discussion, enabling students to tease out issues between themselves. In the majority of lessons, student-student discussions about the mathematical focus were not obviously encouraged, and in a few cases these were actively discouraged.

Catechetic interactions

Some teacher questions were pre-formalised in written worksheets produced by the teacher, or in published materials. These are only considered if they led to some verbal interactions within lessons. Amongst such material, almost no instances of higher- or middle-order questions were noted, almost all written questions being classified as types 9, 10 and 11. Within reform strategy lessons, there is emphasis on an interactive 'mental/oral starter' involving much catechetic teaching, but even in this aspect some case-study lessons had no such interaction. In only 2 lessons was the inverse-catechetic form observed: i.e. *students* leading question-and-answer episodes.

Examples from three lessons involving higher levels of verbal interaction

In a year 9 lesson, students were drawn into a lively introductory discussion, the teacher prompting and probing with questions that were effectively differentiated and embodied high expectations. Tackling probability ideas practically, using empty number lines and problematic spinners, students were able to expose their pre-conceptions about this topic within a genuine class discussion involving contributions from almost all students. Confidence was maintained through both the authentic respect afforded to each student's views and the careful scaffolding of their evolving ideas. Within the rest of the lesson, student-student discussions about these ideas were frequent and productive.

In a year 3 & 4 lesson, the teacher at first engaged students through lively and imaginative exposition in considering place value and rounding to the nearest 10. This lesson involved students in responding to focused, probing and challenging questions, and to general questions where 'if you have the answer, whisper it to the person next to you'. She used a 'think of a number' approach to inject a strong reasoning and conjecturing emphasis. Despite this, the large majority of the verbalising within the lesson remained with the teacher.

In a year 1 lesson, focused upon odd and even numbers to 20, the introductory whole class phase lasted 25 minutes. The teacher used a quiet clear voice to raise questions, including some of a higher order alongside a range of middle and lower order questions. Attention and involvement levels remained high throughout this quite intense catechetic activity. By encouraging student questions as well, she scaffolded inverse-catechetic episodes too. After a period of small group work, the class returned to engage in a plenary, with students contributing strongly to further catechetic episodes. Finally the issue of 'what happens if… we go on beyond 20?' was raised by the teacher, leading to several students imagining and conjecturing.

DISCUSSION

Student Enculturation

In just four of the lessons, the approach seemed conducive to students as king questions, beyond simply seeking information or clarification. Students may need enculturation to believe that it is their responsibility and right to ask questions within mathematics lessons (Gates, 2001). In those few classrooms, the established 'culture 'appeared to welcome students' questions. In most lessons, it seemed that such questions may have been received as an intrusion into the teacher's space, or as a deflection from the focus and purposes the teacher had decided upon for the lesson.

Reform Strategy: forcing espousal of an authoritarian rhetoric?

It is clear that all of the case-study teachers – with varying levels of enthusiasm – espoused the key messages of the reform strategy. However the 'authoritarian' nature of the reform strategy (Brown M, 2000) appears to have led to a number of these teachers espousing its rhetoric, whether or not they plan to enact, or actually enact lessons that exemplify it. In one acute case at one end of the spectrum, the 'oral starter' consisted of students - in silence - tackling 39 written calculations from the board, and the 'plenary' involved students again in an almost identical activity.

Within the reform strategy, much emphasis has been given to oral and verbal interactions, so it might be assumed that this aspect of lessons would play a significant part in observed teaching. This study suggests that a rather more complex set of practices is in use. Prior practices seem more enduring than out-of-classroom discussions and espousals suggest. All these teachers seemed prepared to use reform strategy approaches in prior conversation and in any lesson-planning documents provided, and stated that their lessons would be following such approaches. However, there were catechetic-rich episodes in only three teachers' lessons, and at the other extreme three lessons contained almost no catechetic activity.

Instances within this study where the enactment of classroom strategies authentically engaged the student's voice, or where students themselves recognised the nature of their intended role, or where significant opportunities existed for catechetic and other rich verbal interactions were relatively rare. Therefore this study tends to support the views of those who have previously expressed concerns about the gap between rhetoric and practice.

References

Askew, M. et al. (1997). *Effective Teachers of Numeracy*, London: King's College/TTA.

Black, P. & Wiliam, D. (2004). *Inside the Black Box: Raising Standards Through Classroom Assessment*. London: King's College/NFER Nelson.

Bloom, B. & Krathwohl, D. (1956). *Taxonomy of Educational Objectives: the classification of educational goals, Handbook 1: Cognitive Domain*. Addison-Wesley.

Brown, T. (2001). The social construction of school mathematics curriculums, *Proceedings of 25th Annual Conference PME,* **1**-287, Utrecht.

Brown, M. et al. (1998). Is the National Numeracy Strategy research-based? *British Journal of Educational Studies*, **46**(4), 362-385.

Brown, M. et al. (2000). Turning our attention from the 'what' to the 'how': the National Numeracy Strategy *British Educational Research Journal*, **26**(4), 457-471.

Busatto, S. (2004). What's making the difference in achieving outstanding primary school learning outcomes in numeracy? *Australian Primary Mathematics Classroom* **9** (4) 24-26.

Coles, A. (2001). Listening: a case study of teacher change, *Proceedings of 25th Annual Conference PME,* **2**-281, Utrecht.

Davis, B. (1996). *Teaching Mathematics: towards a sound alternative*. London: Garland.

Denvir, H. et al. (2001). Pupil participation in 'interactive whole class teaching', *Proceedings of 25th Annual Conference PME,* **2**-337, Utrecht.

DFEE (1998). *Numeracy Matters: the preliminary report of the Numeracy Task Force*. Sudbury: DFEE.

DFEE (1998). *The Implementation of the National Numeracy Strategy: the final report of the Numeracy Task Force*. Sudbury: DFEE.

DFEE (1999). *The National Numeracy Strategy: Framework for teaching mathematics: Reception to Year 6*. Sudbury: DFEE.

DFES (2001). *The National Strategy for Key Stage 3: Framework for Mathematics*. Sudbury: DFES.

Dillon, J. (1985). Using Questions to Foil Discussion *Teaching and Teacher Education* **1**(2) 109-121.

Ferreira, R. (2001). Student teachers' reflections on questioning, listening and responding *Proceedings of 25th Annual Conference PME,* **1**-305, Utrecht.

Gardner, H. (1993). *Multiple Intelligences: The Theory in Practice*. New York: Basic Books.

Gardner, H. (1999). *Intelligence Reframed: Multiple Intelligences for the 21st Century.* New York: Basic Books.

Gates, P. (2001). Mathematics teacher belief systems: exploring the social foundations. *Proceedings of 25th Annual Conference PME,* **3**-17, Utrecht

Lim C & Ernest P (1997) Values in mathematics education: what is planned and what is espoused? *Proceedings of 22nd Annual Conference PME,* Lahti, Finland.

Millett, A. et al. (2004). The Impact of the National Numeracy Strategy in Year 4(I): attainment and learning, and (II): teaching. In O McNamara & R Barwell (eds) *Research in Mathematics Education* **6**, 175-206 London: BSRLM.

Nicol, C. (1999). Learning to teach mathematics: questioning, listening, and responding. *Educational Studies in Mathematics,* **37**, 45-66.

Pinel, A. (2002). Catechetics observed in mathematics classrooms. *Proceedings of 26th Annual Conference PME,* **1**-312, Norwich.

QCA/DFEE (1999). *The National Curriculum 2000*. London: HMSO.

Shafer, M. (2001). The Quality of Instruction in the context of reform *Proceedings of 25th Annual Conference PME,* **1**-367, Utrecht.

Sullivan, S. (2001). Effective questioning. In *The Mathematics Co-ordinator's File* **7**, 3-8, London:PFP.

Thompson, I. (2000). The National Numeracy Strategy: evidence- or experience-based? *Mathematics Teaching*, **171**, 23-27.

Wiliam, D. (1999). Formative Assessment in Mathematics, part 1: Rich Questioning, *Equals, Mathematics and Special Educational Needs* **5**(2) 15-18.

Wiliam, D. (2000). Formative assessment in mathematics part 3: The Learner's Role. *Equals: Mathematics and Special Educational Needs,* **6**(1), 19-22.

TEACHING PROJECTILE MOTION TO ELIMINATE MISCONCEPTIONS

Anne Prescott — University of Technology, Sydney

Michael Mitchelmore — Macquarie University, Sydney

Student misconceptions of projectile motion are well documented, but their effect on the teaching and learning of the mathematics of motion under gravity has not been investigated. An experimental unit was designed that was intended to confront and eliminate misconceptions in senior secondary school students. The approach was found to be effective, but limited by the teacher's own misconceptions.

In New South Wales, Australia, projectile motion is taught in Year 11 physics and Year 12 advanced mathematics courses. In the public mathematics examination at the end of Year 12, performance on projectile motion problems is poor. Students appear to learn standard techniques by rote and to resort to intuition when questions become more difficult—often revealing basic misconceptions about projectile motion.

This paper describes an investigation of the role of student misconceptions in the teaching and learning of projectile motion in Year 12 advanced mathematics.

STUDENT MISCONCEPTIONS ABOUT PROJECTILE MOTION

Students develop personal "theories of motion" by generalising the ideas they acquire from observation of objects in everyday situations (Keeports, 2000; McCloskey, 1983b). The research literature shows that many student misconceptions result from a pre-Newtonian *impetus theory* of motion. Put briefly, this theory attributes motion to an impetus that is given to an object initially and is then gradually used up over time. Consequences for student beliefs include the following:

A fired object initially moves in the direction of firing. Only after some impetus has to be used up can gravity act and the object fall towards the ground (McCloskey, 1983a).

An object that is dropped from a moving carrier does not receive any impetus, and therefore tends to drop straight down (Millar & Kragh, 1994). However, air resistance and the speed of the carrier might affect the actual direction of motion.

If an object is moving, then there must be a force in the direction of motion (Tao & Gunstone, 1999).

Falling objects possess more gravity than stationary objects, which may possess none at all (Thagard, 1992; Vosniadou, 1994).

For further misconceptions and detailed references, see Prescott (2004).

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 97-104. Melbourne: PME.

DEALING WITH MISCONCEPTIONS

Only a few studies have attempted to change student misconceptions about projectile motion, and we found only two embedded in classroom situations (Gunstone, Gray, & Searle, 1992; Thijs, 1992). A common method has been that of *cognitive conflict* (Behr & Harel, 1990), described by Liew & Treagust (1995) as a *predict-observe-explain* teaching sequence. In Piagetian terms, the conflict between what was predicted and what is observed may lead to disequilibrium and the construction of a new cognitive structure (Tao & Gunstone, 1999).

A review of the research literature on cognitive conflict suggested that it would be most successful when:

- students are made acutely aware of their misconceptions
- discussion is a major element of the teaching/learning process
- common misconceptions are discussed explicitly in the classroom
- students reflect on projectile motion in a variety of familiar contexts
- teachers are aware of their own misconceptions as well as those of their students.

Again, details may be found in Prescott (2004).

THE TEACHING PROJECT

A research project was designed to test whether a unit on projectile motion based on the above principles would lead to a reduction in Year 12 students' misconceptions. The effect of student misconceptions on teaching and learning was also investigated.

Participants

Two classes in each of two independent girls' schools in the Sydney metropolitan area participated. The relevant classes in School A were graded: Class A1 was the top class. The classes at School B were not graded: Class B1 and Class B2 were parallel. In all, 47 students participated.

One teacher in each school (the teachers of Classes A1 & B1, known hereafter as Teachers A1 & B1) agreed to teach an experimental projectile motion unit. One other teacher in each school agreed to teach the topic as they had taught it for a number of years (Teachers A2 & B2). All four teachers were female, and all had been teaching for between 10 and 20 years. They all had excellent reputations as mathematics teachers within their schools and in the mathematics community.

Assessing misconceptions

A structured 15-20 minutes interview was developed to assess student and teacher misconceptions about projectile motion. The final version (Prescott, 2004) consisted of 16 printed questions (some sourced from the research literature, some original) describing a variety of projectile motion situations and posing a number of questions. Students wrote or drew their answers. The interviewer encouraged students to explain their answers while being careful to avoid any evaluative comment.

The students were interviewed prior to the unit on projectile motion and again three weeks afterwards. The teachers were interviewed after teaching the unit.

In the present paper, we shall concentrate on two scales derived from this interview: one assessing understanding of the motion of fired objects (8 items) and one assessing understanding of the motion of dropped objects (5 items).

The experimental unit

The first author designed experimental materials intended to confront students' misconceptions. The emphasis was on the understanding of concepts—qualitative understanding was sought before the usual quantitative approach was introduced. The materials consisted of the following:

- Detailed information on common misconceptions about projectile motion
- A website that would help the teachers understand the topic (Henderson, 2001)
- Notes on the use of cognitive conflict in predicting the path of dropped and fired objects
- Some suggested activities:
    - Using a videotape of a ball rolling off a bench, students graph the x and y coordinates against time.
    - Students predict the path and time of flight of two coins, one dropped and other fired at the same time (van den Berg & van den Berg, 1990).
    - Students imagine observing a projectile from a long distance, either in the plane of the motion or above it.
    - Students walk across the room and predict where to drop an object so that it lands on a target on the floor.
- Websites where students could explore the effects of varying the velocity and angle of projection as well as air resistance (Fowler, 1998; Stanbrough, 1998)

The teachers were not given lesson plans, but were asked to design their own lesson plans based on the above materials. Teachers were expected to provoke discussion of projectile motion in a number of contexts in the first lessons, and to refer back to these discussions when deriving equations of motion analytically in the later lessons.

Before the unit was taught, the first author held discussions with Teachers A1 and B1 individually to ensure that each teacher was comfortable with the lesson content and, more importantly, understood the ideas herself. During these discussions, both teachers admitted to having misconceptions about the trajectory of an object dropped from a moving carrier and about the time of flight of two objects simultaneously fired horizontally and dropped. These misconceptions were resolved by discussion.

No assistance was given to Teachers A2 and B2.

RESULTS

Firstly, we summarise what happened during the lessons (all of which were either video- or audiotaped). We then analyse changes in student understanding as shown by the fired and dropped objects scales. Finally, we look at teacher misconceptions.

The Experimental Lessons

Teacher A1 wrote a handout for her students based on the experimental materials. Because Teacher B1 was uncertain about her ability to work through the experimental material, these handouts were also offered to her. As a result, Classes A1 and B1 used the same handout and the lessons followed a similar course close to the intended sequence. After each activity, the teachers discussed the implications of the results in terms of the students' expectations and in terms of their knowledge about projectiles.

In introducing the idea of projectile motion, Teacher A1 included dropped objects as well as thrown or fired objects. A discussion followed about Wile E. Coyote and the Road Runner in the Warner Brothers cartoon and how they would in reality not run horizontally off a cliff and then suddenly fall vertically downwards. Teacher B1 only discussed fired objects.

Both teachers illustrated the horizontal and vertical components of motion by considering the vectors at several points along the trajectory of a projectile. They also discussed forces, emphasising that there is only one force (gravity) and that the vertical speed decreases by 9.8 m/s each second as the projectile rises and increases by 9.8 m/s each second on the way down. The vector diagrams also prompted a discussion of the vertical and horizontal velocity at the maximum height.

Discussion of the trajectories in the coins experiment raised many issues, including air resistance, friction, gravity, and the question (again) as to whether there was a horizontal force on the ball. These discussions were robust and gave students many opportunities to confront their misconceptions.

The students spent approximately 30 minutes playing with the suggested websites (Fowler, 1998; Stanbrough, 1998). This activity enabled them to "see" the impact of air resistance, and how different initial velocities and angles lead to different trajectories.

Finally the equations of motion were introduced via a simple problem, starting with the horizontal and vertical acceleration and integrating to find the x and y components of motion. The constants of integration were calculated from the initial conditions. During this part of the lessons and in subsequent work when the students asked about questions from their textbook, the teachers emphasised the link between this more abstract content and the earlier activities.

The lesson sequence of experimentation before formalisation allowed the students to become familiar with a number of different contexts before the introduction of the algebraic techniques. The lesson transcripts show that the graphing, questioning, discussing, and web surfing provided a sound background for the algebraic approach.

The Traditional Lessons

Classes A2 and B2 both covered the topic in a similar manner. Teacher A2 included worked examples in class, while Teacher B2 spent the majority of the lesson time dealing with confusion over the algebra.

In the first lesson, both teachers talked very briefly about balls being thrown into the air. Several students showed that they had an impetus view of projectile motion:

S_1: Is it steeper to start with and then kind of drop a little bit?

S_2: It goes along and then drops

Without noting such misconceptions, both teachers then derived the equations of motion from the horizontal and vertical acceleration, presenting the analysis in general terms rather than using a specific question as a basis. The students were not very involved in the derivation of the equations of motion. The majority of students in both classes did not seem to understand the need for decomposing motion into horizontal and vertical components, and the teachers gave no reason in their "explanations". Students had particular difficulty understanding the constants of integration, and in fact many of them did not seem to understand what was meant by the term "initial conditions". Finally, in exasperation one student in Class B2 said

S_3: Just tell us in words without all the V's and stuff. What's the point of it?

After a long explanation about how the equations were parametric equations, Teacher B2 admitted that it was all "a little bit abstract". To which the student replied:

S_3: I like things more concrete. I like to know what's going on, especially if it's a practical question. I need the practical explanation.

Changes in Students' Understanding

Figures 1 and 2 show students' mean scores on the fired and dropped objects scales in each class. In each case, the left hand column refers to the interview administered before the teaching and the right hand column refers to the interview administered afterwards. Recall that a higher score indicates fewer misconceptions.

Three of the four classes were initially very similar on the fired objects scale (Figure 1), with Class B1 unaccountably lower. Classes A2 and B2 showed no change after the teaching, but Classes A1 and B1 showed a substantial improvement. (Statistical tests could not be made because of the lack of random assignment and the small cell sizes.) However, in both experimental classes, there were still many students with misconceptions about the motion of fired objects.

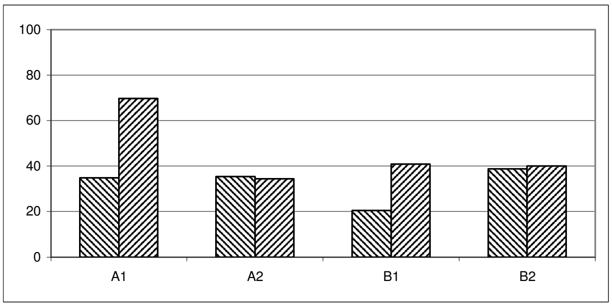


Figure 1: Mean percentage correct in each class for fired objects

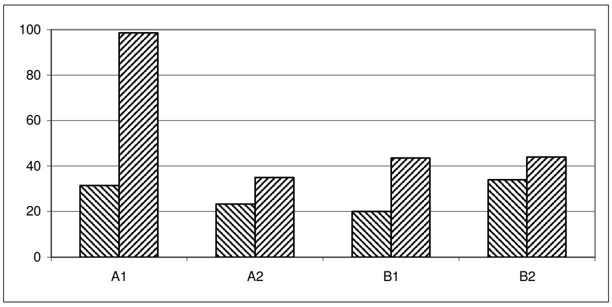


Figure 2: Mean percentage correct in each class for dropped objects

The four classes were also initially similar on the dropped objects scale (Figure 2). Although all classes showed an increase after the teaching, Classes A1 and B1 showed greater increases in mean scores (77% and 24% respectively) than Classes A2 and B2 (12% and 10% respectively). More importantly, all but one student in Class A1 showed they had no misconceptions.

These results indicate that the experimental teaching unit had the desired effect in reducing student misconceptions. But the effect was not uniform across the two scales and, with one exception, it was not very great.

Teacher Misconceptions

There were several times when the teachers revealed their own misconceptions about projectile motion, thus reducing the effectiveness of their teaching.

Although Teacher A1 generally showed an excellent understanding of projectile motion, she did once slip inadvertently into impetus theory: In a discussion about what happens at the maximum height of the trajectory, she incorrectly stated that the projectile "starts to be affected by gravity".

Teacher B1 was not consistent in her understanding of projectiles launched from a moving carrier. She correctly used an example of throwing car keys in the air while walking as an example of projectile motion. However, she became confused when a student mentioned a newspaper article in which a number of stowaways were reported to have fallen from an aeroplane wheel bay into an airport car park. Teacher B1 concluded incorrectly that the wheel bays were opened right above the car park, and no student protested. Through this one example, she negated much of the conflict created by the earlier examples.

Teacher B2 did not seem to understand gravity:

T: Because gravity is the deceleration of an object. Gravity is 9.8 m/s^2. That means I throw something up in the air then this gravity affects it by 9.8 m/s^2.

S: Does that mean that you have to throw something greater than 9.8 m/s for it to go up?

The subsequent discussion did not help the students to differentiate between velocity and acceleration, and appeared to increase their misconceptions about gravity.

The teachers' misconceptions were quite obvious during the interviews. For example, Teacher A2 indicated an impetus approach to projectile motion:

T: If you've given it a push, you must have given it a force.

When asked about objects dropped from a plane, she was torn between the bomb dropping depicted in the film "The Dam Busters" and her misconceptions:

T: I'm thinking of "The Dam Busters" now. In theory, ignore wind, I say it should go straight down but I know in fact it wouldn't. The bomb didn't.

DISCUSSION

The experimental teaching unit set out to show the students by cognitive conflict and "playing" with projectiles that they had misconceptions; then, through discussion and reflection, it was intended that students should deal with those misconceptions. The unit was also designed to help students gain a qualitative understanding before the standard algebraic techniques were introduced.

The results suggest that the unit was partially successful in helping students deal with their misconceptions. Almost all the students in one class eliminated their misconceptions about objects dropped from a moving carrier, but they still had misconceptions about fired or thrown objects. In the other experimental class, there were still many misconceptions about both types of projectiles. On the other hand, the teaching in the comparison classes, which did not confront student misconceptions, had virtually no effect on student misconceptions.

It seems to us that the main reason why teaching was more effective in some cases than others lies in the teachers' own understanding of projectile motion.

Figures 1 and 2 show that Teacher A1 was the most effective teacher, but she was more so for dropped than fired objects. She became aware of her misconceptions during the preliminary discussions, she worked to overcome them, and she was clearly successful in helping her students with their misconceptions. But she seemed less comfortable with fired objects, and an analysis of the lesson transcripts shows that she paid more attention to dropped than fired objects in her teaching.

By contrast, Teacher B1 was only moderately effective on both scales. While she was aware of her misconceptions, this was not sufficient to change her ideas (Viennot & Rozier, 1994). Although she made no errors when she discussed the ideas from the notes, she made several errors when the discussions extended to other examples.

It was clear from the interviews and observations that Teachers A2 and B2 were not aware of their own misconceptions or those of their students. Accordingly, their class activities and discussion did not challenge student misconceptions—indeed, there was some evidence that they actually reinforced student misconceptions. In their classes, the majority of students continued to believe that a projectile had a horizontal force acting on it—but they learned to use equations of motion that assumed there was no such force. It is little wonder that such students should have difficulties solving problems that go beyond routine applications of the equations of motion.

References

Behr, M., & Harel, G. (1990). Students' errors, misconceptions, and cognitive conflict in application of procedures. *Focus on Learning Problems in Mathematics, 12*(3-4), 75-84.

Fowler, M. (1998). *Projectile Motion*. Retrieved 1.7.2002 from the World Wide Web http://www.phys.virginia.edu/classes/109N/more_stuff/Applets/ProjectileMotion/applet.html

Gunstone, R. F., Gray, C. M. R., & Searle, P. (1992). Some long-term effects of long-term uninformed conceptual change. *Science Education*, *76*, 175-197.

Henderson, T. (2001). *Projectile Motion*. Retrieved 10.12.2003 from the World Wide Web http://www.glenbrook.k12.il.us/gbssci/phys/Class/vectors/u3l2a.html

Keeports, D. (2000). Addressing physical intuition—a first day event. *The Physics Teacher, 38*, 318-319.

Liew, C.-W., & Treagust, D. F. (1995). A predict-observe-explain teaching sequence for learning about students' understanding of heat and expansion of liquids. *Australian Science Teachers' Journal, 41*(1), 68-71.

McCloskey, M. (1983a). Intuitive physics. *Scientific American, 248*(4), 114-122.

McCloskey, M. (1983b). Naive theories in motion. In D. Gentner & A. Stevens (Eds.), *Mental Models* (pp. 299-324). Hillsdale, NJ: Lawrence Erlbaum Associates.

Millar, R., & Kragh, W. (1994). Alternative frameworks or context-specific reasoning? Children's ideas about the motion of projectiles. *School Science Review, 75*(272), 27-34.

Prescott, A. E. (2004). *Student understanding and learning about projectile motion in senior high school.* Unpublished doctoral thesis, Macquarie University, Sydney.

Stanbrough, J. L. (1998). *Projectiles*. Retrieved 10.12.2003 from the World Wide Web http://www.batesville.k12.in.us/Physics/PhyNet/Mechanics/Projectiles/HorProPath.html

Tao, P. K., & Gunstone, R. F. (1999). The process of conceptual change in force and motion during computer-supported physics instruction. *Journal of Research in Science Teaching, 36*, 859-882.

Thagard, P. (1992). *Conceptual revolution*. Princeton, NJ: Princeton University Press.

Thijs, G. D. (1992). Evaluation of an introductory course on 'force' considering students' preconceptions. *Science Education, 76,* 155-174.

Van den Berg, E., & van den Berg, D. (1990). Using the laboratory to teach the concept of projectile motion. *Australian Science Teachers' Journal, 36*(4), 78-81.

Viennot, L., & Rozier, S. (1994). Pedagogical outcomes of research in science education: Examples in mechanics and thermodynamics. In P. J. Fensham, R. F. Gunstone, & R. T. White (Eds.), *The content of science: A constructivist approach to its teaching and learning*. London: Falmer Press.

Vosniadou, S. (1994). Capturing and modeling the process of conceptual change. *Learning and Instruction, 4*, 45-69.

AN INVESTIGATION OF A PRESERVICE TEACHER'S USE OF REPRESENTATIONS IN SOLVING ALGEBRAIC PROBLEMS INVOLVING EXPONENTIAL RELATIONSHIPS

Norma Presmeg and Rajeev Nenduradu

Illinois State University

As part of a larger investigation of preservice teachers' use of, and movement amongst, various modes of representing exponential relationships, this report focuses on one case study, that of Mike, whose facility in moving amongst representational registers was not matched by conceptual understanding of the underlying mathematical ideas as he attempted to solve algebraic problems involving exponential relationships. Mike's case casts doubt on the theoretical assumption that students who can move fluently amongst various inscriptions representing the same concept have of necessity attained conceptual knowledge of the relationships involved.

The purpose of the study, including the case reported in this paper, was to identify and characterize preservice teachers' use of representations in solving algebraic problems involving exponential relationships. The significance of the investigation stems from the increasing awareness amongst mathematics education researchers of the role of inscriptions (as some researchers prefer to call representations – Roth, 2003) in the teaching and learning of mathematics. The use of the term inscriptions avoids the ambiguity generated by a distinction between external and internal modes of representation: while both are important in learning mathematics (Presmeg, 1997), we are concerned in this paper with inscriptions, i.e., external representations such as marks on paper. In this paper, using this sense, we shall use the two terms inscriptions and representations interchangeably. From early work on visualization (Dreyfus, 1991b) and on systems of representation (Goldin, 1992), through two Working Groups on the role of representations in the extended PME community (and their resulting publications: Goldin & Janvier, 1998; Hitt, 2002), to a current interest in semiotics as a theoretical framework for studying inscriptions (Sáenz-Ludlow & Presmeg, in progress), it is acknowledged that how mathematical ideas are represented is an issue of importance in mathematics education at all levels.

This importance was reflected in the USA's National Council of Teachers of Mathematics' inclusion of representations in their *Process Standards* for the first time (NCTM, 2000). Learning mathematical representations should provide students the "opportunity to understand the power and beauty of mathematics and equip them to use representations in their personal lives" (NCTM, 2000, p. 364). In this context, students have to familiarize themselves with a diversity of representations and should be able to use these different forms of representations flexibly. In order to accomplish this goal, teachers play a significant role in developing and promoting flexible use of multiple representations during their instruction in mathematics classrooms (National Research Council, 2002). To this end, teacher education programs should include topics in algebra, in particular, those topics that foster preservice teachers' flexible use of representations (Ball, 1990, 2003). Hence our focus was on the inscriptions used by preservice teachers, as a prelude to further research on ways that this flexible use might be fostered. Exponential functions were chosen because many students and preservice teachers in particular find this a difficult topic to grasp and represent (Goldin & Herscovics, 1991). Thus, the purpose of the research was to identify and characterize different representations that preservice teachers use – and how they use them – in solving algebraic problems involving exponential relationships.

THEORETICAL FRAMEWORK

The theoretical perspectives that provided lenses were as follows. Firstly, a theory proposed by Dreyfus (1991a) posited that the learning process evolves in four stages through the use of representations, moving from the use of a single representation at the first stage to the ability to make flexible use of representations at the last stage. Moreover, each stage determines individual levels of understanding of a concept. Secondly, our investigation was informed by Duval's (1999) notion of registers, and his stress on the importance of students being able to work within and among registers, with fluent conversion of representations in this movement.

Dreyfus (1991a) argued that abstracting and representing are complementary processes. He then discussed how these two processes are related in learning. In particular, he suggested that the learning process proceeds through four stages: 1) using a single representation, 2) using more than one representation in parallel, 3) making links between parallel representations, and 4) integrating representations and flexible switching between them. For instance, in learning the concept of function, students can start with any one of numerical, graphical, or algebraic representations. In the second stage, these representations may be used in parallel to learn the same mathematical concept. The following stage is reached when students begin to make links among the representations. Abstraction of the mathematical concept is reached in the last stage where students are able to switch flexibly among different representations as well as being able to integrate those representations. Once the fourth stage is attained, students are said to form an abstract notion of the mathematical concept or to "own" that concept. Thus, the four stages can be considered as increasing levels of understanding with an individual having a limited understanding of a concept at stage one and an abstract or highest level of understanding at the fourth stage.

Duval (1999) proposed a framework for analysing the cognitive functioning of mathematical thinking and conditions of learning. He argued that students work with different registers - forms of representation - that are crucial in understanding students' mathematical thinking. According to Duval (1999), there are three requirements in learning mathematics, as follows:

> to compare similar representations within the same register in order to discriminate relevant values within a mathematical understanding;
>
> to convert a representation from one register to another; and
>
> to discriminate the specific way of working in order to understand the mathematical processing that is performed in this register (p. 24).

Both of these theoretical formulations suggested that "switching flexibly" (Dreyfus), or "converting a representation from one register to another" (Duval), are a *sine qua non* of relational understanding (Skemp, 1987) in learning mathematics. Thus in our analysis of Mike's inscriptions we paid attention to this aspect and the issues associated with it.

METHODOLOGY

The larger investigation, of which the reported research is a part, was characterized as a qualitative instrumental case study (Stake, 2000), because the focus of the research was not the participants themselves, but rather the issue of how their modes of mathematical representation could be characterized. The instrumental case study involved five participants selected from a class (A) designed to promote preservice elementary and middle school (K-8) teachers' use of various mathematical inscriptions. However, in this report we focus on one case, that of Mike (pseudonym). We have chosen to report on Mike's use of inscriptions because his use casts doubt on some possible interpretations of previously published theoretical assumptions.

Participant

Mike was the only male non-traditional middle school major preservice teacher in class A. He was one of the three preservice teachers of average achievement who were selected for the research (the other two – of the five participants – were of above average achievement), from a total of fifteen preservice teachers, based on his previous grades in college algebra courses and a test given on functions at the beginning of the spring semester of 2004. Mike was verbal and asked questions when he did not understand any specific concept discussed in class (a second criterion for selection, the assumption being that more verbal students would be less reticent about their thought processes in interviews).

Data collection, instrumentation, and analysis

Data were collected over the whole spring semester in 2004. The data corpus for the whole study included task-based interviews, classroom observations, interviewer's notes (second author), and one reflective journal. In this paper, we focus on two one-hour audio-taped interview, in each of which Mike was asked to solve two open-ended tasks on exponential relationships. Task-based interviews were used because they are powerful means to focus on "subjects' processes of addressing mathematical tasks, rather than just on patterns of correct and incorrect answers [representations] in the results they produce" (Goldin, 2000, p. 520; our insertion).

The audio-taped interviews were transcribed and transcripts were analysed. A matrix was constructed in order to see patterns in Mike's responses. The matrix was organized according to his responses to each of the four interview items, taking into account Dreyfus' (1991a) theoretical perspective of a hierarchy of levels of representation.

RESULTS

There were two major results from the analysis of Mike's data. Firstly, tabular/numerical and algebraic representations were predominant in Mike's use of representations in solving the given algebraic tasks (table 1). He used graphical representation only once (task #3) for solving the four given tasks. Secondly, in task #3 (Endangered Species), discussed in more detail in what follows, he used tabular, graphical, and algebraic representations to find a solution, whilst interpreting the task as a linear situation instead of an exponential one.

Task	Types of representations used			
#1 (Who wants to become a millionaire? You do!)	Numerical	Tabular		
#2 (Population growth in United States)	Numerical		Algebraic	
#3 (Endangered species)		Tabular	Algebraic	Graphical
#4 (Bank problem)	Numerical		Algebraic	

Table 1: Types of representations used in the four tasks involving exponential relationships.

Table 1 demonstrates that Mike used numerical and algebraic representations more frequently to solve the four tasks given to him during the two interviews. It is interesting to note that in task #3, although he used three different kinds of inscriptions and made connections among them, because he treated the relationship as a linear one it may be deduced that his understanding of the underlying concepts was limited, as shown in the following analysis (see figure 1).

Mike started task #3 (Endangered species) by drawing a table. He wrote the given information in the task under the heading year and number of whales respectively. Mike calculated the first differences between 5000 and 4500, and then between 4500 and 4050 to get 500 and 450 respectively. As these differences were not constant, he calculated the second difference between 500 and 450 to get 50. He then assumed that the population declined in this manner and proceeded to subtract 50 from the first differences to get the number of whales in the years 1997 through 2001 to get 2550 as his answer for the first part of the given task.

ALGEBRA TASK #3

Endangered species!

Several species of whales have been declared endangered. When the population of a particular whale species falls dangerously low, biologists encourage governments to agree to a ban on hunting the species. Suppose that, in 1994, there were only 5000 whales of a particular species and the number of whales in the next two years were 4500 and 4050 respectively. Given the population was predicted to decline in this manner,

(i) What will be the whale population in 2001? 2550

(ii) Suppose the danger point for these whales comes when the population falls below 2000 whales. When will this happen? Explain your answer.

Year	# whales		
1994	5000	> 500	
1995	4500	> 450	> 50
1996	4050	> 400	> 50
1997	3650	> 350	> 50
1998	3300	> 300	> 50
1999	3000	> 250	> 50
2000	2750	> 200	> 50
2001	2550	> 150	> 50
2002	2400	> 100	> 50
2003	2300		
2004	2250		

yrs. 2450

5000 − 4500 = 500 (1 yr)
5000 − 4050 = 950 (2 yr)
5000 − 3650 = 1350 (3 yr)
5000 − 3300 = 1700 (4 yr)
5000 − 3000 = 2000 (5 yr)
5000 − 2750 = 2250 (6 yr)
5000 − 2550 = 2450 (7 yr)

x = year

$y = mx + b$

$= 5000(x) - 50$

$y = mx + b$

$y = -500(x) + 5000 \quad x = 1$

$= -500(1) + 5000$

$-500 + 5000$

$4500 \quad x = 2$

$-1000 + 5000$

$4000 \quad x = 2$

$x = 1.5$

$2450 = 5000 - (500 - 50)^7$

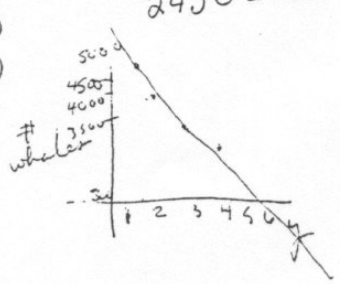


Figure 1: Mike’s inscriptions for task #3.
(Task #3 was adapted and modified from Lappan et al., 1998, p. 74 #1)

When the interviewer (the second author) asked Mike if he could solve the first part in another way, Mike wrote the general form of a linear equation ($y = mx + b$) and then wrote $5000(x) – 50$ to relate to his tabular representation. He then tried to verify mentally whether his algebraic equation held for x equal to one but got 4050 rather than 4500 as the number of whales for the year 1995.

He then tried to take $y = -500(x) + 5000$ as the equation and tried to verify this equation for x equal to 1 and 2 respectively. Although the equation was satisfied for $x = 1$, it was not true for $x = 2$ where Mike got 4000 as his answer. Clearly, Mike was having difficulty relating the tabular representation to the algebraic representation. His difficulties seemed to arise from the fact that Mike assumed the second differences to be a constant (i.e., 50), which therefore should have resulted in a quadratic equation instead of a linear equation. In his first linear equation, Mike used –50 as the y-intercept and later used –500 as the slope. Clearly, Mike was trying to guess the algebraic equation and showed limited understanding of the concept of slope and y-intercept. He did not perceive that the given task involved an exponential relationship with decay factor 0.9.

When the interviewer asked Mike if he could solve the task in yet another way, he drew a straight line as a graphical representation using the values from his table. It is interesting to note that the points from the table do not lie in a straight line but Mike "forced" the graph to be a straight line using his linear equation to relate to the graphical representation. Again, Mike was trying to make connections among the algebraic, tabular, and algebraic representations to solve the first part of the given task. However, his understanding of the related concept involved in the given situation did not match with what he showed in his solution. Also, Mike showed a limited understanding of the concepts of slope and y-intercept through the equations he wrote.

The results showed that Mike was able to use tabular, algebraic, and graphical inscriptions as well as make some links among these representations. Thus, using Dreyfus' perspective on learning, Mike should be in the third level of Dreyfus's hierarchy. However, Mike's interpretation of the related concept in the given task and the inappropriate use of the concepts of slope and y-intercept from his linear equations showed that he had an instrumental understanding (Skemp, 1987). This result may be interpreted as contrary to Dreyfus' perspective of hierarchical levels on learning in the sense that at the third level Mike should have had a relational understanding as demonstrated by his use of inscriptions.

IMPLICATIONS FOR FUTURE RESEARCH

We recognize that Dreyfus did not intend his four levels to be used in an instrumental way to classify whether students had come to "own" an abstract, relational understanding of a mathematical concept: his model takes account of the complexities of individual cognition. However, lest the theoretical assumptions of both Dreyfus and Duval come to be characterized in such a way (according to

Peirce's *Law of Mind*[i], 1992), Mike's case provides a cautionary note that fluency of conversion amongst representational registers is not a sufficient criterion for inferring a robust, relational grasp of the concepts involved.

The research reported in this paper thus suggests that in the quest to find effective ways of fostering flexible movement among various forms of inscriptions by preservice teachers, it will be necessary to pay attention to deeper aspects of the kinds of thinking implicated. Ultimately, the question of what is meant by *relational understanding* is at the heart of such efforts. Students' inscriptions, and how they use and move amongst them, may provide a window into their cognition, on which instruction may build. However, in and of itself, conversion between registers is insufficient as a goal of instruction, as Mike's case demonstrates.

References

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal, 24*, 449-466.

Ball, D. L. (2003). *What mathematical knowledge is needed for teaching?* Washington: U.S. Department of Education.

Dreyfus, T. (1991a). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking (*pp. 25-41). Dordrecht: Kluwer Academic Publishers.

Dreyfus, T. (1991b). On the status of visual reasoning in mathematics and mathematics education. In F. Furinghetti (Ed.), *Proc. 15th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, pp. 33-48). Assisi, Italy: PME

Duval, R. (1999). Representations, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt & M. Santos (Eds.), *Proc. 21st Conf. of the North American Chapter of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, pp. 3-26). Columbus, Ohio: ERIC Clearinghouse for Science, Mathematics, & Environmental Education.

Goldin, G. A. (1992). On developing a unified model for the psychology of mathematical learning and problem solving. In W. Geeslin & K. Graham (Eds.), *Proc. 16th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 235-261). Durham, New Hampshire, USA: PME

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. pp. 517-545). Mahwah, NJ: Lawrence Erlbaum Associates.

Goldin, G. A., & Herscovics, N. (1991). Toward a conceptual-representational analysis of the exponential function. In F. Furinghetti (Ed.), *Proceedings of the Fifteenth PME Conference of the International Group for the Psychology of Mathematics Education*: Volume 2 (pp. 64-71). Assisi, Italy: PME

Goldin, G. A. & Janvier, C. (Guest Eds.) (1998). Representations and the psychology of mathematics education. Parts I & II. *The Journal of Mathematical Behavior, 17*(1&2).

Hitt, F. (Ed.) (2002). *Representations and mathematical visualization.* PME-NA Working Group (1998-2002). Mexico City: Cinvestav-IPN.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (1998). *Growing, growing, growing: Exponential relationships. Teacher's Edition.* Michigan State University, MI: Dale Seymour Publications.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Research Council (2002). *Learning and understanding: Improving advanced study of mathematics and science in U. S. high schools*. Washington, DC: National Academy Press.

Peirce, C. S. (1992). *The essential Peirce: Selected philosophical writings, Volume I (1867-1893)*. Bloomington: Indiana University Press.

Presmeg, N. C. (1997). Generalization using imagery in mathematics. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors and images* (pp.299-312). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Roth, W.-M. (2003). *Towards an Anthropology of Graphing: Semiotic and Activity-Theoretic Perspectives*. Dordrecht: Kluwer Academic Publishers.

Sáenz-Ludlow, A. & Presmeg, N. (Eds.) (in progress). The teaching and learning of mathematics: Semiotic and epistemological perspectives. PME Special Issue of *Educational Studies in Mathematics.*

Skemp, R. R. (1987). *The psychology of learning mathematics.* Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Stake, R. E. (2000). Case studies. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 435-454). Thousand Oaks, CA: Sage Publications.

[i] "Logical analysis applied to mental phenomena shows that there is but one law of mind, namely, that ideas tend to spread continuously and to affect certain others which stand to them in a peculiar relation of affectability. In this spreading they lose intensity, and especially the power of affecting others, but gain generality and become welded with other ideas" (Peirce, 1992, p. 313).

ON EMBODIMENT, ARTIFACTS, AND SIGNS: A SEMIOTIC-CULTURAL PERSPECTIVE ON MATHEMATICAL THINKING

Luis Radford[1], Caroline Bardini[1], Cristina Sabena[2], Pounthioun Diallo[1], Athanase Simbagoye[1]

[1] Université Laurentienne, Canada. [2] Università di Torino, Italy.

The cognitive significance of the body has become one of the major topics in current psychology. However, it is our contention that claims about the embodied nature of thinking must come to terms with the problem of the relationship between the body as a locus for the constitution of students' subjective mathematical meanings and the historical cultural system of mathematical meanings conveyed by school instruction. Referring to episodes from a Grade 9 mathematics lesson, we here sketch a theoretical account of the aforementioned relationship that emphasizes the social and cultural nature of thinking and the cognitive role played by body, signs, and artifacts.

INTRODUCTION: THE EMBODIED MIND

The cognitive significance of the body has become one of the major topics in current psychology. Thus, it is now often claimed that human concepts are crucially shaped by our bodies and our senses (see e.g. Lakoff and Johnson, 1999; Lakoff and Núñez, 2000). The empiricists of the 18th century could not have agreed more. For the empiricists, who asserted that nothing is in the intellect which was not first in the senses, both body and sensuous impressions were indeed the gate to knowledge. For the rationalists, such as Leibniz and Descartes, the gate to knowledge was the realm of innate logic, a realm from which body and sense data were excluded. They continued a long tradition going back to Plato who said that "if we are ever to have pure knowledge of anything, we must get rid of the body and contemplate things by themselves with the soul" (*Phaedo*, 66b-67b).

Current claims concerning the cognitive significance of the body, however, do not necessarily entail a return to empiricism. They should rather be seen as an attempt to relocate the role of the sensual in the realm of the conceptual. Now, was it not precisely the problem of the sensual and the conceptual that Piaget wanted to unravel with his genetic epistemology? Did he not insist that knowledge starts with body actions? In a recent PME research forum, Nemirovsky suggested a list of guiding questions as a basis for a research agenda, among which are the following two: "What are the roles of perceptuo-motor activity, by which we mean bodily actions, gestures, manipulation of materials, acts of drawing, etc., in the learning of mathematics? How does bodily activity become part of imagining the motion and shape of mathematical entities?" (Nemirovsky, 2003). One of the audience's impressions was that Nemirovsky's questions had already been answered by Piaget's epistemology. In a certain sense, this is the case. However, as a good Kantian and follower of 18th century Enlightenment philosophy, Piaget sought to answer the riddle

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 113-120. Melbourne: PME.

of the sensual and the conceptual through a combination of empiricism and rationalism: for Piaget, knowledge starts with sense data but knowledge and ideas cannot be reduced to a combination of impressions distinguished by their intensity, as empiricists like Hume proposed; for Piaget, it is the logic-mathematical structures which will pick sensual knowledge up and transform it into abstract thinking. Nemirovky's questions make sense only in the current context of psychological and educational theories that seek to place the cognitive relevance of the body in a context larger than the limiting one of the sensorimotor stage that marks the beginning of the conceptual development of the child in Piaget's account. Despite their different perspectives, what these theories are claiming is that sensorimotor activity is not merely a stage of development that fades away in more advanced stages, but rather is thoroughly present in thinking and conceptualizing.

This claim for the full insertion of the body in the act of knowing is an indication, we think, of a turning point in contemporary views of knowledge formation. While mainstream 20th century psychology was subsumed in the empire of the written tradition –a tradition that since the invention of printing in the 15th century endowed the written word with an unprecedented degree of authority, a tradition that justified the use of tests and questionnaires to investigate the depths of the mind– we now live at a time when new forms of knowledge representations (such as digital ones) have made a definite incursion in all the spheres of life. Of course, to be fruitful, the plea for an embodied mind has still to be accompanied by theoretical elaborations about knowledge formation. If the *télos* (i.e. the *end* or *final cause*) of conceptual development can no longer be imputed to universally valid logic-mathematical structures only —as was the case in Piaget's genetic epistemology— how then are we to account for it? Piaget and Husserl were certainly right in asserting that the body is a locus for the production of meaning and the first opening of intentionality towards the world. Nevertheless, the world that the body encounters is a cultural world populated by other bodies, objects, signs, and meanings, a world already endowed with ethical, aesthetical, scientific and other values. These values provide the world with specific configurations that, instead of being neutral, qualify the body with the historicity of events and concepts deposited in language, artifacts, and institutions (Foucault, 2001, p. 1011). Hence, there are some theoretical problems that the paradigm of the embodied mind needs to tackle, in order to avoid a theory where the body is considered as the refuge of the transcendental "*I*" of idealism and to overcome critics like Eagleton who sees in the claims of the embodied perspective a theory that in the end is no more than "the return in a more sophisticated register of the old organicism" or else a token of "the post-modern cult of pleasure" and love for the concrete (Eagleton, 1998, pp. 157-58).

It is our contention that an account of the embodied nature of thinking must come to terms with the problem of the relationship between the body as a locus for the constitution of an individual's subjective meanings and the historically constituted cultural system of meanings and concepts that exists prior to that particular individual's actions. In what follows we sketch, from a semiotic-cultural perspective, some general points that might be seen as a preliminary contribution to the problem at hand. Since our main interest is the understanding of the learning and teaching of mathematics, we shall focus on the meaning of thinking and learning and the role of body, signs and objects therein. Our theoretical discussion will be intermingled with comments on a Grade 9 mathematics lesson.

THINKING AS SOCIAL PRAXIS

While traditional cognitive psychology considered thinking as the mental processing of information carried out by an individual, for the semiotic-cultural perspective here advocated, thinking is a form of *social praxis* (Wartofsky, 1979).

Thinking as a form of *social praxis* means that thinking is an active mode of social participation in which *what we know and the way we come to know it is framed by cultural forms of rationality (i.e., by cultural forms of understanding and acting in the world) out of which specific kinds of questions and problems are posed*. To illustrate this idea, let us turn to a passage of a Grade 9 lesson in our ongoing 6-year longitudinal classroom-based research program. The lesson was about the interpretation of graphs in a technological environment based on a graphic calculator TI 83+ and a probe –a Calculator Based Ranger or CBR. The students were familiar with the calculator graph environment, whereas the CBR was new for them. Prior to the passage that we shall discuss ("Pierre and Marthe's walk"), the teacher explained to the students how the CBR worked (its wave sending-receiving mechanism to measure the distance between itself and the target, distance limitations, distortions, etc.). One student was chosen to walk towards a door, holding the CBR and pointing it towards the door, to stop a few seconds and then to walk backwards from the door (the target). The class discussed the graph obtained on the calculator through the CBR. After this introductory activity, the students, in small groups of three, had to describe the motion to be performed by a student walking with the CBR in order to match a given graph (not shown here). Later, the students verified their hypothesis by doing the experiment. Right after this verification, the students worked on the interpretation of a graph related to a story in which two children, Pierre and Marthe walked in a straight line, the latter pointing a CBR to the former. The graph showed the relationship between the elapsed time (horizontal axis) and the distance between the children (vertical axis) as measured by the CBR (see Figure 1). The mathematical problem was intended as a way to introduce the students to a particular mode of thinking –a cultural kind of reflexion about the world conveyed by contemporary school instruction. Through this problem, the students became involved in a *social praxis* that goes back to the 15th century investigation of body motions, a historical revolutionary form of thinking about natural phenomena different from the previous

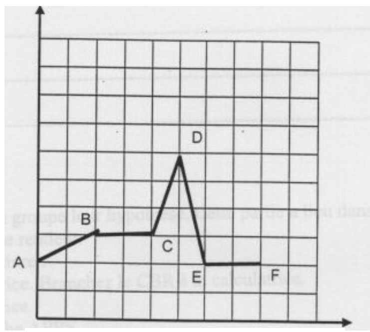


Figure 1. Pierre and Marthe's graph walk.

Aristotelian paradigm of efficient causes. Naturally, we are not asserting that the students are aware of this fact. This is not the point. The point is that, from the outset, the problem on which the students had to reflect was framed by a cultural kind of rationality that legitimizes the problem and endows it with meaning. The relevance of this remark for our understanding of the students' mathematical thinking resides in the fact that what the students will know as a result of their participation in the classroom activity and the way they will come to know it is framed by a historically constituted mode of thinking. However, this remark does not mean that the students are a clean slate or *tabula rasa* on which the classroom activity impresses knowledge. Indeed, what marks the distinctiveness of thinking is its *reflexive* nature, something to be understood not as the individuals' passive reception of the external reality but as a *re- flection*, i.e. a dialectic process between individuals and their reality, a socio-cultural process of active and creative efforts to align subjective meanings with cultural ones. The following excerpt suggests the non-transparency of the kind of mathematical reflexions that were required from the students. The excerpt, that comes from one of the small groups in which the class was divided, starts with an intervention by the researcher to clarify some basic meanings of the graph that were difficult for the students.

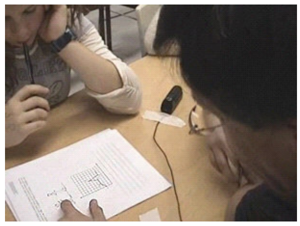


Figure 2. The child with the CBR is indicated through a gesture.

1 Researcher: What is important is that, here (*pointing to segment AB*), it went up … the probe (CBR) is there (*pointing to Pierre, the child on the left on the accompanying drawing; see Figure 2*). So, what does that (*pointing to segment AB*) mean in relation to that? (*indicating the distance between the children in the accompanying drawing*).

2 Karla: They were closer over there? (inaudible)

3 Researcher: Here (*points to (0,0)*) they were at a certain distance. Here (*points to t =1*) were they further apart?

4 Karla : (*with hesitation and doubt*) further apart…?

5 Cindy : closer…? OK, hold on… […]

Although the three students in this group were able to successfully interpret the graph of the previous problem, here the difficulty arose from the fact that the problem required the students to make sense of a graph involving *relative positions*. Thinking mathematically, that is to say, entering here into this *social praxis* of graph interpretation requires an imaginative endeavor aimed at aligning the subjective meaning of signs with their cultural, objective meanings. Signs, indeed, are both social objects and subjective products. Signs in general and mathematical signs in particular (like the graph in our example) are social objects in that they are bearers of culturally objective facts in the world that *transcend the will of the individual*. They are subjective products in that in using them, the individual expresses *subjective and personal intentions*. This double role of signs should not be understood as a dichotomous division. On the contrary, the use of signs rests on *understanding*, that is the transformation or interpretation of a sign into a previous sign (e.g. an interiorized one, in Vygotsky's terms) for which the individual has attained a more or less stable cultural meaning. Let us turn to the next part of the previous excerpt to observe the genesis of understanding.

THE LANGUAGE OF SPACE

Cindy continued her utterance:

5 Cindy: […] OK, hold on. Here (*indicating with her pen the point (0,0)*), when they start at zero … (they) are … closer, right? … They start at zero… Well, there they start here (*she indicates the point A*) … No … that wouldn't make sense.

6 Karla: They're further! (*gesture indicating the segment AB*)

7 Cindy: No, I think that they are closer here (*segment AB; see Figure 3, first picture*), and there (*gesture indicating BC; see second picture*) they are closer, and there (*gesture indicating CD; see third picture*) they move away, and they move away here (*indicating segment DE; fourth picture*) and there (*pointing to segment EF; fifth picture*) they arrive together.

Figure 3. Pointing gestures indicating the segment *AB*, *BC*, *CD*, *DE*, and *EF*.

As we can see from the excerpt, segments were interpreted in accordance to their perceptual inclination as signifying proximity or separation between Pierre and Marthe. The linguistic adverbs "closer", "near", as well as verbs and expressions such as "to move away" and "to arrive together" permitted the students to elaborate an initial understanding. This linguistic-based understanding consisted in the transformation of the graph sign into a verbal sign (made up of verbs, adverbs, etc. See line 7). Language offers each one of us a way to objectify the space in which we live and move. With its rich arsenal of deictics (e.g. "here", "there"), spatial adverbs (e.g. "closer", "near"), etc. language enables us to express and shape our intimate experience of space. By casting our experience in the linguistic categories of our culture, language allows us to do what colors allow the painter to do: to create, so to speak, a personal "linguistic painting". Like the painting of the artist, which carries the historical-cultural experience of colors, the "linguistic paintings" of the students carry deposited sediments of the spatial, historical experience of previous generations who used and refined the language of space that we have come to use. Language makes both our experience of space and our understanding of it simultaneously intimate and cultural –or better still, language makes them culturally intimate. However, it would be inaccurate to say that the students' understanding was linguistic-based only. In the next section, we elaborate upon this idea.

THINKING AS MEDIATED BY, AND LOCATED IN, BODY, ARTIFACTS, AND SIGNS

In the previous excerpt we saw that, along with key linguistic signs (adverbs, deictics, etc.), understanding was achieved through pointing gestures and the kinesthetic action of moving a pen along the graph, all of this synchronized with parts of the students' linguistic utterances. Thus, language (in its various dimensions: lexical, syntactic, pragmatic –e.g., pauses, modes to express doubt, exclamations), the Cartesian graph, the pen, and the students' hands were *mediating tools*. We want to suggest that, because thinking cannot be reduced to mental-cerebral activity, thinking is not only *mediated by*, but also *located in*, body, artifacts, and signs. As the anthropologist C. Geertz remarked after discussing the pitfalls of reducing thinking to an essentially mental intracerebral process, "the human brain is thoroughly dependent upon cultural resources for its very operation; and those resources are, consequently, not adjuncts to, but constituents of, mental activity" Geertz (1973, p. 76).

The most important point, however, is not to acknowledge our cognitive dependence on cultural resources, but to realize that they are an integral part of thinking and that in learning how to use them, the natural-biological line of development of our central psychological functions, such as attention, memory and symbolization are *altered*. Referring to these cultural resources as psychological tools, Vygotsky wrote:

> By being included in the process of behavior, the psychological tool alters the entire flow and structure of mental functions. It does this by determining the structure of a new instrumental act just as a technical tool alters the process of a natural adaptation by determining the form of labor operations. (Vygotsky, 1981, 137)

In terms of our discussion of thinking as *social praxis*, the previous remarks add an important element. The Cartesian graph, for instance, is not merely something to learn to read, something to make sense of. The graph, the calculator, the CBR, are cultural resources which bear an embodied intelligence (Pea, 1993) and carry in themselves, in a compressed way, socio-historical experiences of cognitive activity and artistic and scientific standards of inquiry (Lektorsky, 1984). This embodied intelligence and compressed historical cognitive experience offer an orientation to our cognitive activity.

But again, this orientation presented in the cultural system of ideas and the embodied intelligence carried by the artifacts that the body encounters as it moves in the world, are an overture towards possible paths of conceptual development. Since the child cannot merely be carried along by its social environment and since the latter cannot determine the child's individual thinking, as Vygotsky argued against the behaviorists of his time, the problem, from an educational viewpoint, is to present the child with the occasions to align his or her subjective meanings with cultural ones. Bearing this in mind, let us go back to our classroom episode.

REFINING MEANING

8	Karla:	(*after a pause of ± 3 sec of reflection*) But Cindy, this (CD) goes up [and] this (AB) goes up … that means it is the same thing! […]

9	Celeste:	Huuuh? … It's the same thing?! Then what? (*in a tone denoting confusion)* […] Yeah! They are maybe closer here (*pointing to AB*), closer (*pointing to BC*), further (*pointing to CD*), further (*pointing to DE*), closer (*pointing to EF*).
10	Karla:	But they can't be closer here (*pointing to CD*) […] It (*indicating CD*) is going up, and so is this one (*pointing to AB*)! […]
11	Cindy:	They move away!! So, that (CD), it's you move back; that (DE), you move forward!
12	Karla:	(after a pause of ±3 s) I don't get it!

From the beginning AB was associated with 'being close'. And since AB and CD both were said to 'go up', CD should also mean 'being close'. And this, Karla claims (line 10), is not right. Celeste then adds a new element: the children moved back! Their final written interpretation reads: "At the beginning "A" they moved back slowly and equally together at a distance of 1m. After, they stopped and then began to move back again, but faster. Then they moved forward again with the same speed at which they moved back the second time. After they stopped and they were finished."

Continuing on with our artist metaphor we can say that, by introducing the idea of moving back, the students added a new color to their mathematical painting. Unfortunately, neither the introduction of this idea nor the experiment with the CBR that they carried out later to test their hypothesis were enough to close the gap between the students' subjective meaning and the cultural one. The students still had to refine their manner of thinking about relative motion in a deeper way and to insert the role of the CBR technological artifact into their reflexions. The general learning achievement of the class was still far from the one determined by curriculum expectations. It took the teacher several lessons and general discussions to make apparent for the students the targeted kind of mathematical reflexion.

CONCLUDING REMARKS

Drawing from Wartofsky's work, in this paper we suggested that thinking is an interpretative and transformative *reflexive social praxis* encompassed by a cultural rationality and oriented from the outset towards a cultural objective system of ideas. In contrast to other social praxes, mathematical thinking is characterized by its orientation towards a theoretical, practical, and aesthetical understanding of historical-cultural reality. Thinking, we proposed, is not only mediated by, but also located in signs, artifacts, and body. If it makes sense to talk about embodiment, it is not because we have discovered that the "thinking substances" that we are (to use Descartes' term) have –notwithstanding Plato and the rationalists– a body which is a source of cognition. Body certainly is a locus for the production of meaning and the first opening of intentionality towards the world. But the object that the body encounters is more than a mere *thing*: it is a cultural object. From an educational viewpoint, the problem is that the cultural conceptuality embodied in the object is not necessarily apparent for the students. The cultural object is seen or perceived *in a certain way*, as interpreted through the students' mediated reflexion and as refracted by their particular subjective meanings. Where Grade 12 students see a Cartesian graph, Kindergarten students see a bunch of lines with no connection to a relationship between mathematical variables.

The apprehension of the object in its cultural dimension –i.e. the apprehension of the cultural conceptual content and meaning of the object– requires students to engage in an interpretative and imaginative process whose outcome is an alignment of subjective and cultural meanings. Thus, in the example discussed here, the students dealt with a Cartesian graph – a complex mathematical sign whose objective cultural meaning was elaborated in the course of centuries. The alignment of subjective and cultural meanings involved a profound active re-interpretation of signs by the students, framed by the teacher and the particular context of the classroom, leading to a progressive awareness of significations and conceptual relations that remained, in the beginning, tenaciously inaccessible to the students (e.g. that the segment AB may mean a part of the children's walk in which Marthe moved faster than Pierre or Pierre moved slower than Marthe). The aforementioned alignment of meaning should not be understood, however, as the absorption of the students into their culture: it is only one step in the positioning of the students in this distinctive social praxis that we here called *thinking*. But because of its own reflexive, interpretative, and imaginative nature, thinking also means transformation, a going-beyond, an outdoing of what is given. The subjective dimension of thinking, as something accomplished by historically-situated and unique individuals, makes possible the overcoming of the actual and the expansion and modification of knowledge and culture.

Acknowledgment

This paper is a result of a research program funded by the Social Sciences and Humanities Research Council of Canada (SSHRC/CRSH).

References

Eagleton, T. (1998). Body Work. In S. Regan, *The Eagleton Reader* (pp. 157-162). Oxford: Blackwell.

Foucault, M. (2001). *Dits et écrits*. Paris: Gallimard.

Geertz, C. (1973). *The Interpretation of Cultures*. New York: Basic Books.

Lakoff, G. and Johnson, M. (1999). *Philosophy in the Flesh*. New York: Basic Books.

Lakoff, G. and Núñez, R. (2000). *Where Mathematics Comes From*. N. Y.: Basic Books.

Lektorsky, V. A. (1984). *Subject, Object, Cognition*. Moscow: Progress Publisher.

Nemirovsky, R. (2003). Three Conjectures Concerning the Relationship between body activity and Understanding Mathematics. *Proceedings of the PME27-PMENA25 Conference.* Honolulu.

Pea, R. D. (1993). Practices of distributed intelligence and designs for education. In G. Salomon (Ed.), *Distributed Cognitions* (pp. 47-87). Cambridge: Cambridge University Press.

Vygotsky, L. S. (1981). The instrumental method in psychology. In J. V. Wertsch (Ed.), *The concept of activity in Soviet psychology* (pp. 135-143). Armonk, New York: Sharpe.

Wartofsky, M. (1979). *Models*. Dordrecht: D. Reidel.

GENERALIZATION STRATEGIES OF BEGINNING HIGH SCHOOL ALGEBRA STUDENTS *

Joanne Rossi Becker and Ferdinand Rivera

San José State University, USA

This is a qualitative study of 22 9th graders performing generalizations on a task involving linear patterns. Our research questions were: What enables/hinders students' abilities to generalize a linear pattern? What strategies do successful students use to develop an explicit generalization? How do students make use of visual and numerical cues in developing a generalization? Do students use different representations equally? Can students connect different representations of a pattern with fluency? Twenty-three different strategies were identified falling into three types, numerical, figural, and pragmatic, based on students' exhibited strategies, understanding of variables, and representational fluency.

BACKGROUND

In 1999, with a grant from the Noyce Foundation, San Jose State University and 30 school districts formed a Mathematics Assessment Collaborative (MAC) in an effort to balance state-sponsored multiple-choice tests and to provide multiple measures to evaluate students. The MAC exams are summative performance assessments in grades 3-10. The exams are hand scored using a point rubric and audited for reliability. Student papers are returned to teachers for further instruction and programmatic review. In developing this model system of performance assessment, the MAC spent a year writing Core Ideas for each grade level tested, adapting the National Council of Teachers of Mathematics *Standards* (NCTM, 2000). The assessments are written to match these Core Ideas. MARS results are correlated to state test results and analyzed by various demographic characteristics of students. In 2003, over 60,000 students were tested by the MAC.

At the eighth and ninth grades, one of the Core Ideas tested is that of patterns, relations and functions. Students are asked: to generalize patterns using explicitly defined functions; and, understand relations and functions and select, convert flexibly among, and use various representations for them. Over the five years of MARS data collections, we have found a similar pattern; while students are quite successful in dealing with particular cases of patterns in visual and tabular form, they have considerable difficulty in using algebra to express relationships or to generalize to an explicit, closed formula for a linear pattern. Summary data are shown in Table 1. To gain more insights, we embarked on an in-depth study of a small number of 9th grade students to pinpoint more specifically why they have difficulties in forming

---

*This research was supported in part by a grant from the California Postsecondary Education Commission, grant # 1184. The opinions expressed are those of the authors and do not represent those of the Commission.

	1999	2000	2001		2002		2003	
	9^{th}	8^{th}	8^{th}	9^{th}	8^{th}	9^{th}	8^{th}	9^{th}
Ability to deal with particular	80%	87%	72%	80%	74%	71%	56%	54%
Ability to generalize	15%	21%	22%	12%	21%	17%	5%	9%

Table 1: 8^{th} and 9^{th} Grade Results on Patterns and Functions Items

generalizations so we could help teachers find ways to ameliorate deficiencies in this critical area. Specifically, our research questions were: 1) What hinders students' abilities to generalize a linear pattern? 2) What strategies do successful students use to develop an explicit generalization? 3) How do students make use of visual and numerical cues in developing a generalization? Do students use different representations equally? 4) Can students connect different representations of a pattern with fluency? 5) What can we glean from student work that will inform and improve instruction?

THEORETICAL FRAMEWORK

In everyday situations, children are naturally predisposed to performing generalizations. As bricoleurs, children use whatever is available to them to induce patterns from objects despite developmental physiological constraints and their limited social knowledge, experiences, and expertise (DeLoache, Miller, & Pierroutsakos, 1998). Contrary to either Piaget's (1951) or Bruner's (1966) view that children need powerful hypothetical analytic skills or that they must attain a certain level of conceptual and abstract development prior to being able to induce patterns from objects, developmental psychologists show that children certainly could on the basis of similarity. Medin, Goldstone, and Gentner (1993) perceive similarity as an initial organizing principle, and that similarity is not known a priori and it is not static (Smith & Heise, 1992). It is, however, variable as it is based on children's ability to compare objects and determine what counts as meaningful and relevant features. Medin and Schaffer (1978) claim the significance of context in induction (i.e., a particular sample is a member of a pattern if it resembles some or all of the previously known samples in the pattern). Rosch (1978) demonstrates the role of typicality in assessing for similarity (i.e, a specific instance is a member of a class of objects if it appears to the observer as a typical example and if it resembles the known prototype examples of the class). What is significant for us in this study is Gentner's (1989) classification of three kinds of similarity, namely: analogy, literal similarity, and mere-appearance matches. They differ from one another in terms of the role attributes and relations play in similarity. Attributes "describe properties of entities," while relations "describe events, comparisons, or states applying to two or more entities" (p. 209). Analogical similarity focuses on relations and is not object-dependent; mere-appearance matches focus on object attributes and descriptions;

literal similarity is an overlap between analogy and mere-appearance matches as it utilizes commonalities that exist between attributes and relations. Gentner (1989) claims that young children and novices rely on mere-appearance matches and literal similarity. Also, a relational shift has been documented whereby young children would perform similarity on objects while older children and adults would induce relations with minimal need for surface support.

Küchemann's (1981) study highlights the ease with which beginning algebra students could associate letters as representing particular values versus letters as representing relationships: while these students could correctly deal with particular instances in a table of values that implicitly describe some mathematical relationship involving two quantities, they are unable to easily deal with the additional tasks of generalizing by way of pattern recognition and predicting by way of determining values for the larger cases. A related study by Stacey and Macgregor (2000) provides us with a characterization of the mathematical thinking employed by beginning algebra students on tasks involving pattern formulation: 1) Beginning algebra students could see valid patterns emerging from a given table of values; however, some of those patterns could not be easily translated symbolically. 2) Beginning algebra students perceive patterns as being generated by procedural rules for combining and obtaining numbers in either sequence of dependent and independent values, and not functional relationships. 3) Beginning algebra students have difficulty assigning correct representational meanings to the variables. 4) Beginning algebra students' verbal and algebraic solutions are correlated in such a way that those who could clearly articulate their patterns tend to have greater success at writing the correct rules in symbolic forms. Stacey and Macgregor further insist that students' facility with the properties of numbers and operations could assist them in obtaining a correct description of rules and relationships. Also, students need to know the structural nature of rules such as having only one simple rule for a given table of values.

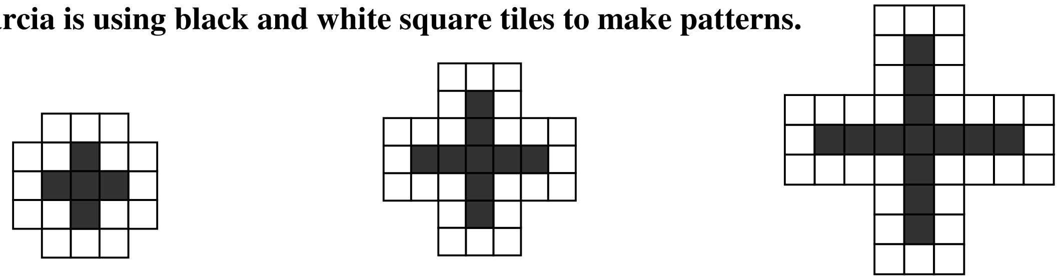
METHODS

Twenty-two ninth grade students (11 males, 11 females) in a beginning algebra course in a public school in an urban setting participated in the study. The students had completed the task (see Figure 1) in December 2002 and were given the same task in May 2003 during an individual interview by the second author that was audio-taped; interviews lasted about 20-30 minutes. Each participant was asked to read the problem and asked to think aloud as they solved the problem. The tapes were then transcribed by a graduate student and analyzed by both authors. The first level of analysis involved several individual readings of each transcript to identify patterns in strategies used for each of the six questions on the item. Then several follow-up discussions and cross-checking followed.

RESULTS

Twenty-three different strategies were used, as shown in Table 2 . The most common strategies are described more fully with portions of transcripts. Of course students used more than one strategy as they solved different portions of the task. Ten of the strategies in the table are primarily visual in nature: 1, 4, 5, 9, 13, 14, 18, and 20-22.

Marcia is using black and white square tiles to make patterns.



Pattern 1 Pattern 2 Pattern 3

1. How many black tiles are needed to make Pattern 4? ________________

Marcia begins to make a table to show the number of black and white tiles she is using.

Pattern Number	1	2	3	4
Number of White Tiles	16	24		
Number of Black Tiles	5	9		
Total	21	33		

2. Fill in the missing numbers in Marcia's table.

3. Marcia wants to know how many white tiles and black tiles there will be in the tenth pattern, but she does not want to draw all the patterns and count the squares.

Explain or show another way she could find her answer.

4., 5., & 6. Using W for the number of white tiles and P for the pattern number, write down a rule or formula linking W with P.

Using B for the number of black tiles and P for the pattern number, write down a rule or formula linking B with P.

Now, using T for the total number of tiles and P for the pattern number, write down a rule or formula linking T with P.

Figure 1: Tiling Squares Problem

Visual Grouping Strategy (S1). Edward provided a prime example of a strategy of counting each "arm" of the pattern and then multiplying to get the total.

> I looked at pattern 3 and I saw the three pattern, three tiles, that are on each side so I thought I looked at the pattern two and it just added one so I multiplied four times four with all the sides and just added one in the middle [for pattern 4].

Visual Growth of Each Arm Strategy (S4). This strategy is similar to #1 except students used an additive rather than multiplicative approach to get the total number of tiles.

Counting ELLs and Adding 4 Center Squares (S14). To find the number of white tiles in pattern #1, Alajandro saw four groups of three white tiles forming an L shape around the center black cross with an additional 4 center white squares on each side.

> You can see here, like, it's three, three, three, three, plus twelve, and four, and the same here [referring to the next pattern]. The thing is you just add four more and if you are doing a table you just add 8, that's 16, 24, 32, 40.

