Lane Andrew

Reasons Why Students Have Difficulties with Mathematical Induction

University of Northern Colorado

Department of Mathematical Sciences

2210D Ross Hall

Greeley, Colorado, 80631

Tel.: (303) 420-6829

Fax: (970) 351-1225

Lane.Andrew@unco.edu

Reasons Why Undergraduates have Difficulty with Mathematical Induction

It is generally agreed among researchers and teachers that students have difficulty understanding the notion of mathematical proof. Unlike more everyday notions of "proof" which are sometimes based on personal experience and empirical checking, mathematical proof in most undergraduate mathematics classes is based on deductive arguments. A deductive argument is one that uses principles of formal logic, where certain assumptions are made, and from these assumptions a conclusion is arrived at. The transition from assumption to conclusion is guided by sequences of implications which are true by logical necessity. This way of thinking is not necessarily natural to students, and although they are often encouraged to think about mathematics in ways that make sense to them, most often they are required eventually to submit a formal deductive argument as proof for a mathematical proposition. Also, there is some evidence to suggest that developmentally, some students are not prepared to think deductively. In fact, students are often persuaded that a mathematical statement is true by checking some cases, appealing to the authority of the text or teacher, or simply accepting work that looks "mathematical." There is also reason to believe that students cannot always tell if a purported proof is indeed a correct or incorrect, which might suggest that if these students were to attempt to write the proof themselves, they would have considerable difficulty. The issues just described pertain to all kinds of proof related tasks, methods, and mathematical contexts; however, it might be helpful to limit our discussion to the context of proofs by mathematical induction, one of the most common proof techniques. This analysis will shed light on why students have difficulty with this specific method of proof.

Mathematical induction is a heavily emphasized when studying discrete mathematics, number theory, and their applications. The proof technique works in the following way: A proposition is put forth which involves a countable, infinite set. For example, $2n! > 5n + 1$ for $n = 4, 5, 6, \ldots$. Let this proposition be called P(n). The goal of the proof is to demonstrate an implication exists between any two consecutive elements in the set. That is, one needs to establish that $P(k) \Rightarrow P(k+1)$ for an arbitrary $3 < k$. This is often called the inductive step. Once this implication is shown, P(n) is shown true for the least element in the set under consideration. In our current example, this would be P(4). This is frequently called the base case. Since P(4) is true, one can reason from the implication that P(5) is also true. Reusing the implication suggests that $P(5) \Rightarrow P(6)$ is true, and so forth. It then follows that P(n) is true for all natural numbers $n > 3$.

As one can see, in order for a student to successfully write and understand a mathematical induction proof, they must accomplish several subtasks (such as working out the base case, and proving the inductive step), as well as grasping how all these subtasks work together to prove the original proposition. The heart of deduction in the proof lays in establishing the inductive step. Most often this task amounts to manipulating algebraic expressions involving factorials, inequalities, series, or rational expressions. With this in mind, the results of Gibson's (1998) work are of interest.

Gibson worked with a cohort of mathematics majors in an Advanced Calculus class as they worked on proving certain statements. Although he did not encourage students to draw pictures or diagrams as they went about the proof tasks, he was interested in the work of students who did use diagrams. He found that students had considerable difficulty when it came to interpreting the symbolic notation often present in the statement to be proved, and also the crafting of proofs using this same notation. In particular, he found that students would often draw diagrams to represent the statement to be proved, so that they might understand what the statement meant and then why it was true. Students commented that when they were thinking of the mathematics of the statement, they were thinking in terms of diagrams, and so they felt it would be helpful to “download” their mental images onto paper. Thus, visual representations were found to be more consistent with students’ internal understandings of mathematics (which tended to consist of visual representations of mathematical situations as opposed to symbolically defined propositions). This could be one reason why mathematical induction is so difficult for students—often times the proposition to be proved is algebraic and not readily converted to a visual representation. This is definitely true of statements like: $2n! > 5n + 1$ and $(7)(8)+(7)(8^2)+\ldots+(7)(8^n) = 8(8^n - 1)$. There are exceptions to this. For example, $1+3+5+\ldots+(2n-1) = n^2$, where square arrays can be created by summing consecutive odd numbers.

