ABSTRACTION AND CONSOLIDATION

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What is involved in consolidating a new mathematical abstraction? This paper examines the work of one student who was working on a task designed to consolidate two recently constructed absolute function abstractions. The study adopts an activity-theoretic model of 'abstraction in context'. Selected protocol data are presented. The initial state of the abstractions and changes that were observed during the consolidation process are discussed. Features of consolidation noted are: reconstruction of the abstractions, increased resistance to challenges, developing a language for the abstractions and greater flexibility.

ABSTRACTION AND CONSOLIDATION

Abstraction and consolidation are important issues in mathematics education but there are differing interpretations as to what these constructs are and involve. An empiricist view of abstraction is that it involves generalisations arising from the recognition of commonalities isolated in a large number of specific instances (see Ohlsson & Lehtinen, 1997 for a critique of this view). An alternative position recognises the importance of the social and contextual factors in abstraction (see van Oers, 2001). This paper adopts an activity-theoretic model of 'abstraction in context' proposed by Hershkowitz, Schwarz & Dreyfus (2001) which we outline shortly. Consolidation is often associated with skill acquisition (Meichenbaum & Biemiller, 1998) and the retention of information (McGaugh, 2000; Spear, 1978). There is little research on consolidating abstractions. We review the research in this area after outlining the Hershkowitz et al. (2001) model of abstraction.

Hershkowitz et al. (2001) view abstraction as an activity of vertically reorganising previously constructed mathematical knowledge into a new mathematical structure ('structure' is their generic term for structures, methods, strategies and concepts – we employ this term in describing their work but avoid it in describing our own work). The new abstraction is the product of three epistemic actions: recognizing, building-with and constructing. Their theory posits that the genesis of an abstraction passes through three stages: (a) the need for a new structure; (b) the construction of a new abstract entity where recognizing and building-with already existing structures are nested dialectically and (c) the consolidation of the abstract entity facilitating one's recognizing it with increased ease and building-with it in further activities. The authors assume *a priori* that recognition of new structures in further activities will consolidate these structures and consequently students will progressively be able to recognise and build-with them with increasing ease. Hershkowitz et al. (2001) and a companion paper (Dreyfus, Hershkowitz & Schwarz, 2001), however, are primarily concerned with the process aspects of abstraction (stages a and b) rather than the outcome and the consolidation of an abstraction.

Three other papers consider consolidation with respect to this theory of abstraction. Tabach, Hershkowitz & Schwarz (2001) and Tabach & Hershkowitz (2002) examine the construction of knowledge and its consolidation. They touch on the importance and necessity of the consolidation after the construction of knowledge but do not analyse the process of consolidation. A major contribution, in our opinion, to understanding the process of consolidation comes from Dreyfus & Tsamir (2001). They analyse the protocol data of one student and conclude that consolidation is a long-term process in which an abstraction becomes so familiar that it is available to the student in a flexible manner. They identify three modes of thinking that take place in the course of consolidation: building-with, reflecting on the building-with, and reflecting. They claim that building-with actions are the most direct and elementary means of consolidation. They characterise the consolidation of an abstraction with the constructs: immediacy, self-evidence, confidence, flexibility and awareness.

Although this work is important further investigations are in order as their claims are based on one case. Our work aims to further understand and characterise the process of consolidating an abstraction. We provide, below, an overview of our study, present protocol excerpts of 17-year-old boy working on the consolidation task and conclude with a discussion of issues raised from the protocol data.

THE STUDY AND TASKS

The work presented in this paper is part of a larger study examining aspects of human interaction and the RBC model of abstraction. The wider study employed an explanatory multiple case study methodology (Yin, 1998). 20 Turkish grade 10 students (16-18 years of age), seven pairs and six individuals, worked on four absolute value of a linear function tasks over four consecutive days. These students were selected from a larger sample on the basis that they could complete the tasks but they had not encountered the mathematical content of the tasks. Four pairs and three individuals were scaffolded by the interviewer (the second author) in their work. All interviews were transcribed. Initial analysis of the protocols was similar to the protocol analysis described in Hershkowitz et al. (2001).