Strategies	No.	Strategy Description	Strategies	No.	Strategy Description
Numerical	2	Numerical use of finite differences in table	Visual	1	Visual grouping by counting each arm; multiplicative relationship
	6	Random trial and error		4	Visual growth of each arm; additive method of counting
	6’	Systematic trial and error		5	Visual symmetry
	7	Numerical finite differences to generalize to closed formula		9	Figural proportioning into pillars; add 4 for external and 4 for internal squares
	8	Implicit recursion		13	Concentric visual counting
	10	Confusing dependent and independent variables		14	Counting Ell shapes and adding 4 center squares
	11	Extending the table		18	Chunking
	12	Missing independent variable		20	Counting by one
	15	Adding two formulas for black and white		21	Visual finite differences after random count
	16	Incorrect use of proportionality		22	Visual finite differences after systematic count by 3s
	17	Get a formula and substitute to get 10th term	DG*	3	Unable to generalize
	19	T in terms of B and W	*Disjunctive Generalization		

Table 2: All Strategies Identified in Solving Tiling Squares Task

Numerical Use of Finite Differences in Table Strategy (S2). Even some of the students who could not generalize, such as Rosendo, were adept at using finite differences in the table. This was obviously a strategy they had been taught. Rosendo showed her work on the paper by drawing a loop connecting the 5 and 9 in the table, then the 9 and the blank, which she filled in with 13.

> Marcia is using black and white square tiles to make patterns. How many black tiles are needed to make Pattern 4? Um, you keep adding 4, 4 plus [5] I think with the pattern.

Trial and Error Strategy (S6/S6’). If one combines the systematic and unsystematic trial and error approach, this was a common strategy. Interestingly, there were two students who used Strategy #2, Finite Differences, yet did not transfer that information into their attempt to generalize to a formula. A third, Jennifer, did not use the table, but was able to get a formula through a guess and check strategy.

S: Black is $4n + 1$.

Interviewer: How did you decide $4n + 1$?

S: I started off with more like 2 and that didn’t work so then I tried to make 5 work and I did the same thing with 2, 3, and 2 and then when I tried it with 4, and I tried to figure a number to make 5 so I add 1, and I tried it on 2 and it still gave me the number.

Individual Patterns. We next graphed each student's strategies on parts 1-6 of the task in Excel so we could examine trends over the course of the task as it ramped up from specific to general. Two examples of graphs are shown here for illustration. Katrina (Figure 2) began the problem with visual strategies, then changed to numerical use of finite differences to get a general formula, which she used to find the values for the 10th pattern. For part 6, the total number of tiles, she indicated to add the number of black and white tiles but did not produce a closed formula.

Rani (Figure 3) also began with a visual strategy, then transitioned into using finite differences and extended the table to answer part 3. However, Rani had to construct the table all the way out to the 10th pattern number in order to correctly answer part 3. Thereafter, he used trial and error to try to get to a generalization but was unable to do so. For example, in part 4:

I: What makes it difficult to figure out that formula?

S: Because I can't find what links them to like equal 16 and then 24 or add up to make it. Number of white tiles goes up by 8. I don't know how I would link to the number of patterns.

Group Patterns. Because of our particular interest in students' ability or inability to generalize, we focused our attention on part 3 of the task, which is the transition point between the specific and the general. In fact, 12 students used Strategy #17, in addition to other strategies, for this part of the task: they tried to get a formula that they could use to find the number of white and black tiles in the 10th pattern. Of those 12, four students were unsuccessful in forming a generalization; one used purely numerical strategies, while the other three used visual or combined visual/numerical strategies. The other eight students were successful in generalizing; of those, three used purely numerical strategies to lead them to a generalization, while the other five used visual or visual/numerical.

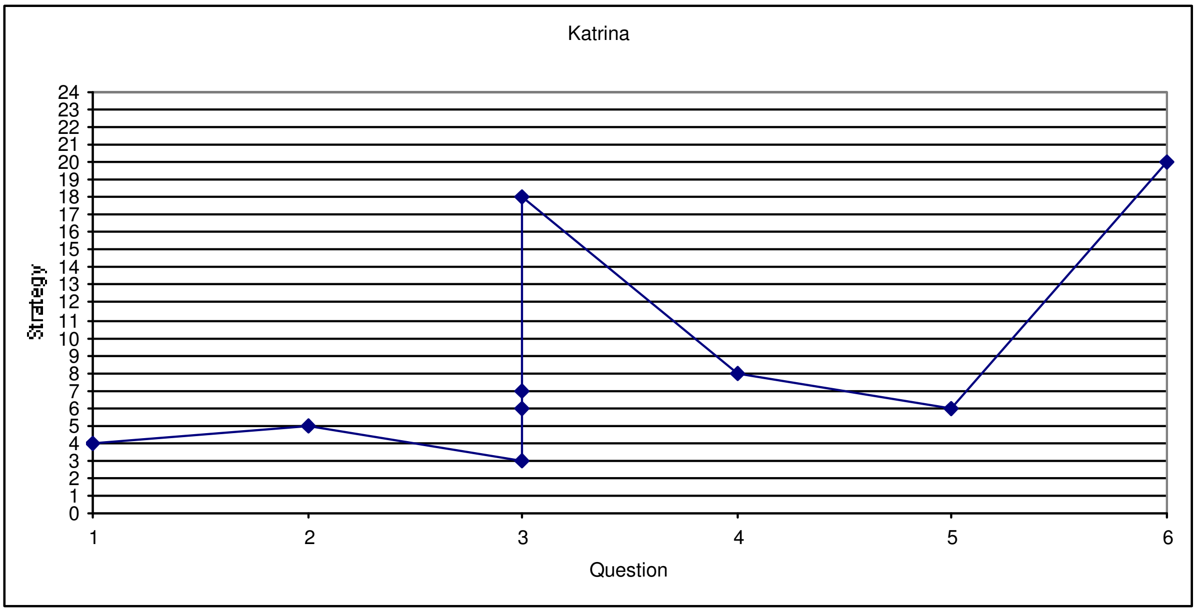


Figure 2: Katrina's Solution Path

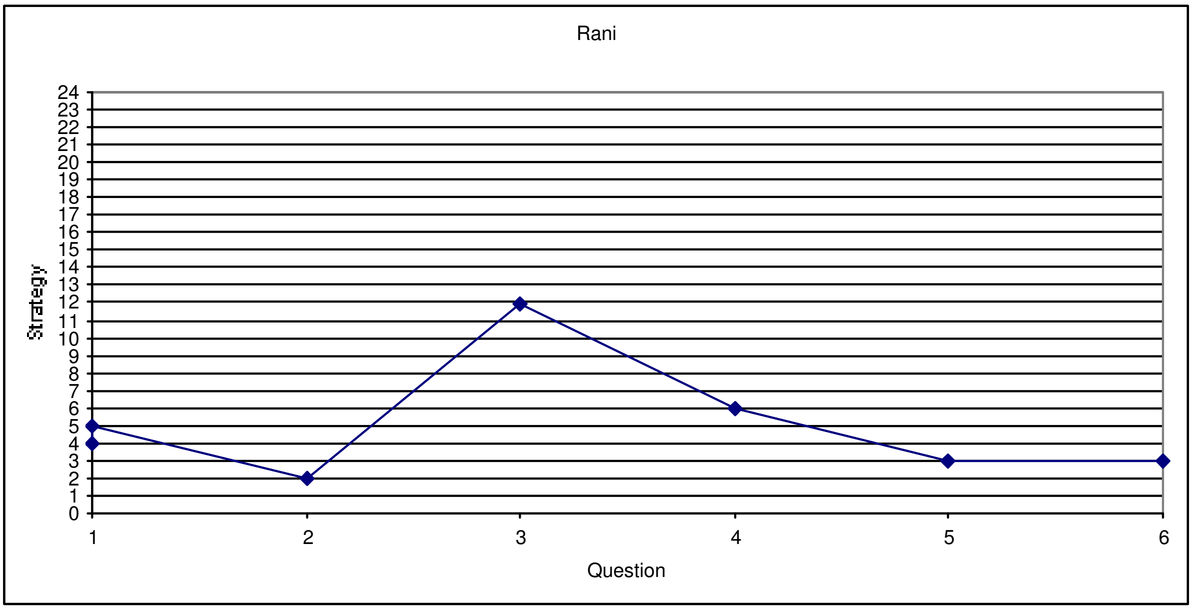


Figure 3: Rani's Solution Path

Inability to Generalize (S3). Table 3 below shows the results on generalization of the 22 students. Two of the 13 classified as unable to generalize had no success on any part of the problem, while the rest were able to do the first three parts of the task.

Category	Number
Able to generalize all parts	5
Able to generalize partially	4
Unable to generalize	13

Table 3: Summary of Results on Generalization

Of the remaining 11 who were unable to generalize, all but one started with a visual strategy but transitioned to a numerical one. At that point, they generally did not return to the visual cues at all, even when they got stuck using their numerical strategies. The most common numerical strategy was to extend the table. One student confused the roles of the independent and dependent variables, and another left out the independent variable. The four students who were able to generalize partially did parts 4 and 5 correctly but completed part 6 by indicating in words or symbols (e.g., B+W) to add the number of black and white tiles; that is, they did not find an explicit formula for the total number of tiles in terms of the pattern number as asked for in the task.

DISCUSSION

This study is consistent with findings from an earlier study we conducted with preservice elementary teachers (Rivera & Becker, 2003) as well as work done by Küchemann (1981) and Stacey & Macgregor (2000). Overall, students' strategies appeared to be predominantly numerical. In this study we identify three types of generalization based on similarity (numerical, figural. and pragmatic) in accord with findings by Gentner (1989) in which children were shown to exhibit different similarity strategies when making inductions involving everyday objects. Students who use numerical generalization employ trial and error as a similarity strategy with no sense of what the coefficients in the linear pattern represent. The variables are used merely as placeholders with no meaning except as a generator for linear sequences of numbers, with lack of representational fluency. Students who use figural generalization employ perceptual similarity strategies in which the focus is on relationships among numbers in the linear sequence. Variables are seen as not only placeholders but within the context of a functional relationship. Students who use pragmatic generalization employ both numerical and figural strategies and are representation-ally fluent; that is, they see sequences of numbers as consisting of both properties and relationships. We see that figural generalizers tend to be pragmatic eventually. Finally, students who fail to generalize (disjunctive generalizers) tend to start out with numerical strategies; however, they lack the flexibility to try other approaches and see possible connections between different forms of representation and generalization strategies.

References

Bruner, J. (1966). On cognitive growth. In J. Bruner, R. Olver, & P. Greenfield (Eds.), *Studies in cognitive growth*. New York: Wiley.

DeLoache, J., Miller, K., & Pierroutsakos, S. (1998). Reasoning and problem solving. In W. Damon, D. Kuhn, & R. Siegler (Eds.), *Handbook of child psychology, volume 2: Cognition, perception, and language* (pp. 801-850). New York: John Wiley & Sons, Inc.

Gentner, D. (1989). The mechanisms of analogical learning. In S.Vosniadou & A. Ortony (Eds.), *Similarity and analogical reasoning* (pp. 199-241). New York, NY: Cambridge University Press.

Küchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics: 11-16* (pp. 102-119). London: Murray.

Medin, D., Goldstone, R., & Gentner, D. (1993). Respects for similarity.*Psychological review,* 100, 254-78.

Medin, D. & Schaffer, M. (1978). Context theory of classification learning. *Psychological review*, 85, 207-38.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Piaget, J. (1951). *Plays, dreams, and imitation in childhood*. New York: Norton.

Rivera, F.D. & Becker, J. (2003). The effects of figural and numerical cues on the induction processes of preservice elementary mathematics teachers. In N. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of PME and PMENA* (pp. 4-63 -70). Honolulu, Hawai'i: University of Hawaii.

Rosch, E. (1978). Principles of categorization. In E. Rosch & B. Lloyd (Eds.), *Cognition and categorization* (pp. 27-48). Hillsdale, NJ: Lawrence Erlbaum.

Smith, L. & Heise, D. (1992). Perceptual similarity and conceptual structure. In B. Burns (Ed.), *Percepts, concepts and categories* (pp. 233-272). Amsterdam: North-Holland.

Stacey, K. & MacGregor, M. (2001). Curriculum reform and approaches to algebra. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives on school algebra* (pp. 141-154). Dordrecht, Netherlands: Kluwer.

SYNCHRONIZING GESTURES, WORDS AND ACTIONS IN PATTERN GENERALIZATIONS

Cristina Sabena[1], Luis Radford[2], Caroline Bardini[2],

[1]Università di Torino, Italy [2]Université Laurentienne, Canada

In this paper we focus on the role of signs in students' perceptive processes underpinning the generalization of numeric-geometric patterns. Based on a video-taped Grade 9 classroom group activity undertaken by three students and framed by a cultural-semiotic theoretical perspective, we carry out a microgenetic analysis of an elementary form of mathematical generalization termed "factual". We present a detailed analysis of the dynamics between oral speech and gestures. Two main results are: the detection of intra-personal and inter-personal synchronizations between different semiotic systems; and the individuation of the key role played by objectifying iconic gestures.

INTRODUCTION AND THEORETICAL FRAMEWORK

The crux of the generalization of patterns lies in the fact that it predicates something that holds for *all* the elements of a class based on the study of a *few* of them. One question that has to be asked in this context is, hence, the following: What is it which enables the generalization to be accomplished? In other words, what is that process that allows the students to see the general through/in the particular? (Mason, 1996). In the case of geometric-numeric patterns, one of the crucial aspects of this process is perception. To *perceive* something means to endow it with meaning, to subsume it in a general frame that makes the object of perception *recognizable*. Because the perceptive process is interpretative, what one student sees in a pattern can be different from what another student sees in the *same* pattern.

The actual possibility of generalization therefore rests on perception and interpretation. In this paper, we are interested in better understanding the role of signs in students' perceptive processes underpinning the generalization of numeric-geometric patterns. As previous research suggests (Radford, 2002), perception as an active ongoing process of adjustments and refinements —a process in which the perceived object takes a progressive shape— is significantly dependent on the use of signs. With a pointing gesture, for instance, a student may indicate a specific part of a particular perceptual object to a colleague and enable him/her to *attend* to something that until then had remained unnoticed. The gesture here plays a specific role: the role of *objectification*, i.e., etymologically speaking, of making something apparent. Naturally, there are also other resources through which to accomplish an objectifying purpose: deictic words (e.g. "this", "that", "top", "bottom"), letters, diagrams, body movements, etc. All such resources that play the aforementioned phenomenological role in knowledge formation have been termed *semiotic means of objectification* (Radford, 2003).

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 129-136. Melbourne: PME.

We are interested here in investigating the microgenesis of an elementary form of mathematical generalization —a generalization termed *factual* (Radford, 2003). For example, in the pattern shown below, a factual generalization enables the students to find the number of circles in *any particular figure* (e.g. fig 100, fig 900) without counting the circles one after the other. Factual generalizations differ from more complex forms of generalization (e.g. contextual and symbolic) in that their level of generality remains confined to the numeric realm. Because of its limited scope, young students using factual generalizations only, cannot answer questions to explain how to find out the number of circles in *any* figure or to find a formula to calculate the number of circles in *Figure n*.

It is our contention that a microgenetic analysis of factual generalizations can shed some light on the way in which perception becomes refined in those crucial moments of the students' mathematical experience leading to the accomplishment of a generalization. In carrying out the microgenetic analysis, we will focus the students' deployment and coordination of semiotic means of objectification. In particular, we shall investigate the dynamics between oral speech and gestures.

METHODOLOGY

Our videotaped data comes from a 6-year longitudinal study, collected during classroom activities. In these activities, which are part of the regular school teaching lessons, the students spend a substantial period working together in small groups of 3 or 4. At some points, the teacher (who interacts continuously with the different groups during the small group-work phase) conducts general discussions allowing the students to expose, confront and discuss their different solutions. In addition to collecting written material, tests and activity sheets, we have three or four video-cameras each filming one group of students. Subsequently, transcriptions of the video-tapes are produced. Video-recorded material and transcriptions allow us to identify salient short passages that are then analysed using techniques of qualitative research in terms of the students' use of semiotic resources (details in Radford, 2000).

We will focus here on the introductory question of a problem in a Grade 9 math lesson. This problem dealt with the study of an elementary geometric sequence (see Fig. 1). In the question that we will discuss, the students were required to continue the sequence, drawing figures number 4 and number 5 and then to find out the number of circles on figures number 10 and number 100.

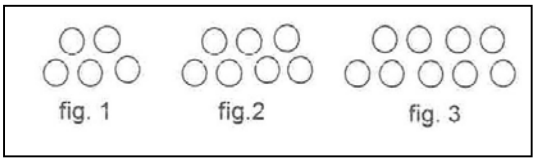


Figure 1.

PROTOCOLS

We will analyse the microgenesis of a factual generalization of one group of students, formed by Jay, Mimi (sitting side by side) and Rita (sitting in front of them). In the videotaped episode, Jay and Mimi keep the worksheet; they begin counting the number of circles in the figures, and realize that it increases by two each time. Then, Jay is about to draw figure 4, with the worksheet and a pencil in his hands:

1 Rita: You have five here… (pointing to figure 3 on the sheet)

2 Mimi: So, yeah, you have five on top (she points to the sheet, placing her hand in a horizontal position, in the space in which Jay is beginning to draw figure 4) and six on the... (she points again to the sheet, placing her hand a bit lower)

3 Jay: Why are you putting...? Oh yeah, yeah, there will be eleven, I think (He starts drawing figure 4)

Figure 2. Mimi's first gesture on line 2.

4 Rita: Yep

5 Mimi: But you must go six on the bottom … (Jay has just finished drawing the first row of circles) and five on the top (Jay finishes drawing the second row)

Although Jay materially undertakes the task of drawing figures 4 and 5, each student is engaged in the action. In line 1, Rita is not merely informing her group-mates that figure 4 contains a row of 5 circles. In fact, through a deictic gesture she is suggesting a qualitative and quantitative way to apprehend the next figures. Pointing to a specific part of figure 3, which is given on the sheet, but referring in her speech to figure 4, Rita provides a link between the two figures. Through her *gesture-speech mismatch* (i.e. through a gesture that refers to something while she talks about something else; see Goldin-Meadow, 2003), she is certainly suggesting a specific way to build figure 4. This is an example of a process of *perceptual semiosis*, that is, a process in which perception is continuously refined through signs.

This apprehension of the figure is easily adopted by Mimi, and properly described through the spatial deictics "top" and "bottom" (lines 2 and 5). It amounts to shifting from blunt counting to a *scheme of counting*. To notice this scheme is the first step towards the general.

In line 2, Mimi's words are accompanied by two corresponding deictic gestures, which accomplish a number of functions: (1) participating in the drawing process, by entering Jay's personal space to offer guidance in carrying out the task; (2) depicting the spatial position of the rows in an iconic way, and (3) clarifying the reference of the uttered words. In line 5, Mimi does not make any gestures; rather, her *words* are perfectly *synchronized* with Jay's *action*, almost directing him in the action of drawing: in fact, to complete her sentence with the description of the second row, Mimi waits until Jay finishes drawing the first row of circles.

Later, the group work is interrupted by an announcement to the class about a forthcoming social activity. While Mimi and Rita pay attention to the announcement, Jay keeps on working, writing "23" and "203" as the answers for the question on the number of circles in figures 10 and 100. So, when the girls return to the task, they ask Jay for an explanation of his results:

6 Mimi: (*Talking to Jay*) I just want to know how you figured it out.

7 Jay: Ok. Figure 4 has five on top, right? *(with his pencil, he points to the top row of figure 4, moving his pencil from the left to the right)*

8 Mimi: Yeah…

9 Jay: …and it has six on the bottom *(he points to the bottom row using a similar gesture as in line 7).* […]

10 Mimi: *(pointing to the circles while counting)* 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. (*Pause*) […] Oh yeah. Figure 10 would have …

11 Jay: 10 there would be like …

12 Mimi: There would be eleven *(she is making a quick gesture that points to the air. Jay is placing his hand in a horizontal position)* and there would be ten *(she is making the same quick gesture but higher up. Jay is shifting his hand lower down)* right?

13 Jay: Eleven (*similar gesture but more evident, with the whole hand*) and twelve *(same gesture but lower).*

14 Mimi: Eleven and twelve. So it would make twenty-three, yeah.

15 Jay: 100 would have one-hundred and one and one-hundred and two *(same gestures as the previous ones, but in the space in front of his face).*

16 Mimi: Ok. Cool. Got it now. I just wanted to know how you got that.

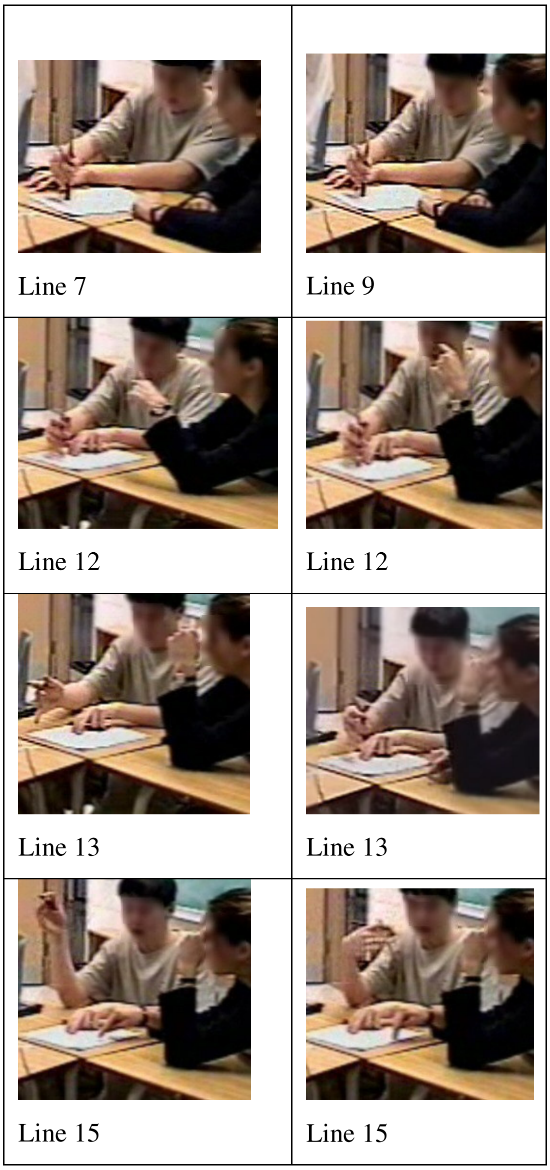


Figure 3. Some gestures occurring in the lines of the dialogue.

To account for his results about figures 10 and 100, Jay (lines 7 and 9) starts talking about figure 4, already drawn on the sheet, and uses the speech-gesture combination previously introduced by Mimi (lines 2 and 5): the same deictic terms "top" and "bottom", and analogous deictic gestures. Turning to figure 10, Mimi (line 12) matches her words with two gestures that refer to the two rows of the geometrical configuration. The same kind of gesture and uttered speech is then used by Jay, who corrects Mimi's answer (line 13). Even if the figure is referred to in two slightly different ways by the two students, starting from the top (Jay) or from the bottom

(Mimi), the words-gesture match is perfectly accomplished in both cases in a very natural way. The same is true for Jay's inference about figure 100 (line 15).

The relevance of the previous remarks is that through a coordination of gestures and speech, the students are accomplishing an *objectification of knowledge* (Radford, 2003), i.e., through signs of different sorts, the students are making *apparent* key traits of figure 100 —a figure that is not directly perceivable. The tight coordination between gestures and speech takes place in a particular segment of the students' mathematical activity. These segments of mathematical activity, characterized by the crucial coordination of various semiotic systems leading to the objectification of knowledge, constitute what have previously been termed *semiotic nodes* (Radford et al. 2003). An index of its presence is the perfect coordination of time, words and movement reached in line 12 (see Fig. 4).

12 Mimi: There would be eleven and there would be ten right?

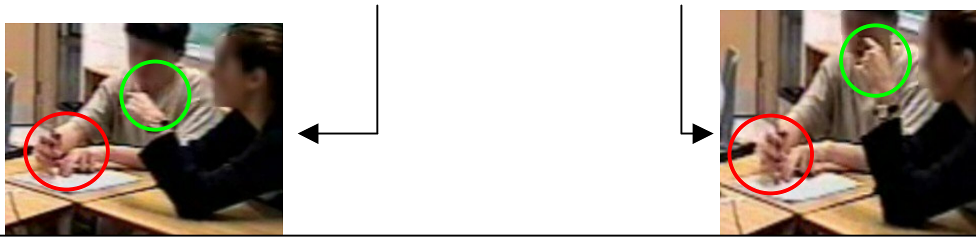


Figure 4. Synchronization between the gestures of the two students

Indeed as Fig. 4 illustrates, Mimi's words are rhythmically beaten not only with her own gestures, but also with those of Jay. In fact, even if the students are not looking at each other, Jay's hands are synchronized with Mimi's words, and, as a consequence, with Mimi's hands.

Let us now focus on the internal dynamics of this semiotic node (which indeed involves the whole episode, going from line 6 to line 16) to disentangle the different specific semiotic components and describe how their mutual relations pave the way for students' generalization.

In Jay's first utterance (lines 7 and 9), the deictic gestures appear endowed with a dynamic feature that clearly depicts the geometric apprehension of the figure as made up of two horizontal rows. Its goal is to clear away any ambiguity about the referent of the discourse, in order to explain a strategy. The figure (number 4) is perceptively present on the scene, and indeed materially touched by Jay through his pencil, a tool that can be actually considered part of his peripersonal space, that is the space immediately surrounding his body. Talking about figure 10, Mimi (line 12) performs two gestures that keep certain specific aspects of those of Jay, i.e. one gesture for each row, and the vertical shift. But now, because the referred figure is not available in the perceptual field, the gestures are made *in the air*. Also Jay's last gestures (line 14), referring to figure 100, appear in the air in the space in front of him, as if pointing to the rows of a non visible figure; indeed, if we pay attention to the position of his hands when he refers to the different figures, we can notice a progressive detachment from the sheet:

Referring to fig 4 Referring to fig 10 Referring to fig 100

Figure 5.

The indexicality of the deictic gesture undergoes a gradual shift from an *existential signification* (referring to figures 4, materially present on the sheet) to an *imaginative* mode of signification (referring to figures 10 and 100). These gestures that mime or "iconize" the referent pinpoint and depict in an iconic way the essential features of the new referent, thus making it apparent. We term *objectifying iconics* these kinds of gestures which, thanks to their iconic features, play an important part in the process of knowledge objectification. Their role is in some way analogous to that of the deictic words previously termed *objectifying deictics* (Radford, 2002).

Notice that the objectifying iconic gestures undergo a process of simplification that involves the loss of movement (along the rows of the figure) and a shortening of their duration. A progressive simplification is also evident in the uttered words: from line ten onward, the deictic terms disappear, leaving a barely numerical semantic content, organized by the conjunction "and". Even if figures 10 and 100 are not materially present, the students can *imagine* them very precisely and would be able to draw them; but, having reached a certain stage in the process of objectification and socialization of the objectified knowledge, they do not need to specify all the details, and the reference to the form of the figure can smoothly remain implicit in their speech. This is also possible due to the role played by gestures. Let us focus on lines 12, 13 and 15 (Fig. 3).

The perfect synchronization between words and gestures allows the students to successfully cope with two intertwined aspects of the problem: one is numerical, discrete, and linear; the other is visual, geometrical, and analogical. The students handle the former through language, and the latter through gestures. They correspond to what Lemke (2003) terms the two fundamental types of meaning-making: meaning-by-kind or "typological meaning" (language) and "the meaning of continuous variation," the meaning-by-degree or "topological meaning" (motor gestures or visual figures). He identifies in this inherent and unavoidable difference a main source of difficulties in learning mathematics, since "In general mathematical expressions are constructed by typological systems of signs, but the values of mathematical expressions can in general vary by degree within the topology of the real numbers" (ibidem, p. 223).

CONCLUDING REMARKS

In this paper we focused on the genesis of a factual generalization. In investigating this kind of elementary form of mathematical generalization our goal was to unravel the semiotic activity that underpins its objectification. Previous research suggested the crucial role of signs in the students' progressive process of apprehension of the pattern. However, the detailed micro study of this process of perceptual semiosis — the interpretative process that enables the students to go beyond the particular and to attain the general— still needs to be better understood. As it has been suggested in earlier work (Radford, 2003), factual generalizations constitute, to an important extent, the basis for more sophisticated forms of generalizations.

Our microgenetic analysis intimates that the process of perceptual semiosis here studied was underlined by two kinds of meanings. On one hand, there is a typological meaning that emphasizes the dimension of "quantitas". On the other hand, there is a topological meaning that highlights the dimension of the "qualitas" induced by the geometric nature of the figures of the pattern. We saw here that the two aspects are inherently merged. Because the goal of the factual generalization is precisely to spare one from counting the circles in a figure one after the other, the numerical and the geometrical dimensions have to be harmonized. To harmonize them, the students activate a number of semiotic systems: oral speech, drawn figures, and gestures whose coordination by the students constitutes a semiotic node of their ongoing activity. Our analysis suggests the occurrence of gesture-speech match and mismatch and the critical role of gestures in the objectification of knowledge. Thus, based on the particular figures (e.g., figures 3 and 4), the students started talking about non-present terms such as figure 10 and figure 100. The latter were *objectified* (i.e., made apparent) thanks to couples of iconic gestures that represent the geometrical components that are essential in a particular figure apprehension —e.g., two horizontal rows of circles (see Fig 5). These objectifying iconic gestures bear the analogical aspect of the problem and allow the students to pair it with its correspondent typological meaning (expressed in the uttered speech), to successfully accomplish the given task.

In addition to this, our results also point to another aspect of the problem. As previously discussed, (Radford et al., 2004), there can be synchronization between different semiotic systems activated by the same individual: lines 3, 7 and 13 show examples of this. However, we also found evidence here of synchronization between individuals. In this case, it can occur between different semiotic systems, as in line 5, where Mimi is almost directing Jay's drawing action, or between different enactments of the same semiotic means by different students, as in line 12, where we observe Mimi's and Jay's simultaneous gesturing actions perfectly coordinated with Mimi's utterance. Thus, besides an intra-synchronization (i.e., an intra-subject or intra-personal synchronization) —of which the gesture speech-match is a particular case— an inter-synchronization (or inter-subject or inter-personal synchronization) also appears.

The hints provided by our micro-analysis need to be investigated further. In particular we need to better characterize the dynamics of semiotic nodes in factual, as well as in other more sophisticated forms of generalization, related not only to the context of patterns but to other domains of mathematics.

Acknowledgment: This paper is a result of a research program funded by the Social Sciences and Humanities Research Council of Canada (SSHRC/CRSH).

References

Goldin-Meadow, S. (2003). *Hearing Gesture. How our hands help us think.* Cambdrige, Ma.: The Belknap Press of Harvard University Press.

Lemke, J. L. (2003). Mathematics in the Middle: Measure, Picture, Gesture, Sign, and Word. In M. Anderson, A. Sàenz-Ludlow, S. Zellweger & V.V. Cifarelli (Eds.), *Educational Perspectives on Mathematics as Semiosis: From Thinking to Interpreting to Knowing* (pp. 215-234). Ottawa: Legas Publishing.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to Algebra* (pp. 65-86). Dordrecht/Boston/London: Kluwer.

Radford, L. (2000). Signs and Meanings in Students' Emergent Algebraic Thinking: A Semiotic Analysis. *Educational Studies in Mathematics, 42*, *3*, 237-268.

Radford, L. (2002). The Seen, the Spoken and the Written: a Semiotic Approach to the Problem of Objectification of Mathematical Knowledge. *For the Learning of Mathematics*, *22(2)*, 14-23.

Radford, L. (2003). Gestures, Speech, and the Sprouting of Signs: A Semiotic-Cultural Approach to Students' Types of Generalization. *Mathematical Thinking and Learning*, *5(1)*, 37-70.

Radford, L., Demers, S., Guzmán, J. & Cerulli, M. (2003). Calculators, graphs, gestures and the production of meaning. In N. A. Pateman, B. J. Doherty, & J. Zilliox (Eds.), *Proc. 27th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol 4, pp. 55-62). Honolulu, USA.

Radford, L., Demers, S., Guzmán, J. & Cerulli, M. (2004). The sensual and the conceptual: artifact-mediated kinesthetic actions and semiotic activity. *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 73-80). Bergen, Norway.

ANALYZING STUDENT MODELING CYCLES IN THE CONTEXT OF A 'REAL WORLD' PROBLEM

Roberta Y. Schorr
Rutgers University-Newark

Miriam Amit
Ben Gurion University

Many students do not apply their real world intuitions and sense-making abilities when solving mathematics problems in school. In an effort to better understand how to help students draw upon these valued resources, we investigate the manner in which the solution to a particular problem activity is repeatedly re-interpreted by a student. This is done within the context of a models and modeling framework in which we discuss the modeling cycles and associated models that were used. We suggest that the nature of the problem activity combined with the time and support needed to cycle through multiple models contributed to this student's ability to move beyond his initial, simplistic solution, toward a more complex solution, one that ultimately fit well within his own "real world" intuitions and experiences.

INTRODUCTION

The fundamental premise of this research is that students should be able to learn mathematics with understanding. A central component of understanding involves reflecting upon one's own solution processes, and then refining and revising the solution as appropriate in order to produce solutions that make sense. (Hiebert et.al., 1997). Unfortunately, this act of understanding does not occur as often as we would hope. Indeed, the mathematics education research literature is replete with instances in which young students provide solutions to mathematical problems that make little or no real world sense (Carpenter, Lindquist, Matthews, & Silver, 1983; Greer, 1997; Yoshida, Verschaffel, & De Corte, 1997; Vinner, 2000). The same is true for college level students. Verschaffel, De Corte, & Borghart, (1997) report that college students "revealed a strong tendency …to exclude real-world knowledge from their own spontaneous solutions of school word problems" (p. 339). Inoue, (2002) also found that college students responded to mathematical problems with unrealistic answers, even when specifically asked to use their real world sense making skills.

Wyndhamn and Saljo (1997) speculate that one reason for the lack of sense making is that students often interpret word problems by "follow[ing] rules and use[ing] symbols without reflecting on, or analyzing, what these rules and symbols imply in the specific context in which they are used." (p. 362). In this research, we suggest that an important component of helping students to make meaningful sense of the mathematics they encounter involves building a learning environment in which meaning is highly valued, and where students are consistently encouraged to reflect on their own problem solving processes, to test their ideas in the context of the problem, and then to refine and revise their solutions accordingly. We contend that this must happen over the course of many cycles (modeling cycles, as will be described later in the paper). In such an environment, simplistic or nonsensical responses can become increasingly refined thereby resulting in mathematically sensible solutions. In this paper, we document one such instance along with the corresponding stages of revision.

FRAMEWORK

A models and modeling perspective was used to guide all levels of this research. We refer to the development of mathematical ideas in terms of "models" and "modeling cycles" (c.f. Lesh & Doerr, 2000; Schorr & Koellner-Clark, 2003). Briefly stated, a model can be considered to be a system for describing, explaining, constructing or manipulating a complex series of experiences. An individual can interpret a situation by mapping it into his or her own descriptive or explanatory system for making sense of the situation. Once the situation has been mapped into the internal model, transformations within the model can occur, which in turn can produce predictions, descriptions, or explanations for use in the problem situation (Schorr & Koellner-Clark, 2003). Models tend to develop in stages where early models are often fuzzy or distorted versions of later, more advanced models. We contend that in many cases, students never cycle through multiple models, and their first or second cut solution reflects that. It is our hypothesis that when given the opportunity to cycle through multiple models in a supportive learning environment, students can develop mathematically more sophisticated and thoughtful solutions (Schorr & Lesh, 2003).

A learning environment consists of at least two critical and interrelated components. The first relates to the classroom atmosphere, and the second relates to the nature and type of problem solving experiences that the students encounter. We contend that classrooms that encourage students to talk about their ideas, reflect on the reasonableness of their solutions (orally and in writing), listen to the solutions of others, discuss different representations of the same problem and the relationship among representations, and share, defend and justify their solutions—orally and in written form, are more likely to result in sense making. In such classrooms, ideas are embraced, reflective activity is expected, and personal experience is valued.

Since a main purpose of this study was to investigate students' modeling cycles, it was not only important to encourage an atmosphere and learning environment in which sense making was valued, but also to find a problem activity that had the potential to elicit a thoughtful, sensible solution. The activity that was chosen was designed to encourage problem solvers to produce products that were not simply answers to specific questions; but in addition entailed constructions, descriptions, and explanations, that revealed many aspects of the thought process that goes into the final solution (Amit, Kelly & Lesh, 1994). Solutions to activities like the one that was chosen often involve sequences of modeling cycles in which the "given" information is systematically re-interpreted in a variety of ways (Lesh & Doerr, 2000; English, 1997).

In the sections that follow, we will provide evidence of the models and modeling cycles that occurred. For each cycle, we will offer our interpretation of the meaning of the particular model, the influence of real world sense making, and the implications of the changes in the final solution.

METHODS AND PROCEDURES

The context for this research was a course that was designed by the author (who was the classroom teacher) in order to help poorly performing students to succeed in college level courses. This particular class consisted of eight students, all of whom were recent graduates of local urban high schools. The students met with the teacher and a teaching assistant twice weekly for approximately one hour per session for a total of 14 weeks.

On this particular occasion, which occurred midway through the term, all students were asked to solve the "Radio Problem" (see below). They were given the option of working alone or with a partner. They were all asked to keep a written journal in which they included reflections on their work, and what, if anything, they might change when they resumed their work. They were also told that there was more than one solution path that could be taken to solve the problem. All students worked on the activity for a total of three hours spread over as many sessions. When the students completed their solutions, they were asked to formally present their work to the class. Selected students were interviewed after their presentations about their solutions and strategies. All sessions, presentations, and interviews were videotaped. The teacher also kept careful field notes. Data include all of the written work, videotapes and field notes.

The Radio Problem Activity: The activity that follows is adapted from a problem developed by the Educational Testing Service as part of the PACKETS® program[1]. The problem was designed to relate to similar experiences that the students might have had when purchasing portable radios with headsets. Note that the final solution is not simply a specific solution that relates to the unique set of data, but rather one that can be generalized to other radios with different attributes.

> The editors of Consumer Reports want to make a new consumer guide for products that are important to teenagers. The first items that they want to rate are portable radio-cassette players with headsets. They need your help to develop a rating system…The editors want a rating system that readers can use to rate any model (even if it is not listed on the attached list), and compare the models to determine which are the "best buys". The editors have also gathered the attached information for some models. They plan to use these as examples to show readers how to use the rating system. To help the editors, please: I) Develop a rating system for these players. Be sure that the system can be used

[1] The problem was taken from the PACKETS® program for Middle School Mathematics which was developed by the Educational Testing Service.

to identify overall "best buys" which take into account the factors that the survey indicates are important. Also, readers should be able to use the rating system with ANY other players, including those not listed in the guide, so include any tables or charts that are part of your system. II) Write clear step-by-step instructions that make it easy for readers to use your rating system. III) Write a letter to the editors explaining why you decided on your rating system and describe its advantages and disadvantages.

Included was a comprehensive data table which listed information for each of 11 brands of radios: The chart below represents only two of the brands, (due to space limitations).

Brand	Price (dollars)	Dimensions (inches)	Weight (ounces)	Tape Sound Quality	Radio Sound Quality	Battery Life (hours)	Number of AA batteries	Comments (on a separate list)
Aiwa	$49	51/4x 31/2 x 1 3/8	good	good	good	12	2	A,B,C,1
Sony	$69	5 3/8 x 4x 1 3/4	14	fair	poor	10 1/2	4	B,F,G,H,J, 1,4,7,9,10

RESULTS

This paper focuses on one particular student, James, whose work is chosen to be representative of the class. James began by constructing a model that represented his solution to the task. The model served as a means by which he could consider the feasibility and utility of his solution as a rating system. As James solved the problem, he often experienced a conflict between his own personal experience of listening to radios and the choices he had made as a result of applying his model (as noted in his reflections and comments). This pattern of considering the solution and assessing its utility in a real world context occurred several times until James reached what he considered to be a useful and generalizable solution. James noted that when he first began, he "…felt this project couldn't be done." but then had what he termed a "breakthrough". He said "Once I got that, it made me want to progress." (noted in his written reflection). Below, we briefly describe what he did, how he reflected on his work (taken from his written reflections, oral comments, and final presentation).

First Model: Rating List. James counted the number of advantages and disadvantages (as provided in the last column of the data table) for each radio. He added the number of advantages (each advantage was assigned the value +1) to the disadvantages (each disadvantage was assigned the value –1) thereby getting a positive number when the number of advantages exceeded the disadvantages and a negative number when the disadvantages exceeded the advantages. He then paired each radio with its corresponding outcome and listed them in an ascending and sequential order. In this "rating list" values ranged from +2 to –6. As part of his written reflection he noted that, "*My first attempt was rushed, kind of a scapegoat,* *and mainly left too many questions open…"* He then discussed the usability of the solution, e.g., the rating list, and as he did, he expressed his dissatisfaction with it. He noted that it would not be "accommodating to the teenage crowd" since it was not " what teenagers are looking for" (per his written reflection on his initial solution). James acknowledged that he had not really found a solution that addressed the needs of the teenaged crowd--the target population for the problem, and those most likely to purchase the radios. More specifically he stated*"…you must take into consideration what teen-agers are looking for including: good sound quality, low price, low cost of running and lightweight."* From this point on, he consistently referred to "the teenage crowd" as being the important factor as he reflected upon the changes that needed to be made.

Second Model: checklist. James built a new physical representation (model) using many of the information provided in the data table. He selected categories such as a radio's weight, price, and battery life to be included in the model, and ignored such categories as size of radio and sound quality. He then sorted the data by magnitude in ascending order, e.g. price was arranged from cheapest to most expensive, also taking into account the frequency of each value's occurrence. James determined what he considered to be a "good" range of data per category as well as a "not good" range, (a price that ranged from $39 to $69 was identified by James as a "good"). A numerical value was allocated to each data range: "good" data received a higher number of points (3) and all the rest received 0 points. James then attempted to test his model by rating a subset of the radios according to the above criteria. This resulted in a "best buy" list where the radio that had the highest rating was deemed as the "best" radio.

The transition from the first to the second "model" was rather dramatic. Instead of continuing to use only one dimension, i.e. the advantages and disadvantages of the radios (as he had previously done), he adopted a multi-dimensional approach in which he selected information from the data (table), intentionally ignored some of the other information (such as brand name), and then defined ranges of "good" with associated numerical values. One piece of information that he chose to ignore, namely, the brand name, proved to be very important as it allowed him to consider the rating of radios in a more generalizeable way (a key aspect of his next model).

The supportive learning environment provided him with an opportunity to take the time to consider "what is important for teenagers" and reflect on his work. Building upon his reflections, he proceeded to select relevant categories and eliminate non-relevant categories. His decision was based on personal beliefs and preferences. For example, he noted that he did not pay attention to the category of "size of radio" because he thought that size did not play an important role in teenagers' purchasing decisions. He also felt that for most people, a less expensive price is a good price. However, a really cheap price may be indicative of poor quality. He stated "if it is too cheap it 's probably not good". Following this logic, he decided exclude the cheaper prices (such as $24, $33 and $35) in the range of "good".

As with his first model, the end result of this new model involved verification and utilization of using a subset of the radios. Although the new model was more comprehensive than the first, James was still dissatisfied. He concluded by saying: *"it does not capture everything, it does not balance*". This statement marks the transition point into the next model.

Third Model: "prototype chart" – refined and expanded checklist. James "fine tuned" and revised the boundaries of the data ranges within the existing category boundaries. He added an intermediate range of "medium" and assigned numbers: 3 point for good, 2 for medium and all the rest 0 points. He continued using a method in which different categories were "weighted" differently. For example, the rating corresponding to the price or weight of the radio was done on a scale of 0 to 3, whereas the rating corresponding to the "life of a battery" category was based on a 0 to 2 scale. These were justified and explained by James as he noted *"price is important and battery life is important, but from a teenagers perspective, price is more important."* Next he expanded the scope of the categories to be included in the decision making process by adding categories with qualitative data. A sound quality category was added to his checklist along with a rating of 2 for good sound quality, 1 for fair sound quality and 0 for poor sound quality were assigned.

In the new scaling system, it appears as if James' allocation of weights to the different categories represented his own way of conceptualizing what is important in the purchasing of a radio. For example, price is more important then battery life; therefore a "good" price contributes 3 points to the rating of the radio while the longest battery life contributes only 2 points. As a final step, James applied the model to the rating of the radios. He was pleased with the results and commented that *"it balances pretty well"*. At this time, James realized that the task called for rating instructions that could be applicable to *any* radio and not limited to the 11 described in the task. James felt that his current model did not fully comply with that (see the statement of the activity). This realization marked the transition point that led James to the fourth and final model.

Fourth Model: General Chart and Operating Manual. James expanded the data boundaries to account for *any* radio's price and weight. He did so by adding the phrases *less* or *over* at the end of the "good" range. In addition, James created a table that included all of the categories, all data values, both quantitative and qualitative, and a list of advantages and disadvantages. He also added a "key" so that the user could easily discern how to use the point value. For 3 points he used brackets, for two points he used a curled line, and so on. He wrote guidelines to account for special case scenarios. For instance, if the overall rating of the product is –6, James decided that the radio should be penalized with the loss of another 2 points. This was done because a rating that was that that low meant that the radio was of poor quality, and should have –2 points added to the overall rating. James also attached a rather detailed manual so that a novice rater could easily use his guide. As part of his finished product he wrote

> You simply place your name of brand walkman into the column (price), scroll right to the next column identifying where the characteristic of your walkman falls under, recognize the point value, and scroll down placing the point value in the void. Continue this process. After all point values for each category are in the voids, add them all together to get your total worth. In regards to tape and radio sound quality, point values are as so: 'Good' is worth 2 points, 'Fair' is worth 1 point, and poor has no worth. All of these components in sync will result in total worth or a "best buy'. Special considerations are present on the chart but hold no real dilemmas.

In addition, James recommended that one should rate the sound quality and the overall tape quality by playing actual music on the radio. After the rating process was complete, James further recommended that all points should be summed up for the different radio brands, and based upon this summation, one could choose the radio of his/her liking. In the end, James checked the final model by rating each of the radios on the original list, and creating a new list. Both the model and the new list were to his satisfaction and he even expanded the targeted audience, claiming that this new "best buy" list could be useful to those interested in purchasing a new radio as well as those who sell radios because the list illustrates and summarizes each product's performance. James summed this up by saying, "With this rating system, the consumer's task will be virtually effortless and seem more inviting, leaving the buyer with no other option but to take advantage of it."

CONCLUSIONS

James went through several cycles in order to solve the problem, cycles that reflected a progression from simplistic to more complex and generalizeable. We suggest that the first or second solutions that James produced are more typical of the solutions that one would expect in many classrooms, solutions that do not fully build upon students' personal, sense making capabilities. It was only through repeated reflection and revision (in which James experienced a conflict between his own personal experience and his mathematical solution), that James was prompted to revise, test, and refine his work. This type of reflection and revision was consistently encouraged within the classroom environment, in conjunction with the use of carefully chosen problem activities. This particular task was specifically designed to capitalize on students' personal experiences with purchasing radios, thereby providing a context in which sense making could be applied. Further, the problem called for a solution that was more than a specific solution for a unique set of data (involving a concrete and local situation), but rather one that could be generalized to include many different types of situations, and whose processes could relate to a whole class of structurally similar problems involving quantifying qualitative data; working with extreme and diversified situations, some of which are directly related (for example, the longer the battery life, the higher the rating), while others within the same problem are not (for example, the higher the price, the lower the rating); the invention and application of weighted scales; etc. We believe that this type of mathematical activity is critical if students are to experience the types of "breakthroughs" that James described.

References

Amit, M, Kelly, E., and Lesh, R., (1994). Characteristics of effective model eliciting problems," In Ponte, J.P., and Matos, J.F., (eds.) *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education*, Portugal, Vol. 3, pp. 160-167.

Carpenter, T.P., Lindquist, M.M., Matthews, W., and Silver, E.A. (1983). Results of the third NAEP mathematics assessment: secondary school. *Mathematics Teacher,* 76, p. 652-659.

English, L. (1997). Children's reasoning processes in classifying and solving computational word problems. In L. English (Ed.) *Mathematical reasoning*. Lawrence Erlbaum, Hillsdale, NJ. P191-220.

Greer, B. (1997). Modelling reality in mathematics classrooms: The case of word problems. *Learning and Instruction.* 7(4) p. 293-307.

Hiebert, J., Carpenter, TP., Fennema, E., Fuson., Wearne, D., Murray, H., Olivier, A., & Human, P. (1997). *Making sense: teaching and learning mathematics with understanding*. Heinemann, Portsmith, NH.

Inoue, N. (2002). The role of personal interpretation in mathematical problem solving. *Unpublished doctoral dissertation.* Teachers College, Columbia University.

Lesh, R., and Doerr, H. (2000). Symbolizing, communicating, and mathematizing: key components of models and modeling. In P. Cobb, E. Yackel, and K. McClain (Eds.) *Symbolizing and communicating in mathematics classrooms.* Lawrence Erlbaum Publishers p. 361-384.

Schorr, R.Y. & Lesh, R. (2003) A models and modeling perspective on classroom-based Teacher Development. In R. Lesh & H. Doerr (Eds.) *Beyond constructivism: a models and modeling perspective on teaching, learning, and problem solving in mathematics education*. Lawrence Erlbaum, Hillsdale, NJ.

Schorr, R.Y. & Koellner-Clark, K. (2003). Using a modeling approach to consider the ways in which teachers consider new ways to teach mathematics. *Mathematical Thinking and Learning*

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.

Verschaffel, L., De Corte, E. and Borghart, I. (1997). Pre-service teachers' conceptions and beliefs about the role of real-world knowledge in mathematical modeling of school word problems. *Learning and Instruction.* 7(4) p. 339-360.

Vinner, S. (2000). Mathematics Education-procedures, rituals and man's search for meaning. Paper presented at the Ninth International Congress of Mathematics Education. Tokyo Japan.

Yoshida, H., Verschaffel, L., and De Corte, E. (1997). Realistic considerations in solving problematical word problems: do Japanese and Belgian children have the same difficulties? *Learning and Instruction.* 7(4) p. 329-338

NEGOTIATING ABOUT PERCEIVED VALUE DIFFERENCES IN MATHEMATICS TEACHING: THE CASE OF IMMIGRANT TEACHERS IN AUSTRALIA

Wee Tiong Seah

Monash University, Australia

This paper reports on a qualitative research study exploring the socialisation experiences of immigrant secondary mathematics teachers practising in Australia. Teacher perception of differences in the ways their respective home and the Australian (host) cultures value aspects of mathematics teaching and learning was observed to lead to dissonance. Their negotiation about these differences highlighted the role played by personally-held values. Although each teacher participant adopts different approaches to mediate the different perceived value differences, the inclusive approaches of amalgamation and appropriation were most widely adopted. Implications towards optimising mathematics pedagogy, and towards meaningful professional support for mathematics teachers in transition, are suggested.

> I strongly believe that Maths is Maths in any culture. I teach Maths my own way, with a great passion and commitment to the students I teach. (Carla, immigrant teacher from Romania)

INTRODUCTION

We live at a time of ever increasing contact and communication across geopolitical boundaries. While the fight against terrorism involves many nations, air travel continues to enjoy increasing growth. As the 2004 Asian tsunami disaster demonstrated nature's ability to unleash its force across countries, affecting the lives of people in even more nations, the coordination and success of the international relief aid effort highlighted the role of cross-cultural communication. Transnational migration continues to be a global phenomenon, and this includes the increase in demographic movement of teachers of mathematics around the world, either directly as a result of responding to mathematics teacher shortage in local schools, or indirectly due to one's family making the move to another country.

It is reasonable to suppose that teachers of mathematics may be less concerned than their colleagues of other subjects to teach in a different country. As the quote above demonstrates, (school) mathematics is often regarded as culture-independent. After all, topics such as the arithmetic operations and Pythagoras' Theorem remain the same in different cultures and these same topics are taught in these different cultures. However, when teachers facilitate the learning process of these supposedly culture-independent topics in mathematics classrooms in different cultures, surely the pedagogy is culture-dependent? For example, a mathematics teacher used to teaching in a teacher-centred manner in her home culture is likely to find it different – even difficult – to teach similar topics in a relatively student-centred learning environment.

In other words, mathematics pedagogy may actually be as culture-dependent as any other school subject. Yet, perhaps because of the belief that school mathematics is culture-independent, educational research into the pedagogical process and professional experiences related to mathematics teaching across cultures is at best scarce.

It is within this context that the study reported in this paper is positioned. In particular, the study inquires into the professional socialisation (Su, Goldstein, Suzuki, & Kim, 1997) of immigrant teachers of mathematics practising in secondary schools in Victoria, Australia, and explores how these teachers negotiate any cultural difference they encounter in teaching and facilitating the learning of school mathematics. This paper will briefly position the construct of values as providing an appropriate framework to examine the socialisation experience of the immigrant teachers. An outline of the research methodology follows, before a summary of the findings is presented. The range of responses adopted by the teacher participants will also be discussed.

THE VALUE-LADENNESS OF (MATHEMATICS) EDUCATION

This study adopts the stance of social-cultural constructivism, acknowledging the development of mathematical knowledge as socialised knowledge, including ethnomathematics (Bishop, 1991; D'Ambrosio, 1985; Knijnik, 1993), and also recognising the socio-cultural context of mathematics teaching and learning in schools (Bishop, 1994; Schmidt, McKnight, Valverde, Houang, & Wiley, 1997). While the consideration of factors such as ethnicity, socio-economic levels and gender in mathematics education research has traditionally reflected the socio-cultural aspect of facilitating school mathematics teaching and learning, this study explores the professional practice of immigrant mathematics teachers using another socio-cultural variable, that is, values.

After all, the very act of educating is by nature value-laden (Gudmundsdottir, 1990). Teachers do — and are expected to — show the values that they themselves embrace (Veugelers & Kat, 2000). In this regard, immigrant teachers bring to the host culture their cultural 'funds of knowledge' (Moll, 1994) pertaining to content and pedagogy, which may be different from the corresponding dominant attitudes, beliefs and values in the host culture. Further, it is hard to discuss cultures and cultural differences without considering the values which constitute the shared meanings understood within groupings of individuals (Hofstede, 2001; Kluckhohn, 1962). Therefore, for immigrant mathematics teachers practising in Australia, the experiencing in the Australian classroom of value differences and the resultant dissonance are inevitable.

VALUES RELATED TO SCHOOL MATHEMATICS EDUCATION

Values related to school mathematics education may be regarded as representing

> an individual's internalisation, 'cognitisation' and decontextualisation of affective constructs (such as beliefs and attitudes) in his/her socio-cultural context. Values related to mathematics education are inculcated through the nature of mathematics, through the

> individual's experience in the socio-cultural environment and in the mathematics classroom. These values form part of the individual's personal value system, which equips him/her with cognitive and affective lenses to shape and modify his/her way of perceiving and interpreting the world, and to guide his/her choice of course of action. They also influence the development of other affective constructs related to mathematics education and to life. (Seah, 2003b)

Bishop (1996) had earlier categorised these values as mathematical, mathematics educational, and general educational. In particular, Bishop (1988) also conceptualised three pairs of complementary mathematical values, being *rationalism* and *objectism*, *control* and *progress*, *openness* and *mystery*. On the other hand, mathematics pedagogy in different classrooms emphasise values such as *technology*, *practice*, and *problem-solving* to differing degrees. At the same time, there are also the general educational values which are espoused in mathematics classrooms differently, examples of which include *neatness*, *creativity*, and *honesty*.

As depicted in the affective taxonomy of educational objectives (Krathwohl, Bloom, & Masia, 1964), values arise from the increasing internalisation of what Raths, Harmin, and Simon (1987) called 'value indicators', which include beliefs and attitudes. Whereas beliefs often deal with truth/falsity (Kluckhohn, 1962) and are thus often expressed in context (e.g., 'All students can achieve good mathematics results'), values tend to be concerned with what is desirable or not (Rokeach, 1973). Thus, values are often expressed as single terms and context-independent. As alluded to in the definition of values above, in the affective taxonomy of educational objectives (Krathwohl et al., 1964), and in Raths et al.'s (1987) valuing process, the acquisition of values is a cognitive process. Even if the arousal of values in an individual may be an affective, emotional response to environmental stimuli, the notion of competing/conflicting values (Hofstede, 1997; Lewis-Shaw, 2001) implies that the act of valuing involves choice and decision-making; that is, the emphasis of values is in itself cognitive.

In this light, what are some of the culturally-based differences in values related to mathematics, mathematics pedagogy and education which immigrant mathematics teachers find in the Australian classroom? More importantly, in the interest of retaining valuable professional resources, and of empowering them to optimise the mathematics learning experience of all students, how do immigrant teachers negotiate about the value differences *they* perceive in the Australian mathematics classroom? What are some of the environmental factors which facilitate or constrain the espousal of particular values in conflict?

CONDUCTING THE STUDY

As a research study which seeks to understand and to generate theory, rather than to test any hypothesis, it is essentially qualitative (Merriam, 1988) in approach. Purposive sampling (Merriam, 1988) from a larger pool of immigrant mathematics teachers (identified earlier through a state-wide postal survey of all secondary schools) had helped to identify eight teacher participants representing the different education systems, both gender, professional placement across different parts of Victoria, and a diverse range of home cultures.

The research method involved the analysis of data collected through semi-structured interviews, lesson observations, and questionnaires and teacher marking of student work. Questions arising from the research questions were cross-referenced across the different data sources so as to achieve triangulation and to enhance validity of findings. Details of the research methodology employed in the study are discussed in an earlier paper (Seah, 2003a).

FINDINGS

The 34 reported differences in cultural values as perceived by the eight teacher participants in their respective secondary mathematics classrooms related not only to mathematics as a scientific discipline, to mathematics pedagogy, to educational aims, but also to differences in the ways in which educational institutions adopt organisational values. In particular, two immigrant teachers (Deanne from Canada, and Betty from England) perceived the valuing of *professional support* and *administrative support* to be emphasised differently in their Australian workplaces.

It is perhaps not surprising from the ensuing discussion that more than half of the perceived value differences were mathematics educational in nature. Interestingly, although value difference is regarded as relative (Hofstede, 1997) across cultures, none of the eight foreign cultures appeared to emphasise the values of *technology* and *numeracy* more than the Victorian mathematics curriculum.

Current knowledge from human resource management (e.g., Hofstede's (1997) 'acculturation curve') suggests that the state of stability is eventually attained after a period of uncertainty and dissonance in cross-cultural transition. There was no evidence in this study, however, that this holds for (mathematics) teaching. Manoj, an immigrant teacher from Fiji who has had 27 years of teaching experience in Australia, continued to perceive value differences during his practice. Analysed data indicated that this is likely because the very nature of students, institution and society change in time, perhaps more frequently and/or more deep-rooted than in the commercial workplace! However, years of experience did help Manoj hone his ability and consolidate his confidence in responding to dissonance brought about by value differences.

Confronted by the perceived value differences, each of the immigrant teachers was observed to adopt a variety of responsive approaches. That is, no one immigrant teacher negotiated about perceived differences in just any one particular way. These approaches are summarised in Table 1, the framework of which was adapted from Bishop (1994).

The observation of the affinity response indicates that perceived value differences need not always lead to dissonance. For example, Betty found that some of the values that were operating in Australia resonated with what she personally embraced, values which were not as valued in the British mathematics classroom. As such, she felt a sense of affinity to these relatively 'Australian' values, namely *application* and *administrative support*.

Culture to which personal value is aligned	Response	Assumption	Teaching
Australian culture	Affinity	There is no culture conflict; my value is aligned with the Australian culture.	The Australian culture supports my mathematics teaching style.
Home culture	Status quo	My home culture should be espoused.	I teach mathematics in the same way I did in my home culture.
	Assimilation	The Australian culture should influence the surface characteristics of my mathematics teaching.	I include the Australian cultural contexts in my teaching, such as in examples and problem sums.
	Accommodation	The Australian culture should be espoused.	Planning and classroom decisions portray the Australian culture.
	Amalgamation	The essence of my home culture and the Australian culture should guide mathematics teaching.	My teaching reflects a synthesis of teaching styles from my home culture and from Australia.
	Appropriation	My home culture and the Australian culture should interact to inform my mathematics teaching.	My mathematics teaching style consistently reflects an adaptation of my home culture to local norms and practices.

Table 1: Responses by immigrant teachers to perceived value differences in mathematics education.

Not all perceived value differences appeared to be mediated successfully by the immigrant teachers, however. For instance, although Carla (an immigrant teacher from Romania) found herself capable of accommodating (see Table 1) to a lesser emphasis of *power distance* in her Australian mathematics classroom, in some cases she appeared helpless and did not know how to respond to the value difference situation.

Thus, attending to the various contextual factors operating at the time when the value differences were perceived had meant the consideration of how the values underlying each of these factors might interact with the values that were already seen to be in conflict. For example, in Rana's perception that *product* was valued more by her students in Australia, and *process* embraced more by her students in India, her responsive approach took into account several contextual factors, such as the nature of work in Australia, student self-esteem, and the relatively heavy teaching schedule in her Australian workplace. Each of these implied that Rana's negotiation involved more than a choice between the valuing of *process* or *product*, to an interplay of values underlying the various contextual factors. Again, the role played by the teacher's personal values in the negotiation of perceived value differences is a significant one.

COMPLEMENTARITY OF VALUES THROUGH THE AMALGAMATION AND APPROPRIATION APPROACHES

Interestingly, however, each and every one of the teacher participants was observed to adopt the amalgamation and/or appropriation approaches. This is noteworthy because these two approaches differ from the affinity, status quo, assimilation and accommodation approaches, in that instead of the values of either the home or Australian cultures being affirmed through the teachers' response, amalgamation and appropriation combine aspects of both these cultures in ways which also re-establish the harmony and equilibrium within the particular teacher's personal value schema. In this way, they may be perceived as being inclusive responsive approaches: approaches characterised by the inclusion, embracing, and mutual support of values from different cultures. On the other hand, the other four approaches may be called exclusive responsive approaches as some values tend to be excluded in the process.

Through this expression of a 'middle way', these inclusive responsive approaches serve to enrich both the home and Australian cultures. In the case of amalgamation, this 'middle way' enables the concurrent emphasis of both cultures' values as they are. Analogically, this is similar to the chemical formation of mixtures (versus compounds): the constituents of the mixture remain distinguishable and separable although they together have produced something new. For example, Betty's response to a relatively higher emphasis of *technology* in the Australian mathematics classroom when she originally subscribed to paper-and-pencil and mental *computations* embraced in the British mathematics classroom (in her opinion), was one of amalgamating the different values: encouraging mental computation for simpler questions and responsible use of technology in more tedious calculations.

On the other hand, in the case of appropriation, the relevant home and Australian cultures are seen to have interacted with each other and redefined each other, such that their individual nature has transformed in the process. Using the analogy of chemical formations again, this approach is akin to the production of compounds: while the properties of the constituents may be distinguishable in the nature of the compound, it would be impossible scientifically to isolate the constituents from it.

For example, when Manoj grew to understand the relative lack of connection between academic performance and personal success in the Australian society (when compared to his Fiji Indian culture), his approach to negotiating the cultures' difference in the valuing of *academic achievement* was one of appropriating it to adapt its relevance in his Australian classroom. While he continued to value *academic achievement*, he no longer expected this to be embodied in the form of absolute assessment scores. Rather, its importance became one of each student performing to her best potential.

Since the balance of emphasis between values changes with each classroom situation, appropriation is an ongoing process as the individual continually assesses how the conflicting values interact with each other in the different situations.

CONCLUSION

This paper has briefly reported on some of the findings relating to immigrant teachers' negotiation of perceived value differences in the Australian mathematics classroom. In mediating the dissonance, the immigrant teachers re-established harmony and equilibrium within the personal value schemas. Although this has led to the adoption of a range of responsive approaches (listed in Table 1), the inclusive approaches of amalgamation and appropriation were by far most commonly used.

This researching process has highlighted how the mathematics learning discourse in the classroom is indeed value-rich. There is thus a need for this aspect of mathematics education to be further researched upon, both for reasons of optimising school mathematics teaching and learning process, and of highlighting the role that mathematics education can – and does – play in the wider good of values education.

The socialisation experience of immigrant teachers can possibly inform similar experiences of mathematics teacher in transition between other kinds of cultures (e.g. public and private schools). This study has shown that successful socialisation does not imply teacher enculturation into the host culture. Neither does it involve ways of preserving the teachers' respective home cultures per se. Rather, an empowering professional development program should focus on enabling teachers to explore the values negotiation in relation to their own personal values. Importantly, this has the potential of developing mathematics teachers' cultural intelligence (CQ), at a time when an individual's capability to complement it with her intelligence quotient (IQ) and emotional intelligence (EQ) is most crucial for personal and professional health and growth.

References

Bishop, A. J. (1988). *Mathematical enculturation: A cultural perspective on mathematics education.* Dordrecht, The Netherlands: Kluwer Academic Publishers.

Bishop, A. J. (1991). Mathematical values in the teaching process. In A. J. Bishop, S. Mellin-Olsen & J. v. Dormolen (Eds.), *Mathematical knowledge: Its growth through teaching* (pp. 195-214). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Bishop, A. J. (1994). Cultural conflicts in mathematics education: Developing a research agenda. *For the Learning of Mathematics, 14*(2), 15-18.

Bishop, A. J. (1996). *How should mathematics teaching in modern societies relate to cultural values --- some preliminary questions.* Paper presented at the Seventh Southeast Asian Conference on Mathematics Education, Hanoi, Vietnam.

D'Ambrosio, U. (1985). Ethnomathematics and its place in the history and pedagogy of mathematics. *For the Learning of Mathematics, 5*(1), 44-48.

Gudmundsdottir, S. (1990). Values in pedagogical content knowledge. *Journal of Teacher Education, 41*(3), 44-52.

Hofstede, G. (1997). *Cultures and organizations: Software of the mind* (Revised ed.). New York: McGraw-Hill.

Hofstede, G. (2001). *Culture's consequences* (2nd ed.). Thousand Oaks, CA: Sage.

Kluckhohn, C. (1962). Values and value-orientations in the theory of action: An exploration in definition and classification. In T. Parsons & E. A. Shils (Eds.), *Toward a general theory of action* (pp. 388-433). New York: Harper & Row Publishers.

Knijnik, G. (1993). An ethnomathematical approach in mathematical education: A matter of political power. *For the Learning of Mathematics, 13*(2), 23-25.

Krathwohl, D. R., Bloom, B. S., & Masia, B. B. (1964). *Taxonomy of educational objectives: The classification of educational goals (Handbook II: Affective domain).* New York: David McKay.

Lewis-Shaw, C. P. (2001). Measuring values in classroom teaching and learning. In D. Clarke (Ed.), *Perspectives on practice and meaning in mathematics and science classrooms* (pp. 155-196). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Merriam, S. B. (1988). *Case study research in education: A qualitative approach*. San Francisco, CA: Jossey-Bass Publishers.

Moll, L. (1994). Literacy research in community and classrooms: A sociocultural approach. In R. Ruddell, M. Ruddell & H. Singer (Eds.), *Theoretical models and processes of reading* (pp. 179-207). Newark, DE: International Reading Association.

Raths, L. E., Harmin, M., & Simon, S. B. (1987). Selections from 'values and teaching'. In J. P.F. Carbone (Ed.), *Value theory and education* (pp. 198-214). Malabar, FL: Robert E. Krieger.

Rokeach, M. (1973). *The nature of human values*. New York, NY: The Free Press.

Schmidt, W. H., McKnight, C. C., Valverde, G. A., Houang, R. T., & Wiley, D. E. (1997). *Many visions, many aims* (Vol. 1). Dordrecht, The Netherlands: Kluwer.

Seah, W. T. (2003a). *Teachers in transition and the negotiation of value differences in the classroom: Perspectives of immigrant teachers of mathematics in Australia: Proceedings of the 2003 NZARE/AARE Conference*, from http://www.aare.edu.au/03pap/sea03594.pdf

Seah, W. T. (2003b). *Understanding mathematics classroom experiences through the values lens.* Paper presented at the 81st Annual Meeting of the National Council of Teachers of Mathematics (Research Presession), San Antonio, TX.

Su, Z., Goldstein, S., Suzuki, G., & Kim, T. (1997). Socialization of Asian Americans in human service professional schools. *Urban Education, 32*(2), 279-303.

Veugelers, W., & Kat, E. D. (2000). *The teacher as a moral educator in secondary education: The student perspective.* Paper presented at the 81st Annual Meeting of the American Educational Research Association, New Orleans, LA.

DEVELOPMENT OF MATHEMATICAL NORMS IN AN EIGHTH-GRADE JAPANESE CLASSROOM

Yasuhiro Sekiguchi

Yamaguchi University, Japan

Mathematical norms are important cultural knowledge of mathematical activities. This paper reports an analysis of mathematical norms in ten consecutive lessons taught by an eighth-grade Japanese teacher. The lessons were located in the unit of simultaneous linear equations. The videotapes of the lessons, their transcripts, and the interview data were analyzed qualitatively. Several major mathematical norms were found in the lessons. The teacher's deliberate strategies to develop them were identified: using students' work, making a comparison, and being considerate of those students who did not follow a norm. Complexities of research on mathematical norm are discussed.

INTRODUCTION

This paper reports an analysis of the ten consecutive lessons taught by one of the eighth-grade Japanese teachers who participated in the *Learner's Perspective Study* (*LPS*), an international research project coordinated by David Clarke (see Clarke, 2004). The analysis focuses on mathematical norms introduced by the teacher.

As Clarke (2004) points out, one of the goals of *LPS* is to complement TIMSS 1999 Video Study. In Hiebert et al. (2003), the analysis of mathematics teaching focused on mathematical knowledge, procedures, and reasoning involved in the problems presented in the lessons. Teaching of mathematical norms was beyond their analysis. Though the mathematical norms are often not explicitly taught by teachers nor written in textbooks, they are crucial when the learning process of mathematics is conceived as mathematical activities.

Mathematical norms are knowledge "about" doing mathematics; therefore, they belong to the domain of metaknowledge in mathematics. It is hypothesized that beginning teachers are often occupied with covering curriculum content, paying their attention to mathematical knowledge and skills: Competent teachers as selected in *LPS* by design would invest more time and effort in teaching of metaknowledge. The major questions guided this analysis of Japanese data are, What mathematical norms would surface in the lessons? How would the teacher introduce, negotiate or establish those norms during the lessons? In the future those questions will be investigated in *LPS*'s lesson data of other countries, too.

THEORETICAL FRAMEWORK

Scientists search for patterns, regularities, rules, or laws in the real world, and try to build causal theories, so as to be able to explain the phenomena in which they are interested; Understanding of causal relationships is useful for prediction and control.

Social sciences, likewise, study those patterns, norms, regularities, rules, or laws appearing in human activities (cultures), so that they can explain and understand human activities. Positivist sociologies are known to have assumed a "normative" conception of human action: It has three main components, "actors," "rules," and "situations," and presumes that "actors know and follow rules in social situations" (Mehan & Wood, 1975, p. 74). This conception closely parallels that of natural phenomenon: "Physical objects follow natural laws in the physical world."

Ethnomethodologists had also studied people's "rule" use in social situations, but they made strong attacks to the positivist's normative conception. They claimed that actors, rules, and situations were mutually shaped in practice, in their terminology, "reflexively" related to each other (Mehan & Wood, 1975, pp. 75-76).

Cobb and his colleagues (Yackel & Cobb, 1996; McClain & Cobb, 2001) introduced the notion of "norms" of classroom process as a device to interpret classroom processes and clarify how children's beliefs and values develop. They identified several classroom social norms working in their project classroom, such as "Students were obliged to explain and justify their reasoning."