Another item Gibson found was that students would often take their diagram and use it to help them generate ideas for starting the proof, because the diagram suggested why the statement itself was true. Once the diagram was understood, they had a better chance of writing a correct formal proof by simply translating components of their diagrammed situation into formal symbolic notation. Thus the visual was used as a guide, and provided the structure for the proof. In mathematical induction, often times proving the implication $P(k) \Rightarrow P(k+1)$ amounts to tricks (such as adding and taking away the same number) and tedious algebraic moves whose only motivation is in arriving at the conclusion. In other words, in some statements the underlying mathematics which explains why the statement is true is not used as a guide when establishing the inductive step. Thus, after one completes the proof by mathematical induction, they have not gained any real insight into why the statement works mathematically.

It must be noted, however, that proofs by induction can be either explanatory or non-explanatory. For example, consider the following statement and its proof by mathematical induction.

Theorem: $6 \mid (7^n - 1)$, for any natural number n.

Proof: By mathematical induction.

Basis: $6 \mid (7^1 - 1) = 6$ is true.

Inductive hypothesis: assume $6 \mid (7^k - 1)$

Inductive step:

$$7^k - 1 = 6q \Leftrightarrow 7^{k+1} - 7 = 6q \cdot 7 \Leftrightarrow 7^{k+1} - 1 - 6 = 6q \cdot 7 \Leftrightarrow 7^{k+1} - 1 = 6q \cdot 7 + 6 = 6(7q + 1) = 6z$$

End of proof.

No substantial insight is given by this proof explaining why 6 should divide $7^n - 1$. One can focus locally on the inductive step of the proof, manipulating the inductive hypothesis using a “trick” (adding 6 to both sides of the equation) and then noticing 6 factors out of both terms on the right hand side of the equation, thus completing the proof.

On the other hand, there are many instances in which mathematical induction proofs are explanatory, elegant, and powerful. Take, for example, the following Heavy Coin Problem, and the following proof:

Suppose you have 3^n coins that look identical, however, one of the coins is heavier than all the others. Show that you can find out which coin is heavier by using a balance no more than n times.

Proof: By mathematical induction.

Basis: When n = 1, we have 3 coins. Put one coin on the left of the scale, one on the right, and one on the ground. With this arrangement, the heavy coin is found in one weighing.

Inductive hypothesis: Assume we have 3^k coins, and that we can find the heavy coin in k weighings.

Inductive step: Since $3^{k+1} = 3 \cdot 3^k = 3^k + 3^k + 3^k$, we can split the 3^{k+1} coins into three piles of 3^k coins each. Put one pile on the left of the scale, the other on the right, and the other on the ground. Thus, in one weighing, we can determine the heavy pile of 3^k coins. By the induction hypothesis, if we have 3^k coins, we can weight them in k weighings. Thus, we can find the heavy coin in k+1 weighings.

End of proof.

This proof gets at the heart of what is going on with the coins—when we have 3^n coins, we can always split them up into three equal piles. A scale can then be used to distinguish the heaviest of three equal piles in one weighing. Although there are other ways to approach this problem, the inductive approach is not only explanatory, but arguably the most elegant.

Although deductive thinking using algebraic expressions does not mirror students' inner thought processes as well as more visual representations of the same situation, students can and do learn to present deductive arguments eventually. It's interesting to note the work of Senk (1989) in this regard, since her work suggests that students must pass through levels of deductive understanding. Senk (1989) studied groups of high school geometry students in regard to their Van Hiele levels of understanding geometry. There are five levels in this scheme, and described very briefly as:

Level 0: students distinguish different shapes based on global features

Level 1: students classify shapes based on specific properties they observe.

Level 2: students understand how shapes are defined based on particular properties of the shape.

Level 3: students deductively prove statements using theorems and definitions.

Level 4: students prove statements across different axiomatic systems.

Senk found that students who entered high school geometry at Van Hiele level 0 or 1 were not able rise to a level 3 understanding by the end of the year. If we assume Van Hiele levels are functions of classroom instruction, and that different students receive different qualities and quantities of mathematical instruction, this might indicate why certain students seem cognitively unprepared to engage in mathematical induction tasks, which clearly reside at Van Hiele level 3.

Another aspect to stages of deductive thinking is the contrasting differences between experts and novices when it comes to writing proofs. For example, Weber (2001) conducted a study involving undergraduates and graduates, where each group was given standard abstract algebra statements to proof during an interview session. The graduate students seemed to instinctively know which theorems would be important to use in constructing the proof, while the undergraduates randomly recalled any and all theorems which were related to the proposition and then used each one, hoping it would be useful in accomplishing the task at hand. In terms of mathematical induction, this suggests that some students might not understand which direction to begin when dealing with the inductive step, or even what theorems might be important to establish the induction step. For example, in working with the expression 2^(k+2), they might not recognize that 2^(k+2) can be split into 2^k times 2 times 2—perhaps an important part of establishing an inductive step.