The objects of the first, second and fourth task were to construct a method to draw the graphs of, respectively, $|f(x)|$, $f(|x|)$ and $|f(|x|)|$, by using the graph of $f(x)$. The third task was designed to consolidate the abstraction of the first and second tasks. This paper reports on a protocol generated in the third task, which had five questions. In the first question, the function $f(x)=2x-2$ is given and students were asked to draw the graphs of $|f(x)|$ and $f(|x|)$. The second question asked students to describe how to obtain the graphs of $|f(x)|$ and $f(|x|)$ from $f(x)=ax+b$. In the third and fourth questions an hypothetical situation was depicted where three fictitious students (Aylin, Cem, and Arzu) made claims about how to obtain the graph of, respectively, $|f(x)|$ and $f(|x|)$ from a given graph of $f(x)$. We present the claims in the appropriate part of the protocols. All of these claims were plausible but not quite correct. The students were expected to evaluate each claim. In the fifth question students were presented with six graphs and asked to identify whether they could be the graphs of $|f(x)|$ and/or $f(|x|)$.

PROTOCOL DATA

We present excerpts from Tugay's protocol for the third task. Tugay (not his real name) was one of the three students who worked alone with the interviewer scaffolding the work. All three of these students successfully achieved the desired abstractions of the first two tasks. In selecting protocol excerpts we attempt to present an overview of the work and include all excerpts referred to in the discussion section. To present data and to discuss issues in depth, in the space available, we focus on the consolidation of $f(|x|)$, though as the third task concerned both $|f(x)|$ and $f(|x|)$ we mention $|f(x)|$ at times. 'I' refers to the interviewer and 'T' refers to Tugay. Each complete utterance was given a new line number. We interleave protocol excerpts with our comments.

Tugay reads question 1. The interviewer asks how he will obtain these graphs.

4T: As far as I remember, I can either first draw the graph of $f(x)$ and then take the symmetries accordingly or I can substitute different values of x and then draw the graphs.

Tugay describes the symmetry for $|f(x)|$ and proceeds to $f(|x|)$.

10T: For the graph of $f(|x|)$, we draw a line parallel to y-axis through the intersection point of $f(x)$ and y-axis. Then we take the symmetry of a ray by flipping up and down accordingly. But I am not too sure how! Maybe according to given graph, I guess.
11I: You told me what you remember. How you could draw these graphs now?
12T: Umm, I think it'd better if I substitute different values of x and then draw the graphs because I feel more secure in that way. Maybe afterwards I can use what I developed before.

Tugay draws the graphs by substituting and moves on to the second question.

44T: I think I use the first question for this. Let's see… but I need to draw the graph of $f(x)$ first so that I can see what happens…

Tugay graphs $f(x)$, describes how to obtain the graph of $|f(x)|$ and moves on to $f(|x|)$.

52T: …after x becomes negative… first we find the intersection point of $f(x)$ and y-axis. Then we draw a line which passes through this point and is parallel to the y-axis. I'd better explain with this graph. We draw the line of y=-2 because it is the intersection of $f(x)$ and y-axis. Then this part [of $f(x)$] under this line will be flipped up to obtain the graph of $f(|x|)$.
53I: You mean you take the symmetry in the line of y=-2?
54T: Hmm yes, symmetry but the line… the symmetry line changes according to $f(x)$.

Between the 56T and 107I Tugay worked on the third question about the graph of $|f(x)|$. We move on to his work on the fourth question which concerns the graph of $f(|x|)$. He reads the first statement (by Aylin who claims that "To obtain the graph of $f(|x|)$, one needs to take the symmetry of the negative $f(x)$ values in the x-axis because this function includes absolute value which makes every negative $f(x)$ values positive and positive values exist only over the x-axis"). The interviewer suggests a graph.

110T: I wonder if a graph of $f(|x|)$ can take negative values, I mean under the x-axis… it could be… let's draw a random graph. In order to draw the graph, we should draw a line passing through the intersection of y-axis and $f(x)$… and then take the symmetry in that line.

