CONCEPT MAP SCORING:
EMPIRICAL SUPPORT FOR A TRUNCATED JOINT POISSON
AND CONWAY-MAXWELL-POISSON DISTRIBUTION METHOD

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Concept map structure, testing, and scoring methods are discussed and a new scoring methodology is introduced using the breadth and depth of individual concept maps. The scoring method proposed here provides advantages of grading "on a curve" such as the ability to estimate and compare the complexity of different concept maps, the ability to measure the statistical distance between individual map scores, the ability to define map-score quantiles, a reduction in grading bias, and an overall improvement in the reliability and validity of concept map evaluation. Empirical support for underlying probability distributions is presented and mathematical aspects are discussed. Tables of concept map score distributions for one branch and one node are provided and may be used to very quickly assign concept map scores based on one branch and one node.

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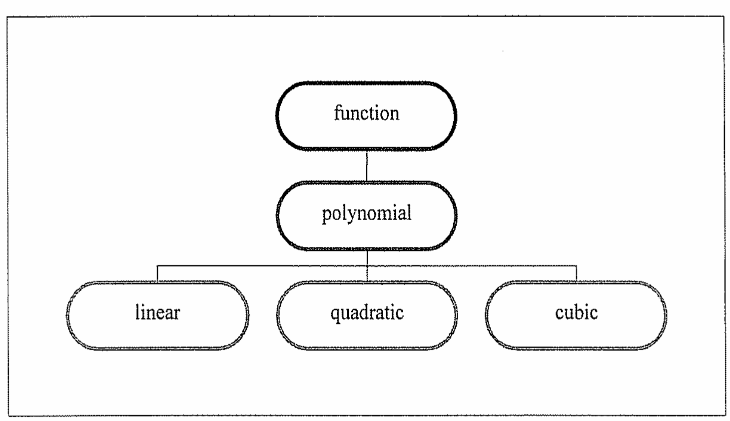
A concept map is a two-dimensional hierachical diagram that reflects how knowledge is organized. Concept maps are often used to evaluate science and math knowledge and are generally accepted as viable evaluation and research instruments. Concept maps may also be used as pre-assessment and post-assessment instruments, to assist in the clarification, consolidation, and reinforcement of knowledge, and in sequencing concepts in lessons and curricula.

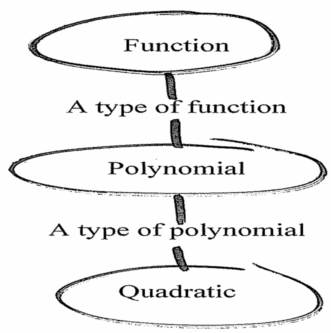
A concept map hierarchy has the most general concepts in the center (or at the top) and has branches that extend outward to the most specific concepts. Because of the hierarchical structure, concept maps reveal students' understandings of relationships between concepts in various areas and provide an alternative to traditional testing instruments. Moreover, concept maps measure a dimension of knowledge that is not usually assessed by traditional tests.

STRUCTURE

Concept maps might contain only concepts and connecting links (or arrows).

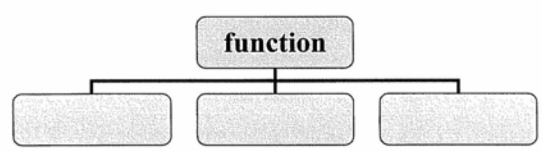


Concept maps might also contain propositions on the links between concepts.



TESTING

Concept map tests range from formats where students start with a blank piece of paper, to formats where lists of concept and relating propositions are provided, to where students start with a pre-arranged map structure with designated spaces for concepts and. In all cases, a key or base concept is provided.

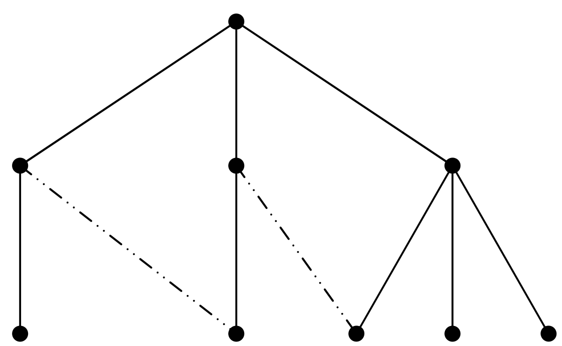


Students develop a list of related concepts from memory, from class notes and/or text book, or students may be given a list of concepts and/or propositions. Usually, the concepts are then organized into groups ranging from the most general, inclusive concepts to the most specific, least inclusive concepts.

SCORING

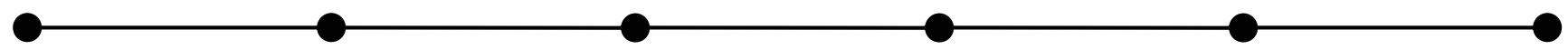
Typically, scores assigned to concept maps are based on the number of times various map features correctly appear in the maps. For example, the number of concepts, the number of hierarchy levels, the number of cross-links, the number of propositions, and/or the number of examples can all be use to compute a score. To help with the scoring process, the scorer might refer to lists of concepts, cross-links, propositions, and examples derived from expert-prepared maps so that a score reflect the similarity between the graded map and an expert map. After frequencies are found, the numbers might be weighted to give more credit to some features over others. For example, features might be weighted by 1 for each concept, 1 for each proposition, 2 for each example, 3 for the number of hierachy levels, and 3 for each cross-link. The weighted numbers are then added together to get a total performance score.

Figure 1.



The map in Figure 1 has eight concepts (not counting the base concept), zero propositions, zero examples, two hierarchy levels, and two crosslinks. Assuming all features are correct, the score for this map would be 1X8 + 1X0 + 2X0 + 3X2 + 3X2 = 20. One problem with this scoring methodology, however, is that highly dissimilar maps can end up with similar scores. For example, the map in Figure 2 has five concepts (not counting the base concept), zero propositions, zero examples, five hierarchy levels, and zero crosslinks. Assuming all features are correct, the score would be 1X5 + 1X0 + 2X0 + 3X5 + 3X0 = 20.