They also pointed out that there were norms specific to mathematics in the classroom, which they called "sociomathematical" norms. By using the prefix "socio-" they seem to be trying to stress that norms of mathematical activities depend on the community (Yackel & Cobb, 1996, p. 461). They contend that the mathematical activity has norms as constituent, and that norms are reflexively related to beliefs and values of mathematical activities.

In this paper I will use a simpler word "mathematical" norm to refer to a norm in the mathematical activity, rather than "sociomathemtical" norm. This is because I consider that mathematics is intrinsically sociocultural activity as current philosophies of mathematics and sociocultural theories inform. The prefix "socio-" of the sociomathematical norm is redundant as far as we accept this understanding.

For the framework of Yackel & Cobb (1996), the notion of sociomathematical norm appears to hold a central position of classroom mathematical activity:

> These sociomathematical norms are intrinsic aspects of the classroom's mathematical microculture. Nevertheless, although they are specific to mathematics, they cut across areas of mathematical content by dealing with mathematical qualities of solutions, such as their similarities and differences, sophistication, and efficiency. Additionally, they encompass ways of judging what counts as an acceptable mathematical explanation. (Yackel & Cobb, 1996, p. 474)

However, this strong emphasis on norms has a danger of leading to the positivist's normative conception. The symbolic interactionism and ethnomethodology do not put norms on the central place in explaining social conduct:

> Rather than the major criterion people employ to regulate their own and other's conduct, social norms are one of several forms of knowledge that people employ in their everyday conduct. ... It should not be thought, however, that norms are constantly implicated in

acts, nor that people behave by finding the appropriate norms that govern each and every social situation. (Hewitt, 1994, p.160)

We should avoid the tendency to explain classroom processes with too much emphasis on abstract norms. Rather, as Waschescio (1998, p. 235) also pointed out, norms should be understood as cultural "tools," which may or may not enhance mathematical activities.

RESEARCH PROCESSES

Unlike TIMSS 1999 Video Study, in *LPS* project, eighth-grade teachers were not randomly selected. Only those who were considered "competent" by local educators were selected. In addition, for each teacher, ten consecutive lessons were videotaped by three cameras (teacher camera, student camera, whole class camera). Students were interviewed by the stimulated-recall method using videotapes of the lessons.

This paper analyzes one Japanese teacher's ten consecutive lessons that were located in the unit of simultaneous linear equations, covering the linear combination method ("addition or subtraction method"), the substitution method, and part of application problems. The videotapes of the lessons, their transcripts, and the interview data were analyzed qualitatively. To let mathematical norms emerge from the data, any piece of the data that appeared to indicate beliefs on how to work on mathematics was coded, and the normative aspects behind those beliefs were repeatedly analyzed.

The eighth-grade students had experienced huge amount of mathematical activities since entering schools. They must have been equipped with many mathematical norms, some of which would have been working when they participated in this research. This report analyzed only the ones that the teacher emphasized during the lessons because I was interested in how the teacher introduced or developed mathematical norms in the classroom.

MATHEMATICAL NORMS IN THE CLASSROOM

Norm 1: Efficiency

The value of pursuing efficient ways of solving problems is generally shared among mathematicians. Many theories, theorems, and formulae in mathematics have been produced to improve efficiency. In this class also, the teacher encouraged the students to pursue efficient ways of solving simultaneous equations.

In the first lesson (L1), the class discussed a simultaneous equations: $5x + 2y = 9$…(1)□$-5x + 3y = 1$ …(2). The teacher asked a student KORI write his solution on the board. He subtracted (2) from (1), obtaining $10x - y = 8$. Solving it for x, he put it into (1), obtaining the value of y. Finally, he put the value of y into (1), and got the value of x. After KORI explained his solution to the class, the teacher asked to the class: "OK, any question? Can you understand? Well, do you have any thoughts as work out this question? Any impressions of this explanation?" (L1 10'54")[This notation indicates that this talk occurred in 10 min. and 54 sec. from the start of L1].

A student SUZU responded to it: "I think there is much simpler one." SUZU wrote his solution on the board: He added (1) and (2), and got an equation without variable *x*. And he solved it for *y*, and got the value of *y*. He then put it into one of the given equations, and got the value of *x*.

The students were then seeing two different solutions on the board. The teacher explained the reason why he asked KORI to write his solution on the board. The teacher intentionally chose KORI because he had observed at the previous lesson that KORI had solved the problem differently from the other students: "Almost, actually almost students have this opinion that I saw the class that we did yesterday. And in fact, the way which KORI did was different so that I wanted them to write on the blackboard" (15'27").

The teacher thought that by comparing solutions with different degrees of difficulties, students would be able to appreciate an easier one well: "I think you can know which point was difficult as you compare the difficult way and the easier one" (15'48"). Finally, the teacher concluded that SUZU's solution was easier and better than KORI:

> Now, actually that way is much better than this way, when we compare the calculations so far. As a result, it is better to notice that this way, which SUZU wrote, is better, you know? (16'24")

He asked the students where they thought KORI's solution was more complicated than SUZU. This question tried to elaborate inefficiency of KORI's solution.

Up to this point, the teacher seems to be putting more value on efficient solutions. The students seem to be encouraged pursuing as efficient solutions as they can. KORI's solution seems to be devalued. This does not mean that inefficient solutions are useless, however. First, the teacher soon pointed out that KORI's method gave the same result as SUZU. Second, he suggested that KORI's method contained an important idea: "There are some important ideas in this [KORI's] process, I think" (19'21"), which I discuss next.

Norm 2: Even inefficient attempts could contain important ideas

Efficiency is not the only value in pursuing mathematics. New ideas for developing new ways of solving problems are equally important in mathematics. Those could be discovered through numerous inefficient, or failed attempts as the history of mathematics shows. In this class, the teacher once gave an opportunity for the whole class to appreciate an important idea found in an "inefficient" solution.

In L1 the teacher pursued "KORI's idea," and went into the idea of the substitution method, which was formally introduced at L7. This pursuit continued well into the next lesson L2. Thus, he seems to believe that *even inefficient attempts could contain important ideas.*

In addition to this normative action, the teacher paid respect and care to both solutions. Devaluing one's idea may hurt his or her feeling. When KORI received negative opinions to his solution, the teacher encouraged KORI: "It's OK. Don't be depressed as it didn't go well. It is better to get some comments, right? Don't worry"(L1 13'42"). By pursuing KORI's idea with the whole class, the teacher showed further care to the student whose idea had been devalued.

Norm 3: In mathematics you cannot write what you have not shown to be true yet

Mathematics is traditionally written in the deductive way: It must begin with axioms, definitions, or already proved theorems, and proceed logically. Therefore, you cannot write what you have not shown to be true yet. This norm is emphasized especially in the teaching of proof in geometry in Japan.

In L3, the teacher reviewed the solution of a simultaneous equation: $3x + 2y = 23$, $5x + 2y = 29$. As homework, he had asked the students to do checking of the solution. First, he asked UCHI to put up his work on the board (Figure 1). As a "different way," he then asked KIZU to put up his work on the board (Figure 2).

By putting $x = 3$, $y = 7$ into $3x + 2y = 23$ and $5x + 2y = 29$

3X3 □ 2X7 = 23   5X3 □ 2X7 = 29
9 □ 14 = 23   15 + 14 = 29
23 = 23   29 = 29

Figure 1: UCHI's writing on the board.

By putting $x = 3$, $y = 7$ into $3x + 2y = 23$ By putting $x = 3$, $y = 7$ into $5x + 2y = 29$

3x3 + 2x7   5x3 + 2x7
= 9 + 14 = 23   = 15 + 14 = 29

Figure 2: KIZU's writing on the board.

The teacher posed the class a question what differences they noticed between them. The students discussed the question with nearby students. After that, UCHI and KIZU explained their work in front. The teacher mentioned that most of the students did the same way as UCHI did. Reviewing the checking of the solution of linear equations studied at previous year, the teacher pointed out UCHI's writing used an unconfirmed fact:

> This is just substituting x as three, and y as seven into the equation, right? It's just substitution, right? It's just substitution but this is already an equality, so the right side and the left side have to be equivalent, doesn't it? But you can't confirm that yet, can you? Right? Which means, if you write it this way, actually,[Writes on the blackboard]you've already shown that the right side and the left side are equivalent at this point. But you haven't confirmed that yet (L3 36'28").

Here the teacher was trying to let the students be aware of a mathematical norm that if you write an equation in your solution, it means that you have already shown the equality, or that in a mathematical explanation you cannot write what you have not shown to be true yet.

Based on this norm, the teacher accepted KIZU's way of checking, and devalued UCHI's way. Again, the teacher did not forget reminding the students of the fact that most of the students did the UCHI's way: You were not the only one who did wrong.

Norm 4: Accuracy is more valued than speed

In mathematics, establishing truths is one of the most important goals. In the history of mathematics, numerous mathematicians have strived to establish the truths of "conjectures." Therefore, the accuracy of the solution is often more valued than the efficiency, though in the application of mathematics to the real world, efficiency is sometimes more valued.

The teacher often emphasized to the students the importance of checking solutions by themselves. When writing a solution process, if one omits to write several intermediate steps, one could save time. But, then it may become harder to check the procedures. In L5, when the teacher was circling among the students, DOEN asked him if he could omit writing calculations in the solution process. The teacher advised him not to omit them:

> Oh, okay, maybe you should write down up to this expression. Because, when you want to check later, if you don't have this part, you suddenly come up with this expression. For example, for this question, negative seventeen y equals to negative fifty-one. So, it will be easier if you have a clue for what you have done by then, but what if you don't have it? I don't think you have to write down the whole process you took, so, maybe this part can come off, but you had better leave the calculations part if you think about the checking. When you try to check, you have another way from always substituting it, but following what you have done. I think that'll be easier for those situations. I recommend you to leave it for a while. In the future, it will be easy to do a sum in your head. [To Class]The thing is, *it's better to be able to do accurate calculations rather than quick calculations* [the italics are added by the author]. (L5 26'31")

PATTERNS IN NORM DEVELOPMENT

From the data discussed above, there seem to be at least three strategies the teacher used to develop mathematical norms.

Using students' work

The teacher explains a norm by using students' work as an exemplification of what it means to follow the norm.

Since any norm has generality, it could be communicated by using only general terms like "in a mathematical explanation you cannot write what you have not shown to be true yet." But the teacher in this study talked about norms almost always by using students' work. In addition, the teacher did not use any artificial example: He always used actual work of students.

Making a comparison

Sometimes the teacher lets the whole class to compare two of their work on the blackboard, and points out that one of them follows a norm properly, and the other do not (see Norms 1 and 3). Then, the teacher asks the students to follow the norm.

This seems to correspond to "neriage," which is an instructional strategy common in Japanese elementary schools. Japanese elementary teachers often ask children to present their own ideas or solutions on the blackboard. Then comparing their writing the children discuss what they notice of them. This process of comparative discussion is called "neriage." (kneading). Since "neriage" accompanies comparative discussion, this is not just "sharing ideas" (cf. McClain & Cobb, 2001, p. 247).

Being considerate of those students who did not follow a norm

The teacher often discussed that a student's work did not follow a norm. When doing it, he took careful measures to reduce psychological and social damage of the student.

DISCUSSION

The present paper identified three patterns in developing mathematical norms. The use of students' work seems very important. Since a norm is about how to work on mathematics, the use of mathematical work is natural for communicating a norm. Also, since students are familiar with their work, the use of students' work would facilitate students' understanding of the norm. Comparison of students' work would also be very helpful for students to produce clear understanding of the norm as well as their metacognition of their own work. Since pointing out students' violation of a norm may hurt their feeling, being considerate of those students who did not follow the norm seems a hallmark of "competent" teachers. In the data on Norms 1-3, the teacher made considerate moves explicitly. For the data on Norm 4, he did so implicitly by *not* pointing out any student's violation.

These three patterns would be found in common strategies of introducing norms. For example, Voigt (1995) discusses an "indirect" way of introducing a norm about "what counts as an elegant mathematical solution." The strategy highlights students' elegant solutions. Thus, it uses students' work, and has students compare their solutions implicitly. Also, avoiding explicit negative evaluation, the indirect way takes care of the feeling of those students who did not follow the norm.

Studying norms requires understanding of relationships between various norms. A classroom in Japanese schools constitutes a community where a teacher and students stay together, negotiate meanings, share common goals, and shape their identities. It forms a "community of practice." A community generates, maintains, modifies, or eliminates various kinds of patterns called norms, standards, obligations, rules, routines, and the like. Consider a mathematical norm that I identified above, "in mathematics you cannot write what you have not shown to be true yet." This is consistent with a general moral "You should not tell a lie to people." The mathematical norm seems to be backed or authorized by the social norm. That is why the norm appeals to educators and students. Also, consider the teacher's considerate treatment with unsatisfactory fulfilment of mathematical norm, which I identified. The teacher's treatment seems consistent with a *social* norm such as "Any attempt to explain his or her thinking should be respected" (cf. McClain & Cobb, 2001, p. 245).

Furthermore, norms may cause a dilemma. In fact, Norms 1 and 2 appear contradictory. Also, Norm 4 indicates that the efficiency is not always given the highest value. Which norm to use seems to depend on the context where participants are situated. As discussed at the theoretical framework, norms cannot prescribe participants' behaviour. Norms are no more than useful cultural knowledge.

References

Clarke, D. (2004). Learner's perspective study: Developing meaning from complementary accounts of practice. In M. J. Høines & A. B. Fuglestad (Eds.), *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, pp. 212-216). Bergen, Norway: PME.

Hewitt, J. (1994). *Self and society: A symbolic interactionist social psychology* (6th ed.). Boston: Allyn and Bacon.

McClain, K. & Cobb, P. (2001). An analysis of development of sociomathematical norms in one first-grade classroom. *Journal for Research in Mathematics Education*, *32*(3), 236-266.

Mehan, H. & Wood, H. (1975). *The reality of ethnomethodology*. Malabar, FL: Robert E. Krieger Publishing.

Hiebert, et al. (2003). *Teaching mathematics in seven countries: Results from the TIMSS 1999 video study*. Washington, DC: U.S. Department of Education, National Center for Education Statistics.

Voigt, J. (1995). Thematic patterns of interaction and sociomathematical norms. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning* (pp. 163-201), Hillsdale, NJ: Lawrence Erlbaum Associates.

Waschescio, U. (1998). The missing link: Social and cultural aspects in social constructivist theories. In F. Seeger, J. Voigt, & U. Waschescio (Eds.), *The culture of the mathematics classroom* (pp. 221-241). Cambridge, UK: Cambridge University Press.

Yackel, E. & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458-477.

SOLVING ADDITIVE PROBLEMS AT PRE-ELEMENTARY SCHOOL LEVEL WITH THE SUPPORT OF GRAPHICAL REPRESENTATION

Ana Coelho Vieira Selva, Jorge Tarcisio da Rocha Falcão,
Universidade Federal de Pernambuco, Brazil

Terezinha Nunes
Oxford Brookes University, UK

This research offers empirical evidence of the importance of supplying diverse symbolic representations in order to support concept development in mathematics. Graphical representation can be a helpful symbolic tool for concept development in the conceptual field of additive structures. Nevertheless, this symbolic tool has specific difficulties that are better dealt with when graphics are combined with symbolic-manipulative tools like building blocks. This combination showed to be effective in the context of a didactic sequence addressed to students in the beginning of elementary school level and aimed to support conceptual development in the domain of additive structures. It provides a theoretical backing for the proposal of using diverse symbolic representations in concept development in mathematics.

The availability of symbolic representations is considered very helpful in conceptual building, since each particular representation (or symbolic model) allows different approaches of conceptual properties (Vergnaud, 1997; Nunes, 1997). In mathematical education, symbolic representation based on concrete artifacts has been considered specially beneficial, since these artifacts are supposed to allow a concrete-metaphorical approach to abstract principles (Selva, 2003; Da Rocha Falcão, 1995; Gravemeijer, 1994; Bonotto, 2003; Bills, Ainley & Wilson, 2003). In fact, the representational power of concrete devices used as didactic tools is not inherent to these devices per se, but is construed in a social and meaningful context of use (Vygotsky, 2001; Meira, 1998). According to this theoretical approach, the "epistemic fidelity" of representational devices is an *essentialist* idea to overcome.

Cartesian graphics are another symbolic representational support for dealing with quantities and their relations. This specific tool has a widespread use in and out of school context; it allows comparisons, demonstration of tendencies in a serial set of data, with the support of visual-cognitive schemas, like "bigger/taller/longer/ is more". Even though the use of graphics is supported by these perceptual schemas, this representational tool is not easy to be used by children at elementary school level (Selva & Da Rocha Falcão, 2002; Bell e Janvier, 1981; Ainley, 2000; Guimarães, 2002). The present research tried then to propose a didactic approach of graphics at pre-elementary school level, in the general context of additive structures. This didactic effort covered two studies, as described below.

The first study was a clinical-exploratory enquiry about the use of building bricks (like those developed by Lego™ - see illustration 1 below) as manipulative auxiliary

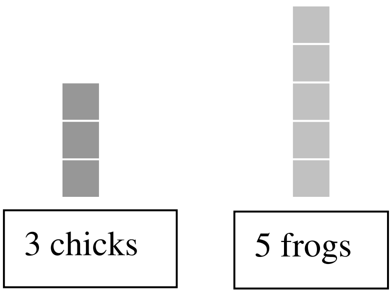
2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 161-168. Melbourne: PME.

tools for graphics comprehension, these graphics being used afterwards as auxiliary tools in solving additive problems.

Twelve pairs of six to seven year-old students (pre-elementary Brazilian public school level) were presented to a set of situations in which they were asked to use the building bricks by organizing them in piles to represent quantities. Each pair of children worked under the supervision of a teacher-researcher in a working-room at school, in a clinical basis, the complete set of activities being covered in seven sixty minutes long meetings. These activities are summarized below:

1. Familiarization with the building blocks: free manipulation; counting of blocks, comparison of piles of blocks.

2. Representing quantities using blocks and solving additive problems:



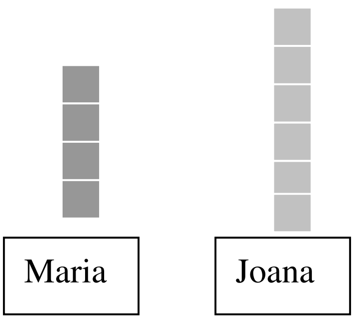
Are there more chicks or frogs?

How many more chicks do we need to have the same amount of chicks and frogs?

How many animals are there in total?

I was told that there were also some beetles in this set of 12 animals. How many beetles were there?

3. Solving comparison problems:

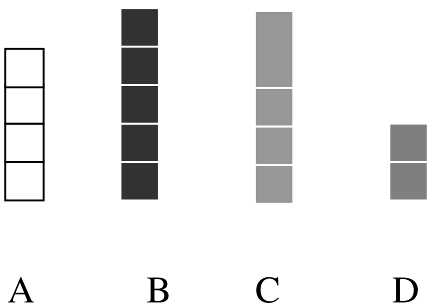


These piles represent the number of school days lost by Maria (4 days) and Joana (6 days).

Who lost more school days? How many school days has Joana lost more than Maria?

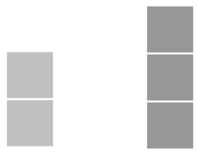
Patrícia, another girl, has had 6 absences. We know she had 2 absences more than Luíza. How many absences has Luíza had?

4. Using different units of measure:



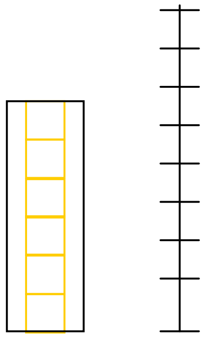
These piles of bricks represent the quantities of absences of four students during the school-year. In the case of C, we used a double-brick that is equivalent to two single bricks. Can we say that B and C had the same amount of absences?

5. Attributing different values to each brick in a pile:

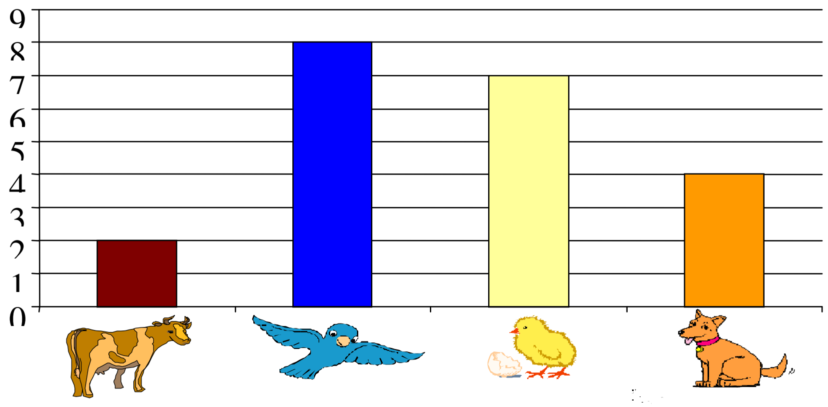


Each brick in the piles representing the pencils owned by Augusto and Pedro stands for 2 units. How many pencils does each boy have?

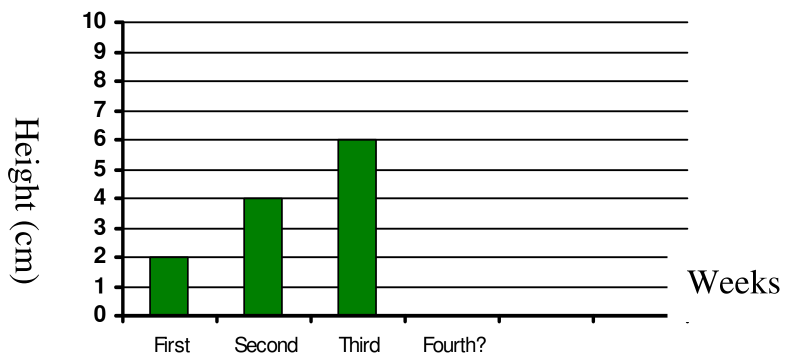
6. Using piles of bricks covered by opaque paper, in order to avoid direct visual inspection of the number of bricks per pile. On the other hand, subjects were introduced to the use of a paper-device for measuring the number of bricks per pile, as shown in the illustration on the right:



7. Using bar graphics in the place of piles of bricks: how many cows are there? What about birds? How many animals in total are there? How many chicks are there more than dogs?



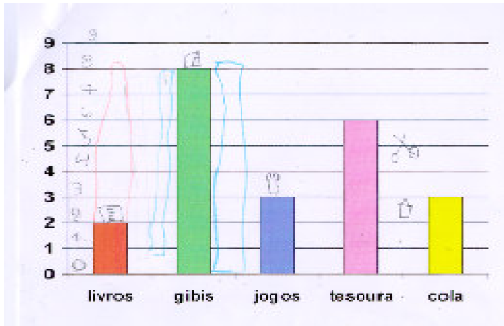
8. Thinking about tendencies in a bar graphic: the growing of a quantity (height in centimeters) during a period (weeks). What is probably going to happen in the fourth week?



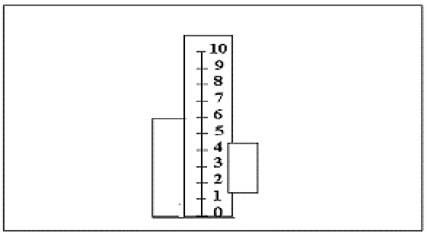
SUMMARY OF RESULTS OF STUDY 1

Clinical analysis of the protocols produced by the pairs in cooperation with the researcher showed that they were able to perform all the activities proposed. Nevertheless, three aspects concerning the use of graphics were sources of difficulty:

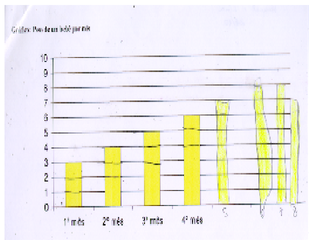
1. Summing up quantities represented by different columns of the graphic: in the protocol on the right, when asked about the sum of books ("livros") and magazines ("gibis"), this pair decided to move the column of magazines to the top of the column of books, in order to see the total amount of books and magazines, but they represented the column of magazines with 6 units (instead of 8), in order to equalize the heights of the original column of magazines and the height of the transported column:



2. Considering different baselines (starting points) to compare columns representing different quantities: this pair of children puts two columns to be compared in different starting points, what makes this comparison task inaccurate.



3. Representing tendencies properly: this pair of children understood that the weight of a baby represented by the graphic on the right was increasing, but could not represent properly the continuation of the tendency (see the two last columns on the right) :

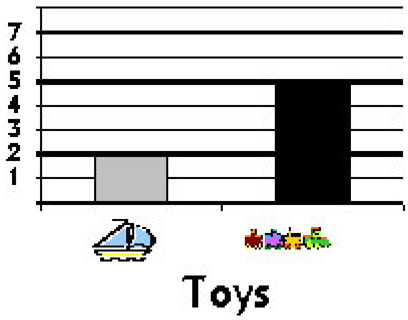


On the other hand, subjects have shown to be able to move from representing quantities through building bricks to doing it through graphics, as suggested by their good performance in activity 7 (see description below). We decided then to test more effectively the didactic importance of building bricks combined to graphics as representational tools for problem solving in the conceptual field of additive structures. In this second study, the research question was the following: is the combination in a didactic sequence between concrete-manipulative representational tools (building bricks) and graphics really helpful in concept development in the conceptual field of additive structures? Or would the proposition of a set of activities concerning the use of graphics (without activities with bricks) allow students to reach an equivalent level of conceptual development? In order to answer this question, twenty-seven children at pre-elementary school level and thirty children in the first year of elementary level, with ages varying between 6 and 8 years, from a private elementary school in Recife (Brazil) took part in this second study. These children were divided, in a controlled way, in three groups: Experimental group 1, submitted to learning activities covering building bricks and the use of graphics, as firstly explored in study 1; Experimental group 2, submitted to activities concerning only the use of graphics, without any exploration of manipulative tools like building bricks; finally, a Control group, submitted only to algorithmic activities involving the same numbers and operations explored by the experimental groups, but without any offer of didactic activity explicitly aimed at conceptual development (for ethical reasons children of control group and experimental group 2 were submitted to the same activity of experimental group 1 at the end of the research). The three groups were submitted to a same pre-test, post-test and delayed post-test (eight weeks after the teaching intervention). Pre and post-tests consisted of a set of thirty problems of combination and comparison, concerning the conceptual field of additive structures. These problems were presented under two forms: verbal-pictorial and graphic. Both forms and structure of problems were randomly presented. Examples of structure and form of representation of the problems are given below:

Structure of problem: comparison

Form of presentation: verbal-pictorial

Problem: A toy-shop has six teddy-bears and two teddy-rabbits. How many more teddy-bears are there than teddy-rabbits?



Structure of problem: combination

Form of presentation: graphic

Problem: A boy has little boats and trains. How many toys has this boy in all?

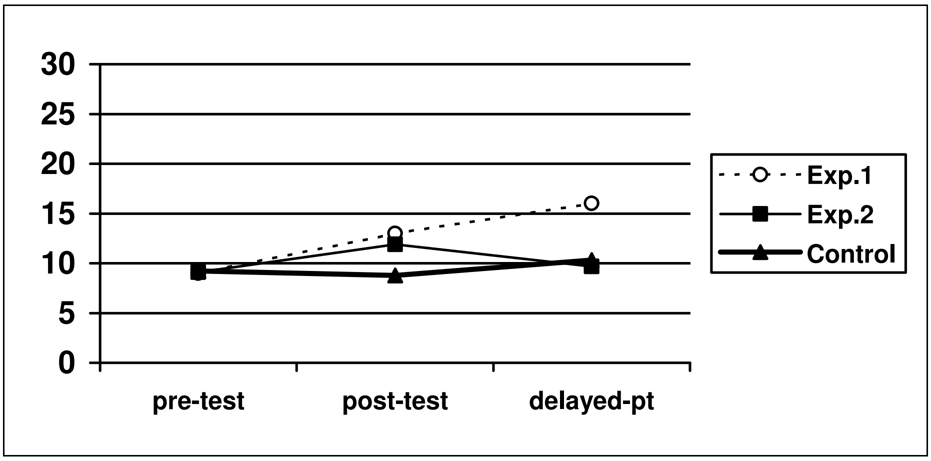
Both pre and post tests were presented collectively, in the classroom. Problems were displayed with the aid of a data-show apparatus, and the children didn't have access to any aid during tests. Each child had a booklet with reproductions of all the questions displayed, where he/she could write their answers.

The didactic intervention proposed for the two experimental groups consisted of the assisted resolution of nine combination problems and eighteen comparison problems. Forms of presentation and structure of problems were randomized, the whole set of twenty-seven problems being presented in three subsets of nine problems in a daily session. Control group, as mentioned before, was invited to solve 27 additive operations in a session (making this calculation activity was a familiar task for them). Children of the three groups were assisted by teachers-researchers during the intervention session, their roles consisting mainly in explaining the activities and encouraging debate and argumentation inside the group.

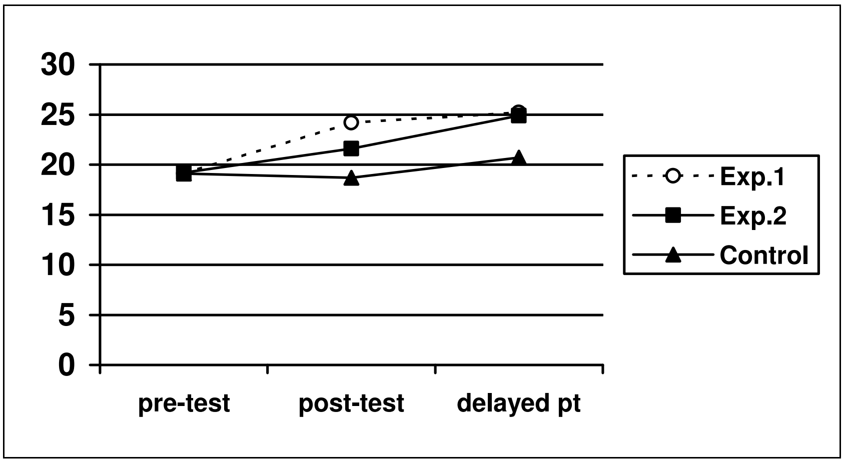
SUMMARY OF RESULTS OF STUDY 2

Performance of children in the post-test was submitted to an analysis of variance having as sources of effect the *school level* (pre-elementary versus first elementary level) and the *group* (experimental 1 or 2 or control). A significant isolated effect of both school level (F=4.61, 1 d.f., p=.037) and group (F=9.552, 2 d.f., p=.000) was observed. Interaction between these two sources of interaction was not observed (F_{inter}= .229, 2 d.f., p> .05). Children from the first elementary level performed significantly better than pre-elementary (difference confirmed by U Mann-Whitney test, U=78.5, one-tailed, p<.000). Children from experimental group 1 (bricks+graphic) and experimental group 2 (graphic) performed significantly better than children from control group (Bonferroni test, p=.000 and p=.033, respectively), but these two experimental groups did not show significant difference when their performance in post-test was compared.

A second analysis of variance was performed having the same factors of previous ANOVA as sources of effect, and performance in a delayed post-test as dependent variable. Results of this analysis was quite similar to those from previous analysis, since isolated effect of school level remains in the same way detected for post-test ($U_{M\text{-}W}$ = 99.5 , one-tailed, p<.000), as well as isolated effect of group, this time with a slight difference: significant difference was noticed only between experimental group 1 and control group (Bonferroni test, p=.014). Interaction effects of both sources of variance analyzed were equally non-significant. A closer analysis of data showed that children from pre-elementary school level, experimental group 2 have had their performance lowered from post-test to delayed post-test, which was not the case for children from first elementary level, as suggested by the graphics below:



Graphic 1: Mean level of right answers in pre, post and delayed post-test, pre-elementary level group.



Graphic 2: Mean level of right answers in pre, post and delayed post-test, first-elementary level group.

CONCLUSIONS AND FINAL REMARKS

Empirical evidences gathered here support the general theoretical hypothesis that symbolic representations are relevant in concept development in mathematics. More specifically, the combination of symbolic tools, including concrete-manipulative tools (like building bricks) as precursors of graphics showed to be effective in conceptual development in the conceptual field of additive structures. Nevertheless, representational aids are not completely effective by themselves, since previous development allows different outcomes for the same didactic tools, as shown by decreasing performance of pre-elementary students from post-test to eight-weeks later delayed post-test. As shown by data from study 2, younger students are those for whom the use of concrete-manipulative representational aids are specially relevant for concept development.

Good didactic effects of the combination of symbolic representations, as shown by both experimental groups when performances at pre and post-tests are compared, do not allow theoretical interpretations in terms of the developmental precedence of concrete, more primitive representations over abstract, more developed ones. Representations allow ways of thinking about information, relations and models; diverse availability of representations can be helpful in concept development, as shown by these data. It does not mean that abstract is based upon concrete because of a “natural” order of acquisition *concrete first, then abstract* (as criticized by Vygotsky, 2001). On the other hand, the interest of combining familiar, practical knowledge with incoming new and formal knowledge, in a *metaphorical* way (Lakoff & Núñez, 2000) receives empirical support by the data presented here.

References

Ainley, J. (2000). Exploring the transparence of graphs and graphing. In *Proc. 24th Conf. of the Int. Group for the Psychology of Mathematics Education* (vol. 2, pp.2-9). Hiroshima, Japan: PME.

Bills, L., Ainley, J. & Wilson, K. (2003). Particular and general in early symbolic manipulation. In *Proc. 27th Conf. of the Int. Group for the Psychology of Mathematics Education* (vol. 2, pp.105-112). Hawai, EUA: PME.

Bonotto, C. (2003). Investigating the mathematics incorporated in the real world as a starting point for mathematics classroom activities. In *Proc. 27th Conf. of the Int. Group for the Psychology of Mathematics Education* (vol. 2, pp.129-136). Hawai, EUA: PME.

Bell, A. & Janvier, C. (1981). The interpretation of graphs representing situations. *For the Learning of Mathematics*, vol.2, nº1, p. 34-42.

Da Rocha Falcão, J.T. (1995). A case study of algebraic scaffolding: from balance scale to algebraic notation. In *Proc. 19th Conf. of the Int. Group for the Psychology of Mathematics Education* (vol. 2, pp. 66-73), Recife, Brazil: PME.

Gravemeijer, K.P.E. (1994). *Development and realistic mathematics education*. Utrecht: CD B Press.

Guimarães, G. L. (2002). *Interpretando e construindo gráficos de barras*. Unpublished doctorate thesis, Graduate Program in Psychology, Universidade Federal de Pernambuco, Brasil.

Lakoff, G. & Núñez, R.E. (2000). Where mathematics comes from: how the embodied mind brings mathematics into being. New York, Basic Books.

Meira, L. (1998). Making sense of instructional devices: the emergence of transparence in mathematical activity, *Journal for Research in Mathematics Education*, 29, 2, p. 121-142.

Nunes, T. (1997). Systems of signs and mathematical reasoning. In: Nunes, T. & Bryant, P. (Eds.) *Learning and teaching mathematics: na international perspective.* London: Psychology Press.

Selva, A.C.V. (2003). Um experimento de ensino sobre a resolução de problemas de estrutura aditiva a partir de gráficos de barra. *Procs. of 26ª Reunião Anual da ANPED* – Santos, Brasil.

Selva, A.C.V., Da Rocha Falcão, J.T. (2002). Understanding and using graphs: a didactic sequence for pre-school children. In Proc. *26th Conf. of the Int. Group for the Psychology of Mathematics Education* (vol. 1, pp. 316). Norwich, UK: PME.

Vergnaud, G. (1997). The nature of mathematical concepts. In Nunes, T., Bryant, P. (1997) *Learning and teaching mathematics: an international perspective*. London, Psychology Press.

Vygotski, L.S. (2001). *A construção do pensamento e da linguagem*. São Paulo, Martins Fontes.

FROM THE EVERYDAY, THROUGH THE INAUTHENTIC, TO MATHEMATICS: REFLECTION ON THE PROCESS OF TEACHING FROM CONTEXTS

Godfrey Sethole

Tshwane University of Technology, South Africa

This paper highlights an attempt by two grade 8 teachers, Bulelwa and Kevin, to draw in the everyday in the teaching of mathematics. Though located in different South African contexts and settings, both teachers tend to enable their learners' access to mathematics by rendering the everyday inauthentic. I argue that inauthenticating the everyday is an unavoidable strategy by which the everyday considerations are silenced and not necessarily teachers' lack of empowerment, as it is sometimes claimed.

Within the mathematics education community, the relative merits and demerits of incorporating the everyday in mathematics remains one unresolved aspect. Whilst a number of studies cite the benefits of summoning the everyday in the teaching of mathematics (e.g., Skovsmose, 1994; Santos & Matos, 2002), others suggest that the everyday, when recruited into the mathematics, tends to conceal or draw some learners' attention away from the latter (Cooper & Dunne, 2000). Therefore, the challenge of balancing access to mathematics with recruitment of the everyday remains one of the central themes regarding the mathematics-everyday relationship.

The new South African education curriculum, Curriculum 2005 (C2005) has offered a definition of mathematics which makes the mathematics-everyday relationship important to reflect on. In its definition, C2005 advances an epistemological position of mathematics as a unique subject, "with its own symbols and language" and suggests the teaching of mathematics which incorporates the everyday. (DoE, 2000:16) This definition presents a practical challenge for teachers with regard to, on the one hand making mathematics accessible and, on the other, recruiting the everyday.

In this paper I draw on the experiences of two grade 8 teachers, Bulelwa and Kevin[*], who were trying to implement C2005. I particularly highlight and reflect on the way the two negotiated the challenge of summoning the everyday in order to enable their learners' access to the mathematics content. I will organize this discussion into four sections. In the second section I will provide the context in which the study took place. I will then, in the third and fourth sections respectively discuss the teachers' motivations for drawing in the everyday and the way in which they moved from these contexts to the mathematics. The final section will provide some reflections and

[*] Not the teachers' real names

implications for the inclusion of the everyday in mathematics. I start this discussion though, by reflecting on studies and opinions expressed in relation to the tension between the everyday and mathematics.

BLURRING THE MATHEMATICS-EVERYDAY BOUNDARY

Bernstein offers, at a more general level, analytical tools by which it is possible to infer that recruitment of the everyday into mathematics entails 'interfering' with the boundary between these two discourses. Using a theoretical construct of classification, Bernstein's (2000) argues that a discourse attains its uniqueness on the basis of the extent to which it manages to insulate itself from other discourses. He draws a distinction between weak and strong classification. "In the case of strong classification, each category has its unique identity, unique voice, its own specialized rules of internal relations. In the case of weak classification, we have less specialized discourses, less specialized identities, less specialized voices." On the basis of this theory, one may infer that that there exists an epistemological boundary between mathematics and the everyday and incorporation of the everyday into mathematics renders mathematics difficult to identify.

That mathematics looses its 'voice' in the face of the everyday contributes towards the suspicion that other researchers have on the value of including the everyday in mathematics. As Gellert, Jablonka and Keitel claim, keeping mathematics insulated from everyday realities is, for the majority of academic mathematicians, "the most powerful aspect of mathematics" (2001: 57). Pimm (1987) observes how the use of everyday English words *some* and *all* tend to confuse students' interpretation of mathematics statements. In particular, students regarded the two terms as contrastive rather than inclusive; that is, some entails not all. Rowlands and Carson (2002) maintain that mathematics transcends cultural practices. The substance of arguments which discourage summoning the everyday is that access towards mathematics is thus made difficult for learners. Other researchers (Verschaffel & De Corte, 1997; Nyabanyaba, 2002) appreciate the challenge brought by the inclusion of the everyday in mathematics, nonetheless, they cautiously argue in favour of blurring the boundary between the two.

Some studies have however highlighted the potential gains of drawing in the everyday in mathematics. For example, drawing from a socio-cultural perspective, Mukhopadhyay (1988: 135) outlines the way in which an activity on a popular doll Barbie helped learners realize that a real-life Barbie would be unnatural and unreal. Within the domain of ethnomathematics, Stillman (1995) has shown the way in which patterns made during the imprinting of tapa can be useful for teaching contents such as matrices vectors and sequences.

The substance of the studies cited above hinges on the relative merits or demerits of blurring the epistemological boundary between the everyday and mathematics. In the next section I turn to one aspect of this particular study; the context.

CONTEXT OF THE STUDY

This paper emerges out of a two-year national collaborative research project, the Learners' Perspective Study (South Africa), in which I participated as a novice researcher. The overall intention of this study was to gain insights into learners' perspectives about their experiences of mathematics lessons. Thirty eight Grade 8 mathematics lessons in three different schools were observed and videotaped. For each lesson we (1) focused on the interactions of a particular group of learners (2) collected their written work and (3) interviewed at the end of the lesson. In addition, teachers shared their impressions about each lesson by completing a questionnaire. At the end of the data collection process at each school, each teacher was interviewed about various aspects of their lessons. All the three schools are situated in Kwazulu-Natal province and are within a 30 km radius of the major city, Durban. Whilst differently resourced; all these schools had access to telephone services and electricity and were thus not a representative sample of schools in the province, the majority of whom have no access to these resourses ((Financial Mail, 02/1999 page 25). For purposes of this paper I focus on two of the three teachers whose lessons I was more exposed to and in whose interview I was able to participate.

Both Bulelwa and Kevin are regarded by their peers and learners in their respective schools as good teachers. Both hold senior positions in their respective schools and by South African standards they are amongst the better qualified teachers. In particular, Bulelwa holds a Barchelor's degree in Science with Mathematics and Statistics majors, a Higher Diploma in education and a B.Ed (Honours). She teaches in Umhlanga High school which is situated in Umlazi township, a single race residential area for blacks. She has been a teacher for over ten years, though she had only been teaching at this particular school for just over one year. All the staff members and learners at this school were black. We observed 9 of Bulelwa's mathematics lessons in a class of 38 learners. Kevin taught at Settlers High school situated in a predominantly white affluent suburban area. He held a four year teaching diploma and had been teaching at Settlers for twelve years. Kevin's class had twenty-eight learners, three of whom were non-white. We observed fourteen of the grade 8 mathematics lessons at Settlers.

Bulelwa and Kevin thus had different backgrounds both in terms of their educational tours, race and teaching settings. In the following section I focus on the way in which the two rationalized the use of the everyday.

INTRODUCING THE EVERYDAY

The mathematics content for all the nine lessons that we observed at Umhlanga was number patterns. It was only in five of these lessons that Bulelwa drew in the everyday. She used worksheets as teaching resourses. The everyday themes that she drew in were *Ancient Societies' practices* (lessons 1 and 2), *AIDS* (lessons 7 and 8) and *Flowers* (lesson 9). For purposes of this paper, I will only focus on Bulelwa's drawing in of the theme AIDS. Of the fifteen lessons we observed at Settlers, it was five in which the everyday was drawn in. Kevin, unlike Bulelwa, did not plan a lesson around a particular theme, instead, he used word problems. Some of these word problems drew from the everyday realities. For this paper I will focus on the experiences of the first lesson.

Bulelwa's rationale for including the everyday: The first paragraph in the worksheet that Bulelwa used for lesson 7 provided a rational for the incorporation of the everyday to the students. It particularly made reference to the relationship between 'mathematics and the natural environment'. The rationale for blurring the boundary between mathematics and the everyday was rooted in the utilitarian value of mathematics for everyday experiences. In other words, Bulelwa presented mathematics as a tool to engage the everyday.

Mathematicians have studied number patterns for many years. It was discovered that there are links between mathematics and our natural environment and sometimes events occurring in our societies. For this reason an understanding of algebra is central to using mathematics in setting up models of real life situations. (Worksheet 5, lesson 7 & 8)

In setting up the scene for lesson 7, Bulelwa (T) reiterated the significance of mathematics with specific reference to engaging or solving the escalation of AIDS. She told learners,

> **7 T**:… we are still looking at the number patterns but now we are trying to relate what is happening in mathematics classes to real life situations. We are actually trying to see whether what you learn in mathematics classes is actually relevant… Do they help mathematicians to figure out what is happening in real life? So I picked that one where mathematicians are trying to use mathematics to solve real life problems. Problem which is actually epidemic… which is big for South Africa. We are having so many people dying of AIDS.

The AIDS context had an emotional appeal to this classroom community because of the proximity of the school to an area where an AIDS activist Gugu Dlamini was stoned to death for declaring her HIV-positive status.

Kevin's rationale for the everyday: In his written reflection about the first lesson, Kevin indicated that his use of word problems was motivated by a desire to illustrate connections between ordinary English and mathematics equations. On the basis of the worksheet he used, 'ordinary English' referred to tasks whose wording referenced both the everyday contexts like the price of chocolate and mathematics context like the area of a rectangle. In setting the scene for the lesson the first lesson, he (K) said to the learners:

> **32: K**: ….. Okay what we're doing now we're doing equations and often you want to know how you can use something in, in maths, in real life. So can you remember the problems I was giving you; things like three CDs cost you three hundred and sixty Rand what does one cost? Okay that's a real life situation you use in equations.

One purpose of the lesson, according to Kevin, was to illustrate the value of 'equations' in real life settings. He, like Bulelwa, argued that there was a place for mathematics in the learners' lived experiences. To illustrate this point, he made up an example by making use of one of the learners, Kelley. He asked for Kelley and her sister's age. On establishing that Kelly was thirteen and her sister fifteen, he phrased this question.

> **38 K:** Okay if we were to say we didn't know you're thirteen and she is fifteen. Okay Kelly's age and her sister's age add up to twenty-eight and if Kelly is two years younger than her sister how old is Kelly? Okay something like that. Okay, out of your minds goes thirteen and fifteen jot down for me on how you would work that out. (At this stage learners take their books out and begin working the sum out).

Bulelwa and Kevin thus drew in contexts which were qualitatively different. The AIDS context was not benign; it had the potential to spark different types of non-mathematical arguments and discussions. Kevin, on the other hand, used a context which was not as emotional. In addition, the visibility of the context in Bulelwa's and Kevin's class differed. Bulelwa followed up a theme of AIDS over two lessons whilst Kevin mentioned a range of contexts in one lesson. Therefore, Kevin's learners spent relatively less time reflecting on the given contexts.

Despite these differences, both Kevin and Bulelwa saw the value of blurring the boundary between the mathematics and the everyday. They made public to their learners, the potential of a dialogue between mathematics and the everyday. Mathematics, as they presented the subject, stood in an "open relationship" with the other realities (Bernstein, 2001:10). Having blurred the mathematics-everyday boundary, the next section focuses on how the two attempt to make mathematics visible.

FROM THE EVERYDAY TO MATHEMATICS

Bulelwa handed out a worksheet, in lesson 7, which showed two tables (Figure 1). The first table depicts the rate of increase in the world population (in millions) and the second shows world increase in the number of AIDS sufferers (in millions).

The first question required learners to 'describe the pattern of population increase every 40 years' and the second required them to 'describe the pattern of increasing AIDS sufferers'. Even though world population trends and the rate of increase of AIDS sufferers are real world phenomena, the figures in the tables are not. The use of these figures renders these contexts inauthentic. Yet, the use of these figures also enable learners to notice a 'describable' pattern. A pattern which fit the exponential functions, $f(n) = 3.2^{\frac{n-1960}{40}}$ (n≥1960) and $16.7x2^{n-1997}$ (n≥1997) for the world population growth and rate of increase in AIDS sufferers respectively. The use of genuine figures would generate a messy data which learners may have found difficult to describe.

Year	1960	2000	2040	2080	2120
World population growth	3 000	6 000	12 000		

Year	1997	1998	1999	2000	2001
World increase in the number of AIDS sufferers	16.7	33.4	66.8		

Figure 1: Two tables in Bulelwa's worksheet

The first task from the worksheet that Kevin asked learners to engage made reference to children's ages. It stated that: "John's age is *p* years. Write down in terms of *p*, Sue's age if Sue is 10 years older than John". Whilst referencing names of real people; this context provides age as *p*, not a useful indicator of a person's age in real life settings. The use of *p* thus renders the task inauthentic from an everyday's perspective. However, the use of *p* also enables the introduction of algebraic expressions and then equations. Secondly, Kevin's use of Kelley (referred to earlier) as a person whose age was to be calculated exemplified the inauthenticity of the activity. Kelley had just highlighted her age as thirteen, it was therefore not meaningful, from an everyday perspective, to embark on a calculation of her age.

In both cases, having recruited the everyday, Bulelwa and Kevin silence it by pruning it off some its attributes. Pruned off its real life attributes, the everyday becomes inauthentic, a "strange real world" which is no more than a see-through into the mathematics content (Cooper & Dunne, 2000). So, access to mathematics is achieved through modifying and thus inauthenticating the everyday contexts.

CONCLUSION

This paper provides a practical challenge faced by two teachers of embracing the everyday whilst at the same time enabling access to mathematics. In embracing the everyday, Bulelwa could refer to AIDS as an epidemic as a result of which many people are dying. Non-mathematical discussions about the "traditional practices" and sexual habits that promote the transmission of AIDS could find legitimacy within this context (Sethole, Adler & Vithal, 2002). Similarly, Kevin could reference non-mathematical aspects about Kelley; her age and her sister's. However, engagement in these discussions conceals distinctions between mathematics and the everyday.

There has been much and substantiated criticisms regarding inauthenticating of the contexts by Bulelwa and Kevin. At a moral level, Bulelwa's figures about AIDS sufferers can be regarded as misleading. Kevin's expectation that learners should calculate Kelley's age, which she publicly announced in the classroom may seem senseless. My view is that expecting access to the formal structure of mathematics through the everyday (as does C2005) needs to be seriously reflected on. However noble, the possible limitations and pedagogic challenges of this expectation needs highlighting. How do teachers move away from the everyday as an object of reflection to the everyday as a see-through towards the mathematics? Is it possible that teachers can enable access to mathematics without rendering the everyday as see-throughs?

Inauthenticating the everyday in order to access mathematics seems to me more a function of wishing to access mathematics through meaningful contexts than the teachers' ability or inability to recruit the everyday in their teaching.

References

Bernstein, B. (2000). *Pedagogic Codes, Class and Control*. London: Routledge.

Cooper, B. & Dunne, M. (2000). *Assessing Children's Mathematical Knowledge: Social Class, Sex and Problem-Solving*. Open University Press: Buckingham.

Department of Education (2000). National Curriculum Statement - Mathematics.

Freudenthal, H. (1991). *Mathematics as an Educational Task*. The Netherlands: Kluwer Academic Publishers.

Gellert, U., Jablonka, E., & Keitel., C. (2001). Mathematical Literacy and Common Sense in Mathematics Education In In B. Atweh., H. Forgasz., & B. Nebres (Eds.) *Sociocultural Research on Mathematics Education – An International Perspective* (pp. 57 – 31). London: Lawrence Erlbaum Associates.

Mukhopadhay, S. (1998). When Barbie goes to classrooms: Mathematics in creating a social discourse. In C. Keitel (Ed.), *Social Justice and Mathematics Education (pp. 150 – 161)*. Berlin: Freie Universitat Berlin.

Nyabanyaba, T. (2002). *Unpublished PHD Thesis*. University of the Witwatersrand.

Pimm, D. (1987). *Mathematically Speaking*. London: Routledge.

Rowlands, S & Carson, R.(2002). Where would formal, academic mathematics stand in a curriculum informed by Ethnomathematics? A Critical Review of Ethnomathematics, *Educational Studies in Mathematics* 50, 79 – 102.

Santos, M. & Matos, J. (2002). Thinking about Mathematical learning with Cabo Verde Ardinas. In G de Abreu, A.J. Bishop and N.C. Presmeg (Eds), *Transitions Between Contexts of Mathematical Practices*, 81-122. London: Kluwer Publishers.

Sethole, G., Adler, J., & Vithal, R.(2002). When AIDS goes to a Mathematics Classroom: Is it a hindrance or a source for revising the mathematics classroom culture? In *ICMI Comparative Study Conference*, Hong Kong, pp. 57 – 68.

Skovsmose, O. (1994). *Towards a Philosophy of Critical Mathematics Education.* The Nertherlands: Kluwer Academic Publishers.

Stillman, G. A. (1995). Ethnomathematics – A source of investigations in later years of secondary school. *Teaching Mathematics and its Applications,* 14(1), 15 – 18.

PERSONAL EXPERIENCES AND BELIEFS IN EARLY PROBABILISTIC REASONING: IMPLICATIONS FOR RESEARCH

Sashi Sharma

The University of Waikato

Data reported in this paper is part of a larger study which explored form five (14 to 16 year olds) students' ideas in probability and statistics. This paper presents and discusses the ways in which students made sense of task involving independence construct obtained from the individual interviews. The findings revealed that many of the students used strategies based on beliefs, prior experiences (everyday and school) and intuitive strategies such as representativeness. While more students showed competence on coin tossing question they were less competent on the birth order question. This could be due to contextual or linguistic problems. The paper concludes by suggesting some implications for further research.

INTRODUCTION

Over the past years, there has been a movement in many countries to include probability and statistics at every level in the mathematics curricula. In western countries such as Australia (Australian Education Council, 1991), New Zealand (Ministry of Education, 1992) and the United States (Shaughnessy & Zawojewski, 1999) these developments are reflected in official documents and in materials produced for teachers. In line with these moves, Fiji has also produced a new mathematics prescription at the primary level that gives more emphasis to statistics at this level (Fijian Ministry of Education, 1994). Clearly the emphasis is on producing intelligent citizens who can reason with statistical ideas and make sense of statistical information.

Despite its decade-long presence in mathematics curricular, statistics is an area still in its infancy. Research shows that many students find probability difficult to learn and understand in both formal and everyday contexts (Barnes, 1998; Fischbein and Schnarch, 1997; Fischbein, Nello, & Marino, 1991; Shaughnessy & Zawojewski (1999). We need to better understand how learning and understanding may be influenced by ideas and intuitions developed in early years.

Concerns about the importance of statistics in everyday life and a lack of research in this area determined the focus of my study. Overall, the study was designed to investigate the ideas form five students have about statistics and probability. This paper presents and discusses data obtained from the probability task involving independence construct. Prior to discussing the details of my own research, I will briefly mention the theoretical framework and some related literature.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 177-184. Melbourne: PME.

THEORETICAL FRAMEWORK

Much recent research suggests that socio-cultural theories combined with elements of constructivist theory provide a useful model of how students learn mathematics. Constructivist theory in its various forms, is based on a generally agreed principle that learners actively construct ways of knowing as they strive to reconcile present experiences with already existing knowledge (von Glasersfeld, 1993). Students are no longer viewed as passive absorbers of mathematical knowledge conveyed by adults; rather they are considered to construct their own meanings actively by reformulating the new information or restructuring their prior knowledge (Cobb, 1994). This active construction process may result in alternative conceptions as well as the students learning the concepts intended by the teacher.

Another notion of constructivism derives its origins from the work of socio-cultural theorists such as Vygotsky (1978) and Lave (1991) who suggest that learning should be thought of more as the product of a social process and less as an individual activity. There is strong emphasis on social interactions, language, experience, catering for cultural diversity and contexts for learning in the learning process rather than cognitive ability only. Mevarech and Kramarsky (1997) claim that the extensive exposure of our students to statistics outside schools may create a unique situation where students enter the mathematics class with considerable amount of knowledge. This research was therefore designed to identify students' ideas, and to examine how they construct them.

PREVIOUS RESEARCH ON PROBABILITY

A number of research studies from different theoretical perspectives seem to show that students tend to have intuitions which impede their learning of probability concepts. Some prevalent ways of thinking which inhibit the learning of probability include the following:

- Representativeness: According to this strategy students make decisions about the likelihood of an event based upon how similar the event is to the population from which it is drawn or how similar the event is to the process by which the outcome is generated (Tversky & Kahneman, 1974). For instance, a long string of heads does not appear to be representative of the random process of flipping a coin, and so those who are employing representativeness would expect tails to be more likely on subsequent tosses until things evened out. Of course, the belief violates independence construct which is a fundamental property of true random sampling.
- Equiprobability bias: Students who use this bias tend to assume that random events are equiprobable by nature. Hence, the chances of getting different outcomes, for instance, three fives or one five on three rolls of a die are viewed as equally likely events (Lecoutre, 1992).
- Outcome orientation: Falk and Konold (1992) point out that the fundamental difference between formal and informal views of probability concerns the

perceived objective in reasoning about uncertainty. Formal probability is mostly concerned with deriving measures of uncertainty, or answering the question How often will event A occur in the long run? On the other hand what most people want is to predict what will occur in a single instance to answer the question Will A occur or not? Thus, the goal in dealing with uncertainty is to predict the outcome of a single next trial rather than to estimate what is likely to occur at the series of events. Konold (1989) refers to this perspective as the outcome approach.

- Beliefs: Research shows that a number of children think that their results depend on a force, beyond their control, which determines the eventual outcome of an event. Sometimes this force is God or some other force such as wind, other times wishing or pleasing (Amir and Williams, 1994;; Truran, 1994).
- Human Control: Research designed to explore children's ability to generalise the behaviour of random generators such as dice and spinners show that a number of children think that their results depend on how one throws or handles these different devices (Shaughnessy & Zawojewski, 1999; Truran, 1994).

Whether one explains the reasoning in probabilistic thinking by using naive strategies such as representativeness and equiprobability or by deterministic belief systems such outcomes can be controlled, the fact remains that students seem very susceptible to using these types of judgements and in some sense all of these general claims seem to be valid. Different problems address different pieces of this knowledge.

OVERVIEW OF THE STUDY

Sample. The study took place in a co-educational private secondary school in Fiji. The class consisted of 29 students aged 14 to 16 years. According to the teacher, none of the students in the sample had previously received any *in-depth* instruction in statistics. Fourteen students were chosen from the class, the criteria for selection included gender and achievement.

Task. The baby (Item 1A) and the coin questions (Item 1B) were used to explore students' understanding of the independence concept and responses demanded both numerical and qualitative descriptions.

Item 1A: The baby problem

The Singh family is expecting the birth of their fifth child. The first four children were girls. What is the probability that the fifth child will be a boy? Please explain your answer.

Item 1B: Coin problem

(i) If I toss this coin 20 times, what do you expect will occur?

(ii) Suppose that the first four tosses have been heads. That's four heads and no tails so far. What do you now expect from the next 16 tosses? Why do you think so?

Interviews. Each student was interviewed individually by myself in a room away from the rest of the class. The interviews were tape recorded for analysis. Each interview lasted about 40 to 50 minutes.

RESULTS

This section describes the patterns of thinking identified in response to the two questions. Extracts from typical individual interviews are used for illustrative purposes. Throughout the discussion, I is used for the interviewer and Sn for the *nth* student.

A few students in my study believed in the independence of events, that is, that each successive trial is independent of the previous trials. For example, Student 12 was able to use the independence concept for both questions. With respect to the baby problem, she explained that since the fifth child could be a boy or a girl, the chance of getting a boy or a girl was 50%. For item 1B, she explained that since getting heads or tails were equally likely, she would expect about 8 heads and 8 tails.

Prior beliefs and experiences played an important role in the thinking of many students. On the baby problem, four students related to their religious beliefs and experiences. The students thought that one can not make any predictions because the sex of the baby depends on God. The religious aspect is revealed in the response of Student 17 who explained:

> We can not say that Mrs Singh is going to give birth to a boy or a girl because whatever God gives, you have to accept it.

It must be noted that the birth order problem is equivalent to the coin question (Item 1B). However, when a different context is introduced, students are comfortable thinking deterministically. For instance, Student 2 was considered statistical on Item 1B but she used the religious perspective on Item 1A. Student 3 tended to draw upon experiences gained from other subjects. She explained that the outcome was managed by the parents and tried to relate her previous knowledge of biology in responding to Item 1A.

Two students in the present study thought that the results depend on individual control (Item 1B). The students said that people can control the outcome by throwing it in a certain direction or throwing it fast. This is reflected in the following interview:

S20: Eh ... it will have 5 heads and 15 tails.

I: Why do you think that there will be 5 heads and 15 tails?

S20: Eh ... because when you throw each time it comes head or tail

I: But you said more tails. Why do you think you will get more tails?

S20: I will throw the coin in one direction so I will get HHH, when I change in another direction I will get all tails.It depends on how fast you throw and how fast the coin swings.

Some students based their reasoning on inappropriate rules and intuitions such as representativenes. Two students applied the n(E)/n(s) rule inappropriately. For example, with respect to Item 1A, student 25 said,

> Chance will be one upon five. Four girls and one to be born; don't know whether it will be a boy or a girl. Like in dice there is one side and the total is six but one is the chance eh.

Although the students had learnt finding probabilities using sample space, they applied this rule inappropriately. The data revealed that while two students used the representativeness strategy for the baby problem, six used it for the coin problem. Students using the representativeness strategy on Item 1B thought that there would be a balancing out so they would expect more tails. Even repeated probing did not produce any probabilistic thinking.

One student drew upon the equiprobability bias on the coin problem. The student reasoned that if one tosses a coin 20 times, one expects to get 10 heads and 10 tails because one does not know which side will fall. Hence equal chance should be given to both events. In three cases, students could not explain their responses. For instance, Student 9 said that there will be equal number of heads and tails but could not explain her reasoning.

DISCUSSION

This section first discusses the results in a broader context. Then limitations of the study are discussed and suggestions made for directions for further research.

Probability: A broader Context

The results show students think that outcomes on random generators such as coins (Item 1B) can be controlled by individuals. The general belief is that results depend on how one throws or handles these different devices. The finding concurs with the results of studies by Amir and Williams (1994), Shaughnessy and Zawojewski (1999) and Truran (1994). It must be noted that the students using the control strategy in this study were boys. One explanation for this could be that boys are more likely to play sports and chance games that involve flipping coins and rolling dice to start these games.

Although this study provides evidence that reliance upon control assumption can result in biased, non-statistical responses, in some cases this strategy may provide useful information for other purposes. For example, student 20's knowledge of physics may have been reasonable. The responses raise further questions. Is there a weakness in the wording of this question in that it is completely open-ended and does not focus the students to draw on other relevant knowledge? Perhaps, including cues such as "fair" in the item would have aided in the interpretation of this question. Are the students aware of the differences in probabilistic reasoning compared with reasoning in other contexts? Although in probability theory we work with an idealised die or coin, deterministic physical laws govern what happens during these trials. It does not make sense to say that the coin has a probability of one-half to be heads because the outcome can be completely determined by the manner in which it is thrown. Additionally, a good Bayesian statistician might not give 50/50 heads/tails as the likely outcome after a run of heads with a particular coin. Such a person might start looking at prior experience to inform a particular situation.

With respect to students' beliefs, experiences and learning, it is evident that other researchers have encountered similar factors. Amir and Williams (1994) note that children's reasoning appeared to be related to their religious, superstitious and causal beliefs. In some respects, the findings of the present investigation go beyond those discussed above. The findings demonstrate how students' other school experiences also influence their construction of statistical ideas. At times the in-school experiences appear to have had a negative effect on the students. An example of negative effect that arose from other school experiences was the student who was deeply convinced that the father decides the sex of the baby. Gal (1998) suggests that such responses constitute what students know about the world, they cannot be judged as inappropriate until a students' assumptions about the context of the data are fully explored. For instance, the students confronted with the problem concerning birth order (Item 1A) may not know which model is appropriate. The statistical model implies that both events are equally probable but the student does not know whether biologically there is some tendency for families to have offspring of a particular gender or the end result of boys to girls should be equivalent. We know now that giving birth to boys and girls is **not** random but affected by things like times of conception and genetic dispositions of the parents. Although the outcomes are independent across births, there are rare occasions of identical or fraternal twins and triplets. In short it is not possible to determine the nature of the error unambiguously on the basis of the students' response.

In the study described here, background knowledge, that is often invoked to support a student's mathematical understanding, is getting in the way of efficient problem solving. Given how statistics is often taught through examples drawn from "real life" teachers need to exercise care in ensuring that this intended support apparatus is not counterproductive. This is particularly important in light of current curricula calls for pervasive use of contexts (Meyer, Dekker, & Querelle, 2001; Ministry of Education, 1992) and research showing the effects of contexts on student' ability to solve open ended tasks (Cooper & Dunne, 1997; Sullivan, Zevenbergen, & Mousley, 2002). Conversely, in spite of the importance of relating classroom mathematics to the real world, the results of my research indicate that students frequently fail to connect the mathematics they learn at school with situations in which it is needed. For instance, Student 2 used statistical principles on Item 1B whereas on Item 1 she refereed to her religious convictions. The findings support claims made by Lave (1991) that learning for students is situation specific and that connecting students' everyday contexts to academic mathematics is not easy.

Limitations

It must be acknowledged that the open-ended nature of the tasks and the lack of guidance given to students regarding what was required of them certainly influenced how students explained their understanding. The students may not have been particularly interested in these types of questions as they are not used to having to describe their reasoning in the classroom. Some students in this sample clearly had difficulty explaining explicitly about their thinking. Another reason could be that such questions do not appear in external examinations. Although the study provides some valuable insights into the kind of thinking that high school students use, the conclusions cannot claim generality because of a small sample. Additionally, the study was qualitative in emphasis and the results rely heavily on my skills to collect information from students. Some directions for future research are implied by the limitations of this study.

Implications for Further Research

One direction for further research could be to replicate the present study and include a larger sample of students from different ethnic backgrounds. Secondly, this small scale investigation into identifying and describing students' reasoning from constructivism has opened up possibilities to do further research at a macro-level on students' thinking and to develop explicit categories for responses. Such research would validate the framework of response levels described in literature (Watson & Callingham, 2003) and raise more awareness of the levels of thinking that need to be considered when planning instruction and developing students' statistical thinking. The place of statistics has changed in the revised mathematics prescription. Statistics appears for the first time at all grade levels (Fijian Ministry of Education, Women, Culture, Science and Technology, 1994). Like the secondary school students, primary school students are likely to resort to non-statistical or deterministic explanations. Research efforts at this level are crucial in order to inform teachers, teacher educators and curriculum writers.

References

Amir, G. & Williams, J. (1994). The influence of children's culture on their probabilistic thinking, In J. da Ponte & J. Matos (Eds.), *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education, 29 July-3 August, 1994*, Vol. II (pp. 24-31). Lisbon, Portugal, The Program Committee of the 18th PME Conference.

Australian Education Council. (1991). *A National Statement on Mathematics for Australian Schools*. Canberra, Curriculum Corporation.

Barnes, M. (1998). Dealing with misconceptions about probability. *Australian Mathematics Teacher,* 54(10), 17-20.

Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematical development' *Educational Researcher,* 23(7), 13-20.

Cooper, B. & Dunne, M. (1999). *Assessing Children's Mathematical Ability*. Buckingham: Open University Press.

Falk, R. & Konold, C. (1992). The psychology of learning probability. In F. Sheldon and G. Sheldon (Eds.), *Statistics for the Twenty-First Century* (pp. 151-164). Washington, Mathematical Association of America.

Fijian Ministry of Education, Women, Culture, Science and Technology. (1994). *Primary Mathematics Prescriptions.* Suva: Curriculum Development Unit.

Fischbein, E. & Schnarch, D. (1997). The evolution with age of probabilistic, intuitively based misconceptions. *Journal for Research in Mathematics Education,* 28, 96-105.

Konold, C. (1989). An outbreak of belief in independence? In C. Maher, G. Goldin & B. Davis (Eds.), *The Proceedings of the eleventh Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* Vol II (pp. 203-209). Rutgers, Rutgers University Press.

Lave, J. (1991). Cognition in practice; Mind, mathematics and culture in everyday life. New York: Cambridge University Press.

Lecoutre, M. (1992). Cognitive models and problem spaces in purely random situations. *Educational Studies in Mathematics*, 23, 557-568.

Mevarech, Z. R. & Kramarsky, B. (1997). From verbal descriptions to graphic representations: Stability and change in students' alternative conceptions. *Educational Studies in Mathematics,* 32, 229-263.

Meyer, M. R, Dekker, T. & Querelle, N. (2001). Context in mathematics curricula, *Mathematics Teaching in the Middle School,* 6(9), 522-527.

Ministry of Education. (1992). *Mathematics in the New Zealand Curriculum.* Wellington: Ministry of Education.

Shaughnessy, J.M. & Zawojewski, J. S. (1999). Secondary students' performance on data and chance in the 1996 NAEP, *The Mathematics Teacher,* 92(8), 713-718.

Sullivan, P., Zevenbergen, R., & Mousley, J. (2002). Contexts in Mathematics teaching: snakes or ladders? In B. Barton, K. C. Irwin, M. Pfannncuch and M. Thomas (Eds.), *Mathematics Education in the South Pacific.* Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia (pp 649-656). Auckland.

Truran, K. M. (1994). Children's understandings of random generators in C. Beesey & D. Rasmussen (Eds.), *Mathematics Without Limits, Proceedings of the 31st Annual Conference of the Mathematical Association of Victoria* (pp. 356-362). Melbourne, Mathematics Association of Victoria.

Tversky, A. & Kahneman, D. (1974). Judgement under uncertainty: Heuristics and biases. *Science,* 185, 1124-1131.

Von Glasersfeld, E. (1993). Questions and answers about radical constructivism. In K. Tobin (Ed.), *The practice of constructivism in science education* (pp. 24-38). Washington, American Association for the Advancement of Science.

Vygotsky, L. (1978). Mind in society: The development of higher psychological processes. Cambridge: Harvard University Press.

Watson, J. M. & Callingham R. (2003). Statistical literacy: A complex hierarchical construct. *Statistics Education Research Journal,* 2(2), 3-46.

ASSIMILATING INNOVATIVE LEARNING/TEACHING APPROACHES INTO TEACHER EDUCATION - WHY IS IT SO DIFFICULT?

Atara Shriki — Oranim Academic College of Education

Ilana Lavy — Emek Yezreel Academic College

Research shows that most training programs aimed at qualifying pre-service teachers (PST) have a slight influence on their beliefs regarding learning and teaching. In order to understand the reasons of this phenomenon we asked our PST to write a portfolio while experiencing learning via a computerized-project-based-learning (CPBL) approach. Analysis of the PST's portfolio raised two main possible reasons for the stagnation of their beliefs: a lack of sufficient success in achieving expected goals, and an inadequate synchronization between the experience of innovative approaches and their implementation. In this paper we present a case study of one of our PST written reflections, in which those two issues are addressed.

INTRODUCTION

During the last two decades there have been intensive calls for implementing reforms in mathematics education (e.g., NCTM's standards, 2000). No doubt teachers should be the ones that put the innovative approaches into practice. Unfortunately real modifications are not as widespread as was expected. Various explanations can be suggested in order to clarify this phenomenon of "stagnation". One of the explanations might be related to what Desforges (1995) had found in his review of literature: teachers are not reflective; they are satisfied with their practices and do not tend to question educational processes. Moreover, they often disregard data that is inconsistent with their beliefs and practice and tend to avoid new experiences. Instead, they prefer to stick to only those practices that match their existing system of believes. Desforges (ibid) findings regarding the characteristics of in-service teachers raise two main questions: the first one concerns the underlying reasons of such behavior, and the second one relates to implication on teacher education. Since we mainly work with pre-service teachers (PST), we were curious about the latter question. It seems that the calls for reform disregard the difficulties experienced teachers might have while trying to adjust themselves to new settings. We were wondering whether experiencing innovative approaches while PST are in their process of training, constructing their pedagogical content knowledge, might raise their ability to adjust to innovative situations.

In this paper we describe our experience with PST of mathematics, in which we attempted to demonstrate the benefits of inquiry-based learning as an example of innovative approach. Though we succeeded in exhibiting some of the advantages of

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 185-192. Melbourne: PME.

that method, still we had to confront several obstacles. As follows we explain some of their sources and nature.

BACKGROUND

Our study examined difficulties PST had in adjusting to an inquiry-based environment aimed at introducing some innovative didactical approaches for teaching junior high-school mathematics. In this paper we discuss some of the PST's dilemmas that might be attributed to the need for creating a new system of beliefs, which is not consistent with the existed one. The theoretical framework of this paper focuses on the meaning of 'system of beliefs', and on social and sociomathematical norms, which are among the constituents of such a system.

