Preservice Elementary Teachers Use Drawings and Make Sets of Materials to Explain Multiplication and Division by Fractions

A Research Study Presented at the 2nd Annual Preparing Mathematicians to Educate Teachers (PMET) Conference at Oswego, New York, June 6, 2006

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Structured Abstract

Background: Multiplication and division by fractions are among the most troublesome concepts in the elementary mathematics curriculum. Recent studies have shown that preservice elementary teachers in the United States do not have deep understandings of these concepts. Effective ways to improve preservice teachers' conceptual understanding of these concepts need to be identified.

Citation: Rule, A. C., & Hallagan, J., editors. (2006). Preservice elementary teachers use drawings and make sets of materials to explain multiplication and division by fractions. A research study presented at the 2nd Annual Preparing Mathematicians to Educate Teachers (PMET) Conference at Oswego, New York, June 6, 2006.

Conclusions: The two activities increased student understandings of multiplication and division by fractions. Although students improved through the activities, many students' understandings were still incomplete. More than two focused activities are needed to ensure deeper understanding of concepts. Preservice teachers need concrete experiences with these concepts in their mathematics classes as well as in mathematics education coursework.

Control or Comparison Group: Both the control group and the experimental group consisted of preservice teachers from several sections of the same instructor's undergraduate mathematics methods courses and were matched on pretest scores. Both groups completed the homework assignment in which they used drawings to illustrate multiplication and division by fractions. The instructor did not present lessons on these concepts to the classes until after the posttest had been completed so that the effects of these activities would not be confounded. The experimental group completed the additional activity of making hands-on materials to model these concepts. The study examined the increase in preservice teachers' conceptual understanding of multiplication and division by fractions through the two activities.

Data Collection and Analysis: Both control and experimental groups were assessed with identical pretest/ posttest instruments constructed by the investigators to determine both procedural knowledge of solving equations involving multiplication and division by fractions and conceptual knowledge of writing equations for story problems and using drawings to illustrate concepts. Posttest scores, student work on the assessments, drawing assignment, and hands-on materials were examined along with student comments on a survey that asked what subjects learned from participating in the intervention activity.

Findings: The two activities improved preservice teachers' understandings of these concepts as revealed by the change in scores from pretest to posttest (50.8% on pretest to 67.5% and 71.4%). Those who completed both assignments scored somewhat higher (71.4% compared to 67.5%) than those who only completed the drawing assignment, but this difference was not statistically significant. Preservice teachers reported that their understandings of these concepts improved through the activities.

Intervention: Both control group and experimental group participated in composing story problems with drawings to illustrate multiplication and division by fractions. The experimental group completed the additional activity of making hands-on materials with accompanying story problems to model multiplication and division by fractions.

Purpose: The purpose of the study was threefold: 1) to investigate the effectiveness of two activities in helping preservice teachers develop deeper understandings of multiplication and division by fractions; 2) to identify typical errors preservice teachers make and identify difficulties they encounter while learning these concepts; and 3) to provide examples of drawings and hands-on materials that effectively model multiplication and division by fractions for others to use in learning and teaching.

Research Design: The study was a pretest - intervention - posttest design with control and experimental groups. Because lower-performing students tended to volunteer for the extra-credit activity (the intervention for the experimental group), blindly matched groups were formed on pretest scores.

Setting: Preservice teachers from three mathematics methods classes of college students majoring in elementary education at a mid-sized college in central New York State during the spring semester of 2006.

Study Sample: Forty-two white preservice elementary teachers enrolled in a mathematics methods course. The experimental group consisted of 18 females and 3 males; the control group consisted of 16 females and 5 males.

INTRODUCTION

Multiplication and division by fractions are among the most difficult concepts in the elementary and middle school mathematics curriculum (Dorgan, 1994). Many students learn these concepts through procedure-oriented, memory-based instruction, attributing little meaning to such operations as "canceling," "reducing," or "inverting and multiplying" (Hanselman, 1997). Students need to develop number and operation sense before learning how to apply these terms through procedures, understanding what the problem means, rather than merely computing an answer. Learning mathematics with understanding is the vision of mathematics reform supported by the National Council of Teachers of Mathematics (2000).

Constructivist teaching holds that students actively construct their own knowledge of mathematics (Mikusa & Lewellen, 1999). An analysis and overview of difficulties students face in learning elementary mathematics (ERIC Digest, 2003) identified the lack of connections between students' informal understandings and knowledge from mathematics instruction as causing students to develop two separate systems of mathematical knowledge. Therefore, connections between concrete work with materials and procedural knowledge must be made for students to understand their mathematical calculations.

Although Groff (1996) presents compelling arguments for dramatically reducing the time spent on teaching fraction concepts, standardized testing exerts a pressure on teachers to devote significant classroom instruction to fractions, including multiplication and division by fractions. The best approach, therefore, is to provide meaningful concrete activities that assist students in connecting their informal knowledge of fractions with more formal instruction to build a foundation of understanding. In this paper, we describe two concrete exercises that helped students understand multiplication and division by fractions. We evaluate the activities through a simple pretest–intervention–posttest design study. In the next section we review the recent literature on multiplication and division by fractions. This is followed by section on methodology, results, discussion, and conclusions. Finally, we provide examples of drawings and concrete materials created by preservice teachers to enhance their understandings of this difficult mathematical topic.

LITERATURE REVIEW

Conceptual Knowledge of Fractions

Conceptual knowledge refers to "knowledge of relationships and interconnections of ideas that explain and give meaning to mathematical procedures," in contrast to procedural knowledge, the "knowledge of the format and syntax of a symbol representation system, and knowledge of the rules and algorithms that can be used to complete mathematical tasks" (Shimizu, 1996, p. 223-224). Conceptual knowledge of multiplication and division by fractions must precede procedural knowledge. Unfortunately, too many students never developed a strong foundation of understanding in this area. Part of the problem is that students have difficulty thinking conceptually (Moss & Case, 1999). This may be a consequence of the amount of time teachers spend on algorithms and encouraging students to apply rules, rather than on constructing meaning. Many elementary teachers do not possess deep understandings of these concepts themselves and provide instruction in procedures rather than teaching the underlying concepts. This approach of teaching algorithms impedes learning because the student is prevented from reasoning through a problem (Kamii, 1985; Kamii & Warrington, 1999).

Students' existing knowledge forms the basis for constructing new knowledge. D'Ambrosio (1990) invented the term, "ethnomathematic" to describe children's existing real-world sense of mathematics that stems from their sociocultural environment. Kieren (1988) claimed that children first build sophisticated knowledge of thought, informal language, and images (their ethnomathematic knowledge) before using more technical language, notation, or algorithms. Teachers must therefore provide realistic situations for students to build the conceptual knowledge base before introducing terms and algorithms. Realistic materials include items from students everyday lives such as fruit, candy bars, containers of water, and bags of cookies, among others.

Pagni (1998) presented some concrete ways of explaining multiplication and division by fractions using folded sheets of paper. For multiplication by fractions, he suggests that the mathematical sentence be interpreted as follows. An equation such as ⅓ x 2/5 should be interpreted as take one third of two-fifths. First fold the paper longitudinally into fifths and color two fifths. Then fold the paper the other way into thirds. Now the student can see how much of the paper is covered by one-third of two fifths. Pagni (1998) also provided a paper-folding model for division of a fraction by a fraction. To represent 2/5 ÷1/3, first determine that the equation is asking, "How many thirds are in two-fifths?" Then, fold the paper into fifth and color two-fifths of the paper. Now, fold the paper the other way into thirds. The paper has now been divided into 15 rectangles, or fifteenths. Because five fifteenths are a third, determine how many sets of five fifteenths are in the colored part of the paper.

