MATHEMATICAL DISCOVERY[1]: HADAMARD RESURECTED

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In 1943 Jacques Hadamard gave a series of lectures on mathematical invention at the École Libre des Hautes Etudes in New York City. These talks were subsequently published as The Psychology of Mathematical Invention in the Mathematical Field (Hadamard,1945). In this article I present a study that mirrors the work of Hadamard. Results both confirm and extend the work of Hadamard on the inventive process. In addition, the results also speak to the larger context of 'doing' and learning mathematics.

What is the genesis of mathematical creation? What mechanisms govern the act of mathematical discovery? This "is a problem which should intensely interest the psychologist. It is the activity in which the human mind seems to take the least from the outside world, in which it acts or seems to act only of itself and on itself" (Poincaré, 1952, p. 46). It should also intensely interest the mathematics educator, for it is through mathematical discovery that we see the essence of what it means to 'do' and learn mathematics. In this article I explore the topic of mathematical discovery along two fronts, the first of which is a brief synopsis of the history of work in this area. This is then followed by a glimpse at a study designed to elicit from prominent mathematicians ideas and thoughts on their own encounters with the phenomenon of mathematical discovery.

HISTORICAL BACKGROUND

In 1908 Henri Poincaré (1854–1912) gave a presentation to the French Psychological Society in Paris entitled 'Mathematical Creation'. This presentation, as well as the essay it spawned, stands to this day as one of the most insightful, and thorough treatments of the topic of mathematical invention. In particular, the anecdote of Poincaré's own discovery of Fuschian function transformations stands as the most famous contemporary account of mathematical creation.

> Just at this time, I left Caen, where I was living, to go on a geological excursion under the auspices of the School of Mines. The incident of the travel made me forget my mathematical work. Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step, the idea came to me, without anything in my former thoughts seeming to have paved the way for it, that the transformations I had used to define the Fuschian functions were identical with those of non-Euclidean geometry. I did not verify the idea; I should not have had the time, as, upon taking my seat in the omnibus, I went on with the conversation already

[1] It should be noted that, although arguments can be made for the distinction between *invention* and *discovery,* Hadamard (1945) made no distinction. As such, for the purposes of this article the two terms will be used similarily and interchangeably.

> commenced, but I felt a perfect certainty. On my return to Caen, for conscience' sake, I verified the results at my leisure. (Poincaré, 1952, p. 53)

Inspired by this presentation, Jacques Hadamard (1865-1963) began his own empirical investigation into mathematical invention. He based this investigation on a 30 question survey authored by psychologists Édouard Claparède and Théodore Flournoy which was published in the pages of L'Enseignement Mathematique in 1902 and 1904. Although impressed with their intensions, Hadamard was critical of Claparède and Flournoy's work. He felt that the two psychologists had failed to adequately treat the topic of mathematical invention on two fronts; the first was the lack of comprehensive treatment of certain topics and the second was the lack of prominence on the part of the respondents. In particular, Hadamard felt that as exhaustive as the survey appeared to be, it failed to ask some key questions – the most important of which was with regard to the reason for failures in the creation of mathematics. This seemingly innocuous oversight, however, led directly to what he termed "the most important criticism which can be formulated against such inquiries" (1945, p.10). This also lead to Hadamard's second, and perhaps more damning, criticism. He felt that only "first-rate men would dare to speak of" (p.10) such failures, and so, Hadamard retooled the survey and gave it to friends of his for consideration – mathematicians like Henri Poincaré and Albert Einstein, to name a few – whose prominence were beyond reproach. The results of this seminal work culminated in a series of lectures on mathematical invention at the École Libre des Hautes Etudes in New York City in 1943. These talks were subsequently published as *The Psychology of Mathematical Invention in the Mathematical Field* (Hadamard,1945).

Hadamard's treatment of the subject of invention at the crossroads of mathematics and psychology was an entertaining, and sometimes humorous, look at the eccentric nature of mathematicians and their ritualistic practices. His work is an extensive exploration and extended argument for the existence of unconscious mental processes. To summarize, Hadamard took the ideas that Poincaré had posed and, borrowing a conceptual framework for the characterization of the creative process in general, turned them into a stage theory. This theory still stands as the most viable and reasonable description of the process of mathematical invention. In what follows I present this theory, referenced not only to Hadamard and Poincaré, but also to some of the many researchers who's work has informed and verified different aspects of the theory.