System of beliefs. Beliefs are basic assumptions regarding perceptions and attitudes towards a certain reality. A System of beliefs does not require external approval (Tillema, 1998). The influence of beliefs is strongest on the meanings which people attribute to occurrences, and on activities they choose to carry out. PST hold beliefs regarding various aspects relating to teaching and learning, among them: their teaching role, students' learning processes, curriculum suitability, and so forth (Van-Dijk, 1998). Their beliefs reflect their values in terms of what is "desirable". As a result of thousands of hours in an "apprenticeship of observation", which inspire school students' perception regarding teaching and learning (Lortie, 1975), PST begin their training with explicit ideas regarding relevant issues (Tilema, 1995). For example, many PST believe that teachers supply knowledge to their students, and learning means memorizing the contents (Richardson, 1996). Their memories of themselves as learners influence their expectations of their future students as well as their views regarding "proper" teaching strategies. The image they possess regarding "good teaching" relates to the kind of teacher they see themselves becoming. As a consequence PST tend to exhibit conservative teaching, replicating their own teachers. Research (e.g. Kagan, 1992) suggests that PST's personal beliefs and images are not affected by their training practice and generally remain unchanged. They tend to utilize the information they are exposed to during their training mainly to strengthen their existing beliefs and perceptions. That means that the contents that are being presented in teacher education programs are subject to interpretations according to PST's pre-existing beliefs (Tillema, 1998). Those interpretations also affect their performance in class (Kagan, 1992), since they rely on their own subjective theories of teaching or on what they believe will work in class. Moreover, many PST expect their educators to tell them explicitly how to teach. Some expect to learn from their own experience. Others believe that teaching is an activity that every one can do and there is little need for training (Calderhead, 1992).

Social and sociomathemtical norms. Norms are among the constituents of system of beliefs. The theme of classroom norms has been largely discussed in recent years. Yackel and Cobb (1996) distinguished between general classroom social norms (for example: the need to explain or justify) and norms that are specific to students' mathematical activities, termed as sociomathematical norms (for example: what counts as mathematically efficient, mathematically sophisticated, mathematically elegant, acceptable mathematical explanation and justification). The teacher's and the students' beliefs serve as key factors for negotiating classroom norms. The teacher-students verbal interactions provide the opportunity to negotiate the sociomathematical norms, which are continually regenerated and modified, and might differ substantially from one classroom to another.

METHODOLOGY

The research data included: (a) Transcripts of videotapes of all the class sessions; (b) Two written questionnaires; (c) Students' portfolios that included a detailed description of the various phases of the project and reflection on the process; (d) Informal interviews. During the class sessions the students raised their questions and doubts, asked for their classmates' advice, and presented their works.

Looking for phenomenological categories in the PST's portfolios, we applied inductive analysis (Goetz & Lecompte, 1984). We studied all the students' utterances through the lenses that concerned their perception regarding various issues relating to teaching and learning.

THE STUDY

The Context. In this paper we present a case study of one PST who participated in an annual course named "Didactical foundations of mathematics instruction". This course focuses on theories and didactical methods implemented in teaching and learning geometry and algebra in junior high-school. One of the main didactical methods discussed in this course is learning via Project-Based-Learning (PBL). PBL is a teaching and learning strategy that involves students in complex activities, and enables them to engage in exploring important and meaningful questions through a continuous process of investigation and collaboration. This process includes posing problems, asking questions, making predictions, designing investigations, collecting and analyzing data, sharing ideas, and so on (Krajcik, Czerniak and Berger, 1999). We termed the approach used in the current study as Computerized-Project-Based-Learning (CPBL) since it rested heavily on the use of computer software. Integrating computer software into the setting of PBL has many benefits. It enables the students to make a lot of experiments, observe stability/instability of phenomena, state and verify/refute conjectures easily and quickly, and so on (Marrades & Gutierrez, 2000).

The Subjects. 25 college students (8 male and 17 female students) in their third year of studying towards a B.A. degree in mathematics education participated in the research. The discussed course was the first didactical course they had taken.

In parallel the PST began their practice in school teaching. In the time they were working on the CPBL they mainly observed experienced teachers.

The CPBL. In order to clarify to the PST what we mean by CPBL and what its phases are, we exhibited a ready-made project which was based on Morgan's theorem (Watanabe, Hanson & Nowosielski, 1996). The PST had experienced CPBL, which included the following phases (Lavy & Shriki, 2003): (1) Solving a given geometrical problem, which served as a starting point for the project; (2) Using the "what if not?" strategy (Brown & Walters, 1990) for creating various new problem situations on the basis of the given problem; (3) Choosing one of the new problem situations and posing as many relevant questions as possible; (4) Concentrating on one of the posed questions and looking for suitable strategies in order to solve it; (5) Raising assumptions and verifying/refuting them; (6) Generalizing findings and drawing conclusions; (7) Repeating stages 3-6, up to the point in which the student decided that the project has been exhausted.

Experiencing the processes that are involved in CPBL enabled most of the PST to realize the benefits the learners gain from working on inquiry assignments (Lavy & Shriki, 2003). Among them: developing mathematical qualifications; increasing self-confidence in the mathematics competence; learning in an exiting and challenging environment. However, we had difficulties in trying to bring the PST to internalize the importance of integrating CPBL into their future classes. Through the reflective process of the PST we tried to find explanations to those difficulties. In this paper we bring parts of the reflection of one representative student. This student was chosen since her expressed beliefs were similar to those of the majority, yet she was more expressive then the others.

RESULTS AND DISCUSSION

In the following section we describe the case study of Ruth, who is a typical student from our class of PST. Ruth's reflection enables learning about the characteristics of the existing system of beliefs PST hold, and the characteristics of new generated beliefs that emerge within an environment that encourages inquiry activities.

Ruth's reflection shows that she experienced the process of learning in two modes in a sequential manner: first she experienced the learning processes as a student and then as a future teacher. In this section we relate to her system of beliefs, and use the abbreviations "eb" and "nb" for designating "existing belief" and "new belief", in accordance. We used "r_1"and "r_2" in order to designate the "repeat" of referring to a certain belief. In addition, we numbered each belief.

The Case of Ruth. Ruth is considered to be an average student; nevertheless her contribution to the class discussions was significant since she often tended to ask for further clarifications to issues that were raised by the students and the teacher. At the beginning of the process Ruth was motivated by her wish to discover a new mathematical regularity, and she kept on saying: "*I want to be like Morgan, I want to discover a new regularity*". At the initial phases of the project Ruth decided to focus on a problem situation in which she changed two of the original attributes. After a period of time, during which she kept on looking for regularities, she had managed to find only marginal discoveries. As follows are some of her reflections during the various phases of her work.

By the end of the first class session in which we explained and demonstrated the components of a project, Ruth wrote:

> At the beginning I asked myself whether there is any connection between what we ought to teach in school and what we have to do in this project. No one at school will ever let us teach in that manner **[eb1]**. Schools do not welcome such an approach **[eb2]**. So at the beginning I was not enthusiastic at all, until I heard about Morgan and his discovery. Only then I felt like I really want to do that - to explore and discover **[nb1]**.

Ruth began working on the project with great enthusiasm. After the second phase of the project she wrote:

> After I wrote the list of various new problem situations I felt good as if I was going to discover something new in mathematics – I really love it! **[$\mathbf{r_1}$nb1]**.

After the 4th phase Ruth reflected:

> The work was very interesting and challenging **[nb2]**. At the beginning I felt a sense of anxiety, afraid I would choose to concentrate on an 'inappropriate' attribute, and it would be a waste of time **[eb3]**. But shortly after, when I worked with the software, I felt confident and it was clear to me that I will gain something meaningful from this project. I believe I will discover a new regularity **[$\mathbf{r_2}$nb1]**.

During the 5th phase, after working without finding anything that seemed to her as a meaningful discovery, she wrote:

> Sometimes during the work on the project I felt a lack of motivation. Perhaps it is because I am not used to activities of this kind **[eb4]**. During my school years we were asked to prove existing mathematical regularities **[eb5]**, and now we are asked to do something different, something that we are not used to – to discover something new. Since when do we have to choose the problem, to solve it and to investigate it? **[$\mathbf{r_2}$nb1]**.

In her final reflection Ruth wrote:

> …Contrarily to what I had said before I must say that when I observe and examine what I had gone through during the work on the project, I realize that only a minor part of the sessions contributed to my professional growth. As part of my educational duties I have to teach in various classes. I don't know yet how to teach and handle class situations in the traditional way [eb6], and you expect that I will adopt and implement innovative teaching approaches which I do not see their relevance to my work.

From the above excerpts it can be seen that Ruth holds beliefs regarding her current state as learner (nb1, nb2), her past experience as school student (eb3, eb4, eb5) and her role as a teacher (eb1,eb2,eb3,eb). Ruth's beliefs regarding herself as a future teacher are in fact a projection of her experience as a school student.

Ruth's beliefs regarding her state as a learner. At the beginning Ruth was enthusiastic. Influenced by the story about Morgan, she was eager to discover a new regularity [nb1]. The work with the interactive software, which facilitated the examination of many problem situations, reinforced her self-confidence in her ability to discover a new regularity [$r_1$nb1]. She began to develop the belief that a discovery process is a challenging and interesting one [nb2]. As long as Ruth felt that she was able to progress in her work, she expressed a tendency towards adopting new beliefs concerning the essence of learning. However, Ruth's enthusiasm began to fade with time, as a result of unfulfilled self-expectations. When Ruth faced a situation in which she did not mange to discover any meaningful regularity she used her initial system of beliefs regarding learning in order to justify her failure [eb4,eb5]. In fact, she does not take responsibility for her lack of success. Instead of searching for new directions in the project, she retreated and used her existing system of beliefs as an "alibi" for her lack of success. Namely, she uses the fact that she is not familiar with this kind of learning, and the fact that it is not the way she believes school students should learn, as causes for not finding a new mathematical regularity. Her attachment to her existing system of beliefs points to the fact that she did not make any genuine links between this system and the new beliefs [nb1,nb2] she was beginning to consider enthusiastically in the initial stages.

Ruth's beliefs regarding her role as a teacher. Ruth started the project with a rigid system of beliefs concerning classroom norms that relate to teaching, learning and school functioning: schools have their own rules regarding "proper" teaching methods, and inquiry-based learning is not part of them (eb1, eb2); teachers should not invest time and efforts in methods that do not guarantee success or lead the student through "vague paths" (eb3), which are time consumers. The rules of the game in the mathematics class, the sociomathematics norms, are clear: teachers provide the problems and the students solve them (eb5).

Due to Ruth's limited experience as a teacher, it can be seen that her beliefs regarding teaching are based on what Lortie (1975) calls "thousands of hours in an "apprenticeship of observation". Indeed, Ruth's memories of herself as learner (Grossman, 1990) influence her willingness to open her mind to new teaching ideas, and in fact inhibit her professional growth. As long as Ruth experienced success she was demonstrating a tendency towards developing new beliefs. However, as can be seen from Ruth's reflection, she did so merely from the learner perspective. Namely, she did not consider any possible change in her beliefs regarding the teacher’s role. Her disappointment caused her to examine the process from the teacher’s perspective as well, using her existing system of beliefs. In the beginning of the process Ruth revealed her beliefs regarding school as a conservative organization (eb1, eb2). In the 4th and 5th phases she related to her beliefs (or sociomathematical norms) regarding her role as teacher (eb3,eb4,eb5) according to which the students should be led in a path that guarantees success or otherwise it is "a waste of time". In addition, the teachers should be the problems providers. Those problems ought to be already known theorems. The students' task is to find the correct proofs.

To summarize, Ruth's past experiences is dominant in determining her views and beliefs regarding learning and teaching. The experiences she gained during the semester were subjected to interpretations in accordance with her already existing system of beliefs. Consequently, it seems that these experiences had slight influence, if any, on changing her beliefs. These findings are consistent with Kagan (1992).

CONCLUSIONS

Lamm (2000) had found that PST's systems of beliefs do not require external approval, and consequently many believe that teacher education programs have a slight influence, if any, on changing those beliefs. In our study we found reinforcement to Lamm's findings. Trying to comprehend the reasons that underlie this phenomenon, we used the analysis of the PST's portfolios. Through the PST's written reflections (with Ruth as a typical case) we managed to identify two main possible explanations:

Experiencing success as a motive for developing new beliefs. As long as Ruth was experiencing success she was willing to adjust her existing system of beliefs to the new learning situations. When Ruth felt that she was not fulfilling her self-expectations she "retreated" to her existing system of beliefs, and utilized them for justifying her failure. It can be assumed that experiencing success can serve as a motive for developing a new system of beliefs. However, a long period of time is needed in order to learn how to implement an inquiry activity and to be able to present a meaningful product. Thus, if teacher's educators wish to assure PST success, they should allow their students to experience this process, as well as other processes that concern innovative approaches, during the whole period of their training.

Choosing the proper timing for experiencing innovative approaches. In her final reflection, Ruth's excerpt eb6, points to the central role of choosing the right timing for introducing innovative approaches. As a "product" of the educational system, the PST had assimilated all the norms that are associated with this traditional organization. Moreover, during the period of their training they get their practical experience within that same system. Adopting innovative teaching/learning approaches requires the ability to adjust the existing system of beliefs to the desirable change. In order to do so, the PST must be convinced that this change is beneficial for them. The question is how to make them realize the necessity for change. Apparently, in order to reach a situation in which a change or an update of an existing system of beliefs regarding teaching and learning, will occurre, this system of beliefs has to be based on an extensive teaching experience and not on theoretical perceptions. It is reasonable to assume that PST would be able to recognize that the methods they are using are not satisfying only following a real practice, which will yield a conflict. Conflict is an essential psychological substance for considering new ideas. Therefore, it might be suggested that the exposure to innovative approaches will be gradual and continuant. PST should be instructed and guided how to implement innovative methods during their practical training. From our experience experiencing innovative approaches in the framework of a didactical course without practicing it in class is to some extent insignificant.

References

Brown , S. I. & Walter, M. I. (1993). Problem Posing in Mathematics Education. In Stephen I. Brown & Marion I. Walter (Eds.) *Problem Posing: Reflection and Applications*, Hillsdale, New Jersey: Lawrence Erlbaum Associates. 16-27

Calderhead, J. (1992). The role of reflection in learning to teach, in I., Valli (Ed.), *Reflective teacher education: cases and critiques*, Albany: State University of New York Press, 139-146

Desforges C. (1995). How does experience affect theoretical knowledge for teaching?, *Learning and Instruction*, Vol. 5, Pp. 385-400

Goetz, J.P. and LeCompte, M.D.(1984). *Ethnography and Qualitative Design in Educational Research.* Academic Press.

Kagan, D. M. (1992). Professional growth among preservice and beginning teachers. *Review of Educational Research*. 62(2), 129-169

Krajcik, J., Czerniak, C. & Berger, C. (1999). *Teaching Children Science - A Project-Based Approach*, McGraw-Hill College, New York

Lavy I. & Shriki A. (2003). Pre-service teachers' transition from "knowing that" to "knowing why" via computerized project-based-learning, in N. A. Pateman, B. J. Dougherty, J. Zilliox (Eds.), *Proceedings of the 2003 joint meeting of PME and PMENA*, Pp. 3-181-3-188.

Lamm, Z. (2000). The ideology and the educational philosophy, in *Pressure and resistance in education: articles and conversation*, Tel Aviv: Sifriat Poalim (in Hebrew), 217-254

Lortie, D. C. (1975). *Schoolteacher*. Chicago: the University of Chicago Press

Marrades, R. & Gutierrez. A. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics*, 44, 87-125.

NCTM - National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM.

Richardson, V. (1996). The role of attitudes and beliefs in learning to teach. In J. Sikula (Ed.), *Handbook of research on teacher education*, N.Y: Macmillan, 102-119

Tilema, H. H. (1995). Integrating knowledge and beliefs: promoting the relevance of educational knowledge for student teachers, in R. Hoz & M. Silberstein (Eds.), *Partnership of schools and institutions in higher education in teacher development*. Beer Sheva: Ben Gurion University of the Negev Press, 9-24

Tilema, H. H. (1998). Stability and change in student teachers' beliefs about teaching. *Teachers and Teaching: Theory and Practice*, 4(2), 217-228

Van-Dijk, T. A. (1998). *Ideology: A multidisciplinary approach*, London: Sage Publications

Watanabe T., Hanson R., & Nowosielski F. D. (1996). Morgan's Theorem. *The Mathematics Teacher*, 89 (5), 420-423

Yackel E. & Cobb P. (1996): Sociomathematical norms, argumentation, and autonomy in mathematics, *Journal for Research in Mathematics Education*, *27*(4), 458-477.

STUDENT THINKING STRATEGIES IN RECONSTRUCTING THEOREMS

Tatag Yuli Eko Siswono

Department of Mathematics, Surabaya State University,

Kampus Ketintang Surabaya, Jawa Timur 60231. Indonesia

A mathematics university student as a future mathematician should have the ability to find "new" mathematics structures or construct theorems based on particular axioms. That ability can be created by using problem posing tasks. To do the tasks, students with different abilities will use different thinking strategies. To understand them exactly, we conducted descriptive research. The high group initiated the process of reconstructing theorems with identifying and understanding axioms, making a visual diagram or making a conjecture and constructing the "new" theorem. The modest group began with understanding information (making a definition), drawing diagrams and calculating the number of lines and parallel lines, then constructing theorems. That pattern is similar for the low group.

INTRODUCTION

The fundamental changing of mathematics structure especially geometry occurred when axioms (i.e., the parallel axiom (postulate)) of Euclid geometry were modified. It fostered the developmental of non Euclid geometry, which was developed by Janos Bolyai, N. Lobachevsky and Rieman. Bolyai and Lobachevsky constructed a new geometry structure by changing the Euclid parallel axiom with the statement: "There exists a line ℓ and a point **A** such that at least two distinct lines pass through A that are parallel to ℓ". The statement of this geometry seems of questionable truth, but in the end it was useful and important in application when modern physics developed rapidly. This parallel postulate is known as the Hyperbolic Parallel Postulate and its Geometry is called Hyperbolic Geometry. In the mathematics deductive structure, this creatively invented axioms of Bolyai and Lobachevsky generated consistent theorems and no contradiction each other. Another creation by changing the parallel axiom of Euclid was done by the Germany mathematician Bernhard Riemann. He changed the parallel axiom of Euclid to the statement: "Given any line ℓ and any point P not on ℓ, there is no line through P that is parallel to ℓ". This axiom is known as the Elliptic parallel axiom and its geometry is called Elliptic Geometry.

Based on that axiomatic system, they derived different theorems such as the following:

Hyperbolic Geometry:	the angle sum of triangle is less than 180°.
Euclid Geometry:	the angle sum of triangle is equal to 180°.
Elliptic Geometry:	the angle sum of triangle is more than 180°.

(Wallace and West, 1992).

The innovation of such a mathematics structure is initially a creation of a mathematician who is challenged to explore a new structure or to prove a truth with a different method. Such a creation may lead to a structure that becomes its own right. Bell (1981) pointed out that the first most important activity of research mathematicians is creating new mathematics and discovering relationships within and among mathematical structures. The second most important work of mathematicians is to demonstrate theorem proving to the satisfaction of the mathematical community. Occasionally, mathematicians are curious to prove theorems with different methods. An example is the Pythagoras' theorem which is already proved with more than 100 methods. Modification of Pythagoras' theorem developed with attempting to discover some positive integers such that $x^n + y^n = z^n$, for $n > 2$. This theorem is known as Fermat's last theorem.

The activity to discover and construct axioms or theorems is very important to students who study mathematics in order to understand and involve them in constructing new mathematical structures. Otherwise, they will ask questions such as: "how did mathematicians discover this theorem or theory? Where does it come from? Is there anything to encourage them constructing a conjecture? What is the next developing?" Our students must make a final project which one alternative is to construct some theorems based on particular axioms or initial theorems. Thereby, when the lecturer teaches about a mathematics structure, they should not just introduce and prove theorems and let student's asking questions go unsatisfied. If it happens, their motivation will decrease and their attention becomes weak. Of course, this situation would disadvantage the development of mathematics. A lecturer should teach students how to discover and generate theorems or mathematics structures. The question is how to train them to develop such skill?

Villiers (1995) suggested problem posing activities or constructing conjectures at regular intervals in classrooms and encouraged students to formulate their own questions and to investigate them. At the heart of making conjectures and problem posing lies the ability to look and ask questions from different perspectives. For example, a good habit is to ask questions such as "what happen if it is changed?", "what happen if…?", "what if not", which will direct the students to form conjectures or new theorems. Lecturers should explain that an intelligent mathematician is not just as a good problem solver but also a creative problem poser. A mathematician's task is never stopped, they continually look back to the original problem or its solution and pose questions related with original or initial problems. Based on Villers's experience, this activity motivated students because they were involved in constructing and proving theorems. Beside that, problem posing gives other benefits, such as to increase problem solving ability, making students be active, and enriching fundamental concepts.

Silver and Cai (1996:292) has noted that the term "problem posing" is generally applied to three quite distinct forms of mathematical cognitive activity:

1. Pre-solution posing, in which one generates original problems from a presented stimulus situation;

2. Within-solution posing, in which one reformulates a problem as it is being solved;

3. Post solution posing, in which one modifies the goals or conditions of an already solved problem to generate new problems.

This investigation uses a form of pre-solution posing. Students pose or generate a problem (theorems) from a stimulus situation (an axiomatic system).

Considering the above argument, I saw that problem posing can be used as an alternative to train student's skill in reconstructing theorems. This activity can be given mainly in the Mathematics Foundation Course, such as logic or set theory. The description of this course is to give understanding and training to set up deductive reasoning which is systematically and regularly based on language and principles of logic and set theory (UNESA Handbooks, 2000). In such a course, mathematical content is divided into 3 parts that are axiomatics, logic and set theory. In axiomatics, students can be taught to construct theorems based on some given axioms and try to prove the truth of these theorems, so that is compatible with deductive-axiomatic structure in mathematics. To understand the result of implementation of the problem posing task, I conducted research which explored the students' ability after they are trained using that task and the difficulties which are experienced by students. The result of that research (Siswono, 2004) pointed out that students encountered some difficulties, such as (a) to determine a number of lines which are constructed, (b) to understand a definition, (c) to understand axioms (stimulus situation/information), (d) to implement a constructed sketch or distinguish between their concepts and sketches, (e) to prove or describe their own theorems and (f) to set up the language of theorems.

I also interviewed some students to grasp their thinking strategy when they were posing a theorem. Thinking is a process by which a new mental representation is formed through the transformation of information with complex interaction of mental attributes of judging, abstracting, reasoning, imagining and problem solving (Solso, 1995). When students face a problem posing task, they are encouraged to think aloud to propose a theorem. They use their skill of judging, abstracting, reasoning, imagining and solving the task as a problem. Occasionally, students use different strategies to transform or interpret information. Krutetskii (1976) explained the term "mathematical cast of mind" which refers to a tendency to interpret the world mathematically. He identified three basic types of mathematical cast of mind: the analytic type (who tends to think in verbal-logical terms), the geometric type (who tends to think in visual-pictorial terms) and the harmonic type (who combines characteristic of the other two). These types can be used as a basis to look at students' thinking strategy. Students' thinking strategy in this research is defined as a process or stage when they are constructing a theorem; how they are understanding the axiom system and what their considerations are to decide on a theorem. To identify their strategies clearly, I focused on three students group: a high, modest, and low group. This classification is based on the score of the pre-test.

METHOD

Subject of this research is 34 mathematics students at The Department of Mathematics, Faculty of Mathematics and Natural Sciences, the Surabaya State University in academic year 2003-2004. This is a descriptive research which tries to describe students' strategies in constructing theorems when they finished the problem posing task. Procedures of this research are as the following:

1. Teaching students to construct theorems based on the presented axiom system. The axioms are about Finite Geometry, known as Four-Point Geometry (Wallace and West, 1992). Because this system has not been taught to them, their ability is authentic.

2. Presenting a problem posing task to students. This task is to understand and identify students' ability in constructing theorems and their difficulties.

3. Analysing the students' problem posing task with the descriptive-qualitative method. Choose some students to be interviewed about their thinking strategy in constructing theorems from high, modest, and low group.

4. Writing the research report

Instruments of this research are a task of problem posing and the interview guidelines. The task of problem posing is as described below.

Consider this axiom system below.

1. There exist exactly four points and no three points in one line.
2. Any two distinct points have exactly one line on both of them.
3. Construct at least two theorems by deriving from the axiom system above. You can determine a definition first about a particular concept.

RESULT

I determined to interview in depth two students of the high group, three students of the modest group and three students of the low group. All groups said that they never have done the task in other mathematics course. However, the task of proving theorems frequently is given by lecturers. Therefore, it is a new model of mathematical activity.

All of the subjects just posed theorems about a number of lines and a number of paired parallel lines. Almost all students just made two theorems. There are 14 students producing true theorems without making a mistake, ten students making one mistake and ten students making two mistakes. Examples of true theorems are the following.

Example 1

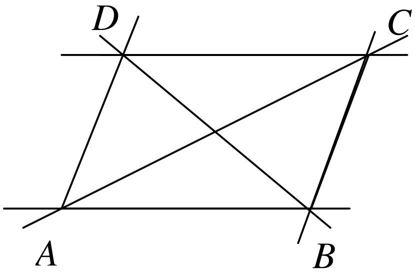
Through 4 distinct points, just 6 straight lines exactly can be constructed.

Proof:

$${}_4P_2 = \frac{4!}{2!(4-2)!} = \frac{4\times 3\times \cancel{2}\times \cancel{1}}{2\times 1\times \cancel{2}\times \cancel{1}} = \frac{12}{2} = 6$$

Example 2

Definition: Two straight lines are parallel if they don't have a common point whatever they are extended.



AB//CD, *AD//BC*, *AC//BD*.

Thus there are only 3 pairs of parallel lines.

An example of a mistaken theorem is the following.

Example 3

If there exist exactly four points and no three points in one line and any two distinct points have exactly one line on both of them, there are just 4 straight lines.

Proof:

$${}_4K_3 = \frac{4!}{3!(4-3)!} \quad \text{(also known in some countries as } {}_4C_3\text{)}$$

$$= \frac{4!}{3!}$$

$$= \frac{4\times \cancel{3}!}{\cancel{3}!}$$

$$= 4$$

That mistake may be caused by difficulties in drawing a sketch or understanding their sketch. Another possibility is that they can create a sketch although they didn't trust the number of lines shown by a sketch. They believe more in analytical procedures actually, but they don't understand the combinatorial concept (${}_4K_3$).

The high group referred to a sketch to bridge them in deriving the theorem. They initiated the process of constructing theorems by identifying and understanding axioms, making a visual diagram (sketch) or making a conjecture and constructing the "new" theorem. When they thought they were making a mistake or a wrong theorem or the statement is not suitable with their sketch, they examined and kept some attention to the theorem and revised or regenerated it without changing a visual diagram. While they constructed a theorem, they never thought about a definition because the task just asked them to construct a theorem. Another student gave the reason that her theorem didn't need a definition. They realized the important role of a definition although they did not write it down. They prefer a task to find a solution of problem or prove a theorem rather than construct theorems because it is difficult to understand axioms. The part of the interview of Iva is the following.

Interviewer: So what did you think when you constructed a theorem?
Iva: Well, Firstly I examined two axioms. Then I tried to draw a sketch by connecting some points and counting a number of straight lines.
Interviewer: When your conjecture was wrong, what did you do then?
Iva: I identify my theorem [conjecture] again and revise it.
Interviewer: Do you not change your sketch?
Iva: No I don't.
Interviewer: Why?
Iva: Because a sketch can be of many different forms, but actually they represent just one structure.

The modest group began by understanding information (making a definition), drawing diagrams and calculating the number of lines and parallel lines, then constructing theorems. When they thought they were making a mistake or a wrong theorem or the statement was not compatible with their sketch, they examine a visual diagram (a sketch) and they changed a diagram, then revised the theorem or regenerated a new one. While they were constructing a theorem, they never thought about a definition because the task just asked them to construct a theorem. A definition comes from axioms. They prefer a task to find a solution of problem rather than to prove a theorem or construct theorems. However, there were some students who thought that constructing theorems is easier than proving theorems. Sonie is one of those students.

Interviewer: When your conjecture is wrong, what do you do then?
Sonie: Change the diagram and see again all the axioms, and make a new theorem.
Interviewer: Do you prove it?
Sonie: No I don't. If it's wrong, I change the sketch.
Interviewer: Do you make a definition? What for?
Sonie: A definition is needed and a sketch is to help in constructing a theorem.
Interviewer: How do you make a definition?
Sonie: Based on two axioms, I combined some words and change with other word. A definition can be based on a sketch but a theorem not.

The low group initiated the process of constructing theorems by identifying and understanding axioms, making a visual diagram (sketch) or making a conjecture and constructing the "new" theorem. The stages seem similar with the high group. Actually, the quality of each step was different. They tried to understand the information even though sometimes they experienced difficulties. When they thought they were making a mistake or a wrong theorem or the statement was not in agreement with their sketch, some students used a different strategy. One student directly revised or constructed a theorem without changing the visual sketch. Some students did not change theorems but they changed the visual diagram in order to make it agree with the theorem statement. While constructing a theorem, they never thought about a definition because they didn't understand the role of a definition. They prefer a task to find a solution of problem to the task of constructing theorems or proving theorems because it was difficult to understand axioms. This can be seen in a part of the transcribed interview of Bayu (the low group):

Interviewer:	When your theorem was wrong, what did you do then?
Bayu:	Changed and revised a diagram; the theorem is not changed.
Interviewer:	What do you think about a definition?
Bayu:	A definition is needed by a theorem.
Interviewer:	But, you don't make it?
Bayu:	Yes.
Interviewer:	Well. What would you do to formulate a definition?
Bayu:	I make a statement from terms of some axioms which is the meaning unclear.

DISCUSSION

This activity actually pointed out student representations of understanding a theorem and learning how to prove theorems. There are different strategies in the three groups, although they can be classified as the visual strategy or geometric type. The high group tends to think systematically. They did not explicitly say that they wrote down a definition before constructing theorems, but they realized the importance of the role of definition. Based on the theorems which were posed, we saw that they used the definitions of Euclidean Geometry about parallel lines, intersections and triangles. The modest group also tends to work systematically, even mentioning about the stage of making a definition. However, they never write down the definition. The actual difference with the high group, is that they sometimes change a sketch to match with the original theorem when they are not sure or are in doubt. This situation causes the possibility of mistakes.

The low group also followed similar stages to other groups. They do not make definitions because they don't understand the role of definitions. When they make a mistake, they frequently change the statement of the theorems or the sketch. These situations created very high potential for them to make other mistakes.

The classification of the students does not show their abilities in constructing theorems. The high group still makes some mistakes, such as formulating a sentence for a theorem. The low group also has some mistakes, basically caused by the weakness of their abstracting or understanding ability.

In terms of the thinking strategy, students tend to use the geometry type. This happened because the task directed them towards this type. However, if we look back at their proving theorems in determining a number of lines, there were two students from the high group using different strategies. One student used analytic type thinking with the combinatorial concept. Another student determined it by calculating the number of lines based on a sketch or diagram, so she used geometric type. The modest groups contain two students using geometric and one student using analytic thinking. All students of the low group apply geometric thinking. To prove parallel lines, all students implement geometric thinking.

According to the students, this task was more difficult than proving theorems or finding a solution of a problem. The reasons are: it needs higher order thinking skill and requires understanding of information and axiom system. However, the problem posing task can still be used as an alternative to teaching thinking mathematically and creative thinking. As noted by Dunlop (2001), problem posing is a valid tool for teaching of mathematically thinking and it can foster a creative thinking.

SUMMARY

Students in the three groups have different strategies for constructing theorems. The difference will impact on their mistakes. Their thinking process leads to the following steps: identifying and understanding axioms, making a visual diagram or making a conjecture and constructing the "new" theorem.

This research has not been understood effectively in the teaching and learning process yet, so it needs further research. The task of problem posing should not only be the straight to geometry type, but also analytic type and harmonic type.

References

Bell, Frederich H. (1981). *Teaching and Learning Mathematics*. Dubuque, Iowa: C. Brown Company Publisher.

de Villiers, M. (1995). Problem Posing Variations on Fermat's Last Theorem. KZN Mathematics Journal, 16-18 December 1995. http://mzone.mweb.co.za/residents/profind/homepage.html. Download November 1 2002

Dunlop, James. (2001). *Mathematical Thinking*.
http://www.mste.uiuc.edu/ courses/ci431sp02/students/jdunlap/WhitePaperII
(Download November 21, 2003)

Krutetskii, V.A. (1976). *The Psychology of Mathematical Abilities in Schoolchildren*. Chicago: The University of Chicago Press

Silver, E. & Cai, J. (1996). An analysis of Arithmetic Problem Posing by Middle School Students. *Journal for Research in Mathematics Education*, 27(5), 521-539

Siswono, Tatag Y. E. (2004). Problem Posing: Melatih Kemampuan Mahasiswa dalam Membangun Teorema. *Proceeding of National Seminar "Penelitian Pendidikan dan Penerapan MIPA", UNY*, Yogjakarta, 2 Agustus 2004

Solso, Robert L. (1995). *Cognitive Psychology*. Needham Heights, MA: Allyn & Bacon

Team. (2000). *UNESA Handbooks* (Buku Pedoman Unesa, Non Pendidikan 2000-2001). Surabaya: Unipress Unesa

Wallace, E. C. & West, S. F.(1992). *Roads to Geometry*. Englewood Cliff, New Jersey: Prentice Hall.

A COMPARISON OF HOW TEXTBOOKS TEACH MULTIPLICATION OF FRACTIONS AND DIVISION OF FRACTIONS IN KOREA AND IN THE U.S.

Ji-won Son

Michigan State University

To illuminate the cross-national similarities and differences in ways of teaching multiplication of fractions and division of fractions, this study compared the lessons on fractions in Korean mathematics textbooks and accompanying teacher's manuals with the corresponding lessons on fractions in one U.S. reform mathematics textbook series and accompanying teachers' manuals [Everyday Mathematics]. This study found that there is a gap between learning goal [intended curriculum] and problems presented in textbooks [potentially intended curriculum].

INTRODUCTION

It is well known that across country, students' learning is highly correlated with curricular treatment of related topics (Ball & Cohen, 1996; Garner, 1992; McKnight et al., 1987; Olson, 1997; Schmidt, McKnight, & Raizen, 1996; Schmidt et al, 2002). A lot of cross-national comparisons of mathematics textbooks including the TIMSS study have reported that U.S textbooks constitute a de facto national curriculum, which has been characterized as superficial, underachieving, and diffuse in content coverage (Fuson, Stigler, and Bartch, 1988; Schmidt, McKnight, Cogan, Jakwerth, and Houang, 1999; Mayer, Sims, & Tajika, 1995; National Council of Teachers of Mathematics, 1989; Schmidt, Houang, and Cogan, 2002).

About the time the TIMSS study was underway, three reform curricula were developed with support from the National Science Foundation: *Everyday Mathematics, Investigations,* and *Trailblazers*. Among them, it has been often reported that *Everyday Mathematics* is used most widely in America. It is reported that *Everyday Mathematics* increases both the depth and the breath of the mathematics taught, focuses on students' mathematical solutions and the examination of alternative strategies, and encouraging students to develop, use, and discuss their own methods for solving problems (Carroll, 1998).

It is well known that many students and adults have difficulty with understanding multiplication of fractions and division of fractions. Algorithms for multiplication of fractions and division of fractions are deceptively easy for teachers to teach and for children to use, but their meanings are elusive (Kennedy & Tipps, 1997). However, students should learn mathematics with understanding (NCTM, 2001). "Instructional programs should enable all students to understand meanings of operations" [with fractions] and how they relate to one another; compute fluently and make reasonable estimates (NCTM, 2001, p. 214).

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 201-208. Melbourne: PME.

This study examined how multiplication of fractions and division of fractions are taught in the reform curriculum presently being used in Korea and *Everyday Mathematics.*

In the TIMSS study, Korean students showed high achievement, ranking number two. Yet, there is little research of how Korean students learn mathematics. Recently, Grow-Maienza and Beal (2003) studied Korean mathematics curriculum. Yet, they focused on the traditional 6^{th} mathematics curriculum. Korean mathematics has been recently changed. There is little research addressing on how the Korean reform curriculum 7^{th} mathematics teaches mathematics and on how problems in mathematics are presented in textbooks.

The purpose of this study is to illuminate the cross-national similarities and differences in ways of conceptualizing and presenting multiplication and division of fractions in Korean reform textbooks with the corresponding lessons on fractions in Everyday Mathematics. According to previous researches, a lot of studies on textbook analysis have focused on either content analysis or problem analysis. They recommend that combining two types of analysis--content analysis and problem analysis--provide richer promises for revealing potential effects of textbooks on students' mathematics achievement. This study focused on two aspects of textbook analysis: content analysis and problem analysis. This study has three research questions:

(a) What are the learning goals related to multiplication of fractions and division of fractions in each curriculum?

(b) When and how are multiplication of fraction and division of fractions introduced and developed in each curriculum?

(c) How many and what types of problems in multiplication of fractions and division of fractions are presented in each curriculum?

METHODOLOGY

This study conducted content analysis and problem analysis. Content analysis is focused on two research questions. Problem analysis is conducted focusing on problems presented in the textbooks.

Textbooks and the Mathematical Problems analysed

EM provides three textbooks (Student Journal 1, 2, and Student reference book) and Korean mathematics provides four textbook (Student Mathematics Ga, Na, and Mathematics workbook ([1]Ga, Na). All textbooks in 5^{th} and 6^{th} are analyzed.

Analysis Plan

Content analysis is conducted focusing on two research questions (a) and (b). In content analysis, both teacher's manuals and student's book were used. First, this study referred to teacher' manuals in order to identify learning goals of multiplication of fractions and division of fractions. Based on the learning goals stated in the teacher's manual, this study explored their emphasis on learning about multiplication of fractions and division of fractions. Second, this study examined when and how multiplication of fractions and division of fractions are introduced and developed in the each textbook series.

Problem analysis is conducted focusing on problems presented in the textbooks. In this study, problem is identified as those mathematical problems or problem components that do not have accompanying solutions or answers presented. Previous studies identified three important dimensions for analyzing mathematical problems: mathematics feature, contextual feature, performance requirement (Li, 1998; Stigler et al., 1986; Tabachneck, Koedinger, & Nathan, 1995). Based on previous studies, by adding some other factors, three-dimensional frameworks were developed in this study. Table 1 shows the dimension of problems analysis: (a) mathematics feature; (b) contextual feature; (c) performance requirement.

Dimensions of Problem Analysis	
1. Mathematics Feature	Single step required (S)
	Multi-step required (M)
2. Contextual Feature	Purely mathematical context in numerical or word form (PM)
	Illustrative context such as visual representation (IC)
3.Performance Requirement	
(1) Response type	Numerical answer only (A)
	Numerical expression required (E)
	Explanation or solution required (ES)
(2) Cognitive requirement	Conceptual understanding (CU)
	Procedural knowledge (P)
	Mathematical Reasoning (MR)
	Representation (R)
	Problem solving (PS)

Table 1. Conceptual Framework for problem analysis

Each problem in all textbooks was coded in terms of the three dimensions stated above. In order to avoid the researcher's subjectivity, a second independent rater who is literate in both English and Korean languages coded problems in textbooks. The interrater agreement was 98 %.

RESULTS

Research Question 1: What are the learning goals related to multiplication of fractions and division of fractions in EM and KM

While EM emphasizes understanding the meaning of multiplication of fractions and division of fractions more than the algorithms for them, KM emphasizes both conceptual understanding and mathematical fluency. EM first provides folding the paper and area model to address multiplication of fractions and then introduces the algorithm of multiplication of fractions. In division of fractions, EM expects students to understand a common denominator method for division of fractions and an algorithm for the division of fractions. In contrast, through the whole learning goals, KM emphasizes both understanding and using algorithms effectively. In addition, KM expects students to understand and formulate various type of multiplication of fractions and division of fractions.

Research Question 2: When and how are multiplication of fractions and division of fractions introduced and developed?

Content organization

First, while EM introduces and develops multiplication of fractions and division of fractions at the same time, KM introduces multiplication of fractions and division of fractions separately. KM first introduces multiplication of fractions and develops it intensively in one unit. Then, it introduces division of fractions and develops it intensively in two units across two grades.

In addition, in EM, several topics are covered in one unit. Almost each lesson has different topics. For instance, 5th graders learn multiplication of fractions and division of fraction with comparing fractions, addition and subtraction of fractions, and percent. In contrast, Korean mathematics curriculum is much more sequentially organized, with almost no repetition. Different topics are taught in different grads. While the sixth grade text in Korea does not duplicate fifth grade topics, the typical EM often duplicates most of the content.

Third, KM devotes more time to developing multiplication of fractions and division of fractions for students to master it. KM devote as twice time as EM does to develop multiplication of fractions and division of fractions. While EM covers multiplication of fractions and division of fractions in a total of 9 lessons, KM covers them in a total of 17 lessons. In addition, there is different intensity of multiplication of fractions and division of fractions.

Content Presentation

First, EM emphasizes understanding first and then algorithm. In particular, EM does not emphasize the algorithm of multiplication of fractions until 6th grade. EM first introduces “many of” and “part of “as indicators of multiplication. Before introducing the algorithm for multiplication of fractions, EM give concrete meaning to finding a fractional part of a fraction part by providing the paper-folding exercise and area-model diagram. However, KM emphasizes understanding and algorithm of multiplication of fractions at the same time. Different lessons teach different types of multiplication. KM, through whole lesson, introduces three activities; understanding the multiplication of fractions, knowing the algorithm of multiplication of fractions in different types, and practice.

Second, EM and KM introduce the algorithm of multiplication of fractions with two same two strategies. However, in the problems of (whole number) × (fractions) or (fractions) × (whole number), EM uses common denominator strategies, which KM does not use. EM asks students to rewrite each fraction in the form.

In division of fractions, both curricula introduce division of fractions from whole number division. However, while EM introduces two strategies of division of fractions—common denominator and invert and multiply method, KM only relies on invert and multiply method.

Research Question 3: How many and what types of problems in multiplication of fractions and division of fractions are presented?

It was found that EM provides more problems in multiplication of fractions than KM in terms of the total number (EM: 251, KM: 190). However, KM provided more problems in division of fractions than EM (EM: 58, KM: 400). Because problem analysis results of division of fractions are similar to those of multiplication of fractions, this study reports the results of multiplication of fractions.

Mathematical Feature

Table 2 and figure 1 show mathematical feature in fractions multiplication.

	Simple computation	Multiple computation	Total
Everyday Math	246 (98%)	5 (2%)	251
Korean 7th Math	158 (83%)	32 (17%)	190
	403	38	

Table 8. Distribution of problem in KM and EM by mathematical feature

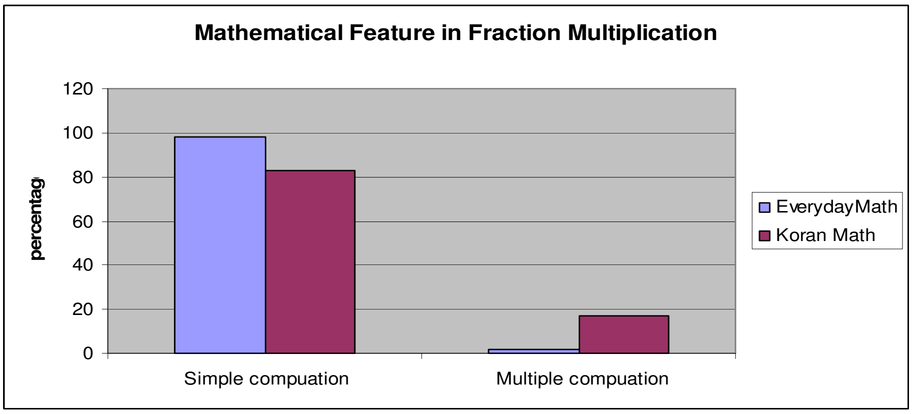


Figure 1. Distribution of problem in KM and EM by mathematical feature

In terms of the number of steps required in the solutions to multiplication of fractions problems, this study revealed that problems in the KM are more challenging than those in EM. It was found that 17 % of the problems in KM needed multi-step to solve, whereas such problems in the EM were around 2%. The less frequent exposure to multiple-step problems for U.S. students might be one reason why they performed not so well on this type of problems, as found in many studies (Carpenter et al., 1980).

Contextual Feature

Table 3 and figure 2 show contextual feature in fractions multiplication.

	Purely Math context	Illustrative context	Total
Everyday Math	175 (70%)	76 (30%)	251
Korean 7th Math	136 (71%)	54 (29%)	190
	311	130	

Table 3. Distribution of problems in KM and EM by different representation forms

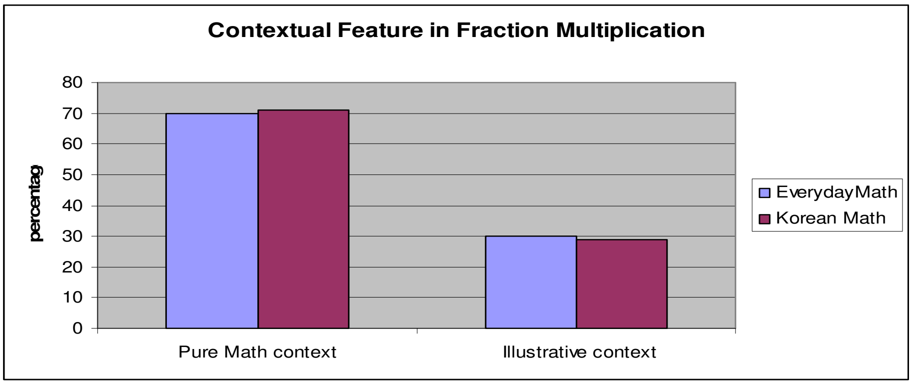


Figure 2. Distribution of problems in KM and EM by different representation forms

It was found that contexts of problems in both Korea and EM textbook were all most same. The majority of problems in all the books were presented in symbolic forms, including mathematical expressions, written words, or a combination of the above two forms (Korea: 71 %, US: 70%).

Performance Requirement

A. Response Type

Table 4 and Figure 3 show the results.

	Numerical answer	Explanation required	Total
Everyday Math	179 (71%)	30 (12%)	251
Korean 7th Math	119 (63%)	27 (14%)	190
	298	57	

Table 4. Distribution of problems in KM and EM by response type

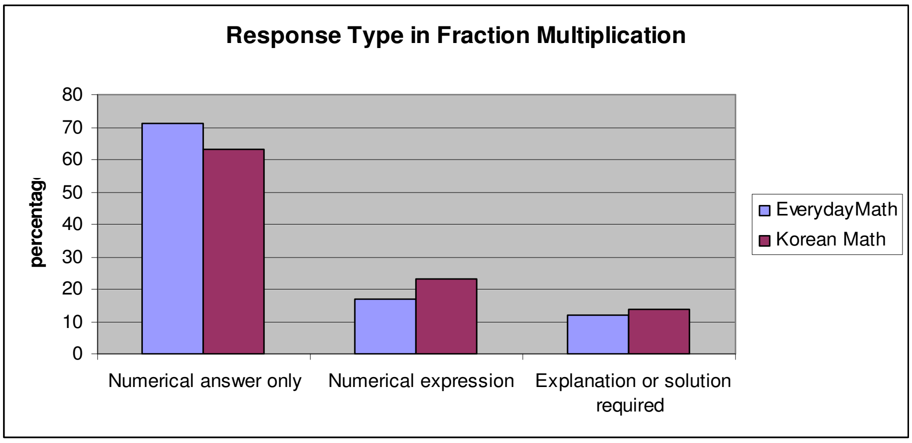


Figure 3. Distribution of problems in KM and EM by response type

It was found that a lot of problems in both Korea and EM textbook were required numerical answer only, numerical expression, and explanation or solution required in order. The distribution of problems-response type in the KM textbooks is more balanced than that in the EM. Clearly, the majority of problems from both textbooks were found to require a numerical answer. However, fewer problems in KM required a numerical answer only (EM: 71%, KM 63%) and more problems required numerical expressions, or explanations or solutions (EM: 29%, KM: 37%).

Cognitive Requirement

Table 5 and figure 4 show the results.

	Conc. Know.	Proc. Fluen.	Math. Reas.	Repre.	Prob. Solving.
EM	1	198 (79%)	6 (2%)	22 (9%)	24 (10%)
KM	1 (1%)	151 (79%)	13 (7%)	7 (4%)	18 (9%)

Table 5. Distribution of problems in KM and EM by cognitive requirement

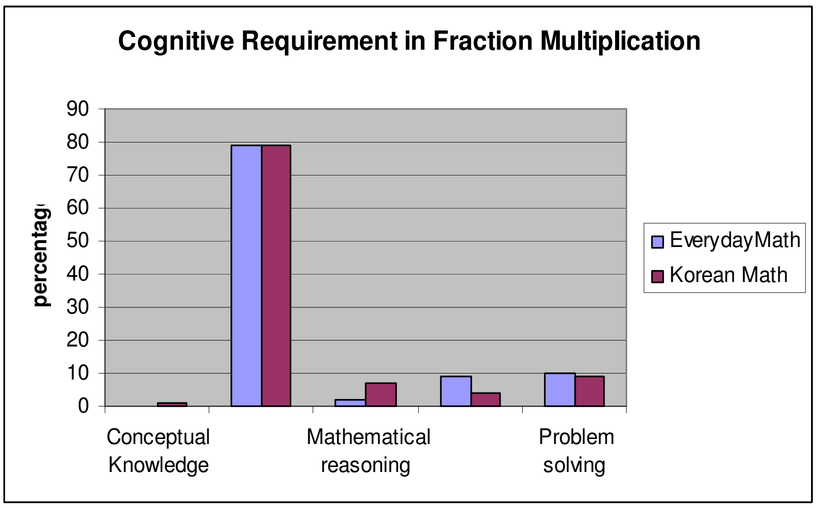


Figure 4. Distribution of problems in KM and EM by cognitive requirement

This study found that procedural knowledge is the most frequent type of knowledge required in the problems about fractions multiplication both curricula. Conceptual knowledge is the least frequent type of knowledge. This result shows that even though both curricula intend to improve conceptual understanding, mathematical reasoning, problem solving, the problems presented in the textbooks ask almost exclusively for procedural knowledge. While more problems in EM require representation and problem solving than KM, more problems in KM require mathematical reasoning.

DISCUSSION

This study examined both Korean 7th mathematics and Everyday Mathematics at 5th and 6th grade. One of important finding in this study is the gap between what is intended and what is presented in textbooks. Both curricula intend to improve students' conceptual understanding of multiplication of fractions and division of fractions. *Everyday Mathematics* seems to provide more opportunities to developing concepts behind algorithms. However, it was revealed that problems in both textbooks are presented in purely mathematical contexts and that a large portion of problems is required single-computational steps and procedural knowledge only. Based on this result, it is not difficult to assume that there is understandably some gap between what described in the textbooks and what actually happen in classrooms. This study has implications to curriculum developers, teachers, and researchers.

References

Bezuk, N. S & Armstrong, B. E. (1995). *Understanding of fraction multiplication and division of Fractions*. Mathematics Teacher, 86(1), 120-131.

Carroll. W. M (1998). *Middle school students' reasoning about geometric situations. Mathematics Teaching in the Middle School* v. 3 (Mar./Apr. 1998) p. 398-403.

Everyday Mathematics, Grade 1-6. (1999). *The University of Chicago School Mathematics Project*. Partially supported by NSF ESI 9252984. Chicago: Everyday Learning Corporation.

Kilpatrick, J. Swafford, J. & Findell, B., Eds. (2001). *Adding it up: helping children learn mathematics*. Washington, DC: National Academy Press.

Korean Ministry of Education (1996). *Mathematics and Teacher's guide for Grade 5,6-Ga and Na*. Seoul: National Textbooks incorporated.

Schmidt,W.H., McKnight, C.C., & Raizen, S.A. (1997). *A splintered vision: An investigation of U.S science and mathematics education*. Boston: Kluwer Academic Publishers.

[1]Ga and Na is Korean own language. Ga means one and Na means two. Thus, 5-Ga and 5-Na are 5-1 and 5-2, respectively.

MATHEMATICAL KNOWLEDGE OF PRE-SERVICE PRIMARY TEACHERS

Beth Southwell Marina Penglase
University of Western Sydney

Seventy eight primary pre-service teachers participated in a survey of arithmetical content knowledge at the conclusion of an elective mathematical content course designed for those with a poor background in mathematics. Not only was the aim of this first stage of a research project to ascertain current knowledge but also to adjust current courses to better suit the students in teacher preparation courses. Analysis of the results of this survey indicate weaknesses in understanding in particular areas including place value, operations with common fractions, multiplication of decimal fractions, percentages and measurement. These areas are related to the curriculum the pre-service teachers will be expected to teach on their graduation.

INTRODUCTION

Several recent reports and movements have emphasized the need to enhance the mathematical content knowledge of students. The *NCTM Principles and Standards for School Mathematics* (2000) states that 'Teachers need different kinds of mathematical knowledge – knowledge about the whole domain; deep, flexible knowledge about curriculum goals and about the important ideas that are central to their grade level …' (p. 17). *The AAMT Standards for Excellence in Teaching Mathematics in Australian Schools* is one of the most recent documents to say that 'excellent teachers of mathematics have a sound, coherent knowledge of the mathematics appropriate to the student level they teach' (2002). If teachers are not confident in their mathematical knowledge, they may find it difficult to ensure that their students gain confidence and competence.

Then, too, the reports arising from the Third International Study of Mathematics and Science (Lokan, Ford & Greenwood, 1996, 1997) indicate that there were deficiencies as well as strengths in student achievement and understanding in mathematical content knowledge. Several researchers (Morris, 2001; Chick, 2002; Amarto & Watson, 2003) have found that pre-service teachers do not always possess the conceptual understanding of the mathematics content they will be expected to teach.

In 2002, the Board of Studies, NSW, released the new *Mathematics K-6 Syllabus* that became mandatory in NSW primary schools from 2004. This Syllabus has many differences from the 1989 *Mathematics K-6 Syllabus* that has been in use in these schools. Included among these differences are many involving different concepts or new approaches to mathematics. Primary teacher education students currently undertaking their teacher preparation courses will be expected to implement the new syllabus when they graduate. Many of them did not complete mathematics courses in

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 209-216. Melbourne: PME.

their senior high school years, often because they had not been successful in their junior years. As a consequence of all of this, it is possible that primary teacher education students start their teacher preparation courses without key mathematical knowledge and with some negative attitudes towards the teaching and learning of the subject.

Over several decades there has been a change in the way mathematics has been taught and in the curriculum that has been followed. Constructivism heralded in a different emphasis on the process of teaching and learning. Unfortunately, many teachers saw it as a way of ignoring their own lack in mathematical content knowledge and concentrated on what they perceived to be the process required in a constructivist based classroom. Von Glasersfeld (1994), however, reminded educators of the possibility of enhancing mathematics achievement and understanding through a constructivist approach. In relation to the learning and teaching of arithmetic he stated:

> … if we really want to teach arithmetic, we have to pay a great deal of attention to the mental operations of our students. Teaching has to be concerned with understanding rather than performance … (von Glasersfeld, 1994, p. 7)

It is important to note that the outcome of learning implied in this statement is understanding and conceptual development.

The aim of this stage of the project is to ascertain the mathematical knowledge of primary teacher education students in a NSW university teacher preparation course at a particular stage in their courses. This will enable the researchers to tailor courses to help fill any gaps that may be found. It will also provide an ongoing measure against which students and university staff can judge the students' learning and the courses being undertaken.

METHODOLOGY

Sample. The 78 participants at a NSW university included both undergraduates in a four year program from second, third or fourth year of their course and students in a one year graduate entry program. They were doing this subject either because they lacked a sound mathematical background or because they had a particular interest in mathematics.

Procedure. The survey was administered during class time at the conclusion of a special elective mathematics content course. A survey methodology was considered most appropriate for this study. McMillan (2004, p. 195) describes surveys as popular because of their 'versatility, efficiency and generalizability'. Their versatility lies in their ability to 'address a wide range of problems or questions, especially when the purpose is to describe the attitudes, perspectives and beliefs of the respondents'. Their limitation, according to Mertler and Charles (2005), is that they do not allow the researcher to probe further as would be possible in an interview. In this current study, the 20 questions used in the survey were designed to ascertain whether the participants had the necessary mathematical knowledge on topics they were expected to teach and any further probing was considered possible if necessary after the initial responses were analysed. The project was approved by the University Ethics Committee.

Analysis. Data were analysed using descriptive statistics only for the first part of this research on the mathematical competence of pre-service teachers. Surveys of attitudes and beliefs will be considered at a later stage. Because the first and the last items had 3 and 2 parts respectively, these were treated in the analysis as separate items, thus making 23 items.

RESULTS AND DISCUSSION

At this stage of the research, there are two main areas that need to be reported. They are the item analysis for the 23 items and the relative difficulty of areas of arithmetic as indicated by responses.

Item Analysis. Table I presents the former of these for items of greatest interest indicated by the difficulties observed.

Table 1. Number and Percentage of Correct Responses and Most Common Incorrect Responses

Item	No. correct	Percentage correct	Most common incorrect responses
3	0	0	11 participants did not answer, one saying he/she could not understand the item. Other responses ranged from 0 to 38 with 7 giving alternatives such as '9 or 18'.
7	1	1.3	19 responses as 46 092 340. 23 did not attempt. 6 gave correct response except for units digit, 2 listed each power but did not join them with +. Three wrote expansions but not expressed as powers of 10.
11	34	43.6	9 did not attempt.
13	48	61.5	19 wrote fractions largest to smallest. See comments below.
14	54	69.2	15 subtracted numerators and denominators respectively. 6 did not attempt. One used addition.
15	45	57.7	8 did not attempt. 12 added numerators and denominators respectively. One added denominators only.
17	45	57.7	13 gave 17/100 as their answer. 3 wrote both 0.17 and 17/100 and 4 gave 17/100 0.17.
19	21	26.9	30 gave correct digits but with decimal point placed the first 2 digits of their answer. 10 did not attempt or crossed out work.
20	58	74.4	11 had each of the algorithms within the brackets correct but gave the wrong answer. 8 had 1 algorithm incorrect but all other working correct. One removed the last bracket, writing the last sign as a minus.
21	48	61.5	6 did not attempt. 10 gave 16 as answer. 14 were completely incorrect though 5 at least started to work a division algorithm.
22	59	75.6	10 gave 623 as response. 4 gave 6023.
23	62	79.5	5 did not attempt. 3 gave 598.7, 3 gave 598700 as response.

Comments on Specific Items

Item 3. No of pairs of numbers that sum to 19. The poor response to this item raises the question of language and its relationship to mathematics. The item was not expressed as clearly as most participants needed. The inclusion of the word ‘whole’, for instance, may have made the item more specific though this particular difficulty had not arisen in previously given similar questions. It could be that this cohort of students had been introduced to inclusive sets of numbers in such a way as to not question the item further and hence the number of non-attempts and the cry “I don’t understand this”. Unfortunately this view is not borne out by the types of responses given as they all refer to whole numbers anyway. It would appear, then, that these participants have been confused with the other wording in the item or are fairly rigid in their arithmetical thinking. ‘Different numbers’ in the item statement could be interpreted as meaning that 0 and 19 are the same numbers regardless of the order. In that case the correct answer would not be the same as it would be if they are considered different. This could explain the number of participants who gave alternative responses, e.g. 9 or 18.

Item 7. Expand 4 609 234 using powers of 10. As this item requires understanding fundamental to the decimal place value system and since numeration was a topic in the course participants followed, it is surprising that this item was so poorly completed. One can understand the omission of the index for the unit figure but again, perhaps the language used was not as familiar as expected. One participant actually wrote that he/she did not understand the word ‘expand’ and this possibly was the case for many others. The response 46 092 340 seems to indicate that participants did not know what was meant by ‘expand’ and thought that by multiplying by ten they were using a power of ten.

Item 11. Calculate 47 x 25 using a different method (N.B. previous question was Calculate 47 x 25) A ‘different method’ caused several problems for participants. Many used the commutative property and did not acknowledge that multiplying 47 by 25 used the same method as multiplying 25 by 47. Only 3 students made use of the distributive law and added each partial product. Several made use of the distributive property but only in part. Only two participants took advantage of the fact that the multiplier was 25 and is therefore one quarter of 100. The picture painted by the responses to this item indicates a fairly inflexible idea of multiplication with the emphasis on the standard algorithm. The NSW Syllabus K-6 Mathematics (1989, 2002) recommends that teachers encourage students to use their own natural methods to complete computations and to explore different ways in which this can be done. In the particular course this group of students has completed is the opportunity for many approaches to algorithms. Only one student used the lattice method and one attempted but was not able to complete the duplation or doubling method. Number sense and flexibility of arithmetical understanding are worthwhile aims at any level and pre-service teachers have the responsibility and opportunity to acquaint themselves with methods of an historical nature as well as being able to accept unorthodox methods created by their students.

Item 17. Convert 17% to a decimal fraction. Participants indicated some degree of uncertainty in their responses to this item. Perhaps there exists some confusion as to the difference between a decimal fraction and a common fraction since so many (13) gave their responses in the common fraction form. This confusion may have arisen from the common (sometimes incorrect) practice of referring to ’fractions' for common fractions and ‘decimals’ for decimal fractions. This practice overlooks the fact that all our numbers are decimal numbers since they are based on ten in the same way as binary numbers are based on two. Decimal fractions are fractions or rational numbers expressed with a decimal point. The difficulty may also have arisen because the method of changing a percentage to a decimal fraction was unfamiliar to the participants. Several knew that 17% meant 17/100 but then became uncertain as to the place value represented by the fraction. Those who put both 0.17 and 17/100 or its reverse may have been ‘hedging their bets’ hoping that one of their responses would be counted correct. Taplin (1998) reported that on medium difficulty items approximately half of the participants in her study with an incorrect answer to a question asking them to find 12% of $68 did not know that 12% meant 12/100.

Item 21. Solve: 1023 students, 63 per bus. How many buses needed? The responses to this item indicated that for some participants there was no need to consider the context of the problem. The answer 16.23 was arithmetically correct but was not a sensible answer to the question. Two participants did realise that they would need a whole number of buses but opted for 16 instead of 17 qualifying their responses by saying some students would stand or be left behind. As the item said that the students were to ‘fit in’ the buses, not necessarily be seated, the responses of 16 were not considered correct. The question of context in word problems - indeed problems of any kind – is one that needs to be considered as a vital aspect of problem solving. Contreras and Martinez-Cruz (2001) report that only 28% of their sample of pre-service teachers were able to give a realistic response to a problem using the same context as this one and only 6 % were able to give an explanation of their answer.

Items 14, 15. Find 5/8 - 2/5. Find 4/5 + 2/3. The error recorded by a number of participants is a recognised common one and indicates not only a faulty algorithm for common fractions but also a lack of flexibility of arithmetical thinking, as a simple check of the reasonableness of the answer would alert the participant to the error. Rational numbers, particularly in the common fraction form, have been recognised for some time as an area of great difficulty for all students. Skemp (1986) claimed that this is partially because students have to apply a process of *accommodation* when they meet rational numbers and this is different to the *assimilation* process that has been possible with all the previous work they have experienced in mathematics. Taplin (1995) found that pre-service primary teachers found difficulty in fractions concepts including operations. One interesting aspect of these items is that a few students were able to complete one of these two items correctly but followed the common error for the other. This seems to indicate an unstable concept of operations with common fractions.

Item 19. Calculate 14.83 x 0.06. Taplin (1995) also identified difficulties in the multiplication of decimal fractions. This same difficulty arose in this study because of the participants' placement of the decimal point.

Items 22, 23. Convert 6.23km to m. Convert 5.987L to ml. These two items seem to indicate a general weakness in measurement which could be linked to a place value deficiency or to a lack of understanding of the metric system of measurement. As the metric system is usually considered an application of the place value system, and is used in that way, it is disturbing that the numbers correct in these items are as low as they are, unless the fact that they are the last items caused participants to think less about them than other items. Morris (2001) also reports a similar deficiency in converting metric measures between units.

Item 13. Put 5/6, 2/3, 4/5 in order. Although responses in the wrong order were accepted as correct with Item 5, they were not in Item 13, mainly because of the possibility that participants do not understand that the magnitude of common fractions is different to that of whole numbers. This point is supported by Leinhardt and Smith (1985, p. 269).

Analysis of Content Areas. The second area of reporting for this stage of this project is in relation to the particular topic that participants found difficult. For this purpose, the 23 items have been linked in groups and each group considered separately. Table 2 shows the items in groups and the relevant statistics related to each group.

Table 2. Number and Percentages of Correct Responses for each Category of Items.

Category	No. of items in each category	No. correct responses	Percentage correct responses
Basic concepts, numeration	3 (5,6,7)	155	65.8 (*98.7)
Basic facts	3 (1,2,3)	154	66.2 (*98.7)
Four operations	5 (8-12)	389	83.1
Order of common fractions	1 (13)	48	61.3
Operations with common fractions	2 (14,15)	99	63.5
Decimal fractions	1 (19)	21	26.9
Percentages	3 (16,17,18)	156	66.7
Measurement	2 (22,23)	121	76.6
Order of operations	1 (20)	58	74.4
Word problems	1 (21)	48	61.5

* Percentage of non-responses to items 3 and 7 are not included.

Surprisingly, the area requiring greatest attention is decimal fractions. Place value concepts seem to have caused most problems. This outcome could be the result of only having one item on decimal fractions, though the other related item on changing percentages to decimal fractions was not well done either. This seems to lend support to the premise that more time needs to be spent on work with decimal fractions. It could be that because the link is so obvious to teachers, they do not spend the necessary time to allow students to construct effective processes of understanding and using decimal fractions.

Language in mathematics is another area that needs more attention. Because mathematics is sometimes spoken about as the universal language, the assumption is made that no matter what language they speak every day, students will be able to understand all aspects of mathematics, including terminology and syntax of problems. This is not necessarily so as Bell and Ho Woo (1998) have found.

CONCLUSION

The study has caused certain possible future research topics to emerge. The concentration of research on specific topics in mathematics is necessary if pre-service teachers are to become properly equipped for their daunting task as teachers. This test needs to be extended to geometry and probability and further testing carried out.

This study reminds teachers and teacher educators in particular, that understanding the way in which learners construct their arithmetical knowledge is of prime importance in all mathematics courses. Much more can be done to analyse thought processes and develop approaches in the classroom that will assist students in their mathematical constructions. Leinhardt and Smith (1985), in a study with elementary teachers, concluded that the 'skills associated with lesson structure and subject matter knowledge are obviously intertwined (p. 247)'. This is a reminder that without sound mathematical knowledge many pedagogical processes are of little benefit. This current study also alerts teacher educators, particularly in New South Wales, to the need to assist pre-service teachers with specific topics as each requires. This can only be done in the time available in pre-service courses by a screening test to identify possible specific areas of weakness and the design of appropriate programs for them. It is anticipated that this study will lead to such a process.

References

AAMT (2002). *Standards for Excellence in teaching mathematics in Australian schools*. Adelaide: AAMT.

Amarto, S. A. & Watson, A. (2003). Improving student teachers' understanding of multiplication by two digit numbers. In N. A. Pateman, B.J. Dougherty, & J. Zilliox (Eds.). *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education and the 25th Conference of PME-NA*. Honolulu, HI: PME

Bell, G., & Ho Woo, J. (1998). Probing the link between language and mathematical conceptualization. *Mathematics Education Research Journal*, 10, 1, 51-74.

Chick, H. L. (2002). Pre-service primary teachers' assessment of and performance on multiplicative numeracy items. In A. D. Cockburn, & E. Nardi (Eds.). *Proceedings of the 26th Annual Conference of the International Group for the Psychology of Mathematics*, (Vol. 2, pp. 233-240). Norwich, UK: PME.

Contreras, J. N., & Martinez-Cruz, A. M. (2001). An investigation of pre-service elementary teachers' solution processes to problematic story problems involving division of fractions and their interpretation of solutions. In M. van den Heuvel-Panhuizen (Ed.). *Proceedings of the 25th Conferenece of the Internaltional group for the Psychology of Mathematics*, (Vol. 2, 289-296). Utrecht: PME.

Leinhardt, G., & Smith, D. A. (1985). Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology*, 77(3), 247-271.

Lokan, J, Ford, K., & Greenwood, L. (1996). *Maths and Science on the line*. Melbourne: ACER.

Lokan, J, Ford, K., & Greenwood, L. (1997). *Maths and Science on the line*. Melbourne: ACER.

McMillan, J. H. (2004). *Educational Research* (4th Ed.). Boston: Pearson.

Mertler, C. A., & Charles, C. M. (2005). *Introduction to Educational Research* (5th Ed.). Boston: Pearson.

Morris, H. (2001). Issues raised by testing trainee primary teachers' mathematical knowledge. *Mathematics Teacher Education and Development*, 3, 37-47.

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics.* Reston, Va: NCTM.

Skemp, R. R. (1986). *The Psychology of Learning Mathematics* (2nd Ed.). Harmondsworth: Penguin.

Taplin, M. (1995). *Difficulties with the basic mathematics among pre-service teachers*. New Horizons in Education, 92, 3-19.

Taplin, M. (1998). Pre-service teachers' problem solving processes. *Mathematics Education Research Journal*, 10(3), 59-76.

von Glasersfeld, E. (1994). A radical constructivist view of basic mathematical concepts. In P. Ernest (Ed.) *Constructing mathematical knowledge*, (pp. 5-7). London: Falmer Press.

ANALYSING LONGITUDINAL DATA ON STUDENTS' DECIMAL UNDERSTANDING USING RELATIVE RISK AND ODDS RATIOS

Vicki Steinle and Kaye Stacey

University of Melbourne

The purpose of this paper is to demonstrate the use of the statistics of relative risk and odds ratios in mathematics education. These statistics are widely used in other fields (especially medical research) and offer a useful but currently under-utilised alternative for education. The demonstration uses data from a longitudinal study of students' understanding of decimal notation. We investigate the statistical significance of results related to the persistence of misconceptions and the hierarchy between misconceptions. Relative risk and odds ratio techniques provide confidence intervals, which give a measure of effect size missing from simple hypothesis testing, and enable differences between phenomena to be assessed and reported with impact.

This paper demonstrates some possibilities for analysing educational data, which draw upon methods that are widely used in reporting results of medical, environmental and epidemiological research. We believe that these measures provide very useful techniques for testing for statistical significance and reporting confidence intervals, which will enhance mathematics education research. Capraro (2004) draws attention to important recent policy changes within the American Psychological Association, evident in their publication manual, that stress the importance of researchers supplementing statistical significance testing with measures of effect size and confidence intervals, with many journals making them mandatory.

For those worried about deep vein thrombosis (DVT) after long flights, BUPA's website (Newcombe, 2003) cites Farrol Kahn as saying that "Several studies have shown [wearing flight socks] to be of benefit and it reduces the risk by up to 90 per cent." However, we can be cheered that "The researchers discovered the risk of developing DVT after a long-haul flight seemed to be low - at about 1 per cent of all long-haul passengers." Some of us can be further comforted by the observation of Runners' World (Reynolds) that "Being athletic accounts for ten times more victims than any other risk factor."

These reports in the popular press, along with reports in research literature, are mostly describing the results in terms of *relative* rather than *absolute* risk. So, for example, instead of commenting that DVT developed in only about 0.10% (10% of 1%) of passengers wearing flight socks, the website reports that the risk is reduced by up to 90%. This puts what might be seen as a tiny reduction in risk (just 90% of 1%) into perspective and shows its importance.

