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What Gesture and Speech Reveal About Students' Interpretations of Cartesian Graphs: Perceptions Can Bound Thinking

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Students' understanding of graphs constitutes a significant sociocultural skill that is explicitly addressed in the national standards for mathematics education (e.g., National Council of Teachers of Mathematics [NCTM], 2000). From a cognitive perspective, our interest is in the *meanings* people ascribe to graphs, not only in people's performance on graph problems. Whereas some students feel free to interpret images of graphs as partial depictions of possibly limitless patterns that are spatially inscribed, others see only the pattern as it exists within the finite bounds of the drawn graphical image. We describe the latter students as holding a *bounded* interpretation of graphs, and we show how this type of interpretation impedes students' abilities to make predictions of how the pattern will behave well beyond the given information (what mathematics educators often call a *far prediction*). The ways that meanings are ascribed to graphs by those with bounded views highlight the powerful impact perceptual and physical attributes of abstract representations have on people's cognitive processes.

In this paper, we report on the results of a study of middle school students' understandings of graphical representations. We examine students' interpretations of graphs through a cognitive science perspective that emphasizes how things take on meanings, and how meaning relates to reasoning and practice. For this analysis, we draw on assumptions from the embodied cognitive and distributed cognition camps, which view cognition as embodied in and mediated by the perceptions, actions, and interactions people have with themselves and others, with physical artifacts, and with inscriptions. Knowledge and reasoning are distributed among mental processes and structures as well as objects and events in the world. It is of particular significance that the object of our study is the Cartesian coordinate system. The Cartesian graph is a highly conventionalized artifact used to represent explicit and implicit relations between quantities, using spatial and geometric relations as well as words and numbers.

Understanding, interpreting, and applying graphical information are core competencies for middle school mathematics students (e.g., NCTM, 2000). Pre-algebra students build facility with graphical representations as a foundation for topics in algebra, statistics, and calculus throughout the physical and social sciences. The primary objective of this study is to describe middle school students' views of these graphical representations. We focus on students' interview responses in speech and gesture and on the metaphors they evoke in their problem-solving explanations.

Theoretical Framework

Much of our understanding of graph use comes from studies of undergraduate students (e.g., Ratwani, Trafton, & Boehm-Davis, 2003). A study by Zacks and Tversky (1999) supported the notion that the forms of graphs influence students' reasoning. When lines were displayed, undergraduates in their study saw the graph as depicting trends. But when the same data were presented using bars, students no longer saw trends extending beyond the page, but rather tended to contrast the data points. If students view graphs as spatially limited, they may struggle to generalize beyond the information presented. In this vein, Stevens and Hall (1998) described a tutoring session on the Cartesian coordinate system in terms of the embodied practices exhibited by the interlocutors. They showed how a student's interpretation of a graph was influenced by its spatial relation to the grid edges, as the student predicted how an entire function would change in appearance when an equation was transformed. The tutor literally blocked the student's view of the grid edges (but kept the line on the graph exposed) to alter the student's perception and strategy: "What happens if I chopped off this paper and you start, and you only saw a little bit of these graphs?" (p. 120). To the tutor's surprise, the student believed this would result in "a different graph" (p. 120), suggesting that his "view" of the function was inseparable from its perceived relationship to the grid.

A significant body of work has exposed how vulnerable cognitive processes are to the geometric and physical properties—the directly perceived affordances—of objects and representations. Roger Shepard and colleagues (Cooper & Shepard, 1973; Shepard & Metzler, 1971) found that people took longer to make similarity judgments about visual displays (novel three-dimensional figures or English letters) as the angular disparity between the two figures or letters increased. Others (Kosslyn, Ball, & Reiser, 1978; Finke & Pinker, 1982) showed that the time it took to scan a mentally stored image of a display (such as a fictitious island with landmarks) matched the time it took to scan an actual picture, even though prevailing information-processing theories of the time suggested that mental encoding of the displays (into, say, propositions) could eliminate the spatial relations. The impact of the form of representation on subsequent perception and action has been called *representational determinism* (Zhang, 1997, p. 213). Novices are particularly vulnerable to salient characteristics of unfamiliar representations (such as shape), and, consequently, these features exert tremendous influence on their perceptual and cognitive processes.