MATHEMATICAL INVENTION

The phenomenon of mathematical invention, although marked by sudden illumination, consists of four separate stages stretched out over time, of which illumination is but one part. These stages are initiation, incubation, illumination, and verification (Hadamard, 1945). The first of these stages, the *initiation* phase, consists of deliberate and conscious work. This would constitute a person's voluntary, and seemingly fruitless, engagement with a problem and be characterized by an attempt to solve the problem by trolling through a repertoire of past experiences (Bruner, 1964; Schön, 1987). This is an important part of the inventive process because it creates the tension of unresolved effort that sets up the conditions necessary for the ensuing emotional release at the moment of illumination (Barnes, 2000; Davis & Hersh, 1980; Feynman, 1999; Hadamard, 1945; Poincaré, 1952; Rota, 1997).

Following the initiation stage the solver, unable to come to a solution stops working on the problem at a conscious level (Dewey, 1933) and begins to work on it at an unconscious level (Hadamard, 1945; Poincaré, 1952). This is referred to as the *incubation* stage of the inventive process and it is inextricably linked to the conscious and intentional effort that precedes it.

> There is another remark to be made about the conditions of this unconscious work: it is possible, and of a certainty it is only fruitful, if it is on the one hand preceded and on the other hand followed by a period of conscious work. These sudden inspirations never happen except after some days of voluntary effort which has appeared absolutely fruitless and whence nothing good seems to have come … (Poincaré, 1952, p. 56)

After the period of incubation a rapid coming to mind of a solution, referred to as *illumination*, may occurs. This is accompanied by a feeling of certainty (Poincaré, 1952) and positive emotions (Barnes, 2000; Burton 1999; Rota, 1997). With regards to the phenomenon of illumination, it is clear that this phase is the manifestation of a bridging that occurs between the unconscious mind and the conscious mind (Poincaré, 1952), a coming to (conscious) mind of an idea or solution. However, what brings the idea forward to consciousness is unclear. There are theories on aesthetic qualities of the idea (Sinclair, 2002; Poincaré, 1952), effective surprise/shock of recognition (Bruner, 1964), fluency of processing (Whittlesea and Williams, 2001), or breaking functional fixedness (Ashcraft, 1989) only one of which will expand on here.

Poincaré proposed that ideas that were stimulated during initiation remained stimulated during incubation. However, freed from the constraints of conscious thought and deliberate calculation, these ideas would begin to come together in rapid and random unions so that "their mutual impacts may produce new combinations" (Poincaré, 1952, p. 61). These new combinations, or ideas, would then be evaluated for viability using an aesthetic sieve (Sinclair, 2002), which allowed through to the conscious mind only the "right combinations" (Poincaré, 1952, p. 62). It is important to note, however, that good or aesthetic does not necessarily mean correct. As such, correctness is evaluated during the fourth and final stage – *verification*.

HADAMARD RESURECTED

As mentioned, my interest in this topic has not been limited to its historical roots. I have also engaged in a number of studies pertaining to the topic (c.f. Liljedahl, 2002). The latest of these studies can best be described as a resurrection of Jacques Hadamard's work. That is, a portion of his original questionnaire was used to elicit from contemporary mathematicians ideas and thoughts on their own encounters with the phenomenon of mathematical discovery[2]. In what follows I present a summary of this study.

Hadamard's original questionnaire contained 33 questions pertaining to everything from personal habits, to family history, to meteorological conditions during times of work (Hadamard, 1945). From this extensive and exhaustive list of questions the five that most directly related to the phenomena of mathematical discovery and creation were selected. They are:

1. Would you say that your principle discoveries have been the result of deliberate endeavour in a definite direction, or have they arisen, so to speak, spontaneously? Have you a specific anecdote of a moment of insight/inspiration/illumination that would demonstrate this? [Hadamard # 9]
2. How much of mathematical creation do you attribute to chance, insight, inspiration, or illumination? Have you come to rely on this in any way? [Hadamard# 7]
3. Could you comment on the differences in the manner in which you work when you are trying to assimilate the results of others (learning mathematics) as compared to when you are indulging in personal research (creating mathematics)? [Hadamard # 4]
4. Have your methods of learning and creating mathematics changed since you were a student? How so? [Hadamard # 16]
5. Among your greatest works have you ever attempted to discern the origin of the ideas that lead you to your discoveries? Could you comment on the creative processes that lead you to your discoveries? [Hadamard # 6]

These questions along with a covering letter were then sent to 150 prominent mathematicians (see below) in the form of an email.

