Measuring Teachers' Mathematical Knowledge

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Abstract

Teachers' knowledge of mathematics is pivotal to their capacity to provide effective mathematics instruction and to their ability to assess student learning (Ball, Hill, & Bass, 2005; Ma, 1999; Schifter, 1999). The National Council for the Teaching of Mathematics (NCTM, 2000) makes it clear that teachers need knowledge of the whole domain as well as knowledge about the important ideas that are central to their grade level. POWERSOURCE is expected, through professional development and job aids, to influence teachers' pedagogical content knowledge and assessment practices. To gauge such effects we have developed teacher measures that focus on three key mathematical principles that are central to POWERSOURCE: the distributive property, solving equations, and rational number equivalence.

Background

The measurement of teachers' knowledge of mathematics has been a problem occupying researchers for several decades. Early efforts, consonant with the assumption that knowledge of and skill with mathematics content is critical to teaching, used characteristics of teachers and their educational background; for example, the number of mathematics courses taken and degrees obtained (Begle, 1979). From his meta-analysis of studies conducted between 1960 and 1976 that examined the effects of teacher characteristics on student performance Begle concluded that the idea that the more subject knowledge a teacher had, the better the teacher, needed 'drastic modification' (p.51).

Later studies, influenced by the notion of pedagogical content knowledge (Shulman, 1986, 1987; Wilson, Shulman & Rickert, 1987) focused on qualitative probing of forms of mathematical knowledge that were more closely tied to teaching (e.g., Ball, 1991; Kennedy, 1997; Ma, 1999). A further approach, developed from this prior work, focuses on mathematics knowledge for teaching (Ball & Bass, 2000; Hill, Rowan & Ball, 2005). Mathematics knowledge for teaching refers to the knowledge that is specific to the profession of teaching (as opposed to the kind of knowledge used by other professions such as accounting and engineering) and is closely linked to student achievement. For example, Ball (1990) describes the difference between a teacher's ability to execute an operation such as division by fractions and the kind of understanding needed for teaching, namely, how to explain what the operation means and why it works.

Researchers from the Study of Instructional Improvement (SII) designed multiple-choice survey instruments to measure growth in teachers' mathematical knowledge used in elementary school mathematics and to provide more information about the ways in which teacher knowledge contributes to student achievement. These instruments were used in a study of the effects of teacher knowledge for teaching on student achievement and the results showed that teachers who scored higher on these measures of mathematics knowledge for teaching produced better gains on student achievement (Ball, Hill & Bass, 2005; Hill, Rowan, &Ball, 2005). Recently, the same researchers have developed similar instruments to measure teachers' knowledge of pre-algebra and algebra.

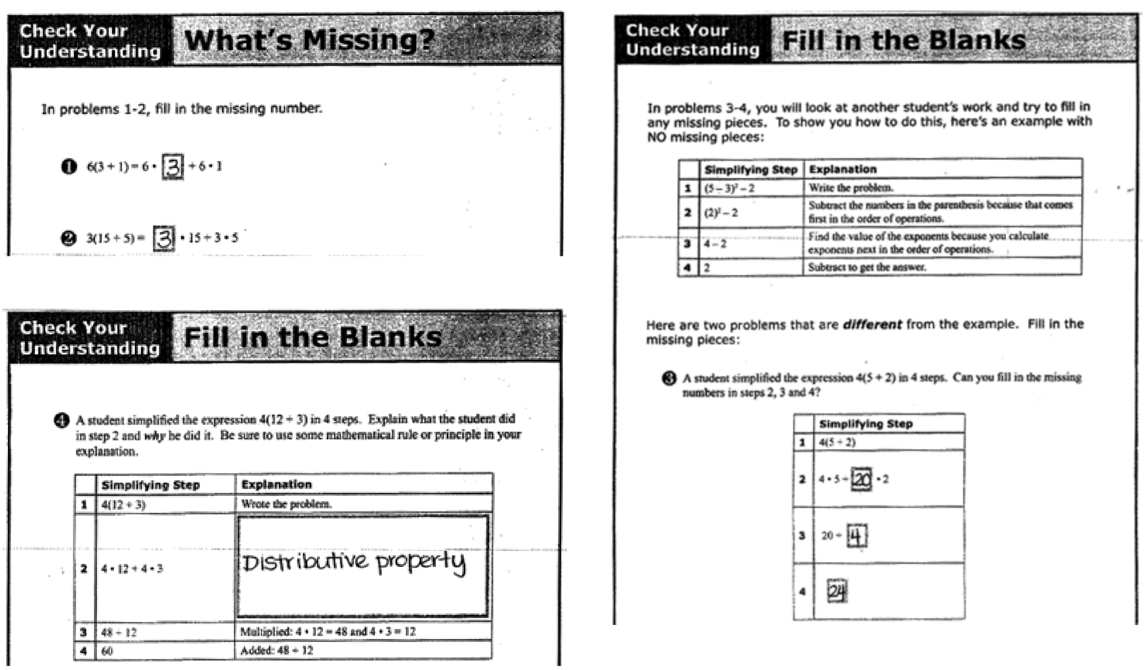
Because of the scarcity of valid and reliable instruments to gauge teachers' mathematical knowledge for teaching, to date, it has been difficult to measure the outcomes of professional development programs designed to improve teachers' knowledge. Recently, Hill and Ball, 2004 used the SII measures to evaluate the outcomes of California's Mathematics Professional Development Institutes and were able to show that 'teachers *can* learn mathematics for elementary school teaching in the context of a single professional development program' (p.345).

