IMAGES OF FRACTIONS *AS* PROCESSES AND IMAGES OF FRACTIONS *IN* PROCESSES

Jan Herman[*], Lucia Ilucova[*], Veronika Kremsova[*], Jiri Pribyl[*], Janka Ruppeldtova[*], Adrian Simpson[†], Nada Stehlikova[*], Marek Sulista[*], Michaela Ulrychova[*]

[*]Charles University, Prague, CZ; [†]University of Warwick, UK.

Within the large range of potential theoretical perspectives on fractions, this paper considers one particular interpretation: fractions' duality as process and object. By considering the number-fractionbar-number composite symbol as simultaneously representing division and rational, some process-object theories imply that fraction-as-process and fraction-in-process should be highly related. Our research studied the images evoked in these two situations across a wide range of learners and shows that while students attempted post-hoc justifications of their fraction-in-process calculations using their fraction-as-process images, these images were rarely compatible with the process of addition. Thus, we suggest that the routes to seeing the fraction symbol as process and as object may be cognitively separate.

INTRODUCTION

There is an impressively wide literature on pupils' interpretation of (and their cognitive roots of) the fraction concept. Much of the literature emphasizes the large range of ways in which symbols, such as $\frac{3}{4}$ might be interpreted: for example, as part-whole, ratio, quotient, operator and measure (Kieren, 1976) or as quotient function, rational number, vector, and composite function (Ohlson, 1987). In this paper, rather than consider the totality of possible students' interpretations of a fraction, we will consider what happens when students are asked to apparently operate with fractions to examine how they interpret the number-fractionbar-number composite symbol when it is involved in a calculation – specifically addition.

While some researchers have examined the iterative development of a fraction scheme as children pass through multiple representations and operations on those representations which will come to form the scheme (Steffe and Olive, 2002), this paper focuses on just one theoretical interpretation of number-fractionbar-number composite symbol which we view as problematic – its duality as process and object.

THEORETICAL CONSIDERATIONS

There are three major alternative theories of process-object duality: procept, APOS and structural-operational theories. When considering the duality of fraction, the three theories have much in common. The notion of the 'procept' puts the symbol as central to this duality – the symbol simultaneously stands for both a process and a concept. In the case of the number-fractionbar-number composite symbol, "the symbol $\frac{3}{4}$ stands for both the process of division and the concept of fraction" (Gray and Tall, 1994). Within the theory of structural-operational duality, rational number is – structurally – a “pair of integers (a member of a specially defined set of pairs)” and – operationally – “[the result of] division of integers” (Sfard, 1991, p5). While most commonly used in the analysis of mathematical ideas at the university level, “some studies such as that of fractions, show that the APOS Theory … is also a useful tool in studying students’ understanding of more basic mathematical concepts.” (Dubinsky and MacDonald, 2001)

These three theories differ, however, in their approach to the relationship between the processes and objects. Gray and Tall make no significant claim about how the development of process and concept aspects of a mathematical idea might be connected. APOS and structural-operational theories, however, indicate that the object conception has its genesis in the process conception: as the encapsulation or reification of the process: “This encapsulation is achieved when the individual becomes aware of the totality of the process, realizes that transformations can act on it, and is able to construct such transformations.” (Cottrill et.al., 1996, p171) and “reification is an instantaneous quantum leap: a process solidifies into object, into a static structure” (Sfard, 1991, p20). In contrast to the theory of procepts, these theories thus posit a fixed learning trajectory for fractions and might be termed ‘process-to-object development’ theories.

Our investigation considers the images associated with the fraction concept on both sides of the reification/encapsulation divide, to examine whether we can see the trace of this trajectory.