Krach (1998) advocated teaching multiplication and division by fractions with two types of models: 1) an area model using manipulatives such as fraction circles, and 2) a measurement model using Cuisenaire rods or centimeter strips. Similarly to Pagni's approach described above, Krach suggests tracing the manipulatives onto paper and partitioning them into equal parts for an area model. For division by fractions with Cuisenaire rods, Krach suggested that students view the orange 10-rod as "one" and select a rod or rods to represent the first fraction. Then smaller rods, representing the division by a fractional part, are measured against these rods to determine how many times the fractional divisor is contained in the dividend.

Elementary and Middle School Students' Understandings

Nagle and McCoy (1999) conducted a small study of nine seventh-grade students. They presented students with "4/5 ÷ 1/2" during private interviews and asked them to solve, explain, and demonstrate a real-life situation for the problem. Three solved the problem incorrectly by either using an incorrect or incomplete algorithm. Of the six who solved the problem correctly, only one could produce a real-world example. Most students explained that their teachers had emphasized procedures rather than explanations and models. Bezuk and Armstrong (1993) suggest some excellent activities for developing upper elementary and middle school students' conceptions of division of fractions that can ameliorate this situation. They use the real-world setting of resurfacing highway and painting road stripes as a setting for several exercises.

Sharp and Adams (2002) conducted a pretest-intervention-posttest study with fifth grade students. They provided students with problems in real world contexts to solve during the instructional intervention. An example is the following problem. My mom wants to make teddy bears. If a bow for a bear takes 1½ feet of ribbon and she has 11 feet of ribbon, how many bows can she supply for bears? Sharp and Adams made some interesting discoveries. Students were more motivated by some problems than others. Their puppy problem in which a photograph of a puppy was supplied along with the story the dog's illness and its need to take 12/3 medicine tablets each day generated tremendous interest and enthusiasm in solving the problem. Sharp and Adams found that their students were able to use drawings and mathematical symbols to solve the problems. Students invented a common-denominator procedure built on their whole number knowledge. None of the students invented an invert-and-multiply procedure.

Sharp (1998) discussed a strategy for enabling students to concretely construct an alternative algorithm to the "invert and multiply" algorithm, the fair-sharing approach, in which students divide objects among a certain number of groups. This approach is used in the hands-on materials created by preservice teachers described in the current study.

Preservice Teachers' Understandings of Multiplication or Division by Fractions

Ball (1990) investigated preservice teachers' knowledge of division by fractions. Although seventeen of the nineteen study participants were able to correctly solve a simple problem involving division with fractions, only five were able to produce an appropriate model of the problem, and eight of the participants were unable to generate a model at all. She found that a common error in the inappropriate models was dividing by 2 instead of ½. Preservice teachers apparently solved the problem by recalling and applying rules rather than understanding what division by a fraction means.

Tzur and Timmerman (1997) also examined preservice teachers' understandings of division of fractions through three case studies. The preservice teachers solved tasks in a computer microworld that allowed them to conceptualize fractions as being broken into even-sized pieces (co-measure units). The researchers based this intervention on previous research on children's conceptions cited in the literature. The researchers suggested that teacher educators use knowledge of children's thinking to create activities that will help preservice teachers develop their understandings.

Another group of mathematics education investigators (Lubinski, Fox, & Thomason, 1998) conducted a case study of one preservice teachers' struggle to construct meaning for $2/3 \div 5/7$. After unsuccessful initial attempts on her own, she asked for help from her husband and daughter. This help turned to frustration, so she tried representing the problem with circles cut into pie pieces. When this too failed, she consulted a math text and observed drawings that she had difficulty transferring to her own problem. Another text provided an algebraic approach, but she wanted to understand the problem rather than just calculate an answer. After trying an analogy between whole numbers and her problem, looking at professional standards, and listening to the reasoning of a friend, the preservice teacher was ready for a question asked by her fourteen-year-old daughter, "How many times does a half go into 2/3?" This led to the preservice teacher developing her own reasoning of the problem. The authors suggest that construction of conceptual understanding is important for preservice teachers to be able to teach these concepts to children in their future careers.

A recent in-depth study by Ma (1999) examined the conceptual knowledge of American and Chinese preservice and inservice teachers concerning multiplication and division by fractions. Ma documented the differences between Chinese teachers' conceptual understandings and American teachers' procedural knowledge without deeper understanding of the concepts.

Object Boxes

The above review of the literature reveals the difficulties both preservice teachers and elementary/middle school students have in conceptualizing multiplication and division by fractions. Several studies clearly found that learners need to have concrete and real-world related experiences to construct meaning for these concepts. The current study involves preservice teachers in creating their own problems with drawings to model the action and in assembling sets of hands-on materials to teach these concepts to elementary students.

The hands-on materials take the form of *object boxes*. An *object box* is a set of materials (objects) and corresponding cards housed in a plastic shoebox (the object box). Object boxes are rooted in Montessori education, as Maria Montessori (1964) first used these materials to teach spelling, reading, and writing to elementary students. Rule expanded this method in the area of literacy (2001) and conducted studies to show their efficacy (Long & Rule, 2004; Rule, Dockstader, & Stewart, in press; Rule, Stewart & Haunold, 2005). Object boxes have also been developed to teach science vocabulary (Rule, 1999; Rule & Barrera, 1999; Rule & Barrera, 2003; Rule, Barrera, & Stewart, 2004; Rule, Young, & Fox, 2003), form and function analogies (Rule & Furletti, 2004; Rule & Rust, 2001), and social studies concepts (Gianetto & Rule, 2005). Object boxes have recently been used in mathematics to teach concepts of numeration, algebra, measurement, and geometry (Rule, Grueniger, Hingre, McKenna, & Williams, in review).

Preservice teachers advance their learning by constructing materials for elementary students. Rule and Lord (2002) found that preservice teacher learning was enhanced by dynamic involvement in peer-tutoring, construction of materials, evaluation of materials, and tutoring of elementary students. Similarly, Rule, Grueniger, Hingre, McKenna, and Williams (in review) showed that preservice teachers significantly increased their knowledge of mathematics by working in small groups to construct boxes of objects with accompanying mathematical clues.

This study employs a new type of object box for modeling multiplication and division by fractions. This box consists of a set of 4-9 identical objects, each of which can be separated into four or more equal parts. The two accompanying cards each have a story problem pertaining to the objects on the front and an explanation of how to solve the problem (for self-checking) on the reverse side. One story problem involves multiplication by a fraction; the other involves division by a fraction.

National and State Standards

The multiplication and division by fractions activities described in this paper support the New York State mathematics core curriculum at the sixth grade level (University of the State of New York, 2005). This is the state in which the study was conducted. Some of the relevant performance indicators are listed below in Table 1.

Table 1. New York State mathematics core curriculum performance indicators for sixth grade related to multiplication and division by fractions.

Performance Indicators for sixth grade related to multiplication and division by fractions
6.N.9 Solve proportions using equivalent fractions
6.N.10 Verify the proportionality using the product of the means equals the product of the extremes
6.N.17 Multiply and divide fractions with unlike denominators
6.N.18 Multiply and divide mixed numbers with unlike denominators
6.N.19 Identify the multiplicative inverse (reciprocal) of a number
6.A.5 Solve simple proportions within context

Besides being addressed by New York State standards for mathematics, multiplication and division by fractions is recognized by the foremost national association of mathematics teachers as being important in students' learning of mathematics. The National Council of Teachers of Mathematics (NCTM, 2000, p. 20), in the *Learning Principle*, states, "Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge." Students who memorize procedures without understanding do not know when to apply their knowledge (Bransford, Brown, & Cocking, 1999). Students in middle grades sixth through eighth grade need to "understand the meanings and effect s of arithmetic operations with fractions, decimals, and integers" (NCTM, 2000, p. 214).