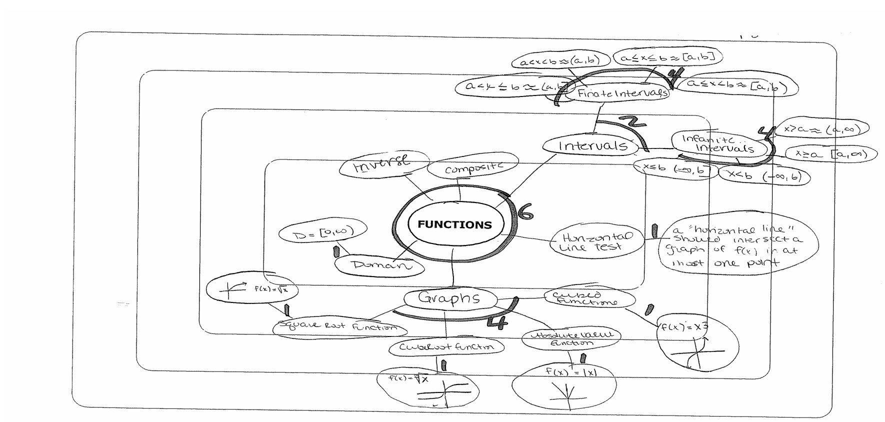
Figure 2.



A SCORING METHODOLOGY

After observing many concept maps drawn by many students, there are two concept map features that stand out in their ability to reflect the level of understanding shown in the concept map construction. These two features are the breadth of the map and depth of the map. That is, students with a better understanding of a particular domain tend to draw maps with a greater number of branches and a greater average branch length than do students with a poorer understanding in the domain. Thus, a straightforeward way to grade concept maps is to base the score on the average number of branches eminating from nonterminal concept nodes, and the average branch length. For example, Figure 3 shows how the average number of branches eminating from nonterminal concept nodes would be calculated for a particular concept map.

Figure 3.



score=(6+1+4+1+1+1+1+1+1+2+4+4)/11 = 2.36 branches per nonterminal concept

The greatest advantage in using the average number of branches eminating from nonterminal concept nodes together with the average branch length is that relative performance information becomes readily available. Specifically, relative performance scoring methods assume there is an underlying probability distribution that describes the concept maps scores. Even though debates continue as to whether relative-performance grading methods are better than performance-based methods in general, the advantages of grading concept maps “on a curve” are many. The relative performance method described below gives the ability to estimate and compare the complexity of different concept maps, the ability to measure the statistical distance between individual map scores, the ability to define map-score quantiles, a reduction in grading bias, and an overall improvement in the reliability and validity of concept map evaluation.

When concept maps with large numbers of both nonterminal and terminal concepts are scored based on the average number of branches eminating from nonterminal concepts and the average branch length, the conditions of the Central Limit Theorem are fairly well met. As the numbers increase, the distribution of the standardized scores approaches a normal probability distribution and all the advantages of “normed” grading method can be realized. However, if the numbers of nonterminal and terminal concepts are small, using a continuous normal distribution to model scores computed from discrete variables is not appropriate. Moreover, scoring large concept maps is generally not accurite and too time consuming. Instead, it is better to test students in very specific knowledge domains so that their maps are compact and manageable.

In the nearly 40 years that concept maps have been used as knowledge-evaluating instruments, no advancements have significantly improved concept-map assessment. However, goodness-of-fit tests reveal that when concept maps are scored using the average number of branches eminating from nonterminal concepts and the average branch length, the scores are accurately described by a certain type of modified joint Poisson distribution. The theoretical argument for why a Poisson model describes the number of branches eminating from nonterminal concept nodes is as follows:

WHY A POISSON DISTRIBUTION

For the number of branches eminating from nonterminal concepts, consider all concepts that could be correctly linked to a specific node. Partition these concepts into classes of equal difficulty. That is, create sets of concepts of the same cognitive level. Call the sets difficulty classes. If n concepts are in a difficulty class, the probability that x out of n are linked to a specific node is given by a binomial probability model provided the following two conditions hold:

(i) the chance of selecting any new concept is constant for all concepts (uniformity), and

(ii) branching to a new concept is independent of any concept already in the map (independence).

In general, condition (ii) above does not hold. However, if a linked concept within a difficulty class is counted only when no other linked concepts are hints for (correlated with) any other linked concept, then the independence assumption is reasonable.

If the uniformity and independence assumptions hold, and if the number of concepts within each difficulty class is large, then, the Poisson distribution approximates the probability that x concepts are selected. Further, if the difficulty classes are independent, then the TOTAL number of concepts selected over ALL difficulty classes also has a Poisson distribution.

An important property of the Poisson distribution is that the mean number of concepts is the sum of the mean number of concepts for each of the difficulty classes. More specifically, let

Y1 + Y2 be the sum of the numbers of concepts from the 1st and 2nd difficulty levels. It follows that if Y1 ~ P(m1) (Piosson with mean=m1) and if Y2 ~ P(m2), then Y1 + Y2 ~ P(m1 + m2).

A similar argument could be made for branch lengths having a Poisson distribution. However, from an examination of concept map data it was found that the variance of branch length was less than the variance of the number of branches eminating from non-terminal nodes. Thus, a standard Poisson distribution (where the mean equals the variance) did not accurately describe the data.. The Conway-Maxwell-Poisson (CMP) distribution gave a fit that was better than the Poisson distribution because this distribution has a second parameter that affects the variance. But as with the Poisson distribution, the variance of the CMP distribution was still too large. A modification to a truncated CMP distribution that reduced the variance (see below) worked well in describing branch length and, moreover, only one parameter was needed.