This paper will present further conceptualizations of measuring sixth- , seventh- , and eighth-grade teachers' mathematical knowledge, describe the development of the measures and a validation study of the measures and conclude with possibilities for using this conceptualization in other contexts.

Further Conceptualizations of Instruments to Measure Teacher Knowledge

The conceptualization of instruments to measure teacher knowledge presented in this paper builds on the prior research from the SII. We have developed and are validating two instruments designed to gauge teachers' mathematical knowledge to teach for understanding key principles underlying mastery of Algebra I, specifically, the distributive property, solving equations and rational number equivalence. The two instruments conceptualize the measurement of teacher knowledge as knowledge that is central to and embedded in the every day practice of teaching, irrespective of teachers' specific curriculum or approach to teaching. This would include the knowledge that teachers draw on to interpret students' understanding of mathematical ideas and plan instruction (Ball, Lubienski & Mewborn, 2001; NRC, 2001a; Shulman, 1986, 1987) to give students feedback (NRC, 2000, 2001b) and to explain, justify and model mathematical ideas to students (NRC, 2001b).

One instrument is a series of performance tasks in which teachers are asked to review student responses to assessments and to respond to a series of questions. Figure 1 shows a student's response to an assessment addressing understanding of the distributive property.


Check Your Understanding
What's Missing?

In problems 1-2, fill in the missing number.

1 $6(3+1) = 6 \cdot$ [3] $+ 6 \cdot 1$

2 $3(15+5) =$ [3] $\cdot 15 + 3 \cdot 5$

Check Your Understanding
Fill in the Blanks

4 A student simplified the expression 4(12 + 3) in 4 steps. Explain what the student did in step 2 and *why* he did it. Be sure to use some mathematical rule or principle in your explanation.

	Simplifying Step	Explanation
1	$4(12+3)$	Wrote the problem.
2	$4 \cdot 12 + 4 \cdot 3$	Distributive property
3	$48 + 12$	Multiplied: $4 \cdot 12 = 48$ and $4 \cdot 3 = 12$
4	60	Added: $48 + 12$

Check Your Understanding
Fill in the Blanks

In problems 3-4, you will look at another student's work and try to fill in any missing pieces. To show you how to do this, here's an example with NO missing pieces:

	Simplifying Step	Explanation
1	$(5-3)^2 - 2$	Write the problem.
2	$(2)^2 - 2$	Subtract the numbers in the parenthesis because that comes first in the order of operations.
3	$4 - 2$	Find the value of the exponents because you calculate exponents next in the order of operations.
4	2	Subtract to get the answer.

Here are two problems that are ***different*** from the example. Fill in the missing pieces:

3 A student simplified the expression 4(5 + 2) in 4 steps. Can you fill in the missing numbers in steps 2, 3 and 4?

	Simplifying Step
1	$4(5+2)$
2	$4 \cdot 5 +$ [20] $\cdot 2$
3	$20 +$ [4]
4	[24]

Figure 1. Student response to an assessment checking understanding of the distributive property

After the teachers have reviewed the student work, they are asked to respond to the following questions:

1. What is the key principle that these assessments address? Why do students need to understand this principle for Algebra I?
2. What inferences would you draw from this student's responses? What does this student know? What does this student not know?
3. If you were this child's teacher what written feedback would you give to this student?
4. If this student were in your class, based on your responses to questions 2 and 3, what would you do next in your instruction?

Also included are some tasks that require the teacher to describe how they would teach certain principles, (e.g., how would you explain the distributive property to students?)