We were also interested in how students use metaphors to describe graphical representations. Our attention to student metaphors was inspired by work from Lakoff and Núñez (2001), who argued that the meaning of mathematical formalisms is best assimilated when abstract mathematical ideas are grounded in familiar, axiomatic concepts through metaphor. These metaphors may give further clues to students' views about the boundedness[1] of graphical representations.

Hypotheses

The following hypotheses emerged through our analysis of interview data on the tasks described below:

1. Students convey views of the boundedness of graphical representations of patterns through speech and gesture;

2. Students' perceptions of boundedness are associated with their reasoning and performance on prediction tasks;

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[1] By *boundedness*, we mean the extent to which students see graphical representations of patterns as bounded or not bounded. For additional discussion, see the Method section below.

3. The complexity of the task further influences the relationship between students' perceptions and performance; and

4. Students' perceptions of graphs are also reflected in the metaphors they use to describe them.

Method

Thirteen students participated in videotaped interviews. The students, in sixth through eighth grades, were from a large urban middle school with high percentages of non-Caucasian students (92%) and students in the free/reduced-price lunch program (86%). During the interviews, the students worked on three problems, each containing several parts. To focus on our hypotheses, we present data from the two far prediction (FP) tasks. Making and justifying FPs are typically regarded as involving one of the most advanced levels of student reasoning about graphs (Carpenter & Shah, 1998; Curcio, 1987; Friel, Curcio & Bright, 2001) since they involve generalization among implicit and explicit relations.

We are aware that the small sample size constrains our ability to make statistical inferences that would allow us to generalize from this data set. However, focusing our efforts on a rich data set from a small sample allows us to examine student thinking and the formation of meaning and perceptions in great detail.

Our analyses of the interview data led us to formulate the constructs of *boundedness* and *unboundedness* to describe students' interpretations of graphical representations. *Bounded* views of a graph are articulated when a student, through speech or gesture, demonstrates that the graphical information is constrained by the drawn boundaries of the graph, so extrapolation is not possible. *Unbounded* views about a graph manifest when a student, through speech or gesture, reasons beyond the boundaries of the graph. Unbounded verbal responses represent an attempt to make an FP outside the numerical boundaries of the graph, regardless of whether the attempt is successful. FP tasks can ask students to reason about values that are beyond both axes (*high complexity*) or beyond only one of the axes (*low complexity*). When examining performance on FP tasks, we took unbounded verbal responses as an indicator of success. Our gesture and speech methodology draws from the theoretical work by Goldin-Meadow (2003) and Alibali and Goldin-Meadow (1993) that shows how gestures can provide an additional window into students' thinking. Gestures can reveal the dominant conceptualization when it differs from that revealed by speech.

Problem 1 (Figure 1) asked students to interpret a two-dimensional, Cartesian coordinate graph with four points plotted along a linear function, $y = 3x + 1$. In Problem 1a, students were asked to determine the cost of making 10 copies of a CD using the information shown in the graph. The correct answer (31) is beyond the range of the y-axis on the graph, so this is considered a one-dimensional (low-complexity) FP task. Problem 1b asked students to determine the cost of making 31 copies of a CD. This task is a two-dimensional (high-complexity) FP task, since the answer (94) is beyond the numerical limits of both of the graph's axes. Problem 1b is more difficult than Problem 1a, as we will show empirically.

In Problem 2a (Figure 2), students were asked to describe a single horizontal axis labeled in units of one from 0 to 14. In Problem 2b (Figure 3), students were asked to describe a single vertical axis labeled in units of 10 from 0 to 280.

Figure 1. Problems 1a and 1b.

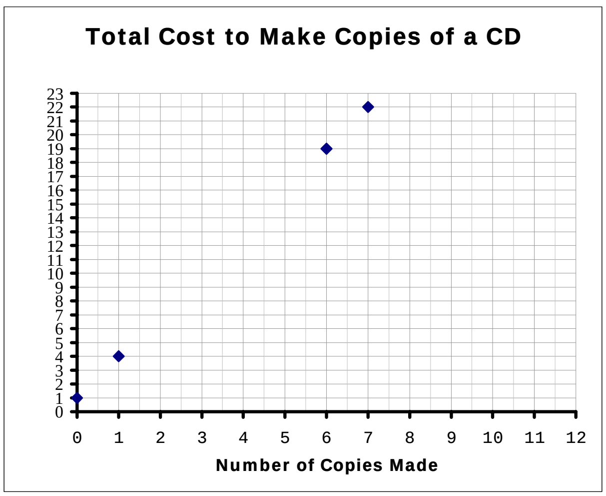


Figure 2. Problem 2a.