TRUNCATED DISTRIBUTIONS

There is one more consideration in using Poisson and modified CMP models to describe concept map scores. A truncated Poisson model is used when the number of zero observations is unknown or unobserved, or if the population size is unknown. In the case of concept maps, the number of concepts that are NOT linked to a terminal node is unknown and unobserved, and in many cases, the number of concepts that might be linked to a given concept is unknown.

Some other examples of when truncated Poisson models might be used are as follows. Consider the number of people with an infectious disease (e.g. measles) per household -- the number of people that do not get sick varies with household size (population size). Consider the number of accidents in a factory -- the number of employees varies per factory, so the number who do not have accidents is unknown. Consider the number of butterflies caught in a net in one pass through a field -- the number of butterflies not caught is unknown. Consider the number of words that are repeated once, twice, three times, etc. -- the number of words repeating zero times is unknown.

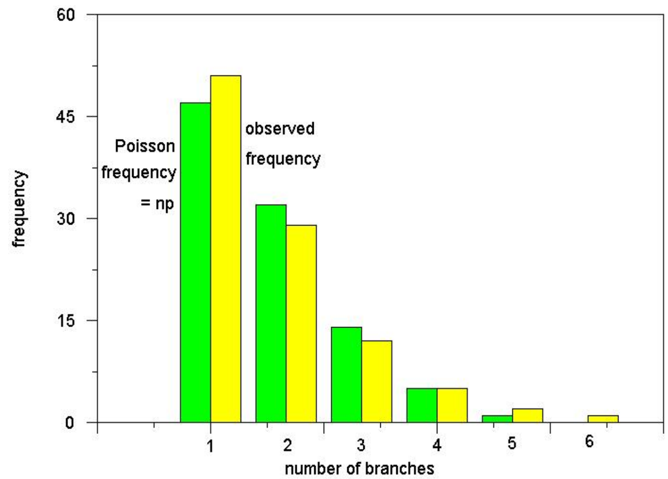
EMPIRICAL SUPPORT FOR TRUNCATED POISSON AND MODIFIED CMP MODELS

The following five tables present the results of goodness-of-fit tests that show how closely truncated Poisson distributions fit the empirical distributions of the number of branches eminating from nonterminal concepts.

The data in Table 1 show the number of links (branches) eminating from nonterminal concepts and the observed frequencies (for example, there were 29 nodes with 2 eminating branches) for students who completed concept maps in Class I. A chi-square test shows the empirical and truncated Poisson distributions are very close. One would have to accept a 70% chance of being wrong to conclude that the empirical and Poisson distributions were different.

Table 1.

			Data I			
No. links	**Obs Freq.**	**Mean**	**Lambda**	**Trunc Poisson**	**Exp Freq**	**(o-e)^2 / e**
1	51			0.471116	47	0.32092
2	29			0.319098	32	0.26535
3	12			0.144089	14	0.40272
4	5			0.048797	5	0.00296
5	2			0.013220	1	0.34762
6	1			0.002984	0	1.64867
Total	100	1.81	1.3546	Chi-sq=	2.9882	Alpha =
					(k-1) = 5 d.f.	0.70

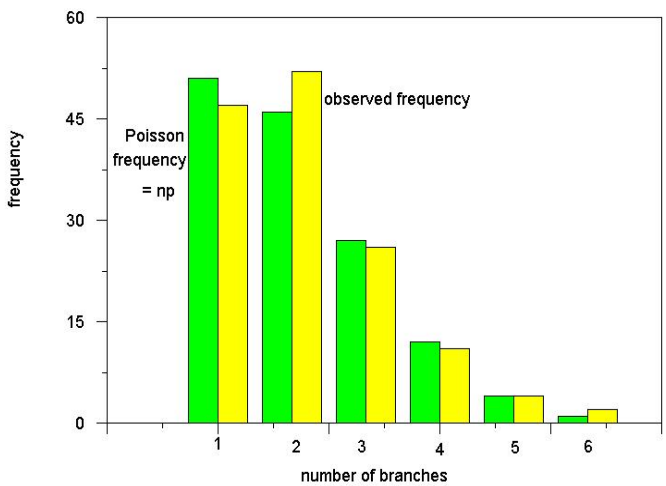


The data in Table 2 show the number of branches eminating from nonterminal concepts and the observed frequency of each number (for example, there were 2 nodes having 6(not shown) eminating branches) for students who completed concept maps in Class II. A chi-square test shows the empirical and truncated Poisson distributions are very close. One would have to accept a 88% chance of being wrong to conclude that the empirical and Poisson distributions were different.

Table 2.

Data II

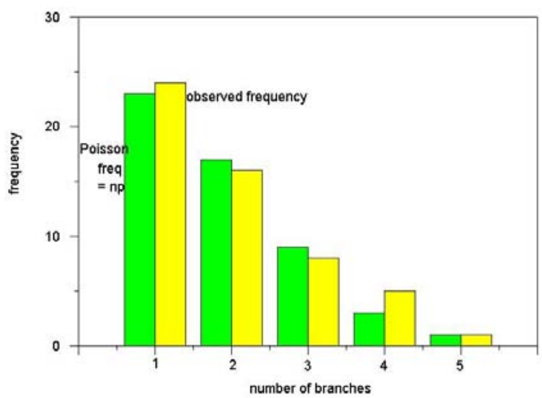
ObsFreq	Mean	Lambda	Trunc Poisson	Exp Freq	(o-e)^2 / e
47			0.357616	51	0.28159
52			0.320953	46	0.90565
26			0.192032	27	0.05902
11			0.086172	12	0.12495
4			0.030935	4	0.03512
2			0.009254	1	0.35793
142	2.1478	1.7949	Chi-sq =	1.7642	Alpha
				(n-1) = 5 d.f.	=0.88



The data in Table 3 show the number of branches eminating from nonterminal concepts and the observed frequency of each number (for example, there were 5 nodes having 4 eminating branches) for students who completed concept maps in Class III. A chi-square test shows the empirical and truncated Poisson distributions are very close. One would have to accept a 91% chance of being wrong to conclude that the empirical and Poisson distributions were different.