Sfard notes that reification requires seeing the newly formed object in the context of processes that act upon it: “There is no reason to turn process into object unless we have some higher level processes performed on this simpler process”(Sfard, 1991, p31). Thus, we suggest, the number-fractionbar-number composite symbol should have the power to evoke, differently, process and object conceptions when it is seen ‘as process’ and ‘in process’ – for example $\frac{3}{4}$ seen in an isolated context and $\frac{3}{4}$ seen within, say, the context of the addition $\frac{3}{4}+\frac{2}{5}$. The test for process-to-object development theories, then, is whether the object conception of fraction has its genesis in the process conception and thus whether the ideas evoked in the two situations are compatible.

RESEARCH CONTEXT

This research was carried out in the Czech Republic, using a wide range of students to provide us with access to as wide a range of imagery of fraction-as-process and fraction-in-process as possible. This included nine 6^{th} grade (7 high and 2 low ability), four 7^{th} grade (2 high and 2 low ability), two low ability 8^{th} grade and four 9^{th} grade (2 high and 2 low ability) school pupils. In addition data was collected from six university students training to be mathematics teachers (two training for lower primary, two for upper primary and two for secondary).

Our methods were adapted from other process-object studies, particularly Pitta and Gray (1996). We gave students two different types of task: one in which they were given different number-fractionbar-number composite symbols individually and one in which they were given these symbols within the context of an addition.

The tasks, presented to the students individually on separate sheets of paper, are shown in figure 1. For each sub-task they were asked three questions

a) Tell me what first comes to mind.

b) How do you represent* this?

c) Can you tell me a story which might involve this?

Task I

i) $\frac{1}{4}$, ii) $\frac{3}{8}$, iii) $\frac{7}{6}$

Task II

i) $\frac{1}{2}+\frac{1}{4}$, ii) $\frac{2}{8}+\frac{3}{8}$, iii) $\frac{32}{45}+\frac{5}{12}$

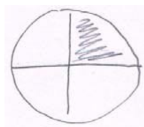
Figure 0: Tasks

Pens were provided and students were informed they could use them if they wished for questions b) and c). After all of the task items had been shown, the interviewer showed the students all of their papers and invited them explicitly to see if they could use their ideas from the task I items to explain the task II items.

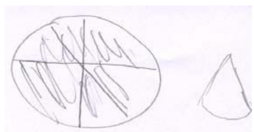
We used semi-structured, clinical, one-to-one interviewing (Ginsburg, 1981) allowing the interviewer to follow the direction set by the students' responses. Each interview was audio taped and transcribed, with the transcripts forming the basis for the classification of the images evoked. The students' responses below are given a code which identifies the student (letter), school grade (number) or 't' for teacher training student, +/- for high or low ability and an indication of the task to which the response relates.

STUDENT RESPONSES

Fractions-as-processes



(a) M9-,I(i)



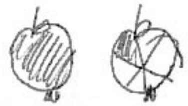
(b) S9+,I(iii)

Figure 0: Standard Images of Fractions-as-processes

The first task asked students to represent fractions when they were given in isolation. We were expecting to see images which related to fraction-as-process, that is, fractions seen as related to the division of a whole object. Indeed, in all cases we saw this. Most commonly we saw standard images in which appropriate portions of a circle, square or bar was indicated by shading (fig 2a). This sense of the image as an association in some cases was heightened when we saw images, even from stronger students, which did not attend

* In the Czech language the word "zna'zornit" is a rough equivalent of 'represent' which does not imply a representation in any particular medium, but is a commonly used word in school.

to the importance of the equal sizes of the parts (e.g. fig 2b). We do not necessarily see this as a failure to develop a satisfactory part-whole scheme (Behr et. al., 1992) but that, for some students at least, the link between the fraction given as an isolated symbol and the act of dividing was a rapid (and no longer crucial) association.