In the sections below, we describe the set-up of the current study focusing on multiplication and division by fractions, discuss the results, and draw conclusions.

METHOD

Sample

Forty-two white preservice elementary teachers (college juniors or seniors) enrolled in three sections of a mathematics methods course taught by the same instructor at a mid-sized college in central New York State participated in the study. The Human Subjects Committee of the State University of New York at Oswego approved this study; students gave written permission for their scores and materials to be included in this publication.

Study Design

The study was a pretest - intervention - posttest design with two groups: 1) a group that participated in the drawing exercise only (Control Group A); and 2) a group that both completed the drawing exercise and successfully made object box materials outside of class (Experimental Group B).

The two sample populations (Group A and Group B) were matched samples. Fewer students volunteered to do the extra-credit object boxes outside of class. Therefore, each group was sorted on pretest scores and matched sets were blindly chosen so that each group had equal numbers of students with the same pretest scores. This resulted in two groups of twenty-one students each.

The same instructor taught both groups and students from both groups were distributed throughout the instructor's three sections of the course. On the first day of class, students took a one-page pretest to assess their skills in multiplication and division by fractions. Then, twelve weeks later, all students took the posttest. The experimental design is shown in Table 2.The pretest/posttest assessment instrument is shown as Appendix 1.

To ensure that all review or learning of the specific mathematical content addressed by the pretest/posttest occurred during the experimental interventions for students, the instructor refrained from discussing multiplication or division by fractions until after the posttest. This study design allowed the investigators to determine the effects of the drawing exercise and the creation of materials upon the learning of math content by preservice teachers.

Table 2. Experimental design

Group	Group A	Group B
Condition	Control	Experimental
N	21	21
Pretest	1st week of class	1st week of class
Intervention	Solve homework problem by drawing two sets of materials and showing how they are multiplied or divided by a fraction	Solve homework problem by drawing two sets of materials and showing how they are multiplied or divided by a fraction. And Create an object box with two story problems for multiplying and dividing by a fraction
Posttest	14th week of class	14th week of class

The Homework and Experimental Projects

Both groups completed a homework activity that involved using clip art or drawings to show multiplication and division by fractions. The assigned activity is shown in Appendix 2.

Preservice teachers in the experimental group completed an additional project on their own for three percentage points on their final course grade of extra credit. This project involved assembling a set of materials that consisted of four to nine identical items that could be separated into four to eight parts each. All items were to be separated into the same number of parts. These items were to be used to solve two story problems, one involving multiplication by a fraction, and the other involving division by the same fraction. Each story problem was to be printed on a card with the answer and explanation of how to concretely solve it printed on the back. An example object box was shown and explained in class. The rubric for scoring this project is shown in Table 3.

Table 3. Rubric for evaluating the extra credit object box projects made by experimental group participants.

Criteria	Yes	No
Materials need to be housed in a **plastic shoebox** or similar box that is labeled with the title of the activity and maker's (your) name.		
The set of materials consists **of four to nine items** that have **been divided into four, five, six, seven, or eight parts each**.		
The items need to be interesting, attractively made, colorful, durable, safe (not sharp, nothing toxic), and three-dimensional (Rule, Sobierajski, & Schell, 2005).		
The "parts" are **detachable**; that is, you can remove them from the items and move them around to figure out the answer to the problem. The items have to be designed so that these "parts" make sense as pieces of the larger item and that they can be assembled back into the original items.		
Two problems have to be presented that can be solved using the set of materials. One problem is multiplication by a fraction. The other problem is division by a fraction.		
Each problem should be written as a **story problem** that fits with the items in the box. Each problem should be printed (typed, word-processed) on a separate card (an index card or similar stiff card will work) with the answer on the back.		
The **fractions** used in the two story problems are limited to any of the following fractions: 3/4, 2/5, 3/5,4/5, 2/6, 4/6, 5/6, 2/7, 3/7, 4/7, 5/7, 6/7, 2/8, 3/8, 5/8, 6/8, or 7/8.		

RESULTS

Example preservice teacher solutions to the homework problems are shown in Appendix 3 (problems involving division by a fraction) and Appendix 4 (problems involving multiplication by a fraction). In general, the solutions were correct and showed preservice teachers' understanding of the assignment. However, any errors were corrected before the example responses shown in the appendices were incorporated into this report.

Example object boxes are shown in Appendix 5. Several of the object boxes had errors that were corrected before being included as examples here; some participants did not receive the full measure of extra credit. These errors included the following problems: 1) not following directions, for example, using more simple fractions than specified; and 2) confusing the meanings of multiplication and division by fractions, for example, equating "division by a half" or "÷½" with dividing the item into two parts – dividing by two. This is similar to the findings of Ball's (1990) investigation.

Table 4. Pretest and posttest scores for individual questions and totals for matched groups. *Standard deviations are shown in parentheses.

Timing	Condition	Percent Correct *						
		Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Total Correct
Pretest	Control Group A: Completed model drawings only	90.5 (30.1)	52.4 (51.2)	76.2 (43.6)	52.4 (51.2)	4.8 (21.8)	28.6 (46.3)	50.8 (20.7)
	Experimental Group B: Completed model drawings and completed object box	100.0 (0.0)	42.9 (50.7)	57.1 (50.7)	66.7 (48.3)	14.3 (35.9)	23.8 (43.6)	50.8 (22.7)
Posttest	Control Group A: Completed model drawings only	90.5 (30.1)	57.1 (50.7)	81.0 (40.2)	76.2 (43.6)	47.6 (51.2)	52.4 (51.2)	67.5 (31.4)
	Experimental Group B: Completed model drawings and completed object box	100.0 (0.0)	52.4 (51.2)	85.7 (35.9)	81.0 (40.2)	52.4 (51.2)	57.1 (50.7)	71.4 (21.8)

Table 5. Reasons why preservice teachers chose to participate or not to participate in the extra credit project of making an object box for teaching multiplication and division by fractions.

Reasons preservice teachers did not participate	Frequency		Reasons preservice teachers chose to participate	Frequency
Did not have enough time.	20		To earn the extra credit points.	21
Wasn't sure what the project required until it was too late to do it.	3		To learn and understand multiplication and division by fractions better	13
Began the project, but encountered problems and did not complete it.	2		To make materials to use in my future classes with elementary students.	4
Did not have money to buy materials	1		To learn how to make object boxes	3
			I had good ideas and excel in hands-on materials making.	2

Table 6. Self-reported learning from preservice teachers who completed the multiplication and division by fractions object boxes.

Responses to question, " What did you learn from participating in making the multiplication and division by fractions object boxes?"	Frequency of Response
Concrete examples help understanding.	17
I learned about multiplication and division of fractions myself.	15
How to write and interpret story problems.	14
How to teach children mathematics with manipulatives.	7
Motivating aspects of manipulatives, such as color and texture.	7
Math can be fun.	5
Everyday items can be used in teaching mathematics.	5
The project allowed me to use creativity and imagination.	4
The project was time consuming.	4
I made real life connections to multiplication and division by fractions.	4
I learned how to make effective hands-on materials.	4

DISCUSSION AND CONCLUSION

Analysis of Pretest and Posttest Performance

Pretest and posttest scores for both groups are shown in Table 4. Because students were blindly selected on total pretest scores to form matched groups, the total scores of the six questions were the same for both groups on the pretest. An analysis of variance (ANOVA) showed that the small differences in student performance on the posttest in total scores favoring the experimental group were not statistically significant (df = 1-40, F = 0.23, p = 0.64). This may be because of the small sample size.