He draws a graph of $f(|x|)$ and focuses on negative values of x and concludes that Aylin is wrong. He moves on to the second statement in this question (by Cem who claims that "There is no difference between the graphs of $f(x)$ and $f(|x|)$ for the positive x values but we cannot say anything about the difference for the negative x values, which depends on the equation of $f(x)$").

124T: Are both graphs the same for the positive x values? Both graphs appear to be the same… for the positive x values… 'There is no difference'… yes… there is no difference…
125I: Do you think he is right?
126T: There seems no difference for now… we have to consider the whole theory to come to a decision… umm, negative x values for the graph of $f(|x|)$… I think we can say something about the graph of $f(|x|)$ for the negative x values… when the x values are negative then we take this part symmetry in a line which parallels to the y-axis… but the difference between $f(x)$ and $f(|x|)$?… Well, the difference is evident… while $f(x)$ is linear, the graph of $f(|x|)$ is something like the shape of V…but one arm of V is the symmetry at the negative x values…

Between 127 & 133 the interviewer challenges Tugay's reasoning.

134T: Well, first of all, I remember that for the positive x values the graph of $f(x)$ remains absolutely the same, well I don't know if I can say 'absolutely'. But for the negative values of x, it was enough to take the symmetry. In fact we made use of analytic geometry for the solutions so…but I am not sure if what I am saying is definite…I am confused…

The interviewer suggests that they return to the first question and examine the graphs. Tugay notes that $f(|x|)$ has the same value for points $\pm n$. The interviewer asks why.

154T: I think due to the absolute value sign, I mean it is outside of the x and that means… regardless of the sign of the values of x, they will have the same value of y.
155I: What does this tell us about the symmetry?
156T: So it tells us perhaps that all of the graphs of $f(|x|)$ are symmetric in the y-axis. In fact I remember that I told something about it on the second task but I did not realise today.
157I: Perhaps? When you say perhaps I feel…
158T: I need to look at once again…[he looks at earlier graphs] yes, all of the graphs must definitely be symmetric in the y-axis because different values of x with different signs must have the same value of y, which is why it must be symmetric in the y-axis.

Tugay restates his confidence in this formulation.

162T: It is evident that we can say that there is no difference between the graphs of $f(x)$ and $f(|x|)$ for positive x values. At the same time we can surely say that the part corresponding to the negative x values must be the symmetry of the ray which is on the right side of the y-axis. So Cem is wrong. We can say how to obtain the graph even without an equation.

They move on to the third statement in this question (by Arzu who claims that "To obtain the graph of $f(|x|)$, one must not change the part of the graph of $f(x)$ at negative x-values and simultaneously the symmetry of this part must be taken in the y-axis").

164T: No, it is not so… I mean when we take the symmetry of the graph of $f(x)$ at positive values of x in the y-axis we obtain the $f(|x|)$…

They discuss this and the interviewer challenges and asks Tugay for a justification.

172T: Because… every value will be positive in the absolute value…it does not matter for positive values whether they are in the absolute value sign or not because it is positive anyway so it does not change. However, the negative values differ if they are in the absolute value I mean when they are in the absolute value sign then they change, they alter into positive and thus result changes…so when one substitutes, for example –2 for x in the $f(x)$, then one would obtain a different result from the result of $f(|x|)$ when one substitutes –2… because |-2| is a positive value and this is –2 in the $f(x)$. So they are totally different

173I: So, for the positive values in the $f(|x|)$…

174T: Let me put it another way. In $f(|x|)$ when we substitute positive values we obtain a result which is the same as $f(x)$. However, if we substitute negative values in $f(|x|)$ we get different results from that of $f(x)$ when the same negative values are substituted in the $f(x)$.

175I: Which shows that…

176T: That proves that the graph of $f(x)$ at the positive x-values is exactly the same graph as the graph of $f(|x|)$. But as the negative x values change in the $f(|x|)$ so does the graph of $f(x)$ when transformed into the graph of $f(|x|)$… I think I made my point, right?

177I: Yes, but say how we can obtain he graph of $f(|x|)$ from the graph of $f(x)$ once again.