Hadamard set excellence in the field of mathematics as a criterion for participation in his study. In keeping with Hadamard's standards, excellence in the field of mathematics was also chosen as the primary criterion for participation in this study. As such, recipients of the survey were selected based on their achievements in their field as recognized by their being honored with prestigious prizes or membership in noteworthy societies. In particular, the 150 recipients were chosen from the published lists of the Fields Medalists, the Nevanlinna Prize winners, as well as the membership list of the American Society of Arts & Sciences. The 25 recipients who responded to the survey, in whole or in part, have come to be referred to as the 'participants' in this study.

The responses were initially sorted according to the survey question they were most closely addressing. In addition, a second sorting of the data was done according to trends that emerged in the participants' responses, regardless of which question they

[2] I would like to acknowledge the contribution made by Peter Borwein with respect to the collection of data for this study.

were in response to. This was a much more intensive and involved sorting of the data in an iterative process of identifying themes, coding for themes, identifying more themes, recoding for the new themes, and so on. In the end, there were 12 themes that emerged from the data, each of which can be attributed to one of the four stages presented in the previous section. Some of these themes serve to confirm the work of Hadamard while others serve to extend it. Given the limited space available here, in what follows I provide a *very brief summary* of four of the themes that extend our understanding of various stages of the inventive process.

The De-Emphasis of Details

A particularly strong theme that emerged from this study was the role that detail does NOT play in the incubation phase. Many of the participants mentioned how difficult it is to learn mathematics by attending to the details, and how much easier it is if the details are de-emphasized.

Stephen: Understanding others is often a painful process until one suddenly goes beyond the details and sees whole what's going on.

Mark: There is not much difference. More precisely, I seldom study or learn mathematics in detail.

In some cases, this also manifested itself as a strategy for problem solving and research.

Carl: Get the basics of the problem firmly and thoroughly into the head. After that, an hour or two each day of thinking on it is all that's needed for progress. [..] For that reason, I started some 20 years ago to ask students (and colleagues) wanting to tell me some piece of mathematics to tell me directly, perhaps with some gestures, but certainly without the aid of a blackboard. While that can be challenging, it will, if successful, put the problem more firmly and cleanly into the head, hence increases the chances for understanding. I am also now more aware of the fact that explaining problem and progress to someone else is beneficial; I am guessing that it forces one to have the problem more clearly and cleanly in one's head.

In presenting his strategy for getting " *the basics of the problem firmly and thoroughly into*" his head, Carl has come up with a strategy that de-emphasizes the details by forcing the transmission of the problem through a medium wherein details are impossible. That is, talking de-emphasizes details and, as a result, will "*put the problem more firmly and cleanly into the head, hence increases the chances for understanding*."

The Role of Talking

Further to this previous theme, it is also clear from the mathematicians' responses that, while working in the initiation phase, they have a much higher regard for transmission of mathematical knowledge through talking than through reading. This is best summarized in the comments of Jerry and George.

Jerry: I assimilate the work of others best through personal contact and being able to question them directly. [..] In this question and answer mode, I often get good ideas too. In this sense, the two modes are almost indistinguishable.

George: I get most of my real mathematical input live, from (good) lectures or one-on-one discussions. I think most mathematicians do. I look at papers only after I have had some overall idea of a problem and then I do not look at details.

Considering these last two themes (de-emphasizing of details and the role of talking) it becomes clear that the painstakingly rigorous fashion in which mathematical knowledge is written, both in journals and in text-books, as well as the detailed fashion of over-engineered curriculums stand in stark contrast to the methods by which mathematicians claim they best come to learn new mathematics.