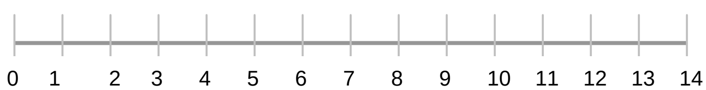
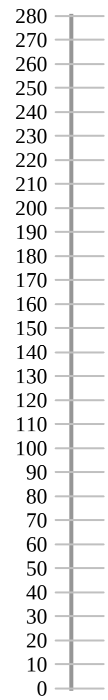


Figure 3. Problem 2b.



Results

We developed a scheme for coding students' speech and gestures. Verbal responses on the FP tasks were coded initially as correct or incorrect and then as bounded or unbounded. A verbal response was coded as bounded if it indicated that the information presented in the graph could not be extrapolated to values outside the graph's boundaries. An unbounded verbal response, on the other hand, indicated that the graph information could be extrapolated to values beyond the limits of the graph. Table 1 provides examples of responses coded as bounded and unbounded. Student gestures were also coded for boundedness.

Student gestures were considered to reflect a bounded view of the graph if they referred to the space within the graph or if they referred to or included the boundary edges of the graph. Gestures were coded as unbounded if they referred to parts of the page that extended past the graph boundaries.

For Problems 2a and 2b, a list of students' metaphorical descriptions were coded as bounded or unbounded depending on the inherent nature of the source concept or object (Table

4). Note that some of the concepts or objects can be thought of in both bounded and unbounded ways.

Table 1
Examples of Unbounded and Bounded Responses From Problem 1b

Unbounded	Bounded
"Um, since it was $31 for 10 and it was asking for 31, I multiplied how much it cost for 10 by 3. Got 93. And added 3 to make 31 copies."	"I did it 5 times and that equals 30 so I counted how, each 6 of these and it equaled 24."
"OK, well 31 copies is kind of a lot of CDs and 310 bucks be for like cheap CDs, but if it'd be like, if you want get this out of the expensive CDs, then it'd be about like 1,000 bucks just to get it done."	"Like, you could like expand the graph to 31 copies and be like, like 23, 24, you know, to see what it would do."

Of the 13 students in our sample, 6 were successful on the low-complexity FP problem (1a), whereas only 3 were successful on the high-complexity FP problem (1b), confirming that the second problem was indeed the more difficult of the two. Recall that, based on our criteria, none of those who expressed a verbally bounded view of the graph were successful.

On the low-complexity FP problem (1a), 9 students (69%) provided an unbounded verbal response and therefore had some opportunity for success. As shown in Table 2, 6 of these 9 students were ultimately successful, even though 5 exhibited a bounded view through gestures. On the low-complexity problem, boundedness as indicated by gestures did not predict success once students exhibited an unbounded view verbally.

Table 2
Performance on Problem 1a

***N* = 9**	No. (%) correct	No. (%) incorrect
Unbounded gestures	**2 (22%)**	1 (11%)
Bounded gestures	4 (44%)	**1 (11%)**
No gestures	0	1 (11%)

On the high-complexity FP task (Problem 1b), 10 students (77%) gave unbounded verbal responses, but only 3 of those 10 (30%) were successful. For this item, the boundedness of the gestures that accompanied students' speech did appear to be predictive of their FP problem-solving performance (Table 3), showing a modest correlation, $r = 0.32$. Even with an unbounded verbal response, when gestures indicated a bounded view, students were 4 times more likely to make an incorrect FP. In contrast, students were twice as likely to produce the correct FP with unbounded gestures.

Table 3
Performance on Problem 1b

***N* = 10**	No. (%) correct	No. (%) incorrect
Unbounded gestures	**2 (20%)**	1 (10%)
Bounded gestures	1 (10%)	**4 (40%)**
No gestures	0	2 (20%)

Although the students who expressed a verbally bounded view were not successful on either FP task, the gestures accompanying their bounded solutions may provide insight into their understandings of the graphical representations in Problems 1a and 1b.

