STRUCTURE SENSE IN HIGH SCHOOL ALGEBRA: THE EFFECT OF BRACKETS

Maureen Hoch and Tommy Dreyfus

Tel Aviv University, Israel

This paper presents an initial attempt to define structure sense for high school algebra and to test part of this definition. A questionnaire was distributed to 92 eleventh grade students in order to identify those who use structure sense. Presence and absence of brackets was examined to see how they affect use of structure sense. The overall use of structure sense was less than expected. The presence of brackets was found to help students see structure.

A student made a minor mistake in solving a problem in a matriculation exam and obtained the following equation: $1-\frac{1}{n+2}-\left(1-\frac{1}{n+2}\right)=\frac{1}{132}$. He solved it by first multiplying both sides by a common denominator then opening brackets and collecting like terms. This solution raised questions about structure sense and inspired the present paper. In this paper structure sense is defined, data that was collected by means of a questionnaire containing equations similar to the above are presented, and the results are discussed in terms of structure sense.

The research described here is part of a study concerning high school students' struggle with algebra. The students in question study mathematics in intermediate or advanced streams. In order to be accepted into these streams they have had to demonstrate a certain proficiency with algebraic techniques. By dealing with students who have been relatively successful in acquiring and using basic algebraic knowledge, at least as it is tested in regular school exams, issues of cognitive level and different approaches to beginning algebra (see MacGregor & Stacey, 1997) are avoided. Nevertheless many of these students do not succeed in applying their basic algebraic knowledge when solving problems in more advanced algebra, trigonometry or calculus.

Linchevski and Livneh (1999) first used the term structure sense when describing young students' difficulties with using knowledge of arithmetic structures at the early stages of learning algebra. Hoch (2003) suggested that structure sense is a collection of abilities, separate from manipulative ability, which enables students to make better use of previously learned algebraic techniques. More precise definitions of structure sense and of algebraic structure are required. The definitions given in this paper are based on interviews with researchers in mathematics education. The full derivation of these definitions will be reported on in a future publication.

DEFINING STRUCTURE SENSE

In order to reach a definition of structure sense it is necessary to discuss what is meant by structure, specifically in the context of high school algebra.

Structure

Why define structure? The term is widely used and most people feel no need to explain what they mean by it. It is used in the field of mathematics education to cover various different meanings (see for example Dreyfus & Eisenberg, 1996). In different contexts the term structure can mean different things to different people. This could cause problems when discussing structure sense. It must be made clear which meaning of the term structure is being used. The following definition of structure will be adopted, for the purposes of discussing high school algebra.

> *Structure in mathematics can be seen as a broad view analysis of the way in which an entity is made up of its parts. This analysis describes the systems of connections or relationships between the component parts.*

Algebraic structure (at high school level)

Algebraic expressions or sentences (equality or inequality relation between two algebraic expressions) can be considered to represent algebraic structures. Examples of two structures from high school algebra are algebraic fractions and quadratic equations. The shape of an algebraic fraction is $\frac{f(x)}{g(x)}$ where *f(x)* and *g(x)* are both polynomial functions. Simplifying or expanding the fraction may reveal an internal order. A quadratic equation is any polynomial equation that can be transformed into the standard shape $ax^2 + bx + c = 0$ where a, b and c are real number parameters. The process of transforming the equation into standard form may reveal the internal order. This may lead to a solution either by factoring, or by using the quadratic formula. The internal order might also lead us to expect that there will be 0 or 1 or 2 solutions, and to know that these solutions are the intersection points of the parabola $y= ax^2 + bx + c$ with the X-axis. Algebraic structure will be defined in terms of shape and order.

