The 28th International Conference

Of the

International Group for the Psychology of

Mathematics Education

Volume 3

14–18 July 2004

Bergen, Norway

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Research Reports - Full-Text

A TEACHER'S MODEL OF STUDENTS' ALGEBRAIC THINKING ABOUT EQUIVALENT EXPRESSIONS

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This research report describes the findings of a study on teachers' ways of interpreting student responses to tasks involving equivalent expressions. The teachers in this study were engaged in model-eliciting activities designed to promote the development of their knowledge and reveal their models (or interpretations) of their students' algebraic thinking about equivalent expressions by creating a library of their students' work. This report focuses on one teacher's model of his algebraic practice. Results showed that this teacher devoted a significant amount of time to the implementation of the algebraic unit. The teacher employed visual strategies for the first time and began to perceive their usefulness in demonstrating the equivalency of two expressions.

INTRODUCTION

Coupled with changes in the past decade advocating a reform-based vision of teaching algebra and the pressing need for effective algebraic instruction in the school curriculum, study about teachers' understanding of algebraic teaching becomes critical. At least a year of high school algebra is now the norm for most students in the United States. Currently, reform efforts advocated by the National Council of Teachers of Mathematics (NCTM) in the United States call for teachers "to analyze what they and their students are doing and consider how those actions are affecting students' learning" (NCTM, 2000, p. 19). Research on the teaching of algebra shows instruction tends to emphasize procedural rather than structural interpretations (Kieran, 1992; Nathan & Koedinger, 2000). Under the reformed vision, instruction should focus on helping students make interpretations between procedural and structural conceptions within algebra. Often the difficulty of students making the cognitive leap from arithmetic to algebra is related to instructional strategies (Kieran, 1992), however, as Kieran noted, "there is a scarcity of research emphasizing the role of the classroom teacher in algebra instruction" (p. 395). A decade or so after Kieran's observation, the research base on teachers' knowledge for teaching algebra is still quite limited (Doerr, in press).

There also exists a need for more effective models of how teachers interpret the learning process (Ball, 1997; Shepard, 2000). The use of student work samples (Chamberlain, 2003; Moskal & Magone, 2000) offers teachers the possibility of detailed information from which to examine students' reasoning processes. However, not all teachers acquire the same information nor interpret it in consistent ways. For example, research on equivalent expressions showed that neither novice nor expert teachers used spatial arrangements to help students see that two expressions might be equivalent (Even, Tirosh, & Robinson, 1993). The primary goal of this study was to focus on the nature of the teachers' models or interpretations about their students' algebraic thinking as the students completed a series of lessons on equivalent expressions. Specifically, the core research question addressed in this study was: How do teachers interpret their practice when they focus on their students' algebraic thinking about equivalent expressions by building a library of student work?

THEORETICAL FRAMEWORK

The theoretical framework guiding this study is that of a models and modeling perspective on teacher development (Doerr & Lesh, 2003). A modeling perspective of teacher development focuses upon the ways teachers think about and interpret their practice. This perspective is based upon the premise that:

> it is not enough to see what a teacher does, we need to understand how and why the teacher was thinking in a given situation, that is, interpreting the salient features of the event, integrating them with past experiences, and anticipating actions, consequences, and subsequent interpretations. (p. 127)

What teachers do is inherently complex. A modeling perspective draws upon the mathematical knowledge teachers currently possess and uses that base to engage teachers in expressing, revising, and refining their knowledge. The intent is to extend that knowledge into increasingly powerful models of classroom teaching. This perspective suggests that teachers be viewed as evolving experts. Doerr and Lesh assert that teachers' models serve as interpretive and explanatory frameworks to make sense of their students' mathematical thinking.

METHODOLOGY AND DATA SOURCES

This study used Lesh and Kelly's (1999) multi-tiered research design, with the following levels:

- *Tier 3: The Researcher Level*. Researchers develop models to make sense of teachers' and/or students' model-eliciting activities.
- *Tier 2: The Teacher Level.* Teachers develop shared tools (such as a library of student work), and then construct and refine models to make sense of students' modeling activities.
- *Tier 1: The Student Level*: Students work on a series of model-eliciting activities that reveal how they are interpreting the situation. (adapted from p. 198)

Involvement with a thought-revealing activity should result in a tool or artifact that can be shared and reused (Schorr & Lesh, 2003). The thought-revealing activity used in this report consisted of asking the teachers to select, analyze, and interpret "exemplary and illuminating" (Doerr & Lesh, 2003, p. 136) samples of student work. The purpose of gathering student work was multifaceted:

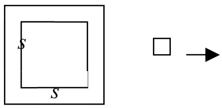
> As the teachers select, organize and compare student work, they reveal how they are seeing the students' mathematical ideas. This may lead to mismatches

> between their expectations of some students... It may lead to seeing students give mathematical interpretations of problem situations that the teacher had not seen. It is the resolution of such mismatches that provided the impetus for the development of teachers' knowledge. (p. 137)

Model-eliciting (or thought-revealing) activities for teachers attempt to help them become reflective of their teaching efforts.

Students were asked to solve problems involving equivalent expressions. The tasks were drawn from the Connected Mathematics Project (CMP) book *Say It With Symbols* (Lappan et al., 1998) from Problem 2.2:

> *Given a square pool as shown, draw a picture to illustrate the border of a square pool in four different ways:*
>
> a. $4(s + 1)$
>
> b. $s + s + s + s + 4$
>
> c. $2s + 2(s + 2)$
>
> d. $4(s + 2) - 4$
>
> e. Explain why each expression in parts a-d is equivalent to $4s + 4$. (adapted from p. 22)



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Teachers were directed to save examples of student work that might be helpful to show a pre-service teacher how students in their own classrooms actually solved these problems. This library also served as the focus of the individual interviews. The primary data source used in this research report was the library of student work created by the teacher. Other data sources included fieldnotes from classroom observations, and transcripts from teacher interviews conducted before the series of lessons was taught, immediately after the series of lessons was completed, and after the algebraic unit on *Say It With Symbols* was taught. The teacher being reported on was implementing CMP for the first time. Bruce had 18 years of experience teaching middle school mathematics in an urban setting. He was one of five eighth-grade teachers in this study.

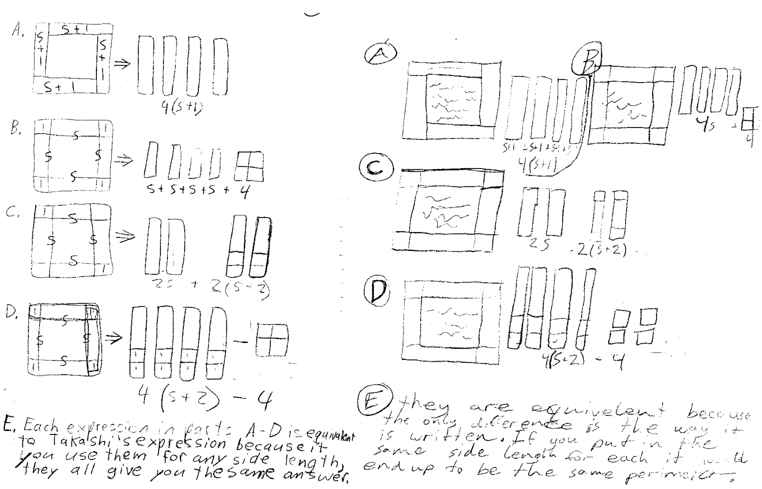
A grounded theory approach (Strauss & Corbin, 1998) was used to analyze the data. I refined my questions and observations based upon emerging themes and patterns of Bruce's actions as the study progressed. First, Bruce selected student work samples for his library. Then, we discussed why he chose each sample, and the similarities and differences amongst the samples in his library. I refined my interview questions based upon Bruce's observations. After Bruce finished teaching the entire *Say It With Symbols* algebraic unit, we discussed his interpretations again. From this process, I developed a profile of Bruce.

RESULTS

Two results emanated from the study. First, Bruce gave a significant amount of time to the implementation of the unit. Second, Bruce was beginning to develop significant insight into student work. These results are intimately connected with one another. Bruce's support of the implementation led to his acceptance of the new ideas about how to teach, which in turn led to his new insights on helping his students learn algebra in meaningful ways. I continue with excerpts of Bruce's responses to interview questions that show support for the implementation of the curriculum, and then excerpts from his library of student work that demonstrate his insight into his students' algebraic thinking on equivalent expressions.

First, Bruce was committed to teaching the sequence in its entirety at the pace dictated by the students' progress. Bruce took about one and one-half times as long to teach the series of lessons under study than recommended by the curriculum developers. He spent more time than any other teacher in this study. This appeared to be due to his willingness to implement the curriculum as intended and to give his students ample time to investigate the problems. This action appeared to enhance the extent of the information that he acquired about his students' thinking and was reflected in his extensive library of student work. But, Bruce commented that the length of time "bothered" him because of his concerns about teaching the entire eighth grade curriculum. Nonetheless, Bruce explained that he was willing to extend the time for this series of lessons because he deemed it important enough: "*Time is always one of the biggest factors, you know. I am always fighting with that...*" At the beginning of the study, Bruce decided not to let time constrain his teaching because he wanted to give his students every opportunity for success with the series of lessons. During the teaching sequence, Bruce passionately commented on how his students worked with more interest, *"The kids are involved… and this leads to a better understanding."* In a final interview, Bruce reflected *"The transition from arithmetic to algebra [is] something to be taken more seriously… I know that I will be doing this [the lesson] again. I will never do it the way I did it in the past. I think this is better."* Bruce came to embrace the curriculum during the course of the study, leading to the next result.

Second, Bruce began to develop significant insight into his students' algebraic thinking. He employed visual strategies for the first time and began to perceive their usefulness in demonstrating the equivalency of two expressions. Bruce reported that in the past he had provided students with procedural examples of the distributive property. Before this study, Bruce had never used spatial arrangements within the context of teaching equivalent expressions. During the study, Bruce was impressed by the subtle differences between the students' papers and wanted each different way the students answered this problem correctly to be part of his library. His library was larger and showed more details within the various student responses than the other teachers in this study. Bruce selected a large number of papers showing a wide variety of student responses. I now provide excerpts from his library. Bruce selected the work of Student A and Student B as exemplary and indicated that he received a great many papers similar to those represented in Figure 1 below:



Student A Student B

Figure 1. Bruce's Library of Student Work--Student A and Student B

Bruce thought that each drawing was clear and the breakdown of the expressions was well laid out. He was quite proud to show me these papers. He selected a number of papers similar to these samples whenever a student drew the representation differently. Bruce was pleased with many of his students' use of mathematical vocabulary in part e. He noted that his students understood the role of the variable *s* to represent the length of the side of the pool, and could distinguish the difference between when to solve for a variable in an equation and when to allow an algebraic expression to represent the solution. At the researcher level, I saw additional subtle differences that Bruce was not aware of. This was Bruce's first time engaged in building a library of student work, and it is likely that his perceptions would grow over time. First, Bruce did not comment about Student A, part c & d, where the student represented both corners of the square at the bottom. A representation in the spirit of the given geometry of the problem might place one square at the top and one at the bottom. I later pointed out these representations to Bruce, but he did not recall the specifics of the instruction. He surmised that perhaps the students were drawing the problem in sequential order, the "$s + 2$." Second, Bruce did not point out that these students did not mark out a section for the "1" representing the corner in part a. And last, Bruce did not comment on the fact that the representations drawn by Students A and B were not necessarily the same size as the original squares. Collectively, these details of analysis indicated that he noticed a variety of solutions, but he did not carefully attend to *all* the details of the solutions.

Another part of Bruce's library included work containing mistakes. (see Figure 2).

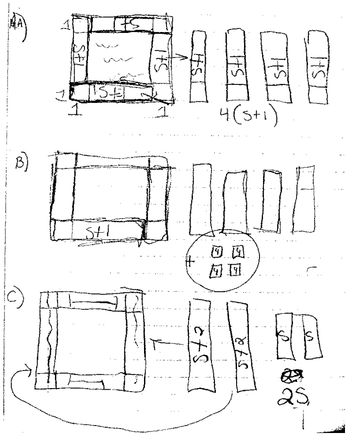


Figure 2. Bruce's Library of Student Work--Student C

Student C drew fairly accurate representations. In part a, the student labeled the individual sections "*s* + 1" and accurately represented the corners. In part b, Bruce circled the use of the 4's in the squares that should each represent a "1." In part c, Bruce noted that the student drew the parts of the expression but did not indicate the position of the corners. This student did not attempt part d or e. These mistakes indicated to Bruce that this student had an incomplete understanding of what the corners represented, and hence, the relationship of the expression to the structural aspect of the problem.

The curricular activities guided Bruce and his students to using variables in contexts different from those that Bruce used in the past. In prior years, Bruce lectured about properties like the distributive property. At the end of the study, he reflected, *"They are using pictures and diagrams and they are labeling the different parts of the diagrams with variable terms, and then expressing those areas in different ways."* Bruce saw that the visual strategies helped his students make sense of the algebraic expressions.

DISCUSSION AND CONCLUSION

Since the quality of algebraic instruction is salient to students' success in algebra, it is important to understand the models teachers develop of their practice. Results of this study suggested that Bruce both embraced the curriculum and developed new insight into the usefulness of the visual strategies while teaching about equivalent expressions. He was able to attend to many details (but not all details) and see the utility of connecting the visual representation with the algebraic expression. In this context, the use of the visual representation generally drove the understanding of the algebraic expression. Contrary to Even, Tirosh and Robinson (1993), Bruce identified visual arrangements as a highly useful instructional tool. This was in part due to the influence of curriculum, and also his personal willingness to implement it. The use of diagrams to represent a variable, such as the side of a square was new to him. That Bruce embraced this particular concept showed a shift in his instructional thinking. The use of visual representations in this manner has the potential to enhance the instructional process by better connecting the procedural and structural relationships within algebraic instruction as advocated by Kieran (1992).

In addition, the study demonstrates the potential effectiveness of the models and modeling framework, and the library of student work, as a tool to study teachers' developing knowledge. Consistent with prior research, Bruce did not attend to all the details of his students' responses. Creating a library of exemplary and illuminating student work may also provide teachers in other settings with examples that are useful for algebraic instruction and assessing the learning process.

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ON MOTIVATIONAL ASPECTS OF INSTRUCTOR-LEARNER INTERACTIONS IN EXTRA-CURRICULUM ACTIVITIES

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In extra-curriculum activities, the nature of the instructor-learner relationship differs from that in class. This constellation, if accompanied by a smaller gap between the mathematical skills of tutor and learner, has an impact on motivational aspects and on the choice of contents in these activities: The use of problem questions of competition type often plays a dominant role as a means to include many individuals in these activities. A survey is presented which indicates two risks of that approach: On the long term, it seems to have a negative effect on the motivation to work on open or more complex problems. Furthermore, it tends to discourage those with individual reference norm and motivation to participate that is intrinsic and directed on the mathematical content.

SOCIAL FEATURES OF EXTRA-CURRICULUM ACTIVITIES IN MATHEMATICS

Extra-curriculum activities give all members of the party the opportunity to break out of every day school life and to leave curricula, classmates, marks, time pressure, etc. behind. Here, an extra-curriculum activity is meant to fulfil the following criteria: The learners take part voluntarily without any immediate rewards such as winning awards or achieving high grades; the activity is of mathematical character or related to mathematics, e.g. as an application; finally, it is assumed that students of different classes join the group because of some interest in this mathematical offer. In particular, we do not focus on courses intensifying regular teaching and private tutoring.

It should be mentioned that activities like these are found at schools where committed teachers offer regular workshops in mathematics – often as a hobby without any extra-salary. Some universities advertise for studies in mathematics with offers like these. Also, one comes across such courses in societies organizing mathematical competitions.

Voluntary activities call for quite a high self-discipline: there are no institutional sanctions looming in case a member misses a session. In view of duties of every-day school life, it may be hard to find a reason for putting work in something that does not seem to pay off for some time. Voluntary work for mathematics is in general not popular – at least to the author's experience in certain countries like Germany and England. It appears to be hard to compete with part-time job opportunities and the tempting offers of the leisure industry designed for adolescents. It is not surprising that the numbers of participants in mathematical competitions decline significantly with advancing age. However, this development is quite drastic[i].

Participants can leave the group easily if they go through a difficult period. When asked for their motivation to do mathematics together, “passion” is often mentioned by both, instructors and learners. The word “passion” is a reminder that activities like these highly depend on the motivational aspects of everybody’s involvement (Roehr-Sendlmeier & Neitzke (1993)).

The organization of a long-term program for mathematically interested young people is a challenge requiring management of knowledge, conceptual design, motivation, and organisation – just to name a few. In mathematics the preparation of those activities is especially difficult. There is hardly suitable literature, let alone a conceptual program for the work with interested high school students. These come as individuals, often with special interests, preferences, and difficulties. A group of mathematically interested students cannot be regarded as a homogeneous constellation – neither in terms of mathematical abilities nor in terms of their expectations, already as school children (Peter-Koop, A. (1998)).

It does not make sense to provide interested students with contents to appear later in the mathematical curriculum. This could work for the moment in the extra-curriculum activity, but it could mean that the participants are bored to death when the very topic is treated later in the regular lessons. An extra-curriculum activity had better concentrate on topics being rather disjoint from or supplementing the curriculum (Renzulli, Reis & Smith (1982)).

Since resources for such topics – especially in an appropriate didactical reconstruction for young students – are still limited an instructor must be willing and able to create a program by himself. Plenty of beautiful mathematics could be made understandable for youngsters, but this takes a lot of energy and time. Let us concentrate on the trainer’s role in this situation: the members of the group hope for a challenge, but do not want to be discouraged if too much is expected from them. There are many different abilities, various levels of mathematical experience, a lot of opinions on how topics should be taught and, as we will see, different motivational aspects to be considered in the planning of the activities.

This could be said about every learning activity with a group of individuals. But the teaching of the mathematics curricula has been developing for generations and has achieved a certain level of sophistication and documentation. Besides, teachers are normally not trained in extra-curriculum training. Apart from those who took part in such an activity at young age themselves, one finds mostly autodidacts in the specialization in this area.

Topics considered interesting by students soon exceed the rigorous mathematical background on the instructor’s side. People who worked with gifted students – a term not to be used here too often for reasons discussed below – know that this requires instructors to cope with the fact that students are sometimes able to think more quickly and thoroughly than the instructor himself.

MATHEMATICAL PROBLEM QUESTIONS OF COMPETITION TYPE

Mathematical problem questions of competition type have many features which are useful for instructors of interested adolescents: These exist in huge numbers, for a variety of topics and degrees of difficulty. They have been collected in a long tradition for different reasons and are, as such, rather well documented and are seen as an important offer for interested students (Kallmann (2002)). Most of these collections contain problems to which a canon of solution strategies can be applied successfully. These are also helpful for instructors because the suggested solutions normally do not leave the area of mathematics that is indicated in the question. For these reasons it is not too difficult to correct written solutions to these problems.

It is widely accepted that those who work on these problem questions make considerable progress in logic, mathematical writing, typical strategies for solving these problems, heuristic thinking, and the basics of the mathematical areas covered by these questions. Furthermore, they appear as an instrument of diagnosis: some problems can be used as indicators of mathematical abilities (Käpnick (1998a)).

The author refers to “mathematical problem questions of competition type” if the following criteria are met:

- The problem is stated with a well-defined task of what is to show.

The learners are sure that the instructor is in possession of a solution.

It is a non-routine problem.

The following observations in a decade of projects as instructor of mathematically interested students were the starting point for investigations in the project presented here:

Instructors tend to choose problem questions of competition type for extra-curriculum activities.

Learners who stay for a longer time in an extra-curriculum activity prefer this approach.

It is difficult for tutors to encourage work on more complex projects once the participants are used to problem questions of competition type.

- There are high school students who are interested in learning more in mathematics than school can offer them, but for whom an emphasis of problem questions of competition type is not attractive in the long term.

The aim of this paper is to check the relevance of these impressions. Therefore, a situation was created to

1. illustrate differences in the instructor's roles depending on the choice of problem types,
2. measure the motivation of students used to competition type problems if these are replaced by open learning environments,

evaluate the learners' expectations and preferences in this setting.

SUBJECTS AND PRELIMINARY DELIBERATIONS

The survey was carried out during an international mathematics camp at Münster, Germany, with 50 students aged between 16 and 17 from 5 countries: the Netherlands, Poland, Czech Republic, Hungaria, and Germany. Each country sent 10 students to the camp. The subjects had been identified for their high abilitites in mathematics in that they were among the best partipants of the Kangourou des Mathématiques competition (the European equivalent of the "Australian mathematics Competition") in their home country in 2002. In Germany, for instance, more than 155.000 students from over 2450 schools took part in the competition in that year. The internationality of the sample is not used for a national comparison analysis. It was considered important to try to work independently of regional and national peculiarities.

From the statistical point of view, a higher sample would be desirable for the future. It should be noted, however, that it is difficult to find opportunities like this camp with an international participation of a comparable level. Since the survey was carried out in order to investigate the relevance of the observations mentioned above, an exact evaluation of statistical characteristics is not in the focus of this aim and would not make sense with this sample size. The results of interviews, which were carried out with 15 students (3 from each country), should be considered equally important.

For the survey, every student took part in two sessions each of 3 hours length. After each session, every participant completed anonymously a standardized questionnaire.

Pre-tests showed high or very high interest in mathematics and more than two thirds took part in regular extra-curriculum activities in mathematics. Almost all (45) claimed to have regular experience with problem questions of competition type. More than two thirds devote regularly their leisure time after school to mathematics.

More than eighty percent of the subjects were male.

EXPERIMENTAL DESIGN

For this experiment, each subject took part in two sessions of three hours each as mentioned above in a fixed group of 12 or 13 members of mixed nationalities. For each of the four groups, the trainer changed on the second day in the second session. Each of the two tutors, a scientist in probability theory with experience in the work with gifted students and the author, had twice a group in the first session and twice in the second.

	Day 1	Day 2
Group A	Tutor I	Tutor II
Group B	Tutor II	Tutor I
Group C	Tutor I	Tutor II
Group D	Tutor II	Tutor I

Table 1: **Organisational plan**

The two tutors worked out guidelines to provide parallel procedures during the sessions. In the first session, the group was given a series of problem questions from competitions around the topic of Markov chains in gambling. In the second session, the other trainer gave problems for which the same mathematical methods at a similar difficulty are useful, but which are lacking a question asking for a definite answer. For example, the subjects were asked to help an insurance company with the decision as to whether a certain life insurance should be offered to a certain number of people. In another problem, the group was asked to work out a simulation to check a certain phenomenon in the theory Markov chains encountered on day 1. None of the problems followed the pattern with a definite question as an objective.

Results

The subjects showed different reactions on the two days. First of all, let us have a look at the participants' interests in the topic:

Statement	Number of this answer on day 1	Number of this answer on day 2
The session was not interesting for me.	7	0
I will try to learn more on the session's topic.	44	35

Table 2: **Interests of the 50 subjects in topics on both days**

Nobody among the subjects considered the second day boring; there is a small number (7 out of 50) who did not like the first day. The first day, however, has a bit more succeeded in encouraging further studies.

The most striking differences between day 1 and day 2 concern motivation and assessment of the trainers' competence.

Statement	Number of this answer on day 1	Number of this answer on day 2
I was highly motivated during the session.	36	17
The tutor is competent.	41	20

Table 3: **Motivation and assessments of trainers' competence on both days**

From the first to the second day, the number of motivated subjects halved. Simultaneously, the number of positive ratings of the tutors' competence halved, too. The latter concerned both tutors equally. Even though the ratings of motivation and of the tutors were not extremely bad on the second day, a clear difference can be observed.

DISCUSSION

During the sessions, the tutors' observations of the subjects' behaviour corresponded to these figures. The motivation on the first day was marvellous. There was both a spirit of competition and a determination to solve as many problems as possible. In the second session, the subjects were challenged to work out suitable objectives for mathematical problems themselves.

A student from the Netherlands rated the first session best of the whole camp (there were 12 sessions altogether) "*because we could do a lot ourselves*". After the second day, a Czech participant said: "*I did not have enough patience: It is tiring to give mathematical arguments all the time. I get a kick out of a pile of clear problems*."

These remarks summarize observations in other programs where mathematical problems are considered as a kind of mental exercise. Among students who prefer an approach with problem questions of competition type, the learning of mathematics has lower priority than the opportunity to show one's own mathematical abilities.

The second day was rather unfamiliar than too demanding. (Only 15 out of 50 considered it as too difficult.) The most irritating fact for the participants was that the tutors could not give definite answers to the – obviously relevant – questions. A student from Germany said: "*If you cannot give a definite answer, why do we work on this?*" Problem questions of competition type seem to be marked by the fact that the learners know that the instructor is in possession of a solution.

It is widely believed that interested and/or gifted students are more demanding and more active than students showing less interest or with average abilities. The results of the experiment underpin the oberservation that their request can be complied with problem questions of competition type. They give them the opportunity to be active as long as they wish and to any degree of difficulty which seems appropriate. Working in a group as in the camp, one can observe that the students co-operate to work off problems given by the tutor. This keeps the level of frustration acceptable because everybody can be sure that sooner or later a bright idea will come up or, in worst case, that the instructor will help.

A group of students with outstanding mathematical abilities was chosen because they should be expected to cope best with new challenges in modelling, simulation and in discussions involving mathematical arguments. To put it much more emotional: Mankind needs them desperately in these areas. But in view of the ratings of the tutors' competence, it seems not to be easy to convince them to put work also on that area of mathematics.

It is interesting to have a closer look at those who appreciated the approach on the second day. Among the 17 students who were highly motivated on the second day, 12 devote their leisure time to reading in mathematics, 8 in natural sciences[ii]. In this group, for these students problem solving of competition type was rated a bit less important than for the average participant. There are found more students among them who intend to study mathematics after school. Among the critics of the second day, one finds almost the whole big group of those who want to study Computer Science at the University (16 out of 19). As a remark aside the author wants to mention that in his project *SamstagsUni* ("Saturday University"), where every high school student is welcome to learn more about an announced topic in the area of mathematics, statistics, physics or engineering in a series of lectures, seminars, exercise classes and talks a significantly lower proportion of participants claims to devote leisure time on mathematical competitions, but a proportion similar to those who favoured the second day to further studies. It would be interesting if the tendency can be confirmed that girls liked the second day more than their male peers.

Apart from these figures, which cannot be claimed to be representative due to the rather small samples, discussions and interviews after the second day indicate that those who were thrilled to show their abilities on the first day were rather unwilling to present themselves on the second day. There are also observations in long term enrichment models that those favouring a competition spirit in mathematics tend to agree with the external reference model.

Since recent results (Plucker & Stocking (2001)) suggest that in the academic self-concept development of academically able students no significant differences among students with strengths in mathematics, verbal areas or both areas can be established, it should be examined further which measures in mathematical enrichment models are suitable for which group of students.

It should be stressed again that problem solving – including competition type questions – has its merits and is not rejected here as an ingredient of enrichment programs. As a possible outlook originating from the results of this survey, it appears desirable to develop areas of problem solving further to learning environments which include the whole variety of mathematical areas like modelling and the construction of mathematical theories and/or which connect different mathematical contents.

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[i] The numbers of the Känguru competition in Germany in 2002 seems representative for the situation: In grades 5/6, there are approx. 57.000, in grades 7/8 approx. 39.800, in grades 9/10 approx. 23.700 and in grades 11/12/13 approx. 9100. It seems difficult to convince older high school students to join in mathematical activities.

[ii] Among all 50 participants of the camp, 18 devote their leisure time to reading in mathematics and 14 in natural sciences.

DEVELOPMENT OF UNDERSTANDING AND SELF-CONFIDENCE IN MATHEMATICS; GRADES 5–8

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This paper presents some preliminary results of the longitudinal aspect of a research project on self-confidence and understanding in mathematics. We have collected a survey data of 3057 fifth-graders and seventh-graders and a follow-up data of ten classes (191 pupils) one and a half years later. The longitudinal data indicates that the learning of mathematics is influenced by a pupil's mathematics-related beliefs, especially self-confidence. Pupils' level of understanding fractions also influences their developing understanding of infinity. These relationships between different variables depend also on pupils' gender and age.

INTRODUCTION

Pupils' conceptions on themselves as learners are strongly connected with what kind of general attitudes they have toward the discipline in question. Mathematics is a highly valued discipline in school, and therefore, pupils experience success in mathematics important. It has been observed, that pupils' beliefs on mathematics and on themselves as mathematics learners have a central role in their learning and success in mathematics (e.g. Schoenfeld 1992). The importance of beliefs in mathematics education is in concordance with the constructivist understanding of teaching and learning. We understand beliefs as "*an individual's understandings and feelings that shape the ways that the individual conceptualizes and engages in mathematical behavior"* (Schoenfeld 1992, 358). Mathematical beliefs can be divided into four main components: beliefs on mathematics, beliefs on oneself as a mathematics learner/applier, beliefs on teaching mathematics, and beliefs on learning mathematics (e.g. Lester *et al.* 1989).