The following transcript describes an episode between the interviewer (*I*) and a student (*S*) in which *S* describes a solution to Problem 1a, for which *S* provides a verbally bounded response. When *I* asks *S* about her solution to Problem 1b, *S* provides a verbally unbounded response, but her gestures remain bounded and constrain her reasoning.

A brief note about transcription conventions: Slashes (/) indicate momentary pauses in speech. Numbers in parentheses indicate the amount of time, in seconds, for longer pauses. The asterisk (*) indicates self-interruptions, self-corrections, and restarts by the speaker. Finally, to identify specific gestures during speech, numerical indexes are given in square brackets and then explicated following each line in the transcript.

Problem 1a:
I: So how did you answer that one?
S: I figured that maybe it's the same as this over here [2] / (2) and I put 23 down. [3]
    [2]: touches pencil tip to dot for (7,22)
    [3]: moves right hand holding pencil off to the right side gesturing in an "I don't know" fashion

Problem 1b:
S: And this one for the 31 [4]
    [4]: touches eraser tip to 31 on question
S: copies, since each [5]
    [5]: touches eraser tip to x *= 6*
S: <uh> 6 of the CDs costs you $19. [6]
    [6]: draws pencil up gridline to dot for (6,19)
I: Hmmmm.
S: I times that [7] times 6
    [7] returns eraser tip to x *= 6*
S: which *is*/ *(1.9)[8]*
    [8]: lifts eraser tip off paper slightly
S: oh no / I did it / (2) [9]
    [9]: points to work on response
S: 5 times and that equals 30 [10]
    [10]: returns eraser tip to x *= 6 and draws pencil up to dot for (6,19)*
S: so I counted how each 6 of these # and it [11] equaled 24.
    [11]: returns eraser tip to response and touches each of the five iterations of (6,19) on response

The reasoning given by this student is representative of that of other students whose verbal responses and gestures were bounded on Problem 1a, but who provided a verbally unbounded response on Problem 1b. For Problem 1a, *S* must find the total cost (y value) to make 10 copies of a CD ($x = 10$). This is an FP because the answer (31) is greater than the maximum value displayed on the x-axis, though the function still includes 10 in its domain. *S* gives an answer of $y = 23$ and describes her solution strategy by referring to a specific point nearest the value of $x = 10$—namely, the ordered pair (7,22) (see gesture motion [2])—and then assuming that the answer is "the same as this over here." Clearly, *S* believes that the value of y when $x = 10$ must be greater than 22, but she relies on reading the graph for information. Throughout *S*'s explanation, her gestures remain within the physical boundaries of the graph grid.

S provides an unbounded verbal response to Problem 1b but still shows bounded gestures. Problem 1b is a more difficult FP task, and the answer lies beyond both the *x*- and *y*-axes as drawn. *S* uses a combined recursive and proportional reasoning strategy in which the x and y coordinates of a given point—namely, (6,19)—are added iteratively a requisite number of times—here, 5 (since $30 = 5 \times 6$)—to obtain the x value sought in the problem. Clearly, there are some errors evident in *S*'s computational process. *S*'s method only allows her to compute the *y*-value for $x = 30$ (not 31; though some students augment this method by adding 1 to x—since $31 = 5 \times 6 + 1$—and multiplying the associated y value by the same amount). *S*'s bounded view of the graph is evident in her gesturing, which remains within the grid area of the graph or moves to the calculation/response area of the second page. Our interpretation (see gestures [7], [8], and [10], above), as supported by the pattern of codes we assigned, suggests that *S* believes the answer must lie in the grid space provided, or perhaps *just* outside it (hence, she selects 24 as her response). The mismatch between the unbounded speech and bounded gestures suggests that *S* is of two minds: she sees a solution path that uses proportional reasoning and recursion from prior points in the pattern, but she also looks to the graph to *contain* the answer.