> *Any algebraic expression or sentence represents an algebraic structure. The external appearance or shape reveals, or if necessary can be transformed to reveal, an internal order. The internal order is determined by the relationships between the quantities and operations that are the component parts of the structure.*

Structure sense

There are structures in high school algebra that are concealed by external appearance. The equation $4x^2 - x^3 + 5(4 - 2x) = (3 - x^2)(6 + x)$ can be transformed into the standard quadratic equation $10x^2 - 13x + 2 = 0$. These two equations are equivalent but whereas in the second the structure is obvious in the first it is less so. Any discussion of algebraic structures will have to involve some discussion on equivalencies. If two algebraic expressions or sentences are equivalent do they possess the same structure? The two expressions $30x^2 - 28x + 6$ and $(5x - 3)(6x - 2)$ are equivalent. Yet the first is clearly a quadratic expression and the second is clearly the product of two linear factors. What is the structure here? Our answer is that "quadratic expression" and "product of two linear factors" are different interpretations of the same structure. Knowing which interpretation is more useful in any given context is a part of structure sense. Structure sense may have a lot to do with experience and something in common with intuition. For example, for a student with structure sense, the need to simplify a fraction is self-evident. After much consideration the following definition for algebraic structure sense is proposed.

Structure sense, as it applies to high school algebra, can be described as a collection of abilities. These abilities include the ability to: see an algebraic expression or sentence as an entity, recognise an algebraic expression or sentence as a previously met structure, divide an entity into sub-structures, recognise mutual connections between structures, recognise which manipulations it is possible to perform, and recognise which manipulations it is useful to perform.

METHODOLOGY

Instruments

A questionnaire was designed with several aims. The main aims were to identify students who display structure sense and to investigate if structure sense is affected by the number of sets of brackets (0, 1, 2) and by the placement of the variable (on one side of equation or on both sides of equation). The researchers could find no reports of research on the effect of brackets on students' success with solving equations. Other aims were to investigate if use of structure sense is more prevalent among advanced students than among intermediate students, and if students are consistent in their use of structure sense. The questionnaire consisted of two equations. Three alternative items were designed for the first question (A, B and C) and three for the second question (X, Y and Z) as follows.

A. $1-\frac{1}{n+2}-\left(1-\frac{1}{n+2}\right)=\frac{1}{110}$

X. $\frac{1}{4}-\frac{x}{x-1}-x=5+\left(\frac{1}{4}-\frac{x}{x-1}\right)$

B. $\left(1-\frac{1}{n+1}\right)-\left(1-\frac{1}{n+1}\right)=\frac{1}{132}$

Y. $\left(\frac{1}{4}-\frac{x}{x-1}\right)-x=6+\left(\frac{1}{4}-\frac{x}{x-1}\right)$

C. $1-\frac{1}{n+3}-1+\frac{1}{n+3}=\frac{1}{72}$

Z. $\frac{1}{4}-\frac{x}{x-1}-x=7+\frac{1}{4}-\frac{x}{x-1}$

The instructions were "Solve for n" (or for x). Each student was asked to solve two equations, A & Y, A & Z, B & X, B & Z, C & X or C & Y. The questionnaire was administered to 92 eleventh grade students in a high school in a well-established Israeli town. These students, aged 16 to 17, learn in intermediate to advanced mathematics streams.

After the questionnaires were examined, four students who were considered to have used structure sense in an unusual manner were interviewed, in an attempt to obtain a clearer picture of their reasoning.

Interpretation

A student who uses structure sense to solve equation A would be expected to do the following. S/he looks at the difference of two terms $1-\frac{1}{n+2}$ as an entity or structure and recognises that the same structure (or sub-structure) appears inside the brackets. The relationship between the two structures is equality, and since they are connected by a minus sign, the result is zero. Then s/he "sees" that the structure of the equation is such that it is in fact equivalent to a much simpler equation.

Of course a student's thought processes cannot be known from a written answer. A student who writes something similar to "zero equals a fraction and so there is no solution" is considered to be displaying structure sense. In a similar manner the hypothetical student, when asked to solve equation X, would be expected to write "-x = 5 and so x equals –5". The items containing two or no sets of brackets can be analysed in an analogous way.

Some students who wrote a line or two of calculations before arriving at the above conclusions were interviewed. They said that they subsequently "saw" the simple equation and thus they were also considered to have used structure sense. Students using other methods, for example opening brackets and/or finding a common denominator, were considered to have displayed a lack of structure sense.