Mathematics can be described as a combination of calculation skill and competence in mathematical reasoning, but neither of these alone characterizes mathematics. There is much research evidence that many pupils learn mathematics as a symbol manipulation without meaning (e.g. Resnick & Nelson – Le Gall 1987). Mathematical understanding can be distinguished from the neighbourhood concepts 'skill' and 'knowledge', for example as follows: Mathematical knowledge answers the question 'What', and one may remember mathematical facts. Mathematical skill answers the question 'How'; which includes, for example, the traditional calculation skill (procedural knowledge). Only mathematical understanding answers the 'Why' - question; it allows one to reason about mathematical statements. These are intertwined concepts, since understanding contains always knowledge and skill. Another view perceives mathematical understanding as a process that is fixed to a certain person, to a certain mathematical topic and to a special environment (Hiebert & Carpenter 1992).

Several studies have shown that beliefs about oneself have a remarkable connection with success in mathematics (e.g. Hannula & Malmivuori 1996, House 2000). However, to establish a causal relationship between self-concept and achievement is more problematic. In a literature review, Linnanmäki (2002) found out that in some studies no evidence for causality could be found, in other studies evidence was found for the causality from self-concept to achievement, while yet others found evidence for an opposite direction. The seemingly contradictory results indicate a developmental trend, where causality is mainly from achievement to self-concept during the first school years, it changes into a reciprocal linkage for the latter part of the comprehensive school, and in the upper secondary school level the causal direction is from self-concept to achievement (Chapman, Tunmer & Prochnow 2000). In her own study on self-concept and achievement in mathematics, Linnanmäki (2002) found evidence for this developmental trend in mathematics for grade 2 to grade 8 pupils. Looking at a more broadly defined concept, attitude, Ma and Kishor (1997) synthesised 113 survey studies of the relationship between attitude towards mathematics and achievement in mathematics. The causal direction of the relationship was from attitude to the achievement. Although the correlations were weak in the overall sample, they were stronger throughout grades 7 to 12, and in studies that had done separate analysis of male and female subjects.

Gender differences favouring males in confidence in mathematics are well recorded. Differences among teenagers have been reported, for example, by Bohlin (1994), Hannula and Malmivuori (1997), Pehkonen (1997), and Leder (1995). Vanayan *et al.* (1997) reported that already in grade 3 boys estimated themselves to be better in mathematics than girls. In mathematics achievement the results on gender differences are less clear. In IEA's large international studies the gender differences previously favoured boys (Husén 1967). In more recent studies the gender differences have decreased and in many countries disappeared completely (Beaton *et al*. 1997). However, robust gender differences are still found, for example, in some tasks on infinity (Hannula *et al*. 2002) and fractions (Hannula 2003).

The focus of this paper is to reveal the development on pupils' understanding and self-confidence from grade five to grade eight. Furthermore, the most important predictors of results are looked for.

METHOD

The study forms a part of a research project "Understanding and Self-Confidence in Mathematics" financed by the Academy of Finland (project #51019). The project contains a large survey with a statistical sample from the Finnish pupil population of grades 5 and 7 with 150 school classes and 3057 pupils. The survey was implemented fall 2001, and the information gathered was deepened with interviews and observation in 10 classes at convenient locations. In spring 2003 the questionnaire was administered a second time in these 10 classes. Altogether 101 pupils in the younger sample and 90 pupils in the older sample have answered our questionnaire twice. Because the number of classes in the longitudinal sample was small we need to control for possible deviations before making generalizations. The questionnaire was planned especially for the project. It contained, in addition to background variables, 19 mathematics tasks, estimations on success expectation and success confidence as well as a belief scale (25 items); see for more details Nurmi *et al.* (2003).

For the analyses of the longitudinal data we recoded the summary variables into three categories (lowest quartile, middle values, highest quartile) and used general linear model multivariate analyses (GLM Multivariate). It provides regression analysis and analysis of variance for multiple dependent variables by several covariates. The first measures of variables were considered as the covariates and the second measures of the same variables as the dependant variables. For each dependent variable, the overall η^2 (eta-squared) statistic is reported as a measure of the proportion of total variability attributable to the covariates. Statistically significant ($p < .01$) η^2 are also reported separately for each pair of covariate-dependent variable as a measure of the proportion of total variability attributable to the specific covariate.

ON RESULTS

In this paper we will report the results of the longitudinal data. There are many published papers on some specific features of the initial results for the research project: on infinity (Hannula *et al*. 2002), on confidence (Nurmi *et al*. 2003), and on number concept (Hannula 2002, 2003).

Here we shall use three sum variables for success in mathematics test (fractions, infinity, other tasks), and three sum variables for beliefs (self-confidence, success orientation, defence orientation). The mathematics variables are based on the analyses made in Hannula (2002). The two first ones (fractions, infinity) represent our indicator for understanding, and the third variable (other tasks) consists mainly of more computational tasks. Belief variables (self-confidence, success orientation, defence orientation) were constructed with the help of factor analyses from the belief scale (cf. Nurmi *et al.* 2003). The self-confidence factor consists of ten statements that were adopted from the self-confidence subscale of Fennema-Sherman Matehmatics Attitudes scales (Fennema & Sherman 1976). For background variables we shall control the effects of gender and grade.

The longitudinal sample of the fifth-graders did not differ much from the overall sample, if we look at the averages of all sum variables (there is only a small effect[1] favouring focus classes in fractions). However, the seventh grade longitudinal sample had better skills in mathematics (small to medium effects) and a slightly lower defence orientation (small effect) than the full sample. The largest effect was found in infinity, where the large sample of 7th graders had a mean score 4.7 (SD 2.9) while the longitudinal sample had a mean score 5.9. In our interpretations of the results we need to be aware of these deviations from the larger sample.

[1] Here we use the *d*-value = $|mean_1 - mean_2| / SD$ as a measure for difference and a convention established by Jacob Cohen (Cohen, 1988) that sets norms for "small," (*d* = 20) "medium," (*d* = 50) or "large" effects (*d* = 80).

Development observed

We noticed a rapid development in all achievement variables from grade 5 to grade 6. The development continued to be rapid in the domain of infinity after grade 7, but it slowed down in other variables, probably due to a ceiling effect[2] (Figure 1). In belief variables, we saw decline in self-confidence and success orientation together with an increase in defence orientation from grade 5 to grade 6 and from grade 7 to grade 8. However, grade 7 measures differed from grade 6 measures to another direction. Here we need to be aware of the differences between the younger and the older sample (Figure 2).

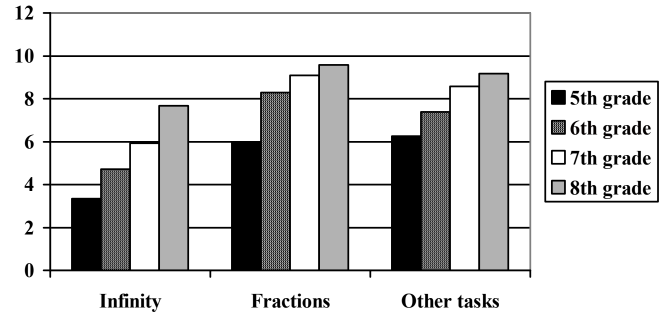

Figure 1. Development in mathematics achievement (infinity, fractions, other tasks).

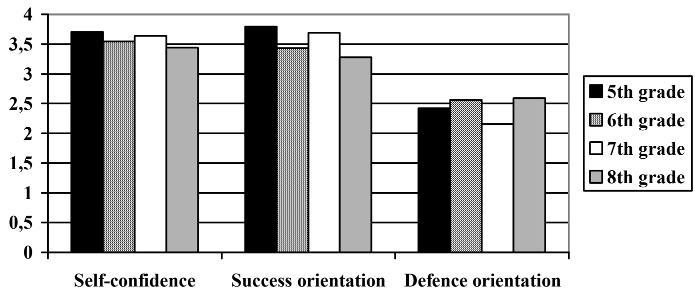

Figure 2. Development in belief variables (self-confidence, success orientation, defence orientation).

Close to half of variation in mathematics achievement was predicted by achievement in the previous test (Figure 3). Only a small part of this variation was attributable to pupil's achievement in the same topic in the previous test. Fractions was an important

[2] The theoretical maximums for variables are 14 (infinity), 13 (fractions), and 12 (other tasks).

predictor for success in infinity and other tasks. For the belief variables we see that self-confidence and success orientation were fairly well predicted by earlier beliefs (Figure 4). Self-confidence was a more important predictor of these two variables. Defence orientation seemed to be relatively unstable variable, and only 8 % of the variation in the later test could be explained.

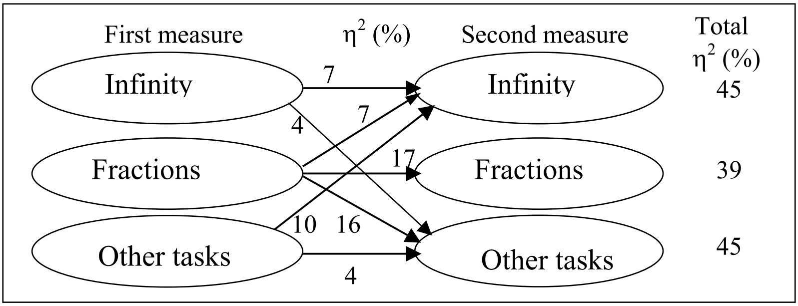

Figure 3. GLM Multivariate analyses of mathematics achievement

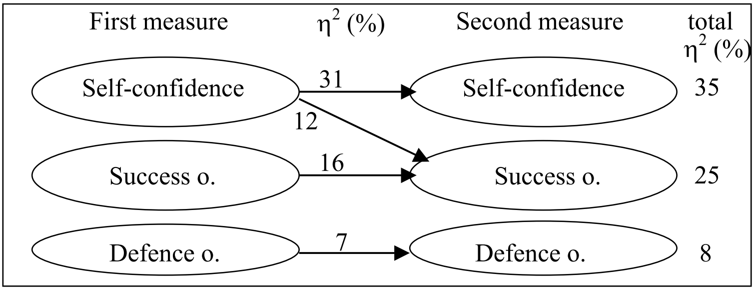

Figure 4. GLM Multivariate analyses of belief variables.

When beliefs and achievement were combined in one model, the explained variance for fractions increased from 39 % to 46 %. Minor increase (1 – 4 %-units) was observed for all other variables except success orientation. A statistically significant new effect was found from self-confidence to fractions (η^2 = 6 %). When analyses were made separately for boys and girls, and for the two age samples, we found some variations in the model:

- Infinity was more strongly predicted by fractions in older samples (η^2 = 13 % (girls), η^2 = 15 % (boys)) and infinity predicted fractions among older girls (η^2 = 16 %).
- Other tasks became a less stable variable in older sample.
- Gender and age had an influence on the stability of beliefs. Self-confidence in the younger sample was more stable among boys (η^2 = 11 % (girls), η^2 = 40 % (boys)), while in the older sample among girls ((η^2 = 37 % (girls), η^2 = 25 % (boys)). Success

orientation was more stable in older samples (η^2 = 35 % (girls), η^2 = 24 % (boys)) and defence orientation was more stable in older boys' sample (η^2 = 21 %).

- Self-confidence was strongly predicted by infinity in older girls' sample (η^2 = 36 %).

We made a more detailed analysis on the connection between fractions and infinity because the connection seemed to be an important one, not easy to explain theoretically, and because the underlying factor constructs were not very strong (Hannula, 2002). We made a GLM multivariate analysis on task level, and found that one fraction task (Figure 5) was by far the most important predictor, and that the effect was on two of the three infinity tasks: "How many numbers are there between numbers 0.8 and 1.1?" and "Which is the largest of numbers still smaller than one? How much does it differ from one?" Looking at the correlations between the two measures of these three variables we found the strongest correlations between the first measure of the fraction task and the later measures of the infinity tasks.

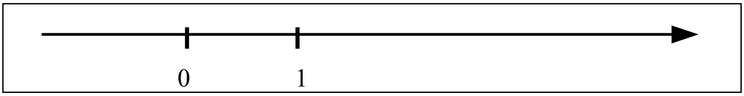

Figure 5. The task 2c: mark 3/4 on the number line.

CONCLUSION

There is significant development during grades 5 to 8 in topics measured in the mathematics test. Most notably the development is rapid both in the topics that are covered in the curriculum (fractions), but also in topics that are not dealt directly with in the curriculum (density of rational numbers). At the same time, there is – somewhat paradoxically – a negative development in beliefs. In both samples self-confidence and success orientation became lower in the second measurement, and defence orientation increased. Confusingly, the beliefs of the older sample at the beginning of the seventh grade were more positive than the results from the younger sample at the sixth grade. Partially the difference can be explained by the differences in the samples (the older sample deviating from the average towards more positive in both achievement and beliefs). This somewhat odd difference in the 6th and 7th graders' beliefs might also be partially due to the effects of time of the measurement, beliefs possibly declining by the end of spring term and increasing again by the beginning of a new term ('a fresh start').

Stability of the measured belief variables seems to be related to pupils' gender and age. Defence orientation is the least stable of the constructed belief variables, and we might even question the validity and usefulness of the variable. However, there seems to be a developmental trend for this orientation to become more stable among older boys. Possibly defensive approach to mathematics is something that develops slowly during school years and more typically for boys.

Mathematics achievement in this specific test can be predicted to a large extent from the pupils' past achievement in the same test. Most notably the pupils' success in fractions becomes an increasingly important predictor for future achievement in the used tasks on number concept, as the pupils grow older. As a specific task, pupils' ability to perceive a fraction as a number on a number line predicts their future understanding of density of number line. This finding highlights the importance of number line as a conceptual tool.

Self-confidence is another variable that seems to be an important predictor for future development. A pupil's self-confidence predicts largely the development of self-confidence in the future, but also the development of success orientation and achievement. A strong connection between self-confidence (and other beliefs on oneself) and mathematical achievement has been found also in earlier research (i.a. Hannula & Malmivuori 1997; Tartre & Fennema 1995).

Regarding the relationship between beliefs and achievement, our analyses suggest that the main causal direction already from grade 5 onwards is from self-concept to achievement. In the older sample we also found achievement in infinity to be a strong predictor for the development of the girls' self-confidence, which supports the hypotheses of a reciprocal linkage. Like the results of Ma and Kishor (1997), also our findings indicate that gender is an important variable in any analyses of causal relationship between affect and achievement.

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MATHEMATICS TEACHERS' KNOWLEDGE BASE: PRELIMINARY RESULTS

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Kien H. Lim
UCSD & San Diego State Univ.

Student learning depends on the teacher's actions, which are, in turn, dependent on the teacher's knowledge base—defined here by three components: knowledge of mathematics content, knowledge of student epistemology, and knowledge of pedagogy. The purpose of this study is to construct models for teachers' knowledge base and for their development in an on-site professional development project.

THEORETICAL FRAMEWORK

Building on Shulman's (1986, 1987) work and consistent with current views (e.g., Cohen & Ball, 1999, 2000), Harel (1993) suggested that three interrelated critical components define teachers' *knowledge base*: (a) knowledge of mathematics content, (b) knowledge of student epistemology, and (c) knowledge of pedagogy:

Knowledge of mathematics content refers to the breadth and, more importantly, the depth of the mathematics knowledge possessed by the teacher, particularly, their *ways of understanding* and *ways of thinking*—terms to be defined in the sequel. The content knowledge is the cornerstone of teaching for it affects both what the teachers teach and how they teach it.

Knowledge of student epistemology refers to teachers' understanding of fundamental psychological principles of learning. This includes knowledge on the construction of new concepts.

Knowledge of pedagogy refers to teachers' understanding of how to teach in accordance with these principles. This includes an understanding of how to assess both students' existing and potential knowledge, how to utilize assessment to pose problems that stimulate students' intellectual curiosity, how to promote desirable ways of understanding and ways of thinking, and how to help students solidify the knowledge they have constructed.

Ways of Understanding and Ways of Thinking

Harel (1998) distinguished between these two categories of knowledge—*ways of understanding* and *ways of thinking*—upon which have been elaborated by Harel and Sowder (in press): Generally speaking, a way of understanding (*WoU*) refers to either a student's (a) meaning/interpretation of a term or sentence, (b) solution to a problem, or (c) justification to validate or refute a proposition. A way of thinking (*WoT*) refers to "what governs one's ways of understanding, and thus expresses reasoning that is not specific to one particular situation but to a multitude of situations." Harel and Sowder (in press) classified WoT into three categories:

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problem-solving approaches, proof schemes, and beliefs about mathematics. The three categories are not mutually exclusive.

Problem-solving approaches: Examples of problem-solving approaches include "look for a simpler problem," "examine specific cases," and "draw a diagram." Unfortunately, some teachers, in attempts to improve problem-solving performance with students, advocate problem-solving approaches that can render sense-making in mathematics unnecessary. "Look for a key word in the problem statement" and "look for relevant relationships among quantities based on their units" are examples of such approaches.

Proof Schemes: Proving is defined in Harel and Sowder (1998) as the process employed by a person to remove or create doubts about the truth of an observation. A distinction is made between two processes of proving: *ascertaining* and *persuading*. "Ascertaining is a process an individual employs to remove her or his own doubts about the truth of an observation. Persuading is a process an individual employs to remove others' doubts about the truth of an observation" (Harel & Sowder, 1998, p. 241) Thus, a person's *proof-scheme* consists of what constitutes ascertaining and persuading for that person. Harel and Sowder provided a taxonomy of proof scheme, which was later refined in Harel (in press).

Beliefs: Here beliefs refer to the teacher's views about the nature of mathematics, of knowing mathematics, and of learning mathematics. Examples of beliefs are "mathematics is a web of interrelated concepts and procedures" and "understanding mathematical concepts is more powerful and more generative than remembering mathematical procedures" (Ambrose et. al., 2003, p. 33). These are obviously desirable beliefs. Examples of undesirable beliefs are "formal mathematics has little or nothing to do with real thinking or problem solving" and "only geniuses are capable of discovering or creating mathematics" (Schoenfeld, 1985, p. 43). On the one hand, one's beliefs influence the way one interprets a situation, understands a mathematical statement, and approaches a problem. On the other hand, one's beliefs evolve as one learns and does mathematics.

METHOD

A two-year, on-site professional development research project is underway to study the evolution of teachers' knowledge base. The site is a public middle/high school that offers an intensive college preparatory education for low-income student populations. The school adopts block schedule in which each class meets five times (four 100-minutes and one 75-minute lessons) in a two-week period. Three teachers have participated in this project.

One class of each teacher was observed by us once or twice each week. We then met for 30-45 minutes with the teacher a few days after each observation to discuss the teacher's goals for the lesson and help the teacher reflect on the activities observed during the lesson. So far, we have conducted a total of 14 such observation-conversation pairs per teacher. The teachers understand that the observations are not to evaluate their teaching ability but a source for them and for us to learn about the learning and teaching of mathematics. All three teachers were eager to participate in this project and have enthusiastically shared their ideas with us.

The classroom lessons (except the first one or two lessons) are videotaped and the conversations are audio-taped. During the teacher-researcher conversation, a teacher shares his (or her) objectives and rationales for his teaching actions, his thoughts about students' WoU and WoT, and her or his plans for subsequent lessons. We pose mathematical and/or didactical situations to test our hypotheses about specific aspects of the teachers' knowledge base.

At present, we have analyzed the first three observation-conversation pairs for one Algebra II teacher whom we call Bud (a pseudo-name). The analysis includes dividing each observation/conversation into segments (roughly speaking a segment is a self-contained episode of an observed classroom activity or of a dialogue in a conversation) and analyzing each segment with the previous observation-conversation pairs in mind.

RESULTS AND DISCUSSION: FOCUS ON THE CONCEPT OF FUNCTION

Conversations

The following is an excerpt from a dialogue between the first author (H) and Bud (B). It reveals certain aspects of Bud's way of understanding the concept of function:

> H: Do you think that they [the students] know what function is?
>
> B: Not really. [My] last year class is an indication that they don't really understand what a function is?
>
> H: … What is for you … understanding function? … What kind of [student] understanding of functions would [make you] happy?
>
> B: Well … if they can … given a variety … Given information in a variety of ways, whether it is a table or a graph, or equation, if they can tell me whether it's a function and why, and if they can give me some [examples of those] that's not a function or explain why something is not ... a function, and explain mathematically why it can't be a function. Then I, then I will be satisfied that they've understood ...

For Bud, understanding functions seems to mean being able to determine whether a graph, a table, or an expression is a function and provide examples and non-examples of function. He further indicated that his "students knew the definition of a function, but they couldn't take it and see it in a graph. … They have problems putting the definition to use." He attributed their difficulties to their lack of understanding the concept of ordered pair of numbers and graph.

When asked to consider the need—from the student's viewpoint—of determining whether a given situation is a function, he said:

> …, much of what they are going to do in math relates to the family of functions. …, and then we talk about non-functions, we talk about, I mean, this is the way I've

always learned it. Here is a function, there is a non-function. The way I learned it, so I am teaching the way I've learned it.

Bud's response suggests that his teaching was content driven rather than student driven. The question of why students would be intrinsically interested in the concept of function is not part of his epistemological or pedagogical consideration. Based on our observations so far, Bud seems to view school mathematics as a fixed set of concepts and procedures that are to be delivered to and remembered by students. These concepts and procedures can be organized systematically into topics and subtopics and can be imparted to students.

In an attempt to advance Bud's knowledge base on the concept of function, we offered him problems whose context can potentially stimulate reasoning in terms of functions. The following problem is an example:

> (I) A pharmacist is to prepare 15 milliliters of special eye drops for a glaucoma patient. The eye-drop solution must have a 2% ingredient, but the pharmacist only has 10% solution and 1% solution in stock. Can the pharmacist use the solutions she has in stock to fill the prescription?
>
> (II) The same pharmacist receives a large number of prescriptions of special eye drops for glaucoma patients. The prescriptions vary in volume but each requires a 2% active ingredient. Help the pharmacist find a convenient way to determine the exact amounts of the 10% solution and 1% solution needed for a given volume of eye drops.

Part II, for example, is likely to help students interpret situations in terms of function. Bud viewed such problem as an application problem appropriate merely for enrichment activities, not to be part of the main curriculum that he is committed to teach. He did use this problem but offered it as the Problem of the Week. He commented that his students had difficulties with the chemistry aspect of the above problem: "I think it wasn't so much the math part that came with the problem, I think more of the problem came from...throwing in chemistry terms into the mix, … the solution and … the terminology." For Bud, the solution of this problem consists of two parts: the process of interpreting the problem statement is not considered mathematics but chemistry. The mathematical part begins when one write the algebraic equations and solves their unknowns.

H: So you have a plan … of how to connect this [the pharmacist problem] to the concept of function.

B: Right, well, umm, I guess I started thinking about more in terms of linear functions instead of functions in general.

H: Oh, linear functions.

B: I don't know if that matters, umm, just cause I originally, as soon as I saw it I just thought two linear equations, that umm, cause I can relate it to, that way I can relate it to slope, I can relate it to y-intercept, I can relate it to solving systems …

Bud viewed this problem as one that can be used to practice linear functions, rather than a situation where one can think in terms of the process conception of function: for any input T (the volume of the prescribed eye-drop solution with 2% active ingredient), one gets the output x (the volume of the 1% solution) from the equation $0.01x + 0.10(T - x) = 0.02T$. Even if Bud did possess the way of thinking of interpreting situations in terms of function, it was not spontaneous for him. As a consequence, he did not attempt to set it as a cognitive objective.

Observations

In his first lesson on the concept of functions, Bud introduced the notions of dependent and independent variables, the definition of function, and the domain and range of a function. He mainly emphasized concept definitions and literal meaning of terms. For example, after discussing the literal meaning of *depend on something*, Bud attempted to relate it to the mathematical meaning of *dependent*. "I depend on the internet [otherwise] I couldn't talk to friends. OK. Just like you guys depend on things, equation has two parts and one part depends on another."

B: Does anybody know what the two parts to an equation are?

S1: The number that makes …

B: Well … No, because we are just talking about the equation. An equation just doesn't have one answer.

S2: Isn't the parts [of both sides of the] equal sign having to be equal to each other?

B: Ummm…

S2: Yes. Say, yes.

B: Kind of yeah but not really what I'm going for.

S3: The independent variable, and the dependent variable (not completely audible)

B: The independent variable and the dependent variable. The independent part and the dependent part.

Bud was particularly focused on his own way of understanding the concept of function that he ignored those of his students. For example, S1 attempted to answer Bud's question "Does anybody know what the two parts to an equation are?" by saying how he understood the meaning of an equation. Rather than trying to build on S1's WoU, Bud chose to reject S1's answer. His style of exchange with students is generally not of a free discussion but of an attempt to deliver his own knowledge.

The following excerpt shows that some students had difficulty with the "uniqueness to the right" property (i.e., for an input value there could be only one output value).

B: … More or less a parabola, a little skewed but that's OK. Is it going to be a function? [S1], why? You are shaking your head.

S1: (inaudible)

B: OK. For each x I have, like say OK, this x right here, [if I'm looking at] this x, how many values of y does it have to match up with?

S2: One.

B: It has one right there. Is there any place on this graph that has more than one y value for the x?

S3: (said something about 2 x values for 1 y value.)

B: Different thing. It's a good question. We will get to that eventually. His question was, what if [we] look at it backwards, I think. What if, you are looking at the y, and say, because this value of the y, you will notice that it has how many x values?

S4: Two.

B: Two. For, being a function, that doesn't matter? Excellent question, we are going to deal with that later. It has to do with inverse functions and things like that. But for now, for functions, all we are looking at, for each x, there is only one y.

Bud did not seem to empathize with students' struggle in understanding why a function must have one y-value for each x-value and not the other way around. Instead of addressing this difficulty, he resorted to a different issue—that of the inverse function—a concept the students had not been exposed to at the time. Student's difficulty with the uniqueness-to-the-right property surfaced again when he discussed whether a line is a function; his students were unable to understand why a horizontal line is a function but a vertical line is not. Bud believed that the concept definition (in the sense of Tall & Vinner, 1981) alone is sufficient for students to overcome their difficulties with the concept of function.

In his lesson on linear function, he discussed the characteristics of linear function, the names and WoU for m and b in $y = mx + b$, the procedure for graphing $y = mx + b$ without plotting points, and the procedure for finding the equation of the line passing through two points whose coordinates are given.

B: What do you think something that is linear is going to look like?

S1: Straight.

S2: Line.

B: Line. So if it is a linear function it could be a straight up and down line then? … It could be a vertical line?

S's: No.

B: Why, why can't I have a vertical line if I want a linear function?

S3: [A vertical line] isn't a function.

B: Right. Vertical lines remember aren't function. When I say … linear functions, I am not talking about vertical lines.

The above excerpt suggests a view that mathematical facts are to be remembered. In the following excerpt, Bud's question "does anybody *remember* how you could do it using slope and y-intercept" suggests that the procedure for sketching is something that one should memorize rather than reconstruct.

B: What, when you graph something, y equals, say $2x$ plus 5, what do you do first?

S1: Plug in the number for x.

S2: Well, I know … (said something about making a table).

B: Well, you could make an in-and-out table. Does anybody remember how you could do it using slope and y-intercept though?

S3: Yes

B: How so?

S3: Get, err, get numbers for x, plug in …(inaudible) …x, you get negative 3.

B: Well, that's making in-and-out table. I want to use; I want to do it without having to make a table. I want to be able to look at the equation and instantly be able to plot points, without having to plug in anything.

Bud chose not to pursue students' suggestions because his goal was to teach the intercept-slope procedure for sketching, a procedure which he considered more efficient. As such, he missed the opportunity to build on students' current knowledge to develop a critical way of thinking, that of appreciation for mathematical efficiency.

CONCLUSION

This preliminary analysis focuses mainly on observations-conversations concerning the concept of function. Bud's ways of understanding and ways of thinking of this concept and the way he taught it give hints as to his knowledge base, which seems to include the following beliefs: (a) mathematics is a fixed set of interrelated concepts and procedures, (b) modeling is not part of algebra, algebra is essentially manipulation of symbols, (c) learning mathematics means essentially remembering what the teacher teaches, (d) content structure, not student need, drive mathematics curricula.

A conceptual framework for teacher's knowledge base would enable us to describe teacher's teaching personalities and the rationale for their teaching actions. We hope that a complete model for Bud's knowledge base would help us explain his preference for teacher-led discussions over lectures, his tendency to disregard students' current ways of understanding, and his choice and sequencing of problems for classroom discussions.

The components of a teacher's knowledge base are inseparable from each other. One's ways of understanding and ways of thinking of mathematical concepts seem to dictate the nature of the other components of knowledge. For example, Bud's way of understanding functions impacted the kind of emphasis he placed on the pharmacist problem. He focused on the procedural aspect of solving the problem rather than the conceptual aspect of modeling the problem situation in terms of functions.