In this paper, we will show how these ideas of relative risk can be applied to educational data and discuss the benefits and issues arising. We illustrate the methods and challenges by some reanalyses of longitudinal data on students' understanding of

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 217-224. Melbourne: PME.

decimals. This was a cohort study, which tracked the developing understanding of over 3000 students in Years 4 – 10 at 12 schools for up to 4 years, testing them with the same test at intervals of approximately 6 months. Details of the sampling, the test and its method of analysis and many results have been described elsewhere; for example, Steinle and Stacey (2003), and Steinle (2004). For the purpose of this paper, it is sufficient to know that students are classified into 4 *coarse codes*, (A, L, S and U) on the basis of their answers to one section of this Decimal Comparison Test. In general terms, students in coarse code A are generally able to compare decimals; students in coarse code L generally treat longer decimals as larger numbers (for a variety of reasons); students in coarse code S generally treat shorter decimals as larger numbers (again for a variety of reasons); and coarse code U contains all remaining students. Answers to the other items in the test refine these coarse codes into 12 *fine codes*, which represent *expertise* (A1, which is a subset of A), various particular misconceptions, or students who cannot be classified. The longitudinal study traced student's understanding in terms of the coarse and fine codes and used this to examine questions such as which misconceptions are prevalent at different ages, whether some misconceptions are better to have than others, how often students appear to lose expertise, and whether students tend to move between misconceptions in predictable ways.

AN EXAMPLE USING RELATIVE RISK AND ODDS RATIOS

The main ideas in this paper will be illustrated by considering the question of whether it is *better* for a student to be in code L or S, i.e. from which of these groups are students more likely to become experts (i.e., move to code A1) on their next test? Table 1 summarises the data. Looking over the whole sample[1], there were 847 occasions where a student completed a test coded as S and then completed another test. On this subsequent test, 230 of the S students became experts and 617 did not, giving a 27% (230/847) chance of an S student becoming an expert and a 73% chance of an S student *not* becoming an expert. Similarly, from Table 1, there were 1257 occasions where a student completed an L test and was followed to their next test. The L students had 20% (251/1257) chance of becoming an expert. It seems that it is better to be an S student[2].

Conditions	Outcome_1 (A1 on next test)	Outcome_2 (not A1 on next test)	Total
Condition_1 (S)	$n_{11} = 230$	$n_{12} = 617$	$n_1 = 847$
Condition_2 (L)	$n_{21} = 251$	$n_{22} = 1006$	$n_2 = 1257$

Table 1: Numbers of A1 and non-A1 tests following S and L tests

[1] More careful analysis, as in Steinle (2004), would define the samples to reduce the effect of confounding variables such as age. The purpose here is to illustrate the procedures; the results here broadly match the refined analysis.

[2] This result is consistent with responses to individual items reported by large-scale studies around the world since the 1980s. See, for example, Foxman et al. (1985).

There are several ways in which this result can be tested statistically. A chi-squared test rejects the null hypothesis that the proportions of L and S students becoming expert are the same ($\chi^2 = 14.82$, d.f.=1, p=0.0001). However, the chi-squared test simply indicates the degree of evidence for association and does not give other information such as a confidence interval.

Analysing absolute differences in proportions becoming experts

A second method is to test whether the proportions of students going to A1 (moving to expertise) from S and L are the same. Assuming the counts for S and L are independent binomial samples, the difference of the proportions from Table 1 is distributed approximately normally with mean (0.27 – 0.20) and standard error 0.019 (Agresti, 1996). Hence a 95% confidence interval for the true difference in probabilities is $0.07 \pm 1.96 \times 0.019$, i.e. the interval (0.03, 0.11). This confidence interval provides more information than the chi-squared test. The interpretation of this confidence interval is in terms of *absolute differences* in the chance of moving to A1 from S and L. With 95% confidence, the percentage of S students becoming expert is between 3 and 11 more than for L students. In other words, if we have 100 L and 100 S students, and if 20 L students become experts on the next test, we can be confident that between 23 and 31 S students will become experts.

Analysing relative risk of becoming an expert

Another approach to testing whether two proportions are the same is to consider the *relative*, rather than the *absolute* difference in the proportions as above. This is especially useful when the proportions are small, as the absolute differences will also be small, although their ratios may be large. Because of its origins in epidemiological studies, the proportions of interest are classically labelled *risk*, but in our circumstance (where becoming an expert is a benefit rather than a harm) *chance* seems a more appropriate label. To answer the question of whether it is better to be S or L, the *relative risk (chance)* of becoming an expert on the next test is calculated as the ratio of the chances of S to A1 and L to A1. Figure 1, where the steps involved are demonstrated and given to two decimal places, shows that the relative chance of becoming an expert (from S and L in that order) is 0.27/0.20 = 1.36. This number indicates that an S student is 36% more likely to become an expert on the next test than is an L student.

Is this a significant difference? As indicted in Figure 1, the natural logarithm of this relative chance (i.e. relative risk) is normally distributed (Agresti, 1996; Bulmer, 2005), and the 95% confidence interval for the relative chance of becoming an expert is (1.16, 1.59). As 1.00 is not inside this interval, we are 95% confident that an S student is more likely to become an expert than an L student. In fact it is reasonable to say that an S student has at least a 16% greater chance of becoming an expert and possibly up to 59% more chance, compared with an L student. The best estimate is 36% more chance since the relative chance is 1.36. This is an intuitive way of presenting the results, with some impact.

Relative Risk (RR)		Odds Ratio (OR)	
Risk of Out_1 given Con_1 *Chance of A1 given S*	$p_{1,1} = \frac{n_{11}}{n_1} = \frac{230}{847}$ $= 0.27$	Odds for Out_1 given Con_1 *Odds of A1 given S*	$o_1 = \frac{n_{11}}{n_{12}} = \frac{230}{617}$ $= 0.37$
Risk of Out_1 given Con_2 *Chance of A1 given L*	$p_{1,2} = \frac{n_{21}}{n_2} = \frac{251}{1257}$ $= 0.20$	Odds for Out_1 given Con_2 *Odds of A1 given L*	$o_2 = \frac{n_{21}}{n_{22}} = \frac{251}{1006}$ $= 0.25$
Relative Risk of Out_1 (Con_1, Con_2) *Relative Chance of A1 (S, L)*	$RR_1 = \frac{p_{1,1}}{p_{1,2}} = \frac{0.27}{0.20}$ $= 1.36$	Odds Ratio for Out_1 (Con_1, Con_2) *Odds Ratio of A1 (S, L)*	$OR_1 = \frac{o_1}{o_2} = \frac{n_{11} \times n_{22}}{n_{12} \times n_{21}}$ $= 1.49$
$Ln(RR_1)$ is normally distributed around $Ln(p_{1,1}/p_{1,2})$ with $SE=$ $\sqrt{\frac{1-p_{1,1}}{n_{11}} + \frac{1-p_{1,2}}{n_{21}}}$	Ln 1.36 = 0.31 $SE = 0.08$	$Ln(OR_1)$ is normally distributed around $Ln(o_1/o_2)$ with $SE=$ $\sqrt{\frac{1}{n_{11}} + \frac{1}{n_{12}} + \frac{1}{n_{21}} + \frac{1}{n_{22}}}$	Ln 1.49 = 0.40 $SE = 0.10$
95% confidence interval for $Ln(RR_1)$ is $Ln(p_{1,1}/p_{1,2}) \pm 1.96 \times SE$	$0.31 \pm 1.96 \times 0.08$ = (0.15, 0.46)	95% confidence interval for $Ln(OR_1)$ is $Ln(o_1/o_2) \pm 1.96 \times SE$	$0.40 \pm 1.96 \times 0.10$ = (0.20, 0.61)
Conclusion based on whether 1 is included in the 95% confidence interval for RR_1 (which is the antilog of above).	95% CI for RR_1 is ($e^{0.15}$, $e^{0.46}$) = (1.16, 1.59) which does not include 1.00	Conclusion based on whether 1 is included in the 95% confidence interval for OR_1 (which is the antilog of above).	95% CI for OR_1 is ($e^{0.20}$, $e^{0.61}$) = (1.22, 1.83) which does not include 1.00

Figure 1: Calculations of Relative Risk and Odds Ratio from Table 1

Analysing the odds ratio for becoming an expert

The right-hand side of Figure 1 also provides an explanation of a related measure of association called the *odds ratio*. The odds of an S student becoming an expert on the next test are 230:617 = 0.37 and the odds for an L student are 251:1006 = 0.25. The *odds ratio* is therefore 0.37/0.25 = 1.49. The calculation of the 95% confidence interval for the odds ratio is (1.22, 1.83) (Agresti, 1996; Bulmer, 2005). As 1.00 is not inside this interval, we can be 95% confident that there is a true difference between the odds for an S and an L student becoming an expert on the next test.

The odds ratio is harder to interpret than the relative risk considered above, but it is widely used because it can be applied to a wider range of research designs than relative risk, and has strong mathematical properties giving it a role in other statistical testing. Moreover, when the risks of the event under both conditions are low (e.g. less than 10%), the odds ratio is a good approximation to the relative risk and can be interpreted as such. SPSS performs odds ratio calculations under the Crosstabs menu, as the *Mantel-Haenszel common odds ratio estimate.*

Features of Relative Risk and Odds Ratio Analyses

In using both relative risks and odds ratios, it is important to think carefully about what is a good comparison to demonstrate an effect. If we had carried out the odds ratio analysis for L compared to S (i.e. swapping conditions in Table 1) then the odds ratio would have been the reciprocal i.e. 1/ 1.49 (i.e. 0.67). Similarly, the relative risk of moving to expertise of L compared to S is the reciprocal of 0.20/0.27, i.e. 74%. When the relative risk is less than one, it is common to use *relative risk reduction* to present the results. Instead of saying that an L student has only 74% of the chance (risk) of becoming an expert that an S student has, it is common to talk about a 26% reduction in the chance of becoming an expert, as was done in the DVT example in the introduction. This again is an intuitive way of presenting the results with impact.

If the odds ratio test gives a significant result for S compared to L, will the test also be significant for L compared to S? The answer is yes: the only disadvantage is that the point estimates less than 100% are harder to describe in words, as indicated above. The formulas in Figure 1 show that the confidence interval would have been obtained from the reciprocals (1/1.83, 1/1.22) = (0.55, 0.82). So, if one of these confidence intervals includes 1.00 (so that the null hypothesis is accepted), then the other will automatically. The same situation applies for relative risk: if S compared to L is significant, then L compared to S will be significant. The choice of whether to discuss condition_1 to condition_2 or vice versa is therefore a choice between interpreting ratios greater or less than one.

Another important question is: if a test of the relative risk (or odds ratio) shows a significant difference in the chances that an event E happens, would these tests show significant differences in the chances that the event not-E happens? In our example, both the relative risk and odds ratios show S students have more chance of becoming expert than L students. Is it also the case that there is a significant difference in the chance of S students, compared with L students, *not* becoming an expert on their next test? Note that in this case, *risk* is a good term because not becoming an expert is a perceived harm. For the odds ratio, this result is true – a significant result for event E implies a significant result for event non-E. This is an advantage of the odds ratio analysis. This situation, however, does not automatically follow for relative risk analysis. For example, the relative difference between risks of 1% and 2% for event E is much larger than the relative difference for risks of 99% and 98% for event non-E.

ESTIMATES OF RISK IN THE LONGITUDINAL DATA

In this section, we apply the techniques described above to questions of hierarchy (which misconceptions are *better* to have), and persistence (are some misconceptions likely to trap students more than others), and consider some of these questions by comparing students in primary school (Years 4 – 6) with secondary school students

(Years 7 – 10). As noted above, Steinle (2004) presents an analysis where confounding variables related to the sampling are treated carefully. The results presented here are in agreement with those from more careful analyses and therefore summarise some of the major results of the refined data analysis.

Hierarchy: which misconceptions are best to have?

The preceding analysis demonstrated that a student in code S is more likely to become an expert (A1) by the next test than a student in code L. The 95% confidence intervals of both relative risks (RR) and odds ratios (OR) determined in Figure 1 are provided graphically in Figure 2 (see the lowest two rows). Confidence intervals which are larger than 1.00 indicate a significant difference with the condition first listed having the larger result. So, it is clear being in L is *worse* than being in S, but which is the *best* code to have: S or U or A?

The intermediate rows of Figure 2 show the confidence intervals for both measures (RR and OR) for a comparison of code U with code S. The RR indicates that a U student is between 1.2 and 1.6 times more likely than an S student to be an expert on the next test. (Typically, students who answer the test inconsistently and hence are not classified by the test, belong to the group U). The top two rows in Figure 2 show that in turn, students in code A are more likely than those in code U to be experts on the next test. This is to be expected, since the numerically largest group in code A is in fact the experts (A1). Note that the confidence interval for OR in row 2 is off the graph to the right (it is between 7.8 and 10.9).

Together these results show that the hierarchy of these four codes is (highest to lowest) A, U, S, then L. It is best to be an expert or near expert (i.e. in A), then it is best to be undecided (U), then to have a shorter-is-larger (S) misconception and worst to have a longer-is-larger (L) misconception.

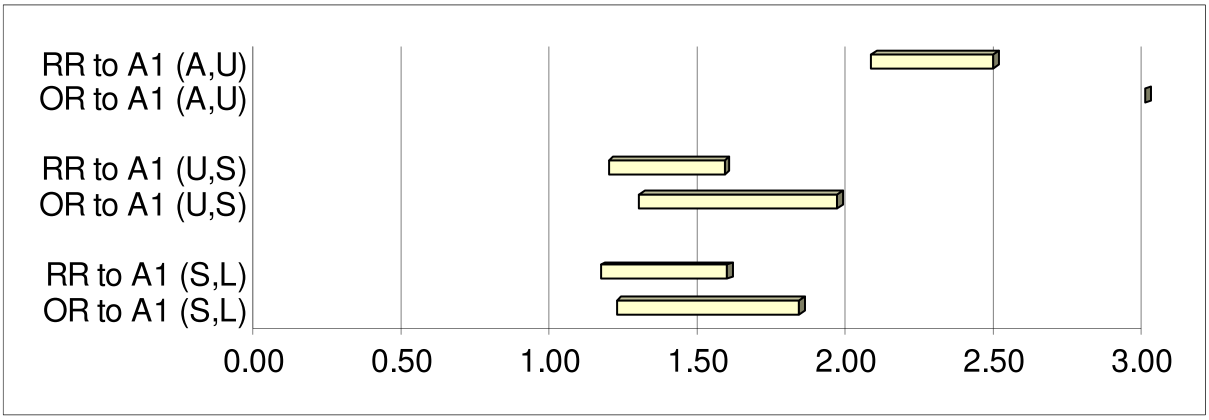


Figure 2: Confidence intervals for RR and OR analyses of the movement to expertise from various codes. (Note. Row 2 is off the scale to the right, so not shown)

Persistence: do some misconceptions keep students longer than others?

Steinle (2004) examined various measures of persistence – how often students retest in the same code on their next test. The basic finding was that 89% of A1 students retest as A1 at the next test (i.e., *persist* in A1), compared to 44% of L students persisting in L, 38% of S students persisting in S and 29% of U students persisting in U. Note that persisting in A1 is desirable, while persisting in other codes is not. Closer analysis showed interesting variations between older and younger students, some of which are summarised graphically in Figure 3.

The top two rows of Figure 3 show that, as both confidence intervals include 1.00, there is *not* a significant difference in the persistence in A1 by students in Secondary school compared with students in primary school. The next two rows are to the left of 1.00 indicating that there is a significant difference and it is the younger L students who have higher levels of persistence than the older L students. Rows 5 and 6 indicate that the opposite result holds true for the S students. In particular, row 5 indicates that older S students are approximately 1.5 times more likely to persist in S than the younger S students. The last two rows indicate that there is no significant difference between older and younger U students in their persistence in U.

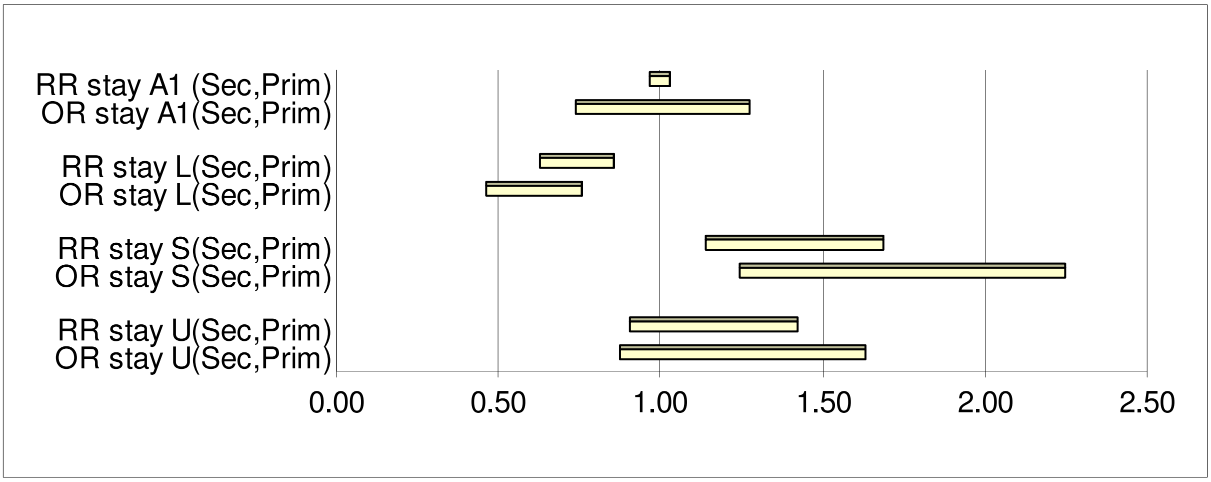


Figure 3: Confidence intervals for RR and OR analyses of the persistence in various codes between students in Secondary and Primary schools

CONCLUSION

The main aim of this paper has been to explore the application of techniques of relative risk and odds ratio analysis to our educational data. Reporting relative risk (or reduced relative risk) is very common in the popular press as well as in the scientific literature in other fields, and so it seems worthwhile investigating it for our context. There are several advantages, which relate to the ease of interpreting the change in risk and the way in which it provides an alternative presentation of results in possibly a more memorable form. Contrast these two statements: *An S student has an extra 30% to 40% chance of becoming an expert, compared with an L student*, with, *The rate that S students become experts (27%) is 7% more than the rate of L* *students becoming experts (20%)*. The difficulty of describing the last absolute, rather than relative, result highlights the inadequacy of ordinary language in distinguishing absolute and relative change, especially when it is a change in rate or percentage that is being discussed. Analysing relative risk approach has advantages here, along with providing confidence intervals.

We expect that some members of the mathematics education community will be uncomfortable when we draw upon the medical context for research methods, even to analyse results. It is inherent in applying these concepts, to take an undesirable outcome (such as a disease or even death) as an implied metaphor for mathematical error or misunderstanding. When using any metaphor, different aspects of the metaphor will be carried across to the target situation by different people. Our position is that we can focus on the positives, as mathematical error as something to be overcome by joint effort of student and teacher. Other people may feel some discomfort in the use of techniques from medical research because of concerns about the way in which medical research has been simplistically held up as the “gold standard” for educational research in debates on funding principles in the USA (NCTM Research Advisory Committee, 2003). We contend that choice of methodology or data analysis techniques should not be judged by political or social associations, but by scientific reasoning. On the other hand, terminology needs to be chosen with sensitivity to the social needs in the area of application. Analysis of odds ratio and relative risk seems to have much to offer, although the language with which they are expressed needs modification.

References

Agresti, A. (1996). *An introduction to categorical data analysis*. John Wiley: New York.

Bulmer, M. (2005). *A Portable Introduction to Data Analysis*. 3rd edition. The University of Queensland Printery.

Capraro, R. M. (2004). Statistical significance, effect size reporting, and confidence intervals: Best reporting strategies. *Journal for Research in Mathematics Education*, 35(1), 57 – 62.

Foxman, D., Ruddock, G., Joffe, L., Mason, K., Mitchell, P & Sexton, B. (1985). *A Review of Monitoring in Mathematics 1978 to 1982. (Vol. 1).* London: Dept of Ed.& Science.

Newcombe, R. (2003). DVT plane risk may be lower than thought. BUPA Health news. http://www.bupa.co.uk/health_information/html/health_news/ Accessed 11 Jan 2005.

NCTM Research Advisory Committee. (2003). Educational research in the No Child Left Behind environment. *Journal for Research in Mathematics Education*, 34(3), 185-190.

Reynolds, M. (n.d.) Air Travel: Runners at Risk? Accessed 11 Jan 2005, from http://www.runnersworld.com/article/0,5033,s6-188-0-0-1392,00.html

Steinle, V., & Stacey, K. (2003). Grade-related trends in the prevalence and persistence of decimal misconceptions. In N.A. Pateman, B.J. Dougherty & J. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 259 – 266). Honolulu: PME.

Steinle, V. (2004). *Changes with Age in Students' Misconceptions of Decimal Numbers*. Unpublished PhD, University of Melbourne, Melbourne.

GIRLS JOURNEY TOWARDS PROPORTIONAL REASONING

Olof Bjorg Steinthorsdottir

University of North Carolina-Chapel Hill

This study focused on 26 girls' development of proportional reasoning in two fifth-grade classrooms in Iceland. The students were used to instructional practices that encouraged them to devise their own solutions to mathematical problems. The results supported four levels of proportional reasoning. Level 1, girls showed limited ratio knowledge. Level 2, they perceived the given ratio as an indivisible unit. Level 3, students conceived of the given ratio as a reducible unit. And at Level 4 students no longer thought of ratios exclusively as unit quantities, but understood the proportion in terms of multiplicative relations. The results suggest that students can reach level 3 reasoning with less struggle than it takes to achieve level 4, which suggests that the knowledge needed to operate on level 3 was within their reach.

OBJECTIVES

This study investigates the developmental of proportional reasoning of girls in two fifth-grade classes in Iceland. The purposes of this study was to further investigate four levels of proportional reasoning identified in a pilot study that the author conducted in collaboration prior to the study reported here (Carpenter et al. 1999). In particular, do the four levels describe the pathway of a population of Icelandic girls before, during, and after they have engaged in a unit focused on proportional reasoning? Secondly, what evidence is there for the existence of Level 2, Level 3, and Level 4 ways of reasoning in students' verbal protocols? And finally how does instruction that is focused on students' reasoning help students make the transition from level to level?

BACKGROUND AND THEORETICAL ORIENTATION

Proportional reasoning represents a cornerstone in the development of children's mathematical thinking (Inhelder & Piaget, 1958; Resnick & Singer, 1993). Ratio and proportion are critical ideas for students to understand; however, although young children demonstrate foundations for proportional reasoning, students are slow to attain mastery of these concepts.

Many studies on children's proportional reasoning provide evidence of various influences on students' thinking about proportion. Among these influential factors are the problem numerical structures[1]. The number structure refers to the multiplicative relationship within and between ratios in a proportional setting. A "within"

[1] Another term commonly used is "numerical structure". I will use the term "number structure" or "numerical relationship" when talking about the multiplicative relationship that is presented in the problem.

relationship is the multiplicative relationship between elements in the same ratio[2] whereas a “between” relationship is the multiplicative relationship between the corresponding parts of the two ratios.

The multiplicative relationship can be integer or noninteger. For example, the problem $\frac{2}{4}=\frac{12}{x}$ has integer multiples both within the given ratio ($2\times 2=4$) and between ratios ($2\times 6=12$). In a noninteger ratio, on the other hand, occur when at least one of the multiplicative relationships (within the given ratio or between the two ratios) is not an integer (Freudenthal, 1983; Karplus, Pulos, & Stage, 1983). For example, the problem $\frac{8}{5}=\frac{48}{x}$ has an integer multiple between the two ratios ($8\times 6=48$) but the within-ratio relationship is noninteger ($8\times \frac{5}{8}=5$ or $5\times 1\frac{3}{5}=8$) (Abromowitz, 1975; Freudenthal, 1983; Karplus et al., 1983; Tourniaire & Pulos, 1985).

From Qualitative to Multiplicative Reasoning

Researchers have hypothesized that students’ learning of proportional reasoning can be described as a learning trajectory[3] (Carpenter et al., 1999; Inhelder & Piaget, 1958; Karplus et al., 1983). The literature on proportional reasoning reveals a broad consensus that proportional reasoning develops from qualitative thinking to multiplicative reasoning (Abromowitz, 1975; Behr, Harel, Post, & Lesh, 1992; Confrey, 1995; Inhelder, & Piaget, 1958; Kaput & West, 1994; Karplus et al., 1983; Kieren, 1993; Noelting, 1980a, 1980b; Resnick & Singer, 1993; Vergnaud, 1983).

Studies of individual cognition and the development of proportional reasoning have identified three categories of strategies that students use in reasoning about proportional relationships: qualitative, additive, and multiplicative (Behr, Harel, Post, & Lesh, 1992; Inhelder & Piaget, 1958; Karplus et al., 1983; Kieren, 1993; Resnick & Singer, 1993). These strategies represent different levels of sophistication in thinking about proportions.

Research with preadolescent students indicates that their representation of situations that involve ratio and proportion occurs on an informal basis long before they are capable of treating the topic quantitatively. A qualitative reasoning strategy is based on an informal or intuitive knowledge of relationships without numerical quantification (Kieren, 1993). Next is additive reasoning, which requires quantification of the ratio relationships. The process of additive reasoning is often referred to as a buildup strategy. For example, consider the following problem:

---

[2] Here I define ratio as the relationship between two quantities that have two different measure units.

[3] By learning trajectory, I am referring to the path that student reasoning travels as students’ understanding of proportion develops. As students reasoning develops, so too does student ability to solve increasingly complex problems. Corresponding to their increasing ability to solve difficult problems, students’ strategies for solving problems also get more complex and more mathematically sophisticated.

It is lunchtime at the Humane Society. The staff has found that 8 cats eat 5 large cans of cat food. How many large cans of cat food would the staff members need to feed 48 cats? (In an algebraic equation, $\frac{8}{5} = \frac{48}{x}$.)

A student might use a buildup strategy to arrive at the solution of 30 cans (Figure 1).

Cats	8	16	24	32	40	48
Cans	5	10	15	20	25	30

Figure 1. Buildup strategy in the form of a ratio table for the problem $\frac{8}{5} = \frac{48}{x}$

Buildup strategies are often observed during childhood and adolescence and appear to be the dominant strategy for many students these ages (Kaput & West, 1994; Tourniaire & Pulos, 1985). Although the buildup strategy can be used successfully in many cases, developing more sophisticated reasoning is crucial for solving more complex problems and understanding the complexity of the multiplicative relationship (Tourniaire & Pulos, 1985). Proportional reasoning is multiplicative and therefore the transition from buildup strategies to multiplicative strategies is considered to be a benchmark of development (Inhelder & Piaget, 1958; Karplus et al., 1983; Noelting, 1980a, 1980b).

When solving simple proportion problems, two types of multiplicative strategies have been identified: "within ratio" and "between ratios" (Karplus et al., 1983; Noelting, 1980a; Vergnaud, 1983). The within-ratio strategy is based on applying the multiplicative relationship within one ratio to the second ratio to produce equal ratios. The between-ratio strategy is based on determining the multiplicative relationship between corresponding parts of the two ratios to create equal ratios. For example, consider the following problem:

A hiking group is organizing a field trip, and they estimate that it will take 3 hours to walk 9 km. How long will it take the group to walk 33 km? (In an algebraic equation, $\frac{3}{9} = \frac{x}{33}$.)

Either ratio strategy—within or between—can be used to find the answer. A student using the within-ratio strategy would notice that the distance is 3 times the hours ($3 \times 3 = 9$) and therefore the same should apply to the target ratio, which would result in 11 hours ($11 \times 3 = 33$). A student using the between-ratios strategy would look for the multiplicative relationship between 9 km and 33 km, realize that $33 = 9 \times 3\frac{2}{3}$, and multiply $3 \times 3\frac{2}{3} = 11$ to get the answer.

While earlier research on students' reasoning relied on within-ratio and between-ratios strategies to analyze students' thinking (Abramowitz, 1975; Karplus et al., 1983; Vergnaud, 1983), Lamon (1993; 1994; 1995) offered a different lens through which to understand students' development of proportional reasoning. Lamon proposed two processes, unitizing and norming, as central to the development of proportional reasoning. Unitizing involves the construction of a reference unit from a given ratio relationship. Norming refers to the reinterpretation of another ratio in terms of that reference unit (Lamon, 1994; 1995). For example, consider the previous problem about the Humane Society (in an algebraic equation, $\frac{8}{5}=\frac{48}{x}$).

Using norming and unitizing, a student might interpret the target ratio as a multiple of the given ratio $\frac{8}{5}$. Therefore, she needs six groups of the 8-to-5 ratio unit in order to get an answer for 48 cats. For their calculations, students might use methods such as buildup strategies or direct multiplication. A student using a between strategy, on the other hand, would consider a single quantity in the given ratio and operate on that quantity, recognizing that the same operation must apply to the second quantity. Referring to the same equation, $\frac{8}{5}=\frac{48}{x}$, the student multiplies 8×6 to get 48 then multiplies 5×6 to get the answer, 30. When unitizing and norming, the student thinks of the ratio as a complex unit. The student can operate on the unit $\frac{8}{5}$ by adding, multiplying, or reducing—but each operation is interpreted as creating a new unit that preserves the relationship within the given ratio. In other words, when unitizing or norming, the operation is performed on the ratio as a unit instead of individual terms in the ratio.

Using Lamon's (1994; 1995) operation of unitizing and norming, the author and team of colleagues identified 4 levels of reasoning in a pilot study conducted in the US in one classroom over a 2 weeks period (Carpenter, et. al. 1999). At Level 1, students showed limited ratio knowledge. The most common strategy was finding the additive differences within and between the ratios. Level 2 is characterized by the perception of the ratio as an indivisible unit. Students at this level are able to combine the ratio units together by repeated addition of the same ratio to itself or by multiplying that ratio by a whole number, but they cannot solve proportion problems in which the given ratio has to be partitioned such as, problems in which the target ratio is a noninteger multiple of the given ratio (e.g., $\frac{8}{12}=\frac{42}{x}$ or $\frac{8}{3}=\frac{2}{x}$). At Level 3, the given ratio is thought of as a reducible unit. Therefore, students at Level 3 can scale the ratio by nonintegers. An example of a Level 3 strategy combines the reduction of the given ratio with a buildup strategy by using either addition or multiplication. Students at Level 4 think of ratios as mere than just as unit quantities. They recognize the relation within the terms of each ratio and between the corresponding terms of the ratios.

METHOD AND ANALYSIS

The subjects of this study are the 26 fifth-grade[4] girls in two classrooms at one of Reykjavik's public schools. I observed every math class throughout the course of the study, taking on the role of "participant observer". During data collection, students worked on 24 problems that were created during 10 weeks of instruction. Each set of problems was composed of three problems with the same contextual structure but with different multiplicative relationships in the proportion. The numbers were chosen to further students' understanding of proportion and to aid their recognition of

[4] Fifth grade in Iceland refers to children that turn 10 years old in the year they start 5th grade.

the multiplicative relationships in the two ratios in the proportion. By varying the multiplicative relationship in the problems, sets of problems were created to distinguish between Level 2 and Level 3 students and between Level 3 and Level 4 students

The pretest and the posttest were created using the same criteria as the instructional problems in regard to the number structure of the problems. The pretest comprised of 18 problems in three sets with different multiplicative relationships. The posttest comprised 12 problems. Students' problems solutions strategies were collected. During instructions students worked both individually and in groups on their problems. All the written work the students produced and artifacts from their work were collected. Also all whole-classroom discussions were videotaped and transcribed.

The criteria for determining what each student's level of reasoning was based on upon which problems she could solve and which problem she could not. Level 1 students use incorrect strategies. The most common strategy is to find the additive difference within the given ratio or between the ratios and apply that difference to the target ratio.

At Level 2 students perceive the given ratio as an indivisible unit and interpret it as a unit whole. They are able to solve only problems that have an integer relationship between the ratios, such as $\frac{2}{6}=\frac{x}{36}$. The numbers in the target ratio have to be bigger than the numbers in the given ratio. At Level 3 students perceive the given ratio as a divisible unit. The given ratio is interpreted as a unit whole. Level 3 students are able to solve problems that involve both an integer and a noninteger relationship, such as $\frac{6}{12}=\frac{15}{x}$. Finally at Level 4 students no longer think of ratios exclusively as unit quantities. They can take into account both the within and between relationships and choose the one relationship that is easier to calculate. They are not limited to building up or partitioning the given unit.

RESULTS

Question 1. The four-level model of proportional reasoning identified in the pilot study proved to be a beneficial tool to analyze their work. Analyzing the pretest the classification of students' solutions resulted in the creation of a transitional level "emerging Level 3". On both pre- and posttest the results show a perfect fit; students on Level 2 were not able to solve any of more complex problems that emerging Level 3 students were able to solve successfully, nor were the emerging Level 3 students able to solve any of the most complex problems that Level 3 students were able to solve with success.

The problems were structured to discriminate between students at different levels of reasoning. Problems that could be solved by students reasoning on Level 2 had an integer relationship between the ratios and involved enlarging (e.g., $\frac{2}{8}=\frac{x}{24}$). Students reasoning on Level 3 could solve problems that were previously mentioned as well as problems that have a noninteger relationship between the ratios ($\frac{5}{6}=\frac{x}{21}, \frac{15}{10}=\frac{6}{x}$).

Problems that proved to be transition problems from Level 2 to Level 3 were the problems that had a scale-down number structure such as $\frac{8}{24} = \frac{2}{x}$. The difference between Level 2 and Level 3 reasoning is the need to scale down the given ratio. During the emerging Level 3 stage, students are able to scale down by whole numbers but they cannot use their knowledge of scaling down within other number structures. Strategies that students used to solve the problem distinguished between Level 3 and Lev el 4 reasoning.

On the pretest, 35 percent of the girls displayed Level 1 reasoning. Around 40 percent exhibited Level 2 reasoning. Twenty-three percent of the girls were emerging Level 3. One girl showed Level 3 reasoning on her pretest. On the posttest only 3 girls reached Level 4 thinking, whereas more than 80 percent reached Level 3 thinking. Therefore, it is evident that reaching Level 4 thinking involves a very complex thinking that most of the girls had not yet adopted.

Question 2. Throughout the course of the study, girls were thinking about the given unit as a single entity that they then operated on to reach their target number. The buildup strategy, the most common strategy, provides clear evidence of the ways in which students understand the given ratio as a single unit that they can then build up or build down. Common explanations from the girls were related to the idea that everything they did had to apply to both terms of the ratio.

Following is an example of a Level 2 girl's explanation of her strategy for the following problem to support that argument: It is lunch hour at the humane society. The staff members have found out that 8 cats need 5 large cans of cat food. How many large cans of cat food would they have to have if they were to feed 48 cats?

Student: I did it—like, here is 8 and then 5 cans of food, and then again—then there is 8 and 16 cans of food until I…reached 48 cats, and then the answer is 30 cans of cat food.

Teacher: How did you know that you should have 8 groups?

Student: Well, I did not know that because I did 8:5 and 8:5 and 8:5 and added the 8s together until I had 48.

She explained her strategy in terms of the unit as an entity. She operated on the unit of 8:5 until she reached her target number of 48. She did not think in advance about the number of groups she had to use; rather, as she is building her units, she is adding on until she know where to stop.

Question 3. Nina represents close to 30 percent of the students. She was a typical Level 1 student at the time of her pretest. In the beginning of the unit, Nina needed a little scaffolding to help her move away from her additive thinking. She quickly solved her first problem ($\frac{2}{6} = \frac{x}{36}$) by finding the additive difference between the ratios. Her first answer was 32. After only a few questions, she was able to get on the right track.

Teacher: What if you had 4 cans of food, how many cats could you feed?

Nina: 8 cats.

Teacher: We know that 2 cans of cat food can feed 6 cats. We get 2 more cans, and can they only feed 2 more cats?

Nina: No, 2 cans can feed 6 cats, not 2.

Teacher: What does that mean, then?

Nina: Well, it is like if 2 cans can feed 6 cats, then another 2 cans can feed another 6 cats.

Teacher: Think about that more and how you can solve your problem differently. I will come back to you.

The teacher left Nina to grapple with her new ideas about the problem. When it came to sharing time, Nina had not yet figured out how to go about solving the problem with her newfound knowledge. A couple of the strategies that were shared were buildup strategies in which students took the given unit $\frac{2}{6}$ and built it up unit-by-unit to reach the target number. Nina really liked that strategy and utilized it with success. When the teacher got to Nina, she had solved the problem by using a buildup strategy. When the teacher asked her to explain what she had done, it became clear that she understood clearly what the numbers in the buildup strategy stood for.

Nina: First there were 2 cans and 6 cats, then next there would be 4 cans for 12 cats and—

Teacher: And why is that?

Nina: It's like first there were 2 cans and 6 cats, then there were 2 more cans and 6 more cats would eat that, and that is like having 4 cans and 12 cats.

When Nina started working on the second problem, she paused a little bit and thought hard before solving the problem with a buildup strategy. The scaffold from the teachers and from the discussion of different strategies provided a basis for girls to attain more advanced levels of proportional reasoning. The case of Nina shows how a student was afforded the opportunity to learn from her teacher and from other students using more advanced thinking. This example also illustrates how less advanced students may learn from listening to other students explain more efficient strategies than they commonly use and how a class may build on ideas that are distributed among members of the class.

Reference

Abromowitz, S. (1975). *Adolescent understanding of proportionality: The effects of task characteristics.* Unpublished Ph.D., Stanford University, Palo Alto, CA.

Behr, M., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296–333). New York: Macmillan.

Carpenter, T. P., Gomez, C., Rousseau, C., Steinthorsdottir, O. B., Valentine, C., Wagner, L., et al. (1999, April 19–23). *An analysis of student construction of ratio and proportion understanding.* Paper presented at the American Educational Research Association, Montreal, Canada.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht, The Netherlands: D. Reidel.

Inhelder, B., & Piaget, J. (1958). The growth of logical thinking from childhood to adolescence. New York: Basic Books.

Kaput, J., & West, M. M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 235–287). Albany, NY: State University of New York Press.

Karplus, R., Pulos, S., & Stage, E. K. (1983). Proportional reasoning of early adolescents. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 45–90). New York: Academic Press.

Kieren, T. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T. Romberg (Ed.), *Rational numbers: An integration of research* (pp. 49–84). Hillsdale, NY: Erlbaum.

Lamon, S. (1993). Ratio and proportion: Connecting content and children's thinking. *Journal for Research in Mathematics Education, 24* (1), 41–61.

Lamon, S. (1994). Ratio and proportion: Cognitive foundations in unitizing and norming. In J. Confrey (Ed.), *The development of multiplicative reasoning in the learning of mathematics*. Albany: Sate University of New York Press.

Lamon, S. (1995). Ratio and proportion: Elementary didactical phenomenology. In B. P. Schappelle (Ed.), *Providing foundation for teaching mathematics in the middle grades* (pp. 167–198). Albany: State University of New York Press.

Noelting, G. (1980a). The development of proportional reasoning and the ratio concept: Part 1. Differentiation of stages. *Educational Studies in Mathematics, 11*, 217–253.

Noelting, G. (1980b). The development of proportional reasoning and the ratio concept. Part 2. Problem-structure at successive stages. Problem-solving strategies and the mechanism of adaptive restructuring. *Educational Studies in Mathematics, 11*, 331–363.

Resnick, L. B., & Singer, J. A. (1993). Protoquantitative origins of ratio reasoning. In T. Romberg (Ed.), *Rational numbers: An integration of research* (pp. 107–130). Hillsdale, NJ: Erlbaum.

Tourniaire, F., & Pulos, S. (1985). Proportional reasoning: A review of the literature. *Educational studies in mathematics, 16*, 181–204.

Vergnaud, G. (1983). Multiplicative structures. In M. Landau (Ed.), *Acquisition of mathematics concepts and processes* (pp. 127–174). New York: Academic Press.

UNIVERSITY STUDENT PERCEPTIONS OF CAS USE IN MATHEMATICS LEARNING

Sepideh Stewart and Michael O. J. Thomas
The University of Auckland

While research has shown many potential benefits of computer algebra system (CAS) calculators in the learning of mathematics, it has also identified a number of obstacles to attaining them. Most research has been conducted with secondary school students, but this study considered the manner in which beginning university students perceive the benefits and difficulties associated with CAS use. The results describe a number of obstacles to such use and the ways students employed them in their study.

BACKGROUND

One of the technologies that has been of special interest over the past decade is the implementation of CAS on calculators. There has been enthusiasm in many quarters for the possibilities that this technology presents, but some research has shown that the potential is not always realised. For example, Hong, Thomas, and Kiernan (2000) have shown there is a problem when students come to rely on CAS since it can then undermine their learning, preventing them from learning important concepts and procedures. Another important area, and one addressed in this study, is the negative reaction of certain students to the use of CAS, as recorded by Bergsten (1996). While for some this attitude stems from their background in mathematics (where, as Artigue (2002) explains, by-hand techniques may have an elevated status), for others it is learned. It may originate in the challenge of the mathematical interface difficulties (particularly in terms of input and output formats), and the 'black box' syndrome, where students are unable to attach meaning to processes carried out within the CAS. While the release of the CAS potentiality depends on both instrumentalisation and instrumentation of the CAS tool (Rabardel & Samurcay, 2001), or how one adapts to the tool and how one adapts the tool to one's mathematical needs, how these processes should proceed is still unclear, as neither of the steps is inherently simple.

A common idea is that because mathematically experienced users can see wide-ranging benefits to CAS use in learning that students will also gain from these. For example Thomas, Monaghan and Pierce (2004) record perceived benefits including the multi-representational nature of CAS, the idea that CAS can reduce user time on procedures freeing them to concentrate more on learning concepts, a propensity for experimentation and generalisation, and a strong focus on algebraic insight, functions and parameters. However, there are obstacles in the path of students engaging in instrumentalisation and instrumentation of CAS in order to get to the point where they can access the benefits of the instrument. Some of the obstacles to use are associated with the tool itself, such as the constraints of the input and the difficulty in translating output to mathematical notation (Drijvers, 2002) and the influence of the social environment (teacher and peers) on student choices.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 233-240. Melbourne: PME.

Describing various categories of student CAS use they found, Thomas and Hong (2004) mention direct, straightforward procedures, direct complex procedures, checking procedural by-hand work, procedures within a complex process, and investigating conceptual ideas. However, they report finding little of the last kind of activity, and a lot of the first three, supporting the idea that instrumentalisation and instrumentation of CAS, where students incorporate CAS into their mathematical thinking, can be a slow process. It takes time and teacher direction for students to learn to decide what CAS is useful for, and what might be better done by hand, and how to integrate the two. Thomas and Hong (*ibid*) maintain that teachers using CAS in the classroom have to be both aware of the possibilities provided by the technology, and be confident in the roles they decide to implement. They should engage students in a discussion of the meaning of the CAS techniques and the conceptions that are being developed (Drijvers, 2002). While most CAS research has concentrated on the secondary school classroom, the nature of the didactic contract (Brousseau, 1997) in the lecture theatre and tutorial room is often quite different from the school situation, and hence instrumentalisation and instrumentation of CAS may proceed differently. In this study we were interested in university students' perceptions of the factors influencing their use or non-use of CAS in a first year mathematics course, the CAS instrumentation, and the attitudes towards it.

METHOD

This research took place in June 2004 and comprised an initial case study of first year Maths 108 mathematics and science students from The University of Auckland. Maths 108 covers both calculus and elementary linear algebra and TI-89 CAS calculators were recently been introduced as an optional component of the course. For the first time in 2004, the department included detailed instructions about use of CAS TI-89 calculators as part of the Maths 108 coursebook. At the end of the last tutorial of the course each student was given a questionnaire, and of the 1014 students, mostly 19-22 years old, enrolled in the paper 167 students (16.5%), from among those attending tutorials, agreed to take part in the study. Of these, 24 (5 females, 18 males, 1 unknown) had been using the TI-89 during the course, while the others had not. This mirrored the total CAS take-up for the course of around 15% of the students. None of the students had ever used a CAS calculator before this course. The questionnaire (see Figure 1—format changed) was divided into two sections (A & B) for those using and those not using the CAS. It comprised both 3-point Likert scale and open questions and addressed student attitudes toward the use of the CAS calculator, the value of the CAS, when they decided to use it and how, and their possible reliance upon it. For those without a calculator we wanted to know why they had decided not to purchase one, and their perspective on its potential value in learning mathematics. Since the input from teachers is a crucial variable in attitudes to technology use we also asked the lecturers from the six streams of the course for observations on the student use of the CAS.

Section A: For each statement below please circle the number which most closely corresponds to your own view. **(For TI-89 users only)**

1. The TI-89 CAS calculators do not improve my understanding of mathematics.
2. I waste a lot of time trying to get the TI-89 CAS calculator going.
3. I am glad that I can use the TI-89 CAS calculator during the exam.
4. TI-89 CAS calculators help me to visualise the problems.
5. I can solve problems using TI-89 CAS calculators even though I don't understand the theory.
6. My answers are usually different from the answers that the TI-89 CAS calculator gives me.
7. I think the TI-89 manual at the back of my book is very helpful.
8. I often check my answers using the TI-89 CAS calculator
9. I would like to learn more about the TI-89 CAS calculators, so I can use them fully.
10. I believe technology is the way to go to learn mathematics.
11. I hope to use my TI-89 CAS calculator in other courses when applicable
12. My lecturers are very supportive and encouraging in using the TI-89 CAS calculators.
13. I explore the TI-89 by myself.
14. I find it difficult to decide when to use the TI-89 in maths problems.
15. Since I have been using the TI-89 CAS calculator, I have forgotten how to do the basic skills.
16. I like to use both TI-89 CAS calculator and pen and paper when working on maths problems.
17. I only use TI-89 CAS calculator when I am stuck using pen and paper for mathematics problems.
18. I bought a calculator at the beginning of the year but never used it, so I think I wasted my money.
19. I find all the TI-89 menus and key presses too difficult to remember.
20. TI-89 CAS calculators make mathematics fun.
21. There is not enough support outside lecture time for using the TI-89 calculator.
22. I believe the TI-89 gives me an unfair advantage in learning mathematics.

Open Questions (For TI-89 users only)

- What do you like using the TI-89 calculator for? (Why?)
- How do you feel about using the TI-89 calculators this year?
- Should the TI-89 calculators be used in the mathematics lectures? If so, how?
- How do you decide when to use the TI-89 calculator?
- Has the TI-89 calculator helped you learn any mathematics? If so, what?
- How much do you feel you rely on your TI-89 calculator? For example, could you still do the problems without having one?
- Do you just try to apply the applications of the TI-89 calculators in the course manual or do you explore for yourself?
- Did you buy the TI-89 calculator at the beginning of the year but never used it? (If yes, why?)
- Why did you buy a TI-89 calculator?
- When did you find out there were notes at the back of your course book (notes on the CD for 150 students) on TI-89 calculators?

Section B: For each statement below please circle the number which most closely corresponds to your own view. **(For those who don't use the TI-89).**

1. I can do everything without a TI-89. In other words I don't need one.
2. The TI-89 CAS calculators are far too expensive.
3. The lecturers are not using them so why should I bother.
4. I would have liked to have one if they were more affordable.
5. I really don't know if they are good or not until I have tried one.
6. I believe students shouldn't be allowed to use them in test and examinations, because it is not fair on those of us who can't afford one.
7. If someone showed me how useful they can be I might consider buying one.
8. I wish I had a TI-89 that I could use.

Please write below any other comments you would like to make.

Figure 1: The Questionnaire

RESULTS

The student results that follow need to be set in the context of the lecturers' attitude to the introduction of the CAS. Lecturing style and approach are not yet prescribed in the department, and in 2004 none of the lecturers used the CAS, probably due to their inexperience with them. This is a crucial point, since as Kendal and Stacey (1999) found, teachers' *privileging* of approaches can differentially affect student learning. It was clear that the Maths 108 lecturers were strongly privileging by-hand work, and so students could be expected to favour this approach. In addition, the majority of tutors on the course were also not familiar with the CAS calculators. However, the first named researcher was a tutor for Maths 108 and was available 4 hours a week in an assistance room to help with the CAS calculators. From the 6 lecturers involved in teaching this course, only 2 (lecturers A and B) replied to our questionnaire. Both said they had encouraged students to consider the calculator, using "written recommendations in study guide, announcements on Cecil [the course management system]" (A) or "No sales pitches here, but reference to manual and encouragement to learn using the beast" (B). Here lecturer B's language clearly shows a stance not entirely in favour of the CAS, although he had encouraged students to look at specific topics and pages in the manual, with announcments such as "learn to use your graphics calculator to find the inverse of a matrix (cf. pp. 284-285)". Both of them claimed, not surprisingly, to have noticed few, if any, examples of students using the TI-89, either in lectures or in small group tutorials. Hence they were not aware of any positive aspects or problems with CAS use. Lecturer A noticed that his students were affected by the price factor (described in more detail below), writing that "Several students have other types of graphics calculator (either from high school or bought elsewhere at half the price). At the start of the semester I promoted the private TI-89 tutorials vigorously, but no student enrolled. I am sure cost is a factor in all this." Put in terms of the didactic contract, Brousseau (1997) explains that, while there are reciprocal obligations, the teacher is the prime mover in the development of the contract. Since the lecturers demonstrated little expectation of CAS use, and some may even have been opposed to its use, this would become part of their contract.

Obstacles to use

A second strong factor to be considered in the development of CAS use is the attitude of the majority of those taking the course, since this forms the social environment in which the use of CAS is situated. The most common reason by far for those not using CAS was the cost. The mean response to the statement "The TI-89 CAS calculators are far too expensive." was a score of 2.78 out of 3. Of the 143 students not using the CAS, 72 wrote something in the open response section, and 22 of these mentioned the high cost as a significant barrier to use. Typical comments included "the price is too high and I can't afford it", "Far too expensive!!!", and "it is very hard for your regular student to shell out hundreds of dollars for a calculator". It appears that a significant number would have liked one had they been cheaper, with a mean response of 2.53 to the statement "I would have liked one if they were more affordable" and 2.21 for "I wish I had a TI-89 that I could use".

Other factors preventing non-users from using the CAS included the view that they are of limited value, especially in terms of helping with understanding of the mathematics, although this may have emanated from the lecturers. The questionnaire statement "I can do everything without a TI-89. In other words I don't need one." elicited a mean agreement score of 2.28, and 12 wrote similar reasons stating that:

- If I can pass the course without one then I don't see the point.
- It is certainly not necessary. Therefore I don't need one.
- I don't think the calculator is very useful because we're familiar with exams without any calculator and we can do nearly all questions without using this.
- If you can use your brain to think and solve the problem, why you have to use calculator.

Another factor was the perceived limited 'shelf life' of the calculator. Students did not want to invest in something that they thought would not be used in future courses: "If I can use TI-89 in all the maths course (not only 108 and 150) exam. I may consider to buy one.", "Not useful after course's finished.", and "It is far too expensive and not worth to be purchased if we only use it for one semester." Interestingly, in contrast to the view that CAS was not useful, there was even stronger agreement with the statement that "I believe students shouldn't be allowed to use them in test and examinations, because it is not fair on those of us who can't afford one." (Mean 2.41). This seems to indicate a clear perception that the CAS was useful, and hence unfair. This was confirmed by 13 of the open responses, including:

- Students should not be encouraged to use TI-89 calculators in exams and tests…It is unfair for those who do not have calculators.
- I really believe that students shouldn't be allowed to use them in test and examinations. It is unfair for us who do not got one.
- I think that these TI-89s should be forbidden to use in tests and exams because with this calculator you basically don't need to know how to do differentiation, integrations, matrices etc. and still do well. Which means the other students will have quite an advantage.

Although, as one person noted, there was also the factor of a perceived time gain: "People without it are at a slight disadvantage in terms of time spent in the exam calculating manually. We'd have less time to do the rest of the test/exam. Few extra mins could be the difference between passing/failing for some people", and another mentioned the ability to check answers "It is kind of unfair, with the advantage of others to be able to check their answers during exams."

A fourth reason, given by 8 of the students, was that they saw the CAS as too complicated to use (maybe they had looked at those belonging to friends or had read the coursebook), and hence too much effort was required to learn its use.

- It's very complicated to use and takes quite a lot of time.
- It takes a long time to learn how to use TI-89 calculators.
- Not easy to use sometimes may make me confused!

- I've got one, but don't know how to use it. Too much time and effort to learn how.

In the light of the lack of compulsion, or privileging of the CAS by the lecturers (described above), it was interesting to note that 5 students picked up on this and interpreted it as meaning that the CAS was not really necessary for the course.

- It is not widely used, and not required, therefore I see no need in getting it.
- If they are good for the course then they should be made compulsory.
- If the lecturers force us to use, I'd like to buy one, otherwise, I prefer thinking by myself.
- It would better that lecturers teach us how to use TI-89, and using in the lecture, that I will think about to buy one.

Positive use of the CAS

In an environment where they were left more or less to their own devices, guided only by the coursebook notes, the 24 students who used the CAS seem to believe that they have got some benefits from it. Table 1 gives their levels of agreement with each of the 22 questionnaire statements (see Figure 1).

Statement	1	2	3	4	5	6	7	8	9	10	11
Mean	1.88	2.00	2.83	2.65	2.21	1.67	2.37	2.50	2.96	2.46	2.92
Statement	12	13	14	15	16	17	18	19	20	21	22
Mean	2.29	2.54	1.83	1.37	2.92	1.78	1.74	1.92	2.17	2.50	1.67

Table 1: Mean level of agreement with the questionnaire statements

Statements 1, 3, 4, 10, 11, 20 and 22 form a subset addressing the value of the CAS. While a few felt that the CAS did not improve their understanding of mathematics (1), and they were unwilling to admit to any unfair advantage (22), they were very glad they could use CAS in the examination (3), believe in the CAS (10) and strongly asserted that they want to use it in future mathematics courses (11). However, statements 9 and 21 give evidence that they want to learn more about the CAS in order to use them more fully. When asked 'How do you feel about using the TI-89 calculators this year?' 11 responded positively, 2 somewhat negatively, 3 were ambivalent because it was "still a learning curve", they lacked confidence or needed more assistance, and 2 claimed to be neutral or "a little dubious".

Considering the picture of the types of use made of CAS given by statements 5, 13, 14, 15, 16 and 17, we find that they agree they can sometimes do the questions without understanding the theory (5), but do not believe that they have lost basic skills by using the CAS (15). It may be that since this course is largely skills-based, and the theory is emphasised less, that the students find this harder. With or without CAS this may well be the case. Statements 13, 14, 16 and 17 consider instrumentation of the CAS, especially in terms of integration with their by-hand working. They agreed that they explore the TI-89 by themselves (13), an essential part of instrumentation, and strongly affirmed that they use both the CAS and pen and paper when working on mathematics problems (16). When doing so they don't have a problem deciding when to use the CAS (14), neither do they just turn to the CAS when by-hand methods fail (17). The responses to the open questions reveal that 9 of the 17 responding found the CAS helpful. However, 8 of the 24 students primarily used the CAS for checking their by-hand working, "Checking answers, because I often make mistakes with signs, and the TI-89 makes it easy to check." Several stated that they used the CAS when the by-hand calculation was too difficult, for "confirmations and when it is hard done with pen" or "basically doing tough calculations" or "I could do most problems, but it's helpful when it comes to nasty algebra simplification". Both of these categories of use confirm the findings of Thomas and Hong (2004). In addition, 7 of the students valued the CAS for drawing graphs or for "visualising the graph of a function" because "visualisation makes solving problems easier". When asked how they decided to use the CAS, 2 of the students provided interesting insight into the process of instrumentation. They replied "If I'm stuck I'll try to get the answer and from there work backwards." and "Yes, from the answers to guess the way." It seems that these students had found a way of working that involved using the direct answer from the CAS to try to work out the by-hand method of solution. In answer to another question, the first of these gave further insight into what he does, saying CAS had helped him "With differentiation, I have been able to recognise the patterns." So he was assisted with the structure of the differentiation results by considering a number of examples on the CAS.

Attending to statements 2, 6, 7, 12 and 19, which examine possible obstacles to CAS use, there is some slight agreement that the complexity of the CAS, including the menus and key presses is a problem, but it is not the major issue that others (Hong, Thomas, & Kiernan, 2000) have found with younger students (2 and 19), and they did not find the format of the CAS output a problem (6). One comment on this was that "It's kind of hard to use because you really need to use and put the brackets, braces at [the] right place." In the open responses 50% said that they explored the CAS by themselves as well as using the coursebook notes, since "both are necessary as the course book manual often lacks details". There is agreement that the CAS manual at the back of the coursebook is very helpful (7), and, surprisingly in view of the above discussion, they thought that the lecturers had been very supportive and encouraging in their use of CAS (12). One student though made the telling comment on the existence of the CAS notes in the back of the coursebook, that "no one told us not even the lecturers, I don't think they even knew about, he was like "oh" when he saw them last week."

In conclusion, we found in this study that the majority of students did not use the CAS because it was too expensive to buy, they saw it as of limited value in doing or understanding the mathematics, the effort to learn it was too great and their was little support in place, and the lecturing staff did not support or promote its use. In spite of this very negative social environment a few positive aspects of CAS use emerged from the small group using them. Even here though most students only employed the

CAS procedurally, using it to check answers, to perform complex but direct calculations, as well as visualise 2-D and 3-D graphs. However, two were able to progress beyond these basic functions to consider conceptually the structure of problem solutions. The students used the CAS alongside their by-hand methods and were making some attempts to integrate it into their learning, although there was an element of resorting to the CAS only when by-hand work was too difficult. Although there are many similarities with results from studies in schools, our results show that university students may be old enough and independent enough for some CAS use to take root even in an adverse social environment. While some obstacles to increased use are currently beyond control (eg price), others, such as the steep functional learning curve, turned out to be less of a problem than non-users anticipated. There would no doubt have been a greater take up of the CAS, improved instrumentation, and hence more prospect of beneficial outcomes, if the department had analysed its support mechanisms and provisions better and put in place systems to coordinate them and foster CAS use. We learn that a piecemeal approach to CAS use, not unexpectedly, produces fragmented results.

References

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7,* 245-274.

Bergsten, C. (1996). Learning by computing? The case of calculus and a CAS. In L. Lum (Ed.), *Proceedings of the 7th annual international conference on Technology in Collegiate Mathematics* (pp. 36-39). Reading, MA: Addison Wesley.

Brousseau, G. (1997). *Theory of didactical situations in mathematics* (N. Balacheff, M. Cooper, R. Sutherland & V. Warfield: Eds. and Trans.). Dordrecht: Kluwer.

Drijvers, P. (2002). Learning mathematics in a computer algebra environment: Obstacles are opportunities. *Zentralblatt für Didaktik der Mathematik, 34*(5), 221-228.

Hong, Y. Y., Thomas, M. O. J. & Kiernan, C. (2000). Super-calculators and university entrance calculus examinations, In M. Chinnappan & M. O. J. Thomas (Eds.), *Mathematics Education Research Journal Special Issue on Technology in Mathematics Learning and Teaching*, *12*(*3*), 321-336.

Kendal, M., & Stacey, K. (1999). Varieties of teacher privileging for teaching calculus with computer algebra systems, *The International Journal of Computer Algebra in Mathematics Education, 6*(4), 233-247.

Rabardel, P. & Samurcay, R. (2001). *From artifact to instrumented-mediated learning*, New challenges to research on learning, International symposium organized by the Center for Activity Theory and Developmental Work Research, University of Helsinki, March.

Thomas M. O. J. & Hong, Y. Y. (2004). Integrating CAS Calculators into Mathematics Learning: Issues of Partnership, *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Bergen, Norway, *4,* 297-304.

Thomas, M. O. J., Monaghan, J., Pierce, R. (2004). Computer algebra systems and algebra: Curriculum, assessment, teaching, and learning. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The teaching and learning of algebra: The 12th ICMI study* (pp. 155-186). Norwood, MA: Kluwer Academic Publishers.

PROSPECTIVE TEACHERS' UNDERSTANDING OF PROOF: WHAT IF THE TRUTH SET OF AN OPEN SENTENCE IS BROADER THAN THAT COVERED BY THE PROOF?

Andreas J. Stylianides*, Gabriel J. Stylianides*, & George N. Philippou**

*University of Michigan, U.S.A., **University of Cyprus, Cyprus

This paper investigates prospective elementary and secondary school teachers' understanding of proof in a case where the truth set of an open sentence is broader than the set covered by a valid proof by mathematical induction. This case breaks the boundaries of students' usual experience with proving tasks. The most important finding is that a significant number of students from both groups who recognized correctly the validity of the purported proof thought that it was not possible for the truth set of the open sentence to include any number outside its domain of discourse covered by the proof. The discussion of student difficulties provides insights into the development of instructional practices in teacher preparation programs aiming to uncover these aspects of students' knowledge fragility and address them accordingly.

Proof is a defining feature of mathematics and, in current school reform recommendations in various countries, is considered a fundamental aspect of instructional programs in all grade levels. However, to have success in the goal to make proof central to all students' mathematical experiences, prospective teachers need to have solid understanding of this mathematical concept. If teacher preparation programs are to develop effective instructional practices that will help prospective teachers cultivate proof in their classrooms, it is essential that these practices be informed by research that illuminates prospective teachers' understanding of proof. Despite the importance of this kind of research, only few studies have investigated in-service or preservice teachers' knowledge of proof (Knuth, 2002; Martin & Harel, 1989; Movshovitz-Hadar, 1993; Simon & Blume, 1996; Stylianides, Stylianides, & Philippou, 2004). Also, these studies have focused more on the logical components of different proof methods than on other important features of the proving process, such as the relationship among the domain of discourse D and truth set U of an open sentence, and a proof that purports to show that the sentence is true in D.

This paper contributes to this research area, focusing on the proof method of mathematical induction. Specifically, we examine what might be some common difficulties that prospective teachers have in dealing with a proof by mathematical induction that is not as encompassing as it could be (D is a proper subset of U). Based on anecdotal evidence that students' normal experience is of being given opportunities to engage in 'universal' proofs (D = U), this study aims to advance the field's understanding of possible issues of knowledge fragility by exposing prospective teachers to a case that falls outside the boundaries of what appears to constitute 'standard practice' for them.

METHOD

The data for this report are derived from a larger study that aimed to examine the understandings of proof held by the undergraduate seniors of the departments of Education and Mathematics at the University of Cyprus. The participants were 70 education majors (EMs) and 25 mathematics majors (MMs). The EMs, prospective elementary teachers, constituted the 50% of the seniors of the Department of Education during the academic year 2000-01. All of them were taking one particular class to which they were allocated randomly. The MMs, prospective secondary school mathematics teachers, were all the seniors of the Department of Mathematics.

Consider the following statement:

For every natural number $n \geq 5$ the following is true: $1 \bullet 2 \bullet \ldots \bullet (n-1) \bullet n > 2^n$ (*)

Study carefully the following proof for the above statement and answer the questions.

<u>Proof:</u>

I check whether (*) is true for $n = 5$:

$1 \bullet 2 \bullet 3 \bullet 4 \bullet 5 = 120 > 2^5 = 32$. True.

I assume that (*) is true for $n = k$: $1 \bullet 2 \bullet \ldots \bullet (k-1) \bullet k > 2^k$ (**)

I check whether (*) is true for $n = k + 1$:

$1 \bullet 2 \bullet \ldots \bullet (k-1) \bullet k \bullet (k+1) > 2^k \bullet (k+1)$ (using (**))

$> 2^k \bullet 2$ (since $k + 1 > 2$)

$= 2^{k+1}$ True.

Therefore (*) is true for every $n \geq 5$.

(A) Choose the best response for the above proof:

1. The proof is invalid.
2. The proof shows that the statement is always true.
3. The proof shows that the statement is true in some cases.
4. I have no opinion.

(B) Use the space below to explain your thinking.

……………………………………………………………………………………………

……………………………………………………………………………………………

(C) State what happens in the special cases where:

$n = 3$	A. The inequality is true.	B. The inequality is not true.
$n = 4$	A. The inequality is true.	B. The inequality is not true.
$n = 6$	A. The inequality is true.	B. The inequality is not true.
$n = 10$	A. The inequality is true.	B. The inequality is not true.

Figure 1: The test item.

The program of study at the Department of Education includes several mathematics courses that emphasize logical thinking. These courses provide EMs with a fair amount of mathematical knowledge about different types of proof including proof by mathematical induction. The preparation of the MMs focuses primarily on mathematical content and abstract thinking.

All 95 participants responded to a specially designed test that included items on different methods of proof: empirical/inductive proof, proof by counterexample, proof by contradiction, proof by contraposition, proof by the use of computer, and proof by mathematical induction. The data from the test were supplemented by semi-structured interviews with a purposeful sample (Patton, 1990) of 11 subjects (eight EMs and three MMs). The interviews were used to investigate further students' thinking and illuminate patterns that arose from the analysis of the tests.

In this paper we focus only on the test item that appears in Figure 1. This item included a statement and a proposed proof for that statement, and was asking the participants to evaluate the validity of the proof and explain their thinking. The subjects were additionally asked to state whether the sentence (inequality marked with *) in the statement to be proved is true or false in four particular cases: $n = 3$, 4, 6, and 10. The purported proof is valid; the best response to Part A is choice '2.' The truth set **U** of the inequality is $\{n \mid n \in \mathbf{N}, n \geq 4\}$), but the domain of discourse is taken as $\mathbf{D} = \{n \mid n \in \mathbf{N}, n \geq 5\}$ that does not include $n=4$. This set up of the test item created a rich context within which we were able to advance our primary goals.

RESULTS

Table 1 summarizes the student responses to Part A of the item. The values represent percentages (rounded to the nearest integer) within major. The vast majority of MMs (92%) said that the proof showed that the statement is always true, while the rest (8%) noted that the proof is invalid. Unfortunately, the students who selected the latter option did not explain their thinking; therefore, we cannot examine further their reasoning. Regarding the responses of the EMs to the same question, approximately half of them (54%) said that the proof showed that the statement is always true, almost one out of three (29%) noted that the proof showed that the statement is true in some cases, 13% considered the proof as invalid, and 4% expressed no opinion.

Of particular interest is the way in which the students justified their responses. Some EMs who supported the validity of the proof faced difficulties in formulating a mathematically accurate explanation. The responses of the students EM24 and EM50 illustrate these difficulties and in addition raise the issue of whether the students' belief about the validity of the proof was well grounded on reason or not.

EM24: The proof shows that the statement is always true. Most of the possible cases have been checked and, therefore, we can conclude that the statement is true in general.

EM50: The statement holds. However, the way mathematical induction is applied is not the best possible, because it has not proved that the statement is also true for n=6 (since 6 is greater than 5) before proceeding with n=k.

On the other hand, the majority of MMs could justify their choice, even though their arguments were mostly limited to saying that the proposed proof followed correctly the steps of the induction method. The argument of the student MM10 is indicative.

Response Option	Education	Mathematics
The proof is invalid	13	8
The proof shows that the statement is always true	54	92
The proof shows that the statement is true in some cases	29	0
I have no opinion	4	0
Total	100	100

Table 1: Percentages for each response option in Part A of the test item by major.

MM10: You used the method of mathematical induction. You checked all the steps of the method and you concluded that they are applied correctly. Therefore, we can conclude that the statement always holds. By saying 'always' we mean 'always' as it is indicated in the context of the statement, that is, for $n \geq 5$.

The last comment of MM10 about the interpretation of the word 'always' with respect to the domain of discourse of the statement in the test item lies at the heart of the concept we wanted to investigate and marks a point that caused considerable trouble to students. Specifically, the data suggest that many EMs considered that the proposed proof showed the statement to be true in some cases, because they thought that when we say that a statement is 'always' true we mean that (a) the sentence in the statement is true for all natural numbers (the most commonly met case in high school and even college mathematics), or (b) the sentence is true for all natural numbers that belong to its truth set (in this particular case, $\{n \mid n \in \mathrm{N}, n \geq 4\}$).

EM51: The proof shows that the statement is true in some cases, because if we check some other numbers, e.g., 3, the statement is false.

EM9: The proof shows that the statement is true in some cases. The statement is always true for $n \geq 5$. I don't know whether it is true for $n<5$.

The same thinking that led some students to conclude that the proposed proof showed that the statement is true in some cases, led others to consider the proof as invalid.

EM20: The proof is invalid. The testing of cases should begin from the first natural numbers: 1, 2, 3, 4. The statement is also true for n=4.

EM49: The proof is invalid because the statement is true for $n \geq 4$.

EM52: The proof is invalid because the statement is false for n=4.

The student EM20 seems to believe that the validity of a proof by mathematical induction depends on whether or not the proof establishes the truth of the mathematical relationship under consideration on the entire set of natural numbers rather than on the specific set to which the relationship refers. The students EM49 and EM52 rejected the validity of the proposed proof based on opposite reasons. EM49 rejected the proof because he found a value for n (=4) outside the domain of discourse for which the inequality *was* satisfied. He seemed to believe that the proof was invalid because it was not as encompassing as it could be, that is, it did not cover the largest subset of natural numbers for which the inequality was true. EM52 failed to see that the inequality was satisfied for n=4 and considered that this violated the assertion ‘the proof shows that the statement is always true.’ He therefore appears to think that a valid proof would show the truth of the inequality over a broader set than its domain of discourse, possibly the set of all natural numbers.

	$n = 3$	$n = 4$	$n = 6$	$n = 10$
Education	83	56	87	91
Mathematics	96	64	100	96

Table 2: Percentages of correct responses to Part C of the test item by major.

Part C of the test item helps investigate further students’ understanding of the relation between the domain of discourse of the statement to be proved and the truth set of the inequality. Table 2 summarizes the percentages of correct responses to each of the special cases in the test item by student major (the values are rounded to the nearest integer). The highlight of the table is the failure of many students from both majors to realize that the inequality is true for n=4; the percentages of success were 56% and 64% for EMs and MMs, respectively. Given the simplicity of the calculations required to check the inequality for n=4, it is plausible to assume that the students reached this conclusion based on an erroneous reasoning. This reasoning was most probably associated with the fact that number 4 was not included in the domain of discourse of the statement. The difference in the percentages of success between the first two special cases, n=3 and n=4, may be attributed to the fact that the latter belongs to the truth set of the inequality whereas the former does not. A student who believed that the inequality could not hold for values outside the domain of discourse of the proved statement would accidentally get the first right and the second wrong. The higher percentages of success for the other special cases, n=6 and n=10, were expected given that many students accepted the validity of the proposed proof and these cases belonged to the domain of discourse of the proved statement. Also, the calculations were not difficult for the students who chose to carry them out.

Table 3 presents a detailed analysis of the results obtained from parts A and C of the test item. In particular, the table provides five response types, each of which corresponds to a different combination of student responses to the two parts.

A significant number of students from both majors, 38 EMs and 23 MMs, recognized the validity of the purported proof, thus responding correctly to Part A of the test item. From these students, only 15 EMs and 13 MMs responded correctly to all four special cases of Part C (Response Type 0). From the same group of students, three

EMs and one MM said that the inequality is true for all special cases (Response Type 1). This response suggests that the students believed that the proof showed the truth of the inequality for values outside its domain of discourse (possibly all natural numbers). Almost all other students who responded correctly to Part A of the test item (18 EMs and eight MMs) said that the inequality is false for n=3 and n=4, and true for the other two special cases (Response Type 2). This response type is most likely associated with the misconception that the truth set of the inequality cannot include natural numbers outside its domain of discourse in the proved statement.

Response Type	Description of Response Types and Frequencies by Major	
	Part A of the test item[a]	Part C of the test item[b]
0	The proof shows that the statement is always true. (38, 23)	The inequality is *false* for n = 3, and *true* for n = 4, 6, and 10. (15, 13)
1	The proof shows that the statement is always true. (38, 23)	The inequality is *true* for all special cases. (3, 1)
2	The proof shows that the statement is always true. (38, 23)	The inequality is *false* for n = 3 and 4, and *true* for n = 6 and 10. (18, 8)
3	The proof is invalid. (9, 2)	The inequality is *false* for all special cases. (2, 0)
4	The proof shows that the statement is true in some cases. (20, 0)	The inequality is *false* for n=3, and *true* for n = 4, 6, and 10. (8, 0)

Table 3: Frequencies of selected student responses to parts A and C of the test item.