RESULTS

Table 1 shows the rate of structure sense used. The result is obviously disappointing. The overall rate of use of structure sense is very low. In the majority of cases, the students solved the equations by first multiplying by a common denominator and only later cancelling like terms. In other words, the majority of students displayed a lack of structure sense. Most of those who didn't use structure sense either made calculation mistakes or failed to cancel an extraneous solution. Those who used structure sense got the answer quickly and accurately.

Table 1 Percentage of questions solved using structure sense

Equation	Advanced stream (45 students)	Intermediate stream (47 students)	Total (92 students)
A: $1-\frac{1}{n+2}-\left(1-\frac{1}{n+2}\right)=\frac{1}{110}$	**18.8%** (3/16)	**6.3%** (1/16)	**12.5%** (4/32)
B: $\left(1-\frac{1}{n+1}\right)-\left(1-\frac{1}{n+1}\right)=\frac{1}{132}$	**23%** (3/13)	**12.5%** (2/16)	**17.2%** (5/29)
C: $1-\frac{1}{n+3}-1+\frac{1}{n+3}=\frac{1}{72}$	**0%** (0/16)	**0%** (0/15)	**0%** (0/31)
X: $\frac{1}{4}-\frac{x}{x-1}-x=5+\left(\frac{1}{4}-\frac{x}{x-1}\right)$	**15.4%** (2/13)	**14.3%** (2/14)	**14.8%** (4/27)
Y: $\left(\frac{1}{4}-\frac{x}{x-1}\right)-x=6+\left(\frac{1}{4}-\frac{x}{x-1}\right)$	**25%** (4/16)	**11.8%** (2/17)	**18.2%** (6/33)
Z: $\frac{1}{4}-\frac{x}{x-1}-x=7+\frac{1}{4}-\frac{x}{x-1}$	**18.8%** (3/16)	**6.3%** (1/16)	**12.5%** (4/32)
Total 184 solutions	**16.7%** (15/90)	**8.5%** (8/94)	**12.5%** (23/184)

Number of brackets

It was expected that in the equations without brackets (C & Z) it would be easier for the students to identify and cancel like terms. This was clearly not the case. Only 6.3% (4/63) of the students used structure sense when no brackets were present as compared to 13.6% (8/59) for one set of brackets (A & X) and 17.7% (11/62) for two sets (B & Y). The lack of brackets seems to have deterred students from recognising like terms. Is it possible that the brackets give the students a clue where to look and that the lack of brackets leaves a long unstructured (in the students' eye) expression? The brackets seem to focus the students' attention and alert them to the possibility of like terms. However one student stated that he mentally opened the brackets in order to cancel out in equation A. Therefore students do not necessarily see an expression inside brackets being the same as one without brackets.

The most glaring result is the total lack of use of structure sense in equation C. Why don't students see what is so obvious to us? It was thought that the use of brackets might confuse students, but here it appears that the absence of brackets seems to be a stumbling block. However this is not so much the case in equation Z although also here the "seeing the obvious" is less than might be expected. What makes equation C harder? Is it the long expression (4 terms) on one side of the bracket? Or do the students just work mechanically from left to right, stopping to "take stock" only when they see the equal sign? Here are some quotes from students who were interviewed.

- *Usually, in my opinion, every student who sees fractions he straightaway deals with them………. Usually you need to get a common denominator.*
- *I don't look at the equation as a whole. I look at each side separately and only then I move things………. I get rid of the brackets. The fewer brackets the better.*
- *First I always open the brackets.*

It seems that in the calculation students aim to "get rid of" the brackets. Yet we found that other students in fact added brackets to "see" the identical terms better (see Table 2). In fact one of the students actually stated this: *with brackets it's easier to see*.

Thus the presence of brackets might help them to see the structure. A feature of using structure sense is "looking" before "doing", something that teachers might be expected to emphasize in the high school classroom when they are reviewing previously learned algebraic techniques.