This has implications to curricula for pre-service mathematics teachers and for professional development programs for in-service mathematics teachers. Focusing on one component of teacher knowledge base in isolation from the other two is unlikely to be effective. It is unrealistic, for example, to expect prospective teachers to change their beliefs and conceptions about mathematics they have formed over the years in one or two courses. Integrated curricula, where the three components of knowledge base are addressed in a synergetic manner, can help teachers grapple with the mathematics and at the same time reflect on their own learning, which, in turn, can help them appreciate epistemological and pedagogical issues.

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THE ROLE OF TOOL AND TEACHER MEDIATIONS IN THE CONSTRUCTION OF MEANINGS FOR REFLECTION

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ABSTRACT

This article reports on a study aiming to design learning systems in which students' knowledge of reflection is brought closer to institutional knowledge of this isometry and to compare how their activities shape and are shaped by different forms of mediation. It presents descriptions of interactions of groups of students (aged 12-13 years) with two computational microworlds, based on either dynamic geometry or multiple-turtle geometry, during attempts to construct and use a tool for reflections and considers how the tools of the microworlds along with the instructional approach adopted by the researcher were important in mediating the passage between meanings emphasizing reflection as property and those emphasizing reflection as function.

INTRODUCTION

Recent considerations of research into the role of technology in mathematics education have stressed the importance of considering the complex inter-relations between all the elements of technology-integrated learning systems, including: the mathematical domain and its epistemological structure; the resources brought by the learners to the system; the affordances and constraints of the technology itself; and the pedagogical structuring of the learning systems by the teacher in the institutionalized setting (see, for example, Herschowitz & Kieran, 2001; Lagrange, Artigue, Laborde & Trouche, 2001). Generally speaking, research into the pedagogical structuring of technology integrated learning systems have been less extensively documented in the research literature than the learning potentials and pitfalls that characterize learners' interactions with technological resources. This paper reports upon a study which analyzed both these aspects of mediation.

The aim of the study was to design learning systems in which learners' knowledge about reflection becomes connected to the institutional knowledge about geometrical transformations that they are intended to learn in school mathematics. Previous research related to geometrical transformations suggests that the majority of school aged students have some knowledge of the properties associated with the isometry reflection, but do not tend to characterize it as function (for a review of this research see Healy, 2002). In short, research suggests that students' analyses during work on reflections are characterized by what Piaget and Garcia (1989) describe as an *intrafigural* perspective generally involving associations with properties of symmetrical designs and that *interfigural* perspectives which might favor emphasis on the functional aspects of the transformation are less evident.

THE STUDY

To address the dual concern of investigating the processes by which mathematical knowledge is mediated while building learning systems which would support students in building from views of reflection based on intrafigural relationships to views of reflection as a function, the study was divided into two phases, the design phase and the comparison phase.

During the first phase, four learning systems were iteratively designed, through a series of successive steps during which tools, tasks and teaching interventions were developed as students' activities with them were observed and analyzed. To focus on tool mediation, two computational microworlds were designed: dynamic-Euclidean Geometry (DEG) and multiple-turtle geometry (MTG). The microworlds presented learners with different models of geometry along with different means for interacting with them: DEG interactions involved direct manipulation of a model of the theoretical field of Euclidean geometry; MTG interaction involved the programming of multiple turtles, whose movements around a two-dimensional surface were controlled by symbolic code. A set of five tasks was developed for use in the learning systems with the condition that the mathematical demands remained consistent regardless of which microworld was in use.

To examine teacher mediations, two instructional approaches were developed on the basis on a major difference between constructivist-rooted approaches on mathematical teaching (didactical engineering and the emergent approach of Cobb *et al.,* 1997, for example) and those guided by sociocultural ideologies (and particularly the work of Davydov as described in Renshaw, 1996). This difference related to the primacy assigned to the individual or the cultural in the learning process. Constructivist approaches emphasize a filling-outwards (FO) flow in which personal understandings are moved gradually towards institutionalized knowledge. A reverse filling-inwards (FI) flow of instruction described in sociocultural accounts stresses moving from institutionalized knowledge to connect with learners' understandings. Teaching interventions in this study were hence designed to allow investigation of these two different instructional approaches: the FO approach aimed to develop general mathematical models from learners' activities; and the FI approach intended to support learners in appropriating general mathematical models previously introduced.

In the comparison phase, an in-depth analysis of the evolution of the four systems was conducted, as a group of six 12-13 year old girls interacted together with the researcher within each system. The students were selected on the basis of their responses to a paper-and-pencil questionnaire, so that response profiles of students were similar across groups and each group consisted of students across the achievement range representative of the inner-city school in which they studied. Data, in the form of audio transcriptions, researcher notes, computer constructions and written work produced by any of the participants, were collected as students worked in pairs during five ninety-minute session of microworld activities and one forty-five minute teaching episode.

During data analysis, the various data sources were synthesized into group profiles, telling the story of the development of each system. To aid in the comparison between systems, data were the organized along the following dimensions: main strategies used in systems incorporating the same microworld; between-pair variations around the main strategies; variation associated with the use of FI or FO approaches; and microworld evolutions. Finally, these dimensions were further analyzed to identify the ways in which the tools, tasks and teaching interventions appeared to constrain and afford the abstraction and concretion of mathematical meanings for reflection in terms of its properties and functional aspects, movements between intrafigural and interfigural perspectives and students' considerations of figures and planes.

RESULTS

In general, the results suggested that students working in all four systems developed new meanings by coordinating intra and interfigural analyses as they built computational models of reflection. The microworld tools had a central role in mediating all aspects of the students' activities: with DEG tools, reflection tended to be represented as a correspondence relationship, usually based on perpendicular distances; MTG tools, in contrast, afforded the expression of reflection as a mapping of one set of turtles onto another and emphasized equal turns and distances. Mathematical meaning-making in all four systems involved the forging of connections between general models of reflection and physical movements on screen, with the support built into the microworlds helped students to see and investigate the generality behind the geometrical figures they were producing. However, results also indicate that the same support system allowed them to find ways of expressing this generality not through formalisation as intended, but through action. The impact of instructional approach on students' meaning-making activities was less marked and also mediated by microworld tools.

To illustrate the overall finding and concentrate on the particular effects associated with the instructional approaches, the remainder of the paper will consider the relationships between students' interactions on two of the five microworld tasks and the mathematical issues discussed in the teaching episodes.

One of these tasks – the third in the sequence – involved students in attempting to build a tool for constructing images of points (in the DEG systems) or turtles (in the MTG systems) under reflection. The second (representing the final of the five tasks involved students in operating with a set of elements to be reflected (see Figure 1).

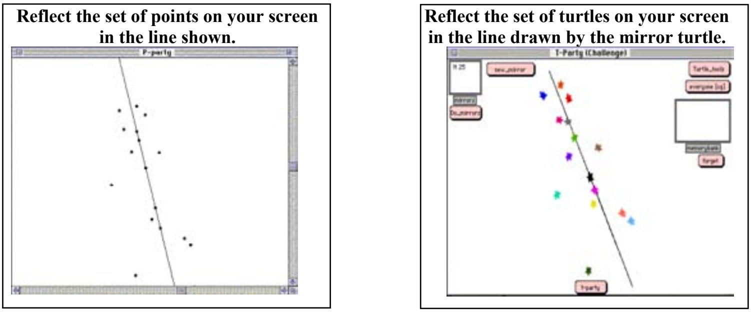

Figure 1: The DEG and MTG versions of the final microworld task

Characterizing the differences in the instructional approaches

The FI and FO instructional approaches differed both in terms of global structuring of the teaching episode and local structuring which related to teaching interventions made during microworld activity.

The teaching episode, which had the aim of emphasizing knowledge of reflection in terms of institutional mathematical practices, was an important aspect of all learning systems. The challenge in designing the teaching episodes was to discuss functional views on reflection in ways that would be meaningful to students and that stressed its connection with what they already knew about reflection. Three foci for discussion were planned: co-ordination of interfigural and intrafigural properties; function as relationship (static view) and as transformation (dynamic view); and meanings for planes and their elements. All discussion took place away from the computer, and was expressed using paper/transparency-and-pencil media. In the two FI systems, this teaching episode occurred before students started on the five microworld tasks and in the FO systems after the tasks had been completed.

In terms of the local structuring, regardless of instructional approach, the intention was that students would be in control of their own solution processes, making decisions and following directions of exploration that they chose for themselves. The differences between instructional approach related to the introduction of new tools at the beginning of each microworld session. In FI systems, the tools were introduced in ways that attempted to emphasis their connection to aspects of the intended knowledge and in particular stressed geometrical objects as sets of points or turtles. In contrast, in the FO systems, emphasis was on encouraging students to connect the empirical effects of a tool with their own knowledge and students were asked to come up with the own descriptions of their output.

Interactions in the FI learning systems

For the FI instructional approach, the teaching episode involved the presentation of models for reflection, in particular, planes were described using a flatland metaphor as two dimensional worlds, with no up or down, consisting of points and the reflection transformation enacted in ways that emphasized the axis of reflection as a perpendicular and angle bisector. To encourage the active interaction of students with this models, four teaching strategies types were adopted, based on those described in Renshaw (1996) *exposition* (presentation of models using mathematical voice), *leading questions* (encouraging students to use general and precise terms), *staging mistakes* (drawing attention to inconsistencies or errors) and *clashing* (provision of different valid representations of the same relations).

Despite participating in identically structured teaching episodes before microworld interaction, the interactions around the task of constructing a reflection tool varied considerable between the two FI systems. In the DEG-FI system all three pairs made use of the equal perpendicular distances to position the image point, although none of them managed to formalize the relationships involved to produce a robust construction. Instead they first constructed a line perpendicular to the axis and passing through the original point, two pairs using the appropriate DEG tool and the third constructing this line by eye and then manually adjusted the image point along the perpendicular line. When the image point was moved, the same procedure was adopted, hence students expressed – at least some elements of – generality through dragging and not through the construction tools.

In the MTG-FI system, one pair managed to formalize a variable Logo procedure for constructing image turtles. The method they used involved hatching a new turtle on top of the turtle to be reflected, sending this turtle so that it had the same location and heading as the mirror turtle (that is, to a special state on the axis of reflection) and then reproducing the remembered distances and angles on the opposite side of the mirror turtle's line. The other two pairs used a second method, which involved the use of an MTG primitive tool which constructed a new turtle at the meeting point of any two turtles (a Logo equivalent of an intersection point). The complete method entailed sending a turtle from the position of the original turtle until it met with the mirror turtle, turning this turtle to have the same heading as the mirror turtle, then repeating the turn and the distance traversed so that the image turtle was in the correct position. The two pairs using this method did not produce variable procedures but expressed its generality method by reusing the same Logo command sequence and manually altering the distances and angles following a change in the position of the original turtle.

When it came to the task of reflecting a set of elements, all the students in both FI systems attempted to transform all the element on both sides of the axis consistently using the same method for each element. However, whereas in the DEG-FI system none of the students operated upon the points located along the axis or discussed the point-set as anything other then the set of specific turtles on screen, those working with the MTG microworld were motivated to begin to connect the idea of reflection to acting upon a more abstract conceptualization of a turtle-set – evident in discussions referring to *"the world of turtles"* and to consider elements invariant under the reflection transformation. One student, for example, described her view thus:

> *"Every turtle has its own reflection turtle with the same distance away from the mirror and the same angle, except for left and rights. This one* (pointing to the mirror turtle) *has no distance away and no angle, but it still has its own reflection."*

Interactions in the FO learning systems

The global structuring of the FO systems did not include the presentation of "ready-made" models for reasoning about the intended knowledge. Instead the aim was that students would construct their own models during computer interactions which would serve as the basis from which they could *(re)invent* models for reasoning about objects of reflection during the teaching episode. The teaching strategies adopted in FO systems involved using the students' *voices* to re-express the intended knowledge from the researcher's *perspective*, along with *matching* (identifying and evaluating identical or overlapping solution approaches) and *contrasting* (identifying and evaluating different approaches to task solution). Although the same three areas for discussion identified above were planned for the two FO teaching episodes, in practice only two areas were covered as students reflected on the intra and interfigural aspects of their constructions and presented their models for reflection. No student descriptions of planes or figures as point (or turtle) sets emerged during the episode of either DEG system, hence this aspect was not discussed.

The differences in the models of reflection discussed in the two FO systems related to differences in the methods used during the task of constructing images. In the DEG-FO group a variety of different methods were constructed – one pair in particular came up with a total of four different construction methods formalized into macro tools (see Figure 2). The other two pairs, like the students in the DEG-FI system, were able to enact but not formalize properties sufficient to define a reflection tool.

a: Using two `angle carry` constructions | b: Using `angle carry` and `compass` | c:Using two `compass` constructions | d: Using `perpendicular line` and `compass`

Figure 2: The four robust image point constructions defined a FO-DEG pair

Students interacting in the MTG-FO system constructed image points using the same two methods as the MTG-FI students, all though none of the three FO pairs formalized the method in a variable procedure, again choosing instead to express generality in the action of reusing a set of commands, altering only the specific values as necessary.

On the final task, the strategies of the DEG-FO students matched those of their DEG-FI counterparts, with none of the students choosing to operate on the points located along the axis for reflection. This proved to be more of an issue for MTG-FO students who did tend to discuss the effect of the transformation on turtles along the axis. However, there was no evidence that the students were thinking beyond the specific set of turtles on screen or conceptualizing figures, let alone the screen, as representable as a turtle-set.

FINAL REMARKS

In summary, the evidence presented in this paper suggests that the efficacy of combinations of microworld interations and instructional approaches is likely also to depend on the specific learning objectives associated with the learning systems. Students in all four systems appeared to extend their knowledge of reflection, but each system had its own particular characteristics.

The DEG-FI system was the only one in which all students referred to the reflection construction traditionally emphasized in school texts, suggesting this system-type might be the most efficient in steering of students to some predetermined set of responses. The greatest variety of reflection constructions were built by a pair in the DEG-FO system, it seemed to offer to them opportunities for exploring equivalent expressions of the same geometry construction. In the two MTG systems, all students invented and explored their own models of reflection. These models were rather different than the traditional school model, especially in that the perpendicular relationship of reflection was not featured. The MTG-FI was the only system in which students connected to the notion of geometrical objects as point-sets, suggesting that connection to this particular abstraction may be facilitated by a system in which students are encouraged to connect the behaviors of geometrical agents in mathematical systems to with those of more animate agents in social systems.

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THE PROVING PROCESS IN MATHEMATICS CLASSROOM – METHOD AND RESULTS OF A VIDEO STUDY[1]–

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Mathematical proof is one of the most difficult topics for students to learn. Several empirical studies revealed different kinds of students' problems in this area. Our own research suggests that students' views on proofs and their abilities in proving are significantly influenced by their specific mathematics classrooms. However, the reasons for these differences in the students' performance remain unclear. Accordingly, we conducted a video study in order to analyse proof instruction in Germany. In this article we will present firstly a method for evaluating instruction and secondly some results that describe proving processes in mathematics classrooms at the lower secondary level from a mathematical perspective.

1. Introduction

Reasoning, proof and argumentation in the mathematics classroom is an important issue in mathematics education research. In the last years one could observe an increasing portion of empirical research on this subject. Moreover, reasoning and argumentation in mathematics was imbedded in international comparative studies like TIMSS or PISA (cf. Baumert et al., 1997; Deutsches PISA-Konsortium, 2001). Our own research adds to this and focuses on proving processes in the mathematics classroom. The aim of the video study, that we will present in this paper, is to describe the classroom conditions under which students learn mathematical proofs. In this article we present a method for evaluating proof instruction and first results of this video study. In particular, we analyse the proving process in the mathematics classroom from a mathematical perspective in a quantitative research design.

2. The role of proof in mathematics and in the mathematics classroom

Concerning the role of proof in mathematics and in the mathematics classroom we want to emphasize three important topics: proof as a social construct, the different functions of a proof, and the distinction between the process of proving and the proof as a product. Since a detailed discussion of these three topics will go beyond the scope of this article, we will restrict ourselves to the main ideas which should give an outline of the framework our research is embedded in.

There is an extensive discussion about the nature of proof in mathematics (e.g., Hanna & Jahnke, 1996). Though mathematics seems to be a strict and exact scientific discipline it cannot be denied that there are no clear definitions for basic notions like for example "proof". There exist probably some necessary conditions for a mathematical proof, but the acceptance of a proof takes place by unwritten rules of the mathematic community: „A proof becomes a proof after the social act of 'accepting it as a proof'" (Manin, 1977, p. 48). This is a fact that under different circumstances might also hold for the mathematics classroom (e.g. Herbst, 1998). The question

[1] This research was funded by the Deutsche Forschungsgemeinschaft in the priority program „Quality of School"(RE 1247/4).

whether a proof is accepted by the community depends on various factors. On the one hand, the proof has to validate whether a conjecture is true. On the other hand, the proof has to satisfy various other functions. As stressed by Hanna & Jahnke (1996), de Villiers (1990) and others, proving in mathematics is more than validation. Hanna & Jahnke (1996) describe eight different proof functions (like explanation, systematisation etc.). The function of validation, which for many people seems to be the most important one, is for mathematicians only one among others.

Mathematicians know through their own work, that the proving process and the proof as a product of this process must be distinguished. Sometimes the process of proving a theorem may take years and may include various approaches which may or may not lead to a success. In general, none of these efforts can be seen in the final product, that is in the formal written proof. Consequently, for the teaching and learning of proof it is not sufficient to show only the product. It is more important to stress the proof process. Boero (1999) described an expert model of the proof process. It is divided into different phases and gives insight into the combination of explorative empirical-inductive and hypothetical-deductive steps during the generation of a proof. We refer to Boero (1999) for the original description of this model; an adapted version for the analysis of proving processes in the mathematics classroom is given in Section 4.2.

3. Students' proof competencies – empirical results

In the last decade several empirical studies were conducted which gave an overview about students' mathematical competencies in different countries. In addition to studies like TIMSS and PISA which showed that in most countries comparatively few students perform well with proof items, there are results of several national studies which focus on the student competence in reasoning and proof (e.g., Healy & Hoyles, 1998; Lin, 2000; Reiss, Klieme & Heinze, 2001).

In an ongoing study with 669 students in grade 7 and 8 of the German Gymnasium (high attaining students) we focussed on the question which cognitive and non-cognitive factors are influencing the students' ability to perform geometrical proofs (cf. Reiss, Hellmich & Reiss, 2002). From the outcomes of two tests we identified three levels of competency: (I) basic competency, (II) argumentative competency (one-step-argumentation) and (III) argumentative competency (combining several steps of argumentation). Low-achieving students were not able to solve any items on level III whereas students from the upper third performed well on level I and level II items and were showing a satisfying performance for level III tasks (cf. Reiss, Hellmich & Reiss, 2002, Table 1 for the grade 7 test). A deeper analysis of students' responses to the test items and additional interviews (Heinze, in preparation) indicate that high-achieving students have problems with combining arguments to a proof and with generating a proof idea whereas low-achieving students in addition have deficits in their declarative and methodological knowledge.

The previous results become even more interesting, if one considers the overall classroom level of performance. In both tests we saw enormous differences in the achievement level of the participating classes (range of the mean scores in grade 7: 22 – 68 % of the possible points, in grade 8: 12 – 59 %). A two-level analysis for the grade 8 test showed that 42.4% of the variance of the individual achievement can be explained by the classroom level. It may be assumed that an essential portion of this influence is based on instruction, since there are hardly any differences in other factors like the number of students per class, their social background etc.

There are not many studies analysing the specifics of the mathematics classroom. One of most important studies in this respect is the TIMS video study comparing classrooms in Germany, Japan, and the United States (Stiegler et al. 1999). The results characterised the typical German teaching style as guiding students through the development of a procedure by asking them to orally fill in relevant information. The teacher generally presents the problem at the black board, eliciting ideas and procedures from the class as work on the problem progresses (cf. Stiegler et al. 1999, p. 133). In the German sample of the TIMS video study proofs occur only in a few lessons. Until now there are no results of video studies focussing on the teaching of proofs in Germany[2]. However, proof in mathematics classroom in Germany is not a blind spot. For example, Knipping (2001) observed the teaching of the Pythagorean theorem and analysed the corresponding proving process on a micro level. The focus of her study was to understand the processes of argumentation and not the connection between the teaching of proof and the student achievement in proof.

4. Research questions and design

4.1. Research questions and sample

The empirical results in the previous section show that students have difficulties dealing with mathematical proofs. Moreover, the statistical analysis indicates that the students' proof competence is substantially influenced by classroom factors. Consequently, we have to focus on the situation in mathematics classrooms and, in particular, the lessons involving proof instruction. This is the starting point of the present video study which is concerned with the process of proving in actual mathematics classrooms. The study is guided by the following research questions:

- How are proofs taught in mathematics classrooms in Germany?
- Which aspects of proof are emphasized by the teachers in the proving process?
- Are there gaps in the proving process or aspects that are underemphasized?

The data of the presented study consists of 20 videotaped mathematics lessons with grade 8 students. The lessons are from eight different classes of four schools (German Gymnasium, i.e., high attaining students). In each class we videotaped between two and four consecutive lessons. The subject of all lessons was reasoning and proof in

[2] In addition to our study there is one other ongoing video project on proofs concerning the teaching of the Pythagorean theorem in German classrooms: http://www.dipf.de/projekte/qualitaetssicherung_pythagoras.htm.

geometry, in particular, in congruence geometry. The participating teachers were asked to provide their typical instruction. Asked by questionnaires after the lessons the vast majority of the students confirmed that the videotaped instruction was the instruction they were used to.

4.2 Design of the study

For the analysis and evaluation of the proving processes in the videotaped lessons we use the model of Boero (1999; cf. Section 2). Since Boero's model gives a detailed description of the mathematical proving process of experts, we found it necessary to modify the phases of Boero to the following coding categories.

1. Phase: The first phase consists of the exploration of the problem situation, the generation of a conjecture and the identification of different types of arguments for the plausibility of this conjecture. This phase is denoted as

treated well: all elements are presented by the students;
treated: either some of these elements are given by the teacher or the phase is very short;
treated badly: the teacher performs the first phase;
not treated: in other cases.

2. Phase: The second phase consists of the precise formulation of the conjecture according to the shared textual conventions. This phase is denoted as

treated well: the students formulate the conjecture (possibly corrected by the teacher);
treated: only the teacher gives a formulation of the conjecture;
treated badly: there are mistakes in the final version of the conjecture;
not treated: there is no formulation of the statement that is to be proved.

3. Phase: This is again an explorative phase that is based on the formulated conjecture. The aim is the identification of appropriate arguments for the conjecture and a rough planning of a proof strategy. We distinguish this phase in four subcategories: (1) reference to the assumptions, (2) investigation of the assumptions, (3) collection of further information and (4) generation of a proof idea. We denote this phase as

treated well: at least three of these subcategories are observed in this phase;
treated: two of these subcategories are observed;
treated badly: there is only one of these subcategories;
not treated: in all other cases.

4. Phase: Based on the proof idea and the selected arguments of Phase 3 it follows the combination of these arguments into a deductive chain that constitutes a sketch of the final proof. This phase can be performed only verbally or in connection with some written remarks; it is denoted as

treated well: students (supported by the teacher) give substantial contributions;
treated: it is presented mostly or exclusively by the teacher;
treated badly: there are big gaps or other deficits in the deductive chain;
not treated: in all other cases.

5. Phase: This is the last phase for the proving process in school mathematics. Here the chain of arguments of Phase 4 is written down according to the standards given in the respective mathematics classroom. In particular, it is important that this phase also gives a retrospective overview about the proof process. It is denoted as

treated well:	all steps are written down and there is a retrospective summary of the proof process;
treated:	the most important steps are written down and there is a retrospective summary;
treated badly:	there are only some arguments written down, but no retrospective summary;
not treated:	in all other cases.

For our study we applied this operationalised model of the proving process to all proofs we identified in the 20 videotaped lessons. For each proof we measured the time that was spent for the different phases and we determined the quality of each phase with respect to the categories described above. Hence, we achieved a characterisation of the proving process with respect to qualitative and quantitative criterions.

5. Results

According to the research questions (cf. Section 4.1) our aim is to describe the teaching of proof in mathematics classroom and, particularly, to describe steps in the proving processes that are emphasized or underemphasized. In the 20 videotaped mathematics lessons we identified 22 proofs. Each proof process was evaluated separately.

5.1. Time-based analysis

Diagram 1 gives an overview about the time portion (in percent) of each phase in the proof instruction (mean values). One can see that most of the time is spent on the fourth phase (about 36%), which is the organisation of arguments in a deductive chain. The first phase as an experimental phase which mainly consists of drawing and measurements takes about a quarter of the time. The important third phase, in which the exploration of the conjecture and the identification of arguments take place summarizes to 22 % of the proof time.

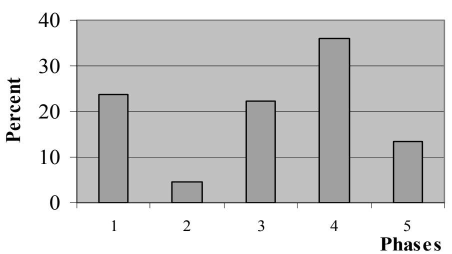


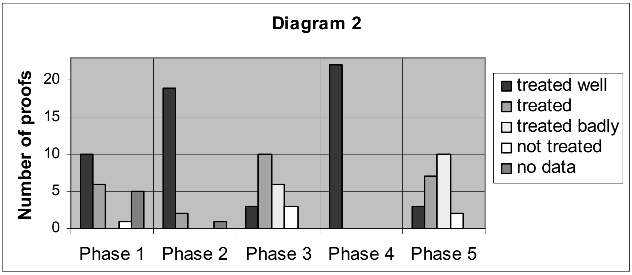
Diagram 1: Percent of time used per phase.

This diagram reflects the typical process of dealing with proof problems in the videotaped lessons. First of all, the students have to draw a geometrical figure and to make some observations on an experimental level. Afterwards a conjecture is discussed and formulated (Phase 2). In the third phase the students have the opportunity to propose some ideas for the proof. If the students are not able to generate a proof idea, then the teacher gives more and more hints. Then the proof is organized step by step on the chalk board. This takes place in a kind of classroom discourse, in which the teacher leads the students through the proof by specific questions. In other words, the students have to follow the proof the teacher has in his/her mind. The writing on the chalk board is frequently a collection of arguments as expected in Phase 4. Sometimes some parts are already very detailed as expected in Phase 5. Only in a few cases a genuine Phase 5 took place, in which additionally a retrospective summary is given.

5.2 Quality of proof instruction

In addition to the time-based results we also collected data about the quality of the proof phases (cf. Section 4.2). In some cases it was not possible to rate the complete proving process, since some phases were partly done in a lesson before that was not videotaped, or they were part of the homework for the observed lesson. This problem affected mainly Phase 1 and here, particularly, the drawings that were prepared at home.

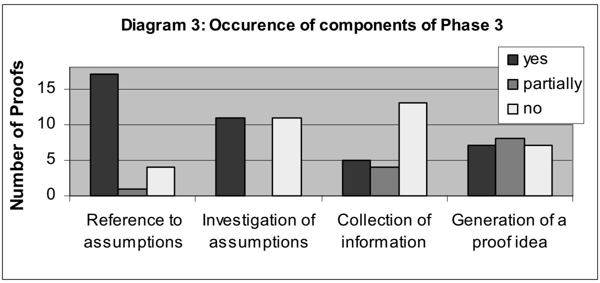
We can see in Diagram 2 that, in particular, the quality of Phase 2 and Phase 4 in the proof processes is quite good. Moreover, one can say, that Phase 1 is treated satisfactorily. Substantial deficits occur mainly in the Phases 3 and 5. The requirements of these phases are not satisfied in 9 and 12, respectively, of the 22 proofs. Their quality in the remaining proofs is in most cases rated as "treated" and only in a few cases as "treated well".



If we combine the time-based results of Section 5.1 with the quality-based results of this Section 5.2 we can identify clear deficits in the proof processes for the Phases 3 and 5. Both phases are important for the learning of mathematical proofs, however, Phase 3 as a phase of exploration of the conjecture, collecting additional information and generating a proof idea seems to be the most crucial phase for proof instruction on the lower secondary level. Therefore, we take a closer look on this phase.

5.3 A detailed analysis of Phase 3

As described in Section 4.2 we split Phase 3 into four different subcategories. In the videotaped lessons we analysed which of these subcategories occurred in the observed proofs. The results in Diagram 3 indicate that in general only one of the four components appeared: in 17 of 22 proofs the teacher and students



referred to the assumptions in Phase 3. Only half of the proof processes observed included an investigation of these assumptions. The collection of additional information did not appear in thirteen proof processes and was only partially included in four cases. Even the generation of a proof idea was given in only seven proofs and given partially in only eight proofs.