[a] The first number in each parenthesis in this column represents the number of EMs who responded the specified way in Part A, and the second represents the corresponding number of MMs.
[b] The numbers in each parenthesis in this column represent the numbers of students who responded the specified way in parts A and C of the item that appear in the same row of the table. The first represents the number of EMs and the second the number of MMs.

The remaining two response types are associated only with EMs. Specifically, from the nine EMs who considered the purported proof as invalid, two said that the inequality is false for all four special cases (Response Type 3). These students most likely believed that, because the proof 'failed' to prove the statement, the truth set of the inequality is the empty set. Finally, from the 20 EMs who said that the purported proof shows that the statement is true in some cases, eight said that the inequality is false for n=3 and true for the other three special cases (Response Type 4). These students most likely thought that the domain of discourse of the inequality in the statement to be proved should be the same with its truth set.

Some of the interviews shed further light on students' thinking regarding the investigation of the special cases. For example, student EM38, whose response in the test belonged to Response Type 0, had difficulty understanding the 'mismatch' between the domain of discourse of the statement to be proved in the test item and the truth set of the inequality. However, after some probing from the interviewers (the first two authors), EM38 appeared to have grasped the relation between these two sets in the context of the given proof ('*I*' denotes the interviewers).

I: Do you find problematic the fact that the statement says that the inequality holds for all $n \geq 5$, but you said here [pointing to his test] that the inequality also holds for $n=4$?

EM38: Oh… Perhaps we have indeed… Seeing what happens for $n=4$ together with the fact that I considered the statement to be true, I believe that there is a problem here. Perhaps the source of the problem is that the proof doesn't specify the value of k. I assumed that k is greater than or equal to 5 and this might be the reason I said the statement is always true.

I: Now that you have the opportunity to think about this problem again, which of the multiple-choice options [referring to Part A of the test item] would you choose?

EM38: I wouldn't choose this option [he refers to choice '1' of Part A] because the statement holds for $n \geq 5$. The issue here is whether the statement also holds for some values smaller than 5.

I: Do you mean to say that proving the inequality for $n \geq 5$ excludes the possibility of the inequality to also hold for smaller values of n?

EM38: Oh… yes. The statement doesn't say '*only* for $n \geq 5$'! Therefore, it leaves open the possibility for other values. Consequently the statement is true.

The student EM38 constantly refers to the correctness or not of the statement rather than to the validity of the purported proof as asked in Part A of the test item. The student seems to believe that there is a strong link between the truth of the statement and the validity of the proof, namely, that the two go together.

DISCUSSION

In this study, we examined prospective teachers' understanding of proof in a case where the truth set of the open sentence in the statement that was to be proved by mathematical induction was broader than the set covered by a valid proof. The analysis of student responses in the test item suggests that a significant number of students of both majors who recognized the validity of the purported proof thought that it was not possible for the truth set **U** of the inequality to include any number outside its domain of discourse **D**. These students incorrectly considered the inequality to be false for $n=4$, and said that the inequality was true only for the two values that belonged to **D** ($n=6$ and $n=10$). The response of a considerable number of EMs that the purported proof shows that the statement is true in some cases seemed to have been influenced by the belief that the domain of discourse of the sentence in the statement to be proved should always coincide with its truth set. These students' observation that the inequality was true for $n=4$ (i.e., $4 \in \mathbf{U}$), coupled with their knowledge of the fact that $4 \notin \mathbf{D}$, seemed to have interfered with their ability to evaluate appropriately the validity of the purported proof.

The results of our analysis highlight difficulties that prospective teachers seem to have in dealing with a proof that is not as encompassing as it could be, thereby uncovering possible aspects of knowledge fragility. The investigation of what might have caused this fragile knowledge requires further research. One possibility is that knowledge fragility has its roots in didactic contracts that possibly prevail in high school and even college mathematics and that promote inappropriately the conception that proofs are always as encompassing as possible.

A related important direction for future research concerns how mathematics teacher educators can organize instruction so that prospective teachers' difficulties in proof surface and become the objects of reflection. Movshovitz-Hadar (1993) suggests one possible way to help prospective teachers reconsider their knowledge, see its problematic aspects, and realize the need for developing a deeper understanding:

> [A]ctivities designed for student teachers should be aimed at accelerating the process of crystallization of their knowledge of particular mathematics notions, such as mathematical induction, by putting them in problem-solving situations which will make them confront their present knowledge and examine it carefully through social interaction with their peers. This process is supposed to reduce the fragility of knowledge. (p. 266)

The test item used in this study has the potential to support the development of learning opportunities that can facilitate the crystallization of prospective teachers' knowledge, as it sets up a situation that breaks the boundaries of what seems to constitute students' normal experience. For example, mathematics teacher educators can use this test item to engage prospective teachers in thinking about whether it is necessary for a valid proof to cover the truth set of an open sentence in its entirety. To manage successfully discussions around issues of this kind, mathematics teacher educators need to be able to anticipate prospective teachers' common conceptual difficulties. A research-based knowledge about these difficulties can support the design of instructional practices aiming to help prospective teachers improve their understanding of proof. Prospective teachers' written and oral responses discussed in this paper can contribute toward this direction.

Author Note

This paper is based on the senior thesis of the first two authors conducted at the University of Cyprus, under the supervision of the third author; the order of authorship of the first two authors is alphabetical.

References

Knuth, E. J. (2002). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education, 33*(5), 379-405.

Martin, W. G., & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education, 20*, 41-51.

Movshovitz-Hadar, N. (1993). The false coin problem, mathematical induction and knowledge fragility. *Journal of Mathematical Behavior, 12*(3), 253-268.

Patton, Q. (1990). *Qualitative evaluation and research methods.* Newbury Park, CA: Sage.

Simon, M. A., & Blume, G. W. (1996). Justification in the mathematics classroom: A study of prospective elementary teachers. *Journal of Mathematical Behavior, 15*, 3-31.

Stylianides, A. J., Stylianides, G. J., & Philippou, G. N. (2004). Undergraduate students' understanding of the contraposition equivalence rule in symbolic and verbal contexts. *Educational Studies in Mathematics, 55*, 133-162.

PLANNING AND TEACHING MATHEMATICS LESSONS AS A DYNAMIC, INTERACTIVE PROCESS

Peter Sullivan · Robyn Zevenbergen · Judy Mousley

La Trobe University · Charles Sturt University · Deakin University

We are researching actions that teachers can take to improve mathematics learning for all students, with particular attention to specific groups of students who might experience difficulty. After identifying possible barriers to learning, we offered teachers mathematics lessons structured in a particular way. Teachers' use of the model outlined in this paper seemed productive and their resulting planning and teaching proved to be dynamic and interactive. This paper uses excerpts from a conversation between two teachers to illustrate specific aspects of the model.

MEASURE CUT MEASURE CUT MEASURE CUT

Many English language sayings urge the listener to plan carefully. "Measure twice, cut once", for example, appeals to dressmakers, carpenters, and everyone else to plan ahead and prepare carefully for their tasks. Our research into inclusive mathematics lessons suggests that, despite the emphasis on linear structures for lesson planning in many pre-service teacher education courses, the process of constructing inclusive and engaging mathematics lessons needs to be more dynamic: more like "measure, cut, measure, cut, measure, cut". The need for dynamic reaction in teaching was addressed early by Brophy (1983), in discussion of ways to overcome self-fulfilling prophecy effects. Brophy urged teachers to be reactive rather than proactive, listening to students and shaping teaching in directions suggested by their responses. In our model of inclusive mathematics teaching we use the terms dynamic interaction to describe this reactive process.

This article reports data from a discussion between two of the teachers who were part of a 3-year project investigating ways to structure lessons in order to include students who are at different stages of readiness. It illustrates ways that the teachers involved adapted our recommended model in their own lesson preparation and illustrates how their planning and teaching could be described as dynamic and interactive.

A MODEL FOR PLANNING AND TEACHING

The framework for our model of planning and teaching inclusive mathematics lessons is based on the work of Cobb and his colleagues (e.g., Cobb & McClain, 2001), who used the terms *mathematical norms* and *socio-mathematical norms* to describe different dimensions of classroom action. We have extended the second of these and use the phrase *mathematical community norms* to encompass not only "classroom actions and interactions that are specifically mathematical" (p. 219) but also norms of practice and other factors that affect learning in mathematics

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 249-256. Melbourne: PME.

classrooms. In particular, our conceptualisation includes elements such as culture, social group, language use and comprehension, and modes of classroom organisation.

Some mathematical norms of the model

While we have found that the model can be applied to most task types, we focus on open-ended mathematics tasks since they are likely to create opportunities for students' personal constructive activity. The open-endedness allows a focus on key mathematical ideas and can be used to encourage students to investigate, make decisions, generalise, seek patterns and connections, communicate, and identify alternatives (Sullivan, 1999). They also generally contribute to teachers' appreciation of students' mathematical and social learning (Stephens & Sullivan, 1997).

Our model for planning and teaching takes *mathematical norms* to be the principles, generalisations, processes, mathematical tasks and sub-tasks, and work products that form the basis of the curriculum. There are three specific aspects of our model of planning and teaching that direct teachers' attention to these mathematical dimensions: mathematical tasks and their sequencing; enabling prompts; and extending prompts (see also Sullivan, Mousley, & Zevenbergen, 2004).

The tasks and their sequence. In building sets of learning experiences, an important aspect of the model is the creation of a notional sequence of tasks that Simon (1995) described as a learning trajectory, made up of three components: a goal determining the desired direction of teaching and learning; the activities to be undertaken; and a hypothetical cognitive process, "a prediction of how the students' thinking and understanding will evolve in the context of the learning activities" (p. 136).

Enabling prompts. We argue that it is preferable to encourage students experiencing difficulty to engage in sub-tasks related to the goal task, rather than requiring them to listen to additional explanations or to pursue goals substantially different from the rest of the class. Enabling prompts temporarily divert students to lower-demand sub-tasks that allow them subsequently to re-join the class learning trajectory. We note that, even though they previously offering contrary advice, the English Department of Education and Skills (2004) now recommends task differentiation that is centred around work common to all pupils in a class, with targeted support for those who have difficulties keeping up with their peers.

Extending prompts. Students who complete planned tasks quickly can be posed supplementary activities, to extend their thinking on those tasks. A characteristic of open-ended tasks is that they create opportunities for extension of mathematical thinking, since students can explore a range of options as well as consider forms of generalised response. The challenge for teachers is to pose prompts that extend students' thinking in ways that do not make them feel that they are getting more of the same or being punished for completing earlier work.

Some *mathematical community norms* of the model

Our model focuses on two characteristics of *mathematical community norms*:

Normative interactions. These are the common practices, organisational routines, and communication modes that impact on approaches to learning, types of responses valued, views about legitimacy of knowledge produced, and responsibilities of individual learners. Through the overt and the hidden curricula of schools, students receive diverse messages (Bernstein, 1996). Sullivan, Zevenbergen, and Mousley (2002) listed a range of strategies that teachers can use to make implicit pedagogies more explicit and so address aspects of possible disadvantage of particular groups. Making these aspects explicit is a feature of our model of inclusive teaching.

Mathematical community norms. These conveys the idea that all students can participate and progress as part of the classroom community, including making individual decisions as well as contributing to class discussions and lesson reviews, at the same time developing a shared knowledge base for subsequent learning. We agree with Wood (2002) who proposed that all students benefit from participation in core mathematical and social experiences, and that rich social interactions with others contribute substantially to children's opportunities for learning mathematics.

MAXIMISING SUCCESS FOR ALL STUDENTS

The overall project, of which the illustrative data below are a part, seeks to identify strategies for teaching mathematics to heterogeneous groups. Initially we identified and described aspects of classroom teaching that may act as barriers to mathematics learning for some students. Next we described strategies for overcoming such barriers (see Sullivan et al., 2002), including creating some scripted experiences, that were taught by participating teachers (see Sullivan et al., 2004). Analysis of these experiences allowed reconsideration of the emphasis and priority of respective teaching elements. It was found that it is possible to create sets of learning experiences that include all students in rich, challenging mathematical learning.

For the most recent stage of the research, we sought to examine whether teachers themselves could use the model to create effective sets of mathematical learning experiences, and have considered ways that specific aspects of the model contribute to the goal of inclusive experience. Our research approach was based in the action research paradigm, with its spirals of planning, acting, monitoring, and reflection (Kemmis & McTaggart, 1988) and emphasis on autonomous decision-making by participants. Essentially this stage involved 20 teachers developing, planning, and teaching sets of experiences of their own design and choice, while considering the aspects of both sets of norms outlined above.

The following illustration of these aspects uses as data a discussion between two female teachers from the same school who planned together and taught various sets of mathematical lessons as part of the project. This report refers to just one of those sets of experiences, planned formally using the model proposed. Their school serves a predominantly lower socio-economic community, and the teachers joined the project because of what they identified as a lack of engagement of their students in learning mathematics. The discussion between the two teachers was moderated by one of the researchers, recorded, transcribed and then analysed by identifying key issues. The discussion below draws mainly on the words on one teacher, Zeta, but refers to the second teacher, Delta, where appropriate. It reports aspects of their planning, their mathematical intent, the goals and sequence of tasks, their emphasis on engagement, and their use of prompts to support students experiencing difficulty as well as those who finished quickly.

Knowing where you are going

The two teachers, planning collaboratively, had chosen the topic of capacity, volume, and surface area. Zeta described the two goal tasks as follows: "Let's get them to think about how big is 360 cm cubed and what do the dimensions of these shapes … really look like", and "If a shape has a surface area of 1000 square cm, what could (the dimensions) be?" Both of these tasks are open-ended and address significant mathematical content; the latter task being well above the normal expectation for a class at the level they were teaching (age 13). The teachers also considered the trajectory of experiences necessary for the students to be able to respond to the goals.

> Zeta: I actually think that we knew … that we were both headed towards what could the dimensions of an object be with the volume of whatever. … We just had to work out what would the teaching be along the way. And so that is what this first [stage of the written plan] was about: empowering them to be able to answer that question.
> (It) talks about volume, capacity, millimetres, millilitres, litres, megalitres—all that stuff—and just gets them to think … about volume. [We said] "Let's start with water, I just want you to start thinking about all the different sizes we can buy water in" , and … got them to list … you can buy it in bottles, what about dams, what about tanks? And so they started to think about water from about this big to however big … to just get them all to the same spot where they are thinking about it.

It is possible to see evidence here of a focus on the mathematical content goals, a sequence of experiences, and the building of community understanding. They were also drawing on student experiences, as the school is in inland Australia where available drinking water in catchments is a community issue, thus facilitating access to the content and allowing key terminology to arise naturally.

> Delta: I don't remember actually saying the word volume. I think the kids came back with the word volume.

> Zeta: Well what word did you say?

> Delta: I said, "In what size can water be purchased?" Exactly like that … but they came back with the word volume and then … I think capacity came from them too. … and I remember writing this down: volume was the stuff inside a container or a dam or whatever, and capacity was when it was filled to the top and we left it hang like that.

> Zeta: I said, "Can you think of …" and obviously I had an answer in my head, but "Can you think of a place or a time where they talk about water capacity?" And I think kids talked about our water storage. "Does anyone know what it is at the moment?" "27%" And I said "Well what does that mean, 27% capacity?" So we talked about it in that way.

These teachers seemed to see teaching as dynamically interactive, responding to children's knowledge of the relevant mathematics.

Engaging with the mathematics

To follow this introductory class discussion, the two teachers had planned a task that required their students to arrange common containers in order of their capacity. In earlier observations we had identified an additional aspect of the model that we are tentatively terming a "hook task" that is interesting to the students and which can be used to engage the students with the mathematical content. Characteristics of such a hook include that it should be within the experiences of the students, that it should be in the form of a problem, and that students should all anticipate success. These characteristics were evident in this first task for the class.

> Zeta: Then we talked about the volume of a container and I pulled out a whole lot of containers. There must have been 30 or 40 containers in the room and I just chose about 10 of them and put them on the table, then asked them to rank them from smallest to largest. We actually did it as a class, because some of them were a little obvious. There were a few times where kids couldn't agree on a shape, and so I sat them on top of each other and I said, "When I get you to list them in an order, I want you to list them in your own order. You can decide then". So I got them to write them down in order and we kind of agreed on some names so that we were all talking the same language. And then I asked them to guess what the volume [of each container] was.

This is an example of actions that explicitly address some mathematical community norms: contributing to a shared knowledge base for subsequent learning. The other teacher also commented on the way that this hook engaged the students. She illustrated how the teachers were alert to a range of students' engagement in the task.

> Delta: I have got all the containers up the front, and kids … coming closer and closer and pushing up to the front of the room; and I became superfluous too, and I ended up saying to these two boys "Well, I'll step out", and stepped back, and they just took over. … and one of them, … can be disruptive and really aggressive at times, about work and about anything, and he was up there … not taking over . … There wasn't any "No, you're wrong" either, which I thought would come up.

There was also clear intention that the students experienced the mathematics that the teachers had in mind, and this was particularly evident in the way the teachers were explicit about the focus of the learning.

> Zeta: I alluded to the fact that I had an ulterior motive. "So what if I told you that there is a mathematical way to calculate the volume of everything?" and "Yes, we can check using water, but I would like to teach you how to calculate it mathematically and then you can check your answer". … So we talked about volume and then I think I picked a box that I hadn't put on the table and we looked at how we calculated the volume of that, and they all did it, and so then they were set the task of calculating the volume of one of the shapes. So in their small groups they each had to choose a shape and go off and calculate the volume of it, bring it back to the table, pick another shape. I think I wanted them all to do 3 or 4, to not calculate all of them … some of them did 5, and others did 1.

While this task was not open-ended in that there was not a range of possible answers for any one object, it did allow variety of activity and the students were able to make individual decisions, a key aspect of the teaching model's mathematical community norms. Also important for the model is the way that the teachers respond to individual students.

> Zeta: But it was interesting, one kid chose the hexagonal vase because he liked the way it looked. But then when it came down to working out the volume he had no idea of how to calculate the area of the hexagon; and I suggested he could break it up into some other shapes. "What shapes could you break it up into?" No idea. He talked about what area was and we talked about area in terms of being centimetres squared or millimetres squared, so he traced around the shapes and he decided to break it up into half centimetre squares, because centimetres squared would be too big for the shape.

This teacher had provided support for an individual student, based on her specific needs. This differential support for students working on common tasks is a vital component of out model. It forms the basis of whole-class shared experience and hence the building of a mathematical community, and also enables learning from whole-class discussion about experience.

> Zeta: Then I had to have a discussion. "Well this is our ranking. How are we going to check them?" Some said "Maybe if you looked at the height, and kind of the width and the length of the shapes, maybe you could work it out". "Say it again: length, by width, by height". And I mean it was just absolutely classic, … and then we went on to a discussion about regular prisms.

Again the focus on the mathematics and dynamic interactivity are evident, although the teacher's agenda is also apparent. These preliminary tasks were leading toward the first of the open-ended goal tasks that allowed the students to calculate volumes, and particularly to recognise that there can be many shapes with the same volume.

> Zeta: I posed this question of the volume being 360 cm cubed, and probably half the class just went for it and just knew what to do, and used the question I had up on the board and just changed the numbers, and so on.

Of course, not all students could readily engage in the task, and the teacher prompted them by posing variations to the initial task with a reduced cognitive demand.

> Zeta: I said, "Well, we know the volume has something to do with the area, it has to do with the area at the base. Where is the base on this picture?" [They were looking at a rectangular prism.] They pointed to it, and I said, "Well, if the volume of the whole thing has to be 360, what could the area of the base be?" And they chose a number. And I said "Well okay, if the area of that base is whatever they chose, … what would the height of that shape be if the base is this and the body is that?"

There were also students who completed the basic task readily. We have noted in many lessons that an important challenge is to engage such students, but we have found consistently that this is much easier said than done.

> Zeta: And there were obviously kids I suggested doing the triangular base, or a cylinder. The cylinder was interesting … The only problem with it probably was that there were some kids that did a couple and then thought "Well I can do any shape now and I don't need to do this any more."

The teachers then followed a similar sequence of interactions and tasks for the second of the goal tasks, that of describing a box with a surface area of 1000 cm2. There was an initial exploration with class discussion, some calculations using actual objects, and the open-ended task with provision of both enabling and extending prompts. Again, the emphasis on engagement, the focus on the mathematics, the interactive pedagogy, the dynamic nature of the teachers' planning, and the differentiation of the task were all evident.

SUMMARY

Basically our model for planning and teaching mathematics includes (a) mathematical norms, including tasks facilitating student engagement in meaningful mathematics, the sequence of the tasks, enabling prompts, and extending prompts; and (b) mathematical community norms, including making normative interactions more explicit and focusing on the development of a mathematical learning community.

We have worked with a number of teachers on a variety of sets of learning experiences, with the above being an example of the type of planning and teaching that have resulted. The above data, along with the other experiences observed, suggest that teachers are able to use the model. Interviews with all of the teachers involved indicate that the model does allow them to focus on the challenge of engaging all students in productive mathematical explorations and provides key principles and strategies for doing this.

In terms of the *mathematical norms*, the teachers were able to create open-ended tasks that addressed important aspects of mathematics, they considered the trajectory of tasks for the class, and they offered suitable variations of the tasks for those experiencing difficulty and those who completed the tasks quickly. (The latter proved the more difficult so the next phase of the research will address this issue.) The *mathematical community norms* were also explicitly addressed by the teachers, not only in the instructions about ways of approaching the tasks and formulating responses, but also in the building of community, by working on common tasks and through the interactive responses and discussions. These particular teachers focused on the engagement of the students in their learning, considering this in their planning and celebrating it in reflection on the experience, just as other participating teachers did.

Two aspects emerged from these data, and indeed observations of other project teachers. The first was the interactivity of the teachers with each other in their planning, and with the students during their teaching. They were clearly willing to observe and listen to the students and to respond accordingly. The second was the dynamic nature of this interaction. Rather than feeling constrained by any preparation or the hypothetical learning trajectory, they were willing to adjust the tasks, the emphases, the timing, and the supports offered. The metaphor is measure, cut, see if it fits, measure, cut, see if it fits, … . We suspect that the model gives teachers a structure that allows them freedom for a dynamic, interactive approach to teaching.

REFERENCES

Bernstein, B. (1996). *Pedagogy, symbolic control, and identity: Theory, research, critique*. London: Taylor & Francis.

Cobb, P., & McClain, K. (2001). An approach for supporting teachers' learning in social context. In Fou-Lai Lin & T. Cooney (Eds.), *Making sense of mathematics teacher education* (pp. 207–233). Kluwer: Dordrecht.

Department of Education and Skills (online November, 2004). The national numeracy strategy. http://www.standards.dfes.gov.uk/numeracy/teaching_resources/reception_to_year6/?leaf=0&subleaf=4

Doyle, W. (1986). Classroom organisation and management. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (pp. 392–431). New York: Macmillan.

Kemmis, S. & McTaggart, R. (Eds.) (1988). *The action research planner*. Geelong: Deakin University Press.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*, 114–145.

Stevens, M., & Sullivan, P. (1997). Developing tasks to assess mathematical performance. In F. Biddulph & K. Carr (Eds.), *People in mathematics education. Proc. 20th Annual Conf. of the Mathematics Education Research Group of Australasia* (pp. 470–477). Rotorua, NZ: MERGA.

Sullivan, P. (1999). Seeking a rationale for particular classroom tasks and activities in J. M. Truran and K. N. Truran (Eds.), *Making the difference. Proc. 21st Conf. of the Mathematics Educational Research Group of Australasia* (pp. 15–29). Adelaide: MERGA.

Sullivan, P., Mousley, J., & Zevenbergen, R. (2004). Describing elements of mathematics lessons that accommodate diversity in student background. In M. Johnsen Joines & A. Fuglestad (Eds.), *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (pp.257–265). Bergen: PME.

Sullivan, P., Zevenbergen, R., & Mousley, J. (2002). Contexts in mathematics teaching: Snakes or ladders? In B. Barton, K. C. Irwin, M. Pfannkuch & M. Thomas (Eds.), *Mathematics Education in the South Pacific. Proc. 25th Conf. of the Mathematics Educational Research Group of Australasia* (pp. 649–656). Auckland: MERGA.

Wood, T. (2002). What does it mean to teach mathematics differently? In B. Barton, K. C. Irwin, M. Pfannkuch & M. Thomas (Eds.), *Mathematics Education in the South Pacific. Proc. 25th Conf. of the Mathematics Educational Research Group of Australasia* (pp. 61-71). Auckland: MERGA.

TEACHER FACTORS IN INTEGRATION OF GRAPHIC CALCULATORS INTO MATHEMATICS LEARNING

Michael O. J. Thomas and Ye Yoon Hong

The University of Auckland

Graphic calculators (GC) have been widely used in teaching for at least 15 years, and yet many teachers still appear to be unaware of their potential for learning mathematics. This research describes a study addressing aspects of the teacher's role in integrating GC technology into their pedagogical approach. It considers issues of didactic contract, pedagogical technology knowledge, procedural and conceptual knowledge in GC integration of 7 secondary teachers. The results describe a number of factors that identify teacher progress toward pedagogical integration of the GC.

BACKGROUND

Brousseau (1997) has described the mutual recognition of the parts that the student and teacher play in the classroom learning process as a *didactic contract*. This term recognises, through the tacit acceptance of each party, that there are *reciprocal obligations* in the relationship. Not least among these is the expectation by students that they will be taught, and by the teacher that the students will want to learn. Of course this social contract is a dynamic entity, changing and adapting to new circumstances that arise in the classroom milieu. Factors influencing a teacher's didactic contract include affective variables (beliefs and attitudes), perceptions of the nature of mathematical knowledge and how it should be learned, mathematical content knowledge, and pedagogical content knowledge (Shulman, 1986). One may assume that if a teacher possesses a limited knowledge of a concept and its related subconcepts (Chinnappan & Thomas, 2003) then they will find it more difficult to provide the kind of environment and experiences that will assist students in the construction of rich conceptual thinking. Instead they may regress to a *process-oriented* approach (Thomas, 1994), presenting students with a toolbox selection of procedures that may be applied to each problem that arises. While such procedures and skills are important, mathematical thinking is clearly much wider than this, and requires procedural and conceptual interactions with the various representational forms of mathematics (Thomas & Hong, 2001). However, teaching is not mediated simply by the mathematical understanding of the teacher (Cooney, 1999), but it is also influenced by the teacher's pedagogical content knowledge. This refers to understanding the mathematical ideas involved in a particular topic and how these relate to the principles and techniques required to teach and learn it, including appropriate structuring of content and relevant classroom discourse and activities (Shulman, 1986; Simon, 1995; Cooney, 1999; Chinnappan and Thomas, 2003).

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 257-264. Melbourne: PME.

The introduction of new technology into the classroom has been shown to be capable of a subversive effect (Thomas, Tyrrell, & Bullock, 1996) radically altering the didactic contract. Thomas, Tyrrell, and Bullock (*ibid*. p. 49) suggest that the introduction of technology requires a new mindset on the part of teachers, a '*shift of mathematical focus'*, to a broader perspective of the implications of the technology for the learning of the mathematics. It has become clearer working with teachers in the years since this study that the *technology knowledge* aspect of instrumentation (Rabardel, 1995), namely how to control the functioning of the tool, is insufficient for a successful mathematics outcome. In addition teachers need to develop what we now call *pedagogical technology knowledge* (PTK), knowing how to teach mathematics with the technology. This arises as they progress through the stages of instrumentalisation and instrumentation of the tool (Rabardel, 1995), gaining a personal appreciation of its role in learning mathematics, and importantly, of ways in which students may be assisted through various teaching approaches to emulate their instrumentalisation and instrumentation of the technological tool.

As instrumentation of the technological tool proceeds teacher beliefs and attitudes are shaped changing their teaching emphasis, and didactic contract, to give increased emphasis to the instrument. In turn, student preferences have been shown to mirror this *teacher privileging* (Kendal & Stacey, 2001). As teachers progress in their belief in the value of the technology in teaching mathematics they have to face the key issue of the level of integration of the technology in learning that they will espouse. This may range from using it at prescribed moments as a teacher-directed add-on, to an ever-present instrument that is an extension of cognition. The research described in this paper followed a group of seven teachers as they began, or continued, their instrumentation of the GC tool, thus extending their PTK. It attempted to understand teacher practice in relation to the congruency of content knowledge, pedagogical content knowledge, instrumentation, PTK and didactic contracts.

METHOD

During August 2004 a GC professional development workshop was arranged over three weeks (3 sessions of 2 hours each) for teachers from two Auckland schools. The course covered both content and pedagogy for algebra and calculus, using the TI–83Plus with several downloadable memory-based FlashApp[lication]s. The teachers who volunteered to attend the workshop had little experience of using the GC to teach mathematics, although two had previously used them. Four teachers attended the workshop from school A, and initially six from school B, but only three of these finished the course, including one trainee teacher. Apart from this trainee, all the teachers were experienced, with between 10 and 24 years teaching. Each teacher was given their own TI–83Plus GC and class sets during the workshop and they kept these for six months after the course. Once the teachers were familiar with the TI–83Plus they asked questions relating to their teaching, and discussed ideas with one another. In the month following the workshop the teachers were given a brief questionnaire on their perspective of the value of the TI–83Plus for teaching mathematics, and all four teachers from school A and two teachers from school B agreed to take part in the classroom-based phase of the research. During this phase we were able, over a three week period of teaching years 10–13 (age 15-18 years), to observe and video their classroom teaching with the GC. They also completed a diary of teaching with TI–83Plus detailing the mathematical content covered and their aims and objectives. The videotapes were transcribed for analysis along with the data from the questionnaire, the lessons and the diaries. Following discussion, the topics the teachers chose included families of curves, linear programming, limit, drawing graphs and derivatives.

RESULTS AND DISCUSSION

One of the first things that a teacher new to using CAS, or indeed any other technology, has to decide is how they will structure its role in their classroom. This is a basic feature of changes to their didactic contract. All the teachers in the study organised their classrooms in a similar manner. They chose to have the students sit in traditional rows and the teacher spent some time at the front of the class, demonstrating examples using a viewscreen while the students followed and copied their working onto their own calculator. This may have been in order to maintain control of the classroom situation. Afterwards the students spent the rest of the time working on problems and tackling exercises as a group, while the teachers circulated and assisted with any difficulties. In spite of this similarity in approach it was soon clear that the teachers were different in the pedagogical advances that they made with the technology. Three of them made good strides forward, while two proceeded more cautiously, and two made little advancement. An analysis follows of some of the differences we perceived between the groups in terms of the variables described above, based on one teacher exemplifying each of the three groupings.

Little advancement

A major factor here influencing the PTK of teacher E from school B, was his lack of confidence with the GC, springing from a lack of instrumentation. He had previously used a scientific calculator in his teaching, and saw value in the GCs, commenting that “Several topics are made ‘easier’ with a GC. But students need to have their own.” He particularly singled out content areas of “Linear programming, simultaneous equations (3 variables), graphing of level 2 graphs” and thought GCs “Very good with ongoing learning. Inequations helpful”. However, when asked if he had problems with the GCs he answered “Yes, lots—need to spend more time using them regularly.” and he found them “Time consuming to learn processes and to remember these processes.” He was conscious of still being in the early stages of instrumental genesis. In class he chose to use the GC to teach year 12 (age 17 years) students “graphing of parabolas, cubics, exponential function, hyperbola, graphing inequalities, linear programming, and solving simultaneous equations” in up to 3 variables. Asked what problems his students faced he said “Same problems. I need to be continually using them or I forget", indicating that like him they often forgot the correct commands, and showing where his emphasis lay. This is a crucial aspect of the instrumental genesis of the GC tool and leads to a lack of confidence in teaching. Evidence of this was observed in the lessons he taught. We can see from statements during a lesson on linear programming, such as, "I think you just have to go 3 down, and push ENTER", "Now what you probably need is… what I've done wrong here" and "I'm not sure what the calculator is going to do because we don't have anything written here", that he was not too confident in his handling of the GC commands, as he freely admitted after the lesson. This lack of familiarity with the operational facets of the GC means that a teacher such as this tends to be tied to the mechanics of operating the tool and due to this heavy cognitive load cannot free up enough thinking space to concentrate on the mathematics. This can often lead to a very procedural, button-pushing emphasis in the lesson, as we observe from the typical quote below.

> So, if you change it from ALPHA F3, we've got X=25, Y=50. I think now, it's going to go around the vertices. We went ALPHA F3, and we were able to move around, and we went ALPHA F3 again, ALPHA F4 initially then ALPHA F3. By doing that we were able to go around.

Cautious progress

In contrast teacher F, a trainee teacher from school B, made cautious progress in her growth of PTK, using the GC in her teaching of mathematics. Teacher F expressed that she "Would like to use [the GC] in teaching" but the potential advantages were seen in a procedural light as "Seems easier in some ways to sketch graphs etc", and when asked if she had problems using the GC she replied "Some difficulty". She taught year 13 linear programming with the GC, but in her interview again focussed on procedural matters as a motivation stating that "I thought it would be a good idea to show different ways for calculating unknowns in linear programming." When asked what she thought it was important for her students to learn she said "Different ways of obtaining the same answers." And that she would introduce the ideas by "By providing examples and working through these." Discussing why students might find the GC helpful she did not refer to mathematical ideas but thought it would be "By providing an interesting way to find solutions." She explained that she believed students "Need a step-by-step explanation." when using the GC, and that their main difficulty would be "Losing track of where they were." So while she did not express a lack of confidence her teaching approach was firmly set in a step-by-step, process- and solution-oriented mode, focussed on the GC rather than the mathematics. This seems to be a feature of this transitional stage of progress in the acquisition of PTK. A second feature of this stage is the inability to take the mathematics and adapt the technology to focus on the content under study.

Teacher F simply took the ideas that were presented in the workshop and tried to present these to her year 13 students. Her instrumental genesis was still very much in progress as shown when she was asked a question early in the lesson, during memory clearance. A student asked ”How do you get rid of inequalities?” and the reply, indicating lack of knowledge, was “Oh, that’s what we are going to putting them on, so don’t worry.” However, she did not concentrate as much on button pushing as teacher E, and so was able to exhibit more occasions when mathematical ideas, even if basic, were encouraged to surface, such as graph intersections corresponding to equation solutions, and testing vertices to find a maximum.

> So, ok, so at the moment we got one that’s a bit we can’t see, so we want to have it shaded where the intersections the graphs are only. So next we will press ALPHA F1 and we want to press 1 for an equation intersection. Press 1 and wait and this could be shading out from the intersections.
>
> So now what we want to do is, we want to record all our intersections for the vertices, because we are working out the maximum, and we know that when we are working out the maximum we want to put all values of vertices into that expression, so you see this one here says POI-trace and that can trace the intersection point, so press ALPHA F3.

However, there was no investigation or discussion of these, or any other, concepts, and, as we see below, the focus was clearly on obtaining the right answer to the problem and then proceeding to the next one.

> Did you get the maximum of 17?…Ok, so this is how we found the maximum so our answer is 17 which is when x=1 and y=5. So I’ll just give you an example to try yourself. That should say 1 and 5. Anyway I’ll give you an example today. So following the instructions and directions on the sheet try doing this question.

Greater strides forward

Teacher A will be used to exemplify the group that had made greater strides forward. She had over six years’ experience of using GCs in her teaching and had been involved in a previous research study with them. In spite of her experience, in the questionnaire she admitted “Sometimes it’s hard to see how to use it effectively so I don’t use it as continuously as I should.” Her motive was a rather pragmatic “We should move with the times” and she had a small reservation about the GC that “It is OK. By now expected better resolution though.” Due to her relative experience she appeared confident in her use of the GC and spoke at length in her interview, describing how “In the past I have also done some exploratory graphs lessons where students get more freedom to input functions and observe the plots.” Further, she explained that she was happy to loosen control of the students and let them explore the GC and help one another: “Students learn a lot by their own exploration…In past lessons I have never had a student get lost while using a graphics calculator. Sometimes friends around will assist someone” However, she acknowledged her need to progress in her types of GC use, “I would like to see them used more frequently and beneficially in class with structured lessons.” While she could perceive imaginative GC usage, part of her difficulty involved the pressure of day-to-day teaching, since she admitted that “I was not relaxed enough with the term coming to an end and other aspects in this year’s teaching to be inspired to use the calculator with imagination for the students." Responding to what she wanted her students to learn she replied in terms of the challenge of the depth of mathematics, "The success for me as a teacher is when they want to learn more and students show a joy either in what they are doing or in challenging themselves and their teacher with more deeper or self posed mathematical problems." She was convinced that the novel and challenging nature of the GC could motivate students—"The calculator puts a radiant light in the class… With a graphics calculator lesson no one notices the time and no one packed up."—and that her perspective on learning mathematics influenced her PTK came out in the comment that "Today we find a lot of Maths does not need underlying understanding… I feel as teachers what we need to really be aware of is what the basics are that students must know manually… when we sit down to work with graphics calculators we need to consider carefully what still should be understood manually." In this way she addressed an important factor of the integration of technology use into mathematics, namely learning what is better done by hand and what could be done better with the technology (Thomas, Monaghan, & Pierce, 2004).

One of her lessons with the GC was with Year 12 students and she considered families of functions with the aim of exploring exponential and hyperbolic graphs and noting some of their features, "we're going to utilise the calculator to show that main graph and then we're going to go through families of $y=2^x$". She was comfortable enough to direct them to link a second representation "Another feature of the calculator I want you to be aware of..[pause] you've got also a list of x and y values already done for you in a table." Teacher A had moved away from giving explicit key press instructions, instead declaring "I want you to put these functions in and graph them and see what's going on.", and "You can change the window if you want to see more detail, and if you want to see where it cuts the x-axis, you can use the "trace" function." Figure 1 shows a copy of her whiteboard working.

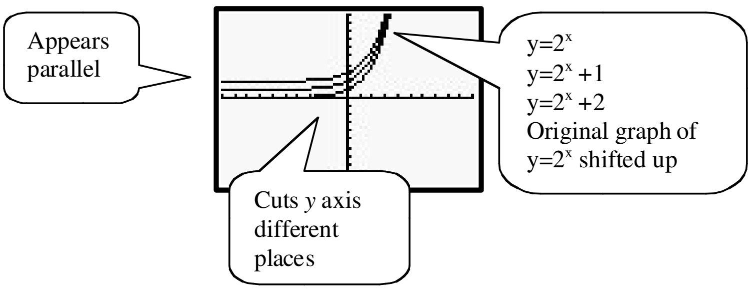


Figure 1: Teacher A's whiteboard working: Viewscreen projection and overwriting.

She was also able to move towards an investigative mode of teaching "if you're not sure where the intercepts are, you can use the "trace" key, remember, and I want you to observe what is happening.", encouraging students to use the GC in a predictive manner, to investigate a different family.

> We want to do some predictions… Looking at the screen try to predict where 3×2^x will go then press "y =…" and see if it went where you expected it to go. You may get a

shock… Can you predict where "$y = 4 \times 2^x$" will be? Now you learned from that, so can you predict where it'll lie. The gap between them gets smaller. If you're interested put in "$y = 100 \times 2^x$". Does it go where you expect?

There was also some discussion of mathematical concepts and how this could help with interpretation of the GC graph. She linked 2×2^x with 2^{x+1} and then during examination of the family of equations $y=2^x$, $y=2^{x+1}$, $y=2^{x+2}$ said of $y=2^{x+1}$ "We expect this to shift 1 unit to the left [compared with 2^x]. Did it?" In this way she made a link with previous knowledge of translations of graphs parallel to the x-axis, and then reinforced this with the comment that "With this family, when you look at the graph can you see that the distance between them stays the same because it's sliding along 1 unit at a time. The whole graph shifts along 1 unit at a time." In addition, there was a discussion of the relationship between the graphs in the family of $y=2^x + k$, and the relative sizes of 2^x and k.

… as the exponential value gets larger, because we're adding a constant term that is quite small, it lands up becoming almost negligible. So, when…all they're differing by is the constant part, you'll find that they appear to come together. Do they actually equal the same values ever? Do they ever meet at a point? No, because of the difference by a constant, but because of the scaling we have, they appear to merge.

The discussion on the relative size of terms in the function continued with "How significant is "+1" or "+2"? We know that 2^5 is 32." and again the use of prediction was evident "I want you to predict where $y=2^x + 3$" would be."

In summary we may describe the differences in the progress of the teachers we have observed in terms of a number of variables that delineate two clearly different groups, with a third progressing between the two. The first group may be identified in terms of their instrumental genesis as teachers who are still coming to grips with basic operational aspects of the technology, such as key presses and menu operations. This leads to a low level of confidence in terms of teaching with the GC in the classroom. In terms of their PTK, this group is characterised by an over-emphasis on passing on to students operational matters, such as key presses and menu operations to the detriment of the mathematical ideas. Furthermore, the mathematics approached through the technology has an emphasis on technology, and work tends to be very process-oriented; based on procedures and calculating specific answers to standard problems. There is little or no freedom given to students to explore with the GC, and it tends to be seen as an add-on to the lesson rather than an integral part of it. These features then become part of the teacher-initiated expectations in the didactic contract.

In contrast to this, the second group have advanced to the point where they are competent in basic instrumentation of the GC and are thus more able focus on other important aspects, including the linking of representations such as graphs, tables, and algebra, and to use other features of GCs. In turn, this better instrumentation of the GC produces a higher level of confidence in classroom use. Considering their PTK they begin to see the GC in a wider way than simply as a calculator. They feel free to loosen control and encourage students to engage with conceptual ideas of mathematics through individual and group exploration of the CAS, investigation of mathematical ideas, and the use of prediction and test methodology. For these teachers the mathematics rather than the technology has again been thrust into the foreground, and the GC has been integrated into the lessons and forms part of the didactic contract. If we think that the approach of this second group is preferable, then we must ask how we assist teachers to progress towards it. One answer is by the provision of pedagogically focussed professional development, relevant classroom focussed resources and good lines of teacher-researcher communication.

REFERENCES

Brousseau, G. (1997). Theory of didactical situations in mathematics: Didactique des mathematiques, 1970-1990. (N. Balacheff, M. Cooper, R. Sutherland & V. Warfield, Trans. & Eds.), Dordrecht: Kluwer Academic Publishers.

Chinnappan, M. & Thomas, M. O. J. (2003). Teachers' Function Schemas and their Role in Modelling, *Mathematics Education Research Journal, 15*(2). 151-170.

Cooney, T. J. (1999). Conceptualizing teachers' way of knowing. Educational Studies in Mathematics, 38, 163-187.

Kendal, M., & Stacey, K. (2001). The impact of teacher privileging on learning differentiation with technology, *International Journal of Computers for Mathematical Learning, 6,* 143-165.

Rabardel, P. (1995). *Les hommes et les technologies, approche cognitive des instruments contemporains,* Paris: Armand Colin.

Shulman, L. C. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15,* 4-41.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Thomas, M. O. J. (1994). A process-oriented preference in the writing of algebraic equations, *Proceedings of the 17th Mathematics Education Research Group of Australasia Conference*, Lismore, Australia, 599-606.

Thomas, M. O. J & Hong, Y. Y. (2001). Representations as Conceptual Tools: Process and Structural Perspectives, *Proceedings of The 25th Conference of the International Group for the Psychology of Mathematics Education*, Utrecht, The Netherlands, 257-264.

Thomas, M. O. J., Monaghan, J., Pierce, R. (2004). Computer algebra systems and algebra: Curriculum, assessment, teaching, and learning. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The teaching and learning of algebra: The 12th ICMI study* (pp. 155-186). Norwood, MA: Kluwer Academic Publishers.

Thomas, M. O. J., Tyrrell, J. & Bullock, J. (1996). Using Computers in the Mathematics Classroom: The Role of the Teacher, *Mathematics Education Research Journal*, *8*(1), 38-57.

STUDENTS' OVERRELIANCE ON LINEARITY: AN EFFECT OF SCHOOL-LIKE WORD PROBLEMS?

Wim Van Dooren [1 2], Dirk De Bock [2 3], Dirk Janssens [2] and Lieven Verschaffel [2]

[1] Research assistant of the Fund for Scientific Research (F.W.O.) – Flanders

[2] Center for Instructional Psychology and Technology, University of Leuven, Belgium

[3] European Institute of Higher Education Brussels (EHSAL), Belgium

Previous research showed students' tendency to improperly apply the linear model when solving non-linear problems about the relation between lengths, area and volume of enlarged figures. Most of these studies, however, were conducted with collective tests containing traditional, "school-like" word problems. The current study shows that students' problem-solving behavior strongly improves when the non-linear problem is embedded in a meaningful, authentic performance task. It is also found that this experience does not affect students' performance on a posttest, where the non-linear problem is offered again as a word problem.

INTRODUCTION

Because of its wide applicability for understanding mathematical, scientific and everyday life problems, linearity (or proportionality) is a key concept throughout primary and secondary mathematics education. Inherent to the attention it receives, however, is the risk to develop an overreliance on the concept: "Linearity is such a suggestive property of relations that one readily yields to the seduction to deal with each numerical relation as if it were linear" (Freudenthal, 1983, p. 267). The tendency to overgeneralise the linear model is repeatedly mentioned in the mathematics education literature, and in recent years it has also been in the focus of systematic empirical research. For example, the phenomenon has been studied in elementary arithmetic (Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005), algebra and calculus (e.g., Esteley, Villareal, & Alagia, 2004) and probability (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003).

The best-known (and extensively studied) case is situated in geometry: many students of different ages believe in a linear relation between the lengths, areas and volumes of similarly enlarged geometrical figures, thinking that if a figure is enlarged k times, the area and volume of that figure are enlarged k times as well (De Bock, Verschaffel, & Janssens, 1998, 2002b; De Bock, Van Dooren, Janssens, & Verschaffel, 2002a; Freudenthal, 1983; Modestou, Gagatsis, & Pitta-Pantazi, 2004). A series of studies has shown that even with considerable support (such as providing drawings, instructing to make drawings, or giving metacognitive hints), the large majority of 12- to 16-year old students failed to solve these problems due to an alarmingly strong tendency to apply linearity (De Bock et al., 1998, 2002b; Modestou et al., 2004). Further research showed that the tendency was due to a set of closely related factors like the intuitiveness of the linear model, shortcomings in students' geometrical knowledge, inadaptive attitudes and beliefs towards mathematical (word) problem solving and a poor use of heuristics (De Bock et al., 2002a).

These last explanatory factors (namely: attitudes and beliefs towards word problem solving and a poor use of heuristics) led us to conduct the study presented in this paper. In most of the previous research the overreliance on linearity was observed in a classical scholastic context by means of collective tests with word problems, i.e. short written descriptions of a problem situation with the task to do some calculations and to write down a short numerical answer. It can be argued that the use of word problems may trigger in students a set of implicit rules and expectations established by the socio-mathematical norms of the classroom setting (Cobb, Yackel, & McClain, 2000; Verschaffel, Greer, & De Corte, 2000). Possibly, the students in our previous studies may not have invested sufficient mental effort in the solution of the problems – assuming that they were dealing with routine word problems –, or may have excluded a number of considerations and problem solving strategies (e.g., checking the viability of a solution by making a sketch of the situation) – assuming that they were not desirable, acceptable or valid in that context. Research evidence shows that students are more inclined to leave their routine word problem solving behavior and include real-world knowledge when the problems are disentangled from their scholastic chains and embedded in more meaningful, authentic "performance tasks" (e.g., DeFranco & Curcio, 1997; Nunes, Schliemann, & Carraher, 1993; Reusser & Stebler, 1997). The current study aimed at investigating whether this would also be effective to break students' overreliance on linearity: Can this tendency be weakened or even eliminated by embedding non-linear problems in meaningful, authentic performance tasks instead of traditional, school-like word problems?

METHODOLOGY

The study was conducted in three steps. First, participants were selected using a pretest. Next, students who made a linear error on the pretest were involved in an individual interview. And third, a posttest was taken of all the interviewed participants. Each part is explained in more detail below.

Selection of participants by pretest

The first step was to select students who were prone to the error under consideration. 93 sixth graders (i.e., five whole class groups in two different schools) solved a pretest that contained six word problems. Five of the word problems acted as buffer items (they were included to avoid revealing the focus of our study). One word problem in the test aimed at detecting whether students tended to give a linear answer to problems about the effect of an enlargement on the area of a square:

> John needs 15 minutes to paint a square ceiling with a side of 3 meters. How much time will he approximately need to paint a square ceiling with a side of 6 meters?

Altogether, 72 students gave a linear answer to this problem (e.g., "3 m × 2 = 6 m → 15 min × 2 = 30 min"). They were involved in the rest of the study.

Interview procedure

Two days after the pretest, these 72 students were taken individually out of the classroom for a semi-structured in-depth interview. During that interview, the students again were asked to solve a non-linear problem, this time about the effect of *tripling* the lengths of the sides of a square on the area of that square. This problem was offered in one of three different ways, depending on the interview condition that the student was assigned to. Assigning students to interview conditions happened by means of matching on the basis of school mathematics performances.

Students in the *S-condition* ("Scholastic" condition, $n = 24$) received a sheet with the following traditional, scholastic word problem:

> Recently, I made a dollhouse for my sister. One of the rooms had a square floor with sides of 12 cm. I needed 4 square tiles to cover it. Another floor of the dollhouse was also a square, but with sides of 36 cm. How many of those square tiles did I need to cover it?

The problem in the *D-condition* ("Drawing"-condition, $n = 24$) was the same as in the S-condition, but this time the sheet also contained a drawing of the small and large figure, as shown in Figure 1.

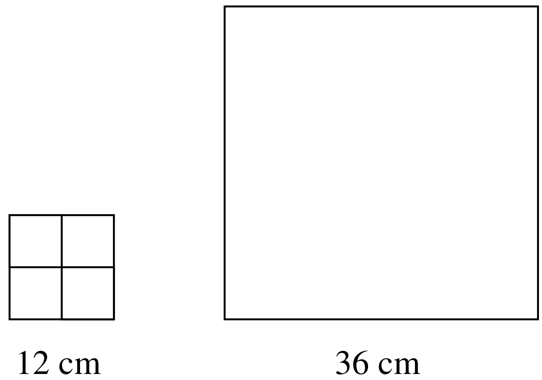


Figure 1: Drawing offered with the word problem in the D-condition

In the *P-condition* ("Performance task"-condition, $n = 24$), the problem was presented as a "performance task": Students were involved in an authentic problem context with real materials (the small dollhouse floor, 4 tiles and a large dollhouse floor) and were asked to perform an authentic action. The interviewer presented the task as follows:

> I have a little sister, and currently I am making a dollhouse for her. Here, you can see the floor of one of the rooms. Can you tell me its shape? [The student tells that it is a square.] Let's measure it. [Student observes that the sides are 12 cm long.] I have some tiles that we can use to cover that floor. Can you do that for me? [Student puts 4 tiles on the small floor.] Indeed, we need 4 tiles to cover this floor.
>
> I also brought another floor of the dollhouse. As you see, it is also square. Let's measure it. [Student observes that the sides are 36 cm.] In a few moments, we will put tiles on this large floor as well. Now, think about how many tiles you will need to do that, and if you have decided, you can go and get exactly enough tiles from the table over there.

The number of tiles brought was registered as the students' final answer. At the end of the interview, students in the P-condition were allowed to put the tiles effectively on the large floor (and could get more tiles if necessary).

Strictly spoken, for our research goal the design only would need to include a S-condition and a P-condition, but nevertheless the D-condition was included, because students in the P-condition not only received the non-linear problem as an authentic performance task instead of a school-like word problem; the problem presentation in the P-condition involved also visual support, while this support was not present in the S-condition. Including a D-condition – which provided the same visual support as the P-condition, but the same scholastic presentation as the S-condition – allowed us to control for this visual support factor.

All interviews were registered on videotape, and students were asked to think aloud while solving the problem. They were told that they could solve the problem in whatever way they wanted and use all materials available (pen, paper, ruler, pocket calculator and in the P-condition also the small and large floors and the 4 available tiles). When necessary, the interviewer asked some additional probing questions to clarify students' thinking. At the end of the interview, the students were asked to indicate on a five-point scale how certain they were about the correctness of their answer ('certainly wrong', 'probably wrong', 'no idea', 'probably correct' and 'certainly correct'), and to justify this[1].

Posttest

One or two days after their interview, students solved a posttest. Besides five buffer items, it again included a non-linear problem that referred to the same mathematical situation as the pretest item (the effect of doubling the sides of a square on its area):

> Carl needs 8 hours to manure a square piece of land with a side of 200 meters. How much time will he approximately need to manure a square piece of land with a side of 400 meters?

Because this problem situation only slightly differred from the one that students had in the interviews (where the sides of the square were tripled), it could be determined whether the experiences during the interview also had a learning effect. For example, manipulating the materials during the interview in the P-condition could be helpful for the student to solve the non-linear problem on the posttest correctly too.

RESULTS

Individual interviews

Table 1 provides a summary of the answers and the solution time (i.e. the time needed to find an answer after the problem was introduced) of the students in the three interview conditions. The table shows that there was a strong impact of the interview condition on students' answers (Fisher's exact test $p < .00015$):

[1] Evidently, P-condition students had to answer the probing questions and the certainty question before they were allowed to put the tiles on the large floor to check their answer.

Nearly all students in the *S-condition* (i.e. 21 of 24 students) erroneously applied linearity to solve the word problem. This confirms – once again – students' very strong tendency to stick to the linear model when solving problems about the area of enlarged figures, as observed in previous studies (De Bock et al., 1998, 2002a, 2002b). Two students committed another error, and only one student found the correct solution (in contrast with the pretest, this student now made a drawing of the problem situation which led him to the correct solution).

In the *D-condition*, the performance was considerably better. Here, 16 students found the correct answer during the interview. In fact, we had expected that the visual support as such would not be helpful for most students, since in previous studies with collective tests (see, e.g., De Bock et al., 1998, 2002b), the provision of ready-made drawings hardly had any effect on students' performance, mainly because students simply neglected them. In the current study, however, many students did actually use the drawing – possibly an effect of being involved in an individual interview context where they felt more obliged to do so – and their solution process clearly benefited from it. Nevertheless, the provision of a drawing was not sufficient to eliminate all linear reasoning: 8 out of 24 students still gave a linear answer to the word problem.

Although presenting a drawing was beneficial for many students, presenting the non-linear problem as an authentic performance task had an even stronger impact on students' reasoning (Fisher's exact test yielded $p = .0412$ for a separate comparison of the D- and P-condition results). In the P-condition, 20 students gave the correct answer, and only 2 students reasoned linearly (and 2 students made another error).

In sum, almost all students in the S-condition made the linear error, which is not surprising considering that they did the same on the non-linear word problem on the pretest. Providing drawings had a positive effect on students' performance, but still one third of the D-condition students made a linear error. Offering the problem as a meaningful, authentic performance task was even more beneficial, since in the P-condition linear errors were nearly absent.

Interviews				Answer on posttest		
Condition	Answer	Freq	Solution time (seconds)	Correct	Linear	Other error
S-condition (n = 24)	Correct	1	120	1	0	0
	Linear	21	50	1	20	0
	Other error	2	115	0	2	0
D-condition (n = 24)	Correct	16	139	0	16	0
	Linear	8	61	1	6	1
	Other error	0	/	/	/	/
P-condition (n = 24)	Correct	20	76	2	17	1
	Linear	2	29	1	1	0
	Other error	2	131	0	0	2

Table 1: Overview of answers and average solution times in each interview condition and of answers on the posttest

A closer look at students' solution times (see Table 1) and procedures (as registered on videotape) was helpful in clarifying how students obtained their answers and in understanding the effect of the experimental manipulations.

First of all, there seems to be a clear relationship between the duration of a solution process and its overall quality. As Table 1 shows, students who gave the linear answer required about one minute less than students who solved the problem correctly, and this difference was found in all three interview conditions. This is not surprising considering our previous research findings: the linear model is self-evident for many students and affects their thinking in an immediate and spontaneous way, whereas they experience the quadratic relation between lengths and area often experienced as counter-intuitive (De Bock et al., 2002a). A quick resposnse to a non-linear problem is often an indication that a student is (mis)led by linear thinking.

More importantly, the analysis of solution times and procedures revealed substantial differences between the interview conditions. Both in the D-condition and in the P-condition, many students did find the correct answer, but the way in which this answer was achieved differed. In the D-condition, the 16 students who found the correct answer needed relatively much time (on average 139 seconds), and they had to rely extensively on the drawing. Often, the idea to work on the drawing came up rather late in the solution process. When students gave the correct answer, many of them were still not very convinced about its correctness. Often, they indicated that reading, interpretation or calculation errors might have occurred or that they might have been overlooking a critical aspect in the problem situation. As an example, we quote from the interview with Deborah (D-condition, solution time of 194 seconds):

Deborah: [Silence of about 60 seconds. Reads the problem several times again.] "So … I think I should see … [Measures sides of small and large square.] The small floor is 12 cm and the large 36 cm. So that's 3 times. Eh … 3 times 4 tiles is 12 tiles. No, it's 3 times here and 3 times there, that's 9 times. You need 9 tiles… Wait, let me read it again [Reads the problem and thinks for a long time.] I don't know how I should calculate it. Maybe here … there's 4 tiles, and … yes! [Draws 9 of the small floors in the large floor.] 9 times more, so I take 9 times those 4 tiles. 36, you need 36 tiles."

Interviewer: "36 is your answer. Can you tell me how certain you are that that is the correct answer?"

Deborah: [Chooses 'Probably correct'] "I'm never sure about myself. I don't trust it, maybe there's something wrong with the drawing. It could be a tricky question."

Such a process contrasts with many solution procedures from the P-condition. Here, students needed on average only 76 seconds to respond correctly. In most cases, the students immediately and spontaneously started to manipulate the materials to find the solution (figuring out rather quickly that 6×6 tiles fit on the large floor, or that the small floor fits 3×3 times on the large one). Remarkably, three students in the P-condition gave the correct answer almost immediately. Once the problem situation was explained to them, they did not require any additional time for thinking or material manipulating at all. They just "saw" the correct solution at a glimpse. Generally, students in the P-condition were moreover very convinced about the correctness of their answer. Consider for example the following fragment from the interview with Marlies (P-condition, solution time of 34 seconds):

Marlies: [Takes the small floor and fits it several times on the large floor.] "It's 9 times this small one, which has 4 tiles, so 36 tiles." [Goes immediately to fetch 36 tiles.]

Interviewer: "So you brought 36 tiles. Wait a moment before putting them on the floor. First, can you tell me how certain you are that that is the correct answer?"

Marlies: [Chooses 'Certainly correct'] "It is correct. I just showed you that it's 9 times more. Why would I need to doubt about it? I am just sure"

Posttest

Table 1 also contains the results on the non-linear posttest item. It aimed at testing whether students who had solved the non-linear problem correctly during the interview would do this on the posttest as well (taken one or two days later). Apparently, this was not really the case. The S-condition student who found the correct solution solved the posttest problem correctly as well (again, by making a drawing). But in the other two conditions hardly any effect of the interview experience was found: While 16 of the 24 students in the D-condition interview profited from the drawings to find the correct answer, all of them again reasoned linearly on the posttest. And whereas 20 of the 24 students found the correct answer in the P-condition interview, only two of them solved the posttest item correctly (and all others, except one, again made a linear error).

Finally, of the four students from the P-group who failed to solve the problem by themselves but who could act out and see the correct answer at the end of the interview task, only one student solved the non-linear posttest item correctly; the other three gave the same answer on the posttest as during the interview.

CONCLUSIONS AND DISCUSSION

In previous research, students' overreliance on linearity was often observed by means of tests containing traditional, scholastic word problems. The current study has shown that this has an important impact on students' solution behavior. When students who made a linear error on the pretest were involved in an interview with more meaningful, authentic performance tasks, they approached the problems very differently, and they were less tended to overgeneralise linear methods. As such, our results confirm those observed by other scholars and for other kinds of modelling problems (for an overview, see Verschaffel et al., 2000).

More importantly, our study has additionally shown that offering meaningful, authentic performance tasks affected students' problem solving behavior only at that moment itself. At a posttest (again with traditional word problems) taken shortly afterwards, nearly all students again gave a linear answer to the non-linear problem. By means of more fine-grained research, focusing on the differences in the mathematical concepts, heuristic and metacognitive strategies, beliefs, assumptions, etc. that students activate when solving meaningful, authentic performance tasks versus traditional word problems, we hope to get deeper insight into the reasons why performance tasks have such a strong but at the same time such a context-specific impact on students' performance.

References

Cobb, P., Yackel, E., & McClain, K. (2000). *Symbolizing and communicating in mathematics classrooms*. Mahwah, NJ: Lawrence Erlbaum Associates.

De Bock, D., Van Dooren, W., Janssens, D., & Verschaffel, L. (2002a). Improper use of linear reasoning: An in-depth study of the nature and the irresistibility of secondary school students' errors. *Educational Studies in Mathematics, 50*, 311–334.

De Bock, D., Verschaffel, L., & Janssens, D. (1998). The predominance of the linear model in secondary school students' solutions of word problems involving length and area of similar plane figures. *Educational Studies in Mathematics, 35*, 65–83.

De Bock, D., Verschaffel, L., & Janssens, D. (2002b). The effects of different problem presentations and formulations on the illusion of linearity in secondary school students. *Mathematical Thinking and Learning, 4*, 65–89.

DeFranco, T. C., & Curcio, F. R. (1997). A division problem with a remainder embedded across two contexts : Children's solutions in restrictive versus real-world settings. *Focus on Learning Problems in Mathematics, 19*(2), 58–72.

Esteley, C., Villarreal, M., & Alagia, H. (2004). Extending linear models to non-linear contexts: An in-depth study about two university students' mathematical productions. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 343–350). Bergen, Norway.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: Reidel.

Greer, B. (1993). The mathematical modelling perspective on wor(l)d problems. *Journal of Mathematical Behavior, 12*, 239–250.

Modestou, M., Gagatsis, A., & Pitta-Pantazi, D. (2004). Students' improper proportional reasoning: The case of area and volume of rectangular figures. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 345–352). Bergen, Norway.

Nunes, T., Schliemann, A., & Carraher, D. (1993). *Street mathematics and school mathematics*. Cambridge, UK: Cambridge University Press.

Reusser, K., & Stebler, R. (1997). Realistic mathematical modelling through the solving of performance tasks. *Paper presented at the Seventh European Conference on Learning and Instruction*, Athens, Greece.

Van Dooren, W., De Bock, D., Depaepe, F., Janssens, D., & Verschaffel, L. (2003). The illusion of linearity: Expanding the evidence towards probabilistic reasoning. *Educational Studies in Mathematics, 53*, 113–138.

Van Dooren, W., De Bock, D., Hessels, A., Janssens, D., & Verschaffel, L. (2005). Not everything is proportional: Effects of age and problem type on propensities for overgeneralization. *Cognition and Instruction, 23*(1), 57–86.

Verschaffel, L., Greer, B., De Corte, E. (2000). *Making sense of word problems*. Lisse, The Netherlands: Swets & Zeitlinger.

A PROCESS OF ABSTRACTION BY REPRESENTATIONS OF CONCEPTS

N.C.Verhoef & H.G.B. Broekman

University of Twente, Utrecht University

The purpose of this article is to describe the integration of epistemological principles, theories about levels of argumentation, and different worlds of abstraction to address secondary school students' development of mathematical concepts. A pilot study concerning 'volume and enlargements' focuses on step-by-step solutions of classified problems to establish the progress of the process of abstraction. The analysis is based on different language use.

INTRODUCTION

Worldwide, but especially in the Netherlands, mathematical education illuminates only a few characteristics of a preparation for a scientific study at a university. In addition mathematical concepts are wrapped in a variety of contexts. Subsequently it is necessary to begin a process of abstraction by releasing these contexts. Otherwise students' intellectual challenge decreases more and more. Casually the attention of (supposed) applications and applicable meanings dominate mathematical education too long. The development of a process of thinking unfastened of applications starts abruptly, without systematically foundation. Dutch textbooks demonstrate a mostly absence of definitions and development of a theory, and an elevation of illustrations and visualizations to mathematical concepts. Students and teachers alienate from their intuition by acquired tricks and bad methods of learning and teaching. Mathematical education, answering this characterization, lacks the intellectual challenge highlighted by a scientifically study in the future.

THEORETICAL FRAMEWORK

Teaching and learning with representations of objects

From a cognitive psychological point of view the development of knowledge takes place by experiencing reality (Piaget, 1972). Experience results from the interaction with objects which on the one hand leads to the development of ideas about these objects and on the other hand by manipulation to the possibility to evaluate whether these ideas are correct or not. Experiences with objects in the real world can be divided into (a) direct experiences with the reality or (b) mediated experiences by the use of media. In the latter hand a medium is used to depict or to describe the reality or both. In education, mediated experience by using a representation of the reality is essential, as real objects are not always available or suitable to use. Besides it is possible to refer to an imaging reality.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 273-279. Melbourne: PME.

The representing medium (the representation) is related to the represented object (the reality) through a set of mapping principles that maps elements of the reality to elements in the representation. Some representations are (almost) similar to the represented object, such as photographs or statues. These are called *pictures.* In these cases, every element in the represented object is represented by a unique element in the representing medium, so that there is a one-to-one mapping or isomorphism between the two. If a representation is an abstraction of the represented object in which the characteristics relevant to the situation emerge and in which the characteristics irrelevant to the situation are left out, this is called homomorphism between the representing medium and the represented object. In this case, two or more elements in the represented object are represented by only one element in the representing medium. An example of this is the figure of a man or a woman on a toilet door. In all cases in which a representation represents the represented object to some extent of similarity, the term *icon* is used. The relationship between an icon and the represented object depends on their 'mode of correspondence'. There are also representations that have no similarity at all with their represented object. These are chosen arbitrarily by convention and are called *symbols*. Examples of these are the letters of the alphabet, or numerals.

From instructional design basic ideas Seel and Winn (1997, p. 298) argued that 'people's thinking consists of the use and manipulation of signs as media for the representations of ideas as well as objects'. Changes in the kind of representations will have a direct effect on learning processes. In this study about the development of mathematical concepts the reference point is the use of signs as: 'pictures' (a video or a photograph); 'icons' (a figure); and 'symbols' (a formula or a definition).

Teaching and learning mathematics by reasoning and argumentation

In mathematics there are frequently used signs at different levels of abstraction. For instance the representation of the concept 'circle' is a photograph of a cup of tea (picture), a round about on paper (icon), or a set of points (symbol).

Particularly the Dutch researcher Pierre van Hiele (1986) was engaged in levels of mathematical thinking, especially in geometry. He introduced levels of reasoning or argumentation to indicate a process of abstraction: the zero-level (visual level) of sensorial perceiving of objects, the first level (description level) of properties of objects, and the second level (theoretical level) of sets, logical operators and formal proofs. There exist some variants with another third level. In that case the second level is called the informal deductive level and the third level is mentioned the deductive level. Later on this variant comes back in this article. Van Hiele characterised the zero-level, or called the ground level, as a lack of relationships. The basic ideas about mathematical concepts rest on intuition. Either at the first level the concepts are founded on the properties of mathematical concepts. At this level it is possible to manipulate concepts because of the understanding properties. By Van Hiele the transition from the zero-level to the first level exists of the determination of sensorial descriptions. So relationships will be built as a net of relations based on knots of mathematical concepts with a pack of properties each. In accordance with Van Hiele it is possible to test the presence of a net of relations by the determination of the usage of language. Van Hiele supposed at the transition of the zero-level to the first level a developmental process of language use from everyday language to formal language. For example the judgment "the direction NNW (bisector) at this map lies exactly between the direction N and the direction NW" in everyday language and "each point at the bisector has the same distances to both sides of the angle" in formal language. The second level of Van Hiele distinguishes a total disconnection of real situations or schemes. The properties of mathematical concepts are logical ordered by which arises a formal relationship between the concepts. At this level it is possible to deduce formal properties from other properties. So proofs as well as the usability of symbols, definitions and formulas are necessary to describe mathematical concepts. By Van Hiele the transition from the first to the second level occurs in a process of analysis and objectivity by a development in language use from formal to symbol language.

Teaching and learning mathematics by a journey through 3 worlds

Also David Tall (2004) is intensively engaged in all about abstraction in mathematics education. He distinguishes explicitly the development of geometric and algebraic mathematical concepts. So he presumes three different worlds: the 'embodied world', the 'symbolic world', and the 'formal world'.

In the first embodied world the development of concepts is realized by a growing process that starts in the real world and consists of our thinking about things that will be perceived and sensed, not only in the physical world, but also in an individual mental world of meaning. By reflection and by the use of increasingly sophisticated language, it is possible to focus on aspects of sensory experiences that enables to envisage conceptions that no longer exist in the real world outside, such as a 'point' that has no thickness, no length and no width. This world mentions the 'conceptual-embodied world' or 'embodied world' for short (Tall & Ramos, 2004). This includes not only mental perceptions of real-world objects, but also intentional conceptions that involve visuo-spatial imaginary. It applies not only the conceptual development of geometrical objects but also other mathematical concepts like algebraically (square root of number two), analytical (derivative) and statistical objects (median).

The second world is the world of symbols that are necessary for calculation and for manipulation in arithmetic, algebra, calculus, statistics and so on. These begin with actions that are encapsulated as concepts by using symbols that allow an effortless switch from processes to do mathematics to concepts to think about. This second world is called 'proceptual (the term procept is a contractation of the terms process and concept) - symbolic world' or 'symbolic world' for short. In this world the central point of view consists of actions with objects.

The third world is the formal world based on properties, expressed in terms of formal definitions that are used as axioms to specify mathematical structures (such as 'group', 'field', 'set'). This third world is termed the 'formal-axiomatic world' or 'formal world' for short. This world considers of mental activities. Properties are used to define mathematical structures in terms of special properties. New concepts can be defined to build a coherent logically deduced theory.

Individual travellers through these three worlds take a unique route. Various obstacles occur on the way that requires earlier ideas to be reconsidered and reconstructed, so that the journey is not the same for each traveller. Either to attain the formal world it is recommended to start experiences in the embodied world, and to continue these experiences in the symbol world. Because of the development of geometrical concepts the journey to the formal world concerns only the embodied world by experiences and thought- experiences. This journey follows a natural growing process of sophistication via four Van Hiele levels of abstraction. The first step is the step of perceiving objects as whole gestalts. The second step is the step of description of properties with language growing more sophisticated. In a way that descriptions in the third step become definitions suitable for process and proof in the fourth step to the formal world. This journey provides a theoretical framework for the developmental process of learning geometry.

THE PILOT STUDY TO DESCRIBE CLASSROOM PRACTICES

In this pilot study the epistemological ideas (Seel & Winn, 1997), theories about levels of argumentation (Van Hiele, 1986) and different worlds of abstraction (Tall, 2004) are combined with each other to design principles of teaching and learning mathematics. For instance the step-by-step developmental process of the geometrical concept 'area of a triangle' can be activated by the usage of signs in Tall's embodied world. That means assignments with representations of this concept at different levels of abstraction. A transparent paper with squares covers a photograph of a flat triangular object. The number of the squares is a measure of the area. Little squares result in another measure than big squares do. At this level the representations are based on sensory perceptions. The arguments are formulated in everyday language use. This is the first step in the developmental process to attain abstraction. The animated joining of the triangle and the encircled rectangle supply a foresight (properties of) the relationship between the area of the triangle and the area of the encircled rectangle. The language use change over to formal language. This is the second step. Afterwards the representation by a drawing of a triangle with base and altitude is enough to understand this concept. The language use reconverts to symbol language use. This is the third step. At the highest level of abstraction even the drawing is not necessary. The representation of the area of a triangle by the formula is enough. This is the last step in the process of abstraction. Finally the representation of the area of a triangle consists of a relationship between variables in Tall's formal world. This is strictly mental and therefore maximal manoeuvrable. In terms of Van

Hiele the argumentation in everyday language use is founded on representations (signs) at the zero-level like photos (pictures), at the first level in formal language use like figures (icons), and at the second level in symbol language use like formulas (symbols).

