ORGANIZING WITH A FOCUS ON DEFINING A PHENOMENOGRAPHIC APPROACH

Amir. H. Asghari

University of Warwick

This paper is based on the preparatory study of a doctoral study in which we learned to consider defining in the realm of organizing. In particular, having engaged students in a situation based on "equivalence relations" (from an expert point of view), we report two different ways of organizing the given situation. One of them results in a "new" definition of equivalence relations, and consequently a new representation for them, that seems to be overlooked by the experts.

INTRODUCTION

Definitions are inextricable parts of higher courses in mathematics. They give definiteness to the concepts to be taught in the course; they designate whether something is an example or not, and they are used in proofs. Embracing those referential and inferential aspects, definitions are tools to organize the content of the course; or in general, as Freudenthal (1983, pp.ix-28) says, they "have been invented as tools to organize the phenomena...phenomena from the concrete world as well as from mathematics...". In addition, incidental to their role as means of organizing, they embody two kinds of the lecturers' (or the mathematicians') choices, first, their choices of what appears to be important to be defined, and second, their choices between possible definitions of what is defined.

Accordingly, researchers examine possible ways of introducing definitions when developing new concepts. According to Freudenthal (ibid, p.32), one possibility is describing definitions in their relation to the situations of which they are the means of organizing; then, "starting from those phenomena that beg to be organized and from that starting point teaching the learner to manipulate these means of organizing."

This study has partly adopted Freudenthal's plan in that students were engaged in a situation that begs to be organized, though it aims at investigating the ways that students organize a given situation, rather than teaching them any particular ways of organizing that. In particular, this study is a phenomenographic investigation of what counts as defining when students organize a situation in which they have an opportunity to experience referential and inferential aspects of definitions in conjunction with their choices of the ways of organizing the given situation.

Adhering to a phenomenographic research approach, the study was conducted by holding individual in-depth task-based interviews, in which a task was used for querying students' referred concepts. All the interviews were audio-taped, and they were analyzed to explore *what students used to organize the situation, and what they did to organize the situation.*

Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 2004 Vol 2 pp 63–70

The importance of the issue can be clearly seen in other researches regarding defining. For example, Mariotti and Fischenbein (1997), in a teaching experiment, brought defining into the realm of students' experience. In their experiment, amongst others, two phases are worth considering, first, introducing a problem situation in which "the concept to be defined functionally emerges from the solution of a problem", and second, the indispensable and involved role of the teacher in their experiment, to guide students to overcome the conflict between "…the spontaneous process of conceptualization and the theoretical approach to definitions". Nevertheless, they repeatedly report the students' unforeseen difficulties to transcend the concrete situation to reach to the intended "systematic organization of concepts".

This and our initial data have led to the idea of looking not for how students define the intended concepts, but which concepts they determine are important for organizing the situation and what part defining play in that organization.

PREPARATORY STUDY

This paper is based on our preparatory study in which a smallish sample of students was engaged in the following tasks (see Table 1&2):

A country has ten cities. A mad dictator of the country has decided that he wants to introduce a strict law about visiting other people. He calls this 'the visiting law'.   A visiting-city of the city, which you are in, is: A city where you are allowed to visit other people.   A visiting law must obey two conditions to satisfy the mad dictator: 1. When you are in a particular city, you are allowed to visit other people in that city.   2. For each pair of cities, either their visiting-cities are identical or they mustn't have any visiting-cities in common.   The dictator asks different officials to come up with valid visiting laws, which obey both of these rules. In order to allow the dictator to compare the different laws, the officials are asked to represent their laws on a grid such as the one below. (See the results section)

Table 1

The mad dictator decides that the officials are using too much ink in drawing up these laws. He decrees that, on each grid, the officials must give the least amount of information possible so that the dictator (who is an intelligent person and who knows the two rules) could deduce the whole of the official's visiting law. Looking at each of the examples you have created, what is the least amount of information you need to give to enable the dictator deduce the whole of your visiting law.

Table 2

When devising this situation the researcher had the standard formulation of 'equivalence relation' and 'partition' in mind (e.g. Stewart and Tall, 2000). And the situation was originally designed with the *intention* of seeing how the students proceed with what was then considered to be *the only way* of organizing the situation in order to come to *the* definitions of ‘equivalence relation'.

