TWO IMPORTANT INVARIANT TASKS IN SOLVING EQUATIONS: ANALIZING THE EQUATION AND CHECKING THE VALIDITY OF TRANSFORMATIONS.

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Our study concerns the analysis of teacher and student activities. Secondary school 6^{th} grade students were confronted, for the firs time, with solving equations. We used our cognitive models of students and experts (in algebraic calculations) for analyzing the teaching process and the cognitive functioning of students. Our model led us to consider the management of mathematical justifications as a fundamental teachers' task. We believe that these models can become a daily tool for teachers, and principally for newer teachers

The analysis of teacher and student activities was identified as an important research object in PME conferences. Boaler, J.(2003) observed : « One interesting observation from our coding of class time was the increased time that teachers spent questioning the whole class in the reform classes. Whereas the teachers in the traditional classes gave students a lot of information; the teachers of the reform approach chose to draw information out of students, by presenting problems *and asking students questions*". A good teacher's activity can be resumed in the following manner: "when students were unsure how to proceed with open problems the teacher encouraged the students to engage in these practices: exploring, orienting, representing, generalizing and justifying. … *rather than deflecting her authority to the discipline : is this correct?"*

Teachers need to analyze problems and student cognitive functioning in order to be able to guide the construction of student competencies in mathematics. Teacher need cognitive models to understand and to anticipate student errors and difficulties. How can students be guided in constructing competencies? We attempt to answer to this question here.

This research is based on our previous researches, principally: "A cognitive model of experts' algebraic solving methods" (Cortés, A. (2003)) and "Solving equations and inequations, operational invariants and methods constructed by students" (Cortés, A. & Pfaff, N. (2000)). In these researches, we observed that most students (10^{th} grade) use transformation rules without mathematical justification and that their solving methods resemble algorithms. In contrast, what makes teachers' solving methods effective is the mathematical justification of transformations. The respect of this fundamental characteristic guided the teaching process that we analyze in this article.

The main goal of our study is to use our cognitive models in constructing a tool for analyzing student activities and the teaching process. To discuss the relevance or the best manner of teaching pre-algebra is not our goal.

The experimental class and the research. The official curriculum for sixth grade in France includes the solving of equations. We followed this prescription and our sixth grade students were confronted, for the first time, with solving equations of the type: x+a=b; x-a=b; x/a=b; a/x=b; ax+b=c; b-ax=c; where a, b and c are positive decimal numbers. Our students had never worked with numbers with sign (oriented numbers), so the solutions we asked them to calculate were never negative numbers. The sixth grade is the first year of the secondary cycle , the average age of our thirty students was eleven years old and students' level of knowledge was good. The experiment was conducted in a secondary school situated in the north-east of Paris. We experimented for three hours with solving of equations and the teacher of the class was Mrs. Kavafian. The data analyzed were the students' written work and the recorded interaction between the teacher and the students. During the experiment, students also solved word-problems that we do not analyze in this article.

THEORETICAL FRAMEWORK

The students explicitly explored the concept of the equation, the concept of the unknown and they used transformations for the first time. We focused our work on two tasks that we consider essential in algebraic calculations: the analysis of the mathematical object and the checking of the validity of transformations. In this theoretical framework we only included the used results of our previous researches and the theoretical concepts used in constructing the learning courses and in the analysis of data.

In Cortés A. (2003) five invariant tasks were identified. These are the tasks that the expert always carries out (implicitly or explicitly) in performing transformations. Each task is carried out by means of some specific piece of knowledge or by means of a competence, that we call the operational invariant of the task. The concept of operational invariant was introduced by Piaget (for example, Piaget (1950) considered the conservation principles in physics as operational invariants of physical thought). The invariant tasks of algebraic calculations, adapted to the solving of equations in our experiment are:

1- Analyzing the equation and choosing a transformation. The operational invariant is the concept of the equation.