Pretest scores for both groups were similar because the groups were blindly matched on pretest scores. The pretest scores indicate that students in general grasped computation of multiplication by a fraction as shown by the high scores for problem 1, but had considerable difficulty when it came to dividing by a fraction (question 2). Question 2 asked students to calculate $90 \div \frac{1}{3}$. The most common error was division by three, resulting in a quotient of 10 rather than the correct response of 90. This confirms Ball's (1990) findings of preservice teacher difficulties. Students in both groups made gains on question 2 during the posttest. The posttest scores indicated that students in both groups gained proficiency in questions 3 and 4, which asked students to determine an equation for solving a story problem involving multiplication or division by fractions. Posttest results for questions 5 and 6, where students were asked to use pictures to concretely show what multiplication or division by a fraction means, indicated that more students in the experimental group were able to do this. This is probably because of their extra practice in creating a set of materials to model multiplication and division by fractions.

For questions 5 and 6 on the pretest/ posttest instruments, more drawings were provided than necessary to illustrate the problem solutions. This was made clear in the instructions that stated, "Use all or part..." and when the assessments were administered, the instructor repeated the warning that it was entirely possible that more drawings were provided than needed. However, many students incorrectly attempted to use all the drawings to illustrate each of the answers to questions 4 and 5. This shows a very fundamental lack of understanding of the problems that wee presented. However, the most common error on these questions was to merely show the results of computation and not to illustrate how the answer was obtained.

Unfortunately, the posttest results show that most students' understandings of multiplication and division by fractions, although improved, are still incomplete. This indicates that more than two exercises are needed to build a strong knowledge foundation in this area of mathematics. Because most mathematics methods courses are only one-semester courses with much pedagogy to teach, it is difficult to incorporate more exercises on this concept into the course. Perhaps this concept could be addressed in a more concrete rather than procedural manner in the mathematics courses preservice elementary teachers take.

Preservice Teacher Responses to a Survey

Participants were surveyed after the study had been concluded to determine their reasons for participation or non-participation in the extra credit object box activity and to determine the perceived benefits of those who had participated. Table 5 shows that the main reason for non-participation was lack of time and the main reason for participation was the opportunity for extra credit. All of the students in this course are also enrolled in two other elementary education courses and a practicum experience and had similar demands upon their time. Therefore, those who chose to complete the extra credit assignment probably felt somewhat more compelled to try to earn extra credit. Participants in the experimental group probably felt less confident in their mathematical abilities and decided to ensure a better grade in the course by completing the extra credit assignment.

Experimental group participants were asked to describe what they learned from participating in making the multiplication and division by fractions object boxes. The responses to this question are shown in Table 6. Most reported that making the concrete materials helped them understand the concepts themselves, indicating the usefulness of the intervention. They also recognized several positive aspects of the project, including having materials with which to teach future students and the motivating aspects of using colorful materials with real-world connections. This latter comment can be connected to another study that found both preservice teachers and fourth graders performed better mathematically when the materials with which they were provided were perceived as colorful and attractive (Rule, Sobierajski, & Schell, 2005).

Conclusion

We have shown two activities that helped preservice teachers understand the difficult concepts of multiplication and division by fractions. Example drawings and example sets of hands-on materials created by preservice teachers are provided in Appendices 3, 4, and 5 for readers to use in teaching learners at the K-12 or college level.

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Appendix 1. Pretest and Posttest

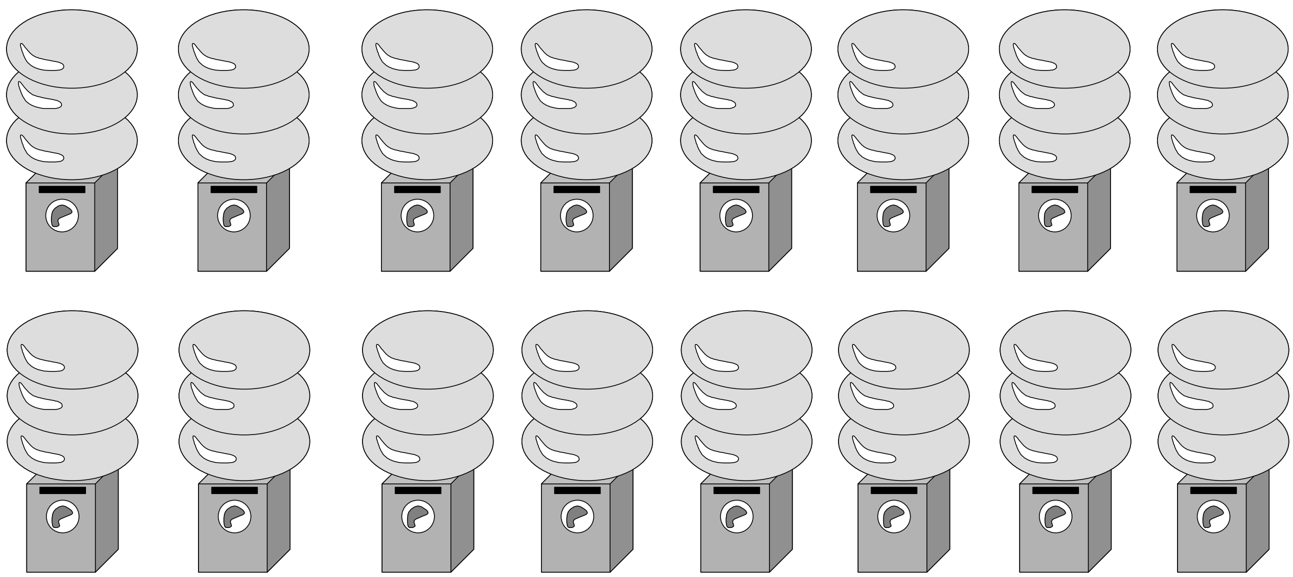
Solve the following two problems.

1.) 1/2 x 20 = 2.) 30 ÷ 1/3 =

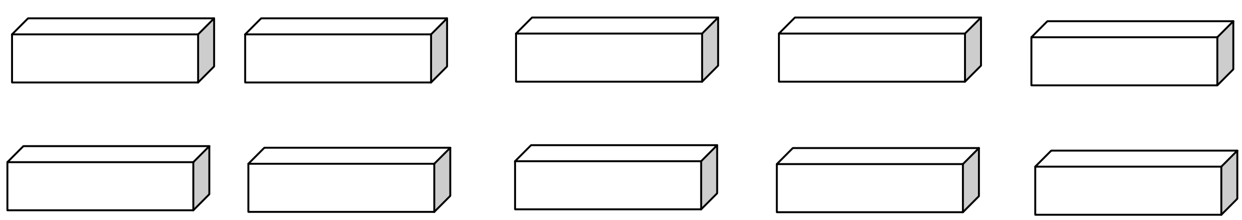
3. Translate this sentence into an equation that contains a fraction: **How many 1/2 cup servings are there in 2 cups of ice cream?**

4. Translate this story problem into an equation that contains a fraction: Kenosha and Todd washed cars to make money. Kenosha worked two days, but Todd only worked one day. They made $60. Kenosha wants 2/3 of the money because she worked more than Todd. How much money does she want?