In detail, having captured the reflexivity in the first condition of a visiting law, the situation aimed at leading students to the symmetry and transitivity through creating their own examples demanded in the first task on the one hand, and giving the minimum amount of information demanded in the second task on the other hand.

The study started with a small opportunistic sample of students comprising one graduate mathematics student, two first year mathematics students, two second year physics students (initial sample) , and then with one computer science student, and one sixth-form student (the last two will be used to describe the merits of the study).

The initial data revealed that the students spontaneously created their own way of organizing the given situation which are not necessarily those intended by the situation designer. Accordingly, the *intention* of the study became an investigation of *the ways* that students organize the given situation. In addition, those results led the study to the phenomenographic methods to provide the study with a conceptual framework for describing the variation of ways of organizing the given situation.

Methodology

As mentioned this study adhered to a phenomenographic approach. According to Marton and Booth (1997) phenomenography is a research approach that aims to reveal and describe the variation of ways of experiencing a phenomenon or a situation. Having this in mind, we elaborate our methodology in the context of two interviews with two students having no formal previous experience of equivalence relations and related concepts usually used to define it. Tyler is an undergraduate computer science student and Jimmy is a sixth form student studying mathematics.

The interviews had a simple structure; the two tasks (Table 1&2) were posed in order, but the timing and questions were contingent on students’ responses. The interviews aimed at reaching a mutual understanding between interviewer and interviewee (in the sense of Booth et al, 1999, p.69) of the situation and the ways that interviewee organized it. Therefore the interviewer did not judge the interviewees' utterances as to his own understanding, and insisted on the students giving transparent reasons for their decisions, mainly, as Marton and Booth (1997,p.130) say, “through offering interpretations of different things that interviewee has said earlier in the interview”. Tapes and written work were treated as data; and they analyzed according to the phenomenographic analysis method in which, as Booth (2001, p.172) says, ‘the data is pooled, temporarily losing the individual context in which it was gathered and gaining a collective context of the voices of other individuals who have contributed to the data. The researcher engages with this pool of data and seeks critical differences that can act as catalysts for an understanding of the whole'.

Results

Regarding Jimmy’s work and Tyler’s work, two differences can be identified:

- The difference in what they did to organize the situation
- The difference in their outcomes

The difference in what students did to organize the situation

To satisfy the first condition of the given situation (Table 1), Jimmy and Tyler blacked the diagonal and continued as follows (Table 3):

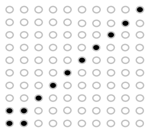
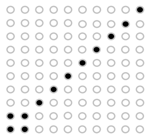
J- Now we have to satisfy the second condition, for each pair of cities, either their visiting-cities are identical, if you have the city one, if you can visit two, you have to, in city two either you can visit city one, like that, you have to because otherwise, they have something in common already, so you have to be able to visit. 	T- If I am in city one, and we allow to visit city two, how the other things need to change, to keep the rules consistent and see either they are completely the same or completely different, so aha, so city two now have to be able to visit city one… 

Table 3

Jimmy "has a rule to apply"; he suspends his reasoning and replicates the result. In other words he replicates a two by two block-square (table 4). On the other hand, Tyler considers two things, "mirroring in y equals x" and "box" (square), and then "to see what was happening" he decides to make city one visit city ten (table 4).

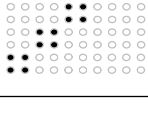
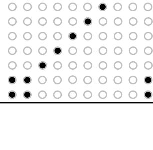
J- And likewise, if you go like that in pairs…It's like paired-up, so if you compare one and two, they have every thing in common, identical, if you compare one and three, one and four, one and five, or one *and six, they have nothing in common…* 	T- … and I realised first that, city ten has to visit city one…so that the second law …city ten has to visit city two…now I look at the city two, now I realised they are different from city one…so I copy number one on to number two also just to keep them the same… 

Table 4

As a result, Tyler abandons the "block square", keeps the "mirroring" and *proves* it as a "general pattern of these dots" (if (x, y) then (y, x)). In addition, the way that he proves "mirroring", gives him a new insight, i.e. considering the relationship between any two individual cities:

Tyler- If you allow a city to visit any other city, then it's gonna end up with having the same visiting-rules as that city that's allowed to visit and vice versa...

Jimmy still keeps the "block square" to generate his next examples, while Tyler uses "mirroring" and its proof.