6. Discussion

Our analysis points up that essential phases in the proof process are neglected by the teachers. As already mentioned in Section 5.1 the typical proof process in the mathematics classroom is planned and controlled by the teacher. This means that the teacher leads the students through the “labyrinth” of the proof situation. The role of the students is to guess which direction the teacher has in mind. This kind of instruction is caused by the so called “fragend-entwickelnd” (questioning-developing) teaching style, which the German TIMSS video group identified as the most popular form of mathematics instruction in Germany (cf. Klieme, Schüner & Claussen, 2001). The problem with this kind of teaching is that there is no place for in-depth phases which are necessary in the proof process, e.g., for the exploration of the problem situation or the collection of additional information. In the videotaped lessons the first exploration phase (Phase 1) consists mainly of making drawings and measuring the lines or angles. The second in-depth phase (Phase 3) the students have no time for a deeper investigation or exploration of the situation (see Section 5). The consequence is that they get no real chance to solve the proof problems on their own. They have to follow the hints and questions of their teacher. As shown in Diagram 1 Phases 4 and 5 take most of the time in the proof processes (apart from the time for drawings in Phase 1). However, in general, these two phases do not coincide exactly with the description in the model of Boero. We observed more or less a mixture of the ideal Boero phases: the teacher elaborated on the proof step by step at the chalk board by asking questions and giving hints (i.e., applying the “fragend-entwickelnd” teaching style). From the students’ perspective the proof splits into small steps which they have to deal with successively. Many students will finally loose the overview, since a retrospective summarizing of the proof was lacking in most lessons.

Taking into account the results of our video analysis, we are not surprised that the proof competence of German students on the lower secondary level is poor (Section 3). We think that the students’ problems are significantly influenced by the proof instruction as the results of the video study given in this article indicate. A systematic investigation of this “correlation” is in process. It requires a deeper analysis of the video data than the one described in this article.

We have to emphasize that we analysed the teaching of proof in the videotaped lessons only from a mathematical perspective. This means that in this study other questions and problems like the teaching style, the combination of classwork and seatwork, the participation of students etc. were not discussed. However, already the mathematical point of view discovered problems in proof instruction, since important topics concerning the proving process were not covered sufficiently.

Finally, we want to stress that we could identify examples of good practice in the lessons observed. In two classrooms all proof phases were treated well, and for each phase the teacher spent an appropriate portion of time.

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STRUCTURE SENSE IN HIGH SCHOOL ALGEBRA: THE EFFECT OF BRACKETS

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This paper presents an initial attempt to define structure sense for high school algebra and to test part of this definition. A questionnaire was distributed to 92 eleventh grade students in order to identify those who use structure sense. Presence and absence of brackets was examined to see how they affect use of structure sense. The overall use of structure sense was less than expected. The presence of brackets was found to help students see structure.

A student made a minor mistake in solving a problem in a matriculation exam and obtained the following equation: $1-\frac{1}{n+2}-\left(1-\frac{1}{n+2}\right)=\frac{1}{132}$. He solved it by first multiplying both sides by a common denominator then opening brackets and collecting like terms. This solution raised questions about structure sense and inspired the present paper. In this paper structure sense is defined, data that was collected by means of a questionnaire containing equations similar to the above are presented, and the results are discussed in terms of structure sense.

The research described here is part of a study concerning high school students' struggle with algebra. The students in question study mathematics in intermediate or advanced streams. In order to be accepted into these streams they have had to demonstrate a certain proficiency with algebraic techniques. By dealing with students who have been relatively successful in acquiring and using basic algebraic knowledge, at least as it is tested in regular school exams, issues of cognitive level and different approaches to beginning algebra (see MacGregor & Stacey, 1997) are avoided. Nevertheless many of these students do not succeed in applying their basic algebraic knowledge when solving problems in more advanced algebra, trigonometry or calculus.

Linchevski and Livneh (1999) first used the term structure sense when describing young students' difficulties with using knowledge of arithmetic structures at the early stages of learning algebra. Hoch (2003) suggested that structure sense is a collection of abilities, separate from manipulative ability, which enables students to make better use of previously learned algebraic techniques. More precise definitions of structure sense and of algebraic structure are required. The definitions given in this paper are based on interviews with researchers in mathematics education. The full derivation of these definitions will be reported on in a future publication.

DEFINING STRUCTURE SENSE

In order to reach a definition of structure sense it is necessary to discuss what is meant by structure, specifically in the context of high school algebra.

Structure

Why define structure? The term is widely used and most people feel no need to explain what they mean by it. It is used in the field of mathematics education to cover various different meanings (see for example Dreyfus & Eisenberg, 1996). In different contexts the term structure can mean different things to different people. This could cause problems when discussing structure sense. It must be made clear which meaning of the term structure is being used. The following definition of structure will be adopted, for the purposes of discussing high school algebra.

> *Structure in mathematics can be seen as a broad view analysis of the way in which an entity is made up of its parts. This analysis describes the systems of connections or relationships between the component parts.*

Algebraic structure (at high school level)

Algebraic expressions or sentences (equality or inequality relation between two algebraic expressions) can be considered to represent algebraic structures. Examples of two structures from high school algebra are algebraic fractions and quadratic equations. The shape of an algebraic fraction is $\frac{f(x)}{g(x)}$ where *f(x)* and *g(x)* are both polynomial functions. Simplifying or expanding the fraction may reveal an internal order. A quadratic equation is any polynomial equation that can be transformed into the standard shape $ax^2 + bx + c = 0$ where a, b and c are real number parameters. The process of transforming the equation into standard form may reveal the internal order. This may lead to a solution either by factoring, or by using the quadratic formula. The internal order might also lead us to expect that there will be 0 or 1 or 2 solutions, and to know that these solutions are the intersection points of the parabola $y= ax^2 + bx + c$ with the X-axis. Algebraic structure will be defined in terms of shape and order.

> *Any algebraic expression or sentence represents an algebraic structure. The external appearance or shape reveals, or if necessary can be transformed to reveal, an internal order. The internal order is determined by the relationships between the quantities and operations that are the component parts of the structure.*

Structure sense

There are structures in high school algebra that are concealed by external appearance. The equation $4x^2 – x^3 + 5(4 –2x) = (3 – x^2)(6 + x)$ can be transformed into the standard quadratic equation $10x^2 – 13x + 2 = 0$. These two equations are equivalent but whereas in the second the structure is obvious in the first it is less so. Any discussion of algebraic structures will have to involve some discussion on equivalencies. If two algebraic expressions or sentences are equivalent do they possess the same structure? The two expressions $30x^2 - 28x + 6$ and $(5x - 3)(6x - 2)$ are equivalent. Yet the first is clearly a quadratic expression and the second is clearly the product of two linear factors. What is the structure here? Our answer is that "quadratic expression" and "product of two linear factors" are different interpretations of the same structure. Knowing which interpretation is more useful in any given context is a part of structure sense. Structure sense may have a lot to do with experience and something in common with intuition. For example, for a student with structure sense, the need to simplify a fraction is self-evident. After much consideration the following definition for algebraic structure sense is proposed.

> *Structure sense, as it applies to high school algebra, can be described as a collection of abilities. These abilities include the ability to: see an algebraic expression or sentence as an entity, recognise an algebraic expression or sentence as a previously met structure, divide an entity into sub-structures, recognise mutual connections between structures, recognise which manipulations it is possible to perform, and recognise which manipulations it is useful to perform.*

METHODOLOGY

Instruments

A questionnaire was designed with several aims. The main aims were to identify students who display structure sense and to investigate if structure sense is affected by the number of sets of brackets (0, 1, 2) and by the placement of the variable (on one side of equation or on both sides of equation). The researchers could find no reports of research on the effect of brackets on students' success with solving equations. Other aims were to investigate if use of structure sense is more prevalent among advanced students than among intermediate students, and if students are consistent in their use of structure sense. The questionnaire consisted of two equations. Three alternative items were designed for the first question (A, B and C) and three for the second question (X, Y and Z) as follows.

A. $1-\frac{1}{n+2}-\left(1-\frac{1}{n+2}\right)=\frac{1}{110}$

B. $\left(1-\frac{1}{n+1}\right)-\left(1-\frac{1}{n+1}\right)=\frac{1}{132}$

C. $1-\frac{1}{n+3}-1+\frac{1}{n+3}=\frac{1}{72}$

X. $\frac{1}{4}-\frac{x}{x-1}-x=5+\left(\frac{1}{4}-\frac{x}{x-1}\right)$

Y. $\left(\frac{1}{4}-\frac{x}{x-1}\right)-x=6+\left(\frac{1}{4}-\frac{x}{x-1}\right)$

Z. $\frac{1}{4}-\frac{x}{x-1}-x=7+\frac{1}{4}-\frac{x}{x-1}$

The instructions were “Solve for n” (or for x). Each student was asked to solve two equations, A & Y, A & Z, B & X, B & Z, C & X or C & Y. The questionnaire was administered to 92 eleventh grade students in a high school in a well-established Israeli town. These students, aged 16 to 17, learn in intermediate to advanced mathematics streams.

After the questionnaires were examined, four students who were considered to have used structure sense in an unusual manner were interviewed, in an attempt to obtain a clearer picture of their reasoning.

Interpretation

A student who uses structure sense to solve equation A would be expected to do the following. S/he looks at the difference of two terms $1-\frac{1}{n+2}$ as an entity or structure and recognises that the same structure (or sub-structure) appears inside the brackets. The relationship between the two structures is equality, and since they are connected by a minus sign, the result is zero. Then s/he “sees” that the structure of the equation is such that it is in fact equivalent to a much simpler equation.

Of course a student’s thought processes cannot be known from a written answer. A student who writes something similar to “zero equals a fraction and so there is no solution” is considered to be displaying structure sense. In a similar manner the hypothetical student, when asked to solve equation X, would be expected to write “-x = 5 and so x equals –5”. The items containing two or no sets of brackets can be analysed in an analogous way.

Some students who wrote a line or two of calculations before arriving at the above conclusions were interviewed. They said that they subsequently ”saw” the simple equation and thus they were also considered to have used structure sense. Students using other methods, for example opening brackets and/or finding a common denominator, were considered to have displayed a lack of structure sense.

RESULTS

Table 1 shows the rate of structure sense used. The result is obviously disappointing. The overall rate of use of structure sense is very low. In the majority of cases, the students solved the equations by first multiplying by a common denominator and only later cancelling like terms. In other words, the majority of students displayed a lack of structure sense. Most of those who didn’t use structure sense either made calculation mistakes or failed to cancel an extraneous solution. Those who used structure sense got the answer quickly and accurately.

Table 1 Percentage of questions solved using structure sense

Equation	Advanced stream (45 students)	Intermediate stream (47 students)	Total (92 students)
A: $1-\frac{1}{n+2}-\left(1-\frac{1}{n+2}\right)=\frac{1}{110}$	**18.8%** (3/16)	**6.3%** (1/16)	**12.5%** (4/32)
B: $\left(1-\frac{1}{n+1}\right)-\left(1-\frac{1}{n+1}\right)=\frac{1}{132}$	**23%** (3/13)	**12.5%** (2/16)	**17.2%** (5/29)
C: $1-\frac{1}{n+3}-1+\frac{1}{n+3}=\frac{1}{72}$	**0%** (0/16)	**0%** (0/15)	**0%** (0/31)
X: $\frac{1}{4}-\frac{x}{x-1}-x=5+\left(\frac{1}{4}-\frac{x}{x-1}\right)$	**15.4%** (2/13)	**14.3%** (2/14)	**14.8%** (4/27)
Y: $\left(\frac{1}{4}-\frac{x}{x-1}\right)-x=6+\left(\frac{1}{4}-\frac{x}{x-1}\right)$	**25%** (4/16)	**11.8%** (2/17)	**18.2%** (6/33)
Z: $\frac{1}{4}-\frac{x}{x-1}-x=7+\frac{1}{4}-\frac{x}{x-1}$	**18.8%** (3/16)	**6.3%** (1/16)	**12.5%** (4/32)
Total 184 solutions	**16.7%** (15/90)	**8.5%** (8/94)	**12.5%** (23/184)

Number of brackets

It was expected that in the equations without brackets (C & Z) it would be easier for the students to identify and cancel like terms. This was clearly not the case. Only 6.3% (4/63) of the students used structure sense when no brackets were present as compared to 13.6% (8/59) for one set of brackets (A & X) and 17.7% (11/62) for two sets (B & Y). The lack of brackets seems to have deterred students from recognising like terms. Is it possible that the brackets give the students a clue where to look and that the lack of brackets leaves a long unstructured (in the students' eye) expression? The brackets seem to focus the students' attention and alert them to the possibility of like terms. However one student stated that he mentally opened the brackets in order to cancel out in equation A. Therefore students do not necessarily see an expression inside brackets being the same as one without brackets.

The most glaring result is the total lack of use of structure sense in equation C. Why don't students see what is so obvious to us? It was thought that the use of brackets might confuse students, but here it appears that the absence of brackets seems to be a stumbling block. However this is not so much the case in equation Z although also here the "seeing the obvious" is less than might be expected. What makes equation C harder? Is it the long expression (4 terms) on one side of the bracket? Or do the students just work mechanically from left to right, stopping to "take stock" only when they see the equal sign? Here are some quotes from students who were interviewed.

- *Usually, in my opinion, every student who sees fractions he straightaway deals with them………. Usually you need to get a common denominator.*
- *I don't look at the equation as a whole. I look at each side separately and only then I move things………. I get rid of the brackets. The fewer brackets the better.*
- *First I always open the brackets.*

It seems that in the calculation students aim to "get rid of" the brackets. Yet we found that other students in fact added brackets to "see" the identical terms better (see Table 2). In fact one of the students actually stated this: *with brackets it's easier to see*.

Thus the presence of brackets might help them to see the structure. A feature of using structure sense is "looking" before "doing", something that teachers might be expected to emphasize in the high school classroom when they are reviewing previously learned algebraic techniques.

Placement of variable

The results show that 9.8% (9/92) of the students used structure sense when the variable appeared only on one side of the equation (A, B, C) as compared to 15.2% (14/92) when the variable appeared on both sides (X, Y, Z). Thus the placement of the like terms on opposite sides of the equation seems to enable students to identify them more easily. It is worth considering whether the fact that equations X, Y, Z have a non-empty solution set while equations A, B, C have an empty solution set has an effect on the results. Since structure sense is more concerned with how students approach an equation and less with what they do with the final solution the effect should be minimal. One student wrote $0 = \frac{1}{110}$ as his solution to A. In the interview he said that it did not make sense. When asked what was the solution of the equation he said that there was no solution. He was considered to have used structure sense.

Use and consistency

Overall only 19.6% (18/92) of the students displayed structure sense in at least one of the questions. As expected the advanced students used structure sense more: 24.4% (11/45) than the intermediate students: 14.9% (7/47). Of the students who used structure sense, only 27.8% (5/18) were consistent, using it in both equations.

Eighteen students used structure sense to answer a total of 23 questions. In Table 2 these 23 questions are examined to see how structure sense was used.

Table 2 Different uses of structure sense (N = 23)

Method	Example	
Immediate	$0 = \frac{1}{132} \rightarrow$ no n (from B)	**39.1%** (9)
Minimal working	$\frac{1}{4} - \frac{x}{x-1} - x = 5 + \left(\frac{1}{4} - \frac{x}{x-1}\right)$ $\left(\frac{1}{4} - \frac{x}{x-1}\right) - x = 5 + \left(\frac{1}{4} - \frac{x}{x-1}\right)$ x = -5 (from X)	**43.5%** (10)
Calculation	$\left(\frac{(x-1)-4x}{4(x-1)}\right) - x = 6 + \left(\frac{(x-1)-4x}{4(x-1)}\right)$ x = -6 (from Y)	**17.4%** (4)

The minimal working shown here involved the addition of brackets, but in some other cases it involved the removal of brackets. The student who wrote the calculation in Table 2 was interviewed. His reaction on being asked what he was thinking when he wrote this was: *Oh. Now I see. Simply, I did common denominator. And now I see it was completely unnecessary.*

A teacher's suggestion

After the disappointing results from the students it was decided to show some teachers the student's solution to the equation mentioned in the first paragraph of this paper. About half of these teachers noticed immediately that the left-hand side is zero. The others had to be prompted to "see" what had seemed very obvious to the researchers. This lack of structure sense among teachers may be a clue to the disappointingly low incidence of structure sense among the students.

One teacher suggested that tenth graders would perform better due to the proximity of learning about equations. The questionnaire was subsequently administered to a tenth grade advanced class. In fact the rate of use of structure sense among the tenth graders was higher than among the eleventh graders. Several students substituted a new variable in place of a longer algebraic term, a method they had recently learned. However the eleventh graders had learned this technique a year ago and did not use it. This indicates that it might be possible to teach attention to structure but better methods must be found to ensure retention of this knowledge.

CONCLUSION

What does the above data tell us? Very few of the students in this study used structure sense. Those who did so were not consistent. As expected, the advanced students used it more than the intermediate ones. Those who used structure sense got the answer quickly and accurately, avoiding opportunities for mistakes that often occur in long calculations. The presence of the variable on both sides of the equation helped in identifying like terms. The presence of brackets also seemed to help students see structure, focussing their attention on like terms and breaking up the long string of symbols. However the evidence about the effect of brackets is inconclusive, as some students seem to prefer to eliminate them from the equation.

What abilities might be present in the students who used structure sense to solve the equations? The ability to see an algebraic expression or sentence as an entity necessitates stopping to look at the equation before automatically applying algebraic transformations. The ability to recognise mutual connections between structures, in this case equality, could lead to choosing the appropriate manipulations.

Not all teachers seem to use structure sense. Presumably these teachers don't encourage their students to use it. Is structure sense something that can be taught? Should it be taught? We feel that the last two questions should be answered in the affirmative but are not yet ready with an answer for the obvious next question: How should structure sense be taught?

We are convinced of the importance of drawing students' attention to structure. The above definitions should be useful as guidelines for further research and didactic design.

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EXPLAINING VARIABILITY IN RETRIEVAL TIMES FOR ADDITION PRODUCED BY STUDENTS WITH MATHEMATICAL LEARNING DIFFICULTIES

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Predictors of retrieval times produced by students having difficulty developing a reliance on retrieval for simple addition were discovered. The findings support the notion that separate limitations operate in working memory when retrieval occurs and call into question the use of the term' retrieval deficit' to explain difficulties experienced by these students.

INTRODUCTION

The inability to develop, strengthen, and access associations in memory that allow for the rapid and accurate retrieval of answers to problems such as 3+2, 4+5 and 8+7 is a distinguishing and persistent characteristic of a mathematical learning difficulty (Jordan & Montani, 1997; Ostad, 1997; Geary, Hamson, Hoard, 2000; Robinson, Menchetti, & Torgesen, 2002). When students with mathematical learning difficulties do retrieve answers to simple addition problems, retrieval times appear unsystematic. Factors accounting for variability in reaction times (RTs) for retrieval trials remain unidentified for children with learning difficulties (Geary, 1990; Geary and Brown, 1991).

To date, investigations of retrieval times have been examined with reference to problem-specific variables only, such as problem size. This paper outlines a study that set out to explain variability in RTs produced by students with learning difficulties as they retrieved answers to simple addition problems, using performance-specific and learner-specific variables. The results based on a 3-level hierarchical linear model indicate that retrieval time for solving an addition problem will increase if a student has previously performed the problem incorrectly or if the student has applied a different strategy to solve the problem, on a different occasion. The findings support the notion that sources of increase in RTs are two separate working memory limitations, one relating to source activation and the other relating to a response selection mechanism. These sources of limitation are the same as those identified for normally achieving children.

THEORETICAL FRAMEWORK

The 'problem-size effect' is an important aspect of research in the area of simple arithmetic given the robustness of findings supporting this phenomenon (Ashcraft, 1992). The problem-size effect refers to the finding that as the 'size' of the problem increases, the time taken between the presentation of a problem and a response (RT) also increases. The explanation of the problem-size effect in simple addition performance amongst children is straight forward: If children use a counting strategy then RTs will increase as the size of the problem increase, as more counts are required. However, the problem-size effect is also evident when answers to simple addition problems are directly retrieved by children (Geary, 1990) and adults (LeFevre, Sadesky & Bisanz, 1996). Hopkins and Lawson (2002) explain the problem-size effect in retrieval times to addition problems as follows:

> As the size of the problem increases, the time taken to retrieve the answer increases because the link associating the problem with the answer is less frequently used and is more prone to interference from links to incorrect facts associated with similar problems or with previous incorrect performance. (p.144)

Thus problem size is indirectly related to activation strength.

To identify possible performance-specific variables that could be used to predict retrieval times, theory relating to processing limitations was reviewed. Based on a cognitive science perspective, the concept of 'working memory' is commonly used to depict the limiting aspect of the processing system. However, researchers view the limited nature of working memory differently. Based on explanations of dual-task interference, at least three explanations are evident in the literature. Working memory is thought to encompass: (1) <u>a limited mechanism</u> that creates a bottleneck in processing when more than one response is required for selection (Pashler, 1994); (2) <u>a source activation limitation</u> relating to the amount of knowledge that can be activated at any one time by the information currently held in working memory (Anderson, Reder & Lebiere, 1996); and (3) <u>a limited set of processing resources</u> that fuel the executive component of working memory so the greater demand placed on the executive, the more its efficiency at performing certain functions will be reduced (Gathercole & Baddeley, 1993).

It is possible that the speed of retrieving an answer to a simple addition problem is influenced by a source activation limitation. Siegler's strategy choice model (Siegler & Jenkins, 1989; Siegler & Shipley, 1995) is based on the assertion that both correct and incorrect answers are stored in LTM memory each time a simple addition problem is performed. If source activation has to be divided amongst correct and incorrect answers, then the time taken to activate or discriminate the correct answer will be affected. In the present study we test the premise that retrieval times are influenced by a source activation limit.

It is also possible that the speed of retrieving an answer to a simple addition problem is influenced by a bottleneck effect as demand is placed on a limited response selection mechanism. In a review of studies comparing the processing speed of younger and older children, Chi and Gallagher (1982) found that a major limitation of children's processing speed occurred at the response selection stage. When task complexity increased, as the number of possible choices of response increased, they reported that children were particularly disadvantaged compared with adults. There are many strategies that can be applied to perform simple addition, including Count All strategies, Count On strategies and Decomposition Strategies (Hopkins & Lawson, 2002). These different strategies may compete for expression so that the problem stimulus activates not only candidate answers but also representations (schemas) for the procedural knowledge associated with the different strategies (Siegler & Jenkins, 1989). In the present study we test the premise that retrieval times are influenced by a limited response selection mechanism.

Furthermore, it is possible that demand placed on limited resources fuelling the executive could influence retrieval times. A retrieval model for addition presented by Campbell and Oliphant (1992) introduced the idea that a single problem can activate multiple responses (including, for example, responses relating to the operation of addition, multiplication and naming). When this occurs, the executive component is required to manage the candidate set of answers and inhibit the processing of irrelevant information. It follows then that the greater number of associations activated, the greater the demand placed on the executive, which could result in slower RTs. However, due to the difficulty in identifying when irrelevant associations are activated but inhibited, this aspect of a working memory limitation was not examined in the present study.

METHODOLOGY

Participants and procedure

Six students aged between 13yrs 9 months and 17 yrs agreed to participate in the study. All attended a secondary college (catering for students in Years 8 to 10) in suburban Adelaide and had previously been taught by the principal researcher. All students were performing poorly in mathematics, five of the six being in remedial mathematics classes. The remaining student was in Year 8 for which there was no remedial class. The students' scores on the Standard Progressive Matrices Test (Raven, 1938) were all below age means. Selection of students was also based on classroom observations that identified them as regularly using fingers to count on for simple addition.

Addition performance was assessed using a problem set consisting of 65 addition problems, written in the form m+n, where n,m>0 and n≥m. Problems included: all 45 single digit addition problems where m, n<10, 10 addition problems where m<10 and n=10, and 10 addition problems where m<10, 10<n<15.

Students individually performed 26 problems each school day, until the problem set was completed. This procedure was then repeated until the problem set was performed five times (recorded as occurring in time interval 1 to 5). Problems were shown one at a time on a computer programmed to display problems in random order, as well as record correct responses and the number of seconds taken to respond (correct to one decimal place). After each problem was solved (referred to as a trial), the student was required to describe the strategy they had used. Based on self-report plus observation (student performance was videotaped) the strategy used on each trial was coded as either a counting strategy, a decomposition strategy, retrieval (commonly reported as "I just knew it"), or as undefined. The approach is similar to that first adopted by Siegler (1987). Only RTs to correct trials where students reported using retrieval were analyzed in the present study.

Analysis

A number of assumptions are made when interpreting a traditional linear regression equation for predicting RTs. Regarding performance based on a retrieval strategy, it is assumed that the processes of encoding the problem, selecting a response and stating the answer require a constant amount of time. We refer to these collectively as the production component of processing. The activation component is dependent on a problem-specific variable relating to size and indirectly relating to activation strength. The PROD variable (the value of the product of the addends) is typically the best predictor of retrieval times (Miller, Perlmutter and Keating, 1984; Geary & Brown, 1991). The traditional equation used to predict RTs to retrieval trials for simple addition is represented by Equation (i).

$$\underbrace{RT = a}_{\text{\{production component\}}} + \underbrace{b(PROD)}_{\text{\{activation component\}}} \quad \ldots\ldots\ldots\ldots(i)$$

In the multilevel analysis framework adopted in this study, RTs to correct retrieval trials were identified as having predictors that could operate on three different levels: an occasion level, a problem level and a student level. The focus of interest in the analysis was to test whether a variable proposed to be related to a source activation limit (labeled Previous Error or PE) and a variable proposed to be related to a limited response selection mechanism (labeled Transformation or TRANS) would emerge as significant predictors of variation in student retrieval times. A description of the variables constructed at the different levels is given below.

At the occasion level, a dummy variable representing the occurrence of a previous error (PE) was tested. A trial was coded with PE=1 if a previous error was made on the same problem by the same student on any preceding trial, otherwise PE=0. At the occasion level, a dummy variable representing practice (PRAC) was also tested to account for expected variation in RTs over time, given that each problem was presented five times to students. A trial was coded with PRAC=0 if the trial occurred in either time interval 1, 2 or 3, or it was coded with PRAC=1 if it occurred in time interval 4 or 5.

At the problem level, the traditional PROD variable was tested as well as a dummy variable, labeled TRANS, representing multiple strategy use (indicative of the transformation process where a reliance on retrieval is in an early stage of development). For each student, a problem was coded with TRANS=1 if they had correctly used a strategy other than retrieval on the problem, otherwise it was coded with TRANS=0 (indicating only retrieval had been used).

At the student-level, the variable RAVEN was used to predict retrieval times, representing the raw score each student achieved on the Standard Progressive Matrices Test.

Hierarchical Linear Modeling (HLM) (Bryk & Raudenbush, 1992) was used to analyze retrieval times because of its appropriateness for data that have a multilevel structure. The outcome variable was the time taken to correctly retrieve an answer to a simple addition problem. The equation used in the present study to predict RTs to retrieval trials is represented by Equation (ii). Interaction variables at the same level were created by multiplying two predictor variables together (Jaccard, Turrisi & Wan, 1990).

$$\begin{aligned}
RT= \; & a_1 + a_2(\mathrm{RAVEN}) + a_3(\mathrm{PRAC}) + a_4(\mathrm{TRANS}) + a_5(\mathrm{PE}) \\
& + a_6\underbrace{(\mathrm{TRANS})(\mathrm{PRAC}) + a_7(\mathrm{PExPRAC}) \qquad\qquad}_{\textit{production component}} \\
& + b_1(\mathrm{PROD}) + b_2(\mathrm{RAVEN})(\mathrm{PROD}) + b_3(\mathrm{PROD})(\mathrm{PRAC}) \\
& + b_4(\mathrm{TRANSxPROD}) + b_5(\mathrm{PROD})(\mathrm{PE}) \\
& + b_6\underbrace{(\mathrm{TRANSxPROD})(\mathrm{PRAC}) + b_7(\mathrm{PROD})(\mathrm{PExPRAC})}_{\textit{activation component}} \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}
\end{aligned}$$

Using multilevel analysis it was possible to test the supposition that the variables TRANS and PE represent situations where different working memory limitations are operating: If TRANS is related to a limited response selection mechanism then it should influence the production component of processing (which incorporates the time taken to select a response) but not the activation component; if PE is related to a source activation limit then it should influence the activation component of processing but not the production component.

SUMMARY OF FINDINGS AND DISCUSSION

The mean percentage of trials where retrieval was correctly performed by each of the six participants was 41.9%, ranging from 20% to 62.2%. This data set (including 817 RTs) was analyzed using the HLM procedure previously described. (The steps made to construct equations at each of the three levels and a detailed description of the results is not given in this paper but is available upon request.)

An analysis of the fully unconditional model indicated that 54.2% of variance in retrieval times could be attributed to problem-level differences, 33.7% to occasion-level differences and 12.0% to student-level differences.