Table 3.

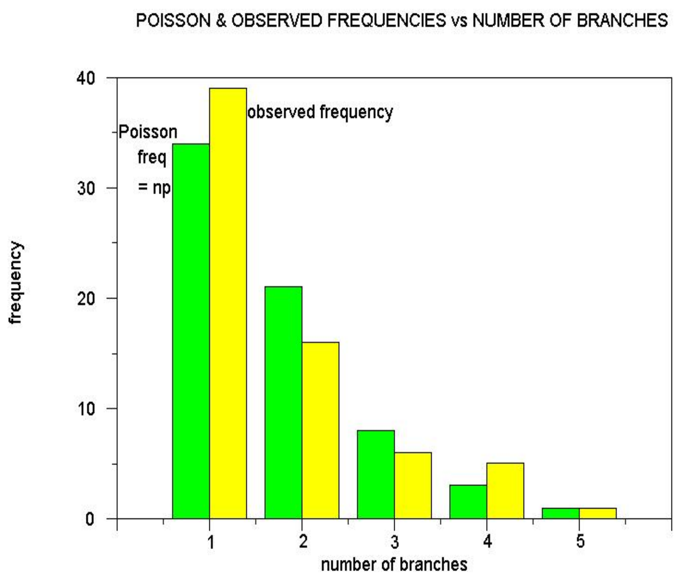
			Data III			
No. links	Obs Freq	Mean	Lambda	Trunc Poisson	Exp Freq	(o-e)^2 / e
1	24			0.4223	23	0.062380
2	16			0.3235	17	0.123623
3	8			0.1651	9	0.095032
4	5			0.0632	3	0.733935
5	1			0.0193	1	0.002089
Total	54	1.944	1.5319	Chi-sq =	1.0170	Alpha=
					(n-1) =4 df	0.91



The data in Table 4 show the number of branches eminating from nonterminal concepts (x) and the observed frequency of each number (for example, there were 16 nodes having 2 eminating branches) for students who completed concept maps in Class IV. A chi-square test shows the empirical and truncated Poisson distributions are very close. One would have to accept a 30% chance of being wrong to conclude that the empirical and Poisson distributions were different.

Table 4.

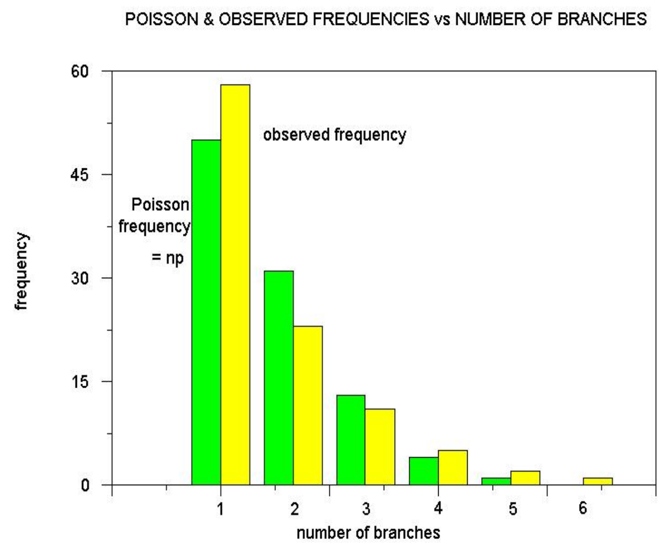
	x	Poisson	exp freq	obs freq	$(o-e)^2/e$
x-bar=					
1.70149	1	0.514071	34	39	0.60298
	2	0.311063	21	16	1.124577
	3	0.125482	8	6	0.689302
	4	0.037965	3	5	2.372122
lambda=	5	0.009189	1	1	0.23994
1.2102					
N=67			chi-sq=	5.02892	alpha=0.30



The data in Table 5 show the number of branches eminating from nonterminal concepts (x) and the observed frequency of each number (for example, there were 58 nodes having 1 eminating branch) for students who completed concept maps in Class V. A chi-square test shows the empirical and truncated Poisson distributions are very close. One would have to accept a 17% chance of being wrong to conclude that the empirical and Poisson distributions were different.

Table 5.

	x	Poisson	exp freq	obs freq	$(o-e)^2/e$
x-bar=					
1.73	1	0.502475	50	58	1.196113
	2	0.313601	31	23	2.22867
	3	0.130482	13	11	0.3215
	4	0.040718	4	5	0.211608
	5	0.010165	1	2	0.951581
	6	0.002115	0	1	2.940282
Lambda =					
1.248226					
n=99		chi-sq =	7.849754	k-1 = 5 d.f.	alpha= 0.17



BRANCH LENGTH

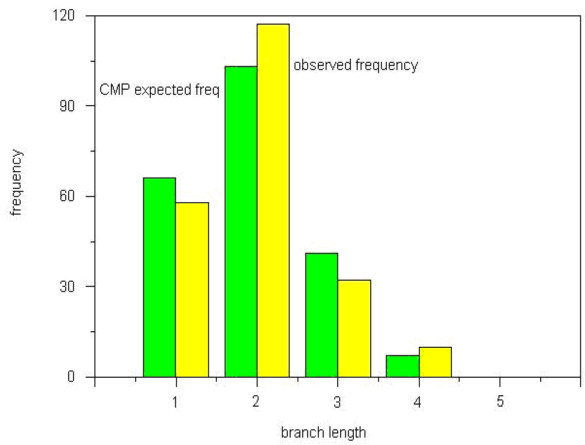
The following two tables present the results of goodness-of-fit tests that show how closely truncated and modified CMP distributions fit branch length data collected from two classes.