The concept of the equation enables subjects to carry out the analyses which lead to choosing the right transformation. However, our students were being confronted with equalities in which the unknown was represented by a letter for the first time. So, during the experiment they explicitly began the construction of the concept of the equation. The teacher guided the students in conceptualizing the letter as an unknown number, the notation of multiplication as a juxtaposition of the letter and the coefficient and, notably, the meaning of the equal sign: an equivalence which can either be "true" or "false".

The teacher guided the students in analyzing the equations appropriately according to the solving method. First, equations were solved by substituting numbers for the unknown. Later, equations were solved by means of transformations.

Frequently, the analysis of the equation lead the students to make inferences like "It's an addition, so I should subtract", "It's a multiplication, so I should divide", etc.; which are false in certain situations. These transformation rules are theorems in action (Vergnaud G. (1990)); they are mathematical properties that the students use automatically without mathematical justification.

2- Identifying the operation to be given priority.

The identification (usually implicit) of the operation to be given priority allows choosing a relevant transformation. There is a multiplicity of operations and the priority of operations depends on the situation. Subjects need to have knowledge concerning each pair of operations involved in a particular situation; in this sense the operational invariant is composite. Similar operational invariants were found by Pastré, P. (1997), in the area of cognitive ergonomics. In the present research, students were confronted with the priority of multiplication over addition and subtraction.

3- Checking the validity of the transformation. A mathematical justification of the transformation is the operational invariant of the task.

A mathematical justification of the transformation chosen establishes a link between the mathematical properties of the equation and the transformation. A mathematical justification allows subjects to check the validity of transformations. In this research the mathematical justifications used were, principally, the students' knowledge of arithmetic.

a- Operational invariants of the "principle of conservation" type: the conservation of the truth-values of equations.

At the beginning of the experiment equations were solved by substituting numbers for the unknown: the truth-values of the equation allowed students to identify the solution. This mathematical property is "self-justified" or "self-evident" for the students and is the mathematical justification of the procedure. Students implicitly used the conservation of the truth-value when they verified solutions obtained by transformations (by substitution in the given equation).

All authorized transformations conserve the truth-values of mathematical expressions and this property constitutes the more general mathematical justification and filiation of transformations in solving equations, inequations, systems of equations... We believe it is relevant to use this principle in teaching algebra from the very outset as well as in all types of algebraic calculation in secondary school.

b- *Operational invariants of the "self-evident mathematical property" type which students are able to justify*.

Mathematical justifications of additive transformations. Students summarized the addition of two decimal positive numbers in the following manner: "the addition of two little numbers makes a bigger number". Then, the biggest number was identified and, for students, this mathematical property was "self-evident". This property allowed students to analyze subtraction: the addition of two numbers, for example 67+22=89, makes two subtractions possible "the biggest minus a smaller number" (89-22=67; 89-67=22). Likewise, the subtraction of two numbers, for example 89-22=67, makes an addition possible, 67+22=89, whose truth-value allows checking the validity of the result of the subtraction. It is possible to check an addition by means of a subtraction but it is not usual.

Mathematical justifications of multiplicative transformations. The students were competent in multiplying and dividing integers. Their identification of the biggest number allowed transforming a multiplication into a division and vice-versa, for example, 7*8=56 leads to 56/7= 8 and 56/8=7. This property lead students to make errors with decimal numbers: 9.1=0.7x13 sometimes lead to 13/9.1=0.7. The main goal of our research was to explicitly introduce the tasks of analysis and the checking of the validity of transformations. To this end, we used division without any particular difficulties.

4- Checking transferred terms in a new written expression. This task is not relevant to solving the equations used in this research.

5- Numerical calculations. Most students could do numerical calculations without any particular difficulty. For other students, the solving of equations allowed them to reconstruct lost arithmetical knowledge.