Placement of variable

The results show that 9.8% (9/92) of the students used structure sense when the variable appeared only on one side of the equation (A, B, C) as compared to 15.2% (14/92) when the variable appeared on both sides (X, Y, Z). Thus the placement of the like terms on opposite sides of the equation seems to enable students to identify them more easily. It is worth considering whether the fact that equations X, Y, Z have a non-empty solution set while equations A, B, C have an empty solution set has an effect on the results. Since structure sense is more concerned with how students approach an equation and less with what they do with the final solution the effect should be minimal. One student wrote $0 = \frac{1}{110}$ as his solution to A. In the interview he said that it did not make sense. When asked what was the solution of the equation he said that there was no solution. He was considered to have used structure sense.

Use and consistency

Overall only 19.6% (18/92) of the students displayed structure sense in at least one of the questions. As expected the advanced students used structure sense more: 24.4% (11/45) than the intermediate students: 14.9% (7/47). Of the students who used structure sense, only 27.8% (5/18) were consistent, using it in both equations.

Eighteen students used structure sense to answer a total of 23 questions. In Table 2 these 23 questions are examined to see how structure sense was used.

Table 2 Different uses of structure sense (N = 23)

Method	Example	
Immediate	$0 = \frac{1}{132}$ → no n (from B)	**39.1%** (9)
Minimal working	$\frac{1}{4} - \frac{x}{x-1} - x = 5 + \left(\frac{1}{4} - \frac{x}{x-1}\right)$ $\left(\frac{1}{4} - \frac{x}{x-1}\right) - x = 5 + \left(\frac{1}{4} - \frac{x}{x-1}\right)$ x = -5 (from X)	**43.5%** (10)
Calculation	$\left(\frac{(x-1)-4x}{4(x-1)}\right) - x = 6 + \left(\frac{(x-1)-4x}{4(x-1)}\right)$ x = -6 (from Y)	**17.4%** (4)

The minimal working shown here involved the addition of brackets, but in some other cases it involved the removal of brackets. The student who wrote the calculation in Table 2 was interviewed. His reaction on being asked what he was thinking when he wrote this was: *Oh. Now I see. Simply, I did common denominator. And now I see it was completely unnecessary.*

A teacher's suggestion

After the disappointing results from the students it was decided to show some teachers the student's solution to the equation mentioned in the first paragraph of this paper. About half of these teachers noticed immediately that the left-hand side is zero. The others had to be prompted to "see" what had seemed very obvious to the researchers. This lack of structure sense among teachers may be a clue to the disappointingly low incidence of structure sense among the students.

One teacher suggested that tenth graders would perform better due to the proximity of learning about equations. The questionnaire was subsequently administered to a tenth grade advanced class. In fact the rate of use of structure sense among the tenth graders was higher than among the eleventh graders. Several students substituted a new variable in place of a longer algebraic term, a method they had recently learned. However the eleventh graders had learned this technique a year ago and did not use it. This indicates that it might be possible to teach attention to structure but better methods must be found to ensure retention of this knowledge.

CONCLUSION

What does the above data tell us? Very few of the students in this study used structure sense. Those who did so were not consistent. As expected, the advanced students used it more than the intermediate ones. Those who used structure sense got the answer quickly and accurately, avoiding opportunities for mistakes that often occur in long calculations. The presence of the variable on both sides of the equation helped in identifying like terms. The presence of brackets also seemed to help students see structure, focussing their attention on like terms and breaking up the long string of symbols. However the evidence about the effect of brackets is inconclusive, as some students seem to prefer to eliminate them from the equation.

What abilities might be present in the students who used structure sense to solve the equations? The ability to see an algebraic expression or sentence as an entity necessitates stopping to look at the equation before automatically applying algebraic transformations. The ability to recognise mutual connections between structures, in this case equality, could lead to choosing the appropriate manipulations.

Not all teachers seem to use structure sense. Presumably these teachers don't encourage their students to use it. Is structure sense something that can be taught? Should it be taught? We feel that the last two questions should be answered in the affirmative but are not yet ready with an answer for the obvious next question: How should structure sense be taught?

We are convinced of the importance of drawing students' attention to structure. The above definitions should be useful as guidelines for further research and didactic design.

References

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