The second instrument is a survey that asks teachers to rate their level of expertise, on a five-point scale ranging from novice to expert, in teaching key principles for understanding. For example, rating their expertise in 'showing and justifying how defining multiplication as repeated addition can be used to multiply fractions by integers' and in 'explaining how to use the distributive property to add fractions with the same denominator.'

Prior CRESST research found that teachers' own survey ratings of their subject knowledge were related to student achievement (Boscardin et al, 2005) and suggests that self-rating could be a reliable way of measuring teacher knowledge.

Developing the measures

The foundation for both measures is a map of the domain of Algebra I, developed at CRESST in collaboration with algebraists from the University of Chicago. The map represents the key principles of Algebra I, and the relationship between and amongst them. The key principles have not been derived from any particular curricula, but they are reflected in a range of the state's mathematics standards and in the NCTM standards.

To design the survey, a group of expert mathematics teachers and mathematicians analyzed the three key principles and several curricula to determine what was involved in teaching the key principles for understanding. Survey items for each key principle were developed from this analysis to reflect the actual knowledge that teachers would need to teach for student understanding.

The performance tasks were developed from a pilot study of formative assessments to assess students' understanding of key principles. A group of mathematicians and expert mathematics teachers analyzed student responses and selected those that showed gaps in knowledge, misconceptions, and understanding of the key principles. The group then designed tasks for teachers to complete that were related to the student responses and that required teachers to draw on their mathematical knowledge for teaching the principles for understanding. For example, from the student's response to an assessment of rational number equivalence the teachers were asked, what do you think this student understands about rational number equivalence? What does the student not understand? What would you do next in your instruction?

A group of eight mathematics teachers of varying levels of experience and expertise completed the performance tasks, and to capture the range of knowledge represented four point scoring rubrics were derived from an analysis of their written responses. For example, in response to the question about the principle addressed by the assessment shown in Figure 1 and why it is essential for algebra, responses ranged from 'order of operations – essential for solving problems' to 'distributive property, which is needed in Algebra I to solve algebraic equations and multiply monomials and polynomials.'

The rubric for the task of explaining what the teachers' next instructional steps would be is shown in Figure 2. This rubric assesses the degree to which the teacher responses show detailed and principle-based knowledge of distribution. A score point of 4 shows that the teacher understands the distributive property as repeated addition, factoring as distribution in reverse, using the distributive property with whole numbers, and generalizing the distributive property to other numbers and variables; score point of 3 shows that the teacher has an understanding of either the distributive property as repeated addition or of factoring as distribution in reverse and of using the distributive property with at least whole numbers; a score point of 2 shows the teacher has a rudimentary, procedural, rather than principle-based understanding of the properties of arithmetic; and a score point of 1 indicates that the teacher has no knowledge or understanding of the distributive property.

4	• Explain the distributive property as repeated addition • Explain factoring as distribution in reverse • Model the use of the distributive property with whole numbers • Model generalizing to other numbers and variables
3	*Either* • Explain the distributive property as repeated addition *Or* • Explain factoring as distribution in reverse • Model the use of the distributive property with at least whole numbers
2	• Explain procedures for how to use the distributive property, equating procedures with the order of operations
1	• No explanation of the distributive property

Figure 2. Scoring rubric for teaching the distributive property.

Validation Study

Currently, a study is being conducted to validate the self-rating survey and the performance tasks. One hundred, sixth-grade teachers have been recruited to participate in the study and they will complete the survey and performance tasks, and a multiple-choice measure of mathematics knowledge for pre-algebra and algebra, developed by researchers from the SII that address the key principles. The multiple-choice items will be scored according to the answer key for the measure, and after rater training, retired mathematics teachers will score the performance assessments using the rubrics for each task.

Reliability of scoring and the consistency of the responses to the various measures will be examined with generalizability studies. Factor analysis and correlation will be used to examine the distinctness and/or generality of the various measures. Results will be used to provide validity evidence for teacher measures used in the POWERSOURCE study and to define the most efficient set of measuring changes in teacher knowledge.

Conclusion

Currently, few valid and reliable measures exist to assess outcomes of professional development. Until recently, measures of teacher knowledge, such as licensing tests for example, focused on computational knowledge rather than on the kind of mathematical knowledge that teachers need to use in the context of the classroom. Therefore, professional development programs designed to improve teachers' knowledge for teaching lacked the appropriate instruments to measure growth.

The measures described in this paper have been designed to gauge growth in teachers' mathematical knowledge for teaching three key principles underlying mastery in algebra. However, we foresee that the conceptualization of measuring teacher knowledge that we have presented has the potential to adapt to other areas of mathematics and other subject areas, and to be used to evaluate teacher professional development programs.

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