The expert's algebraic solving methods. We summarize the main characteristics of expert's solving methods (Cortés A. (2003)). The analysis of the particularities of the mathematical object allows subjects to choose relevant transformations. For most exercises, teachers immediately choose a relevant transformation. They have reached a very high degree of expertise and they do not need to explicitly justify transformations: transformation rules are self-evident (they have the intimate conviction that they are true); it allows them to work quickly. But teachers are able to check the validity of transformations explicitly: they know the mathematical justifications of the transformations they use.

Some situations are not self-evident for teachers and when they are confronted with the choice of one transformation among several, they explicitly check the validity of transformations by means a "self-evident " mathematical property. This aspect of the experts' functioning constitutes a relevant model for the analysis of learning and teaching processes.

EXPERIMENTAL WORK

In solving equations in which the unknown was represented by a letter, students explicitly explored the concept of equation and used two solving methods.

EXERCISE 1 - Find the numerical value of the number n: 56/n=7; n= ; 22-n=11, n= ; n+8=18, n= ; 108/n=12, n= ; 25/n=5 n= ; 200-2*n=88, n= ; n*1.2=12, n= ; 50-n=46, n= ; 3*n+5=23, n= ; n*42=0, n= .

The analysis of the equation: the teacher asked the class to read the equations together before doing any calculations. The first equation was read as a question, this allowed analyzing the meaning of the letter. Students read the equation as an unknown result, for example "how much is n?"; "which result is n?". One student managed to express the question as "what number represents the unknown, n".

The analysis and the solving of the equation. The first equation was read as a numerical operation: a student said "56 divided by n makes 7". Students were given the solution "n is equal to 8" and were asked to check the validity of the result by the truth-value of the equation (56/8=7). The teacher summarized the reading of the second equation as "22 minus a number makes 11". Students substituted numbers for the unknown (22-11=11) and answered "n is equal to 11". For easy equations, most students proceeded by substituting numerical values until they could find the value that satisfied the truth value of the equation, the truth value being what allowed them to identify the solution.

Solving the equations through transformations. Analyzing equations lead students to make inferences: "It's an addition, so I should subtract", "It's a multiplication, so I should divide", etc. These rules, at that time, were theorems in action. For example, some students mentally transformed the equation 56/n=7 into 56=n*7 (they did not write the new equation), and thus used multiplication for solving the equation. Similarly, for the equation 22-n=11 which was transformed into 22= n+11, the solution was calculated by addition. In the solving of the equation 108/n=12, some students suggested "the solution is 9; 108 must be divided by 12".

Identification of the operation to be given priority. Transformation rules lead to errors. For example, in order to solve 200-2n=88, two students proposed to the class: "200-2=198 and 198-88=110". The teacher analyzed the equation taking into account the priority of the multiplication: "200 minus two times something(2n) is equal to 110"; "give numerical values to n and see what you can find". The students made the correct calculation: "200 minus something is 88... so it's 112...and 112/2 is 56". The teacher asked them to check the validity of this solution: "Are you sure, what do you have to do to be sure?". The students suggested that "200-(2*56) must be equal to 88". The students implicitly used the conservation of the truth-value as justification.

Exercise 2: The solution and the truth-value of the equation: only one solution is possible.

The following equations: a) 3n+15=27; b) 78-5x=46; c) 42-3n=2n+27, were implicitly analyzed as the equality of two functions. Students completed tables in which several values were given to the unknown. The two members of the equation were compared: the equality was either "true" or "false".

Students explored the variation of the two members of the equations and were able to determine and justify the fact that there was only one solution (we used equations which have a single solution). For de first equation the justification was "…it keeps going up and will never be 27". For the second equation the justification was "When x is bigger than 24.8, it keeps going down and never will be 46". For the third equation the justification was more elaborate: "One side of the equation goes up and the other goes down, they meet only once". Students wrote these justifications down.

Exercises 3 and 4 - Using transformations and checking the validity of the transformations.