Classified problems to structure the concept developmental process

The concept developmental process would be activated and stimulated by problems related to each level of abstraction, or each representation (picture – icon – symbol).

To structure the process of concept development the problems are classified in three different types: (1) categorisation problems, (2) declaration problems, and (3) design problems (Van Merriënboer & Dijkstra, 1997). In case of categorisation problems, mathematical objects must be assigned to unknown real or imagined situations. The strategy to solve categorisation problems underpins the use of a variety of visual and dynamic representations of mathematical objects.

In case of declaration problems, the cognitive constructs or declarative knowledge are principles, as well as casual networks and explanatory theories. The strategy to solve declaration problems is based on the declaration of relationships between mathematical concepts, manipulations with mathematical concepts, and applications with mathematical concepts. In this strategy, the focus is based on students' predictions concerning what will happen in specified situations, test predictions indicating whether it is confirmed or falsified, and, if relevant, specify the range of probabilities of occurrences of certain events. Explanatory theories predict changes of objects and relationships and lead to understanding of the casual mechanics involved. In the case of design problems, an artefact must be imagined and a plan has to be constructed to solve arising mathematical problems in the real world. The strategy to solve design problems concerns the construction of a model and the interpretations of models, or more models if necessary. For simplifying the reality and constructing a cognitive mathematical content, the label *mathematical model* is used. The mathematical model of the reality needs mathematical techniques (e.g. computer simulations) to be solved. Statements about the mathematical model will be retranslated in the modelling reality.

Step by step solutions to establish any progress in the developmental process

Problem solving skills are related to language use. To establish any progress in the concept developmental process the web-based problems were supplied to the - 12 or 13 years old - students with empty formats to preserve digital answers. The format was, based on principles of problem solving skills in heuristic mathematics education, divided into five steps (Van Streun, 1989): (1) application of the concept in the reality, (2) to order data, (3) to describe the approach, (4) to execute the approach, and (5) to reflect at the solution and at the approach. So students' notes could be sampled and compared step by step with everyday language use, formal language use, and symbol language use. For instance the following notes to assignment 14, paragraph (9.2):

Calculate the volume of a soup cup with the diameter 11,4 cm and the altitude 5,23 cm. Complete the answer at one decimal.

(1) volume of a circle, (2) diameter (11,4 cm) and altitude (5,23 cm), (3) to decide the volume of a cylinder by using the area of the ground plane (circle with radius 5,7 cm) and altitude (5,23 cm), (4) volume (soup cup) = $(\pi * 5{,}7^2) * 5{,}23 \approx 530{,}8\ cm^3$, (5) 530,8 cm^3 is a little bit more then half a litre can (reflection on the solution), the volume of a cylinder is like the area circle * altitude (reflection on the approach).

The pilot study to students' concept developmental processes

Twelve groups of secondary school students – 12 or 13 years old- of about thirty students each were involved in a three-week period where they applied a website with problems. All the problems were classified as categorization problems. The dynamical geometrical environment Cinderella functioned as a workspace. The subject was all about 'volumes and enlargements'. The paragraph was divided into sub-paragraphs: units of volume, volume of prisms and volume of cylinders, volume of pyramids, to enlarge and to reduce objects, area at enlargements, volume at enlargements, summary, mixed assignments, repetition, and extra, more creative assignments. All assignments were copied from an original, most common textbook, without any change. A picture illustrated each assignment. Ten participated teachers had the following instructions to support students' concept developmental learning processes: (i) to switch zigzag with students to representations of concepts at different levels of abstraction, and (ii) to search for other representations of concepts at the same level of abstraction (Verhoef, 2003). The data consisted of: (i) electronic students' step-by-step responses in the last (third) college time in the first week (pre-test) and in the ended (twelfth) college time in the last week (posttest), and (ii) a written test, the same test and the same teacher as the previous year. The electronic responses were transcribed and categorised by two student assistants into catchwords in everyday language, formal language and symbol language. The data were analysed at the language use. The expectation was (i) less everyday language use in the posttest in relation to the pre-test, en (ii) higher results in the written test than the results at the same test in the previous year.

The findings of this pilot study are related to epistemological principles as well as theories about teaching and learning in mathematics education.

In the pre-test students were inclined to annotate all the five steps. The possibility to answer step-by-step was a new phenomenon for these students. The solution (step 4) was in de post-test correctly annotated in formal language use and symbol language use. Students didn't use everyday language anymore.

The results of the written test were higher than the test results of previous year in three groups. Teachers of these groups required only students' step-by-step solutions. In the other groups the test results were averagely lesser. The low results were attributed to the incorrect use of symbol language use (formulas) without formal language use (descriptions).

DISCUSSION

This pilot study involved analysis of secondary school students' development of mathematical concepts by categorization problems and the possibility to answer step-by-step. Firstly it is necessary to investigate types of different problems like categorization problems, but also declaration and design problems. Each problem type has original skills to solve these problems, prepared on different representations (picture – icon – symbol), different language use (everyday – formal- symbol), and different (thought-) experiments.

Secondly, it seems advisable to answer step-by-step, because of higher results in written tests. Reflection (step 5) will be the most important step to attain a higher level of abstraction. Naturally students follow their intuition, they don't reflect (De Bock, Van Dooren, Verschaffel, & Janssens, 2002). So teacher's support focuses on the emphasis at the process of reflection highlighted by the use of representations.

References

De Bock, D., Van Dooren, W., Verschaffel, L., & Janssens, D. (2002). Improper use of linear reasoning: An in-depth study of the nature and irresistibility of secondary school students' errors. *Educational Studies in Mathematics, 50* (3), 311–334.

Piaget, J. (1972). *The Principles of Genetic Epistemology* (we. Mays trans.) London: Routledge & Kegan Paul.

Seel, N. M., & Winn, W. D. (1997). Research on media and learning: Distributed cognition and semiotics. In S. Dijkstra, N. Seel., F. Schott & R.D. Tennyson (Eds.), *Instructional design: International perspectives, Volume 1* (pp. 290-327). Mahwah, NJ: Lawrence Erlbaum.

Tall, D.O., & Ramos, J.P.M. (2004). Reflecting on Post-Calculus-Reform. Plenary for *Topic Group 12: Calculus, International Congress of Mathematics Education*, Copenhagen, Denmark.

Tall, D.O. (2004). Thinking through Three Worlds of Mathematics. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Bergen, Norway.

van Hiele, P.M. (1986). *Structure and Insight.* Orlando: Academic Press. Inc.

van Merriënboer, J.J.G., & S. Dijkstra (1997). The four-component instructional-design model for training complex cognitive skills. In Tennyson, R.D., Scott, F., Seel, N., & Dijkstra, S. (Eds.). *Instructional design: international perspectives, Volume I* (pp. 427-446). Mahwah, NJ: Lawrence Erlbaum Associates.

van Streun, A. (1989). *Heuristisch wiskundeonderwijs*. Proefschrift Rijksuniversiteit Groningen.

Verhoef, N.C. (2003). *Implementatie van ICT, een probleem voor docenten? De scholing van hbo-docenten bij de implementatie van een elektronische zelfstudiemodule voor het vak wiskunde.* Proefschrift Universiteit Twente.

ARGUMENTATION PROFILE CHARTS AS TOOLS FOR ANALYSING STUDENTS' ARGUMENTATIONS

Jill Vincent — University of Melbourne
Helen Chick — University of Melbourne
Barry McCrae — Australian Council for Educational Research

Traditional argument theories focus on how the structure of statements determines their contribution to an argument. Such theories are useful in analysing arguments as products, or for analysing the sub-arguments that are generated during an argumentation. This paper outlines a method for analysing an argumentation as a process, focusing on the social interactions between pairs of Year 8 students and the teacher-researcher in the context of geometric reasoning.

CONJECTURING, JUSTIFYING AND ARGUMENTATION

Over recent decades concern has been expressed that school mathematics focuses on product rather than process, with the result that many students are unable to justify or explain their reasoning. In 1991, for example, the Australian Education Council asserted that

> the systematic and formal way in which mathematics is often presented conveys an image of mathematics which is at odds with the way it actually develops. Mathematical discoveries, conjectures, generalisations, counter-examples, refutations and proofs are all part of what it means to do mathematics. School mathematics should show the intuitive and creative nature of the process, and also the false starts and blind alleys, the erroneous conceptions and errors of reasoning which tend to be a part of mathematics. (p. 14)

Mathematics curriculum statements in many countries (see, for example, National Council of Teachers of Mathematics, 2000) are now emphasising the need for students to engage in conjecturing and to justify their reasoning.

Argument and argumentation

An *argument* may be defined as a sequence of mathematical statements that aims to convince, whereas *argumentation* may be regarded as a process in which a logically connected mathematical discourse is developed. Krummheuer (1995) views an *argument* as either a specific sub-structure within a complex argumentation or the outcome of an argumentation: "The final sequence of statements accepted by all participants, which are more or less completely reconstructable by the participants or by an observer as well, will be called an *argument*" (p. 247). We can therefore distinguish between argumentation as a process and argument as a product. Krummheuer notes that argumentation traditionally relates to an individual convincing a group of listeners but may also be an internal process carried out by an individual. He uses the term 'collective argumentation' to describe an argumentation accomplished by a group of individuals.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 281-288. Melbourne: PME.

Some researchers, for example, Boero, Garuti, Lemut, and Mariotti (1996), assert that it is only by engaging in conjecturing and argumentation that students develop an understanding of mathematical proof. Boero et al. use the term 'cognitive unity' to signify the continuity that they assert must exist between the production of a conjecture during argumentation and the successful construction of its proof:

> During the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices. During the subsequent statement-proving stage, the student links up with this process in a coherent way, organising some of the justifications ('arguments') produced during the construction of the statement according to a logical chain. (p. 113)

Boero et al. claim that the reasoning which takes place during the argumentation plays a crucial role in the subsequent proof construction—"it allows students to consciously explore different alternatives, to progressively specify the statement [of the conjecture] and to justify the plausibility of the produced conjecture" (p. 118).

Critics of this conjecturing/argumentation approach to proof assert, however, that the natural language of students' argumentation is in conflict with the logic associated with deductive reasoning. Balacheff (1991), for example, regards argumentation in the mathematics classroom as an invitation to convince, by whatever means the students choose. He asserts that argumentation implies the freedom to convince by whatever means one chooses and hence that there is a contradiction between the natural language of students' argumentation and the logic associated with deductive reasoning:

> The aim of argumentation is to obtain the agreement of the partner in the interaction, but not in the first place to establish the truth of some statement. As a social behavior it is an open process, in other words it allows the use of any kind of means; whereas, for mathematical proofs, we have to fit the requirement for the use of some knowledge taken from a common body of knowledge on which people (mathematicians) agree. (p. 188−189)

More recently, Balacheff (1999) again makes the strong assertion that argumentation is an obstacle to the teaching of proof because of this inherent conflict between mathematical proof [démonstration], which must "exist relative to an explicit axiom system", and argumentation, which implies freedom to choose how to convince:

> The sources of argumentative competence are in natural language and in practices whose rules are frequently of a profoundly different nature from those required by mathematics, and carry a profound mark of the speakers and circumstances. (p. 3)

Responding to Balacheff's views on argumentation and proof, Boero (1999) focuses on the distinction between 'proving' as a process, that is, argumentation, and 'proof' as a product. He notes that from this perspective that the nature of arguments used by students depends on the establishment of a culture of theorems in the classroom, on the nature of the task, and the specific kinds of reasoning emphasised by the teacher.

Boero regards Balacheff's (1999) reference to "the freedom one could give oneself as a person in the play of an argument" as inappropriate, as strong teacher intervention should ensure that students' arguments are based on sound mathematical logic. Hanna (1995) also emphasises that teacher intervention must be a part of any learning methods which encourage students to interact with each other. She asserts, though, that where classroom practice is informed by constructivist theories, evidence indicates that in many cases teachers are not intervening:

> … teachers tend not to present mathematical arguments or take a substantive part in their discussion. They tend to provide only limited support to students, leaving them in large measure to make sense of arguments by themselves. (p. 44)

ANALYSING THE STRUCTURE OF ARGUMENTS

Argument theories such as those of Toulmin (1958) provide a theoretical framework for analysing the structure of written arguments, particularly deductive arguments, as well as the structure of the reasoning that occurs during a process of argumentation. Toulmin asserts that the foundation for the argument (*data*) and the *conclusion* based on this data must be bridged by a *warrant* that legitimises the inference. Toulmin describes warrants as "inference-licences", whose purpose is to show that "taking these data as a starting point, the step to the original claim or conclusion is an appropriate and legitimate one" (p. 98). Toulmin notes that his model for an argument layout is focusing on a micro-argument: "when one gets down to the level of individual sentences" (p. 94). Micro-arguments form part of the larger context of a macro-argument. Krummheuer (1995), for example, applies Toulmin's model to an argument where the conclusions from two subordinate arguments form the data for the main argument.

PROVIDING A CONTEXT FOR ARGUMENTATION

As part of a research study of the role of argumentation in supporting students' deductive reasoning in geometry (see Vincent (2005); Vincent, Chick & McCrae, 2002), 29 above-average Year 8 students at a private girls' school in Melbourne, Australia were presented with a range of conjecturing/proving tasks. Some of these tasks were pencil-and-paper proofs, some were computer-based (using Cabri Geometry II TM), and others involved the investigation of the geometry of appropriate mechanical linkages. For the linkage tasks, the students worked with physical models of the linkages as well as with teacher-prepared Cabri models. During the video-recorded lessons, the students worked in pairs to formulate conjectures and to develop geometric proofs. In the context of this research, argumentation was viewed as a social process and the extent to which each participant benefited from engaging in an argumentation was influenced by the level of peer interaction.

Deductive reasoning was a new experience for these students, and teacher intervention was of paramount importance in the argumentations. Some interventions were merely to clarify the content of the students' statements, answer non-geometric queries, or assist with software related difficulties. Other interventions, however, assisted the students in some way—re-directing the students' thinking if they had reached an impasse (designated *guidance*), for example, "What other things do you know about parallelograms?"; correcting false statements (*correction*), and ensuring that the students' arguments were based on sound mathematical logic. Boero (1999) notes that "the development of Toulmin-type … argumentations calls for very strong teacher mediation" (p. 1). Interventions which I termed *warrant-prompts* were intended to provoke deductive reasoning by asking the students to justify their statements. An example of a warrant-prompt is: "Why do you say that?" in response to a students' claim: "Those two angles are equal".

In general, four different phases of activity could be identified in the argumentations. An initial *observation* phase generally commenced with task orientation, where the students familiarised themselves with the task by referring, for example, to the given data or noting how the mechanical linkage moved. Following this *observation* phase, or sometimes associated with it, was a *data gathering* that led into *conjecturing* and *proving* phases. The phases were not always distinct, and observations and data gathering often continued throughout the conjecturing phase, and statements of deductive reasoning occasionally occurred in the task orientation phase.

ARGUMENTATION PROFILE CHARTS

Toulmin's model was used to analyse the structure of the students' arguments. In order to provide a visual display of the features of each argumentation, however, I devised an argumentation profile chart (for example, see Figure 1). The charts were constructed as X-Y scatter graphs, with speaking turns on the x-axis. Each characteristic to be displayed—the two students and the teacher-researcher, the phases of the argumentation associated with each statement (task orientation, data gathering, conjecturing, proving), and the medium in which the students were working (computer environment, pencil-and-paper, or a physical model of a linkage)—was given a unique y-value.

Figure 1 depicts the argumentations of two pairs of students, Jane and Sara, and Anna and Kate, during their first conjecturing-proving task—an investigation of Pascal's angle trisector (referred to as Pascal's mathematical machine to avoid disclosing its geometric function). The students had access to a physical model of the linkage as well as to a Cabri model, where they were able make accurate measurements and drag the linkage to simulate its operation. Although Anna and Kate's argumentation is more condensed, the structure is similar in each case, with both pairs of students engaging in a large number of observations and requiring substantial guidance. In both argumentations, initial tentative steps of deductive reasoning were supported by further data gathering and conjecturing. Jane and Sara, however, made many unproductive observations and incorrect statements, for example, "… so these two [two angles which formed a straight line] added together would have to equal 90 or something like that" (Sara, turn 58). Both pairs of students moved between the physical model, the Cabri model, and their pencil-and-paper drawings. Anna and Kate, however, did not return to the physical model once they began exploring the Cabri model.

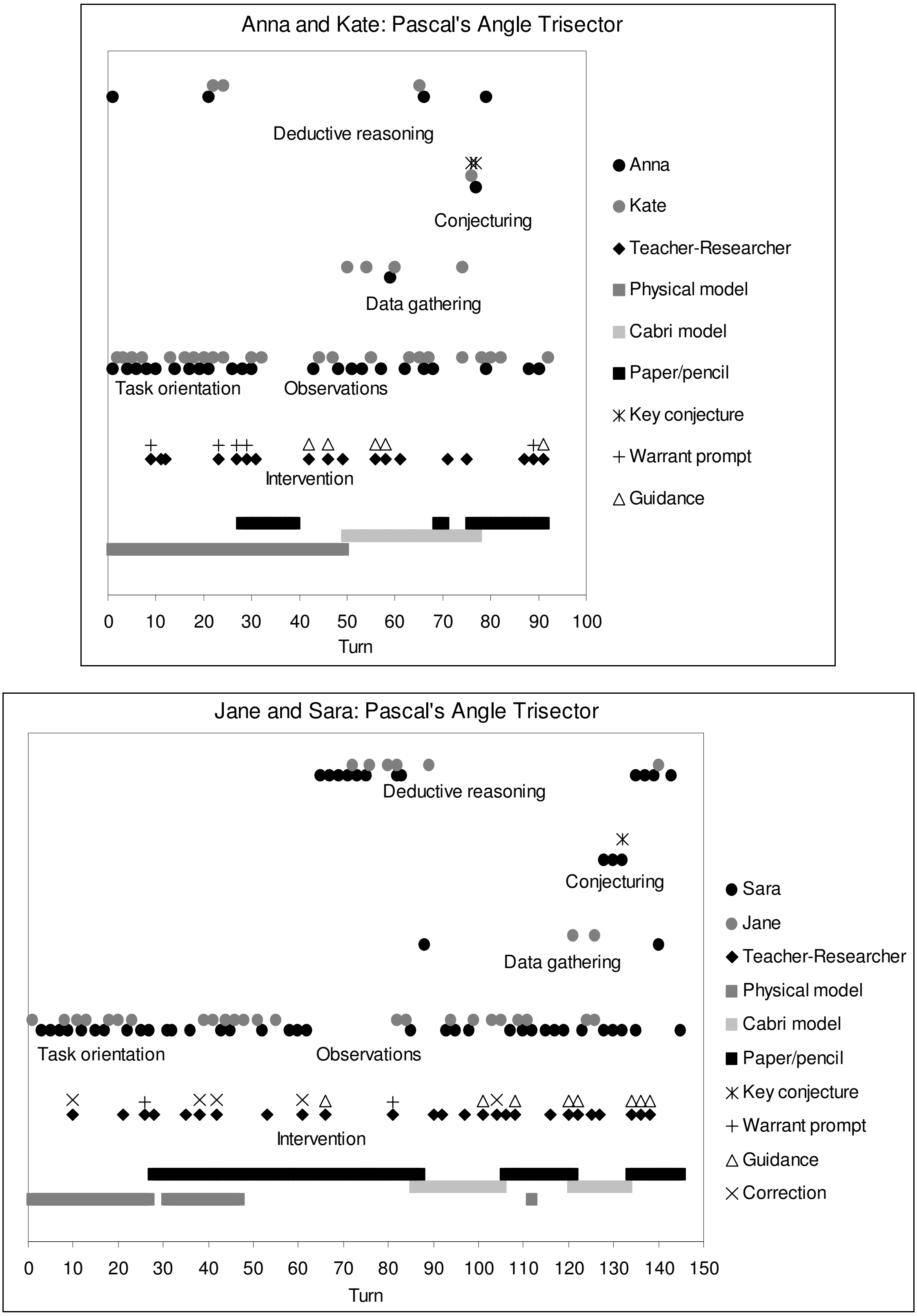


Figure 1

Figure 2 shows the argumentation profiles for the students' fourth conjecturing-proving task—a Cabri-based task in which the students investigated the joining of the midpoints of the sides of a quadrilateral.

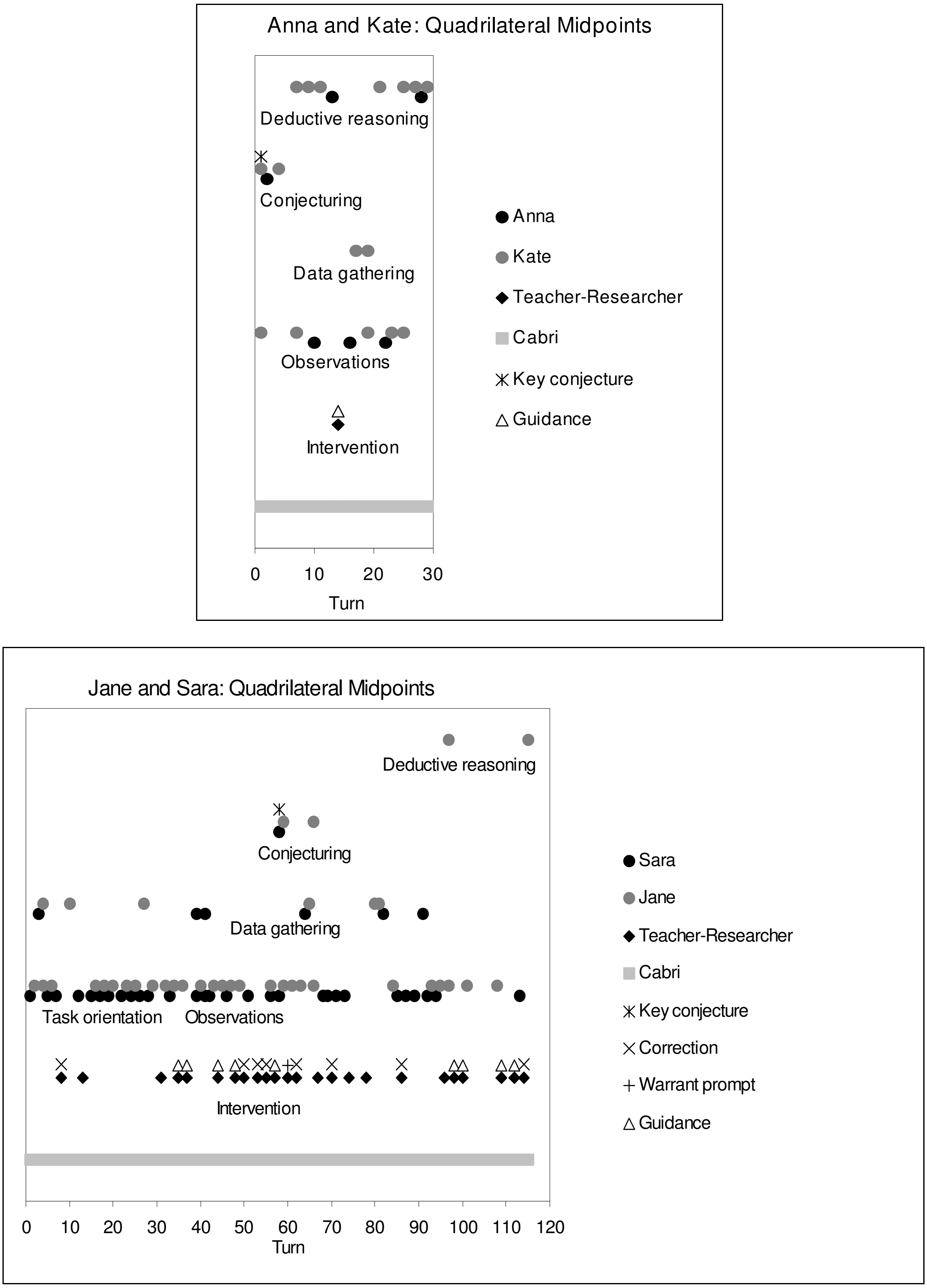


Figure 2

Jane and Sara's argumentation contrasts sharply with that of Anna and Kate, who immediately recognised the parallelogram formed when the midpoints were joined. Anna and Kate completed their proof without the need for teacher guidance, although it was Kate who dominated the deductive reasoning. Jane and Sara, however, focused on other features of the figure and failed to notice the parallelogram until their attention was drawn to it by intervention at turns 37 and 44. They were also handicapped in their conjecturing and arguing by frequent incorrect observations and their lack of confidence with quadrilateral properties and relationships:

69	Sara:	I think it's … um … because the midpoint always stays the same and if the angles of the triangle are always joined to the shape …
70	TR:	Which triangle?
71	Sara:	I mean of the square … sorry … of this … the parallelogram … this parallelogram is always … it's centred … it's in the very centre of the whole shape because of the lines … therefore it stays there.

DISCUSSION

A comparison of Anna and Kate's argumentation profile charts for their first conjecturing-proving task (Pascal's angle trisector) and for their fourth task (Quadrilateral midpoints) demonstrates the development of the deductive reasoning ability of these two students. Further evidence for this development was provided by an analysis of their argumentations and written proofs for other tasks which they completed. The ability of Anna and Kate to engage in argumentation was largely due to their facility with the language of geometry and their understanding of basic properties of triangles and quadrilaterals. It was, however, the process of argumentation which provided these two students with a sense of ownership of their proof. During the argumentation, deductive reasoning statements became ordered so that production of the written proof followed naturally, supporting the claims of cognitive unity by Boero et al. (1996).

By contrast, Jane and Sara were hindered by a poor knowledge of geometric language and properties and substantial teacher intervention was required. However, the process of argumentation did create an environment in which Jane and Sara were able to develop some understanding of the nature of deductive reasoning and to gain a sense of satisfaction from their proof construction.

CONCLUSION

Argumentation profile charts facilitate comparisons of the extent of collaboration between students during the argumentation; the efficiency of the students' data collection, conjecturing and deductive reasoning; and the level of intervention required by different pairs of students, or by the same pair of students in different tasks. By focusing on interactions and the overall structure of an argumentation, that is, on conjecturing and proving as a *process*, the argumentation profile chart can provide valuable insight into how students approach problem-solving tasks.

References

Australian Education Council (1991). *A national statement on mathematics for Australian schools*. Carlton, Australia: Curriculum Corporation.

Balacheff, N. (1999). L'argumentation est-elle un obstacle? Invitation à un débat. *International Newsletter on the Teaching and Learning of Mathematical Proof*. May/June.

http://www.lettredelapreuve.it/Newsletter/990506Theme/990506ThemeUK.html

Balacheff, N. (1991). The benefits and limits of social interaction: The case of mathematical proof. In A. Bishop, S. Mellin-Olsen, & J. van Dormolen (Eds.), *Mathematical knowledge: Its growth through teaching* (pp. 175−194). Dordrecht, The Netherlands, Kluwer Academic Publishers.

Boero, P. (1999). Argumentation and mathematical proof: A complex, productive, unavoidable relationship in mathematics and mathematics education. *International Newsletter on the Teaching and Learning of Mathematical Proof*. July/August.

http://www.lettredelapreuve.it/Newsletter/990708Theme/990708ThemeUK.html

Boero, P., Garuti, R., Lemut, E., & Mariotti, A. M. (1996). Challenging the traditional school approach to theorems: A hypothesis about the cognitive unity of theorems. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 2, pp. 113−120). Valencia, Spain: PME.

Hanna, G. (1995). Challenges to the importance of proof. *For the Learning of Mathematics*, 15, 42−49.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229−269). Hillsdale, NJ: Lawrence Erlbaum.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. NCTM. Electronic format: http://standards.nctm.org

Toulmin, S.E. (1958). *The uses of argument*. Cambridge: Cambridge University Press.

Vincent, J. (2005). Interactive geometry software and mechanical linkages: Scaffolding students' deductive reasoning. In W. Masalski (Ed.), *Technology-Supported Mathematics Learning Environments* (2005 National Council of Teachers of Mathematics Yearbook). Reston, VA: NCTM.

Vincent, J., Chick, H., & McCrae, B. (2002). Mechanical linkages as bridges to deductive reasoning: A comparison of two environments. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4 (pp. 313–320). Norwich, UK: PME.

CHARACTERIZING MIDDLE SCHOOL STUDENTS' THINKING IN ESTIMATION

Tanya N. Volkova

Illinois State University, Normal, IL, USA

The goal of the research reported was to develop a framework for describing students' thinking in situations involving computational estimation. Case-study methodology was used to investigate 8th grade students' abilities to estimate. The developed Estimation Thinking Framework is empirically based, and consists of four developmental levels: (1) predimensional, (2) unidimensional, (3) bidimensional, and (4) integrated bidimensional. These levels are hierarchical in nature and based on the data collected during the study, analysis of existing classifications of estimation strategies, as well as Case's (1996) theory of cognitive development.

Mathematics education reforms of the recent decades in the United States have recommended that school mathematics curricula include topics on estimation. The National Council of Teachers of Mathematics (NCTM), in its Principles and Standards for School Mathematics (NCTM, 2000), acknowledged the importance of developing students' ability to estimate. The attention given by NCTM to the necessity of developing estimation skills has resulted in the expansion of research on how students develop computational estimation strategies and how they reason in problem situations in which the context calls for an estimate.

Over the last three decades, a substantial volume of research has been accumulated and provides a theoretical foundation for understanding the development of students' abilities to estimate. However, further investigation of learning, understanding, and teaching estimation is necessary in order to provide educators with insights into practical applications of this research knowledge. The goal of the current study was to develop a framework for characterizing levels of students' thinking based on their choice of computational estimation strategies. The research, stemming from the analysis of existing classifications of estimation strategies (e.g., Reys, Rybolt, Bestgen, & Wyatt, 1982; Rubenstein, 1985; Levine, 1982) and Case's (1996) theory of cognitive development, addressed the construction of an empirically based framework which can be used to identify a level of middle school students' thinking with regard to estimation. Thus, the findings of the study shed light on why a student chooses a particular estimation strategy.

REVIEW OF RELATED LITERATURE

There seems to be a consensus among mathematics educators and researchers that instructional decisions should be grounded in research-based knowledge of student thinking (e.g., Carpenter & Fennema, 1989, 1993; Mack, 1995; Lamon, 1993). Carpenter et al. (1989) and Fennema et al. (1993) posited the need for cognitive frameworks that would provide researchers and educators with detailed knowledge about children's thinking in each knowledge domain. This approach has been adopted by several studies targeting various mathematical knowledge domains. Carpenter et al. (1989) used this approach in a study focused on whole number arithmetic; Mack (1995) applied the same approach in the domain of fractions; Lamon (1993) utilized this principle in the domain of proportional reasoning; and Fuys, Geddes, and Tischler (1988) have proven this approach to be effective in the geometry domain. While the body of knowledge on development of children's ability to estimate is growing, there is still a need for a framework that will allow characterizing levels of students' thinking with regard to estimation.

Several researchers devoted their studies to identification and classification of estimation strategies. Rubenstein (1985) examined the computational estimation abilities of eighth-graders, and developed an estimation test to measure four types of computational estimation (open-ended, reasonable versus unreasonable, reference number, and order of magnitude). Whereas Rubenstein (1985) classified estimation strategies based on the type of problems the students were given, Levine (1982) suggested a model for classification of estimation strategies based on the type of estimation technique. The model consisted of 8 categories: (1) using fractions; (2) using exponents; (3) rounding both numbers; (4) rounding one number; (5) using powers of 10; (6) using "known"/ "nicer" numbers; (7) incomplete partial products; and (8) proceeding algorithmically.

Levine's (1982) model is limited in its application: it can be used for strategy identification purposes only, and provides no way to account for all the possible estimation strategies. However, the nature of computational estimation is such that the number of estimation strategies is limited only by one's level of cognitive development. Dowker (1992) conducted a study on estimation strategies used by professional mathematicians. The results showed that mathematicians used a great variety of strategies, as many as 23 for a single problem. This presented a challenge for the researcher, who tried to encompass all the possible estimation strategies in one comprehensive classification. In the attempt to overcome such shortcomings, Dowker (1992) modified Levine's (1982) model by changing several categories and including the catch-all category named "other." This category included those strategies that were used only for one or two problems.

Reys et al. (1982) focused on "good estimators" in order to identify cognitive processes they use to solve problems that require computational estimation. The researchers administered a 55-item computational estimation test to over 1,200 students in grades 7–12 and to selected adults in order to identify a group of 59 "good" estimators. The people from the selected group were interviewed to determine strategies and processes they use to solve estimation problems. The three key cognitive processes people use in computational estimation were identified as: reformulation, translation, and compensation.

It is noteworthy that these classifications and models listed above, while providing some insight into the variety of estimation strategies and cognitive processes that students use, do not focus on the students' progression through the levels of thinking in computational estimation. The current study developed a framework that can be used to characterize students' thinking with regard to computational estimation. The framework provides mathematics educators and teachers with descriptions of the levels of students' thinking with regard to computational estimation.

In 1996, Case put forth a theory of cognitive development with regard to quantitative thought. According to this theory, children's cognitive growth proceeds through two stages during the school years: the dimensional stage (approximate ages 5-10) and the vectorial stage (approximate ages 11-18). More importantly, Case (1996) identified the four sub-stages of each stage of the cognitive growth: (1) predimensional, (2) unidimensional, (3) bidimensional, and (4) integrated bidimensional. Based on the Case (1996) model of cognitive development the Estimation Thinking Framework was developed for describing and characterizing the development of student's thinking in computational estimation.

METHODOLOGY

Participants

For the current research, a case-study methodology was selected to construct descriptors of the developmental levels of the framework. Eight students in grade eight at a public school in Normal, Illinois, formed the population for the study. All of the students were chosen from the top level of their mathematics classes, on the assumption that the high-level achievement students could be expected to exhibit a greater variety of strategies and techniques (Reys et al., 1982; Dowker, 1992) in solving problems involving computational estimation.

Data Collection

This study utilized an original interview protocol, comprised of 19 tasks designed to assess students' ability to estimate across the four constructs: whole number, fractions, percent, and decimal fractions. The tasks allowed children to respond orally; students were not allowed to use paper and pencil to solve problems. The Interview Protocol was administered individually to each student in one 40-50 minute session; interview sessions were audio taped and transcribed. Students' responses to the Protocol tasks, along with the researcher's field notes, provided a basis for evaluating their levels of thinking in computational estimation.

Data Analysis

Transcripts of audiotapes and researcher's field notes on each student's thinking comprised the data for this study. Data were entered into a meta-matrix containing all students' responses to each question. A double coding procedure (Miles & Huberman, 1994) was used to code students' responses. First, the data were labeled, partitioned and clustered into the four categories (see Figure 1) according to the level of complexity of their estimation strategies: Level 1 (Predimensional), Level 2 (Unidimensional), Level 3 (Bidimensional), or Level 4 (Integrated Bidimensional).

Level	Descriptors of levels of cognitive development
Level 1 Predimensional	Students who operate at this level: • have a propensity to compute an exact answer, applying written or mental computation procedures and algorithms. • demonstrate inability to compare numbers using "benchmarks" (to identify which of two numbers is closer to a third number), to order numbers, to find or identify numbers between two given numbers • possess a limited understanding of the relations between numbers (e.g., multipliers vs. product, addends vs. sum, etc.) in arithmetic sentences.
Level 2 Unidimensional	Students who operate at this level: • use in addition to standard algorithmic procedures (mental or paper-and-pencil) other estimation strategies. For instance, the students might adapt and use rounding or front-end techniques with whole numbers. In problems involved fractions they might accommodate techniques that require converting fraction to decimal fractions. • tend to treat decimal fractions as whole numbers, employing rounding and truncating techniques. • might see the relationship between percents and decimals; however, tend to fall short of seeing connections between percents and fractions. • are limited in their use of the strategies, applying the same successful strategy over and over again for different types of problems.
Level 3 Bidimensional	In contrast with students who function at Level 2 (unidimensional), students who operate at Level 3 (bidimensional): • tend to use a greater variety of estimation strategies. • choose computational strategies in accordance with the situation described in the problem. • do not rely solely on a successful strategy that worked for other problems; become more flexible in their choice of strategies. • However, in contrast with students who function at Level 4 (integrated bidimensional), students who operate at Level 3 (bidimensional) • tend to use the estimation strategies that are available for them in isolation for each particular situation. • cannot easily switch from one strategy to another, which is illustrated by their inability to come up with different strategies for a single problem. • do not look for different possible techniques or strategies to verify their estimate
Level 4 Integrated bidimensional	Students who operate at the integrated bidimensional level • are able to coordinate two complex and multi-dimensional components (mental computation and number nearness task), • show their ability to switch between strategies easily. • are no longer applying the same strategy to "check" their estimates; they use different strategies to confirm the results. Moreover, once they find an estimate, they tend to try another strategy to yield a closer estimate.

Figure 1. Estimation Thinking Framework.

Next, students' responses to items within each category were described to yield criteria for each level of the initial framework. To assure the quality of the study, data were collected from two sources: (1) students' responses to the interview protocol tasks; and (2) researcher's field notes on each student's thinking strategies; the data from both sources were later tested for consistency.

RESULTS

The following subsections provide a parallel comparison of the descriptors of the Estimation Thinking Framework based on the data collected in the current study to the descriptors of sub-stages of the Case (1996) theory of cognitive development with regard to quantitative thought.

Level 1 – Predimensional

With regard to quantitative thought, Case (1996) found that the students who operated at the predimensional level were able to generate number tags (e.g., 2, 3, etc.) and to make qualitative judgments about quantities (e.g. more or less). However, they fell short of integrating these two aspects of knowledge into a meaningful structure, and therefore tended to respond at chance level when asked to decide whether, say, 4 was larger or smaller than 5. Results of the current study showed that students who operate at this level tend to use estimation strategies that are limited to standard algorithmic procedures. They have a propensity to compute an exact answer, applying written or mental computation procedures. They also show inability to compare numbers using "benchmarks" (to identify which of two numbers is closer to a third number), to order numbers, to find or identify numbers between two given numbers; as well as limited understanding of the relations between numbers (e.g., multipliers vs. product, addends vs. sum, etc.) in arithmetic sentences. For example, they are unable to predict what happens to the product if a multiplier is less than one, or if the multiplier is greater than one. Furthermore, when asked to estimate, students tend to respond at chance level by guessing the estimate.

The following is an excerpt from the interview with Masha (pseudonym), whose responses exemplify Level 1, "predimensional," of computational estimation.

Interviewer: Estimate the answer to "3 1/8 +2 4/5.

Masha: I'll start by converting this to improper fractions, so that'll be 25/8 plus…

Interviewer: Do you think it's easier to work with improper fractions?

Masha: Yeah.

Interviewer: Can you just add these numbers, 3 and 2?..

Masha: Yeah. So, that would be 5 and then find the common denominator… I think it would be 40.

Interviewer: You don't need to find a common denominator, but think about fractions… 4/5 and 1/8. If you add these two fractions, is the sum greater or less than 1?

Masha: Greater…

It is clear from the above responses that Masha cannot think of any estimation strategies besides standard algorithmic procedure. Moreover, her predictions can be described as a "lucky guess" strategy, since her answer is incorrect and she could not provide an explanation of her reasoning.

Level 2 – Unidimensional

According to Case (1996), students who exemplified thinking at unidimensional level were able to use a mental counting line to count backwards and forwards. However, they were unable to use one number line to locate two numbers and then compute differences simultaneously using the second number line. With regard to estimation, the results of the current study showed that, in addition to standard algorithmic procedures (mental or paper-and-pencil), students are able to use other estimation strategies. For instance, the students might adapt and use rounding or front-end techniques with whole numbers. In problems that involve fractions they might accommodate techniques that require converting fractions to decimal fractions. These students usually treat decimal fractions as whole numbers, employing rounding and truncating techniques. Even though the students operating at this level might see the relationship between percents and decimals, they fall short of seeing connections between percents and fractions. Despite the growing number of strategies available to students at the bidimensional level, they are still limited in their use of the strategies, applying the same successful strategy over and over again to different types of problems.

Level 3 – Bidimensional

Students who operated at Case's bidimensional level showed proficiency at performing more than one mental operation, which Case calls vectors. Results of the current study showed that, in contrast with students who function at Level 2 (unidimensional), students who operate at Level 3 (bidimensional) tend to use a greater variety of estimation strategies. They choose computational strategies in accordance with the situation described in the problem. In other words, they do not rely solely on a successful strategy that worked for other problems; they become more flexible in their choice of strategies. However, the strategies are not connected and are used in isolation for each particular situation. The students cannot easily switch from one strategy to another, which is illustrated by their inability to come up with different strategies for a single problem. When they get an estimate, they stop looking for other possible techniques or strategies to verify their result.

Level 4 – Integrated Bidimensional

With regard to quantitative thought, Case (1996) found that the students who operated at the integrated bidimensional level were able to use multiple mental counting lines to do whole number arithmetic. With regard to computational estimation, the results of the study suggest that students at the integrated bidimensional level are not only able to coordinate two complex and multi-dimensional components (mental computation and number nearness task), but to do this they apply more than one strategy for a single problem. Thus, they show their ability to switch between strategies easily. At this level they are no longer applying the same strategy to "check" their estimates; they use different strategies to confirm the results. Moreover, once they find an estimate, they tend to try another strategy to yield a closer estimate.

The following is an excerpt from the interview with Victor, who demonstrated his ability to apply two different strategies to find an estimate for the problem.

Interviewer:	Estimate the answer to "3 1/8 +2 4/5.
Victor:	I think five and something… a little less than six.
Interviewer:	Why is that?
Victor:	Because, uhm… these two [1/8 and 4/5] don't look like they are enough to create one, so it would be more than five, because three and two make five…
Interviewer:	But how do you know that 1/8 and 4/5 will not be enough to make a whole?
Victor:	Just because one-fifths is greater than one-eighths.
Interviewer:	Can you think of another way to justify that?
Victor:	Uhm, because, 4/5 is, like, 80% of 1.
Interviewer:	Yes…
Victor:	and then 1/8 is, like, uhm… about 10% of 1.
Interviewer:	Uh-huh…
Victor:	So these two together are about 90%…So the answer will be close to 6…

It is clear from Victor's responses that, first, he applied a "benchmark" strategy – he estimated how far the two numbers are from the number 1, and how much it takes to complete another number to get the whole unit. Then Victor estimated what percent of the whole unit the fractions represent, and by adding two percents found that the fractions are not enough to make a whole. Thus, Victor's responses exemplify Level 4 thinking in computational estimation.

DISCUSSION

Data collected for the current research showed that children of the same age (in this case – middle school children) can be on different levels of thinking with regard to computational estimation. This finding is more in line with the van Hiele (1959/1984) model than with the Case (1996) model. While Case focused on students' age or maturation, the van Hiele model – although it does not specifically deal with the domain of computational estimation – places the emphasis on students' progression through the levels of understanding due to the instructional intervention. The levels of estimation thinking framework are not age-specific, and progress from level to level will depend more on the content and methods of instruction received by the student, than on their age. The presented framework was developed to characterize middle school students' thinking in estimation; however, further research is necessary to explore whether the framework is applicable to other age groups.

REFERENCES

Carpenter, T.P., & Fennema, E. (1989). Research and cognitively guided instruction. In E. Fennema, & T.P. Carpenter (Eds.), *Integrating Research on Teaching and Learning Mathematics* (pp.2-17). Wisconsin: Wisconsin Center for Education Research.

Case, R. (1996). Reconceptualizing the nature of children's conceptual structures and their development in middle childhood. *Monographs of the Society for Research in Child Development, 61, (1-2, Serial No. 246),* 1-26.

Case, R., Sowder, J. T. (1990). The development of computational estimation: A neo-Piagetian analysis. *Cognition and Instruction, 7,* 79-104.

Crowley, M. L. (1987). The van Hiele model of the development of reflective thought. In M. M. Lindquist (Ed.), Learning and Teaching Geometry, K-12, (pp. 1-16). Reston, VA: National Council of Teachers of Mathematics.

Dowker, A. (1992). Computational estimation strategies of professional mathematicians. *Journal for Research in Mathematics Education, 23 (1),* 45-55.

Fuys, D., Geddes, D., & Tischler, R. (1988). The van Hiele model of thinking in geometry among adolescents. *Journal for Research In* Mathematics Education, Monograph Series, Number 3. Reston, VA: National Council of Teachers of Mathematics.

Lamon, S. J. (1993). Ratio and proportion: Connecting content and children's thinking. *Journal for Research in Mathematics Education, 24,* 41-61.

Levine, D. R. (1982). Strategy use and estimation ability of college students. *Journal for Research in Mathematics Education, 13,* 350-359.

Mack, N. K. (1995). Confounding whole number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education, 24,* 41-61.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook (3rd ed.).* Newbury Park, CA: Sage Publications.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics.* Reston, VA: Author.

Reys, R. E., Rybolt, J. F., Bestgen, B.J., & Wyatt, J.W. (1982). Processes used by good computational estimators. *Journal for Research in Mathematics Education, 13,* 183-201.

Rubenstein, R. N. (1985). Computational estimation and related mathematical skills. *Journal for Research in Mathematics Education, 16,* 106-119.

van Hiele, P. M. (1984). A child's thought and geometry. In D. Geddes, D. Fuys, & R. Tischler (Eds.), *English translation of selected writings of Dina van Hiele-Geldof and Pierre M. van Hiele*. Washington D.C.: NSF.

REVIEWING AND THINKING THE AFFECT/COGNITION RELATION

Margaret Walshaw — Massey University, NZ

Tânia Cabral — UERGS, Guaĭba, RS, Brazil

This paper is a theoretical discussion about learning. In it the core question of learning is first reviewed, assessed, then reworked to offer a new sensibility about what it is that prompts us to learn. Central to the reframing is the affective domain and the role that affect plays in learner outcomes. Our intent is to develop a theory of learning that foregrounds the non-rational and often unexplained aspects of learning. The general strategy taken draws upon the work of Lacan and uses that framework and language for developing a coherent explanation of some affective aspects of learning that are ordinarily overlooked in mathematics education.

INTRODUCTION

This paper is a theoretical discussion about learning. In it the core question of learning is first reviewed, then rethought and reframed to offer a new sensibility about what it is that prompts us to learn. Theoretical insights about learning are not new in mathematics education and an enduring history has mapped out robust explanations about what it is that prompts us to take up new ideas. The approach taken in this paper takes as its central plank the affective domain and the role that affect plays in learner outcomes. Going against the grain of much contemporary scholarly work on affect (see Hannula, Evans, Philippou, & Zan, 2004), we look at one way in which the affect/cognition is currently being worked through within social science. In arguing for the usefulness of our approach for learning theory, we contend that it offers a fresh and helpful way to explain the relationship between the individual and the social. Arguably the approach presents a challenge to classic ideas about learning, yet the potential of such work to move forward current understandings of learning is not to be underestimated.

Research interest in the affective domain has proceeded through quite different theoretical viewpoints. Characterisations of affect are inclusive of “a wide range of beliefs, feelings and moods that are generally regarded as going beyond the domain of cognition” (McLeod, 1992, p. 576). Those characterisations go by the name of: anguish, anxiety, attitudes, autonomy, beliefs, confidence, curiosity, disaffection, dislike, emotions, enthusiasm, fear, feelings, frustration, hostility, interest, intuition, moods, panic, perseverance, sadness, satisfaction, self-concept, self-efficacy, suffering, tension, viewpoint and worry. All these categories have come under scrutiny (e.g., Goldin, 2000; Hannula, 2002; Ma, 1999; Martinez & Martinez, 2003;

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 297-303. Melbourne: PME.

McLeod, 1992) and from those investigations important conclusions have been drawn with respect to the affect/cognition relation.

Contrary to McLeod's (1992) contention that research on affect lacks a strong theoretical basis we believe that what we are witnessing now is a plethora of groundings, drawn from theories of discursive practice (e.g., Evans, 2000; Walshaw, 2004a), of embodiment (Drodge & Reid, 2000), of somatic markers (e.g., Brown & Reid, 2004), of neuroscience (Schlőglmann, 2002), of representation (e.g., Goldin, 2000), and of situated practice (e.g., Lave, 1988). We develop our own theory of affect using Lacanian ideas, in the hope that it might contribute towards the centering, rather than the marginalisation, of research on affect within the field. A Lacanian treatment like ours is not entirely foreign in mathematics education (see Breen, 2000; Brown, Hardy & Wilson, 1993; Cabral, 2004; Evans, 2000; Walshaw, 2004b) and our work builds on that recent tradition.

IDEAS ABOUT LEARNING

Learning in mathematics education is by no means a unified theory. In attempting to produce a rigorous method and a satisfactory explanation of learning, theorists have proceeded with different emphases from alternative starting points and have often been in contest with one another. Since Gagné's (1965) classic interpretation of learning as behavioral change, new paradigms, influenced by cross-disciplinary practices, have tended to problematise conditions of learning as 'holding good' for learners, irrespective of the learner's history, interests and circumstances. Yet for all the inclusiveness in this exacting scholarship, the new paradigms tend to overlook affective aspects that we consider fundamental to the pedagogical encounter.

In the constructivist approaches influenced by Piaget and the post-Piagetian work of von Glasersfeld, it is the autonomous individual, and more specifically, the individual's developing internal representation within the mind (Goldin & Shteingold, 2001) that becomes the central unit of analysis. Drawing on humanist sensibilities about the individual, constructivists' accounts of learning necessarily rely on the autonomous learner, understood as the stable, core, knowing agent. In opposition to the constructivists' privileging of interior mental processes, sociocultural perspectives, mark up social contexts and experiences. They give priority to shared consciousness, or intersubjectivity, arguing that conceptual ideas proceed from the intersubjective to the intrasubjective. Semiotic mediation theory is proposed to account for intersubjective arrangements and the part those arrangements play in the development of internal controls in the learning process. Emotive and unconscious aspects are ignored by that learning mechanism.

In claiming that learning comes about from ongoing participation within a community, situated theorists offer ideas about learning that are relational and connectivist (Greeno, 2003). From a stress on the mutually relational effects of the social and individual, the idea is developed that learning is constituted socially. Lave's social practice theory, in particular, offers an insightful critique of the central processor model of the mind. She foregoes description of a learning mechanism to explain learning as participation in social practices. Similarly, in embodied mathematical learning theory, learning is generated mutually and relationally from active and ongoing engagement within a community. Mathematical ideas are "not held by institutions or individuals but are embodied by human beings with normal human cognitive capacities living in a culture" (Lakoff & Núñez, 2000, p. 359).

Those evolving practices, and the adaptations people make to maintain coherence within complex, dynamic systems, are brought to the fore in enactivist theory. "Learning is understood in terms of ongoing, recursively elaborative adaptations through which systems maintain their coherences within their dynamic circumstances" (Davis & Simmt, 2003, p. 138). In these formulations of learning, it is not the autonomous individual that is the principal unit of analysis; nor is a collective understanding the focus. Rather, what are at stake are the evolving relationships between people and the settings made through the "nested learning systems" (*ibid*, p. 142) within which both the individual and collective are mutually constituted. In the next section we offer a development of a mechanism that is able to explain how learning emerges between people and settings and how it evolves within the dynamics of the spaces people share and within which they participate.

TAKING AFFECT INTO ACCOUNT

Each of the learning theories discussed above offers important insights (as well as important criticisms of others) about how it is that we come to learn. However, in valorising, in turn, the rational aspects of learning and promoting shared consciousness and the realization that experience is always conscious, all these viewpoints have a tendency to sidestep important affective aspects that we believe are integral to learning. As has been argued (e.g., Britzman, 1998; Ellsworth, 1997; Jagodzinski, 2002), when experience is synonymous with rational consciousness, the complex affective situations and conditions in which learning takes place inevitably are glossed over. A different perspective would foreground the importance of non-rational and unexplained aspects of learning and for us, Lacan provides a suitable theoretical framework and a language for doing that. In this section we elaborate some aspects of his critical work on psychoanalysis, and draw upon them to suggest theoretical and empirical directions for an analysis of how we learn.

Psychoanalytic theories presents complex and well-developed ideas about subjectivity (Grosz, 1995) and offer instructive lessons about knowledge that have the potential to inform understandings about learning (Britzman; Jagodzinski). In Lacanian thinking, unconscious levels of awareness, as well as conscious ones, are central to the human psyche. This understanding points to a different set of presuppositions from those upon which the disciplinary theories of learning discussed above are built. In those theories, cognitive know-how rests upon the modernist conception of the conscious and rational knower. Subjectivity, for Lacan, on the other hand, is not constituted by consciousness alone; unconscious processes will always interfere with conscious intentionality and experience (Britzman). In Lacanian thinking, the subject is always 'already rhetorically marked.'

Lacan maintains that the subject's very existence consists of desire. However, rather than conflating desire with conquest and attainment, desire in the Lacanian formulation revolves around the quest for a secure identity. The learner in the classroom could not be that person without relationships, location, networks and history that allow her to fabricate a presence of self-coherence and effectivity. The desire for self-presence, however, will always be subject to the constant deferral of satisfaction. Marked by both conscious and unconscious intentionality that actualise the learner's talk and actions, desire takes shape in the margins (Lacan, 1977). As the "*reality* of the unconscious" (Grosz, 1995, p. 67), language plays a key role in its dynamics.

It is in Lacan's three psychic registers of subjectivity—the Symbolic, the Imaginary, and the Real—that we see potential for understanding what it is that prompts learning to take place. In the classroom setting the psychic registers work together to inform the learner's experience and sense of perception. It is the responsibility of the learner to negotiate through any conflict that might arise from the forms of recognition that each offers. In particular, the symbolic for Lacan is the domain of laws, words, letters, and numbers that structure our institutions and cultures—the 'Law of the Father' and the 'Big Other.' For example, in the school, the Big Other might include the mathematics curriculum, the rules and procedures of the school community, and the norms of the classroom as well as the sociomathematical norms established by the classroom learning community. Students desire recognition from each other and from their teacher, as they work at embodying those signifiers. When they succeed, the recognition becomes a motivator and learning is made possible.

Lacan's Imaginary register is the realm of visual-spatial images and illusions of self and world. Lying at the limits of perception, the Imaginary register works to undermine the individual learner's sense of self. In the pedagogical relation the teacher and the students look for an image with which they choose to identify themselves—an image with which they feel comfortable and hope to be liked by others. For example, many students work hard to construct a sense of self and bodily appearance. That sense of self may or may not be in opposition to the contents of the Symbolic register and it is the successful learner who is able to resolve any conflict between the 'data' from the Imaginary and the Symbolic registers.

Lacan's Real Register is an indicator of our socio-psychical growth; in our understanding, it can also be a measure of a productive pedagogical encounter. Desire for recognition in the Real register is concerned with the mirroring of affect and emotion. A learner may want to mirror the teacher's desire on the basis of a range of impressions and feelings that pass through memories and unconscious desires. Those memories can be triggered by, among other things, a gesture, or the tone, pitch, or resonance of the teacher's voice (Britzman). Lacan (1973) claims that *language* *constitutes the subject of desire*, and in this he is saying that when the subject─either teacher or student─speaks he or she is trying to be recognised and liked.

In the classroom student's desire for recognition from the teacher plays a crucial part in the learning process. It is our contention that desire for the teacher's desire is what attaches the psychical to classroom practice, and classroom practice to the psychical. Role modelling is not at stake in the teacher/learner relation, precisely because the learner's talk and actions go beyond the proposals of role model pedagogies. What we want to stress is that when the learner secures the emotional resonance she desires it is precisely that time when a mathematical idea is able to attach itself and enable the student to learn productively in the mathematics classroom. It is through investigating repeated performances of the learner's strategies of self construction, in connection with others (Britzman), and explaining where the learner locates spaces of personal advantage, that the process of learning can be laid bare.

The Lacanian idea, then, that the subject's very existence consists of desire for a secure identity might be observed as those strategic projects by which, through resolving conflict between psychical registers, the learner personalises rules of conduct in order to optimize existence in the classroom. When there is no struggle over meanings between the learner and the teacher about what it means to be a learner in this classroom, the classroom becomes a safe place in which to speak and act. Inevitably that secure identification will produce new knowledge for the leaner. Self-construction is, of necessity, part of a dynamic and complex interchange with knowledge. It is fundamental to learning.

CONCLUSION

This paper has explored ideas about learning. It first mapped out conventional and current ideas about learning as proffered within the discipline. It traced an engagement with questions of how learning takes place in constructivist, sociocultural, situated, embodied, and enactivist formulations of learning and proceeded to assess those viewpoints in relation to work being undertaken within social science. Noting how all these theories offer important insights (as well as important criticisms of other ideas) about how it is that we come to learn, the assumptions propping up the respective theories were unpacked. A reliance on, in turn, the rational autonomous learner, a conflation of experience with consciousness, a unequivocal acceptance of shared consciousness, and a lack of a learning mechanism were all noted as critical shortcomings to a productive understanding of the affect/cognition relation.

We have outlined some fundamental concepts from Lacanian theory and have drawn on these concepts to consider the affect/cognition relation. Although these concepts challenge central assumptions within mathematics education, the choice of psychoanalytic concepts has been deliberate to fill in the gaps and the inconsistencies in current formulations and to account for previously unexplained aspects of learning. In offering sights about how the unconscious is structured, we suggest that Lacan offers a useful way of considering how knowledge is constituted. Drawing on his ideas about unconscious desire we suggest theoretical directions for thinking about learning as a psychic event and hint at the implications of those ideas for classroom research.

Learning in this perspective becomes a question, not about conscious experience with self and others, but rather to do with the way in which unconscious processes, working at different levels and with different kinds of information, undermine experiential knowing. The place of the unconscious, and hence the non-rational learner, then become crucial to the learning process. Arguably the approach presents a challenge to classic ideas about learning, yet the potential of such work to move forward current understandings of learning is not to be underestimated. It is our belief that it offers a fresh and helpful way to explain the relationship between the individual and the social.

References

Breen, C. (2000). Becoming more aware: Psychoanalytic insights concerning fear and relationship in the mathematics classroom. In T. Nakamara & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for Psychology of Mathematics Education* (Vol. 2, pp. 105-112). Hiroshima: Hiroshima University.

Britzman, D. (1998). *Lost subjects, contested objects: Toward a psychoanalytic inquiry of learning.* New York: State University of New York Press.

Brown, T., Hardy, T., & Wilson, D. (1993). Mathematics on Lacan's couch. *For the Learning of Mathematics, 13*(1), 11-14.

Brown, L., & Reid, D. (2004). Emotional orientations and somatic markers: Implications for mathematics education. In M.J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol 1, pp 123-126). Bergen: PME.

Cabral, T. (2004). Affect and cognition in pedagogical transference: A Lacanian perspective. In M. Walshaw (Ed.), *Mathematics education within the postmodern* (pp. 141-158). Greenwich, CT: Information Age.

Davis, B. & Simmt, E. (2003) 'Understanding learning systems: Mathematics education and complexity science', *Journal for Research in Mathematics Education 34*(2), 137-167.

Drodge, E.N., & Reid, D.A. (2000). Embodied cognition and the mathematical emotional orientation. *Mathematical Thinking and Learning, 2*, 249-267.

Ellsworth, E. (1997). *Teaching positions.* New York: Teachers College Press

Evans, J. (2000). *Adults' mathematical thinking and emotions: A study of numerate practices.* London: Routledge Falmer.

Gagné, R. (1965). *The conditions of learning.* New York: Holt, Rinehart and Winston.

Goldin, G.A. (2000). Affective pathways and representations in mathematical problem solving. *Mathematical Thinking and Learning, 17*, 209-219.

Goldin, G., & Shteingold, N. (2001). Systems of representations and the development of mathematical concepts. In A.A. Cuoco & F.R. Curcio (Eds.), *The roles of representation in school mathematics: 2001 Yearbook* (pp. 1-23). Reston, VA: NCTM.

Greeno, J.G. (2003). Situative research relevant to standards for school mathematics. In J. Kilpatrick, W.G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 304-332). Reston, VA: NCTM

Grosz, E. (1995). *Jacques Lacan: A feminist introduction.* London: Routledge.

Hannula, M. (2002). Attitude towards mathematics: Emotions, expectations and values. *Educational Studies in Mathematics, 49*(1), 25-46.

Hannula, M., Evans, J., Philippou,G., & Zan, R. (2004). Affect in mathematics education: Exploring theoretical frameworks. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol 1, pp 107-136). Bergen: PME.

Jagodzinski, J. (Ed.) (2002). *Introduction. Pedagogical desire: Authority, seduction, transference, and the question of ethics.* Westport, CT: Bergin & Garvey.

Lacan, J. (1973). *Le Séminaire de Jacques Lacan. Livre XI. Les quatre concepts fondamenteaux de la psychanalyse.* 1964. Paris: Editions du Seuil.

Lacan, J. (1977). *Ecrits. A selection.* London: Tavistock.

Lakoff, G. & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life*. Cambridge: Cambridge University Press.

Ma, X. (1999). A meta-analysis of the relationship between anxiety toward mathematics and achievement on mathematics. *Journal for Research in Mathematics Education, 30*(5), 520-540.

Martinez, J.G. R., & Martinez, N.C. (2003). Raising middle school math standards without raising anxiety. *Middle School Journal, 34*(4), 27.

McLeod, D.B. (1992). Research on affect in mathematics education: A reconceptualization. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 575-596). New York: Macmillan.

Schlőglmann, W. (2002). Affect and mathematics learning. In A.D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th conference of the international group for the Psychology of Mathematics Education* (Vol. 4, pp. 185-192). Norwich, UK: University of East Anglia.

Walshaw, M. (2004a). A powerful theory of active engagement. *For the Learning of Mathematics, 24*(3), 4-11.

Walshaw, M. (2004b). The pedagogical relation in postmodern times: Learning with Lacan. In M. Walshaw (Ed.), *Mathematics education within the postmodern* (pp. 121-140). Greenwich, CT: Information Age.

YOUNG CHILDREN'S ABILITY TO GENERALISE THE PATTERN RULE FOR GROWING PATTERNS

Elizabeth Warren

Australian Catholic University

A common approach used for introducing algebra to young adolescents is an exploration of visual patterns and expressing these patterns as functions and algebraic expressions. Past research has indicated that many adolescents experience difficulties with this approach. This paper explores teaching actions and thinking that begins to bridge many of these difficulties at an early age. A teaching experiment was conducted with two classes of students with an average age of nine years and six months. From the results it appears that young children are capable of not only thinking about the relationship between two data sets, but also of expressing this relationship in a very abstract form.

INTRODUCTION

Mathematics activity is seen as the domain of reasoning about objects and their relations, and involves examining and investigating the truth of claims about those objects and relations (Carpenter, Franke & Levi, 2003). The power of mathematics lies in relations and transformations which give rise to patterns and generalisations. Abstracting patterns is the basis of structural knowledge, the goal of mathematics learning in the research literature (Jonassen, Beissner & Yacci, 1993; Sfard, 1991). Thus mathematics teaching should focus on fostering fundamental skills in generalising, and expressing and systematically justifying generalisations (Kaput & Blanton, 2001). Such experiences give rise to understandings that are independent of the numbers or objects being operated on (e.g., $a+b = b+a$ regardless of whether a and b are whole numbers, decimals, or variables). Ohlsson (1993) names such understanding *abstract schema* and argues they are more likely to promote transfer to other mathematical notions than a schema based on particular numbers or content.

Traditionally, elementary schools give little emphasis to relations and transformations as objects of study. It appears that, as Malara and Navarra (2003) argued, classroom activities in the early years focus on mathematical products rather than on mathematical processes. Strings of numbers and operations in arithmetic are not considered as mathematical objects but as procedures for arriving at answers (Kieran, 1990). Fundamental to relations and transformations is the concept of the function, a schema about how the value of certain quantities relate to the value of other quantities (Chazan, 1996) or how values are changed or mapped to other quantities, referred to in the literature as co-variational thinking.

A common activity that occurs in many early years' classrooms in the Australian context is the exploration of simple repeating and growing patterns using shapes, colours, movement, feel and sound. Typically young children are asked to copy and continue these patterns, identify the repeating or growing part, and find missing

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 305-312. Melbourne: PME.

elements; a focus on single variational thinking where the variation occurs within the pattern itself. Approaches for introducing algebra to young adolescents (12-13 years) build on early explorations of visual patterns, using the patterns to generate algebraic expressions (Bennett, 1988). Such patterns are predominantly growing patterns. Students are asked to form the functional relationship between growing patterns and their position, and use this generalisation to generate other visual patterns for other positions, that is, they are asked to reconsider growing patterns as functions (i.e., as a relationship between the pattern and its position) rather than as a variation of one data set (i.e., as relationship between successive terms within the pattern itself). This often involves generating the visual representation, recording data in a table (the position and number of elements at that position), and from the table identifying the relationship between the two data sets. Past research has indicated that many young adolescents experience difficulties with the transition to patterns as functions (Redden, 1996; Stacey & MacGregor, 1995; Warren, 1996, 2000). These difficulties include the lack of appropriate language needed to describe this relationship, the propensity to use an additive strategy for describing generalisations (i.e., a focus on a single data set), and an inability to visualise spatially or complete patterns (Warren, 2000). However young children are believed to be capable of thinking functionally at an early age (Blanton & Kaput, 2004). This research investigates teacher actions that begin to assist young children to view and describe growing patterns in terms of their positional relationships, that is, to begin to bridge the gap between single variational thinking and functional thinking before they commence formal algebraic thinking. The specific aims of this research were to: (i) investigate models and instruction that help young students to create unknown steps/positions in growing patterns, and (ii) articulate the generality of the growing pattern in terms of its position in the pattern.

METHOD

Two lessons were conducted in two Year 4 classrooms from two middle socio-economic elementary schools from an inner city suburb of a major city. The sample, therefore, comprised 45 students (average age of 9 years and 6 months), two classroom teachers and 2 researchers. The lessons reported in this paper were those conducted by one of the researchers (teacher/researcher). The lessons were of approximately one hour's duration. The first lesson focused on copying and continuing simple growing patterns, describing the patterns in terms of positional language, and using this relationship to predict and create the pattern for other positions, for example, the 10^{th} position. In this instance the patterns chosen were those where the links between the pattern and its position were visually explicit. (e.g., a pattern where its width is its position and its height is always 2). The second lesson entailed re-examining some of these patterns, extending young children's language and thinking to describe and predict the patterns for any position, and reversing the thinking (i.e., identifying the position when given the pattern). It was decided not to record the data in a table but to ascertain if children could link the generalisation with the construction of the pattern itself. Using Halford's structural mapping theory, Warren (1996) found that converting a visual pattern to a table of values increased the processing load, making the task more difficult. In fact, recording data sequentially in a table appeared to encourage single variational thinking, that is finding relationships along the sequence of numbers instead of finding the relationship between the pairs, hence the omission of this step in this research.

Data gathering techniques and procedures

During the teaching phases, the other researcher and classroom teacher acted as participant observers. The lessons occurred sequentially. In each instance the other researcher and classroom teacher recorded field notes of significant events including student-teacher/researcher interactions. Both lessons were videotaped using two video cameras, one on the teacher and one on the students, particularly focussing on the students that actively participated in the discussion. At the completion of the teaching phase, the researcher and teacher reflected on their field notes, endeavouring to minimise the distortions inherent in this form of data collection, and come to some common perspective of the instruction that occurred and the thinking exhibited by the children participating in the classroom discussions. The video-tapes were transcribed and worksheets collected. A pre and post-test were administrated before the first lesson and two weeks after the completion of the second lesson. The two tests comprised three questions as shown in Figure 1.

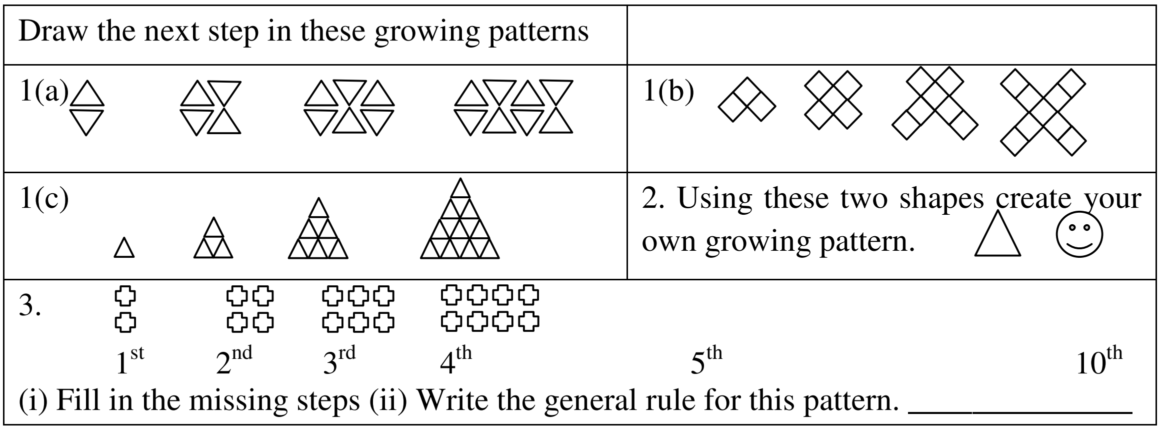


Figure 1. Growing pattern questions on the Pre & Post test.

Questions 1 and 2 were included to ascertain children's understanding of growing patterns while Question 3 probed their ability to predict further positions in the pattern and describe, in general terms, the relationship between the pattern and its position. These tests mirrored the types of activities and discussions that occurred during the teaching phase. The delayed post test served to ensure that the responses reflected children's own thinking, rather than simply recalling the discussions that ensued during the teaching phase.

RESULTS

The results of the pre and post-tests (see Table 1) indicated that there was growth in children's understanding of growing patterns and in their ability to describe in general terms the relationship between the pattern and its position.

Table 1. Frequency of response to growing pattern questions on the Pre & Post test

	Pre test					Post test				
	1a	1b	1c	2	3(i)	1a	1b	1c	2	3(ii)
Incorrect	16	21	21	24	5	8	16	13	20	1
Correct	29	22	21	19	31	36	28	31	24	44
No answer		2	3	2	9	1	1	1	1	0

As these results indicated, at the beginning of the teacher phase many children experienced difficulties in simply continuing and creating growing patterns. Even though after the two lessons, many more children were successful in these activities, there were still many who exhibited some difficulties with these tasks. Responses to Question 3(i) indicate that by the completion of the two lessons more children were able to correctly draw the pattern when given differing positions.

The responses to Question 3(ii), the question relating to writing the general rule for a simple growing pattern, were categorised. The responses fell into 7 broad categories ranging from descriptions that gave no indication of the relationship between the pattern and its position to responses that specifically related the pattern to its position. The next section describes each category with a typical response for each.

Category 1. No response.

Category 2. No direct relationship to the question asked (No relationship)

Typical response:You do your original number.

Category 3. Stating that the pattern is simply growing (Single variation)

Typical response: *The patterns keep on growing and growing.*

Category 4. Describing a relationship within the pattern itself

Typical response: *Always the same as the tops as the bottom.*

Category 5. Stating that the pattern is growing in 2 (Quantifying the single variation)

Typical response: *Goes up by two. One more on each end.*

Category 6. Relating the position to the total number of tiles required for that position

Typical responses: Each step number x 2 = number of *'s or It's double the step number.

Category 7. Relating the position to a description of the pattern

Typical response: *The top and bottom row of the stars is the same number as the step*

Table 2 summarises the frequency of responses for each category for the pre-test and the post-test.

Table 2. Responses to Question 4(ii): Write the general rule for this pattern

	Category	Pretest	Posttest
1.	No response	12	4
2.	No relationship	2	4
3.	It grows – Single variation	6	3
4.	Relationship within the pattern itself	0	3
5.	It grows in twos – Quantifying single variation	22	12
6.	Relationship between position and pattern – total number of tiles	2	7
7.	Relationship between position and pattern – visual description	1	12

The results indicated that there was significant growth in these children's ability to describe the pattern in general terms, that is, specifically relate the pattern to its position (co-variational thinking).