The analysis of the final model indicated that three factors significantly predicted RTs to correct retrieval trials. By substituting in the estimated coefficients (msecs) and including only significant predictors for the conditional model, the final equation is shown by Equation (iii).

$$\mathrm{RT} = \underbrace{1{,}727 + 706(\mathrm{TRANS})}_{\textit{production component}} \underbrace{+15(\mathrm{PROD})(\mathrm{PE}) - 16(\mathrm{PROD})(\mathrm{PExPRAC})}_{\textit{activation component}} \ldots\ldots\text{(iii)}$$

The TRANS variable was a significant predictor of retrieval times. Thus the time taken to correctly retrieve an answer to an addition problem increased if a student had previously solved the problem using a strategy other than retrieval. In our theoretical analysis the use of additional strategies is argued to impact on retrieval times due to a bottleneck effect, where a range of responses compete to be selected. The finding that the TRANS variable impacted the production component of processing and not the activation component supports this theoretical analysis.

The PROD variable explained a significant amount of variation in retrieval times for problems that had been previously performed with error (indicated by the PE variable). Thus the time taken to correctly retrieve an answer to an addition problem increased (in proportion to the size of the problem) if a student had previously given an incorrect answer to the problem (using either a counting strategy, decomposition or retrieval). The finding that the PE variable impacted the activation component of processing and not the production component was hypothesized and supports the assertion that a previous error increases the time taken to activate a correct answer because the energy source for activation is dispersed to an incorrect association. The effect the source activation limitation had on retrieval times was found to reduce with practice.

Estimated variances for the final model indicated that significance residual variance remained to be explained at the problem and student levels (particularly the student level, given that the RAVEN variable was not a significant predictor of retrieval times).

The findings in this study suggest that the retrieval times of students who have difficulty developing a reliance on retrieval for simple addition may not be as unsystematic as we have previously thought. A significant amount of variance in the retrieval times of these students can be explained by factors that are thought to underlie performance limitations of normally achieving children. This result suggests that it may be misleading to describe students with learning difficulties as having a 'retrieval deficit' (Geary, 1993; Robinson, Menchetti & Torgesen, 2002) because this label implies that limitations outside the bounds of normal variation are operating. The sources of limitation impinging on retrieval are the same for students with learning difficulties as those identified for normally achieving students. Further research is needed to test the significance of the magnitude of the limitations experienced by students with learning difficulties.

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EARLY GENDER DIFFERENCES

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In a longitudinal study of children in grades 0-4 gender differences are developing very early in some number domains but there do not appear to be similar differences showing in the measurement and geometry domains studied.

INTRODUCTION

Gender is still an issue in mathematics education. There has been much change, both in terms of achievement differences and in the nature of the research, since the issue became one of research interest. Many though seem to feel that the gender problem has been solved. Indeed in some circles the focus has changed as reflected in a recent report in Australia called *Boys, Getting it Right* (Commonwealth of Australia, 2002), which expressed concern about the drop out rate of boys and raised particular concerns about their language development. What is clear is that the nature of the questions in relation to gender have changed and so have the research findings.

Recent research has shown that the achievement gap has been reduced in many countries, but that there are still areas of concern. The differences are not now as straightforward as they were in the 1970s and 1980s. In the TIMMS study at middle primary school (age 9) in most countries there were no significant gender differences in performance (Mullis, Martin, Fierros, Goldberg & Stemler, 2000). In Australia and New Zealand this was also true at junior secondary school (age 13). The follow-up PISA study showed no clear gender differences, however, although not of statistical significance, it is of note that New Zealand was one of three countries where the girls out-performed the boys in mathematics (Lokan, Greenwood, & Cresswell, 2001). Other New Zealand studies show higher achievement for girls throughout the early and middle years of schooling (Flockton & Crooks, 2002). In Australia, during the primary school years (Grades 0-6), state-wide and national testing has generally found no gender differences favouring boys (Collins, Kenway & McLeod, 2000; Doig, 2001; Yates, 1999). As noted by Forgasz, Horne and Vale (in press), gender differences in achievement in mathematics continue to decline, but are not consistent across year levels societal groups and assessment instruments.

While achievement tests show no significant difference in outcomes there have been differences shown in thinking (Fennema, Carpenter, Jacobs, Franke & Levi, 1998; Horne, 2003). Both of these studies were with children in grades 1-3 and showed that in addition and subtraction, girls tended to continue using counting strategies while boys were more likely to move to more sophisticated strategies.

Gender differences have also shown in responses to some types of questions, such as the amount of interpretation of diagrammatic information (Lokan et al, 2001).

THE STUDY

The Early Numeracy Research Project (Clarke, 2001) collected data at the beginning and end of each year from over 11,000 children in Grades 0-2 from 1999-2001. Of these, 1237 children (672 boys, 565 girls) formed a longitudinal cohort who started Grade 0. A subset of 572 (295 boys, 277 girls) of these children were also assessed at the end of Grade 3 in 2002 and Grade 4 in 2003. The students were assessed individually through a task based interview of about 30-40 minutes duration near the beginning and end of each of their Grade 0-2 years and at the end of the Grade 3-4 years. This interview investigated mathematical growth in the curriculum areas of number, measurement and geometric thinking. Within number there were four separate domains: Counting; Place Value, Addition and Subtraction Strategies; and Multiplication and Division Strategies. Longitudinal data were also collected in measurement in the domains of Time and Length, and in geometry (space) in the domains of Properties of Shape and Visualisation. The geometry domains were not used until 2000 so the data for them cover one less year of development. The focus of the interview was not just on achievement but also on the strategies used. In designing the interview care was taken to have a high ceiling in each domain so that there was an appropriate challenge for each child, but in a particular domain, when the child started to be not successful the interviewer moved to another domain. This meant that the children did not attempt all questions and experienced little failure.

Figure 1 shows an example of questions from near the beginning of the Place Value domain.

11) Calculator Tasks

Get the calculator please. Have you used a calculator before? Please turn it on.

a) Type these numbers on the calculator (7, 47, 60, 15, 724, 105, 2469, 6023)
(*Stop when the child is not successful. Ask the child to clear the calculator between numbers.*)

b) STEP 1: Pick any number from 2 to 9, and type it on the calculator. Read the number. (*Don't clear the calculator.*)

STEP 2: Type in a different number from 2 to 9 (*thus forming a 2-digit number*). Read the number.

STEP 3: Type in a different number again from 2 to 9 (*giving a 3-digit number*). Read the number.

[*Continue to limit of correct answers: STEP 4, STEP 5, etc.*]

Figure 1. Question from early part of Place Value domain

A question from the end of the Place Value domain is shown in Figure 2 to illustrate the ceiling. The full interview schedule is available (Department of Education, Employment & Training, 2001). The questions in the interview were linked to a framework of 4-6 Growth Points for each domain that reflected milestones in a common learning trajectory. For a full description of the development of the framework and the growth points see Clarke, Sullivan, Cheeseman, and Clarke (2000).

17) Interpreting the Number Line

Show the child the orange page of number lines. Point to the first one, pointing to the relevant numbers as you read the question.

[at any stage]: K $\rightarrow$ C

a) The numbers on this line go from 0 to one 100.. . . . (*pointing to the little mark*) Round about what number would this be? (*acceptable range: 55 to 75*).

b) The numbers on this line go from 0 to 2000. . . . (*pointing to the little mark*) Round about what number would this be? (*acceptable range: 400 to 600*).

c) The numbers on this line go from 39 to 172. . . . (*pointing to the little mark*) Round about what number would this be? (*acceptable range: 65 to 95*).

d) The numbers on this line go from zero to one million. . . . (*pointing to the little mark*) Round about what number would this be? (*acceptable range: 700 000 to 800 000*).

Figure 2. Sample questions from the end of the Place Value domain.

ANALYSES, RESULTS AND DISCUSSION

Following the interview, trained personnel assigned each child a growth point for each domain. These growth points did not form an interval scale. Horne and Rowley (2001) describe a statistical approach that was used to create an interval scale from the growth points in each domain. These interval scales are close to the growth points and were set to cover the same range but they allow statistical comparisons to be more easily made as parametric statistics can be used to identify differences.

For each of the eight domains mentioned above, means and standard deviations were calculated separately for the male and female students from the longitudinal cohort for their entry into the project at the beginning of Grade 0, then at the end of that and each subsequent year until the end of Grade 4.

	First three years			Five years		
Domain	**df**	***F***	***p.***	**df**	***F***	***p.***
Counting	1/1235	46.09	0.000	1/570	19.38	0.000
Place Value	1/1235	9.30	0.002	1/570	1.316	0.252
Addition & Subtraction	1/1235	18.83	0.000	1/570	16.172	0.000
Multiplication & Division	1/1234	9.01	0.003	1/570	12.691	0.000
Time	1/1177	0.13	0.721	1/570	1.835	0.176
Length	1/1179	1.30	0.254	1/570	.162	0.688
Properties of Shape	1/2433*	4.58	0.033	1/570	1.352	0.245
Visualisation	1/2449*	1.19	0.277	1/570	.301	0.584

* Since this was only a two year period there was a larger cohort.

Table 1. Analysis of variance for change by gender

An analysis of variance of the growth in each domain (See Table 1) shows that over the first three years, significant differences developed in each of the number domains (p<.01) but generally not in either the measurement or geometry domains. Over the five years with the reduced cohort all the number domains but Place Value showed significant differences (p<0.01) for gender and again measurement and geometry domains showed no differences. The growth points for the domains for Addition and Subtraction Strategies and for Multiplication and Division Strategies are linked to strategies students use in achieving a correct solution to the task. However the domains for Counting and Place Value are not as closely linked to strategies. While clearly solving any problem requires strategies the actual assignment of Growth Points for the number domains using the word "strategies" were directly linked to the nature of the strategies used.

This growing separation of the genders is shown more clearly in the graph in Figure 3 for the counting domain. The two central lines show the means for the boys (squares) and girls (triangles) from the beginning of their Grade 0 year until the end of their Grade 4 year. It illustrates that there were no differences on arrival at school but that there were differences which developed and continued throughout the five years. The other lines on the graph illustrate the spread of the data by showing the boundaries of the band representing 90% of the children. The assessment was specifically designed for Grades 0-2. The ceiling for these grades was high, and because of this it was used at grades 3 and 4 though it is clear that by grade 4 a number of children have reached the ceiling.

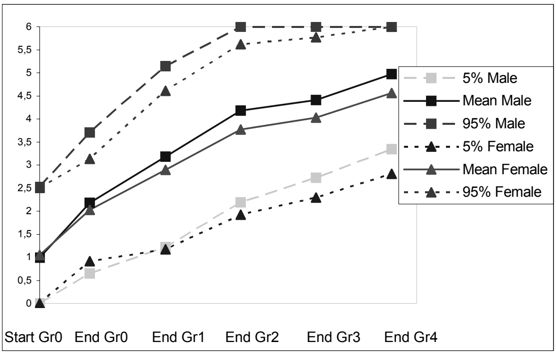

Figure 3. Gender differences in growth in the Counting domain

Figure 4 shows the patterns of growing separation of the genders for the other number domains and the similarity between them. The shape and separation of the two central lines representing the means is the critical aspect of the graphs. The legend and horizontal scale are the same for each of the graphs.

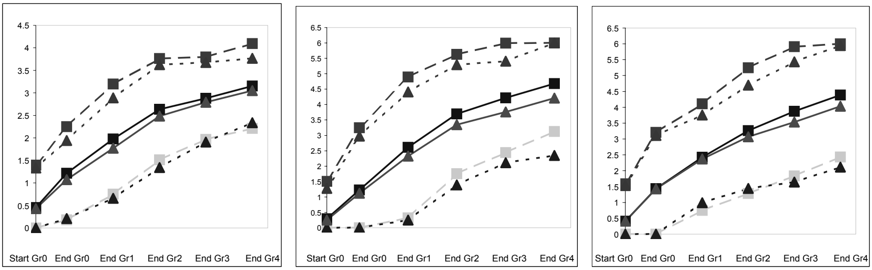

Figure 4. Gender differences in growth in the domains of Place Value, Addition and Subtraction Strategies, and Multiplication and Division Strategies

Of the four domains in Measurement and Geometry there appeared to be the greatest differences in growth in the Properties of Shape domain (Figure 5). This domain was not assessed for the longitudinal cohort when they were in Grade 0 so the data are shown here from the start of Grade 1 to the end of Grade 4. While at the start of Grade 1 the girls were ahead of the boys this was not significant. There were slight differences developed during the four years but the separation did not continue to grow and the change was not significant by the end of grade 4.

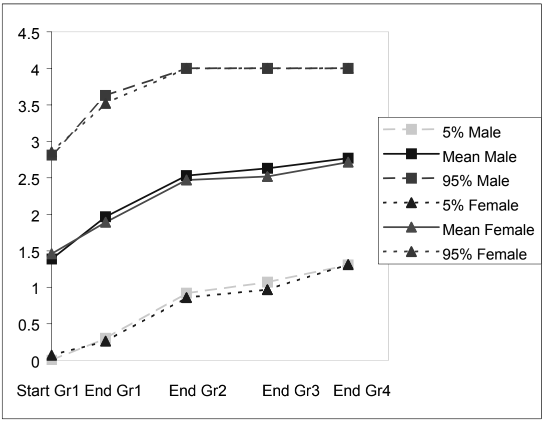

Figure 5. Gender differences in growth in the Properties of Shape domain

What is of interest is that the other measurement and geometry domains show a similar pattern of little to no gender difference, in contrast to the number domains where most of the differences were significant and continued to increase during the five year period. Figure 6 shows the growth graphs of the remaining three domains.

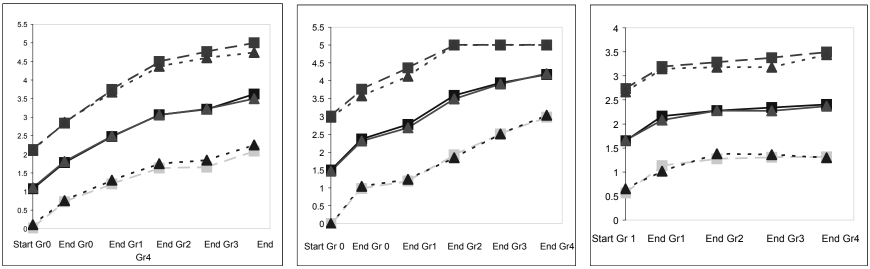

Figure 6. Gender differences in growth in the domains of Time, Length and Visualisation.

Horne (2003) raised the concern that the gender gap in the Addition and Subtraction Strategies domain was increasing as the children became older. It is of concern that other number domains are showing similar patterns of increasing separation. Fennema et al. (1998) also focused particularly on addition and subtraction. The differences showing here in other number areas need to be replicated.

In contrast, the lack of gender difference in measurement and geometry is of interest and supports the general findings on achievement tests in middle primary schools. Owens (1996), commenting on gender and spatial differences, indicated that differences seemed to show in favour of boys when three dimensions were involved. The lack of differences here may be related to the fact that both the geometry domains focused on two dimensions. There has been very little written about gender differences in the area of measurement in recent years. The choice of other domains, such as Area and Volume, may have lead to different results just as the choice of three dimensional examples may have lead to different results in geometry.

Hyde and Jaffee (1998), in reacting to the findings of differences in strategies in addition and subtraction suggest that one explanation could be the stereotyping, either by the teachers or the children themselves, of girls as compliant and boys as resistant to rules and as risk-takers. It could be that in the interview situation the girls, more than the boys, are trying to give the teacher the responses that they perceive the teacher is seeking. There is clearly a need for a closer look at gender differences in approaches in all areas of mathematics at the earliest levels of schooling.

Analysis of data like this raises more questions than it answers. Why are the differences showing in number and not in measurement and geometry? How might the classroom culture and the teacher affect this? Would the same differences show in strategy use between boys and girls in other mathematical areas? Gender is still an important area of research and there are still many unanswered questions.

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PERCEPTUAL AND SYMBOLIC REPRESENTATIONS AS A STARTING POINT OF THE ACQUISITION OF THE DERIVATIVE

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In this paper we study how a student begins to acquire the concept of the derivative, what kind of representations he acquires and how he connects these representations. A teaching period, in which different perceptual and symbolic representation were emphasized, was carried out and task based interviews conducted to five students. The students used several different kinds of representations to process the derivative. They were all able to consider the derivative as an object by using perceptual representations at an early stage of the acquisition process. Also they all could calculate the derivative at a point by using symbolic representations. They all formed perceptual representations of the limiting process of the difference quotient but they would still need guidance to connect them to symbolic representations.

INTRODUCTION

Within the mathematics education research community there has been a lot of discussion about students' conceptions of mathematical concepts and about the development of these conceptions. There has been discussion about process and object conceptions (Asiala & al. 1997, Gray & Tall 2001, Sfard 1991) and about different representations (Goldin 2001 & 1998, Gray & Tall 2001). The concept acquisition process can start from perceptions of objects or from actions on objects (Gray & Tall 2001). Asiala et al. (1997) have presented a genetic decomposition corresponding to the APOS-theory in which they described students' analytical and graphical ways to construct the concept of the derivative. The students whose course was based on this analysis may have had more success than students of traditional courses (Asiala & al. 1997). The same result was found in Repo's research (1996), in which she implemented a calculus course planned on the basis of the APOS-theory and in which different representations of the derivative were emphasized. According to the research of Kendal and Stacey (2000) teacher's emphasis on certain representations of the derivative influence on how students can deal with representations. According to Watson's and Tall's research (2002) students attending teaching based on perceptual representations and process-object development had more success than students attending standard teaching in the subject of the vectors.

This research is part of the author's ongoing work on his PhD thesis, in which students' acquisition process of the derivative is studied. This paper is focused on the beginning of that process. Data was collected by conducting task based interviews after a five-hour teaching period to get information on with which kind of representations a student can start to acquire the derivative. Especially student's perceptual and symbolic representations and connections between them is studied.

THE CONCEPT ACQUISITION PROCESS

A student can make internal representations of a mathematical concept which is presented to him by using external representations. According to Goldin (2001) the internal representation systems can be a) verbal/syntactic, b) imagistic, c) formal notational, d) strategic and heuristic, and e) affective. According to him the study of student's conception and understanding of a concept should focus on studying student's internal representations. This is done by interpreting student's interaction with, discourse about, or production of external representations (ibid. 5-6). A concept is learned when a variety of appropriate internal representations have been developed with functioning relationships among them (ibid. 6).

According to the APOS-theory the student constructs a mathematical concept so that an action performed to an object is interiorized to a process which then encapsulates to an object. A schema is a collection of processes, objects and other schemas. (Asiala & al. 1997.) According to Gray and Tall (2001) the concept acquisition can start by an action performed on an object, but also by making a perception of an object. Gray and Tall call this kind of perceived objects embodied objects. The embodied objects are mental constructs of perceived reality, and through reflection and discourse they can become more abstract constructs, which do not anymore refer to specific objects in the real world (Gray & Tall 2001). Hence student's conception can start to develop from perceptual or from symbolic representations, and it is important to connect these representations. The conception of a mathematical object, formed by encapsulation, already has a primitive existence as an embodied object (Gray & Tall 2001).

In Goldin's classification verbal/syntactic and formal notational representations are symbolic representations and imagistic representations are perceptual representations. According to Goldin (1998, 156) representation systems are proposed to develop through three stages, so that first, new signs are taken to symbolize aspects of a previously established system of representation. Then the structure of the new representation system develops in the old system and finally the new system becomes autonomous.

THE TEACHING PERIOD OF THE DERIVATIVE

In the five-hour teaching period, planned according to theoretical framework, the derivative was introduced by using different representations and open approach. At first, the rate of change of the function was perceived from the graph. Moving a hand along the curve, placing a pencil as a tangent, looking how steep the graph was and the local straightness of the graph were used as perceptual representations. Then, the average rate of change was calculated by difference quotient and as the slope of the secant. After that the students were given the following problem: How to determine the instantaneous rate of change at a certain point? Finally, the derivative was defined as the limit of the difference quotient.

THE RESEARCH METHODS AND DATA COLLECTION

The teaching period was carried out by the author in the autumn of 2003 as a part of a Finnish high school course "Differentiaalilaskenta 1". There were 14 about 17-year old students in the course. The data was collected by a pretest, by videotaping the lessons and by conducting videotaped task based interviews to five students. The students were directed to think their solutions aloud. In five tasks they were asked to tell in their own words what the derivative is, make observations of the derivative of the function from its graph (Fig. 1), estimate the derivative of the function 2^x at the point $x = 1$, interpret the form (new to them) of the difference quotient and the limit of it from the graph of an unknown function, sketch the graph of the distance and acceleration from the graph of the velocity, and determine the average and instantaneous accelerations from the graph of the velocity.

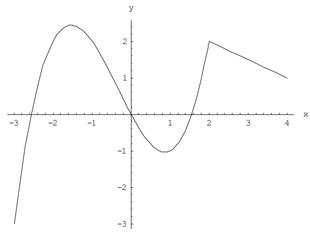

Figure 1: The given graph of function

The interviews were analyzed by using the constant comparative method to find the representations that the students used. After that it was analyzed how each student had connected the representations together and how strong each representation was. Special attention was paid to perceptual and symbolic representations. When appropriate it was analyzed whether the representation used was an action, a process or an object.

Interviews of the subjects Tommi and Niina were carried out right after the teaching period. Samuel was interviewed one, Susanna three and Daniel five lessons after the teaching period. Under that time the teacher of the course continued with the concept of the derivative function and with differentiation rules.

Based on their success on mathematics before the course the students could be classified so that Niina and Susanna are weak (w), Tommi and Samuel are average (a) and Daniel is good (g). In the pretest all these students could determine the average velocity from the graph of the distance, but only Daniel estimated the instantaneous velocity. They all, except Susanna, determined correctly also the sign of the velocity. Niina and Susanna had some difficulties with functions and they could not draw a tangent. The other three could also draw a tangent of slope zero and, except Samuel, determine the slope of a general tangent. Only Daniel could interpret the difference quotient as the slope of a secant and estimate how it changes when the base interval decreases.

THE STUDENT'S REPRESENTATIONS OF THE DERIVATIVE

All interviewed students were able to consider the derivative qualitatively as an object by using perceptual representations and quantitatively by using symbolic representations. For example they could determine the sign of the derivative from the graph of the function and, except Niina, estimate the derivative of the function 2^x at the point $x = 1$. Table 1 summarizes the most common representations used by the student and classifies them as actions, processes or objects.

Representations		**Niina** (w)	**Susanna** (w)	**Tommi** (a)	**Samuel** (a)	**Daniel** (g)
Perceptual	Tangent	Object	Object	Object	Object	Object
	Rate of change	Process	Process / Object	Object and symbolic process	Verbal	
	Steepness	Object	Object		Object	Object
	Concrete objects or processes		Pencil as a tangent	Pencil as a tangent	Ruler as a tangent	Tangents in the air
	Limiting process	Local straight-ness	Secants approach tangent	Average rate of change over a small interval	Secants approach tangent	Average derivative and secants
Symbolic	Limit of the difference quotient	Action	Action	Process	Process	Action
	Slope of the tangent	Process	Process	Process	Verbal	Process
	Differentiation rule		Action			Process

Table 1: The student's most common representations of the derivative

The perceptual representations of the derivative

All the students had an imagistic representation of the tangent as an object. This was especially strong for Tommi, Samuel and Daniel. Niina, Susanna and Tommi had also the rate of change as an important representation. Tommi explained that when the rate of change is three, it means instantaneously that "if we move one step forward, we must go three steps upward". It seems that Tommi has understood the rate of change as a perceptual object and as a symbolic process which is closely connected to the process of defining the slope of the tangent. He understood also that acceleration is "the rate of change of the velocity". Niina's and Susanna's representations were a bit weaker and Samuel's only verbal and not connected to the other representations.

As the most concrete representations Susanna, Tommi and Samuel used pencil or ruler as a tangent to embody the derivative. In addition, Daniel used tangents freely drawn in the air. They used these concrete representations only in individual cases when arguing or examining problematic points. So obviously these representations were connected closely to other perceptual representations which are more abstract and with them the derivative can be examined mentally. Only in some cases they needed to use concrete representations. For example Samuel argued his observations of the derivative made from the graph of the function by placing the ruler as a tangent: “because the tangent goes like this”. Tommi’s pencil-representation seemed to be connected to the rate of change. When trying to sketch the acceleration from the graph of the velocity, he moved pencil as a tangent along the graph and when asked explained: “If it points up, then accelerating.”

All the students, except Tommi, used the imagistic representation of the steepness of the graph when making observations of the derivative of the function from its graph (Fig. 1). They used this representation especially when defining the point where the derivative is at the largest and where at the smallest. Probably the extreme values are easy to examine with this representation. For example Susanna explained, that the derivative is at greatest, when “the graph rises most steeply” and she used the pencil as a tangent to find this point. On the other hand, the point where the derivative is at the smallest was more difficult to find:

> “Where it goes most steeply downward, hmm (places pencil as a tangent). Somewhere hmm. Its a bit difficult to look, but maybe somewhere there (points the graph approximately at a point 0,8).”

Also Niina failed to determine this point and proposed the same point as Susanna. It seems that the negative rate of change as a more abstract concept causes difficulties. Niina and Susanna seemed to consider the rate of change more as a process of change than as an object of one point. Niina was the only student who demonstrated the representation of the local straightness when arguing why the derivative is not zero at the point where there is an angle in the graph (Fig. 1):

> “If you would zoom in on here (x = 0,8) for example, it would be straight for a while (draws a line with a finger), but not there (points at the angle, x = 2)”

However, all the interviewed students determined correctly from the graph of the function (Fig. 1) the points where the derivative is zero, the sign of the derivative, the points where the derivative is at the greatest and the interval where the derivative is constant. Tommi, Daniel and Samuel determined correctly also the point where the derivative is not defined and the point where it is at the smallest. In addition, Tommi and Daniel made observations correctly about the rate of change of the derivative (that is the sign of the second derivative which was not taught). Daniel made his observations of the derivative from right to left:

> “When we start to go forward down here (from x = 2 to x = 0,8), it (the derivative) is all the time actually increasing, since it’s steepest there (x = 2). No, it decreases, because it’s positive there. It goes here (x = 0,8), down here it’s zero. Then here it becomes negative, we are going upward. It starts to increase again somewhere in the middle (x = -0,4) and there (x = 2,5) it’s zero again.”

All the students except Susanna sketched almost correctly the graph of the distance from the graph of the velocity. Incorrectly Niina’s graph was formed from line segments and Daniel’s graph was steepest at a wrong point. Tommi, Samuel and Daniel sketched also the graph of the acceleration correctly.

The symbolic representations of the derivative

All the students except Samuel had interiorized the process of determining the slope of the tangent to a symbolic process and this was connected to the tangent as a perceived object. For example Niina explained that the value three of the derivative means the slope of the tangent which is calculated so that "the change in y is divided by the change in x, and you'll get three". On the other hand, Samuel had only a verbal representation for the slope of the tangent, since he could determine the slope only by using the derivative. Apparently it would be easy to guide Samuel to acquire the missing process, since he already could determine the slope by the limit of the difference quotient, connect difference quotient to the secant and could determine the average acceleration from the graph of the velocity.

Only Tommi and Samuel had interiorized the limit of the difference quotient as a process and were able to describe this process without performing it. When trying to estimate the derivative of the function 2^x at the point $x = 1$ Samuel first tried to determine the limit of the difference quotient, but did not figure out how to simplify the quotient. Since he could not determine the exact value, he estimated it by calculating the difference quotients over the intervals [0,9; 1], [0,99; 1] and [0,999; 1]. Samuel has connected this symbolic process to the perceptual process of secants approaching the tangent:

Interviewer:	What do these (difference quotients) tell about the function? If this is the derivative and these aren't quite the derivative, what do they mean?
Samuel:	(Draws a graph and a tangent.) It would really be that. (Draws the secants approaching the tangent.) They approach constantly the correct derivative.
Interviewer:	Ok. Ok. Do you have more to say about that?
Samuel:	No, or well, that this is because you can't substitute one here, because it would be zero here, but you can put it however close to... mm close to one, but not however one, then there will be no zero and you can calculate this and this is why it approaches.

The other students were only at the action level in determining the limit of the difference quotient and they only tried to remember the formula. However, Daniel was able to interpret the form (new to him) of the difference quotient and the limit of it from the graph of the unknown function by using his representations of the slope of the line and the "average derivative": "It would be the slope of that line (secant), that is average, how to say it, average derivative at that interval." Also Niina and Susanna tried to interpret, but did not proceed very well. In order to really understand the limit of the difference quotient, one should also have some other representations than only the symbolic representation (like Samuel and Daniel have). Tommi instead had interiorized the limit of the difference quotient to a process, but he had not connected it to the perceptual representations. When trying to estimate the derivative of the function 2^x at the point $x = 1$, he had the following representation of the limiting process:

> "You could calculate the average rate of change of the function for example at points 1,1 and 0,9 and continue to approach 1. Finally it would become very close to that correct one. – – I don't remember at all how it's calculated."