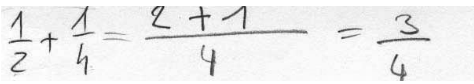
(a) R6+,I(iii)

(b) Bt+,I(ii)

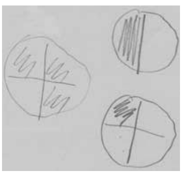
Figure 0: Non-standard images of fractions-as-processes

There were very few other images for these early items, though we did see apples (fig. 3a) and musical notation (fig. 3b) described as “There is a quaver in music. When there are three quavers, so there can be three eighth notes in one time”. One student (Nt+) used the idea of time as an image for fractions: “And then I remembered such a typical one quarter such as a quarter of an hour. Those classical clocks.” and 7/6, “Seven sixths, I realised that it would ... no, it is a stupid thing... well, it would be nicely, say, done on the clock, maybe. The six suggests the sexagesimal system so the seven sixths from an hour could be an hour and ten minutes”.

Fractions in Processes



(a) F9+, II(i)



(b) F9+,II(i)

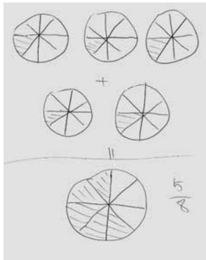
Figure 0: Post-hoc justification

It is in task II, however, that we see the most significant issues raised about the problems of the adapting the image of the fraction-as-process to be an image of fractions-in-process. In almost all cases, students faced the problem presented as one of working directly with the symbolism of the number-fractionbar-number composite symbol (fig. 4a). However, when asked to represent or tell a story about the addition, they attempted what we saw as *post hoc justifications* for their calculations by trying to *adapt their images* from task I (fig. 4b). The imagery which might be well suited to representing the particular fractions is not, we suggest, suitable for representing the *process* of addition.

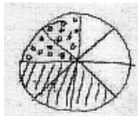
For example, in task II(ii), we saw many examples of the type reproduced in fig. 5. The circles use shading to appropriately represent each fraction, but it is far from clear whether the process of adding is being represented since the shading for the two addends overlaps. In only one case (fig 5c) did a student explicitly highlight how the two parts fit together in the process of adding. This post-hoc justification based on the attempt to adapt their images is also highlighted in the language they use to describe their process of fraction addition. In one instance student Z6+ had already completed task II(ii) in symbolic form. In describing how she would represent this she draws two circles (cakes) and says: “We will divide it into eighths. In one I will colour two, in one three, so the result is five eighths” and draws figure 5d. In this she makes no reference to how the act of adding is represented and we again see a representation of the addends, but no representation of the process of adding.

In two further cases, we see the separation of representing the fractions from representing the process on the fractions even more clearly.

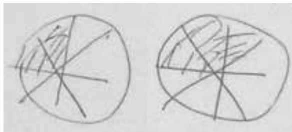
(a) A6+,II(ii)



(b) Vt+,II(ii)



(c) Bt+,II(ii)

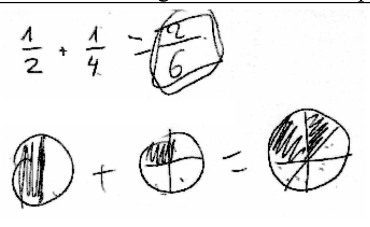


(d) Z6+,II(iii)

Figure 0: Adapting Imagery

In fig 6, a student has acted upon the number-fractionbar-number composite symbol without regard to its composite nature, thus getting an incorrect answer. However, she still represents, in standard imagery, the three fractions from her calculation.

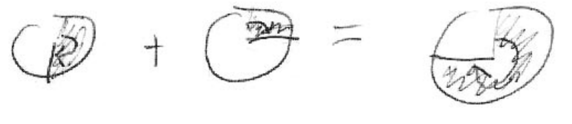
In fig 7, the trainee teacher using the time image (Nt+), first draws a picture appearing to represent a minute hand moving from the hour to half-past, a picture of the hand moving from the hour to quarter-past and, to represent the sum, a picture of the hand moving from the hour to quarter-to. The interviewer asked him explicitly to consider the images and he appears to realize that the imagery, as drawn, does not adapt well to the act of adding:



F7-,II(i)