Examination of the lessons gave some insights into the teaching actions that assisted this growth in understanding. The next section describes some teaching actions that supported this growth and thinking that hindered the generalisation process.

Supporting processes

The use of concrete materials The use of concrete materials appeared to assist many children ascertain the missing steps in the pattern. A number, when completing the accompanying pen and paper worksheet, recreated the pictorial pattern with the tiles and then used the tiles to create the 5th and 10th step. They then drew a picture of their solution on the worksheet.

Patterns where the relationship between the pattern and position were explicit These types of patterns appeared to assist children to verbally describe the relationship between the pattern and the position, for example, it is twice the step number, it is the same as the step number, it is one more than the step number.

Explicit questioning to link the position to the pattern For the pattern presented in Question 3, when asked to describe the 4th step, one child responded that is was 8 tiles. Explicit questions needed to be asked to ensure that the children connected the pattern's shape to its position. These questions were of the form – What does the pattern look like? How many rows? How many in each row? For the 3rd step, how many on the bottom, how many on the top? The questions explicitly related the position to the pattern's visual components.

Generalising from the pattern in small position numbers, to large position numbers. It was found that to articulate the relationship between position number and the visual pattern in general terms, children needed to discuss the relationship for increasingly larger positions, for example, describe the 10th step and the 20th step. Most children successfully completed this task. To ensure that children were linking the pattern's position to the pattern itself, several more discussion ensued, with each time the step number increasing, for example, what would the 100th step look like? 1000th step, 3000th step? While most children appeared to successfully complete this task, on the post-test over half the sample reverted to a single variation description of the pattern.

Hindering processes

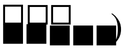
Language used to describe the generalisation

Most children experienced difficulty in precisely describing a visual pattern. For example, when they created a 20 by 3 array, most described this as 60. With probing, some indicated it was 3 across, 3 rows of 20 and eventually 3 columns of 20.

Writing the generalisation as compared with saying it orally

The classroom discussions indicated that these children found it much easier to verbalise the generalisation than to provide a formal written response. When asked to share their written responses for the pattern delineated in Question 3, one child shared 'it increases by 2 every time' another 'always the same number on the bottom and the top' and two more said 'Each step has step number on the bottom and top' and 'Just put them in groups of 2 one on each other'. The range of responses indicated that even though many could verbally say the generalisation, when it came to writing it many experienced difficulties, with the tendency to give responses that focused on the single variation of the pattern.

Completing patterns – single variation

Their propensity to think of growing patterns as adding on the growing part to the preceding step affected their ability to create missing steps within the pattern and to create the step number when given the total number of tiles. For example, we presented visual patterns giving only 1st, 2nd and 5th steps; children were asked to complete the missing steps. The most common strategy was to simply compare the 1st and 2nd step and continue adding on tiles to reach the 5th step. This single variation thinking (Additive strategy) was best exhibited in the following example where they were given the 1st and 3rd step (& ). Nearly all of the children gave  as the 2nd step. When challenged they simply recreated the 3rd step to fit their pattern.

Reversing the thinking We also presented the total number of tiles and asked which step this represented. Most children found this very difficult, perhaps for two reasons. First, it relied on linking the position to the step number, which many struggled with, and second in some instances it required a good understanding of number patterns.

Expressing the generalisation in language Many children could not express the pattern in general language, and when using the language there was confusion between the ordinal language and number of tiles.

T.	What if I had the *n*th position? What would the pattern look like?
Cl	*n*th on the top and *n*th on the bottom.
T.	Describe the pattern in terms of the number of tiles.
C2	*n* tiles on the top and *n* tiles on the bottom.

On the more positive side, there were at least five children in each class that could not only describe the generalities in correct mathematical language but also write these generalities using abstract notation systems (e.g., for the *n*th step there are n blue tiles and $n + 1$ yellow tiles).

DISCUSSION AND CONCLUSION

As indicated by the results of the pre-test nearly half of the children could not complete the next step in simple growing patterns nor create their own growing pattern. This could be for two reasons. First, they had had limited experiences with growing patterns in the early years, or second, growing patterns are not as easy as they first appear. An examination of curriculum documents and commonly used classroom texts would suggest that the predominant focus in the early years is on repeating patterns. These children certainly did not experience the same difficulties with repeating patterns (due to space restrictions this data cannot be reported in this paper). By the completion of the teaching phase there had been some improvement in their ability to complete and create growing patterns, indicating that perhaps the difficulties did indeed stem from a lack of experience in this area. As indicated by past research, many young adolescents experience difficulties with the transition to patterns as functions. The inability to visualise spatially or complete patterns (Warren, 2000) is a key impediment to this. The impact that earlier classroom experiences have on this thinking requires further investigation. This research appears to indicate that such experiences purposely built into elementary classroom experiences may indeed commence to address this impediment.

The results confirm the conjecture of Blanton and Kaput (2004) that young children are capable of thinking functionally. They also suggest that there are a variety of teaching actions that support this thinking, namely, using concrete materials to create patterns, specific questioning to make explicit the relationship between the pattern and its position, and specific questioning that assist children to reach generalization with regard to unknown positions. Young children are not only capable of thinking about the relationship between two data sets but also of expressing this relationship in a very abstract form.

While young children are capable of thinking functionally, it appears from this research that single variational thinking is perhaps cognitively easier or so entrenched in early experiences that a propensity to revert to this thinking is understandable. It was conjectured that not recording the data in a table would reduce the probability of this occurring. However, instead of looking for patterns in sequences of numbers, they appeared to look for patterns in the sequence of tiles, that is, instead of saying we keep adding on 2 for the sequence of numbers in the table, they said "we add on two tiles as we proceed along the steps". This thinking was so entrenched that some children were even willing to change the examples given to make them fit their sequential thinking pattern.

The interaction between oral description of patterns and putting this description in written form also requires further investigation. Many children exhibited an ability to express the generalisations orally, but such descriptions often lacked precision. While their oral responses appeared 'correct', one wonders how much 'filling in' the listener does when hearing the responses to questions asked. A review of the video-tapes indicated that this was indeed the case, suggesting that the precision needed for correct written responses can be missing from classroom conversations. In this instance, gestures and manipulation of materials add to the conversations, elements that are missing from written responses. These children also appeared to lack some of the mathematical vocabulary needed to give precise responses, words such as row and column and describing an array as 2 rows by 4 columns. Thus on many occasions they could model the functional relationship with concrete materials and could attempt to describe this relationship using imprecise language embellished with gestures, they often reverted to 'lower level' responses when asked to write their generalization in a written form (e.g., "add on 2" instead of "the number of tiles is double the step number"). This could begin to explain the large variations in responses on Question 3(ii) on the post test, a problem that nearly all could complete and describe orally within the context of the classroom discourse.

This research commences to not only identify teacher actions that support examining growing patterns as functional relationships between the pattern and its position, but also delineate thinking that impacts on this process. Many of the difficulties these children experienced mirror those found in past research with young adolescents. This suggests that perhaps these difficulties are not so much developmental but experiential, as these early classroom experiences began to bridge many of the gaps.

References

Bennett, A. (1988). Visual thinking and number relationships. *Mathematics Teacher*, 81(4), 267-272.

Blanton, M., & Kaput, J. (2004). Elementary grades students' capacity for functional thinking. In M. Jonsen Høines & A. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 135-142). Oslo: PME.

Carpenter, T. P., Franke, M. L., & Levi, L. W. (2003). *Thinking Mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth: Heinemann.

Chazan, D. (1996). Algebra for all students. *Journal of Mathematical Behavior, 15*(4), 455-477.

Halford, G.: 1993. *Children's understanding: The development of mental models.* Hillsdale, NJ: Erlbaum.

Jonassen, D. H., Beissner, K, & Yacci, M. (1993). *Structural knowledge: Techniques for representing, conveying, and acquiring structural knowledge.* Hillsdale, NJ: Erlbaum.

Kaput, J., & Blanton, M. (2001). Algebrafying the elementary mathematics experience. In H. Chick, K. Stacey, J.Vincent & J.Vincent (Eds.), *The Future of the Teaching and Learning of Algebra. Proceedings of the 12th ICMI study conference* (Vol 1, pp. 344-352). Melbourne: Australia.

Kieran, C. (1990). Cognitive processes involved in learning school algebra. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition: A research synthesis by the International Group for the Psychology of Mathematics Education* (pp. 97-136). Cambridge: Cambridge University Press.

Malara, N., & Navarra, G. (2003). *ArAl Project: Arithmetic pathways towards favouring pre-algebraic thinking*. Bologana, Italy: Pitagora Editrice.

Ohlsson, S. (1993). Abstract schemas. *Educational Psychologist, 28*(1), 51-66.

Redden, T. (1996). "Wouldn't it be good if we had a symbol to stand for any number": The relationship between natural language and symbolic notation in pattern description. In L. Puig & A. Gutierrez (Eds.), *Proceedings of the 20th International Conference for Psychology of Mathematics Education* (Vol. 4, pp. 195-202). Valencia, Spain: PME.

Sfard, A. (1991). On the dual nature of mathematical concepts: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 191-228.

Stacey, K., & MacGregor, M. (1995). The effect of different approaches to algebra on students' perceptions of functional relationships. *Mathematics Education Research Journal*, 7, 69-85.

Warren, E. (1996). *Interaction between instructional approaches, students' reasoning processes, and their understanding of elementary algebra*. Unpublished dissertation. Queensland University of Technology.

Warren, E. (2000). Visualisation and the development of early understanding in algebra. In M. v. d. Heuvel-Panhuizen (Ed.), *Proceedings of the 24th conference of the international group for the psychology of mathematics education* (Vol. 4, pp. 273-280). Hiroshima.

CONSOLIDATING ONE NOVEL STRUCTURE WHILST CONSTRUCTING TWO MORE

Gaye Williams

Deakin University

This study reports the cognitive processing of a Year 8 female student (Kerri) during a test, and during her subsequent homework as she consolidated as part of abstracting; a topic of recent research interest. This case adds to the body of knowledge about how constructing and consolidating can occur simultaneously. The analysis captured the complexity of the cognitive processing, and their intertwined nature. Data was captured through lesson video, and post-lesson, video-stimulated reconstructive student interviews. It was found that Kerri's constructing and consolidating included characteristics previously identified by others, and an additional feature. Her constructing included "branching" (Kidron & Dreyfus, 2004, p. 159); but unlike the case cited, one of these branches related to a new goal.

INTRODUCTION

The process of consolidating has been studied using tasks specifically designed to elicit constructing and consolidating activity at junior and senior year levels in secondary mathematics, through clinical interviews (Dreyfus, Hershkowitz & Schwarz., 2001; Dreyfus & Tsamir, 2004, Monaghan & Ozmantar, 2004) and during classroom learning (Tabach, Hershkowitz, & Schwarz, 2004). In each of these cases, the constructing and consolidating processes were directed towards a particular mathematical goal. The present study contributes an illustration of a different nature; Kerri was undertaking her usual learning and assessment, associated with her mathematics classes, and was not engaged in a task specifically designed for constructing purposes. Her constructing process 'branched out' (Dreyfus & Kidron, 2004) into two separate constructing processes when she suddenly recognised the new mathematics she had constructed could serve another purpose. Kerri extended her understanding of interrelationships between algebraic, geometric, and numeric representations of linear graphs as a result of the overall constructing and consolidating she undertook. This abstracting should be of use in her future studies.

THEORETICAL BACKGROUND

The process of 'abstracting' contains three stages: (a) a need to know; (b) 'constructing' a new entity; and (c) 'consolidating' that entity so it can be recognized with ease and built-with in future activities [1](Hershkowitz, Schwarz, & Dreyfus, 2001). 'Constructing' (**C**) has 'recognizing' (**R**) and 'building-with' (**B**) nested within it. These epistemic actions are observable through dialectic discourse: and

[1] Hershkowitz, Schwarz, & Dreyfus (2001) is henceforth referred to as HSD (2001).

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 313-320. Melbourne: PME.

together form the **RBC**-model (HSD, 2001). Williams (2002) integrated Krutetskii's (1976) mental activities ('comprehending', 'analyzing', 'synthetic-analyzing', 'synthesizing', and 'evaluating'), and an additional activity she empirically identified ('evaluative-analysis'), into the **RBC**-model (see Figure 1). The types of thinking represented in Figure 1 are now described, and are later elaborated through the illustration in this paper. Recognizing involves seeing the relevance of previously known procedures, processes, and strategies (a process requiring an understanding of the mathematics involved, or comprehending, and which can also involve an analysis of structure). Building-with (**B**) involves using known mathematical ideas, concepts, and strategies in an unfamiliar combination, or an unfamiliar sequence to solve a problem. **B** can include finding patterns (analysis), simultaneously considering more than one aspect of a problem (synthetic-analysis), and using this synthetic-analysis for purposes of judgement (evaluative-analysis). Constructing involves integrating mathematical concepts to form a new mathematical structure (synthesis), and can include immediate recognizing of the relevance of a newly abstracted entity for a new purpose (evaluation). The shaded parts of Figure 1 are used to distinguish spontaneous constructing from constructing guided by an expert other. For more information about differences in these ways of constructing, see Tabach, Hershkowitz, and Schwarz (2004) and Williams (2004) respectively.

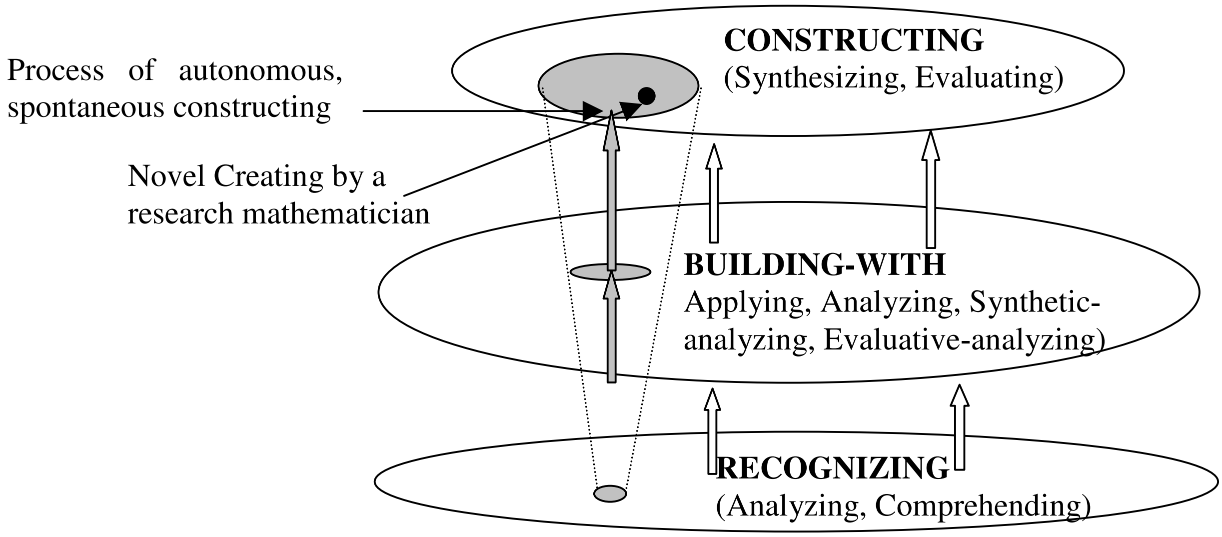


Figure 1. Williams' (2002) integration of observable cognitive elements (HSD, 2001) with Krutetskii's (1976) mental activities illustrating their nested nature.

Consolidating possesses three characteristics (**B**, Building-with; **RfB** Reflecting upon building-with, and **Rf**, reflecting generally) (Dreyfus & Tsamir, 2004) although Monaghan & Ozmantar (2004) question whether **Rf** always occurs. Consolidating a novel structure whilst constructing another has been identified in several contexts including Year 7 students working in a pair (Tabach, Hershkowitz, et al., 2004) and a learner of tertiary mathematics thinking alone (Dreyfus & Kidron, 2004). Ivy, abstracting in tertiary mathematics (Kidron & Dreyfus, 2004), constructed and consolidated through a "branching" constructing process which later reunited as she pursued her ultimate goal. Her branching process involved simultaneously considering two directions, continuing along one, and later linking the two branching constructing processes in pursuit of her overall goal. In the present study, constructing is examined through the cognitive activity represented in Figure 1, and examination of consolidating is informed by the research above.

The research question: By what process did Kerri construct and consolidate new knowledge, and how does her activity further illuminate these processes?

RESEARCH DESIGN

This study was part of a broader study of autonomous, spontaneous, and creative student activity during mathematics learning in Year 8 classrooms in Australia and the USA. Data was generated as part of the international Learners' Perspective Study. Each class was video-taped for ten or more lessons using three cameras that focused respectively on the class, the teacher, and different focus students each lesson. A mixed-image video was produced in class with the focus students at centre screen, and the teacher as an insert in the corner. In addition to providing evidence of social influences on cognitive processing, this video provided salient stimuli for individual, post-lesson reconstructive student interviews undertaken to generate valid data (Ericcson & Simons, 1980) associated with student classroom activity including student thinking. The student operated the video remote to find the parts in the lesson that were important to her, and then discussed what was happening, her thinking, and her feelings. Kerri's interview focused on the mathematical activity of interest to the US research team (mathematical entity developed, and social influences on that development), rather than specifically on the student's cognitive processing (interview focus in Australia for other cases in the author's broader study). Due to Kerri's ability to articulate her thoughts, and her desire to do so, the interview captured rich data about Kerri's cognitive processing as well.

THE CONTEXT

Just prior to the start of the research period, the class were taught to find equations of linear graphs by plotting two points, ruling the line between them, drawing a right-angled triangle to measure 'rise' and 'run' (called a 'slope triangle' by the teacher), taking the ratio of these lengths to find the gradient, finding the y-intercept by inspection, and substituting the gradient and y-intercept into the equation **y = mx + b**. Kerri's interview reconstruction of the test on this procedure just prior to the research period showed she forgot to bring graph paper so created a novel solution process. After the test, and prior to research period, students undertook a homework exercise to find the equation to a line given the co-ordinates of two points, and to find the length of the line segment between the points. The length of line segment formula had not been taught; the teacher expected students to plot graphs, measure lengths, and find the y-intercepts by inspection. In class the next day (Lesson 1 of the research period) the teacher demonstrated 'finding the equation of a line without graphing' (when two points were given). Evidence for this case is presented below as a narrative. Data sources are indicated by '[SI]' for Kerri's interview after Lesson 1, and '[L1]' for video of Lesson 1.

ANALYSIS AND RESULTS

It was inferred that Kerri's constructing activity prior to Lesson 1 was spontaneous (self-instigated and self-directed, see Williams, 2004) because the teacher awarded 200% to Kerri's test thus suggesting the originality of the work. During her interview, Kerri reconstructed her thinking in her test. Her novel method entailed sketching and recognizing that a slope triangle lay between the two points given "cuz you can picture a line in a little right triangle on it" [SI]. She then used her knowledge of the Cartesian Axis System to find the lengths of the horizontal and vertical sides in this triangle by subtraction, and then recognized she could substitute **x** and **y** as well as the constant **m** into the general linear equation **y = mx + b** to find another constant (this complex substitution method had not been previously taught).

> if you find the slope and the … difference of the points and … then we can substitute, oh perfect. So I just wrote the equation. [SI]

Kerri demonstrated her pleasure in this discovery of how substituting could be applied by adding "oh perfect" to illustrate her affective state when she found the way to proceed. When reflecting on this process in her interview, she commented on the quality of the newly developed process "Actually I thought like- I thought it was kind of a big idea… it wasn't too big for me" [SI]. The next evening, whilst doing her homework (using the teacher's graphical method), Kerri developed a generalized understanding of the new method she had developed in the specific numerical question in the test. (Notation for transcript: Three dots indicate omissions that do not alter the meaning. Square brackets: researcher additions clarifying the context):

> I was doing my graph [during homework], and then I like realized like- *really solidly*, … I got the same answer, … [by measuring as] if you do the subtraction. [SI]

Kerri thought about both methods as she did her homework, and realized each gave the same answer. I firstly provide evidence about what Kerri realized, then later describe Kerri's further novel thinking during that homework session. What Kerri knew '*really solidly*' was evident in class the next day [L1]. In class, Kerri drew attention to the teacher's sketch, used as she demonstrated 'finding the equation of a line without a graph': "You still graphed it" [L1-Kerri]. Kerri demonstrated she could find the gradient by operating on elements of the ordered pairs representing the points on the line without referring to the slope triangle: "I would just be like the difference in Y is two, and the difference in X is one. So that's your slope" [SI]. In Lesson 1, in preference to using a plot or a sketch, Kerri used her fast method of subtracting relevant elements of ordered pairs to find the side lengths to take the ratio to find the gradient. She continued using her new method even though most students relied on graphical representations to support their thinking, and also queried Kerri's method.

The additional constructing Kerri undertook during her homework session, after she had found the relationships between operations on elements of ordered pairs, and the rise and the run is now described.

and then also [during my homework] … we had to find the distance between the two plots, and it was supposed to graph them too-… I was using Pythagoras' Theorem. [SI]

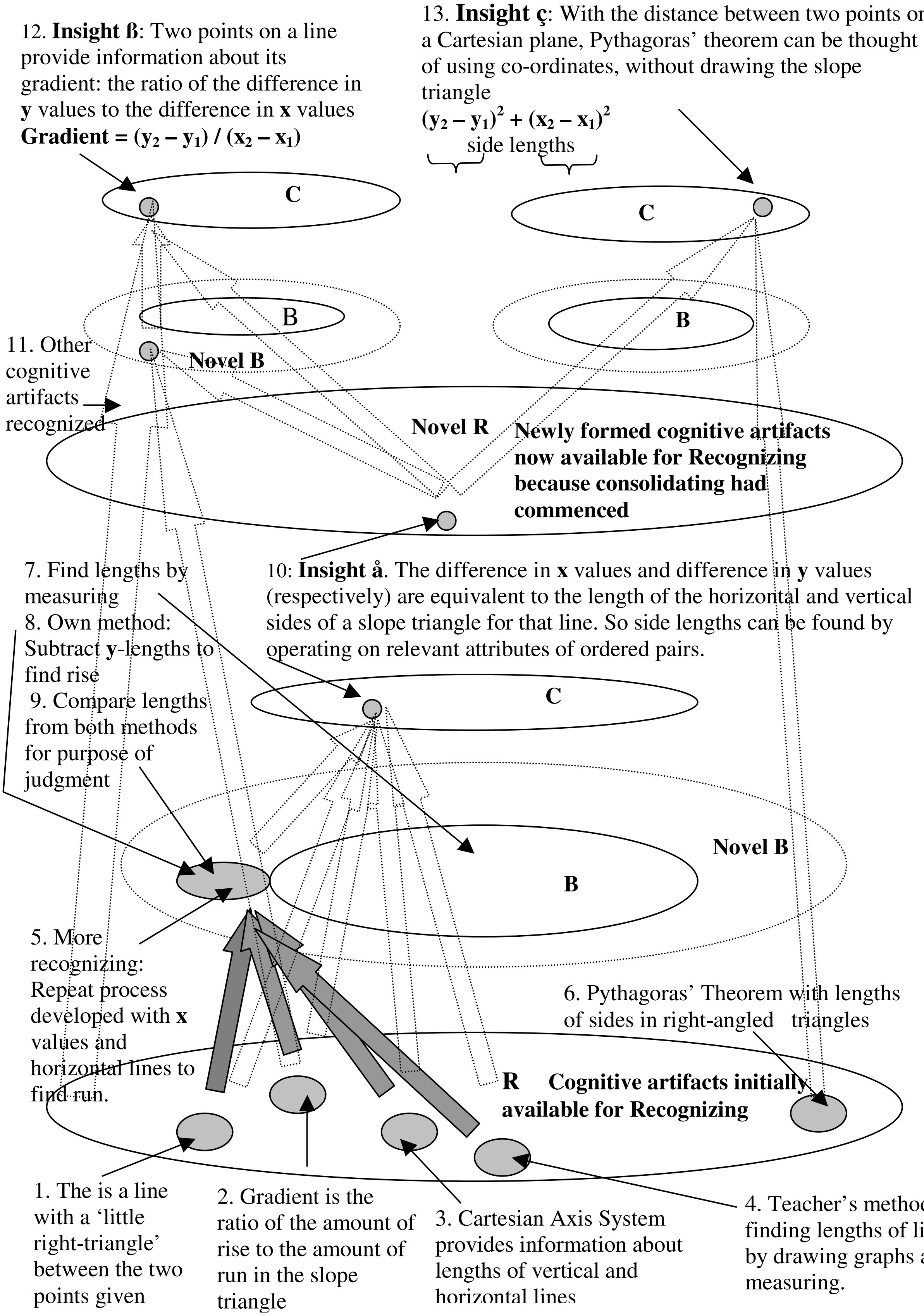


Figure 2 Kerri's Constructing and Consolidating Activity

Once Kerri realized how to find the vertical and horizontal side lengths in the slope triangle, she recognized it was useful for another purpose—finding the length of the line segment. In her interview after Lesson 1, Kerri described what she understood (and used in class) that the students at her table did not [SI]:

Interviewer	So what do you think that you discovered, that other people didn't?
Kerri	[the question] said graph and find the distance- and most people would graph the line, and then do the little thing [slope triangle]. But I would find what- see that'd be two and then one [subtracting **y** values and **x** values], so you do um, **a** squared plus **b** squared equals **c** squared. … if you make it a right triangle- it's the hypotenuse- not just the distance

In doing so she demonstrated an ability to shift flexibly between representations and use both specific numerical values and general formulae as she justified her method.

Figure 2 (above) diagrammatically represents Kerri's spontaneous cognitive activity described below, and aids communication of this activity. Figure 2 superimposes Kerri's cognitive activity on the diagrammatic representation used in Figure 1. In this way, the cognitive elements Kerri displayed are evident through both the diagram and the text. Each element of cognitive activity in Figure 2 is represented by a number to add clarity to the text. The Figure 1 convention of shading all spontaneous activity has not been followed with arrows that would obscure Figure 2 (these arrows were left unshaded). Figure 2 includes three copies of Figure 1; the original Constructing of Insight å (Figure 2; 10) at the bottom of the page, and the additional constructing of both Insights ß and ç (Figure 2; 12, 13), that use the outcome of Insight å as a cognitive artefact. This further constructing is represented by the two smaller copies of Figure 1 positioned side by side at the top of Figure 2. They both share the same 'Recognizing ellipse' containing Insight å, but draw also from the original 'Recognizing ellipse' (base of diagram, cognitive artefacts possessed prior to Kerri's initial constructing). Both Figures 1 and 2 inform the interpretation below.

Kerri analyzed her sketch and recognized cognitive artefacts (right-angled triangle in sketch [1], co-ordinates associated with a Cartesian Axis System [3]), and built-with these cognitive artefacts (subtracted **y** values, and **x** values in the two ordered pairs [8, 5]) to develop a novel way to solve the problem. When Kerri undertook practice exercises for homework, she simultaneously considered her new method [8] and the teacher's method [7] (synthetic-analysis, as part of building-with). She compared the lengths obtained by each method and found they were the same [9] (evaluative-analysis, part of novel building-with). As a result of this process she constructed new insight [10]—she realised she no longer needed the triangle to find the lengths. In constructing new insight, Kerri synthesized (sub-category of constructing) by subsuming the attributes of the Cartesian Axis (that enabled her to find the lengths of the vertical and horizontal lines) into relevant attributes of the ordered pairs representing points on the line [9]. She operated upon these ordered pairs without needing a sketch to aid her thinking. She then found the gradient through additional operations on these ordered pairs [12]. Kerri then recognized a new purpose for

Insight å (evaluating as part of spontaneous constructing). Kerri synthesized Pythagoras' Theorem [6], with her novel mathematical structure [10], to construct Insight ç—the length of a line segment can be found by operating on elements of the ordered pairs at the end of the segment [13]. In developing Insights ß and ç [12, 13], Kerri Built-with Insight å [10]; thus consolidating it during novel building-with.

DISCUSSION AND CONCLUSIONS

Figure 2 draws attention to the increased complexity in thinking that occurred during Kerri's sustained exploratory activity associated with three insights. The whole of Figure 2 captures Kerri's overarching constructing of interconnections between different aspects of, and representations of function. Kerri's cognitive activity illustrated the progressive connecting of different representations that occurred through synthetic-analysis and evaluative-analysis. Further research is required to explore whether, and how, these cognitive processes could be elicited during classroom learning, and whether doing so would increase constructing opportunities.

Kerri demonstrated the three stages of abstracting (HSD, 2001): (a) she needed to know a way to answer the test question; (b) constructed as a result of this need; and (c) consolidated her new learning as she recognized a way to satisfy another need. She also exhibited the three identified stages of consolidating (Dreyfus & Tsamir, 2004); novel building-with; through her test solution (**B**), reflecting on novel B when she compared it with the teacher's process during homework (**RfB**), and reflecting generally about associated positive affect, what she thought about other students' querying her approach, and the quality of the construction achieved (**Rf**). Kerri was consolidating her novel B as she constructed her first insight thus illustrating the intertwined nature of consolidating and constructing processes. In addition, Kerri demonstrated that consolidating was occurring through the characteristics she displayed that have previously been identified by Dreyfus and Tsamir (2004): "impetus", "self-evidence", "confidence", "flexibility", and "awareness". Kerri displayed these characteristics as she: (a) recognized her novel structure for other activity; (b) justified her thinking; (c) changed from tentative knowing to knowing "really solidly"; (d) flexibly moved between representations; and (e) knew other students required visual support that she did not, and that she had found a "big idea".

Kerri, like Ben, in Dreyfus and Tsamir (2004), was a gifted student, so although these findings support the presence of the identified consolidating characteristics during the activity of gifted students (**R, RfB, Rf**), they do not address the absence of **Rf** found by Monaghan & Ozmantar, (2004) in another case. As Kerri was in a class of gifted students, and the other students did not spontaneously construct in this instance, the study does point to features of the situation that provided impetus for Kerri's novel thinking. In this case, the *absence* of a 'tool' gave impetus to novel thinking. This is contrary to findings in other contexts where tools were found to enhance learning (e.g., Tabach, Hershkowitz, et al., 2004). There was also opportunity to sustain novel thinking during reflection undertaken with 'hands on activity' that preceded 'telling rules'. Aspects of the idiosyncratic learning situation in which Kerri's thinking occurred could inform research into factors that promote learning in which complex mathematical thinking and high positive affect can co-exist during the generation of new knowledge. Although some aspects of Kerri's constructing process were not explored (e.g., whether she explored unproductive pathways), Kerri's reconstruction of her thinking was sufficiently rich to demonstrate 'branching for another purpose' as a possible outcome of consolidating during further constructing.

Acknowledgements: Thanks to Mary Barnes for her useful comments that contributed to the final version of this paper.

This data forms part of the US data set for the international Learners' Perspective Study directed by David Clarke at the University of Melbourne http://www.edfac.unimelb.edu.au/DSME/lps/. Data was analyzed using the facilities and technical assistance available through the International Centre for Classroom research.

References

Dreyfus, T., Hershkowitz, R., & Schwarz, B. (2001). Abstraction in context: The case of peer interaction. *Cognitive Science Quarterly, 1*(3), 307-368.

Dreyfus, T. & Kidron, I. (2005). Interacting parallel constructions: a solitary learner and the bifurcation diagram. Technical report, Tel Aviv University, Israel.

Dreyfus, T., & Tsamir, P. (2004). Ben's consolidation of knowledge structures about infinite sets. *Journal of Mathematical Behavior, 23*(3), 271-300.

Ericsson, K., & Simons, H. (1980). *Verbal reports of data. Psychological Review*, *87*(3), 215-251.

Hershkowitz, R., Schwarz, B., & Dreyfus, T., (2001). Abstraction in Context: Epistemic Actions. *Journal for Research in Mathematics Education, 32*(2), 195-222.

Kidron, I., & Dreyfus, T. (2004). Constructing knowledge about the bifurcation diagram: Epistemic actions and parallel constructions. In M. J. Høines & A. B. Fuglestad (Eds.), *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 153-160). Bergen, Norway: PME.

Krutetskii, V. (1976). *Psychology of Mathematical Abilities in Schoolchildren.* J. Kilpatrick & I. Wiszup [Eds.], J. Teller (Translator), Chicago: The University of Chicago Press.

Monaghan, J., & Ozmantar, M. (2004). Abstraction and consolidation. In M. J. Høines & A. B. Fuglestad (Eds.), *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 353-360). Bergen, Norway: PME.

Tabach, M., Hershkowitz, R., &Schwarz, B. (under review). The construction and consolidation of mathematical knowledge within dyadic processes: A case study, Educational Studies in Mathematics.

Williams, G. (2002). Identifying tasks that promote creative thinking in mathematics: a tool. In B. Barton, K. Irwin, M. Pfannkuch & M. Thomas (Eds.), *Mathematics Education in the South Pacific* (Vol. 2, pp. 698-705). Auckland New Zealand., Mathematical Education Research Group of Australasia Conference.

Williams, G. (2004). The nature of spontaneity in high quality learning situations. In M. J. Høines & A. B. Fuglestad (Eds.), *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 433-440). Bergen, Norway: PME.

SPREADSHEETS, PEDAGOGIC STRATEGIES AND THE EVOLUTION OF MEANING FOR VARIABLE

Kirsty Wilson[1], Janet Ainley[1] and Liz Bills[2]

[1]Institute of Education, University of Warwick, UK

[2]School of Education and Lifelong Learning, University of East Anglia, UK

We report on one aspect of a longitudinal study which seeks insight into the ways in which spreadsheet experience and teachers' pedagogic strategies shape pupils' construction of meaning for algebra. Using data from stimulated recall interviews we analyse the evolution of meaning for variable through the mediation of the variable cell and the mediation of naming a column. We discuss metaphors of change and dragging, together with the process of naming.

BACKGROUND

Research suggests that spreadsheets can support pupils in developing an understanding of variables. In a longitudinal study of two groups of 10-11 year old pupils working on traditional problems, Sutherland and Rojano (1993) conclude that 'a spreadsheet helps pupils explore, express and formalise their informal ideas' (p.380), moving from thinking with specific numbers to symbolising a general rule. Moreover, it is claimed that spreadsheet notation 'can ultimately be used as cognitive support for introducing and for sustaining the more traditional discourse of school algebra' (Kieran, 1996, p.275). Dettori et al. (1995) discuss the benefits and limitations of such use of spreadsheets, suggesting their value in supporting pupils' understanding of what it means to solve an equation, for example, and their limitation in the formal manipulation of relations.

Sutherland (1995) found that low achieving 14-15 year olds, who had worked on a unit which required them to write an algebraic version of a spreadsheet formula, were able to use their knowledge in a paper and pencil test. One pupil drew a spreadsheet on paper and was able to represent the relationships using letters when subsequently interviewed. Sutherland concluded that 'the spreadsheet symbol and the algebra symbol came to represent "any number" for the pupils' (p.285). Mariotti and Cerulli (2001) similarly report that in a paper and pencil environment, pupils used signs derived from their symbolic manipulator 'L'Algebrista.' Several researchers point to the important role of the teacher in guiding pupils' construction of meaning when working with technological tools (for example Dettori et al, 1995).

The framework of semiotic mediation is useful for interpreting the role of technological tools in a didactic situation. Rooted in the work of Vygotsky, semiotic mediation refers to the mediating function of signs and tools on the learners' construction of meaning. Mariotti (2002) identifies two levels of semiotic mediation.

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 321-328. Melbourne: PME.

At the first level, meanings emerge directly from pupils' activity, and hence the tool functions as a semiotic mediator. At the second level

> '(Since) the *mathematical* meaning incorporated in the artifact may remain inaccessible to the user … evolution is achieved by means of social construction in the classroom, under the guidance of the teacher' (Mariotti, 2002, p.708)

Spreadsheets offer access to the meaning of algebra through the use of formulae and graphing and specifically to the meaning of variable through the notion of a 'variable cell' and 'variable column' (Haspekian, 2003). In this paper we consider how a teacher guides the evolution of meaning for variable, focusing in particular on guiding meaning for variable in paper and pencil activity.

This study builds upon the Purposeful Algebraic Activity project[1] which aimed to explore the potential of spreadsheets as tools in the introduction to algebra and algebraic thinking. The project involved the design and implementation of a spreadsheet-based teaching programme with five Year 7 classes (aged 11-12). Two of the tasks in particular involved moving away from the spreadsheet and making some links to standard notation. Of relevance here is the finding that different tasks offered different opportunities for pupils to construct meaning for variable (Ainley, Bills and Wilson, 2004) but that not all pupils seemed to construct this meaning for spreadsheet notation. We have also found that spreadsheet affordances can support pupils' paper and pencil generalising (Wilson, Ainley and Bills, 2004).

These emerging findings have informed the development of this study which focuses more closely on the guidance of the teacher. It seeks insight into the ways in which spreadsheet experience and teachers' pedagogic strategies mediate pupils' construction of meaning for algebra. As part of the study, one of the classes who participated in the teaching programme was traced into Year 8 (aged 12-13). The class of high attainers was taught by Judith, an experienced mathematics teacher who was familiar with the content of the teaching programme (having taught it to other classes in Year 7). During this year, the class participated in follow up work, planned in collaboration with Judith and driven by the demands of Year 8 curriculum together with the affordances of the spreadsheet. Judith also taught some additional algebra lessons which focused on simplifying expressions and solving equations.

PEDAGOGIC STRATEGIES

Judith employed specific pedagogic strategies with the aim of guiding the evolution of meanings for algebra and making links to the paper and pencil activities in the curriculum. Such strategies were employed at various times during the year but mainly fell into four series of lessons within algebra units of work. An overview of these lessons is given below. Some lessons took place in a computer room, others took place in the classroom, often utilising the projected image of the spreadsheet.

[1] Both the Purposeful Algebraic Activity project and this study are funded by the Economic and Social Research Council

Series of lessons	Pedagogic strategies
‘Odd one out’ lessons used the spreadsheet as an environment for generating equivalent expressions. Pupils were given the task of identifying which formula or expression was the odd one out and then were invited to make up their own odd one out games.	• Emphasising idea of the variable cell and the variable column • Making links between spreadsheet notation and standard notation
‘Myphone’ lessons built upon and extended work from Year 7. Pupils set up a spreadsheet to show the cost of calls under different tariffs and solved various problems on the spreadsheet and by solving equations.	• Naming a column on a spreadsheet • Making links between spreadsheet formulae, equations and graphs • Focusing on what it means to solve an equation
‘Sum and product’ lessons built upon and extended work from Year 7. Pupils used trial and improvement in both environments to solve quadratic equations, one esoteric and one about a sheep pen of a given area.	• Naming a column on a spreadsheet • Focusing on what it means to solve an equation
‘Generalising’ lessons involved writing spreadsheet and algebraic formulae to represent various relationships, and then solving problems using various methods.	• Writing formulae and using substitution through work on printed screen snaps • Using spreadsheet affordances - focus on calculations, use of notation, feedback • Considering different solution strategies

This paper focuses on three strategies which specifically relate to the meaning of variable: emphasising the idea of the variable cell and column; making links between spreadsheet notation and standard notation; and naming a column on a spreadsheet.

DATA COLLECTION

In each of the lessons, a range of data was collected including field notes, audio and video recordings of Judith’s teaching, and video and screen recordings of the activity of a pair of pupils. Following each series of lessons, a small group of pupils was interviewed using the technique of stimulated recall (Calderhead, 1981). The pupils were invited to watch short video clips from the lessons, some with and some without sound, and asked questions such as: Can you remember what was happening here? What do you think Judith meant when she said …? Can you remember what you were thinking? One focus of the discussion was pupils’ construction of meaning for variable. The discussions were characterised by the openness of the pupils who had known the researcher for over a year and who represented the range of attainment within the set. Although the pupils were not always confident that they could recall what they were thinking at the time of the lesson, the interviews provided some valuable insights into their interpretations of notation and pedagogic strategies.

Alongside the data from the lessons, pupils were also interviewed in pairs at the beginning and end of the year, although there is not space to refer to this data here. Judith was also interviewed at the end of the year. The data was semi-transcribed and transcripts were interwoven with non-verbal behaviour, any written work and interaction with the computer as appropriate. In our initial analysis, we outline the affordances offered by the spreadsheet and then consider the evolution of the meaning for variable within the field of experience of pupils' spreadsheet-based activity. We focus in particular on the mediation of the variable cell and of the process of naming a column in making links to standard algebraic notation.

SPREADSHEETS

The spreadsheet environment offers three important affordances related to variable which we consider in this paper: the variable cell, the variable column and the named column. Haspekian (2003) identifies four features of a 'variable cell,' such as A2. The first feature corresponds with the use of a letter to stand for a variable:

- 'an abstract, general reference: it represents the variable
- a particular concrete reference: it is here a number
- a geographic reference
- a material reference' (p.6)

Spreadsheets also have the facility for filling down a formula through a range of cells and generating a 'variable column' (Haspekian, 2003). A name for a column, or indeed a cell or row, can also be defined and that name used in a formula (the 'A' column is defined as 'n' in the example below). When the formula is filled down, each new formula then includes the same name. This facility has not been widely researched in terms of pupils' algebraic thinking.

	A	B
1	1	=2*A1+3
2	2	=2*A2+3
3	3	=2*A3+3
4	4	=2*A4+3

	A	B
1	1	=2*n+3
2	2	=2*n+3
3	3	=2*n+3
4	4	=2*n+3

EVOLUTION OF MEANING FOR VARIABLE

In the spreadsheet environment, the activity of writing formulae to solve problems involves using notation, such as A2, but pupils may or may not recognise that the notation represents each of the features of the variable cell, and in particular an 'abstract, general reference,' a variable (Haspekian, 2003). This meaning emerges from the pupils' activity and reflection guided by the teacher. In this case, Judith emphasised the idea of the variable cell and variable column, making links to the use of standard notation in the paper and pencil environment.

Mediation of the variable cell

In a number of the lessons in the teaching programme and then in the follow up work described earlier in the paper, the pupils had worked on various tasks which involved writing formulae and then changing the value in a cell. In the 'Odd one out' series of lessons, pupils were given various spreadsheet expressions such as '6*A2+12', 6*(A2+2), '2*(A2+2)+4*(A2+2)' and '5*(A2+1)+8' and were set the task of finding the odd one out by 'testing' various values in A2. Judith made links between the spreadsheet notation of A2 and standard algebraic notation which the pupils had already been introduced to ("denotes emphasis on the following syllable).

Judith	This A2 that we've been thinking about has been a "cell (hand gesture, box) on the spreadsheet that we were looking at last lesson (…) … Could I put any number into that A2 box? (substitutes 3 and then 7 for A2 in the formula written on the whiteboard) … When we can change a number … we're thinking to write maths down … what do we normally do? Yeah
Pupil	Put *x*
Judith	We usually put an *x*, a missing number, *x*. Just because we were thinking of this as a cell on the spreadsheet it's the "same as using algebra, it's the same as putting a missing number, which we usually put *x* … Could I use any other letter? Could I have put a *q* in there? …
Pupil	You can use any letter
Judith	You can use any letter. So instead of using A2 now we're gonna look at can you do it with any other letter, okay, because that letter just represents any number … So if I change all of these to *x*'s, does it still work? (erases each A2 from the formula on the whiteboard and replaces each with *x*)

Judith emphasised the idea of a variable cell, firstly using the image of the cell as a 'box,' literally replacing A2 with different numbers, and then making the link to standard notation. In their subsequent work on equivalent expressions using letters pupils could draw upon this image of variable. When asked in the stimulated recall interview about the values that A2 and *x* could take for expressions to be equivalent, Jason referred to the link that Judith had emphasised.

Researcher	Could it [*x*] be a decimal number? …
Jason	Yeah, it's the same as A2

Beatrice said that she thought that *x* meant 'any number' but she wasn't sure whether the expressions were equivalent if *x* was a decimal or a negative number. In contrast, when asked about similar work using A2 in her book, Beatrice felt strongly that A2 *could* be a decimal or a negative number in the equivalent expressions. This is an indication of the complexity of pupils' interpretations of notation.

In terms of the rationale for replacing A2 with *x* in the paper and pencil environment, Judith referred to the socio-mathematical norm of using a letter as 'what we normally do' and referred in later lessons to potential confusion if A2 was used on paper. Indeed, in one of the 'Odd one out' lessons, Erin had tried to multiply 4 by A2, and 2 by A2 on paper, but had written the products as 8A2 and 4A2 respectively. She commented on this possible source of confusion in the interview:

Erin	Because in the classroom they might think it's A times 2 … normally in algebra, when you're doubling something, you should put (.) say if it was *x*, it would be 2*x*. So they might think A is just another *x*

Other pupils, however, felt that using A2 on paper would not be confusing because in standard algebra 'it's 2*a*' (rather than '*a*2') and 'algebra letters are never capitals.'

Analysis of the interview with Judith indicates that she sees the variable cell as important in mediating meaning for variable. Further, she pointed to the distinction between changing a number in a cell and filling down a series of calculations.

Judith The "single cell … I think it reinforces the idea of a variable … having one number that can be changed for anything is slightly different than having a number, and an answer, and a "different number and an answer … I can't put my finger on "exactly what, but a few times I've got the impression from what the children have said. They don't "quite see that this calculation here is the same as the next one above but with a different number in your variable position, if you see what I mean. They think of it as a different position almost and therefore not the same, not quite the same variable

Judith's perception of the pupils' construction of meaning is insightful in two important and complementary ways. The variable cell offers the metaphor of kinaesthetic and visual *change*. Pupils *see* that a range of values can be entered into the 'material reference' (Haspekian, 2003) of the cell. They can also *see* that the formula which includes the variable cell *does not change*.

The variable column offers the metaphor of kinaesthetic and visual *dragging*. When a formula is filled down through dragging, pupils *see* a range of values as a list in the variable column. The formulae then include different variable cells: A2, A3, A4 etc. which, in Judith's words are 'not quite the same variable.' In terms of making links to standard algebraic notation, the inter-relation between this series A2, A3, A4 etc. and a single letter is more complex than for the variable cell. Yet an understanding of this relationship can potentially support pupils in their work with literal symbols through the metaphor of dragging. In the context of activity involving the variable column pupils tended to talk about 'the number that is in A2,' focusing on the 'particular concrete' (Haspekian, 2003) interpretation of A2. This is consistent with the fact that pupils tend not to change the number in A2 when they fill down.

Mediation of naming a column

In the 'Myphone' and 'Sum and product' lessons, Judith taught the class how to name a column on the spreadsheet. The aim was twofold: to make clearer the links between spreadsheet notation and standard algebraic notation; and to encourage pupils to see the notation as representing a variable. We do recognise however, as some pupils did, that there is no real reason to name a column on the spreadsheet.

Judith [Discussing the formula =A2*0.16+15] We don't wanna call those cell A2, B2, er, A3, A4, A5 … We want to give them a letter like we would do in the classroom if we're gonna do that algebraically … Now instead of having to write down it's that cell times nought point one six plus fifteen, you can now say it's nought point one six times *x* if you've used *x* or nought point one six times *m* if you've used *m* and add fifteen

Our analysis suggests that naming a column mediated meaning in two ways. Naming a column involves highlighting the column and 'defining' a name. This engages pupils in the *naming* process and provides an image for variable as a range of numbers. The pupils' language reflects this active process, for example 'we had to minus the *a*, because we defined that as the *a* thing.' When asked to explain what *a* or *m* meant in a formula, pupils such as Julian clearly drew upon this image:

Julian A2 is just that individual column (points to a single point), column, like cell thing (makes box shape with hands), the *a* is the actual whole thing column (moves hand up and down)

When writing a formula, rather than clicking on a cell and perhaps ignoring the notation, pupils type the name such as *a* or *m*, engaging directly with the notation.

As well as the image associated with *naming*, the named column (like the variable column) also offers the metaphor of *dragging*. Here, unlike the variable column, the links between notation are clearer, although there are different conventions in standard notation, such as omitting the * for multiply and writing 6*a*, for example, rather than *a*6. For these high attaining pupils the use of a single letter seemed to be a helpful bridge and the differences in convention were unproblematic.

Judith was also positive about naming a column, seeing it as a valuable strategy in making links to paper and pencil algebra and curriculum demands.

Judith The naming of the column I think is successful in terms of them linking the algebra, certainly from my point of view … I think it helps them to see, instead of it's, well it's anything in that column, that it's "n, if you see what I mean … that column is a "variable column as opposed to just somewhere where you put a sum, if that makes sense

Judith also perceived that naming a column helped some pupils to identify the variable and to set out their work.

DISCUSSION

We have discussed the evolution of meaning for variable in one class of high attainers, drawing upon data in which pupils and their teacher were invited to discuss their learning and teaching. In particular, we attempted to analyse the mediation of the variable cell and the process of naming a column under the guidance of the teacher. The dynamic metaphors of *change* and *dragging* together with the process of *naming* appeared to support the evolution of meaning for variable.

Most pupils interpreted A2 as 'any number' in the context of work with variable cells, supporting the findings of Sutherland (1995). However, analysis of discussion around activity involving the variable column suggests that pupils' interpretations are context specific. We recognise the complexity of researching pupils' interpretations and the limitations of our analysis given that there was not space to refer to interview data and observations of pairs. Nonetheless, we suggest that naming a column can potentially support pupils in developing a clearer sense of the notation as a variable.

Whilst some pupils comment that 'when you do spreadsheets then you do algebra,' others, perhaps with a narrower conception of what algebra is, say that 'it doesn't really feel like algebra on the spreadsheet.' Does this matter? Is it a good thing? Is it important to make links to paper and pencil activity? We would not want to suggest that spreadsheet activity is valuable only as a preliminary to introducing the traditional discourse of algebra, but we do suggest that rich spreadsheet activity can be invaluable in supporting pupils' construction of meaning for algebra. Our data points to the important role of the teacher in guiding this evolution of meaning, as illustrated in the words of Erin talking about spreadsheets and algebra:

Erin I see links between them when she [Judith] talks about links between them but when they're like separate then I think they're separate (laughs)

In this ongoing research we are also analysing the mediation of writing formulae and graphing activity in the evolution of meaning for algebra.

References

Ainley, J., Bills, L., & Wilson, K. (2004). Constructing meanings and utilities within algebraic tasks. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education*, (Vol. 2, pp. 1-8). Bergen, Norway: PME.

Calderhead, J. (1981). Stimulated recall: a method for research on teaching. *British Journal of Educational Psychology*, 51, 211-217.

Dettori, G., Garuti, R., Lemut, E., & Netchitailova, L. (1995). An analysis of the relationship between spreadsheet and algebra. In L. Burton & B. Jaworski (Eds.), *Technology in mathematics teaching - a bridge between teaching and learning*. (pp. 261-274). Bromley: Chartwell-Bratt.

Haspekian, M. (2003). Between arithmetic and algebra: a space for the spreadsheet? Contribution to an instrumental approach. *Proceedings of the Third Conference of the European Society for Research in Mathematics Education*. Pisa: Universita Di Pisa.

Kieran, C. (1996). The changing face of school algebra. In C. Alsina, J. M. Alvares, B. Hodgson, C. Laborde, & A. Pérez (Eds.), *Proceedings of the 8th International Congress on Mathematics Education: selected lectures*. (pp. 271-290). Seville: S.A.E.M. 'Thales'.

Mariotti, M. A. (2002). The influence of technological advances on students' mathematics learning.In L. English, D. Tirosh & M. Bartolini-Bussi (Eds.), *Handbook of international research in mathematics education*. (pp. 695-723). Mahwah, NJ: Lawrence Erlbaum

Mariotti, M. A., &. Cerulli, M. (2001). Semiotic mediation for algebra teaching and learning. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education*. (Vol. 3, pp. 343-350.). Utrecht, The Netherlands: PME.

Sutherland, R. (1995). Algebraic thinking: the role of the computer. In L. Burton & B. Jaworski (Eds.), *Technology in mathematics teaching - a bridge between teaching and learning*. (pp. 275-287). Bromley: Chartwell-Bratt.

Sutherland, R., & Rojano, T. (1993). A spreadsheet approach to solving algebra problems. *Journal of Mathematical Behavior*, 12, 353-383.

Wilson, K., Ainley, J., & Bills, L. (2004). Spreadsheet generalising and paper and pencil generalising. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education*, (Vol. 4, pp. 441-448). Bergen, Norway: PME.

A STUDY OF THE GEOMETRIC CONCEPTS OF ELEMENTARY SCHOOL STUDENTS AT VAN HIELE LEVEL ONE

Der-bang Wu — National Taichung Teachers College, Taiwan

Hsiu-lan Ma — Ling-tung College, Taiwan

This study presents partial results from the project "A Study of perceptual apprehensive, operative apprehensive, sequential apprehensive and discursive apprehensive for elementary school students (POSD)", funded by National Science Council of Taiwan (NSCTW, Grant No. NSC92-2521-S-142-004). It was undertaken to explore the geometric concepts of the elementary school students at the first level of van Hiele's geometric though. The participants were 5,581 elementary school students, randomly selected from 23 counties/cities in Taiwan. The conclusions drawn from this study were: (a) It was easier for students to identify straight and/or curved lines due to the obvious distinctions; (b) Students had difficulties in judging rotate figures because of the direction and position concepts; (c) Identifying circle was the easiest for students, triangle next; quadrilateral was the most difficult one.

INTRODUCTION

Geometry is one of the most important topics in mathematics (Ministry of Education of Taiwan (MET), 1993, 2000, 2003; National Council of Teachers of Mathematics (NCTM), 1989, 1991, 1995, 2000). Geometry curriculum is developed and designed according to the van Hiele model of geometric thought (MET, 1993, 2000, 2003).

In 1957, the van Hiele model was developed by two Dutch mathematics educators, P. M. van Hiele, and his wife (van Hiele, 1957). Several studies have been conducted to discover the implications of the theory for current K-12 geometry curricula, and to validate aspects of the van Hiele model (Burger & Shaughnessy, 1986a; Eberle, 1989; Fuys, Geddes, & Tischler, 1988; Mayberry, 1983; Molina, 1990; Senk, 1983; Usiskin, 1982, Wu, 1994, 1995). Most of researchers focus on the geometry curricula of secondary school. To discover the implications of the van Hiele theory for elementary school students. However, it is also very important. The focus of this study is at the elementary level. This research report is one of the six sessions from the project "A Study of perceptual apprehensive, operative apprehensive, sequential apprehensive and discursive apprehensive for elementary school students (POSD)", funded by National Science Council of Taiwan (NSCTW, Grand No. NSC92-2521-S-142-004).

The main objectives of this study were as follows:

1. To determine the passing rate of each geometric shape.
2. To determine the passing rate of each geometric type.

THEORICAL FRAMEWORK

There are five levels of the van Hiele's geometric thought: "visual", "descriptive", "theoretical", "formal logic", and "the nature of logical laws" (van Hiele, 1986, p. 53). These five levels have two different labels: Level 1 through Level 5 or Level 0 through Level 4. Researchers have not yet come to a conclusion of which one to use. In this study, these five levels were called Level 1 through Level 5, and the focus of this study was on Level 1, visual.

At the first level, students learned the geometry through visualization. "Figures are judged by their appearance. A child recognizes a rectangle by its form and a rectangle seems different to him than a square (Van Hiele, 1986, p. 245)." At this first level students identify and operate on shapes (e.g., squares, triangles, etc.) and other geometric parts (e.g., lines, angles, grids, etc.) according to their appearance.

METHODS AND PROCEDURES

Participants

The participants were 5,581 elementary school students who were randomly selected from 25 elementary schools in 23 counties/cities in Taiwan. There were 2,717 girls and 2,864 boys. The numbers of participants, from 1st to 6th grades, were 910, 912, 917, 909, 920, 1,013 students, respectively.

Instrument

The instrument used in this study, Wu's Geometry Test (WGT), was specifically designed for this project due to there were no suitable Chinese instruments available. This instrument was designed base on van Hiele level descriptors and sample responses identified by Fuys, Geddes, and Tischler (1988). There are 25 multiple-choose questions of the first van Hiele level (Part 1); 20 in the second (Part 2); and 25 in the third (Part 3). The test is focus on three basic geometric concepts: triangle, quadrilateral and circle. The result of the first part of WGT was used in this research report.

Twenty-five questions at level one were characterized into nine types based on its geometric attributions. They are Type 1: identification of open and closed figures, Type 2: identification of convex and concave figures, Type 3: identification of straight line and curve line, Type 4: identification of rotate figure, Type 5: identification of figures of different sizes, Type 6: identification of extremely obtuse figures, Type 7: identification of wide and narrow figures, Type 8: identification on the width of contour line, Type 9: identification on filled and hollow figures.

The scoring criteria were based on the van Hiele Geometry Test (VHG), developed by Usiskin, in the project "van Hiele Levels and Achievement in Secondary School Geometry" (CDASSG Project). In the VHG test, each level has five questions. If the student answers four or five the first level questions correctly, he/she has reached the first level. If the students (a) answered 4 questions or more correctly from the second level; (b) reached the criteria of the first level; and (c) did not correctly answer 4 or more questions, from level 3, level 4, and level 5, they were classified as in second level. Therefore, using the same criteria set by Usiskin (1982), the passing rate of this study was set at 80%.

Validity and Reliability of the Instrument

The attempt to validate the instrument (WGT) involved the critiques of a validating team. The members of this team included elementary school teachers, graduate students majored in mathematics education, and professors from Mathematics Education Departments at several preservice teacher preparation institutes. The team members were given this instrument, and provide feedback regarding whether each test item was suitable or not. They also gave suggestions about how to make this test better.

In order to measure the reliability of the WGT, 289 elementary school students (Grades 1-6) were selected to take the WGT. These students were not participants in this study. The alpha reliability coefficient of the first part of WGT was .6754 ($p <$ .001) using SPSS® for Windows® Version 10.0.

Procedure

The one-time WGT was given during April 2004. The class teachers of the participants administered the test in one mathematics class. The tests were graded by the project directors.

The distribution of the questions is in Table 1.

	Triangle	Quadrilateral	Circle
Type 1: open and closed figure	Q 1	Q2	Q3
Type 2: convex and concave figures	Q 4	Q5	Q6
Type 3: straight line and curve line	Q 7	Q8	Q9
Type 4: rotate figure	Q10	Q11	Q12
Type 5: figures of different sizes	Q13	Q14	Q15
Type 6: extremely obtuse figures	Q16	Q17	
Type 7: wide and narrow figures	Q18	Q19	
Type 8: identification on width of the contour	Q20	Q21	Q22
Type 9: identification on filled and hollow	Q23	Q24	Q25

Table 1: The type and distribution of questions in level one

RESULTS

The passing numbers and passing rate for each type and each geometric shape at level 1 were reported in Table 2.

Overall performance on basic figures

From the data of Table2, the total passing rate was 77.5%. The overall passing rates of the triangle concept were 75.88%, 71.49% for quadrilateral, and 85.14% for circle. It seemed that the circle concept is the easiest one for students, followed by triangle concept, and quadrilateral concept.

	Total		Triangle N=5581		Quadrilateral N=5581		Circle N=5581	
Type 1	12289	73.40%	4072	72.96%	3976	71.24%	4241	75.99%
Type 2	14308	85.46%	4750	85.11%	4181	74.91%	5377	96.34%
Type 3	15642	93.42%	5307	95.09%	4932	88.37%	5403	96.81%
Type 4	13213	78.92%	4522	81.02%	3723	66.71%	4968	89.02%
Type 5	11401	68.09%	3047	54.60%	3713	66.53%	4641	83.16%
Type 6	6122	54.85%	3675	65.85%	2447	43.85%		
Type 7	6537	58.56%	3232	57.91%	3305	59.22%		
Type 8	14088	84.14%	4940	88.51%	4706	84.32%	4442	79.59%
Type 9	13686	81.74%	4570	81.88%	4928	88.30%	4188	75.04%
Total	107286	77.50%	38115	75.88%	35911	71.49%	33260	85.14%

Table2: The numbers passed and passing rate of each type and shape

Overall performance on each type

The overall passing rates, from Type 1 to Type 9, were 73.40%, 85.46%, 93.42%, 78.92%, 68.09%, 54.85%, 58.56%, 84.14%, and 81.74% respectively. It seemed that Type 3 is the easiest one for students, followed by Type 8, and Type 9. Type 6 was the most difficult one, followed by Type 7, and Type 2.

Type 1 (Identification of open and closed figure)

The example of Type 1 questions is shown in Fig. 1. The passing rates of the triangle concept were 72.96%, 71.24% for quadrilateral, and 75.99% for circle. It showed that students could easily identify the open and closed figures in circular concept and have difficulties on quadrilateral.

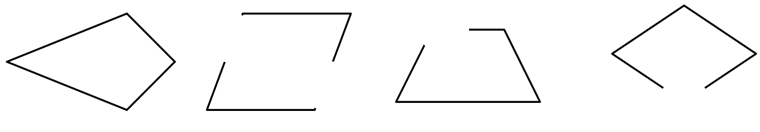


Fig. 1: The identification of open and closed figure

Type 2 (identification of convex and concave figures)

The example of Type 2 questions is shown in Fig. 2. The passing rates of the triangle concept were 85.11%, 74.91% for quadrilateral, and 96.34% for circle. It showed that students could easily identify the convex and concave figures in circular concept and have difficulties on quadrilateral.

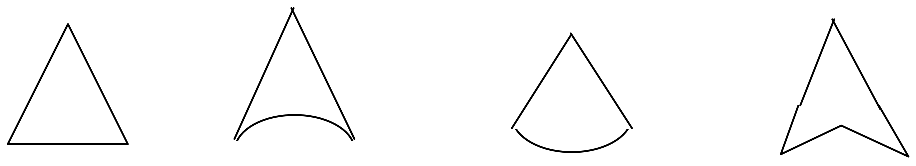


Fig. 2: The identification of convex and concave figure

Type 3 (identification of straight line and curve line)

The example of Type 3 questions is shown in Fig. 3. The passing rates of the triangle concept were 95.09%, 88.37% for quadrilateral, and 96.81% for circle. It showed that students could easily identify the straight line and curve lines in circular concept and have difficulties on quadrilateral.

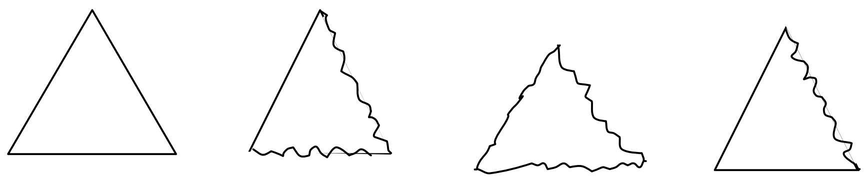


Fig. 3: The identification of straight line and curve line

Type 4 (identification of rotate figure)

The example of Type 4 questions is shown in Fig. 4. The passing rates of the triangle concept were 81.02%, 66.71% for quadrilateral, and 89.02% for circle. It showed that students could easily identify the rotate figures in circular concept and have difficulties on quadrilateral.

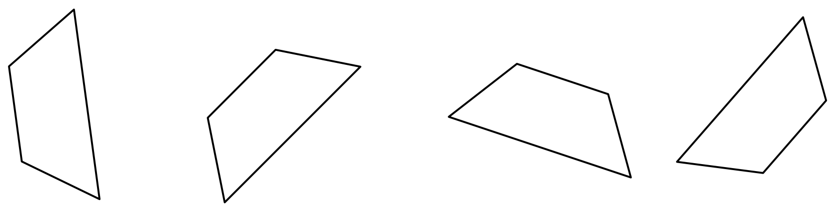


Fig. 4: The identification of rotate figures

Type 5 (identification of figures of different sizes)

The example of Type 5 questions is shown in Fig. 5. The passing rates of the triangle concept were 54.60%, 66.53% for quadrilateral, and 83.16% for circle. It showed that students could easily identify the figures of different sizes in circular concept and have difficulties on quadrilateral.

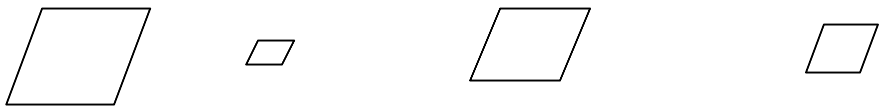


Fig. 5: The identification of figures of different sizes

Type 6: identification of extremely obtuse figures

The example of Type 6 questions is shown in Fig. 6. The passing rates of the triangle concept were 65.85% and 43.85% for quadrilateral. It showed that students could easily identify the figures of extremely obtuse figures in triangular concept and have difficulties on quadrilateral.

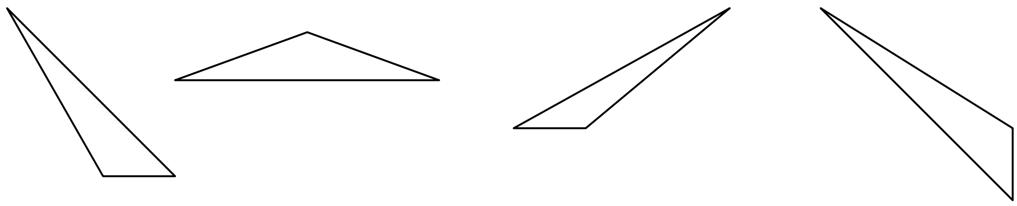


Fig. 6: The identification of extremely obtuse figures

Type 7 (identification of wide and narrow figures)

The example of Type 7 questions is shown in Fig. 7. The passing rates of the triangle concept were 57.91% and 59.22% for quadrilateral. It showed that students could easily identify the figures of wide and narrow figures in quadrilateral concept and have difficulties on triangular.

Fig. 7: The identification of wide and narrow figures

Type 8 (identification on width of the contour line)

The example of Type 8 questions is shown in Fig. 8. The passing rates of the triangle concept were 88.51%, 84.32% for quadrilateral, and 79.59% for circle. It showed that students could easily identify the width of the contour line in triangular concept and have difficulties on circle.

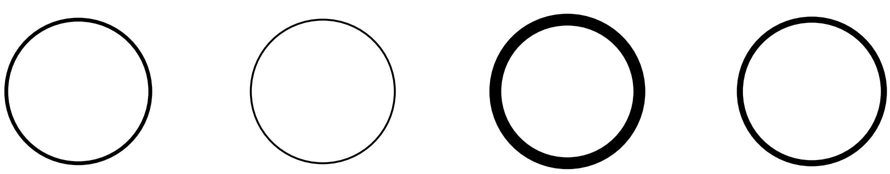


Fig. 8: The identification of width of the contour line

Type 9 (identification on filled and hollow figures)

The example of Type 9 questions is shown in Fig. 9. The passing rates of the triangle concept were 81.88%, 78.30% for quadrilateral, and 75.04% for circle. It showed that students could easily identify the filled and hollow figures in triangular concept and have difficulties on circle.

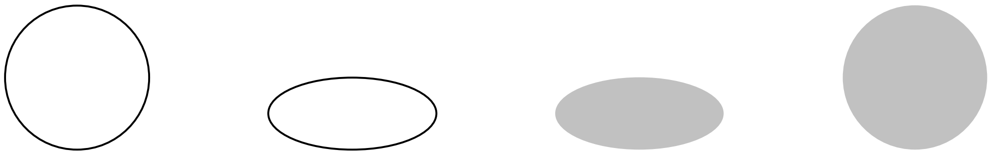


Fig. 9: The identification of filled and hollow figures

CONCLUSION:

At the first van Hiele level (visual), students judged the figures by their appearance. Among these nine different types of figures in this study, Type 3 (identification of straight line and curve line) is the easiest for students and Type 6 (extremely obtuse figures) is the most difficult one. The circular concept is the easiest for students; on the other hand, the concept of quadrilateral is the most difficult to students.

The results of this study identified the easiest and the most difficult concepts for students, it is important to investigate the reason(s) behind this result. The authors of this study are interested to investigate why elementary students have difficulties in identifying extremely obtuse figures. One reason might be that extremely obtuse figures are rarely shown in the textbook, and in their daily lives. Researchers might consider this as their research interests.

ACKNOWLEDGEMENTS:

The research reported in this paper was supported by the National Science Council of Taiwan under Grant No. NSC92-2521-S-142-004. Any opinions, viewpoints, findings, conclusions, suggests, or recommendations expressed are the authors and do not necessarily reflect the views of the National Science Council, Taiwan. The investigators of this research appreciate Dr. K Hsieh for her help to correct this paper of English version. We appreciate Dr. Kai-ju. Hsieh, a co-worker of Dr. Der-bang Wu, who gave many comments on this paper.

References

Burger, W. F., & Shaughnessy, J. M. (1986a). *Assessing children's intellectual growth in geometry* (Final report of the Assessing Children's Intellectual Growth in Geometry project). Corvallis, OR: Oregon State University, Department of Mathematics.

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial Reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420-464). Reston, VA: National Council of Teachers of Mathematics.

Eberle, R. S. (1989). *The effects of instruction on the van Hiele levels of geometric reasoning in preservice elementary teachers.* Unpublished master's thesis, The University of Texas at Austin.

Fuys, D., Geddes, D., & Tischler, R. (1988). *The van Hiele model of thinking in geometry among adolescents.* Reston, VA: National Council of Teachers of Mathematics, Inc.

Ma, H. L. & Wu, D. B. (2000): An Introduction to the van Hiele Model of Geometric Thought. *Journal of Ling-Tung College, 11*, 289-310.

Mayberry, J. W. (1983). The van Hiele levels of geometric thought in undergraduate preservice teachers. *Journal for Research in Mathematics Education, 14*(1), 58-69.

Ministry of Education of Taiwan (MET) (1993). *Curriculum standards for national elementary school in Taiwan.* Taipei, Taiwan: The Author. (In Chinese)

Ministry of Education of Taiwan (MET) (2000). *Grade 1-9 Curriculum Provisional Guidelines—Mathematics.* Taipei, Taiwan: The Author. (In Chinese)

Ministry of Education of Taiwan (MET) (2003). *Grade 1-9 Curriculum Guidelines—Mathematics.* Taipei, Taiwan: Author. (In Chinese)

Molina, D. D. (1990). *The applicability of the van Hiele theory to transformational geometry.* Unpublished doctoral dissertation, The University of Texas at Austin.Senk, S. L. (1983). Proof writing achievement and van Hiele levels among secondary school geometry students (Doctoral Dissertation, The University of Chicago, 1983). Dissertation Abstracts International, 44, 417A.

National Council of Teachers of Mathematics (NCTM) (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA.: Author.

National Council of Teachers of Mathematics (NCTM) (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics (NCTM) (1995). *Assessment standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics (NCTM) (2000). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

Usiskin, Z. P. (1982). *van Hiele levels and achievement in secondary school geometry* (Final Report of the Cognitive Development and Achievement in Secondary School Geometry Project). Chicago, IL: University of Chicago, Department of Education. (ERIC Reproduction Service No. ED 220 288).

Van Hiele, P. M. (1986). *Structure and insight: A theory of mathematics education.* Orlando, FL: Academic Press.

Wu, D. B. (1995). A study of the use of the van Hiele model in the teaching of non-Euclidean geometry to prospective elementary school teachers in Taiwan, the Republic of China. *Journal of National Taichung Teachers College, 9,* 443-474.

Wu, D. B. (1994). *A study of the use of the van Hiele model in the teaching of non-Euclidean geometry to prospective elementary school teachers in Taiwan, the Republic of China.* Unpublished Doctoral dissertation, University of Northern Colorado, Greeley.

Wu, D. B. (2003). *A study of perceptual apprehensive, operative apprehensive, sequential apprehensive and discursive apprehensive for elementary school students* (Final Report of the A study perceptual apprehensive, operative apprehensive, sequential apprehensive and discursive apprehensive for elementary school students Project). Taichung, Taiwan: National Taichung Teachers College, Department of Mathematics Education. (National Science Council under Grant No. NSC91-2521-S-142-004. In Chinese).