Tommi could make the missing connection by combining the perceptual representations of the average rate of change and the slope of the secant to the symbolic representations of them and to the difference quotient. He almost did this when trying to interpret the form of the difference quotient (new to him):

> "Could it then be the rate of change over that interval. (Draws a secant.) No, it is like the average rate of change over that interval."

Daniel should instead practice to determine the limit of the difference quotient, so that this representation would become stronger and interiorize to a process. Niina and Susanna should both connect the limit of the difference quotient to the other representations and try to interiorize it to a process. Susanna had a perceptual representation of the secants approaching the tangent and Niina the local straightness of the graph which could be connected to the limit of the difference quotient.

Susanna and Daniel were the only students to whom the differentiation rule of the function x^n had been taught. They had a very strong representation of it and used it as the first method to determine the derivative even when it was not appropriate. Susanna was easily guided to use her perceptual representations to notice that she had not applied the differentiation rule correctly to the function 2^x:

> Susanna: Actually the derivative of 2 would be zero. Would this be then zero?
>
> Interviewer: How could you figure this out?
>
> (Susanna draws under the guidance the graph on paper and also with a calculator.)
>
> Susanna: Actually it can't be zero, since it's after all increasing at point one.

It seems that students like Niina and Susanna could base their learning of the derivative to the perceptual representations if they do not succeed in connecting the limit of the difference quotient to the perceptual representations.

CONCLUSIONS

All the interviewed students seemed to be able to begin their concept acquisition of the derivative by developing different perceptual representations. By using these perceptual representations they could understand the derivative as an object. At the beginning of the course they examined the derivative qualitatively at the level which is the goal in terms of differentiation rules and sign considerations at the end of the course. Especially well the perceptual representations seem to suit for beginning to develop understanding of the relation between function and its derivative function.

Students may have very different representations and as Kendal and Stacey (2000) stated, representations which are emphasized in the teaching influence on the construction of students' internal representations. Usually the most important perceptual representations are considered to be the slope of the tangent and the rate of change, but one should consider also emphasizing some other representations. For example, steepness of the graph could be used to examine the derivative as an object and local straightness to demonstrate the limit of the difference quotient.

Perceptual representation of the tangent was usually connected to the symbolic process of determining the slope of the tangent. Instead, it was difficult to understand the limit of the difference quotient by using other representations though all the students had some kind of perceptual representation which could be used for this. Apparently students would need some individual guidance in this acquisition process. As Gray and Tall (2001) underline it's very important to connect perceptual representations to symbolic representations. Because it seems to be easy to deal with the derivative with perceptual representations, one should consider in how these could be used along the course for example to intuitively derive differentiation rules.

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IMPLEMENTATION OF A MODEL USING AUTHENTIC INVESTIGATIVE ACTIVITIES FOR TEACHING RATIO & PROPORTION IN PRE-SERVICE TEACHER EDUCATION

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Abstract - *In this research we examined and assessed the impact of implementation of a model using authentic investigative activities for teaching ratio and proportion in pre-service teacher education. The model was developed following pilot studies investigating the change in mathematical and pedagogical knowledge of pre- and in-service mathematics teachers due to experience in authentic proportional reasoning activities. The conclusion of the study is that application of the model, incorporating theory and practice leads to a dramatic positive change in the pre-service teachers' content and pedagogical knowledge. In addition, an improvement occurred in their attitudes and beliefs towards learning and teaching mathematics in general, and ratio & proportion in particular.*

THEORETICAL BACKGROUND

Proportional reasoning is at the heart of mathematics in the upper grades of the elementary schools and in the middle schools. In principle, proportional reasoning deals with mathematical relations, which are multiplicative in nature, in contrast to additive mathematical relations, that are typical for many young children in elementary schools (see Harrel and Confrey, 1994). According to the NCTM curriculum and Evaluation Standards (1989), "The ability to reason proportionally develops in students throughout grades 5-8. It is of such great importance that it merits whatever time and efforts that must be expended to assure its careful development." (p.82). Therefore, the topics of ratio and proportion should have central part in mathematics curriculum for children in school as well as for pre-service mathematics teacher education.

Nevertheless, research has consistently shown that "…relatively few junior high students of average ability use proportional reasoning in a consistent fashion" (Post, Behr, and Lesh, 1988, p.78), or successfully cope with it (Ben-Chaim et al., 1998). The topic even "remains problematic for many college students" (Lawton, 1993), and "There is evidence that a large segment of our society never acquires fluency in proportional thinking." (Hofer, 1988, p.285). Furthermore, recent research findings all over the world have indicated many gaps in the content knowledge of pre- and in-service teachers in mathematical subjects taught in elementary and middle schools, including the topics of ratio & proportion. Frequently, existing knowledge is technical, schematic, unconnected, and incoherent. As a result, difficulties arise which are evidence of the pre-service teachers' lack of understanding of mathematical concepts, including ration & proportion, and feeling that they are incapable of both, coping with the material, and teaching it (Keret 1999; Fischbein et al., 1994; Ben-Chaim, Ilany, and Keret, 2002).

In a pilot study by Ben-Chaim, Ilany, and Keret (2001), nineteen activities were developed and conducted in order to assess the influence of exposing pre-service mathematics elementary teachers to authentic investigative proportional reasoning activities. Many of the activities are in the appropriate form for presentation to children in the upper classes of elementary school and in the classes of middle school.

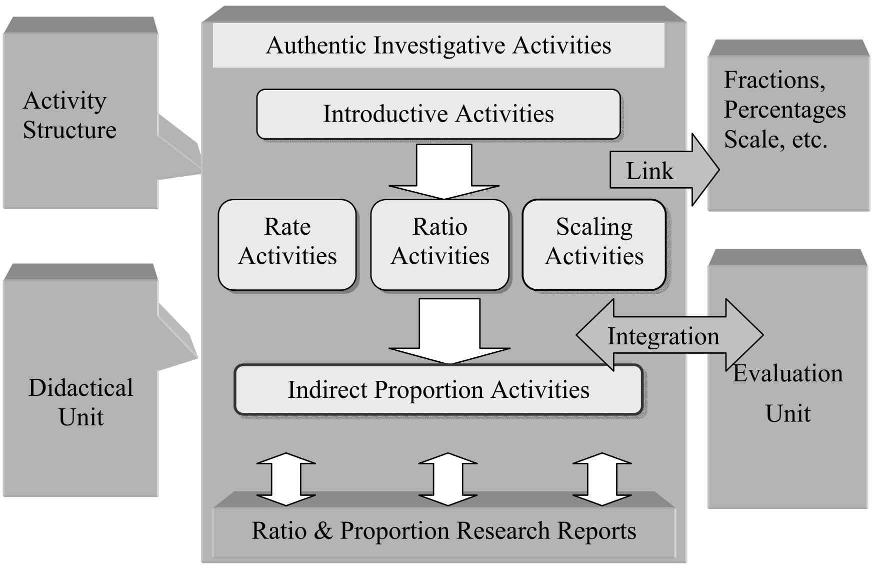
The activities include mathematical tasks, that require quantitative and qualitative numerical comparisons between ratios and finding a missing value. The tasks involve small and large integer numbers, fractions, decimals, and percents. The activities establish the understanding of many concepts related to ratio & proportion topics and are focused on the three main categories of proportional reasoning problems: Rate and Density, Ratio, and Scaling (see examples in figure 2).

The findings of the pilot study showed that after exposing and experiencing with authentic ratio & proportion investigative activities, the pre-service teachers were more successful in solving ratio & proportion problems, exhibited different strategies for solving the problems, and were more capable of providing a good quality of written and oral explanations (during the interviews) to their work. In addition, they improved their attitudes toward mathematics in general and all the components and aspects of ratio & proportion in particular (Ben-Chaim et al., 2002). A replication of this study with several classes of pre-service teachers in 3 different colleges showed similar findings. From these research findings, the conclusion was that it is necessary to teach the ratio & proportion topics in pre- and in-service mathematics elementary and junior teacher education, by applying authentic investigative activities, such as those developed and conducted through the pilot study. As a result, a special model for teaching the ratio & proportion topics in mathematics teacher education was developed.

The model incorporates the main areas in the training of pre-service teachers, particularly in the areas of content knowledge, pedagogical content knowledge, pedagogical reasoning, training, and beliefs. The model is comprised of 4 components with interaction between them (see figure 1). The first component - the core of the model includes the authentic investigative activities with 5 types of activities: Introductive activities, investigative activities dealing with ratio, dealing with rate, dealing with scaling, and dealing with indirect proportion. For each activity there is a basic part, to build the basic knowledge needed regarding ratio & proportion. Then, there is an extension part to further develop understanding (see figure 2).

In parallel to the engagement in the investigative activities, the pre-service teachers are referred to articles, dealing with the topics of ratio & proportion. The articles are related to the mathematical as well as to the didactical-pedagogical aspects of the topics. The analysis of the research articles, enables the students to be aware of their mathematical knowledge, while the presentation of the research findings might lead to a wide perspective and deeper discussion regarding suitable teaching strategies for teaching the subject. In this case, the intention was to create an integration between theory and practice (Greeno, 1991,1994), or as Lienhardt et al. (1995) notation of "…knowledge learned in the academy vs. in practice". They claim that professional knowledge can vary by the location of the learning (in the academy or in practice), the type of knowledge (declarative or procedural), the generality of knowledge (abstract or specific), and the nature of principles (conceptual or pragmatic).

Figure 1: A Model Using Authentic Investigative Activities For Teaching Ratio & Proportion



The second component includes the structure of the activities. They are structured as authentic investigative problems related to content and context familiar to the prospective teachers and to children in the elementary and junior level. In general the activities include types of tasks reported in the professional literature as appropriate for assessment for proportional reasoning (Cramer, Post, and Currier, 1993): (a) Missing value problems, where three pieces of information are given and the task is to find the fourth or missing piece of information; (b) Numerical comparison problems, where two complete ratios/rates are given and a numerical answer is not required, but the ratios or rates are to be compared; (c) Qualitative prediction and comparison problems which require comparisons not dependent on specific numerical values.

The third component includes the structure of the didactical unit. It includes a unit around a concept; for example: A didactical unit that its role to impart the concept of ratio. The structure of the didactical unit includes: (a) Working by groups; (b) Discussion of results with the whole class; (c) Mathematical and didactical summary; and (d) Homework.

The fourth component includes the evaluation unit with several types of assessment instruments: (a) An instrument to evaluate student's mathematical knowledge, and mathematical didactical knowledge; (b) Attitude questionnaire; (c) Portfolio - An instrument for alternative assessment related to learning processes of pre-service prospective teachers during their studies; and (d) An instrument for assessing research reports during and after the course studies. For a detailed description of the model see Keret, Ben-Chaim, and Ilany (2003).

The main goal of this paper is to report the results of an implementation of this model, in a pre-service teacher education course, in an Israeli teacher college, during the second semester of the academic year 2002/2003.

METHODOLOGY

Sample: 11 Pre-service teachers from the Israeli Teacher College as part of their training to teach mathematics in elementary and middle school.

Research instruments: In addition to the model, the activities and the theoretical material (articles and research reports), the research instruments included:

1) A proportional reasoning questionnaire that was administered twice: First at the outset of the course and second at the end of the semester (14 weekly teaching sessions, each of 90 minutes). The questionnaire included 5 rate and density problems, 5 ratio problems, and 6 scaling problems. In each problem, the participants were asked to provide support work by giving reasons for their answers. For a full description of this instrument, see Ben-Chaim et al., (2002).
2) Attitude questionnaire, that includes 22 items and 3 open questions. The questionnaire was administered twice before and after the instruction. The items of the attitude questionnaire belong to four categories as presented in Table 4. For more details see Ben-Chaim et al. (2002).
3) Observations that were aimed at the formative evaluation of the model, and at the follow up of the pre-service teachers' procedures and change in behavior.

4) Personal interviews that were aimed at in-depth examination of the participants opinions on the impact of the course. For this purpose, a representative sample of 3 participants were interviewed.

The instructor of the course was one of the authors (Dr. Ilany). She implemented the model including all of its components using the teaching method of inquiry.

RESULTS AND DISCUSSION

The results of the quantitative analysis of the participants performance on the proportional reasoning questionnaire are presented in Table 1. The results indicate a very significant progress from pre- to post-test: 45% to 90%. The range on the pre-test was 24%-74% and on the post-test 80%-100%. On each one of the subtests: Rate, Ratio, and Scaling the improvement was remarkable, 69% to 97%, 30% to 88%, and 36% to 86% accordingly. Even though the participants in this study started with lower achievement than in the previous study (Ben-Chaim et al., 2002), their improvement was much better. Another indication is a better quality of the explanations provided by the participants in this study, for their methods to solve the problems, especially after instruction.

Table 1: Overall Pre-Post Results on 16 Problems (5 Rate, 5 Ratio, and 6 Scaling), N=11, Pre-service Mathematics Teachers

	Rate	Ratio	Scaling	Total
Pre	69%	30%	36%	45%
Post	97%	88%	86%	90%

The attitude questionnaire included items dealing with attitudes toward: Mathematics teaching in general, confidence in the ability to deal with ratio & proportion, difficulties in teaching ratio & proportion, and the importance of teaching ratio & proportion. Table 2 presents the pre-post results, that are quite similar to those presented and discussed in the previous study (Ben-Chaim et al., 2002). The data in Table 2 indicates that the pre-service teachers of this study improved their attitude toward teaching mathematics in general, even though they started with quite a positive level. In addition, they were much more confident in their ability to deal with ratio & proportion at the end of the course.

Nevertheless there is a difference which is noticeable by the responses to the three open questions at the end of the attitude questionnaire, asking them to mention a situation of ratio, a situation of proportion, and to mention concepts and words that are related to the topics of ratio & proportion. In most cases, in the pre-test the pre-service teachers wrote: "I don't remember", or "I don't know", especially for proportion.

Figure 2a: Investigative Problem 1: Preference of Cola

BOLA-COLA OR COLA-NOLA?

The following results are related to surveys of preference between BOLA-COLA and COLA-NOLA:

A. The ratio of those who preferred BOLA-COLA than COLA-NOLA is 3 to 2.
B. The numbers of those who preferred BOLA-COLA than COLA-NOLA are in ratio of 17,139 to 11,426.
C. 5,713 more participants preferred BOLA-COLA than COLA-NOLA.

1. Decide if the above three statements are necessarily extracted from the same survey? Explain!
2. Which statement describes most accurately the results of comparison between BOLA-COLA and COLA-NOLA? Explain!
3. If you need to advertise the results, which statement seems to be the most effective for advertisement? Why?
4. Suggest other possible ways for comparison between the popularity results of the two kinds of cola.

Figure 2b: Investigative Problem 2: Sharing Pizza

Every month Danny's friends meet at a restaurant for a pizza party. Danny as usual is late, but his friends like him a lot and they are waiting for him to come. They reserved for him a seat in each of the two tables 1 and 2. Finally he arrived and then he had to decide where to sit: should he join his friends at Table 1 in which there are 4 large pizzas and 9 people or Table 2 in which there are 3 large pizzas and 7 people?

Table 1

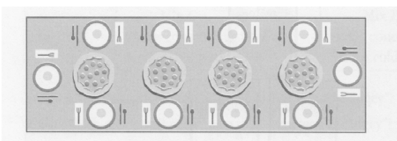
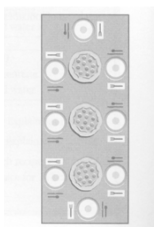


Table 2



1. What is your suggestion? Which table should Danny choose? Explain your reasoning.
2. The ratio of large tables (Table 1 with 10 seats) to small tables (Table 2 with 8 seats) in the restaurant is 9 to 5. There are exactly enough seats for 390 people. How many tables of each kind are in the restaurant?

On the post-test, every one of the 11 participants provided several correct examples for ratio & proportion situations with indication of proper related concepts and words, and even using mathematical notations such as $a/b = c/d$, $a,b,c,d \neq 0$.

Table 2 : Summary of Attitudes Toward Ratio and Proportion Scale (1-5)

Number of items		Mean Before Instruction N = 11	Mean After Instruction N = 11
4	Items relating to attitudes toward teaching mathematics in general.	4.46	4.72
7	Items relating to confidence in ability to deal with ratio and proportion.	2.82	3.85
3	Items relating to attitudes toward difficulties in teaching ratio and proportion.	3.26	3.21
6	Items relating to attitudes toward the importance of teaching ratio and proportion.	4.65	4.40

The in-class observations and the in-depth interviews with the three representative participants, strongly supported the findings, suggesting that an implementation of the model with all of its components leads not only to acquiring mathematical content knowledge and pedagogical-didactical knowledge, but also, to a profound change in the pre-service teachers' opinions on teaching and learning mathematics. For example, the interviewees indicated that they liked the strategy of "not just to provide the answer, because it is possible to get there by a wrong procedure, hence one needs to add verbal explanations", or "I liked the strategy of inquiry, challenges, cooperation with colleagues, the activities were very interesting and can be presented to school children, the theoretical materials (articles and research reports) were very helpful", or "The activities, the theoretical material, and the guidance during the teaching sessions helped me to move from technical knowledge to a knowledge of understanding". It is interesting to note that most of the participants felt that the course should be longer, since they were not confident enough in their knowledge.

In summary, the findings of this study prove that the implementation of the proposed model is appropriate for application in pre-service mathematics teacher education.

During our presentation, the full model will be introduced and discussed; in addition, more data will be presented and the implication of the implementation of the model will be drawn.

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MATHEMATICIANS AND THE SELECTION TASK

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Learning to think logically and present ideas in a logical fashion has always been considered a central part of becoming a mathematician. In this paper we compare the performance of three groups: mathematics undergraduates, mathematics staff and history undergraduates (representative of a 'general population'). These groups were asked to solve Wason's selection task, a seemingly straightforward logical problem. Given the assumption that logic plays a major role in mathematics, the results were surprising: less than a third of students and less than half of staff gave the correct answer. Moreover, mathematicians seem to make different mistakes from the most common mistake noted in the literature. The implications of these results for our understanding of mathematical thought are discussed with reference to the role of error checking.

LOGIC IN MATHEMATICS

Learning to think logically appears to be at the heart of almost every university level mathematics course. Stewart & Tall, for example, explain that

> everyday language is full of generalities which are vaguely true in most cases, but perhaps not all. Mathematical proof is made of sterner stuff. No such generalities are allowed: all the statements involved must be clearly true or false [... we must] be sure that our mathematical logic is flawless. (Stewart & Tall 1977, p.110)

The mathematics education literature agrees, Devlin, for example, notes that

> the ability to construct and follow fairly long causal chains [and] a step by step logical argument [...] is fundamental to mathematics. (Devlin 2001, p.15)

Previous work on logic in the mathematics education literature has largely concentrated upon schoolchildren. Hoyles & Küchemann (2002), for example, found that even high achieving Year 9 students often "failed to appreciate how data can properly be used to support a conclusion as to whether $P \Rightarrow Q$ is true or not" (p. 217). This finding mirrors similar results from experiments conducted on the general population (e.g. Oakhill, Johnson-Laird & Garnham 1989). Despite these results, the assumption that the ability to use logic is an essential ingredient in becoming a successful mathematician has remained unchallenged. Perhaps the most famous experiment that demonstrated the lack of logical thought in the general population was conducted by the psychologist Peter Wason.

THE SELECTION TASK

The 'selection task' was first reported by Wason (1968). His experimental setup was elegant and deceptively simple. Participating subjects were shown a selection of cards, each of which had a letter on one side and a number on the other. Four cards were placed on a table:

The participants were given the following instructions:

> Here is a rule: "every card that has a D on one side has a 3 on the other." Your task is to select all those cards, but only those cards, which you would have to turn over in order to discover whether or not the rule has been violated.

The correct answer is to pick the D card and the 7 card, but across a wide range of published literature only around 10% of the general population do. Instead most – Wason (1968) suggested about 65% – incorrectly select the 3 card.

The selection task has spawned a phenomenal number of investigations in the psychological literature that have closely replicated Wason's findings in these abstract, non-deontic settings, but which have given a wide range of different explanations for the results. They include confirmation bias, matching bias, Bayesian optimal data selection and relevance theory (see Sperber, Cara & Girotto (1995) for a review). It has even been suggested that the fundamental issue was the 'defectively' educated participants (Bringsjord, Noel & Bringsjord 1998).

This paper does not explore these theories in any depth. Instead we merely note that no explanation is generally accepted, and that the reasons behind Wason's results remain unclear. However, given that no existing study has noted a significant difference in performance between those of differing subject backgrounds (and none has investigated mathematicians' performance), we note that the main theories explain the *uniform* poor performance.

The goal of the current study was to compare the performance of mathematicians and non-mathematicians on the task. If the received view of logic's place in mathematical thought is accurate, one might expect the mathematics undergraduates to perform significantly better than the general population, and the mathematicians to perform nearly flawlessly.

METHODOLOGY

In order to maximise the sample size available to us, we used an internet based survey. There were three categories of participant: mathematics undergraduates, mathematics academic staff and history undergraduates. All were from the University of Warwick. The historians were selected as representatives of the general population, as it was assumed their degrees would have little or no explicit teaching of logic. It is worth noting that this is in keeping with the practice of other researchers in the field: in many studies the population consists of psychology or other undergraduates. We are not aware of any studies conducted with more representative samples of the general population.

E-mails were sent to all members of the stated populations at Warwick, asking them to participate. If they agreed, they accessed a website which contained the following instructions:

> Four cards are placed on a table in front of you. Each card has a letter on one side and a number on the other.
>
> You can see:

> Here is a rule: *"every card that has a D on one side has a 3 on the other."*
>
> Your task is to select all those cards, but only those cards, which you would have to turn over in order to discover whether or not the rule has been violated.

This wording is identical to that used by Wason (1969). Once the subjects had submitted their answers, the webpage recorded five pieces of data: the subject's answer, whether or not they had seen the task before, which group the subject was from, the time, and their IP address. The answers from people who had seen the task before were deleted – very few (<2.5%) fell into this category. Participation rates were high: 260 maths students (34% of the whole population), 21 maths staff (24%) and 123 history students (23%) took part. These figures are particularly impressive when compared to the limited sample sizes available to Wason (1968, 1969) and other pre-web experimenters.

Using the internet to conduct research brings problems as well as benefits. The experimenter has severely limited control on the conditions the subject took part in. Whether they were in a quiet office or a busy internet café is uncertain. Perhaps the biggest problem with the method, however, is that of multiple submissions. There is no foolproof way of preventing subjects submitting their answers more than once. We followed Reips (2000) who suggested logging the IP address of the subject. This isn't a watertight method; often users have dynamic addresses – each time they go online they are assigned a different one. But by logging the time and the IP address of subjects, it was possible to catch those who resubmitted in quick succession (there seemed to be only one case of this, and his or her answers were deleted).

In the end the main defence against resubmissions is simply that subjects have no incentive to do so, it offers them nothing and, to avoid being caught by the IP address log, it is very costly in terms of time. Indeed, one experiment (in the days when dynamic IP addresses were rare) put the resubmission figure at 0.5% of total submissions (Reips 2000, p.105). In another study, Krantz & Dalal (2000) compared the results of twenty internet based surveys with their laboratory counterparts and

	Maths Students		Maths Staff		History Students	
D	92	35%	5	24%	27	22%
DK	1	0%	0	0%	0	0%
D3	15	6%	1	5%	41	33%
D7	76	29%	9	43%	10	8%
DK3	0	0%	1	5%	2	2%
DK7	34	13%	3	14%	1	1%
D37	8	3%	2	10%	8	7%
DK37	21	8%	0	0%	23	19%
non-D	13	5%	0	0%	11	9%
n	260		21		123	

Table 1: Answer Selections.

found a remarkable degree of congruence between the two methodologies. As a result of these factors, it seems clear that the benefits of using the web in this piece of research substantially outweigh the disadvantages.

RESULTS

The results of the study are shown in table 1.

The results for the historians are distributed in a similar fashion to those from Wason's (1968) original research on the 'general population'. This fact can only boost confidence in the methodology that we used.

It can clearly be seen that the maths students do indeed perform significantly differently to the history students (χ^2=95.9, p<0.001). However, although the mathematics students have a significantly higher success rate (χ^2=20.8, p<0.001), they still don't perform at all well. Less than a third of students – and less than half of staff – managed to identify the correct answer. Interestingly, a χ^2 test does not reveal a significant difference between the performance of the mathematicians and that of the mathematics students (χ^2=1.21, 5% level $\approx$ 3.8), though this may be due to the small numbers of maths staff taking part.

Looking carefully at the results reveals that not only did the mathematicians perform better than the non-mathematicians, but that they seemed to make *different* errors. This result is easy to see when the number picking each selection is expressed as a percentage of their group's incorrect answers only (see table 2). The history sample followed the pattern set by previous work: those that failed to choose the correct answer tended to pick D3, D or DK37. Those in the mathematics sample

	Maths Students	Maths Staff	History Students
D	50%	42%	24%
DK	1%	0%	0%
D3	8%	8%	36%
D7	-	-	-
DK3	0%	5%	2%
DK7	18%	25%	1%
D37	4%	17%	7%
DK37	11%	0%	20%
non-D	7%	0%	10%

Table 2: Incorrect Selections.

(both staff and students) who failed to find the correct answer were much more likely to select the D card on its own.

In the Wason Selection Task, the choice of each card corresponds to one of four logical inferences or common fallacies. Given the statement *every card that has a D on one side has a 3 on the other* (corresponding to $P \Rightarrow Q$), choosing the D card (corresponding to *P*) in the expectation of 3 (*Q*) on the other side suggests an appreciation of modus ponens. Choosing the K card (not-*P*) in the expectation of something other than a 3 (not-*Q*) suggests the fallacy of denying the antecedent. Choosing the 3 card (*Q*) in the expectation of a D (*P*) suggests the fallacy of affirming the consequent and choosing the 7 card (not-*Q*) in the expectation of something other than a D (not-*P*) suggests an appreciation of modus tollens. As well as comparing the frequency of each selection of cards, the results can be analysed in terms of the suggested inferential appreciations or fallacies (see table 3).

	Maths Students	Maths Staff	History Students	Inference or fallacy
D	95	100	91	modus ponens
K	25	19	26	denying the antecedent
3	20	19	62	affirming the consequent
7	57	67	40	modus tollens
n	260	21	123	

Table 3: The percentage of each group selecting each card.

The differences between the populations in their ability to recognise the relevance of the 3 card (and therefore their awareness of the logical fallacy of affirming the consequent) are stark. Nearly two thirds of historians selected it, whereas only a fifth of mathematicians thought it necessary. The other main difference between the groups was in recognising the validity of the modus tollens argument (this is the logical form of a contrapositive argument).

So, while it appears that mathematicians are significantly better at the selection task than non-mathematicians, from the point of view of their experience of learning and using logic, their performance is remarkably poor. Less than a third of maths students – and half of maths staff – answered correctly. These findings are somewhat surprising; no existing theory of performance on the selection task would seem to explain them. Our results thus raise two important questions:

- What are the features of mathematical cognition that allow mathematicians to perform significantly better than the general population? Why are they much less likely to make the standard mistake of selecting D and 3?

- What accounts for the unexpectedly poor performance of the mathematicians? If the role of logic in mathematics is as crucial as undergraduate textbooks would suggest, why didn't more respondents find the correct answer?

We suggest the role of error checking in mathematics may provide a potential answer to the first of these questions.

ERROR CHECKING IN MATHEMATICS

In his celebrated essay on mathematical invention Jacques Hadamard wrote:

> Good mathematicians, when they make [errors], which is not infrequent, soon perceive and correct them. As for me (and mine is the case of many mathematicians), I make many more of them than my students do; only I always correct them so no trace of them remains in the final result. (Hadamard 1945, p.49)

The importance of mathematical error checking was confirmed by Markowitz & Tweney (1981). In an empirical study of the behaviour of mathematicians when testing a conjecture, they found that 'disconfirmatory strategies' play a much greater role in mathematics than in the physical sciences. Thus we can claim that while mathematicians frequently make errors, in contrast to non-mathematicians they are highly skilled at detecting and correcting them. This error-correcting might provide a tentative explanation for our results.

We suggest that, along with the rest of the population, the initial reaction of the mathematicians to the Wason Selection Task would be to choose D and 3. If Hadamard is correct with his idea that mathematicians are significantly more adept at error checking, the typical mathematician would check their answer carefully, and quickly see that the 3 card was unnecessary. At this point they could do one of two things. Happy that they had corrected an error, they might stop checking and select just the D card (35% of students and 24% of staff selected the D card only). Or; checking their new answer carefully, they might realise that the 7 card was crucial and amend their answer accordingly. Perhaps after a further error check revealing no mistakes, D and 7 would be selected (29% of students and 43% of staff made this selection).

This chain of events would explain the two biggest selections by the mathematicians; that of D and D & 7. We conducted a brief pilot qualitative study involving clinically interviewing students as they attempted to solve the task. The small amount of data we have collected provides some support for our hypothesis: mathematics students initially choosing D and 3, pausing, rejecting the 3, pausing again and then choosing the 7. We are currently working on a larger scale qualitative study.