5. Use **all or part** of the water tanks drawing below to show the answer to this problem. Shade in, circle, or otherwise mark parts to illustrate the answer. Add brief explanations. Cross out any water tanks not used or needed for the problem. **2/3 x 4 water tanks = ?**



6. Use **all or part** of the drawing below to show the answer to this problem. Shade in, circle, or otherwise mark the parts to illustrate the answer. Add brief explanations. Cross out any sticks of butter not used or needed for the problem. **2½ sticks butter ÷ ¼ = ?**

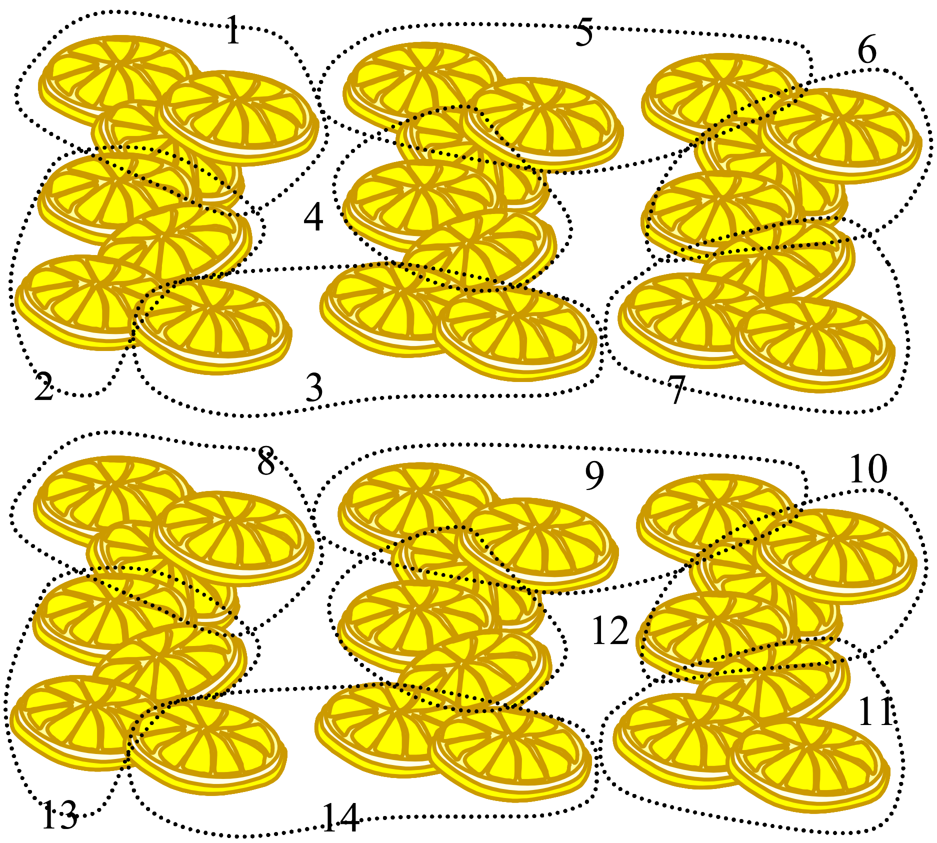


Appendix 2. Directions and example problems for modeling multiplication and division by fractions with drawings.

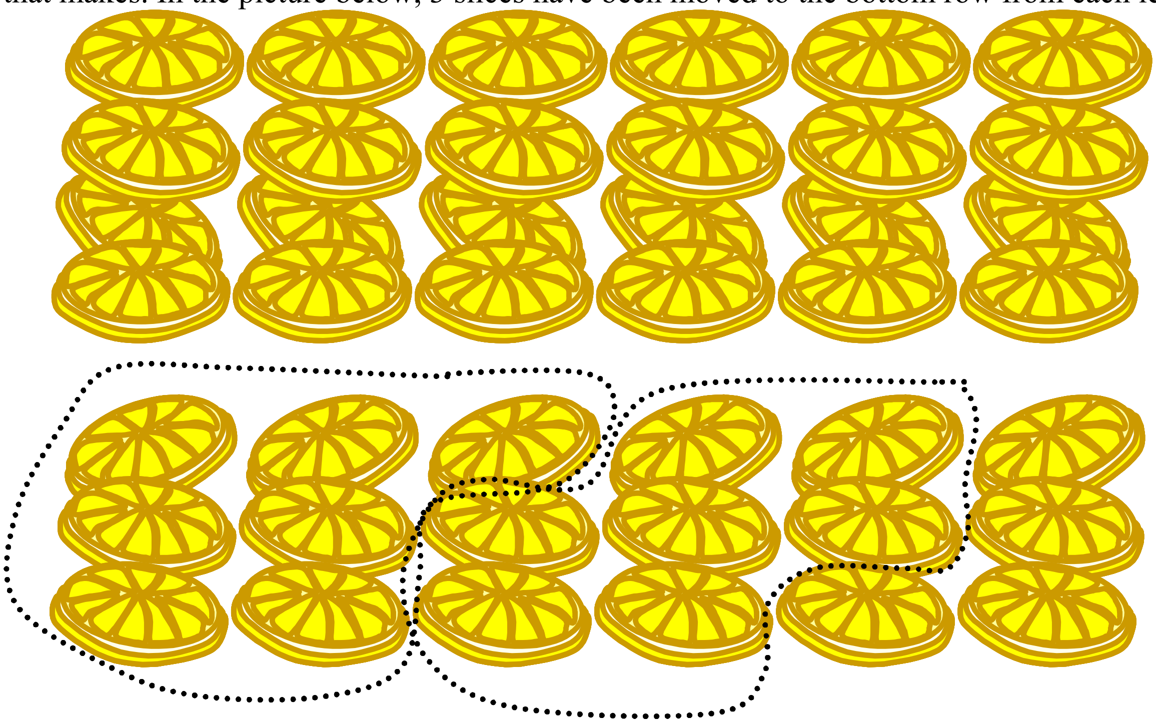
Create two story problems for the same set of manipulative pictures that involve multiplication and division by fractions. The fractions you use can be any of the following: 2/3, ¾, 2/5, 3/5, 4/5, 4/6, 2/7, 4/7, 5/7, 6/7, 3/8, 5/8, 6/8, 7/8, 2/9, 3/9, 4/9, 5/9, 6/9, 7/9, or 8/9. The number of items can be from 3 items to 10 items. **Use the same fraction and same number of items for both problems.**
The set of manipulative pictures can be clip art, digital photographs, images from the Internet, or drawings. All the necessary items must be shown on the page. Besides making the story problem and picture set, you need to provide, on a second page, the answers and an explanation. I recommend creating this assignment electronically because of the ease of duplicating and positioning images. However, I will accept work that has been done manually as drawings or cut-and-paste. **All work must be neat.**
In the space below, I have provided an example using 6 items and the fraction 3/7 so that you will understand how to complete this homework assignment.

Lemon Problem 1. Lemons can be bought in bags of six. If a restaurant cook wants to make a sauce recipe that calls for 3/7 of a lemon per batch, how many batches could be made with a bag of lemons? The drawings show six lemons, each cut into 7 slices.

This a division problem because you want to see how many 3/7 are in 6 lemons: 6÷ 3/7. I have drawn enclosures around the sets of three-sevenths in the picture below. There are 14 sets, so 14 batches.