The data in Table 6 show branch lengths and the observed frequency of each length (for example, there were 117 branches of length 2) for students who completed concept maps in Class VI. A chi-square test shows the empirical and truncated Poisson distributions are close. One would have to accept a 12.4% chance of being wrong to conclude that the empirical and modified CMP distributions were different.

Table 6.

Branch Length	Obs Freq	Mean	Lambda	Dist	Exp Freq	(o-e)^2/e
1	58			0.3	66	0.88
2	117			0.47	103	1.93
3	32			0.19	41	2.02

4	10			0.03	7	0.92
5	0			0	0	0
Total=	217	1.97	1.57	Chi-sq=	5.76	Alpha =
					3 degrees of freedom	0.124

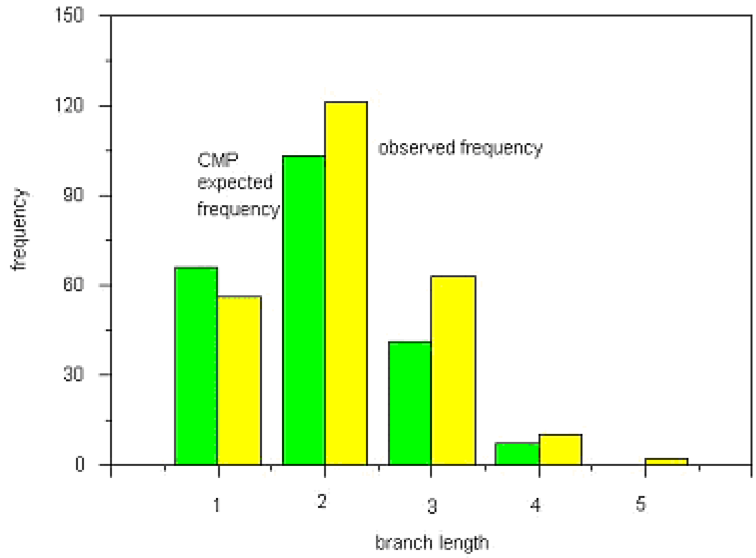


The data in Table 7 show branch lengths and the observed frequency of each length (for example, there were 10 branches of length 4) for students who completed concept maps in Class VII. A chi-square test shows the empirical and truncated Poisson distributions are close. One would have to accept a 11.8% chance of being wrong to conclude that the empirical and modified CMP distributions were different.

Table 7.

Branch Length	**Obs Freq**	**Mean**	**Lambda**	**Dist**	**Exp Freq**	**(o-e)^2/e**
1	58			0.28	72	2.58
2	121			0.48	122	0.01

3	61			0.2	49	2.83
4	10			0.03	9	0.24
5	2			0	1	1.69
Total=	256	2.09	1.72	Chi-sq=	7.35 4 degrees of freedom	Alpha = 0.118



MATHEMATICAL BACKGROUND

The complete Poisson distribution may be expressed as

$$f(x, \lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}, x = 0, 1, 2, \ldots$$

Where $f(x, \lambda)$ is the probability of observing exactly x occurrences of the event under consideration and λ is the mean and variance of the number of occurrences.

The probability of zero occurrences is

$$f(0, \lambda) = e^{-\lambda}$$

so the probability of one or more occurrences is

$$1 - e^{-\lambda}$$

If x=0 is not being considered, the remaining probabilities for x=1, 2, ... are normalized by multiplying by

$$\frac{1}{1 - e^{-\lambda}}$$

Then the Poisson distribution, truncated below 1 is

$$f(x, \lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} \cdot \frac{1}{1 - e^{-\lambda}}, \; x = 1, 2, \ldots$$

Or

$$f(x, \lambda) = \frac{\lambda^{x}}{x!(e^{\lambda} - 1)}, \; x = 1, 2, \ldots$$

Then

$$\sum_{x=1}^{\infty} \frac{\lambda^{x}}{x!(e^{\lambda} - 1)} = 1$$

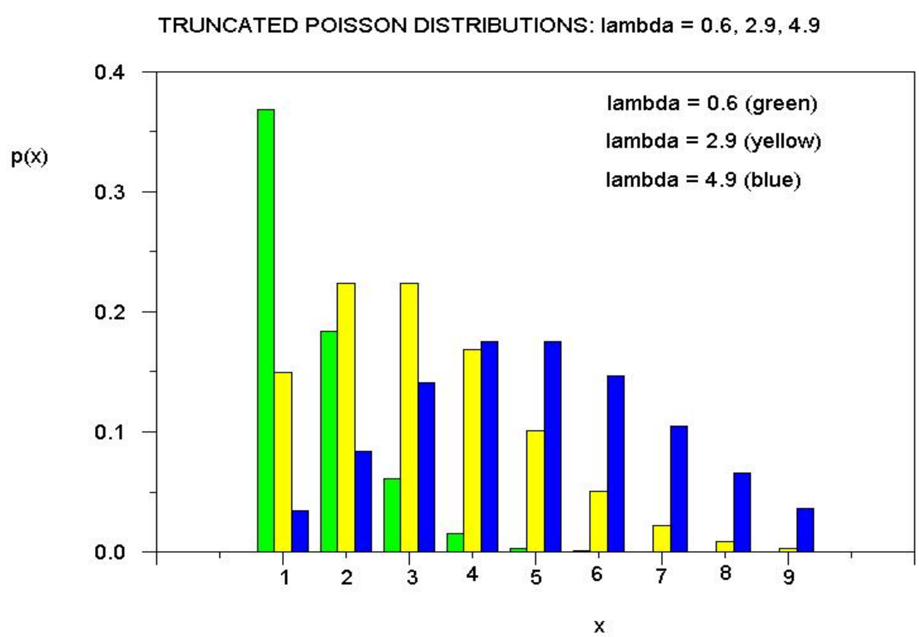
and

$$\frac{f(x, \lambda)}{f(x-1, \lambda)} = \frac{\lambda}{x}$$

and, just as with the complete Poisson distribution

$$f(x, \lambda) = \frac{\lambda}{x} f(x-1, \lambda)$$

The shape of the truncated Poisson distribution is shown in the figure below for various values of lambda.