A reasonable way of testing the error checking hypothesis would be to time the responses of subjects. If it was true that an extra level of error checking caused the subjects to answer correctly, then one might expect them to take slightly longer. Such an experiment might prove to be a useful test of our tentative theory.

The second of our questions is related to the first. Faced with the initial selection of D and 3, there are two errors to be spotted: the incorrect selection of 3, and the failure to select 7. The data would seem to suggest that our sample was much better at finding the first of these errors than the second. The reasons for this are rather harder to pin down.

Amongst others, Johnson-Laird & Byrne (1991) suggest that human deduction fits remarkably poorly with formal logic. However, with the exception of Markowitz & Tweney's (1981) work, there has been little empirical research into exactly how professional mathematicians use fundamental logical ideas (such as disconfirmation) in mathematics. Previous work that describes the mathematical discovery process has mostly relied upon personal experience (Hadamard 1945, Tall 1980) and historical analysis (Lakatos 1976). Our data suggests that the role of logic in mathematicians' reasoning may be somewhat more subtle than previously thought.

When replying to our request for comments after the experiment one senior mathematics lecturer wrote:

> I don't think many of us think about the logical definition of $P \Rightarrow Q$ when writing out a proof in a research paper. The truth table for $P \Rightarrow Q$ is not very intuitive.

Could it be the case that there is a significant difference between the intuitive grasp of logic and the formal theory in the case of highly successful, professional mathematicians? If so, one has to ask why such emphasis is placed upon formal logic in first year undergraduate courses.

Acknowledgments:

We would like to thank Giles Taylor for his assistance with technical matters.

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INSIGHT INTO PUPILS' UNDERSTANDING OF INFINITY IN A GEOMETRICAL CONTEXT

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Abstract. *School students' intuitive concepts of infinity are gained from personal experiences and in many cases the tacit models built up by them are inconsistent. The paper describes and analyses a series of tasks which were developed to enable the researchers to look into the mental processes used by students when they are thinking about infinity and to help the students to clarify their thoughts on the topic.*

INTRODUCTION

Students' understanding of infinity is determined by life experiences, on the base of which they have developed tacit models of infinity in their minds. These tacit models (Fischbein, 2001) deal with repeated, everlasting processes, such as going on forever, continued sub-division etc. which are considered mathematically as potential infinity, or by the consideration of a sequence Boero et al (2003) called sequential infinity. Basic concepts of mathematical analysis are usually introduced through limiting processes, which also reinforces the idea of infinity as a process. However, the symbol ∞ is used as a number when tasks about limits and some other concepts of mathematical analysis are solved.

When we consider the historical development of infinity, philosophers and mathematicians were dealing with this concept only in its potential form for over 1500 years. Moreno and Waldegg (1991) noted how a grammatical analysis of the use of infinity could be applied to Greek culture. The Greeks used the word 'infinity' in a mathematical context only as an adverb, indicating an infinitely continuing action, which is an expression of potential infinity and it was not until 1877that actual infinity, expressed as a noun in mathematics, was defined by Cantor. His ideas were only accepted years later by the general community of mathematicians, following the initial strong denial of its existence. Nevertheless secondary and university students usually go from their intuitive ideas on infinity, which are not purposefully developed during their primary and secondary school years, to lectures which require their understanding of actual infinity. This lack of development must contribute to the difficulties the students find with this concept. Actual infinity contradicts many of their intuitive ideas of infinity. The historical development of the understanding of the concept of infinity gives us the clear answer to the question 'What intuition of infinity is more natural for pupils and why do students have such problems with accepting actual infinity?'

THEORETICAL FRAMEWORK

Most previous research in the area of infinity has been with older students, upper secondary and university students (Tirosh & Tsamir, 1994; Monaghan, 1986; Tsamir, 2002). Most researches have looked for difficulties students have in understanding actual infinity and consequently difficulties with the understanding of such mathematical concepts as infinitesimal calculus, series, limits, continuity (Tirosh, Fischbein & Dor, 1985, Nuñez, 1993).

Monaghan (2001) argues that pupils mainly think of infinity as a process and that even when pupils use the phrase 'Going towards infinity' they are not necessarily thinking of infinity as an object which could be interpreted as actual infinity. In our earlier research (Jirotková & Littler, 2003) we have found it difficult to determine whether or not the students who use the noun infinity have any idea of the difference between actual and potential infinity; we believe they do not. They use the words infinity and infinitely often in the same statement with no real differentiation between the two.

We believe that initially pupils get their main information about infinity away from the classroom environment. They live in a space exploration age and hear about satellites going into deep space 'which is never ending'. It is such experiences which set up the tacit models, which Fischbein (2001) states are not consciously controlled and which may lead to wrong interpretations and contradictions later. We believe that these tacit models themselves are often internally contradictory. The pupils live in a real world, which is perceived as finite and there is little opportunity for them to relate to or discuss about infinity in school. We are not aware of any secondary school syllabus in the Czech Republic or the UK in which infinity is given as part of the curriculum.

Boero, et al (2003) undertook research with fifth grade classes and one of their findings was that more than a third of the students undertaking the research changed their understanding from a finite position to an infinite position or visa-versa when discussing infinity related to number problems. This would indicate that the pupils' knowledge is not firm at this stage and that listening to arguments within a group can easily alter their perceptions. This finding supports our aim of challenging teachers to make use of any opportunity to discuss the concept of infinity with their pupils of any age. The concept of infinity is difficult; the ontogenesis of this concept gives evidence for this. It is important that the phylogenesis of the concept should correspond to the ontogenesis.

In our previous paper on this topic (Jirotková & Littler, 2003) we gave the results of analysing the written definitions of the concept of a straight line given by 72 teacher-students. The method of analysis, which we developed during the initial stage of our research, was based on taking simple mental units, which involved the word infinity and any words derived from it out of students' responses, and classifying them on three levels. The criterion for the second level of classification was the grammatical use of infinity – noun, adjective or adverb, and indirect expression of infinity (Moreno & Waldegg, 1991). This classification proved to be a useful tool to help us to identify phenomena which characterised and described students' understanding of infinity. We were aware that our analysis was dependent on our interpretation of the student's writings and that the students themselves might not have the communication skills to express what was in their minds succinctly (Jirotková &

Littler, 2002). This is why we tried to find as many plausible interpretations as possible, developed additional tasks to diagnose the most likely interpretation and went on with our research.

METHODOLOGY

Our aims for the follow-up research were 1) to use the initial task given to university students with school students, and do a comparative analysis of the results of Czech and English students; 2) to make use of the tasks we had developed as a research tool to help us to look at and analyse the mental processes which school students used when writing about infinity.

Pilot Study

In March 2003 we asked 27 Czech and 35 English school students, of 11-12 years of age, to write down their definition of a straight line. No mention was made of the word infinity because we wanted the students to use the term spontaneously in their responses. In most research in this field almost all the experiments have been carried out on a one-to-one basis with the researcher interacting with the pupil. Boero et al (2003) used class discussion to counteract this phenomenon, whereas we used the written form of response to avoid distorting the results by an adult.

The English students did not provide any phrases which included the word infinity which meant that we were unable to do any comparative analysis. We looked for the reason for this unusual and surprising phenomenon. The main reason found was in the differences in geometrical syllabuses for the two countries. The Czech geometrical syllabuses since 1976 are based on a deductive approach starting with concepts of point, which was considered as the simplest concept from the point of view of the axiomatic building of geometry, line segments, ray and straight line and so on. This approach required the pupils to speak about extending bounded objects into infinity (line segments into straight line etc.). Moreover the word infinity is used in common language in Czech. The English pupils had received a much more Euclidean approach to straight line, in which the difference between straight line and line segment is not stressed which corresponds to Euclid's concept 'euthea' used in his work Stoichea.

Main Study

The pilot study resulted in the development of a series of 7 tasks in which students' ideas from our previous research were used. We gave them to 44 Czech and 54 English pupils of 11-15 years of age in June 2003. The series involved some tasks which we devised following the analysis of the student-teachers' responses from our previous research (Jirotková & Littler, 2003) for those students who provided phrases in their definitions, which contradicted each other or phrases which could be interpreted in different ways. These tasks helped us to clarify our interpretations of students' phrases and provided us with good evidence about the respondents' intuitive ideas of infinity. We used a geometrical context in all but one of the tasks because we believe that the geometrical world provides unique and exceptional possibilities to develop the understanding of the concept of infinity and infinite processes (Vopěnka, 1989).

The seven tasks were set out on paper so that the pupils could record their answers on the same sheet. The tasks were set during their normal mathematics lessons and there was no time limit. Generally the time needed was between 20 and 40 minutes.

Task 1. Two boys are talking. Decide which of them is correct.
Adam: *A straight line has two 'infinities'. If I go in one direction I'll reach infinity. If I go in the opposite direction I'll also reach infinity.*
Boris: *Those two 'infinities' are the same, so there is only one infinity on a straight line. It is the place where both ends join together like a circle.*
Write the name of the boy who you think is correct. Explain why.

Task 2. Three girls are talking. Decide which of them is correct.
Cecilie: *Two parallel rays end in two different infinite points.*
Dana: *I do not agree. They end in the same infinite point.*
Eva: *Neither of you is correct. A ray goes on and on and never ends.*
Write the name of the girl who you think is correct. Explain why.

Task 3. Given a line segment *AB*. We extend it twice, three times, …, one thousand times, …, infinitely many times. Describe the resulting object and give its name

Task 4. Given a line segment *AB* whose end points *A* and *B* have been cut off. We extend it twice, three times, …, one thousand times, …, infinitely many times. Describe the resulting object and give its name.
Are the resulting objects from the above tasks 3 and 4 the same? If not, explain why not.

Task 5. Frank states that he can write down the smallest positive decimal number. Is this possible?
If YES, write down the number. If NO, give reasons.

Task 6. Given a semicircle with diameter *AB* [diagrams were drawn]. Consider all the possible triangles *ABC* having the vertex *C* on the circumference of the semicircle. Draw two triangles *ABC* such that the height *h* (from *C* to *AB*) is (a) the greatest possible, (b) the smallest possible.
Describe as precisely as possible, the position of point *C* in each case.

Task 7. Given a straight line *b* and a point *A* not lying on *b*. Consider all squares *ABCD* whose vertex *B* is on the straight line *b*. Draw square *ABCD* with **(a)** the smallest possible area, (b) the greatest possible area. Describe position of the point *B*. Draw the diagonals *AC* and *BD* and mark the centre *S* of the square *ABCD*. If you do not have enough room on your paper to mark any of the points, draw arrows to indicate the direction in which they lie. [Diagrams were provided.]

Task 1 gives two statements which argue for either one or two infinities. We are aware that both ideas are supported by some mathematical theory so that no answer is unambiguously correct. First we will describe what each child's hypothetical statements in tasks expresses.

Three ideas about a straight line occur in the five responses given in tasks 1 and 2. A straight line is like:
1) A line segment extended in both directions so that its end points are put into infinity. It is homomorphic to a closed line segment. This idea is expressed by Adam.
2) A circle which is created by extending a line segment into infinity where the two end points are joined. It is homomorphic to a projective straight line. This idea is explicitly expressed in Boris's statement.
3) A line segment without its end points infinitely extended in both directions. It is homomorphic to an affine straight line. This idea is expressed in Eva's statement.

Eva's understanding of a straight line is potentially infinite whereas Boris's and Adam's are quite clearly actually infinite. For Adam infinity exists even at both ends of a straight line, and it is reachable by man although it is very far away. Saying *I'll reach infinity* he finalises the process and by that he actualises it. Infinity is placed before the horizon and it is possible to handle it as if it is a final phenomenon. For Boris, Cecilie and Dana infinity also exists but they do not mention the possibility of being able to reach it. Only Eva refuses the existence of infinity for a ray and is the only one who supports potential infinity. Boris expresses his interpretation of Adams's statement – a straight line has two <u>different</u> infinities. He does not refute anything that Adam said but just gives his own interpretation. It is possible to derive Cecilie's understanding of two infinite points very naturally: Let us consider a rectangle *ABCD*. Fix the side *AB* and take the side *CD* and go on and on into infinity. Then infinite points *C* and *D* are still different. Dana states that the infinite end points are placed at the horizon. When we look at two parallel rays we can see that when they are reaching the horizon they are closer and closer to each other and when they arrive at the horizon and we lose them from our sight they reach infinity where they join together.

We did not expect students to express anything new about infinity in tasks 3 and 4. These tasks as well as task 7 were included to provide respondents with other contexts in which to express their ideas about an infinite object such as a straight line. The different contexts for the same mathematical idea were a rich source for inconsistencies in the students' answers.

In our experience most pupils felt more comfortable speaking about infinity even if it is infinitesimally small, in the context of numbers. Task 5 and 6b) were included to enable us to compare the responses to the same mathematical problem, task 5 in the context of numbers and task 6 in a geometrical context.

Tasks 6a) and 7a) played the role of warming up exercises so that the students could start to think about the task in the way they normally thought in geometry. The core of the tasks are the speculative parts b). In 6b) the smaller the height of triangle *ABC*, the shorter the side *AC* is. So the task is transferred to a simpler one, that is to find such a point on the semicircle that is as close as possible to the point *A* or *B*. The solution to this task is connected with the solvers understanding of an infinitesimally small line segment which is impossible to get just by using our perception. We will call this type of infinity, *infinitesimal infinity.*

In task 7 the area of the square is dependent on the length of the line segment *AB*. The problem is thus transferred into looking for the shortest or longest line segment *AB*, where *B* is an element of the given line *b*. The line segment *AB* can extend without limit when the point *B* goes further and further in any direction. The problem is to think and then communicate about the process and its finalisation.

ANALYSIS AND DISCUSSION OF TASKS 1 AND 2

We have used the method of analysis for the present data which we developed in our previous research (Jirotková & Littler, 2003) and partly comparative analysis as well.

It was surprising that other than two boys (aged 15) out of all the respondents everyone else accepted one of the offered answers as being correct. We feel that this is the consequence of the teaching approaches and textbooks used, which do not offer open-ended tasks to pupils and do not challenge them to think speculatively.

In both countries more students considered the idea of two infinities correct, 75 % of Czech students and 59 % of English students choosing Adam. In spite of clear indications that Adam's answer was right we found inconsistencies between their choice of this sentence and the explanation for their choice. For instance a student who chose Adam then wrote "When I go in any direction I will reach the infinity" and another "... because a straight line goes on for infinity". Because the word infinity was used in the task, students from both countries used it, more Czech pupils, 50 % compared with 41 % of English pupils.

Task 2 produced an interesting result in that most students of both countries chose the sentence spoken by Eva, 61 % of Czech and 63 % of English students. This would indicate that these students were happier thinking of infinity as a process which never ends, as a potential infinity. Choosing Eva's statement as correct, the respondents contradicted any choice they made in task 1. Let us consider each possible combination of answers from the two aspects – potential/actual infinity, affine/projective straight line.

Adam, Eva (38 %) – These students do not perceive any contradiction between actual and potential infinity. We can say that their understanding of infinity is not clear.
Boris, Eva (24 %) – A similar situation to the one above, the only difference being the students' understanding of straight line. The idea of a projective straight line is closer to these students.
Adam, Cecilie (16 %), Adam, Dana (10 %) and Boris, Dana (8 %) – These pairs of answers are consistent. There is no contradiction in the two observed aspects.
Boris, Cecilie (4 %) – a contradiction exists only in the understanding of straight line. This contradiction comes probably from the students seeing the distance between two parallel lines as a constant number which they project to the distance between the two infinities. Quality such as distance does not change itself when approaching infinity.

The experiences of these students are linked just to finite or bounded objects. They do not have any experience with phenomena beyond their perception. That is why they speak about these phenomena as if they were in the 'illuminated' world reachable by our senses. All who answered Dana accept the change in the quality distance when it comes to infinity, and all who answered Cecilie do not accept any change. They are ready to extend the rule *distance between two parallel lines does not change* from perceivable world to infinity. Consideration about the change of quality of phenomenon when coming from finite world to infinity is very difficult for students. We have evidence about finite phenomena, about perceivable objects and we project these to our consideration about phenomena beyond the boundary of the real world. Tall & Tirosh (2001) note the 'epistemological distinctions and contradictory aspects from extending finite experiences into the infinite case'.

COMMENTS ON TASKS 3 TO 7

57 % of the Czech and 64% of the English students said that Frank could write the smallest positive decimal citing answers such as $0,\overline{0}1$ or $0,0000000000000....1$ with the comment "the number includes an infinity of zeros". Those who said it was not possible gave reasons such as "this number can have an infinite number of zeros". As can be seen from these answers the problem from smallest object was transferred to one of how they could write infinitely many zeros. Only 3 % of students did not attempt a solution to task 5 comparing 35 % in task 6, which looked at infinitesimally small object in a geometrical context. The most frequently given solution in task 6 was "*C* is the point closely neighbouring to the point *A* or *B*".

We can divide the images about the furthest point of a straight line which we met in the students' solutions in task 7 into three groups:

1. Any solution to the task is accepted. There are three cases of this represented by students' authentic statements:
   a) "Point *B* does not exist, nor does the square *ABCD*."
   b) "It is not possible to determine point *B*, nor the square *ABCD*."
   c) "I cannot describe/find the point *B*, nor the square *ABCD*."
2. Point *B* is situated beyond the horizon (Vopěnka, 1996). It means that the point *B* is unreachable to our perceptions and also the square *ABCD*. It is an object that exists but our perceptors cannot grasp it. All we can perceive is the tendency of the point *B* to go far away when the point *B* is still in front of the horizon. The tendency is expressed by:
   a) the verb denoting a movement, eg leaving, going, running, ….
   b) a phrase - as far as possible, the biggest possible, the furthest possible.
3. The point *B* lies on the fixed locality of the line *b*, marked by ∞, which we call infinity.

CONCLUSIONS

We analysed the students' responses from two didactical points of view:

- to get our explanation of the mechanism of the creation of their images,

- to find a suitable tool to further develop the images.

We prefer to use the phrase "further develop' rather than 're-educate' because we do not agree with teachers' frequent opinion that if the students have a different approach to some concept, it is wrong. It is not possible for this demanding and abstract concept of infinity to be explained to students within a short time; the understanding has to develop over a long period in the cognitive net of a student and the more abstract the concept is, the wider the spectrum of experiences an understanding requires. There is research evidence (Monaghan, 2001) to show that a student's view about infinity differs with the context in which it is expressed, therefore we cannot say that some students' understanding is incorrect. The students are only at a particular developmental stage in their understanding. This implies that we have to think about what other experiences we should provide to the cognitive net of these students to raise their quality of understanding.

Fischbein, Tirosh & Hess (1979) found that the intuition of infinity was relatively stable from 12 onwards. However, the pupils we tested did not show the stability of intuition of infinity when the contexts were changed.

We believe that investigating the process of the understanding of the concept of infinity in different contexts enabled us to approach the world of very special and qualitatively different mental activities, which are related to non existing phenomena which cannot be founded on direct empirical experiences. The important point is the concept building process of the concept of infinity. This is based on introducing the concept in many different contexts and solving the contradictions which come to our minds as a consequence of it.

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The research was supported by grant GAČR 406/02/0829.

TEXTUAL DIFFERENCES AS CONDITIONS FOR LEARNING PROCESSES

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The aim of the study (Johnsen Høines 2002) reported here was to shed light on the process through which student teachers express and interpret their understandings about mathematical notions and thereby gain insight into them. This paper focuses on how students cooperate and move between different ways of understandings. It emphases in particular on how differences in the ways of explaining a mathematical content make different conditions for the dialogue and learning processes that develop.

INTRODUCTION

The study to which this paper relates (Johnsen Høines 2002), had the purpose of shedding light on the processes employed by student teachers to communicate mathematical content; on how they do gain insight into mathematical notions by exploring different ways of expressing, interpreting and investigating them. I wanted to learn about those processes whereby students engage in learning in ways that can be described as taking ownership of it (Mellin-Olsen 1989, Skemp 1971, Goodchild 2001). Such situations can be characterised by the behaviours exhibited when students are in charge, making decisions about what to do and how to do it. They develop authorship (Burton 1999). I had, as a teacher and a teacher educator, experienced such processes within situations that were characterised by energy, determination, communication and independence. I wanted to study how the students, within the frame of such situations, move between knowledge and knowing, and how they use *differences* within the ways of expressing content as tool for their goal related activities.

THEORETICAL BACKGROUND

The dialogical perspective on learning referred to in this paper is related to the work of M.M.Bakhtin (1981,1998). Understanding is seen as constituted by ways of understanding. To develop understanding is, in the study on which this paper is based (Johnsen Høines 2002) accordingly seen as moving between ways of understanding, conducting the ways of understanding in opposition to one another, seeing them in the light of the other(s). Bakhtin (1981:12) states that *"Language throws light on each other: one language can, after all, see itself only in the light of another language."* The differences get vital. Without the differences the interactions would not have any function. The understanding would not develop. *"Different voices are not enough to create meaning; the tension and struggle between them create understanding"* (Dysthe

Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 2004 Vol 3 pp 105–112

1999:76) The dialogical approach used in the study is a text-theoretical approach where I see texts as constituted by texts interacting with each other.

METHODOLOGY

The empirical data was gathered from a situation where student teachers were working on a task within the theory of functions:

> Five 14-year olds have started a bicycle-club. They repair bicycles, they arrange outings and they discuss traffic problems. The organisers of the club wish to recruit more members in order to make the club grow. They set the goal that each member should recruit three more members every quarter of the year until they reach a set limit of 1000 members. A waiting list is established from which to pick new members should someone drop out. Express the aspired number of members as a function of time and by using the premises set for how to reach the aspired goal.

A particular episode developed in interaction between two of the students, Mette and Kari. It was chosen as a research situation because of four criteria: the students' intentionality, the different ways of understanding that were communicated, the interactions between the students and the way they persisted because they wanted to gain insight into the processes.

I subscribed the communication on the basis of observation. The students were subsequently interviewed. I invited them to comment on the subscript and to tell about what they thought happened in the episode. The interviews and comments were transcribed. These transcripts served as the empirical data for the study.

To get in touch with the complexity underneath, I wanted to make a detailed analysis of a short situation. A text-theoretical perspective underpinned the analyses in that I studied the learning situation as a text. I understand the concept of *text* in a broad way by regarding text as what is interpreted as well as the interpretation. When I refer a written text, text is not only seen as what is written. This is taken together with a reader's interpretation. A text can also be oral or it might be a picture. A (teaching/learning-) situation can also be regarded as text. To summarise, text is conceived as a combination of what is being interpreted together with the interpretation. Texts are developed in interaction between texts through a process of confronting each other (Bakhtin 1981).

The task referred to above is in itself referred to as a text. It serves as an illustration for the individual and social approach. It is individual in the sense that the students' interpretations will differ. Each student has her own interpretation. The social dimension implies that their interpretations are related to the interpretations of the other students' interpretations and of earlier interpretations done.

When analyzing the differences, I found that three aspects achieve importance: *First,* I regard the texts as related to the individual; Kari's text and Mette's text differ. The students make different texts relevant as a basis for their interpretations, for further developing.

Second, this illustrates the social perspective. They discuss. They have (partly) shared basis for their references. This might be conscious and shared knowledge. They talk, for instance, about "this is how we understand it". In the context of Bakhtin's approach, we have social references also without being conscious of them. Our personal interpretations imply social dimensions. Thinking is seen as dialogic; in itself it is social. I look into how the differences between the students' understandings can be understood in relation to a social dimension.

Third, the students relate to texts that predict a kind of objectivity. The task is formulated in a certain way. It has a specific genre. When reading, the students make other tasks and other kinds of mathematical texts relevant to their interpretations. The task is ordered in a specific way. The processes of understanding can be seen as attempts to make a structure of what is structured; make an order of the order one observes is there. This can also be seen as describing the learners' struggle to know, by moving between knowing and knowledge. (Burton 1999) According to the concept of genre I relate to in this study, the order carries in itself a content. Ordering a content in different ways, make different meanings. (Bakhtin 1981, Johnsen Høines 2002)

I see the discussion Hewitt (2003) makes about how the visualisation of a mathematical content constitutes challenges for the teaching- learning processes relevant to these aspects. "Learning to use formal notation involves not only developing meaning for symbols but also developing meaning for the positioning of symbols in relation to other symbols. The symbols within an algebraic equation and their relative positions reproduces a visual impact which affects the way those equations are manipulated when re-arranging the equation." (p.3-68) I learnt from my study, that Kari and Mette saw the content and the way it was expressed as interwoven. They struggled to understand the mathematics by investigating what was expressed by other persons and by themselves. As mentioned above, I regard their way of expressing as a way of ordering the content. There is a content embedded within the way of ordering, decided within the genre. In this paper, when learning is seen as developing text, it is seen relational, related to an individual, social and the genre-related perspective. (Johnsen Høines 2002)

It becomes a methodological challenge to uncover interpretations that are legitimated in the context of the dialogic process, to visualise them in the light of each other and to evaluate them as plausible interpretations from the perspectives of the individual, social and genre related aspects mentioned above. I experience it as challenging and vital to develop the analyses by moving between apparently conflicting aspects: regarding the text as both nearly not personal and personal. This implies moving between my interpretations of what is said and what is written a) as little as possible in the context of what I know about the person and her intentions and b) in the context of the students, the relationships between them and the personal intentionality embedded in their actions. I find it, within the limitation of this paper, impossible to bring forward the complex analyses that in the study are underpinning my interpretations. When the data is referred to in the following, it is mostly presented as my interpretations and it has to a certain degree an illustrative function.

STRUGGLING TO UNDERSTAND WHY THE ANSWERS DIFFER

The situation turned into a research situation when Kari and Mette communicated: *"We don't understand this. It does not fit. We have different answers and we cannot understand why! "* The students were intensively searching for insight into the problem. They did not ask: *"What is the correct answer?"* They communicated their questioning with the teacher and a few other students just being there. They commented on their failure to understand why the answers differed. They were deeply engaged, looking at each other's written work. They talked slowly in an investigative way, were listening and questioning.

Mette's and Kari's contributions can be seen as if they brought two different texts into the field. Mette's notes showed the solution as f (x)= $5 \cdot 2^x$. Kari showed her result as $f(x) = 5 \cdot 4^x$ and argues that it had to be correct. They investigated the calculations done. It seemed that each managed to follow the other's reasoning, but neither understood why the answers differed. Kari seemed convinced of her own solution, and Mette told that she agreed. In this respect Kari's model seemed unproblematic to them. Accordingly they focused on Mette's solution f (x)= $5 \cdot 2^x$. They told that they could not understand why it was wrong. The calculations done by Mette seemed reasonable to them.

The students tried to understand the differences by investigating the two solutions in light of the other. It is my interpretation that the insight they had developed, and the insights they were trying to get about the differences, provoked them to continue (Renshaw & Brown 1999). *"I have used a month as the variable, Kari has used a quarter of a year. Is it possible that both solutions are correct? Could it be two ways of describing the same development?"* Mette asked. By doing some calculations, however, they showed to be convinced: $f(x) = 5 \cdot 4^x$ had to be correct and f (x)= $5 \cdot 2^x$ could not be. Kari seemed satisfied; they had found the answer. Mette, however, was not comfortable. She was convinced that the solution $f(x) = 5 \cdot 4^x$ was correct, but she communicated that she could not understand why f(x)= $5 \cdot 2^x$ was not.

I see this interaction as a conflict between the students' understandings; they develop different understanding in interaction with each other. It could be seen as comprising personal contributions and one could reflect on the process provoked by conflicts between personal perspectives. However, it seems more appropriate to focus on the different ways of understanding that are brought forth. Analyses on how the students investigated the two models make me argue that they to a minor degree are defending each individual model. It appears as conflicts between understandings more than conflicts between the two students' *personal* understandings. The students positioned the different understandings in opposition to each other. They investigated the differences by seeing one in light of the other. The analyses developed in the study argue that Mette's main goal became to understand why f(x)= $5 \cdot 2^x$ was not correct (and why the other one is correct). It is in this context when Mette turns to Kari and points at her notes:

> "But if we think quarter, why is it $f(x) = 5 \cdot 4^x$? I cannot understand why it has to be $f(x) = 5 \cdot 4^x$!" Kari writes: 5 + 5·3. She talks when she writes: *"They were five and by the end of a period of one quarter each of them has recruited three." "Yes,"* Mette says*, "it corresponds with my thinking."* (She points at some notes in her book.) Kari writes: 5+5·3 = 5+5+5+5= $5 \cdot 4^1$ *"This makes five plus five plus five plus five; that means five multiplied by four……raised to the first, it makes five multiplied by four to the xth."* [1] Mette looks at Kari and says *"Yes…."* It is obvious that she does not accept this as an explanation. *"I see that it is correct, but…."* They look at each other. They both seem confused. I interrupt by commenting: *"You see that it is correct, but you do not understand why it is?"* Mette looks gratefully at me.