Figure 0: Adapting imagery to incorrect solutions

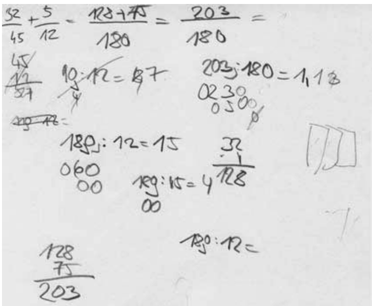
“First the half hour is going on, we will look at the position of the hand, and then the quarter will follow. … they wouldn’t be able to add it like that”. His image as drawn, has the fractions ½ and ¼ represented as he had represented them when presented as isolated symbols, but to adapt this successfully to the process of addition, the addends would have to be represented as abutting. His phrase "they wouldn't be able to add it like that" suggests he is beginning to recognize the problem of his own imagery here.



Nt+,II(i)

Figure 7: Adapting time imagery to addition

The most difficult subtask (II(iii)) proved to be the most problematic for students' attempts at adapting the fraction-as-process imagery for a fraction-in-process situation. Many of the students, particularly the higher ability and the trainee teachers, could correctly solve the problem working with the symbolism (fig. 8). When it came to trying to represent it, most again attempted to adapt the imagery from task I to provide a post hoc justification (fig. 9a, 9b). However, in this case, many of the students were able to see the difficulties caused by their fraction-as-process images. In some cases the problems were caused by the nature of the components of the number-fractionbar-number composite symbol. The denominator 45 was seen as "too big".



F9+,II(iii)

Figure 8: Working Symbolically

In other cases, however, students became explicitly aware that their imagery did not allow them to represent the processes they went through in adding the fractions. Student F9+ was able to correctly work with the symbolism, but when faced with attempting to adapt her imagery said:

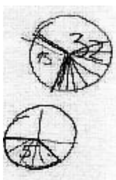
F: I would do it for example like that [drawing a rough bar picture], those 45 and 32 from them coloured, and here [pointing under the picture] I would do 12 frames and I would colour 5 of them.

Interviewer: How would you add it?

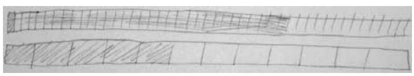
F: I would have to find the common divisor, I would draw those 180 frames again, and colour 45 of them...[long pause]

Interviewer: And then?

F: [Long pause] … I don't know how to represent it. I will take those 180 houses and from them, I don't know if 32 or 45 of something, I don't know...[trails off].



a) Bt+,II(iii)



b) U6+,II(iii)

Figure 0: Task II(iii) imagery

In all three fraction-in-process tasks we saw, across the wide variety of students we worked with, the same features:

- working with the symbols (either correctly or incorrectly)
- attempting a post hoc justification
- adapting fraction-as-process imagery

In most cases, the images used adequately represented the addends and the sum as individual fractions-as-processes (normally the division of a cake, a bar).

However, we suggest that, even though they make significant efforts to adapt this imagery beyond the realms in which it is suitable, even the strongest students were not able to use the imagery they have associated with particular fractions to represent the *process* of addition (which, for those able to perform the process, is a process only on symbolic objects).

DISCUSSION

The data suggests that even those fluent in acting upon fractions in symbolic form and whose language indicates they are able to think about these as representing elements of the field of rational numbers, are unable to adapt imagery they associate with fractions to fractions-in-processes. This suggests that, for them, the developmental route to fraction as an object which can be acted upon may not be immediately linked to their notions of division.

The data supports the duality of the number-fractionbar-number composite symbol as being able to represent both process and object (upon which processes act) for some of the students in the sample. However, it suggests that the routes to the two parts of this duality may be cognitively separate.

Dubinsky claims that the construction of objects from processes is extremely tricky: "I … have often stated that one of the most important (and difficult) general mathematical activities consists in encapsulating processes to objects and de-encapsulating objects back to the processes from which they came" (Dubinsky 1997). This difficulty may just be because (in the domain of fractions at least) objects are not the encapsulation or reification of processes at all.

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