Table 4 lists the descriptions of the graphs students generated for all problems, along with their corresponding frequencies and boundedness codes. For Problems 1a and 1b with the Cartesian coordinate system, only one student described the graph using a metaphor—namely, a line plot. The line plot metaphor is considered bounded since it is most frequently used with a finite set of data. For Problems 2a and 2b, students used a wide range of metaphors to describe the graphs. One salient feature of the data is that there are an equal number of metaphors coded as bounded or unbounded. Students seem to be equally divided in their interpretations of the boundedness of single axes.

Table 4
Frequency of Metaphors Used to Describe Graphs for FP Tasks

Description	Problems 1a & 1b	Problem 2a	Problem 2b	Code
Graph	6	0	0	n/a
Describe features	6	0	0	n/a
Line plot	1	0	0	Bounded
Ruler	0	2	1	Bounded/unbounded
Finite number line	0	1	0	Bounded
Number line	0	4	1	Unbounded

Description	Problems 1a & 1b	Problem 2a	Problem 2b	Code
Axis	0	1	3	Bounded/unbounded
Finite timeline	0	1	1	Bounded
Timeline	0	1	0	Unbounded
Positives	0	2	0	Bounded
Thermometer	0	0	3	Bounded
Bar graph with #'s	0	0	1	Bounded
Times table	0	0	2	Unbounded

Discussion and Conclusions

The evidence suggests a relationship between gestural unboundedness and correct performance on FP tasks with two-dimensional graphs for the high-complexity problem (Table 3). Although students who exhibited unbounded speech and gesture were only slightly more likely to make a correct prediction than those who did not, students who exhibited *bounded gestures* were much more likely to predict incorrectly. Goldin-Meadow (2003) defines the combination of bounded gesture and unbounded utterances as a mismatch in coding between gesture and speech and argues that this shows students can be of "two minds" about the phenomena. While we are not making developmental claims here, as Goldin-Meadow and her colleagues do, we are suggesting that students' gestures may provide valuable insight (above and beyond speech) into how students conceptualize graphical representations. The bounded gestures in the high-complexity problem are most indicative of students' FP performance, suggesting that gestures may reveal their dominant views about the graph's meaning.

What needs to be explored further is how students' choices of strategies affect their performance on these FP tasks. In the low-complexity problem (Problem 1a), more students with bounded gesture answered the question correctly than students with unbounded gesture. Preliminary analyses of strategy choices for this problem reveal that adopting a strategy using a computational method (e.g., recursion), rather than a spatial strategy (e.g., following the linear relationship) helped students circumvent their bounded views of the graph to obtain a correct answer. Observations of strategy choices for the high-complexity problem (Problem 1b) indicated that students were less likely to use recursive or numerical patterns, perhaps due to the labor-intensive nature of applying such strategies to the high-complexity FP task. Strategy choice may be an especially important component of successful performance on graphical interpretation tasks for students with bounded views of the graph.

The range of metaphors students provide to describe graphs may offer further insight into how they link graphs to their prior experiences. When students relate graphs metaphorically to other objects or ideas, the inherent boundedness of such objects (e.g., thermometers) and ideas (e.g., number lines) as source domains may also influence the boundedness of students' views of graphs.

These analyses identify perceptions and interpretations used by students that appear to influence student reasoning. Instructional sensitivity to the gestural modality during students' solutions and explanations may uncover deeper understandings about graphical representations that may be masked by their verbal responses. In addition, it may be useful for instructors to select mathematical activities that are designed to distinguish between bounded and unbounded interpretations of graphs. Awareness of the role and meaning of students' gestures contributes to teachers' pedagogical content knowledge, and can be used to inform mathematics instruction and assessment practices. The metaphors for graphs evoked by textbooks, lessons, and students themselves also merit focus, as metaphors, too, may shape and indicate students' perceptions of mathematical representations.

Generalization of this work is limited by the small sample size, necessitated by the intensity of the micro-level analyses. In future work with more students, we will explore the effects of discrete versus continuous representations and the ways in which graphical interpretation develops from a bounded to an unbounded view. There is a rich body of work that describes how student gesture conveys developmental transitions by identifying gesture-speech mismatches (Goldin-Meadow, 2003). Drawing on our coding scheme, we hope to identify these transition points and extend work (e.g., Stevens & Hall, 1998) on how perception informs students' understanding of graphs and other external representations.

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