Furthermore, the graduate students in Weber's study knew what techniques or proof methods would be the quickest and most efficient when faced with certain types of questions. In the context of mathematical induction, this might suggest that when students are given a proposition, they might not even realize that induction is a technique for proving the statement. For instance, suppose the one needs to show 7^(k+1) – 1 is divisible by six, given that 7^k – 1 is divisible by six. The student may not realize that a common strategy in problems like this is to add and subtract the same number.

Students might be at different levels in terms of how able they are to reason deductively, and furthermore, evidence suggests that they also differ in how they perceive certain types of proof. What convinces an individual student that a statement has been proved mathematically is not necessarily the same as what would convince a mathematician or instructor. For example, Martin and Harel (1989) demonstrated that preservice teachers held empirically-based notions of proof; a statement is true if it is verified for several random examples. They also found a certain element of distrust for deductive arguments. Thus, even when students were shown a correct deductive proof, they felt they needed to check more examples just to be sure the theorem was really true. Sowder and Harel (2003) have formulated a taxonomy of what they call student "proof schemes.” These descriptions consist of the types of arguments that students find convincing. In the context of mathematical induction, at least two proof schemes are noteworthy. First, the so-called empirical proof scheme is, as the name suggests, based on verifying that certain numbers “work.” A student with such a scheme might find it unnecessary to prove a statement such as $2n! > 5n + 1$, since after checking several examples it is “obviously” true. Secondly, the symbolic, non-quantitative proof scheme relies on the arrangement of purely symbolic statements which may be manipulated (according to personal or common-sense transformations) into what is desired. This proof scheme often contributes to errors made in which students perform operations on symbols with no thought as to what those symbols represent (e.g. $[3x + 4]/ 3 = x + 4$). Students with this proof scheme might see mathematical induction as an exercise is symbol-pushing, instead of logical deduction, and their work prone to a host of illegal algebraic transformations that are passed over when they are examining their own work. In fact, this proof scheme could potentially contribute to another phenomenon discovered by Selden and Selden (2003) in their work with proof validation.

These authors found that university students have considerable difficulty determining if a given proof is valid. They noted that many students “talked a good line” when it came to how they said they validated purported proofs, but when it came down to actual validation, these students focused almost exclusively on local algebraic manipulations. This suggests that these students had a symbolic, non-quantitative scheme of proof, in that a proof was correct as long as each step, and each chain of inference was correct in isolation to an overall proof structure or strategy. That is, a proof that was correct in each step, yet proved the converse of the theorem, it still seen as correct.

It appears than, that students often see a mathematical proof as an exercise in algebraic gymnastics, or simply the ability to follow proper mathematical form. Mingus and Grassl (2006) noted such a phenomenon in their examination of practicing secondary teachers when they graded student proofs. The researchers found that large numbers of teachers gave full credit to induction proofs which did not even use the induction hypothesis, but displayed sophisticated (and correct) symbolic manipulations.

In summary, there are many reasons why students have difficulty with mathematical induction proofs. In particular, deductive reasoning appears to be unnatural to many students would prefer visual representations. Also, individual students within the same class appear to arrive at the beginning of the semester at different levels of cognitive thinking ability, and are somewhat limited in the progress they can make within normal educational situations. This is intensified by personal student views of what makes a convincing argument, such thinks as verifying by plugging in numbers, or graphical pictures, or sophisticated, yet incorrect, symbolic manipulations. This is also evidenced in a different way when students validate proofs based on surface features, and ignore the overall structure of the proof. Considering the importance of proof within mathematics, these misconceptions held by students need to be resolved.

References:

Gibson, D. (1998). Students’ use of diagrams to develop proofs in an introductory analysis course. *Issues in Mathematics Education, 7*, 284-307.

Martin, W., & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education*, *20*, 41-51.

Mingus, T., & Grassl, R. (2006). An examination of practicing a prospective teachers' validation of purported proofs by mathematical induction. Submitted for publication.

Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education*, *34*, 4-36.

Senk, S. (1989). Van Heile levels and achievement in writing geometry proofs. *Journal for Research in Mathematics Education*, *20*, 309-321.

Sowder, L., & Harel, G. (2003). Case studies of mathematics majors' proof understanding, production, and appreciation. *Canadian Journal of Science, Mathematics & Technology Education*, *3*, 251-267.

Weber, K. (2001). Student difficulty in constructing proof: The need for strategic knowledge. *Educational Studies in Mathematics, 48*, 101-119.