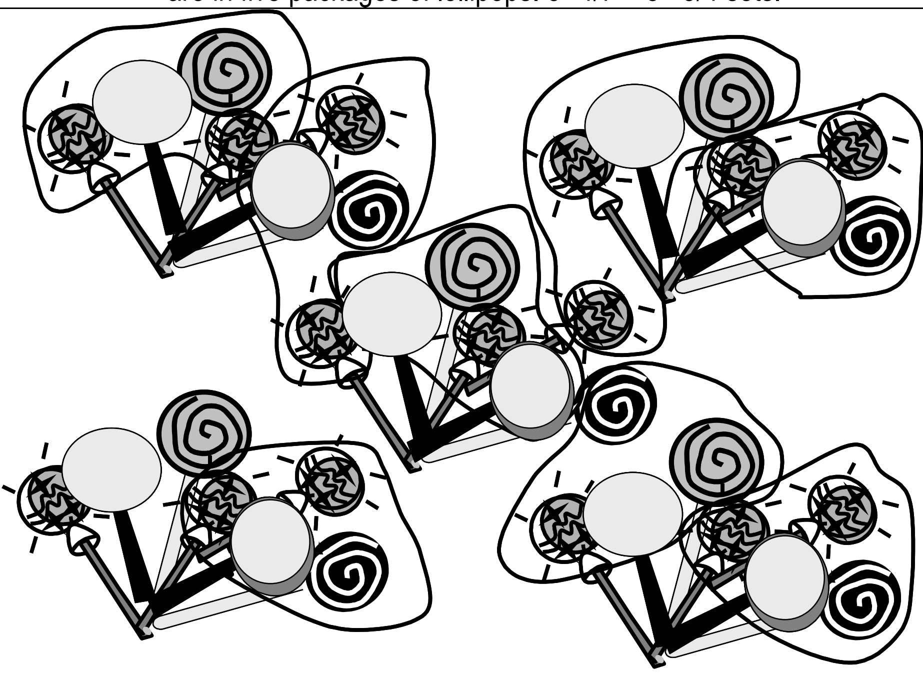
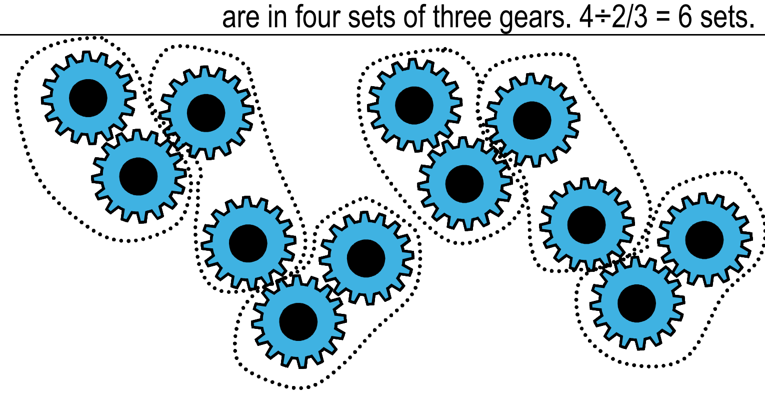


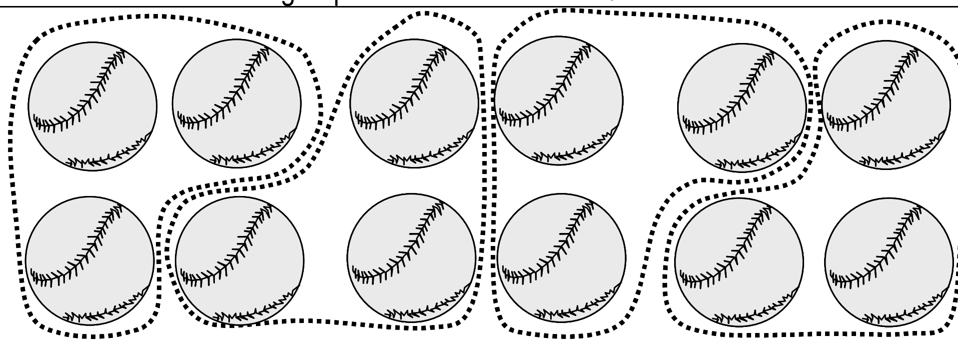
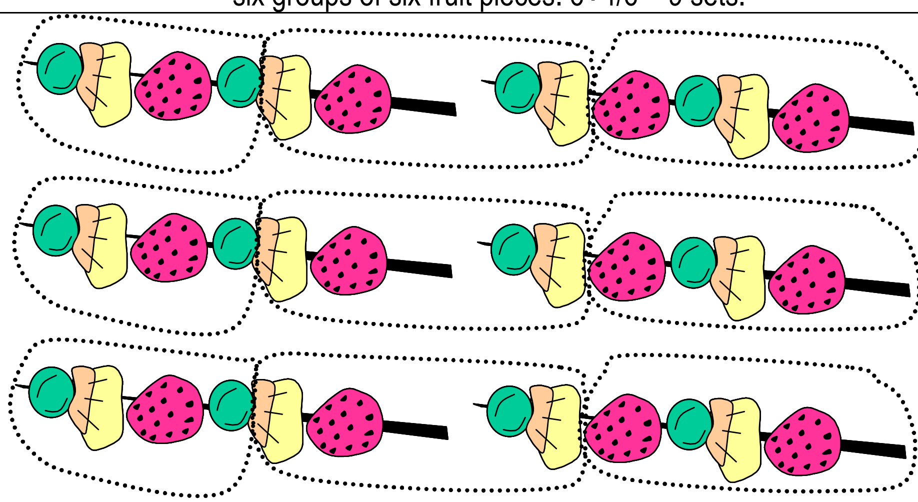
Lemon Problem 2. A cook buys a bag of six old lemons on sale, slices each into seven slices, but finds that 3/7 of the lemons are unusable because of bad spots. If the rotten slices are reassembled to form complete lemons, how many lemons have to be discarded?
This is a multiplication problem to find out what three-sevenths of six is or 6 x 3/7. Because the lemons are cut into 7 slices each, assemble 3 slices from each lemon to see how many lemons that makes. In the picture below, 3 slices have been moved to the bottom row from each lemon.

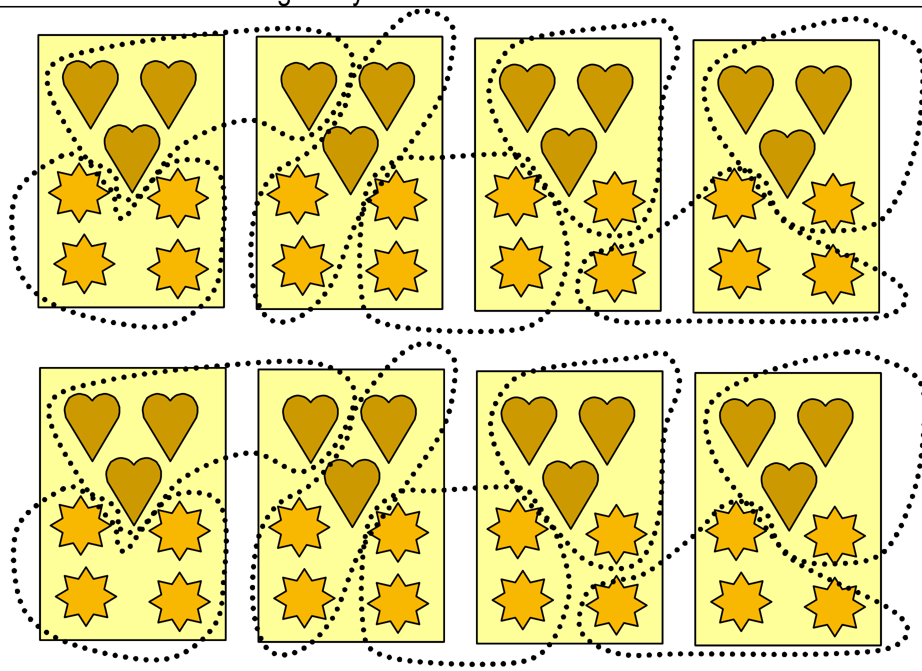
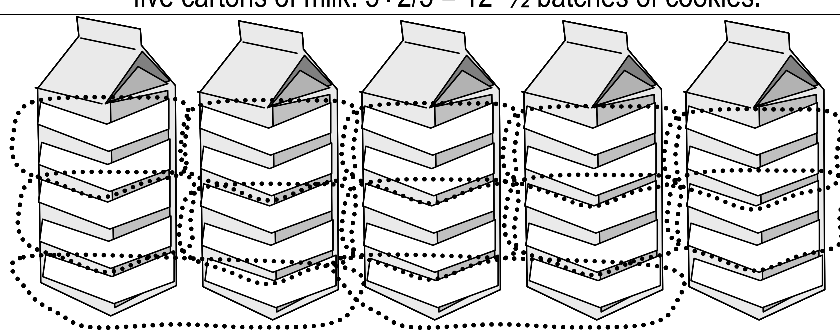


The slices at the bottom can be combined to make two and four-sevenths lemons. That is how many lemons out of a full bag that were bad.