178T: Well in fact we can obtain the graph of $f(|x|)$ in two different ways. The first one is that …we can draw a parallel line to the y-axis through the intersection point of $f(x)$ and y-axis. And then for the negative x-values we can take the symmetry of that part of graph in this line. Secondly, well we can take… umm the graph of $f(x)$ at the positive x values remains the same; I mean we can take the symmetry of this part in the y-axis and cancel the part of $f(x)$ at the negative x-values… and so this is $f(|x|)$… yes definitely so.

DISCUSSION

We discuss the initial state of the abstraction of $f(|x|)$ and the changes that were observed during the consolidation process. We also briefly relate our findings to the model proposed by Dreyfus & Tsamir (2001).

The initial state of the new abstractions

Tugay's new abstractions did not appear to be firmly consolidated when he started the third task in that he was not confident in their validity. He was, for example, able (10T) to describe how to obtain the graph of $f(|x|)$ from the graph of $f(x)$, but his comment "I'm not too sure" suggests that he was not certain about this construction. He also expressed feelings of insecurity (12T) with regard to this construction as a means of obtaining the graph of $f(|x|)$. His comment "if I substitute" suggests that he is uncertain about the validity of the abstractions constructed in the first and second tasks. A hesitancy in defending formulated abstractions for a considerable period after their constructions was common in all the protocols of students who made these abstractions. In Tugay's case we can see his uncertainty reappearing as the interviewer probes different aspects of the graphs. In 134T, for example, he states that he was not sure if his symmetry argument for negative values of x were correct and stated "I am confused".

In the early parts of the protocols of the third task students made extensive use of specific examples and these examples were used as a basis for formulating their ideas. Only students who consolidated the abstractions in this task went beyond specific examples and then only in the latter parts of the protocols. This is not surprising but it does draw attention to an apparent need to ground the new abstractions in concrete examples. In Tugay's case he states, (44T), "I need to draw the graph of $f(x)$ first so that I can see what happens". He did not talk about the relationship between the graphs of $f(x)$ and $f(|x|)$ until he had drawn them (44T - 52T). The points and lines he constructed were cojoined with demonstrative adjectives in his discourse: "this point … this line" (52T) – he appeared to be unable to formulate his constructions in general mathematical terms free from specific lines and points. We take this, and the uncertainty noted above as evidence that the new abstractions are fragile and need to be consolidated.

Changes in the course of consolidation

As Tugay worked on the third task he appeared to consolidate his abstractions from the first two tasks. We focus on: reconstruction of the abstraction, increased resistance to challenges, developing a language for the abstraction and greater flexibility.

It appeared to us that Tugay reconstructed his abstractions of $|f(x)|$ and $f(|x|)$ in the initial stages of the third task. Reconstruction is a process in which abstractions are derived from past constructions, i.e. the abstractions are not simply recalled. In 44T-54T, we see Tugay combining and manipulating various bits of information about absolute values, symmetries and graphs. This process continues throughout the third task. For example, later in the protocol (172T) we see his justification that the graph of $f(|x|)$ is the same as the graph of $f(x)$ for positive values of x by combining bits of information and actively reorganising them. We do not equate reconstruction with consolidation but reconstruction appears to be an important part of consolidation.

The fragility of new abstractions, we believe, makes students reluctant to use them to counter challenges. In the course of consolidation, however, students begin to resist challenges by establishing interconnections between the new abstractions and established mathematical knowledge and by reasoning with these abstractions.

Tugay established interconnections between the graph of $f(|x|)$, absolute values, symmetry and linear functions (154T, 158T, 172T & 174T). In 172T, for example, Tugay explains why $f(x)$ and $f(|x|)$ are the same for positive values of x by establishing connections between the graph of $f(|x|)$, absolute values and $f(x)$. Shortly after (176T) a change in the tone of his assertions can be noted, "that proves … I think I made my point". This change to a confident tone continues from 176T, e.g. compare "yes definitely so" (178T) with "I don't know if I can say absolutely" (134T). This aspect of this abstraction, that $f(x)$ and $f(|x|)$ are the same for positive values of x, appears to be fully consolidated as he used this to confidently elaborate how to obtain the graph of $f(|x|)$ (174, 176, & 178). We posit that Tugay's initial insecurity in his claims about his new abstractions partly stem from the fact that the interconnections between the new abstractions and existing knowledge were not sufficiently well established. The more connections students make between the new abstractions and existing knowledge, the more meaningful and accessible the new abstractions become, and students become more confident and resistant to challenges.