Parameter Estimation

If $\bar{x}$ is the mean of a sample from a Poisson distribution with the zero class missing, then the maximum likelihood estimate of the population mean is

$$\widehat{\lambda} = \bar{x} - \sum_{j=1}^{\infty} \frac{j^{j-1}}{j!}(\bar{x}e^{-\bar{x}})^{j}$$

then the maximum likelihood estimate of the population mean is

$$\widehat{\lambda} = \bar{x} - \sum_{j=1}^{\infty} \frac{j^{j-1}}{j!}(\bar{x}e^{-\bar{x}})^{j}$$

Or

$$\bar{\lambda} = \bar{x} - \bar{x}e^{-\bar{x}} - (\bar{x}e^{-\bar{x}})^{2} - \frac{3}{2}(\bar{x}e^{-\bar{x}})^{3}$$

$$-\frac{3}{2}(\bar{x}e^{-\bar{x}})^{3} - \frac{8}{3}(\bar{x}e^{-\bar{x}})^{4} - \frac{125}{24}(\bar{x}e^{-\bar{x}})^{5} \dots$$

The CMP distribution is given by

$$f(y, \lambda, \nu) = \frac{\lambda^{y}}{(y!)^{\nu}} \cdot \frac{1}{\sum_{k=0}^{\infty} \frac{\lambda^{k}}{(k!)^{\nu}}} \qquad y = 0, 1, 2, \dots$$

The truncated CMP distribution is given by

$$f(y, \lambda, \nu) = \frac{\lambda^y}{(y!)^\nu} \cdot \frac{1}{\sum_{k=1}^{\infty} \frac{\lambda^k}{(k!)^\nu}} \qquad y = 1, 2, \ldots$$

The joint distribution has the form

$$\left(f(x, y, \lambda_1, \lambda_2, \nu, c) = \frac{(\lambda_1 - c)^x (\lambda_2 - c)^y}{x!(y!)^\nu} \right) \times normalizing \qquad term$$

Where X has a truncated Poission distribution, Y has a truncated CMP distribution, and c is the covaiance of X and Y. Note that because the joint distribution must be positive, c must be less than the minimum of lambda_1 and lambda_2. It is clear that the number of branches and branch length are positively correlated but the standard error of the covariance c may be small for all math and science concept maps. The scoring process would be simplified if a universal constant could be used for c. This is being investigated by the author but for now, the population covariance should be computed in the traditional way.

The above distribution may be used to construct tables of cummulative score distributions. Concept map scores may be assigned based on the cummulative values of the above joint truncated distribution for various population mean branch length and population mean number of branch values are given in the tables below. The tables below give cummulative percentages for the joint Poisson / Conway-Maxwell-Poisson distribution. Tables for sums of branches from nodes, and sums of branch lengths (that may be used to assign more valid concept map test scores) are being investigated. The tables below may be used to quickly assign scores for individual concept maps based on one branch and one node selcted from each map.

CONCEPT MAP CUMMULATIVE PROBABILITY TABLES (for one node and one branch)

P(X<=x and Y<=y)

Mean number of branches (X-bar) = 1.6
Mean branch length (Y-bar) = 1.6
Covariance (X,Y) = 0.2

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	47	60	62	62	62	62
number	**2**	67	86	89	90	90	90
of	**3**	73	94	97	98	98	98
branches (one node)	**4**	74	96	99	99	99	99
x	**5**	75	96	99	100	100	100
	6	75	96	99	100	100	100

Ave number of branches (X-bar) = 1.6

Mean branch length (Y-bar) = 1.6
Covariance (X,Y) = 0.4

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	56	68	70	70	70	70
number	**2**	75	92	93	94	94	94
of	**3**	79	97	99	99	99	99
branches (one node)	**4**	80	98	100	100	100	100
x	**5**	80	98	100	100	100	100
	6	80	98	100	100	100	100

Ave number of branches (X-bar) = 1.6
Mean branch length (Y-bar) = 1.6
Covariance (X,Y) = 0.6

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	66	77	78	78	78	78
number	**2**	82	95	96	96	96	96
of	**3**	85	98	99	99	99	99
branches (one node)	**4**	85	98	99	100	100	100
x	**5**	85	98	99	100	100	100
	6	85	98	99	100	100	100

Ave number of branches (X-bar) = 1.6
Mean branch length (Y-bar) = 1.6
Covariance (X,Y) = 0.8

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	79	86	87	87	87	87
number	**2**	90	98	99	99	99	99
of	**3**	91	99	100	100	100	100
branches (one node)	**4**	91	99	100	100	100	100
x	**5**	91	99	100	100	100	100
	6	91	99	100	100	100	100

Ave number of branches (X-bar) = 1.6
Mean branch length (Y-bar) = 1.6
Covariance (X,Y) = 1

	length of one branch y					
	1	**2**	**3**	**4**	**5**	**6**
1	94	96	96	96	96	96

number	**2**	97	100	100	100	100	100
of	**3**	97	100	100	100	100	100
branches (one node)	**4**	97	100	100	100	100	100
x	**5**	97	100	100	100	100	100
	6	97	100	100	100	100	100

Ave number of branches (X-bar) = 1.8
Mean branch length (Y-bar) = 1.6
Covariance (X,Y) = 0.2

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	40	51	53	53	53	53
number	**2**	63	81	84	84	84	84
of	**3**	71	92	95	95	95	95
branches (one node)	**4**	74	95	98	99	99	99
x	**5**	74	96	99	99	99	99
	6	75	96	99	100	100	100