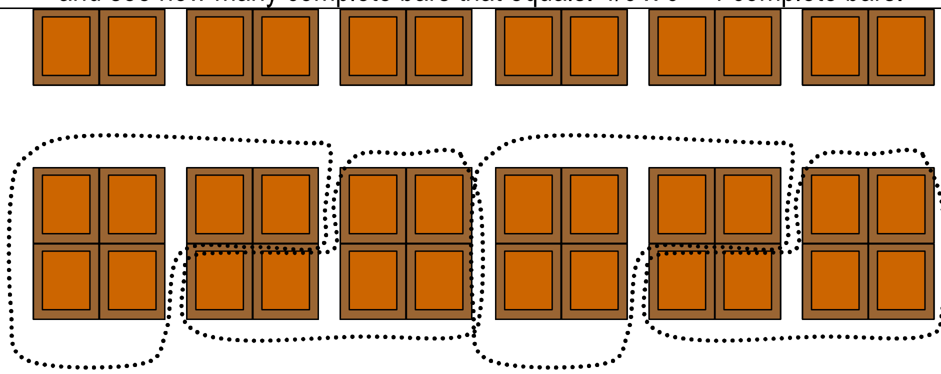
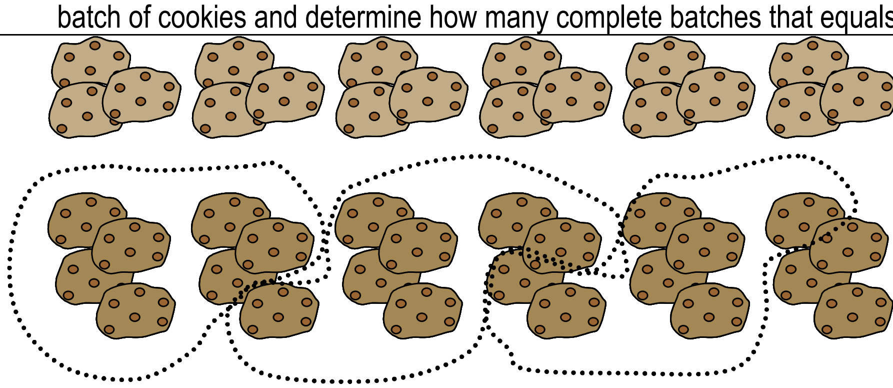
Appendix 3. Drawing Models of Division by a Fraction

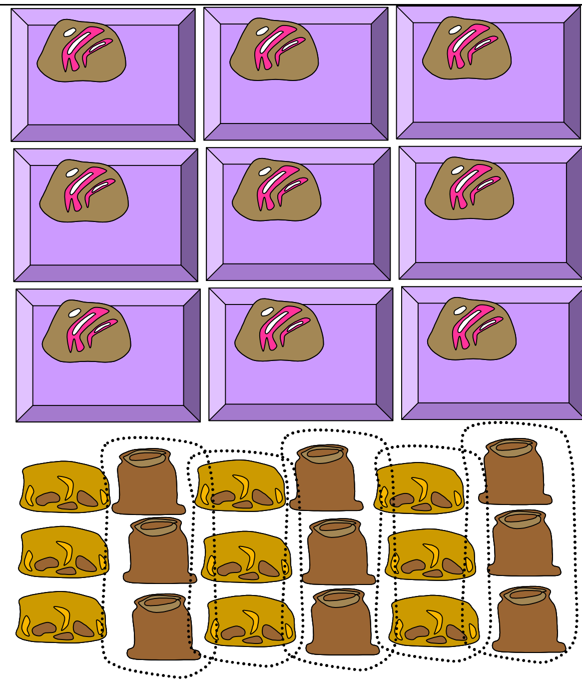
Lollipop Problem Adam DeSantis	Division by a Fraction The problem is a division problem because you want to see how many 4/7 groups are in five packages of lollipops. 5÷4/7 = 8 3/4 sets.
Michael buys five packages of seven lollipops each. He has to reserve at least 4/7 of a package for each of his cousins. Including Michael, how many people can the lollipops serve if each person gets 4/7 of a package? Are there any left-over lollipops? If so, how many people can receive an extra lollipop?	
There are 8 sets of 4/7 of a package of lollipops. There are 3 lollipops left over, so three people get an extra lollipop.	
Gear Problem John Michael Grosso	Division by a Fraction The problem is a division problem because you want to see how many two-thirds are in four sets of three gears. 4÷2/3 = 6 sets.
Gears must be purchased in groups of three. If a mechanic only needs 2/3 of the gears in each group for conveyor belt mechanisms, how many conveyor belts can be fixed if four sets of gears are shipped?	
There are six sets of two thirds of a package, so six conveyor belt mechanisms can be fixed.	

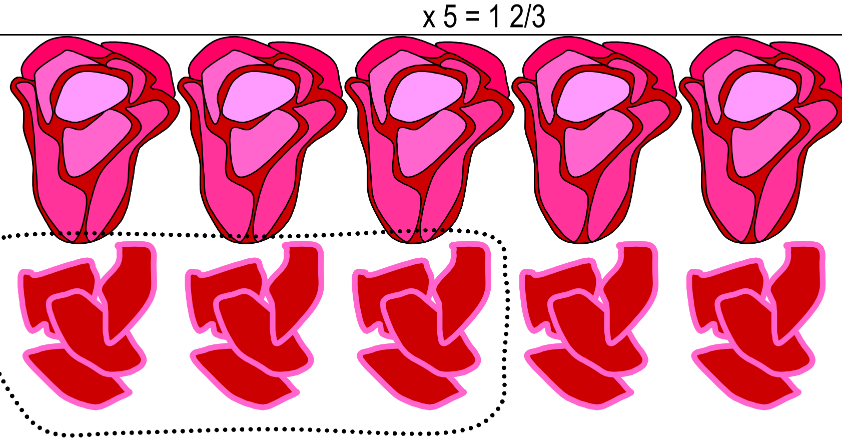
Baseball Problem Greg Lavery	Division by a Fraction The problem is a division problem because you want to see how many ¾ are in three groups of four baseballs. 3÷¾ = 4 sets.
Baseballs are manufactured in groups of four. Sporting goods stores sell the baseballs in packages of three. Therefore, 3/4 of a group from the factory is used to make a store package. How many store packages can be made from three factory groups of baseballs?	
There are four sets of three balls, so four store packages can be made.	
Fruit Kabob Problem Ashley Millerd	Division by a Fraction The problem is a division problem because you want to see how many 4/6 are in six groups of six fruit pieces. 6÷4/6 = 9 sets.
Fruit kabobs come from the store with six different fruit pieces on each sticks. A woman who is throwing a party wants to change the fruit desserts. If the woman wants to put only 4/6 of the fruit on a kabob to make smaller kabobs, how many new fruit kabobs can she make with six kabobs from the store?	
There are nine sets of four fruit pieces, so six store packages will make 9 smaller kabobs.	