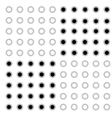
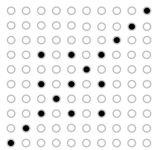
J- …I think there is something to do with square along this line of one and one, two and two, three and three, four and four, five and five; along this line …if you draw a square…people from this city, this city and this city are able to visit each other, they will have identical connection, but other people will not be able to visit them…so people from this group and this group haven't anything in common, but inside, then, they are identical. 	T- city three can visit city five and seven…so I think of course it's gonna be reflected in y equals x…No, this is not I want to finish, because now I have cities that have dots in common and they aren't the same…what I'm missing…what I'm saying here is a sort of square… 

Table 5

Then Tyler draws out, from the big block squares and "a sort of square" appeared in his last example (presented in table 5), the concept of the group of cities:

Tyler- …I completely lost of this sort of way of representing the laws (on grid) because I think they start showing what cities are reachable…in sort of groups you can reach one of the other by travel down the road, you allow to pass the cities between to get from one to other…

Although Jimmy uses the group of cities to organize the given situation, his way is qualitatively different from Tyler's. As it can be seen in the table 4, Jimmy divides cities into two groups, one of them (focal group) includes identical visiting-cities and another one includes the rest (except for his example illustrated in the table 5 in which two views coincide), while Tyler divides cities into groups so that, each group includes identical visiting-cities. That is, from our perspective, Tyler has the notion of partitions, while Jimmy has a split in the set of cities into 'the group I'm currently working with' and 'the rest'.

The difference in their outcomes

While the result of Jimmy's work is many individual examples, Tyler transcends the situation by introducing new concepts. Particularly, he introduces a new concept with general applicability (the 'box concept'):

Tyler- How do I say that columns must be the same mathematically? (He writes)

If (x_1, y_1) and (x_1, y_2) and (x_2, y_1) then (x_2, y_2)

Interviewer- Could you explain.

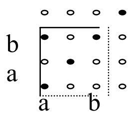
Tyler- I think it's a mathematical way of saying …if a column has two dots, and there is another column with a dot in the same row, then that column must also have the second dot in the same row…I take maybe a box of four dots…I use the coordinate because that makes it very general, and so if I made that my second law, for a mathematician might be easier to follow.

Given this, an equivalence relation can be understood as a relation having the reflexive property and the box property (If (x_1, y_1) and (x_1, y_2) and (x_2, y_1) then (x_2, y_2)). That is, Tyler has explicitly generated a new (and, for us, unexpected) definition (which happens to be mathematically equivalent to the standard definition of equivalence) in order to organize this situation.

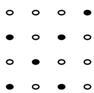
As an ongoing study, we are seeking a more detailed picture of such differences in the light of our new data. Thus let us at the moment focus our attention on different ways of organizing related concepts as informed by the present data.

Equivalence relations, revisited

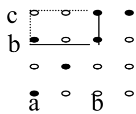
The normative definition of equivalence relation, based on reflexivity, symmetry and transitivity, is widely used to introduce the subject. So let us have a look to the other definitions of it, learned through our data: one based on box concept, and the other based on "triangularity". The following diagrams shows how, having reflexivity and box concept, we can deduce symmetry and transitivity.



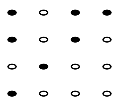
(a, b), (a, a), (b, b) are three corners of the box



(b, a) is the forth corner



(a, b), (b, b), **(b, c)** are three corners of the box



(a, c) is the forth corner

Although the normative way of defining equivalence relations and its definition based on the box concept are logically equivalent, they have dramatically two different representations that could affect students' understanding of the subject. For example, Chin and Tall (2001) suggested "the complexity of the visual representation" as to the transitive law as a source of a "complete dichotomy between the notion of relation (interpreted in terms of Cartesian coordinates) represented by pictures and the notion of the equivalence relation which is not". Accordingly, they suspected that that dichotomy inhibits students from grasping the notion of relation encompassing the notion of equivalence relation. However, the above figures show that the stated dichotomy, to a large extent, depends on the standard way of defining equivalence relation, i.e. if we define equivalence relation as a relation having the reflexive property and the box property, that dichotomy would disappear.

It is worth saying that the notion of equivalence relation defined by the box concept and its normative definition reveal two different ways of organizing the related concepts. While the former provides us with a simpler visual representation, the latter endows the subject with a seemingly more comprehensive quality in which two important types of relations, equivalence relations and order relations can be seen as particular types of transitive relations. Leaving a concept suitable for organizing a local situation in favour of grasping a more global picture is a particular aspects of mathematics that once again appear as to "triangularity".