Equations were solved by writing transformations, for example: 67+n=125; n=125-67; n=58. The written transformations allowed the teacher to perform the checking of the written numerical operations using a student's arithmetical justifications. The teacher established a strong tutorial activity: she asked the class to write the transformations down before performing the numerical calculations. In exercise 3, students solved the following equations: 33+n=150; 4.5+n=9.2; n-22=67; x-33=99; 5y=135; 10m=66; n/11=33; z/6=56. In exercise 4: 67+n=125; 220+2n=9000; 4z-16=8; 200-n=88; 6.6y=132; 250=5t-35; n/3.1=7; n/6+20=90.

Checking the validity of additive transformations. In solving the first equation 33+n=150, the teacher analyzed the equation in a particular manner: "n is the number which we must add to 33 to obtain 150, is this the definition of a subtraction?" The students answered, "n is the difference between 150 and 33". The equation n-22=67 was analyzed as a subtraction, which can be verified by an addition: 67+22=n. *The analysis of the equation as a numerical operation, which can be transformed into another operation, allows choosing a relevant transformation* (notably, the numerical calculation of the solution). At the same time, arithmetical justification allows checking the validity of the transformation.

Certain students applied transformation rules which lead to errors. For example, the equation 200-n=88 was transformed into n=200+88. The teacher guided the students in constructing another equation, an addition (the verification of the subtraction) "200 is equal to 88+n". In order to solve the equation 4z-16=8, one must first calculate the number 4z. The teacher guided students' analysis, which ended in "4z is the biggest number. If I subtract 16, the result is 8; so 4z is equal to 8+16" ...."z is equal to 24/4".

Checking the validity of multiplicative transformations. For solving equations such as 5y=135, students used rules like "It's a multiplication, so I should divide". They proposed that "y is equal to 135 divided by 5". They did not check the transformation, they checked the solution by verifying the truth-value of the equation:

"Once y has been calculated, you can multiply 5 by y and should get 135)". The teacher analyzed the equation appropriately, this allowed checking the transformation: "In the equation 5y=135, we have the numerical value of 5 times y and we need to know the value of only one y, so y=135/5". The equation was considered as a particular case of proportionality. To solve the equation n/11=33, students proposed, "n=33x11". When the teacher asked for a more analytic reading of the equation, students came up with "If you divide n into eleven equal parts, the value of each part is 33"; "you multiply n by the number of parts". *The teacher's tutorial activity obliged students to perform only those transformations that they could justify.*

A month later, the students were given a post-test. We observed seven errors in the solving of 3,7-y=0,1 (y=0,1+3,7) and x/2+5=8 (x=4/2). Students made small number of errors in the solving of the ten other equations (similar to Exercise 4).

CONCLUSION. In research and in constructing teaching courses, it is fundamental to analyze both the problems and the subject's solving activity. Identifying the invariant tasks constitutes a tool for the analysis of a subject's solving methods.

An operational invariant was associated to each invariant task, this is the mathematical knowledge that allows subjects to perform the task. This definition of the theoretical concept of the operational invariant enables us to explore and to understand the nature of operational mathematical knowledge.

We focussed our research on the analysis and the justification tasks, which are essential to solving mathematical problems. The students' arithmetical knowledge was used to justify the transformations performed. In order to solve equations, the students analyzed numerical operations in different manners, thus constituting another approach to arithmetic which enabled most of them to develop their mathematical knowledge, and, enabled still other students to reconstruct or to learn some of those arithmetic properties. In general, in teaching new concepts, it is possible to develop or to reconstruct students' previous knowledge.

We consider that students must be made to realize the advantage of working with mathematical properties that they are able to justify through daily work in class. The transformation of student theorems in action into justified mathematical knowledge is the way to build operational thought. This was the main goal of this research, because algebraic calculations will be presented differently later. Indeed, solving equations, inequations and systems of equations that include other types of numbers (numbers with sign, fractions…) is made possible by means of more general transformation rules and justifications. But the tasks of analyzing and checking the transformations remain invariant.

The teacher of the class had much experience and our research did not radically change her work. But our theoretical framework allowed her, notably, to refine the questions which guided students in conceptualizing new procedures.

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