Ave number of branches (X-bar) = 1.8
Mean branch length (Y-bar) = 1.6
Covariance (X,Y) = 0.4

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	48	59	60	60	60	60
number	**2**	71	86	88	88	88	88
of	**3**	78	95	97	97	97	97
branches (one node)	**4**	79	97	99	99	99	99
x	**5**	80	97	99	100	100	100
	6	80	97	99	100	100	100

Ave number of branches (X-bar) = 1.8
Mean branch length (Y-bar) = 1.6
Covariance (X,Y) = 0.6

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	58	67	67	68	68	68
number	**2**	79	91	93	93	93	93
of	**3**	84	98	99	99	99	99
branches (one node)	**4**	85	99	100	100	100	100
x	**5**	86	99	100	100	100	100
	6	86	99	100	100	100	100

Mean number of branches (X-bar) = 1.8
Mean branch length (Y-bar) = 1.6
Covariance (X,Y) = 0.8

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	69	75	76	76	76	76
number	**2**	88	96	96	96	96	96
of	**3**	91	99	100	100	100	100
branches (one node)	**4**	92	100	100	100	100	100
x	**5**	92	100	100	100	100	100
	6	92	100	100	100	100	100

Ave number of branches (X-bar) = 1.8
Mean branch length (Y-bar) = 1.6
Covariance (X,Y) = 1

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	82	84	84	84	84	84
number	**2**	96	98	98	98	98	98
of	**3**	97	99	100	100	100	100
branches (one node)	**4**	97	100	100	100	100	100
x	**5**	97	100	100	100	100	100
	6	97	100	100	100	100	100

Mean number of branches (X-bar) = 1.6
Mean branch length (Y-bar) = 1.8
Covariance (X,Y) = 0.2

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	45	60	62	63	63	63
number	**2**	65	86	90	90	90	90
of	**3**	71	94	98	98	98	98
branches (one node)	**4**	72	96	99	100	100	100
x	**5**	72	96	100	100	100	100
	6	72	96	100	100	100	100

Mean number of branches (X-bar) = 1.6

Mean branch length (Y-bar) = 1.8
Covariance (X,Y) = 0.4

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	53	68	70	70	70	70
number	**2**	71	91	93	93	93	93
of	**3**	75	96	98	99	99	99
branches (one node)	**4**	76	97	99	100	100	100
x	**5**	76	97	99	100	100	100
	6	76	97	99	100	100	100

Mean number of branches (X-bar) = 1.6
Mean branch length (Y-bar) = 1.8
Covariance (X,Y) = 0.6

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	63	76	78	78	78	78
number	**2**	78	95	96	96	96	96
of	**3**	80	97	99	99	99	99
branches (one node)	**4**	81	98	100	100	100	100
x	**5**	81	98	100	100	100	100
	6	81	98	100	100	100	100

Mean number of branches (X-bar) = 1.6
Mean branch length (Y-bar) = 1.8
Covariance (X,Y) = 0.8

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	74	86	87	87	87	87
number	**2**	84	98	99	99	99	99
of	**3**	85	99	100	100	100	100
branches (one node)	**4**	85	99	100	100	100	100
x	**5**	85	99	100	100	100	100
	6	85	99	100	100	100	100

Mean number of branches (X-bar) = 1.6
Mean branch length (Y-bar) = 1.8
Covariance (X,Y) = 1

	length of one branch y					
	1	**2**	**3**	**4**	**5**	**6**
1	88	96	97	97	97	97

number	**2**	91	100	100	100	100	100
of	**3**	91	100	100	100	100	100
branches (one node)	**4**	91	100	100	100	100	100
x	**5**	91	100	100	100	100	100
	6	91	100	100	100	100	100

Mean number of branches (X-bar) = 1.8
Mean branch length (Y-bar) = 1.8
Covariance (X,Y) = 0.2

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	39	51	53	54	54	54
number	**2**	61	81	84	84	84	84
of	**3**	69	92	95	96	96	96
branches (one node)	**4**	72	95	99	99	99	99
x	**5**	72	96	99	100	100	100
	6	72	96	100	100	100	100

Mean number of branches (X-bar) = 1.8
Mean branch length (Y-bar) = 1.8
Covariance (X,Y) = 0.4

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	46	59	60	60	60	60
number	**2**	68	86	88	89	89	89
of	**3**	75	95	97	98	98	98
branches (one node)	**4**	76	97	99	100	100	100
x	**5**	76	97	100	100	100	100
	6	77	97	100	100	100	100

Mean number of branches (X-bar) = 1.8
Mean branch length (Y-bar) = 1.8
Covariance (X,Y) = 0.6

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	54	66	67	67	67	67
number	**2**	75	90	92	92	92	92
of	**3**	80	97	98	98	98	98
branches (one node)	**4**	80	98	99	99	100	100
x	**5**	81	98	100	100	100	100
	6	81	98	100	100	100	100

Mean number of branches (X-bar) = 1.8
Mean branch length (Y-bar) = 1.8
Covariance (X,Y) = 0.8

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	65	75	76	76	76	76
number	**2**	82	95	96	96	96	96
of	**3**	85	99	100	100	100	100
branches (one node)	**4**	86	99	100	100	100	100
x	**5**	86	99	100	100	100	100
	6	86	99	100	100	100	100

Mean number of branches (X-bar) = 1.8
Mean branch length (Y-bar) = 1.8
Covariance (X,Y) = 1

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	76	84	84	84	84	84
number	**2**	89	98	99	99	99	99
of	**3**	91	100	100	100	100	100
branches (one node)	**4**	91	100	100	100	100	100
x	**5**	91	100	100	100	100	100
	6	91	100	100	100	100	100

Mean number of branches (X-bar) = 2
Mean branch length (Y-bar) = 1.8
Covariance (X,Y) = 0.2