Triangularity is the name that for the sake of this paper is given to one of the most common way that our students tackled the situation, i.e. relating the cities in a group of related city without any particular *order* or any *direction*, or referring to equivalent columns without any particular order. Beyond this particular situation, triangularity means when two things are related to a third, the first two are related to each other too. In detail, it is a disjunctive concept, that, if a is related to b and b is related to c then a is related to c *or* if a is related to b and a is related to c then b is related to c (As it can be seen the first part of this or condition is what is known as transitivity). It is the concept that is seemingly behind Euclid's account of equality (far long before having any account of relations or equivalence relations), the first among common notions, that, "things which are equal to the same thing are also equal to one another" (Heath, 1956, p.155). And it is the concept that is clearly behind Freudenthal's account of equivalence relations (in the years of having transitivity as one of the distinct concept comprising equivalence relations). Freudenthal (1966, p.17) defines equivalence relations as a relation possessing the following two properties: first, "every object is equivalent to itself (reflexivity)", and second, if "two objects are equivalent to a third, then they are also mutually equivalent (transitivity)", and shortly after that he notes that those two indicate symmetry property, that, "If an object is equivalent to a second object, then the second object is also equivalent to the first (symmetry)"; but he emphasizes that "actually, the first two properties are sufficient" to define equivalence relations. While, in the course of defining equivalence relations, he uses the term transitivity for what we call triangularity, a few pages on (ibid, p.19), when considering *order* he uses the term transitivity for what is usually known as transitivity:

> …and if, for every three different members a, b, c, of Z it follows from a< b and b< c, that a <c (transitivity of the <-relation).

Having a group of "equivalent elements" in mind, there is no way to separate transitivity from triangularity; that is probably why Freudenthal exploits the term transitivity where he uses triangularity, and in the same vein, Skemp (1971, p. 175) does so:

> The importance of the transitive property is that *any* two elements of the same sub-set in a partition are connected by the equivalence relation.

And that is why no student in our study (not even in the initial interviews where interviewer had a bias toward the standard definition) could notice transitivity as a distinct property. In general, not only in our situation, but also *in any other situation based on splitting a set into disjoint sub-sets by using a particular relation, there is no way to bring the transitivity up unless it is taught.* That is probably why Stewart and Tall (ibid, p.73), *right after* comparing two relations, one splits a certain set into disjoint pieces, and the other does not, "take account of three very trite statements" (including transitivity)as what makes the former work.

Deep down, while by standard account of equivalence relations and order relations, they fall into our hands as special cases of transitive relations, as a drawback, we impose something extra on the equivalence relations, i.e. a sense of direction or order.

Conclusion

As it can be seen in the Mariotti and Fischenbein's study (ibid), it is widely taken for granted that there is a fixed concept that the students are trying to negotiate, but the present study (and implicitly Mariotti and Fischenbein itself) suggests that these open tasks (that designed around an intended concept) can be organized in different ways. Thus a categorization of a) how they are organized and b) what is organized, is of clear value.

References

Booth, S. (2001). Learning Computer Science and Engineering in Context. *Computer Science Education*, 11(3): 169-188.

Booth, S. Wistedt, I. Hallden, O. Martinsson, M. and Marton, F. (1999). Paths of Learning- The Join Constitution of Insights. In Burton, L (Ed). *Learning Mathematics, From Hierarchies to Networks.* London: Falmer.62-82.

Chin, E-T. and Tall, D. O. (2001). Developing formal mathematical concepts over time. In M. van den Heuvel-Pabhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics,* 2, 241-248.

Freudenthal, H. (1966). *The Language of Logic*. Elsevier Publishing Company.

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures. Dordrecht:* Reidel

Heath, T. L. (1956). *The Thirteen Books of Euclid's Elements*. Dover Publications, Inc.

Mariotti, M.A. and Fischbein, E. (1997). Defining in Classroom Activities. *Educational Studies in Mathematics*, 34 : 219-248

Marton, F. and Booth, S. (1997). *Learning and Awareness*. Mahwah: LEA.

Skemp, R. R. (1977). *The Psychology of Learning Mathematics*. Penguin Books.

Stewart, I. and Tall, D. (2000). *The Foundation of Mathematics*. Oxford University Press.