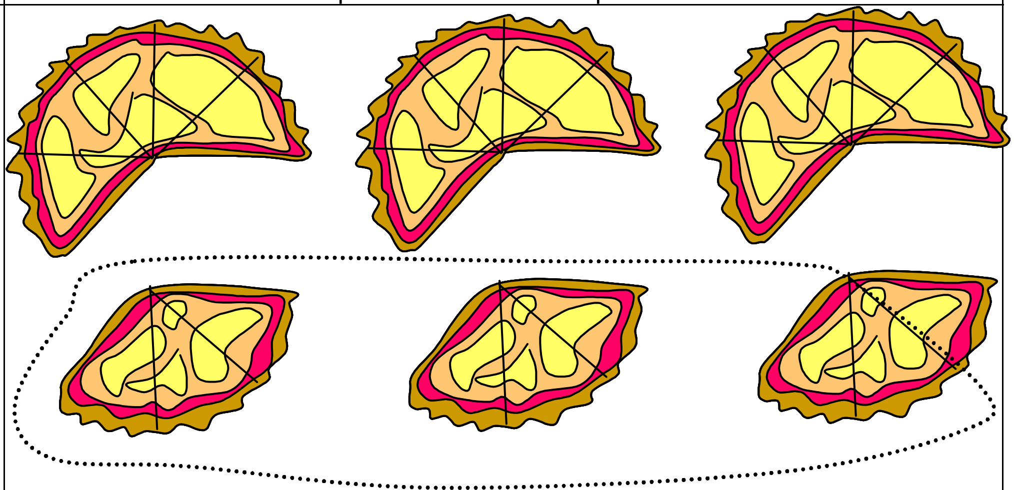
Cookies on Trays Problem Michelle Pollino	Division by a Fraction The problem is a division problem because you want to see how many 4/7 are in eight trays of cookies. 8÷4/7 = 14 sets.
Exactly seven cookies fit on each tray for baking. Four-sevenths of a tray is used to make a small box of cookies. If a baker has made eight trays of cookies, how many boxes can be filled?	
There are fourteen sets of four cookies, so fourteen small boxes of cookies can be made.	
Milk Cartons Patrick McCarthy	Division by a Fraction The problem is a division problem because you want to see how many 2/5 are in five cartons of milk. 5÷2/5 = 12 ½ batches of cookies.
A class wants to bake cookies for a bake sale. The class has five small cartons of milk. It takes 2/5 of a carton of milk to make a batch of cookies. How many batches of cookies can be made with five cartons of milk?	
There are twelve and a half sets, so the class can make 12 batches with a little milk left over.	

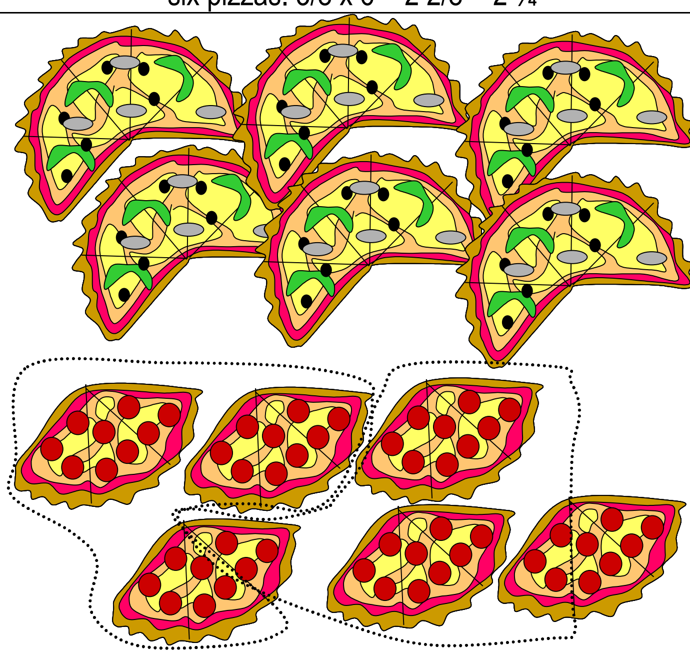
Appendix 4. Drawing Models of Multiplication by a Fraction

Chocolate Bar Problem Meghan Wheeler	Multiplication by a Fraction The problem is a multiplication problem because you want to take 4/6 of each bar and see how many complete bars that equals. 4/6 x 6 = 4 complete bars.
Chocolate bars can be bought in bags of eight bars. On a hot day, Cory bought a bag of chocolate bars. Each bars was divided into 6 pieces. When Cory opened the bag of bars, he noticed that 4/6 of each bar had melted. The equivalent of how many complete chocolate bars had melted?	
Take 4/6 of each of the six chocolate bars. Then determine how many complete bars that makes. In the picture above, the 4/6 have been removed to the bottom row and have been regrouped into complete bars.	
Cookie Problem Aaron Pascale	Multiplication by a Fraction The problem is a multiplication problem because you want to take 4/7 of each batch of cookies and determine how many complete batches that equals.
Jenny bakes six batches of seven cookies each. She finds that the back f the oven was too hot and 4/7 of the cookies of each batch are burned. How many batches in all have to be discarded?	
There are six batches of seven cookies each. Remove 4 cookies from each batch to represent the burned cookies. Then reorganize the se burned cookies into whole batches. There are 3 3/7 batches.	

<table>
<tr><td>Chocolate Bonbons Problem
Ashley Hughto</td><td>Multiplication by a Fraction
The problem is a multiplication problem because you want to take 2/3 of each box of bonbons. 2/3 x 9 = 6</td></tr>
<tr><td>Mark makes homemade chocolate bonbons. He puts three candies in each small box. A costumer doesn't like nuts or raisins. Two-thirds of each box of candies is made with nuts or raisins. If Mark opens nine boxes of candies and removes 2/3 from each box, how many full boxes of candies will Mark be removing?</td><td></td></tr>
<tr><td colspan="2">Remove 2/3 of the candies from each box. Then group those into sets of three to determine how many full boxes this makes. There are 6 full boxes that have been removed..</td></tr>
</table>

<table>
<tr><td>Rose Petal Problem
Tori Sivers</td><td>Multiplication by a Fraction
The problem is a multiplication problem because you want to take 1/3 of five roses. 1/3 x 5 = 1 2/3</td></tr>
<tr><td>A florist is pulling apart rose petals for a flower girl basket for a wedding. One batch of five roses is old and 1/3 of the petals cannot be used. If each rose has 12 petals, how many rose-equivalents have to be discarded?</td><td></td></tr>
<tr><td colspan="2">There are five roses. Remove 1/3 of the petals from each rose (4 petals). Then reassemble them into groups of 12 petals. There is one complete group and 2/3 of another group.</td></tr>
</table>

<table>
<tr><td>Pizza Slice Problem
Erin Fitzgibbons</td><td>Multiplication by a Fraction
The problem is a multiplication problem because you want to take 3/8 of each pizza. 3/8 x 3 = 1 1/8 pizzas</td></tr>
<tr><td>A group of friends ordered three pizzas. Each pizza was cut into 8 slices. At the end of the meal, the friends had eaten 3/8 of each pizza. The equivalent of how many pizzas were consumed?</td><td></td></tr>
<tr><td colspan="2">Remove 3/8 of the slices from each pizza. Then group those into sets of eight to determine how many full pizzas this makes. There are the equivalent of one and one-eighth pizzas that have been eaten..</td></tr>
<tr><td colspan="2"></td></tr>
</table>

<table>
<tr><td>Pizza Problem
Andrea Tucker</td><td>Multiplication by a Fraction
The problem is a multiplication problem because you want to take 3/8 of six pizzas. 3/8 x 6 = 2 2/8 = 2 ¼</td></tr>
<tr><td>Bob bought six pizzas for his family. These pizzas had several different toppings. If 3/8 of each pizza is covered with pepperoni and no one likes that, the equivalent of how many total pizzas are not eaten?</td><td></td></tr>
<tr><td colspan="2">There are five roses/ Remove 1/3 of the petals from each rose (4 petals). Then reassemble them into groups of 12 petals. There is one complete group and 2/3 of another group.</td></tr>
</table>