Apart from the confidence of students' language as the abstractions of the first and second tasks are consolidated in the third task, there was a qualitative shift in the clarity and precision of their language in the course of the third task. It seems that language development (to describe new abstractions) has a dialectical relationship with the consolidation of the abstraction. For example, in Tugay's protocol, his description in 52T of the graph of $f(|x|)$ lacks precision and is slightly ambiguous whereas in 178T his mathematical language is precise and unambiguous. A lack of precision in the initial part of the third task is not surprising, but language development during the task is significant with regard to consolidation in that the language of the new abstractions needs time to develop.

Students' use of examples is closely related to this development in their language of the abstraction. Prior to consolidation students appear to need concrete examples to formulate their thoughts but after consolidation they appear to use examples to demonstrate assertions. Tugay, for example, in 52T, articulates his thoughts by referring to specific properties of graphs. In 172T, however, he uses examples to clarify, to convince the interviewer, and to justify his assertions.

The use of specific examples to articulate thoughts suggests to us that the new abstractions are somewhat inflexible. When Tugay, for example, was asked to give an account of the graphs of $f(|x|)$ (52T) he appears to begin stating a general rule, "after x becomes negative", but later he focuses on a specific graph. Later in this protocol (174T), however, he quickly provides an alternative way to view $f(|x|)$ – and does so without recourse to a specific example. The phrase "let me put it another way" along with the confident and precise way he states this other way, suggest to us that he has consolidated this abstraction and is using it flexibly.

Comments on Dreyfus & Tsamir's consolidation model

We deliberately chose not to employ Dreyfus & Tsamir's (2001) constructs in this analysis, to avoid a narrow line of enquiry in a new area of research. Their paper, however, is an important one and it is useful to make some comparative comments.

They isolate three distinct modes of thinking in the consolidation process: building-with, reflecting on the building-with and reflecting. Our analysis of Tugay's protocol shows that building-with was the dominant mode of thinking throughout the third task. He occasionally reflected on building-with, for example, when he says in 134T, "it was enough to take the symmetry. In fact we made use of analytic geometry". We did not, however, note what Dreyfus & Tsamir call 'reflection', "an impressive reflection on a wide range of mathematical and psychological issues" (ibid., p.27). We do not think it is an essential part of consolidating an abstraction.

Dreyfus & Tsamir claim that consolidation occurs both in using new abstractions and while reflecting on them. Our data supports this. In the early stages of the third task Tugay reconstructed his new abstractions and later developed convincing arguments to defend his claims. During the reconstruction he used the abstractions. When challenged he developed convincing arguments where he both used and reflected on the new abstractions. This helped him to establish interconnections between his established mathematical knowledge and the new abstractions.

Dreyfus & Tsamir put forward five psychological and/or cognitive constructs associated with the progressive consolidation of an abstraction: immediacy, self-evidence, confidence, flexibility and awareness. Our data broadly supports this. We have already discussed confidence and flexibility. Regarding immediacy, there are clear indications in Tugay's protocol that this develops during consolidation. At the beginning of the third task (12T) Tugay was rather slow in describing how to draw the graph of $f(|x|)$ and somewhat hesitant in evaluating the initial two propositions in the fourth question (110T, 124T, & 126T). However, towards the end of the task he was quickly describing (162T & 178T) ways to obtain the graph of $f(|x|)$ and reacted to the third proposition in the fourth question (164T) almost immediately after reading it. Regarding self-evidence and awareness, these appear to be natural consequences of this consolidation process and it may not be always possible to refer to particular utterances to exemplify.

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