Deriving and Re-Creation

A third theme that resonates with the two previous themes, while again speaking to the initiation phase of the inventive process, is the importance of deriving mathematics for oneself. In part, this theme reflects the personal practices of assimilating the work of others through active re-creation of the mathematics rather than passive reading of it.

Debby: I need to work out examples, specific instances of the new ideas to feel that I have any real understanding. Anything one creates oneself is much more immediate and real and so harder to forget.

Mark: When there is a need to fully assimilate something, I must redo everything in my own way.

Dick: I've forgotten what Hadamard had to say on this, but for me there's no difference, - in order to 'understand', I have to (re)create. To be sure, it's much easier to follow someone else's footsteps, i.e., it is much easier to prove a result one knows to be true than one that one merely guesses to be true.

However, the theme also manifested itself as advice to young mathematicians about how to best approach doing mathematics in the expansive response from Peter.

Peter: If you have an idea, develop it on your own for, say, two months, and only then check whether the results are known. The reasons are: (1) If you try to check earlier, you won't recognize your idea in the disguise under which it appears in the literature. (2) If you read the literature too carefully beforehand, you will be diverted into the train of thought of the other author and stop exactly where he ran into an obstacle. This happened to a friend of mine, who started three Ph.D. theses in totally unrelated fields, before he finished one.

Again, this stands in stark contrast to the conventional methods by which mathematical knowledge is often conveyed.

The Contribution of Chance

Up until now, the themes that have been presented all pertain to the initiation phase of the inventive process. Indeed, seven of the 12 emergent themes deal with this stage. The theme presented here, however, concerns the illumination stage. From the responses provided by the participants it became clear that, for them, *chance* plays a very large role in illumination and insight. There are two types of chance, *intrinsic chance* and *extrinsic chance*. Intrinsic chance deals with the luck of coming up with an answer, of having the right combination of ideas join within your mind to produce a new understanding. This was discussed by Hadamard (1945) as well as by a host of others under the name of "*the chance hypothesis*" and is nicely demonstrated in Dan's comment.

> Dan: And relevant ideas do pop up in your mind when you are taking a shower, and can pop up as well even when you are sleeping, (many of these ideas turn out not to work very well) or even when you are driving. Thus while you can turn the problem over in your mind in all ways you can think of, try to use all the methods you can recall or discover to attack it, there is really no standard approach that will solve it for you. At some stage, if you are lucky, the right combination occurs to you, and you are able to check it and use it to put an argument together.

Extrinsic chance, on the other hand, deals with the luck associated with a chance reading of an article, a chance encounter, or a some other chance encounter with a piece of mathematical knowledge, any of which contribute to the eventual resolution of the problem that one is working on. This is best demonstrated by the words of Mark and Carl.

> Mark: Do I experience feelings of illumination? Rarely, except in connection with chance, whose offerings I treasure. In my wandering life between concrete fields and problems, chance is continually important in two ways. A chance reading or encounter has often brought an awareness of existing mathematical tools that were new to me and allowed me to return to old problems I was previously obliged to leave aside. In other cases, a chance encounter suggested that old tools could have new uses that helped them expand.

> Carl: But chance is a major aspect: what papers one happens to have read, what discussions one happens to have struck up, what ideas one's students are struck by (never mind the very basic chance process of insemination that produced this particular mathematician).

Once again, the idea that mathematical discovery often relies on the fleeting and unpredictable occurrences of chance encounters is starkly contradictory to the image projected by mathematics as a field reliant on logic and deductive reasoning. Extrinsic chance, in particular, is an element that has been largely ignored in the literature.

CONCLUSIONS

Mathematical discovery and invention are aspects of 'doing' mathematics that have long been accepted as standing outside of the theories of "logical forms" (Dewey, 1938, p.103). That is, they rely on the extra-logical processes of insight and illumination as opposed to the logical process of deductive and inductive reasoning. This study confirms this understanding as well as adds to this cohort of extra-logical processes the role of chance. In addition, this study also comments on the initiation phase of the inventive process in showing that it is best facilitated through the de-emphasizes of details and transmission of knowledge through talking and (re)creation. As such, this study also contributes to our understanding of the larger contexts of 'doing' and learning mathematics.

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