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	33	44	46	46	46	46
number	**2**	56	75	78	78	78	78
of	**3**	67	89	93	93	93	93
branches (one node)	**4**	71	94	98	98	98	98
x	**5**	72	96	99	100	100	100
	6	72	96	100	100	100	100

Mean number of branches (X-bar) = 2
Mean branch length (Y-bar) = 1.8

Covariance (X,Y) = 0.4

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	39	50	52	52	52	52
number	**2**	63	80	83	83	83	83
of	**3**	73	92	95	95	95	95
branches (one node)	**4**	76	96	99	99	99	99
x	**5**	76	97	100	100	100	100
	6	76	97	100	100	100	100

Mean number of branches (X-bar) = 2
Mean branch length (Y-bar) = 1.8
Covariance (X,Y) = 0.6

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	47	57	58	58	58	58
number	**2**	70	85	87	87	87	87
of	**3**	78	95	97	97	97	97
branches (one node)	**4**	80	97	99	99	99	99
x	**5**	81	98	100	100	100	100
	6	81	98	100	100	100	100

Mean number of branches (X-bar) = 2
Mean branch length (Y-bar) = 1.8
Covariance (X,Y) = 0.8

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	56	64	65	65	65	65
number	**2**	78	90	91	91	91	91
of	**3**	84	97	98	98	98	98
branches (one node)	**4**	85	98	99	99	99	99
x	**5**	85	99	100	100	100	100
	6	85	99	100	100	100	100

Mean number of branches (X-bar) = 2
Mean branch length (Y-bar) = 1.8
Covariance (X,Y) = 1

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	66	73	73	73	73	73
number	**2**	86	94	95	95	95	95

of	**3**	90	99	99	99	99	99
branches (one node)	**4**	91	99	100	100	100	100
x	**5**	91	100	100	100	100	100
	6	91	100	100	100	100	100

Mean number of branches (X-bar) = 1.8
Mean branch length (Y-bar) = 2
Covariance (X,Y) = 0.2

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	38	52	54	54	54	54
number	**2**	60	81	84	85	85	85
of	**3**	68	92	96	96	96	96
branches (one node)	**4**	71	95	99	100	100	100
x	**5**	71	96	100	100	100	100
	6	71	96	100	100	100	100

Mean number of branches (X-bar) = 1.8
Mean branch length (Y-bar) = 2
Covariance (X,Y) = 0.4

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	45	58	60	60	60	60
number	**2**	66	86	89	89	89	89
of	**3**	73	95	97	98	98	98
branches (one node)	**4**	74	97	100	100	100	100
x	**5**	75	97	100	100	100	100
	6	75	97	100	100	100	100

Mean number of branches (X-bar) = 1.8
Mean branch length (Y-bar) = 2
Covariance (X,Y) = 0.6

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	53	66	68	68	68	68
number	**2**	72	91	93	93	93	93
of	**3**	77	97	99	99	99	99
branches (one node)	**4**	78	98	100	100	100	100
x	**5**	78	98	100	100	100	100
	6	78	98	100	100	100	100

Mean number of branches (X-bar) = 1.8
Mean branch length (Y-bar) = 2
Covariance (X,Y) = 0.8

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	62	74	75	76	76	76
number	**2**	79	95	96	96	96	96
of	**3**	82	98	100	100	100	100
branches (one node)	**4**	82	99	100	100	100	100
x	**5**	82	99	100	100	100	100
	6	82	99	100	100	100	100

Mean number of branches (X-bar) = 1.8
Mean branch length (Y-bar) = 2
Covariance (X,Y) = 1

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	72	83	84	84	84	84
number	**2**	85	97	98	98	98	98
of	**3**	86	99	100	100	100	100
branches (one node)	**4**	86	99	100	100	100	100
x	**5**	86	99	100	100	100	100
	6	86	99	100	100	100	100

Mean number of branches (X-bar) = 2
Mean branch length (Y-bar) = 2
Covariance (X,Y) = 0.2

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	33	44	46	46	46	46
number	**2**	55	75	78	78	78	78
of	**3**	66	89	93	93	93	93
branches (one node)	**4**	70	94	98	99	99	99
x	**5**	71	96	100	100	100	100
	6	71	96	100	100	100	100

Mean number of branches (X-bar) = 2
Mean branch length (Y-bar) = 2
Covariance 0.4

(X,Y) =

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	38	50	52	52	52	52
number	**2**	62	80	83	83	83	83
of	**3**	71	92	95	95	95	95
branches (one node)	**4**	74	96	99	99	99	99
x	**5**	74	97	100	100	100	100
	6	74	97	100	100	100	100

Mean number of branches (X-bar) = 2
Mean branch length (Y-bar) = 2
Covariance (X,Y) = 0.6

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	45	57	58	58	58	58
number	**2**	68	85	87	87	87	87
of	**3**	76	95	97	97	97	97
branches (one node)	**4**	78	97	99	100	100	100
x	**5**	78	98	100	100	100	100
	6	78	98	100	100	100	100

Mean number of branches (X-bar) = 2
Mean branch length (Y-bar) = 2
Covariance (X,Y) = 0.8

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	53	64	65	65	65	65
number	**2**	75	90	91	91	91	91
of	**3**	81	97	98	98	98	98
branches (one node)	**4**	82	98	100	100	100	100
x	**5**	82	99	100	100	100	100
	6	82	99	100	100	100	100

Mean number of branches (X-bar) = 2
Mean branch length (Y-bar) = 2
Covariance (X,Y) = 1

		length of one branch y					
		1	**2**	**3**	**4**	**5**	**6**
	1	63	72	73	73	73	73
number	**2**	82	94	95	95	95	95
of	**3**	86	99	99	100	100	100

branches (one node)	**4**	86	99	100	100	100	100
x	**5**	86	99	100	100	100	100
	6	86	99	100	100	100	100