I see this as representing, in several ways, conflicts between the arguments for the different solutions that was articulated:

a) They got two *different answers*, and tried to find the correct one or to document how both of them were correct.

b) They were searching for *different kinds of understanding* underpinning the solutions. It is obvious that Kari does not understand what Mette is searching for, neither is it easy for Mette to explain her intention. Kari commented on this in the following way: *"This happens over and over again. We do not listen well enough. I did not understand what she did not understand. I wanted her to understand it in my way!"* (Johnsen Høines 2002: 146)

c) The two students *expressed themselves differently*. Kari's expressions can be characterised as a 'linear' text. The utterances were structured in a predictable way, as "we have this – this follows this, then this, and therefore this…." This can be seen in her writing and also in the oral language. The voice was distinct. The written symbolization seemed prior to the oral expressions. (Pimm 1987: 20). Kari repeated herself, she used the same words over and over again, she told how to do it in a

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[1] In this paper I use parts of the subscript in order to underline some aspects of the research. Other parts, as for instance 'why four to the xth?' were important in the study, but not emphasised here.

rhythmic way. It is as if she let her oral language follow the written symbols. As she moved on, she tried to be more distinct, clearer. She tried to get Mette to understand.

Bakhtin (1981,1998) shows the importance of identifying the voices of the traditions or of the culture. He claims that the words are not just ours, we cannot use them freely. The voices from earlier users are implied within the words. This is realised in the interpretation. When listening to Kari's explanations, I could identify what I characterise as the teacher's voice, telling how to do or how to think. It is likely that Mette identified the teacher's voice, and that even Kari did so. Kari moves within a genre that is constituted by certain ways of expressing. The way of thinking is restricted and is given possibilities within the genre. The argumentation and solution are to be interpreted in the context of the genre.

Mette's way of expressing herself was not as easy to follow. She talked slowly, she sounded inquiring, moving, comparing. Late in the situation she made a drawing:

Thoughtful and tarrying she said: *"Oh.. yes..one..becomes.. four..that's ..why..it.. has.. to.. be.. four.. times.. (long pause)..yes, I understand."* Mette organised the use of language in a different way. Accordingly she has got other limitations and other potentialities. The two texts appear as different due to the different genre within which they are expressed.

The situation can be described by how the two texts interact with each other. They are different and it becomes clear to me that the appearance of the other is a vital condition for the development.

AUTHORITATIVE TEXT

The utterances of Kari did not necessarily express the process in which Kari came to understand the relationships on which she is focusing. It was however the explanation she made in order to get Mette to understand. I found the teacher's voice a vital part of Kari's text. I have characterised it as linear, algebraic, rhythmic and repetitious. It showed an authoritative and monological tendency. It demanded "to do like this", or "this is how to understand". It did not invite for discussions. (Wertsch 1991)

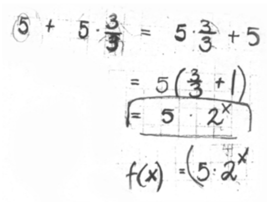
I characterise Mette's text differently. It was as if she turned to and fro, searching. The utterance did not show the result of her thinking. The drawing was made slowly and investigative. It seemed to be a part of the process of getting insight. Mette's voice was a silent voice, though persisting. The way Mette structured her investigation is not easily grasped. Her process was not predictable.

Mette initiates a communicative turn when she after having got a new repetition from Kari, questioned the explanation. With energy in her voice, she attacked: *"Then it* *could be 4·5 as well, and then* $4{\cdot}5^x$ *"* Kari looked at her, paralyzed: *"Yes….. but…"* They stopped, looking at each other, silent. I intervened*: "Can you explain what you do not understand?"* Mette, without hesitation: *"That it is four times, that it is four times every time."*

It appeared to me that Mette had a competent objection. She showed insight into the logical structure presented by Mette. She identified a logical gap. When she attacked the logical structure she also attacked the way of explaining, she attacked the teacher's voice. Kari was not able to answer. In the interview she confirmed that she did not have any alternative explanations. She had not seen the logical gap Mette had pointed to. Kari was limited within the frame of ways of express herself. Kari did not expect Mette's question; it was not invited for. They moved within different limitations.

A TEXT WITHIN THE TEXT

Looking into how the texts were contradicting with each other, particularly in the context of Mette's attack, made me identify another text within Mette's text (Mette 1 and Mette 2.)



Her first solution (Mette1) seems similar to Kari's when it comes to the written and oral structure. They seem close in terms of genre. It has the same weakness that she attacks in Kari's explanation. This argues for seeing the process as interaction between three texts, between Kari and Mette1 and Mette2. In the study I find it reasonable to assume that when inquiring and attacking Kari's text, Mette attacked parts of her own text (Mette1). The analyses show the possibility that Mette's listening to Kari's explanations helps to make some connections to her own reasoning. The authoritative text within her own text might be powerful condition for investigating the explanations she is offered by Kari. By the insight she has into the differences, she knows what kind of understanding she is (and is not) heading for. Her knowledge about the authoritative texts might be a condition for attacking them.

CONCLUSION

Investigating the students struggle to know, by moving between knowing and knowledge, helped me to get insight into how understanding develops as interactions or conflicts between different understandings. Words as *struggle*, *conflict* and *attack* is in this paper used to describe the processes in which the students conduct different ways of expressing in opposition to one another. The differences in the students' expressions are seen in an individual, social and genre perspective. The focus has been to show the importance of that the order carries in itself content, there is content embedded within the genre. I argue for seeing the learners struggle to know in context of this.

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STUDENTS' GENDER ATTITUDES TOWARDS THE USE OF CALCULATORS IN MATHEMATICS INSTRUCTION

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The study analyzed gender attitudes of students in learning mathematics by using a calculator, in one of Botswana's Junior Secondary Schools. Students' attitudes were sought using a questionnaire and data was analyzed by both quantitative and qualitative methods. Attitude variables used were usefulness of calculators, enjoyment and anxiety in using calculators. The findings indicated that students of both sexes did not realize the benefits of using a calculator in mathematics learning. Generally, no gender differences were noted in the variables used. While most students were accessible to calculators, they were not accessible to calculator technology.

INTRODUCTION

For many years now (since early 1970s), hand-held calculators have been widely used in mathematics instruction in many educational systems of the world. Advances in technology have improved the use of the calculator and mathematics curricula have been designed to involve this technology. Many studies have indicated that the use of calculators can enhance students' ability to learn basic facts and that students who used calculators frequently exhibited more advanced concept development and problem solving skills than those who did not use them (Cockroft, 1982; Suydam, 1982; Howson, 1991; Hembree and Dessart, 1992). These studies reduced some earlier fears that calculators could affect students' mastery of computational skills acquired from traditional paper-pencil methods (NCTM, 1974; Cockroft, 1982).

Some educationists consider the use of technology in instruction as the only way to go if not a substitute for conventional teaching and learning methods (Broekman et al, 2002). Its interactive testing and review mechanism, together with "a let-s-go-back and look-at-that-again-loop" is believed to offer the best of all worlds of learning (The Star Newspaper, 2000). Broekman (2002) argues that the use of such technologies promise power for students to control over their own learning, and promises to give "voice" to learners. The latter arguments are considered essential for a smooth learning environment for both girls and boys, as it is believed that such interactive ways of learning could minimize or eliminate gender differences in learning.

Studies on gender differences show how different ways and methods have been used to minimize the gender gap, not only in mathematics instruction, but in many fields of study especially in science, engineering and technical fields. Studies have shown that gender attitudes towards mathematics tend to influence students' performance in the subject as well as their future careers involving mathematics (Finn, 1980;

Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 2004 Vol 3 pp 113–120

Clegg, 1984; Gilbert, 1986; Duncan, 1989; Ngemeza, 1991; Clifford, 1998; Fennema, 1990 & 2000; and many others).

The interactive nature of calculators could provide the opportunity for girls especially, to work independently and become more confident in learning. Some studies have shown that the nature of interaction in class involves classroom practices that tend to influence female attitude towards the subject (Kaino, 2002). Practices in the classroom, involve for example the boys' use of verbal and non-verbal language to command more of the teacher's time in both attention and classroom control, and boys being more mobile in class than girls, which tend to influence some teachers' beliefs that boys were more competent than girls (Serbin, 1978; Barry, 1981; Jungwirth, 1991; and Lee, 1990). Furthermore, it is argued that some teachers tend to believe that boys' contributions were more impressive than those of boys (Goddar-Spear, 1989; Fennema et al, 1990; and Fennema, 1990) because of such classroom practices.

While the use of technology such as the use of hand calculators is considered an important tool to impart knowledge, critics of such technologies argue that some areas of the content that aim at developing critical thinking and reflective practice cannot be satisfied by this technology, as learning entails not only of conceptual tools but also the ability to use the tools in arguments, discussions, research and practice (Broekman, 2002). Access to such technologies has also been criticized for lack of epistemological access to students. It is argued that the theories of social construction of knowledge require the relationship between the person and the social phenomena as central to epistemological access i.e. knowledge is to be constructed and developed in and through social mediation (Vygotsky, 1987). Broekman (1992) argues that enabling epistemological access to technology requires the consideration of developmental aspects of affinity with technology, and that affinity depends on confidence which itself develops social mediation.

As part of a larger study on the use of Information and Communication Technology (ICT) in Botswana, the study on the use of calculators was considered essential at this stage, after being in use for many years in schools. Hand-held calculators are accessible to most students in schools and presently ICT materials are used in mathematics teaching, especially some new computer programs. Assessment of gender attitudes in the use of technologies in learning is important for future introduction of ICT materials in mathematics instruction. This paper presents part of the findings of the study.

Aim of the study

The study analyzed students' attitudes by gender on the use of hand-calculators in learning mathematics.

Research questions

What were the students' views by gender on :

1) the usefulness of calculators

2) enjoyment when using calculators in learning mathematics

3) anxiety when using calculators

DEFINITION OF TERMS

In this study, students' attitudes have been restricted to three variables, i.e. usefulness of calculators, enjoyment and anxiety in using calculators.

Usefulness is referred to as whether students viewed calculators as helpful tools in learning mathematics, in their daily activities and for their future careers.

Enjoyment is referred to as whether students liked using calculators during mathematics activities or not.

Anxiety is referred to as students' worries when using or when have to use calculators in mathematics class activities.

METHODOLOGY

The target of the study was students in Junior secondary schools in Botswana. Junior level consists of forms one, two and three. In this study, form three level was considered. As a case study, one district was selected at random, from which one school was also selected at random. One stream of students was again selected at random to involve 20 girls and 20 boys. Data was collected using a questionnaire. The questionnaire was validated before it was administered and the reliability was tested using Cronbach's alpha. Cronbach's alpha estimates the internal consistency by determining how the items used in the instrument are related to each other. Both qualitative and quantitative methods were employed for analysis. The quantitative data involving closed-ended questions was analyzed using the Statistical Package for Social Sciences (SPSS). Responses were analyzed using the 4-point Likert scale. The t-test method was used to determine any differences that existed between boys' and girls' responses. Furthermore, an estimation of the standard error of the difference between the two gender samples was tested by using Levene's method of one-way analysis of variance on absolute deviation of scores. The significance or probability value was set at less than or equal to 0.05.

From the qualitative data, involving open-ended responses, individual responses were analyzed and dominant views determined for comparison with responses from the closed responses in the qualitative part.

A 4-point Likert scale used the following: Strongly Agree (SA), Agree (A), Disagree (D) and Strongly Disagree (SD).

Quantitative variables used are as follows:

Usefulness of calculators:

(i) I do not have any use for calculators on a day to day basis

(ii) I do not think a calculator will be useful to me in my future job

(iii) Anything that a calculator can be used for, I can do just as well some other way.

(iv) I do not see how a calculator can help me to learn some new skills.

Enjoyment of using calculator:

(i) I enjoy using a calculator in mathematics activities

(ii) A calculator is interesting, fascinating and easy to use

(iii) I enjoy investigating mathematics problems using a calculator

(iv) A calculator is very interesting and challenging to use..

Anxiety in using calculators:

(i) I do not feel comfortable when I learn mathematics using a calculator

(ii) The thought of using a calculator in mathematics activities frightens me

(iii) I am worried about using a calculator because I do not know what to do if something goes wrong

(iv) The use of a calculator confuses me.

FINDINGS

Usefulness of calculators (Q1)

The average score for usefulness of calculators in mathematics learning (from the 4-point scale) was 3.10 for girls and 2.81 for boys, indicating that majority of students of both sexes felt calculators were not useful in mathematics learning. This shows that more girls than boys felt calculators were not useful. Levene's test showed no significant differences in variables (i), (iii) and (iv), Table 1. The test on variable (ii) showed significant differences on the view that the calculator was useful to students' future job. The main reasons given by students of both sexes were that; calculators were confusing, not easy to use and needed much time to understand. The common view among students of both sexes was that the calculator was useful in addition, subtraction, multiplication and division operations.

		Levene's test for equality of variances				
		F	Sig.	T	DF	Sig. (2-tail)
(i)	Var.[1]	0.340	0.564	0.690	35	0.495
	Var.[2]			0.684	33	0.499
(ii)	Var.[1]	3.064	0.089	2.593	35	0.014
	Var.[2]			2.515	28	0.018
(iii)	Var.[1]	4.667	0.038	1.118	35	0.271
	Var.[2]			1.085	28	0.287
(iv)	Var.[1]	0.001	0.979	0.000	35	1.000
	Var.[2]			0.000	34	1.000

Table 1: Summary of t-test for female and male students' mean responses on usefulness of calculators (Q1)

*Var.[1] Assumed equal variances *Var.[2] Not assumed equal variances

* (i), (ii), (iii) and (iv) are Usefulness of calculators

Enjoyment of using calculators (Q2).

The average score for boys was 3.11 and 2.96 for girls, which indicated that majority of students enjoyed working with calculators, with boys enjoying more than girls. The Levene test showed no significant differences in the on four enjoyment variables between boys and girls, Table 2. The general scores on variables (i) and (iii) for both sexes were low (2.2 and 2.1 respectively) compared to other variables, indicating that most students enjoyed working with calculators on things other than mathematics activities. Many students said that calculators needed a lot of skills and knowledge in learning, that they were time consuming and that they did not indicate whether the answer was right or wrong.

		Levene's test for equality of variances				
		F	Sig.	T	DF	Sig. (2-tail)
(i)	Var.[1]	2.261	0.142	1.198	35	0.239
	Var.[2]			1.151	26	0.260
(ii)	Var.[1]	3.632	0.065	-0.920	35	0.364
	Var.[2]			-0.895	29	0.378
(iii)	Var.[1]	2.917	0.096	1.301	35	0.202
	Var.[2]			1.275	30	0.212
(iv)	Var.[1]	5.456	0.025	0.902	35	0.373
	Var.[2]			0.868	25	0.394

Table 2: Summary of t-test for female and male students' mean responses on Enjoyment of using calculators (Q2)

*Var.[1] Assumed equal variances

*Var.[2] Not assumed equal variances

* (i), (ii), (iii) and (iv) are Enjoyment of using calculators

Anxiety in using calculators (Q3).

A mean of 3.14 and 2.75 for girls and boys respectively was obtained from the anxiety scale. This revealed that most students were of the view that calculators caused anxiety, with more girls being affected. The Levene test on variables (i), (ii) and (iv) showed no significant differences students' anxiety in using calculators. There were significant differences between boys and girls on variable (iii) on students' worries when something went wrong when using calculators. Views from many students of both sexes indicated that students were afraid when required to use calculators as they were not sure of the methods to use, that the process by the calculator would not tell whether the solution was correct or wrong, that the calculator could terminate the solution without a track of the process, that without using a calculator could mean the solution was wrong.

		Levene's test for equality of variances				
		F	Sig.	T	DF	Sig. (2-tail)
(i)	Var.[1]	2.235	0.144	0.982	35	0.333
	Var.[2]			0.949	27	0.351
(ii)	Var.[1]	1.595	0.215	1.399	35	0.171
	Var.[2]			1.357	28	0.186
(iii)	Var.[1]	20.542	0.000	2.214	35	0.033
	Var.[2]			1.085	28	0.287
(iii)	Var.[1]	11.250	0.002	- 0.173	35	0.864
	Var.[2]			-0.165	24	0.870

Table 3: Summary of t-test for female and male students' mean responses on Anxiety in using calculators (Q3)

*Var.[1] Assumed equal variances

*Var.[2] Not assumed equal variances

* (i), (ii), (iii) and (iv) are variables of Anxiety in using calculators

SUMMARY

While most students, of both sexes, did not consider the calculator to be useful, with more girls than boys holding this view, majority of students from both sexes enjoyed working with calculators. Boys enjoyed more to work with calculators than girls. However, students enjoyed working with calculators in other things other than mathematics. Students' dislike of working with calculators in mathematics is realized from their anxiety when on calculators. Anxiety of majority students to use or when using calculators to learn mathematics, with girls affected more than boys, can well be explained by students' lack of enjoyment in using this technology to learn the subject. Generally, the findings showed no gender differences in the three considered variables, i.e. usefulness, enjoyment and anxiety between girls and boys. Component variables that showed some differences, were on students' view whether the use of calculator was useful to their future careers, and worries when using calculators to solve mathematical problems.

CONCLUDING

The findings provide some clues on the nature of perception held by students on using calculators in mathematics instruction. Some students' views that calculators required skills and knowledge to work on them, was an indication that students lacked knowledge to apply the technology in learning. Lack of knowledge to apply calculators in learning, could have led students to find no use of calculators in mathematics and in their daily activities as well. The latter could also have led students to believe that calculators were not useful to their future careers as they dealt with only arithmetical operations, that could not necessarily be required to their careers. Lack of enjoyment to work on mathematical problems using the calculator should have greatly contributed to their anxiety in learning.

While the findings of this study indicate that boys were more advantaged in using this technology, it is concluded that most students of both sexes did not benefit from the use of calculators in mathematics learning. It can further be argued that accessibility to technology did not necessarily mean accessibility to technology use and application. As realized from this study, most students in Botswana are accessible to calculators, but it did not necessarily imply their access to calculator technology. The mathematics syllabus (of Botswana) stipulates clearly the objectives of using calculators in teaching, and as teachers were aware of the role of calculators (Masimaneotsile, 1999), teachers' knowledge on the use of calculators in instruction has to be revisited.

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TEACHING ARITHMETIC AND ALGEBRAIC EXPRESSIONS

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A teaching intervention study was conducted with sixth grade students to explore the interconnections between students' growing understanding of arithmetic expressions and beginning algebra. Three groups of students were chosen, with two groups receiving instruction in arithmetic and algebra, and one group in algebra without arithmetic. Students of the groups that learnt arithmetic developed a strong understanding of the concept of term and applied it to reason about equal expressions. They performed better at some questions in algebra, especially those that required a sense of the structure and meaning of the expression.

INTRODUCTION

Many of the difficulties that school students face in learning algebra may have their source in the poor understanding of two important concepts – the variable and the algebraic expression. Sfard (1991) and Tall (1999) have pointed to the difficulty in understanding the process-product duality of algebraic expressions, which encode both operational instructions as well as denote a number that is the product of these operations. The difficulty in understanding the multiple meanings encoded by expressions may underlie the inability of many students to operate with unclosed expressions (Booth 1984).

It has been recognized that students' understanding of arithmetic and algebraic expressions are interconnected. For example, students who make errors in manipulating algebraic expressions repeat some of these errors when dealing with arithmetic expressions (Linchevski and Livneh, 1999). Many students have a poor sense of the structure of arithmetic expressions and are unable to judge the equivalence of expressions like 685–492+947 and 947–492+685 without recourse to computation (Chaiklin and Lesgold, 1984). Algebra, as generalized arithmetic, symbolizes and exploits the structural aspects of arithmetic. However, it is not clear whether instruction oriented to developing the structure sense of arithmetic expressions transfers to algebra. It might well be the case that learning algebra paves the way for a better understanding of arithmetic expressions since the algebraic symbolism enhances the structure of the expression. Linchevski and Livneh (1999) have recently raised doubts about whether focusing on teaching structured arithmetic as a preparation for algebra is a good pedagogic strategy.

The present research study is aimed at exploring the interconnections between students' growing understanding of arithmetic expressions and beginning algebra. Sixth grade students were taught the topic of arithmetic expressions using a novel approach that emphasized structural aspects. They were also taught beginning algebra broadly following the approach used in school classrooms. The study was carried out in two phases. The first phase was exploratory and was aimed at developing suitable instructional material that allowed students to develop a better understanding of the meaning and structure of arithmetic expressions. The second phase of the study had a two-group design, where one group was taught arithmetic expressions in conjunction with algebra, while the other group received instruction in algebra without instruction in arithmetic.

THEORETICAL FRAMEWORK AND ASSUMPTIONS

In the primary mathematics classroom, children encounter arithmetic expressions as ‘questions’, i.e., instructions to carry out certain operations and produce an answer. The transition to algebra requires pupils to understand the three-fold meaning of arithmetic expressions: process, product and relation. The relational meaning of an expression is expressed when we say that 12+7 denotes a number which is `seven more than twelve’ (or alternatively, `twelve more than seven’). Thus 12+7 and 10+9 denote the same number but express different relations. In algebra, the relational meaning is generalized to the concept of a function. From a pedagogic point of view, the relational aspect is intuitively meaningful for children and can serve as an introduction to the structure of arithmetic expressions.

The equality of expressions is the core notion around which the structure of expressions can be grasped. The instructional units aimed at developing a structure sense in arithmetic were organized around this notion. Pupils were required to compare expressions without computation, by examining the terms in the expression. This provided a meaningful situation for leaning the concept of term and allowed students to appreciate the importance of this concept.

The concept of term and the concept of equal expressions may function as possible loci of the transfer of knowledge from arithmetic to algebra. Thus they may be called ‘bridge concepts’. The emphasis on terms allows students to perceive a number in an expression together with the attached sign. The concept may be extended to include product terms and to bracketed terms, which are again useful concepts in both arithmetic and algebra.

The ability to deal with brackets is a critical component of the understanding of symbolic arithmetic and algebra and underlies the capacity to manipulate unclosed expressions. From the experience of the first, exploratory phase of the study, it was felt that this topic could be more effectively taught as a set of clearly specified and connected rules for removing brackets and rewriting an expression. Rules must be connected to concepts in order to enhance their learning and retention. Since concepts occur as referents in the statement of rules, conceptual misunderstanding may lead to incorrect learning of rules. Pupils need to be flexible in their application of rules and conceptual understanding mediates such flexibility. The evidence that we have gathered so far is not sufficient to throw light on the relation between concepts and rules. Further analysis and study is being planned to explore this aspect.

METHOD

The first phase of the study was carried out in the summer of 2003 over 13 one-hour instructional sessions. It was an exploratory study carried out with a small group of about 15 students about to enter grade 6 (about 11 years of age). The focus of the first phase was to prepare an instructional module aimed at developing a sense of the structure of arithmetic expressions among the students. The students belonged to an urban school and were from a low socio-economic background. Although the medium of instruction was English and the students could follow instructions in English, they were not fluent at communicating in English. Students volunteered to enrol in the program and were short-listed on the basis of a random draw.

The pre-test showed that most of the students could not correctly use the rules of order of operations, or find the value of the unknown in simple linear equations or write expressions for sentences presented verbally. These topics are cursorily presented in the grade 5 textbook, but are usually not emphasized during teaching in grade 5. So the instructional module included exercises where the students learnt to express the relational meaning of arithmetic expressions, translate a sentence into an expression, evaluate expressions, and learnt the meaning of the '=' sign as a relationship between two expressions having the same value. This was essential before the students could judge the equality of arithmetic expressions.

Some students could on their own explain why two given arithmetic expressions were equal. For example, the reason given by a student in order to justify that 27+32 = 29+30 was "take 2 from 32 and give it to 27". In the more difficult examples requiring reasoning about expressions, students needed to be able to deal with brackets. It was evident that some repository of rules is essential to symbolic manipulation and reasoning about expressions. Accordingly, instructional units on rules of order of operations and removing brackets were included in the module.

The second phase of the study was conducted in October-November 2003 to explore the connection between students' knowledge of arithmetic expressions and of beginning algebra. This study had a two-group design. The students were selected in a manner similar to the first phase (by a random draw from applicants) from two nearby English medium schools. These students were studying in grade 6 and had recently been introduced to integers but had not done any algebra. One of the schools was the same as the one that participated in the summer course. The other school caters to students coming from varied socio-economic backgrounds. The students from the second school came from the middle socio-economic group and were relatively more comfortable with the English language although many were not fluent in conversing in English.

A pre-test was given to all the selected students and two equivalent groups were formed on the basis of their performance in the pre-test. Group A had 27 students and group B had 26 students to start with. Some students dropped out and the final analysis was done on 25 students of group A and 21 students of group B. An additional group, group C of 35 students from school using the local language (Marathi) as a medium of instruction, was also included in this study. These students were from the low and middle income group.

The programme was conducted during a mid-term school break and had 11 instructional sessions of one and a half hours each spread over three weeks. Group A and C were taught arithmetic expressions and algebra whereas group B was taught only algebra. Group B had a few sessions on an unrelated topic: activities in geometry.

The arithmetic expressions module taught to groups A and C included the following topics: meaning of arithmetic expressions, meaning of '=' sign, evaluating expressions using the rules of order of operations, the concept of term, comparing arithmetic expressions without calculation and rules for removing brackets. The algebra module taught to all the groups included the following topics: letters and algebraic expressions as standing for numbers, concepts of term, like and unlike terms, rules for removing brackets and rewriting, simplifying expressions and using algebraic manipulation to justify the outcome of guess-the-number games. In the guess-the-number game, students did a series of operations with a chosen number and recorded the final result. The operations were chosen so that all the students obtained the same answer. The students then had to use algebraic manipulation to justify why the answer was the same.

The same instructor taught algebra to students of groups A and B. The arithmetic lessons for Group A were taught by a different instructor. Group C had a separate instructor who taught both arithmetic and algebra in the Marathi language. The lesson plans were carefully drawn up for the three groups, with groups A and C following the same plans. The overall instruction time for the three groups was equal. Group B received more instruction time in algebra than the other two groups. The additional time was spent mainly on simplification of algebraic expressions and applying it to explain the outcome of guess-the-number games. Students of all the groups were regularly given practice exercises and feedback on their learning was collected through tests and worksheets. The lessons were video-recorded for further analysis of the teaching-learning sequence and students' responses. In order to assess the gains in students' understanding in arithmetic and in algebra and to obtain information about the possible interconnections between these, a post-test was administered to all the students. The post-test had components similar to the pre-test, as well as some additional questions testing their ability to simplify algebraic expressions, compare arithmetic expressions without calculations and justify the outcome of a guess-the-number game using algebra.

PRELIMINARY ANALYSIS OF RESULTS

The preliminary analysis reported here is of the performance data from the pre- and post-tests for the three groups. In addition, the justifications that students wrote for their answers to questions on comparing arithmetic expressions have been analyzed.

These have been taken from the post-test and from worksheets that were filled by the students during the instructional sessions.

Arithmetic expressions

The pre-test contained questions mostly on arithmetic. Groups A and B were formed as equivalent groups on the basis of their overall performance in the pre-test. The performance of Group C was slightly lower than the other two groups in most questions in the pre-test. Students from all the groups did well on comparing simple arithmetic expressions using calculation. In these questions students had to fill in the correct sign from '=', '>' or '<', for example, $15 - 5 \ \underline{?}\ 5 \times 3$. In the questions on evaluating expressions using the order of operations rule, many students from the three groups did poorly. In the question asking them to rewrite expressions by removing brackets, students appeared to be unaware of the rule for doing so, and answered such questions by evaluating the expression and doing the bracketed operations first.

In the post-test, groups A and C, which received instruction in arithmetic, improved their performance considerably on the arithmetic questions. As might be expected, students in Group B did not significantly improve their performance on the arithmetic questions, except in the questions requiring them to rewrite the expression by removing the brackets, where there was substantial improvement. In their algebra lessons during the program, the students had learnt the rule for removing brackets. Group B students were able to transfer their new knowledge to arithmetic expressions because the form of the question items in algebra and arithmetic were similar. This transfer may hence have been enabled by superficial features. Notably, group B did not improve their performance on the questions requiring students to equalize expressions by filling in the blanks, on which 80% of the students from the other two groups answered correctly. None of the groups had directly worked with such questions during the instructional sessions.

	Group A		Group B		Group C	
	Pre	Post	Pre	Post	Pre	Post
Comparing expressions by calculation	97	94	83	85	83	93
Equalizing expressions by filling blanks	82	86	69	65	63	80
Evaluating expressions	51	74	58	60	44	70
Removing brackets and rewriting	22	85	28	71	37	91

Table 1: Percentage of correct responses in arithmetic

Structure of expressions and the concept of term

The lessons done with groups A and C on comparing expressions without calculation proved interesting to the students. Such exercises were new to them. The students had to justify their answers in their own words. Their spontaneous reasoning produced responses that we had not anticipated. As described earlier, one student in the first phase of the study spontaneously justified that 27+32 = 29+30 because we can "take 2 from 32 and give it to 27". In the second phase, one of the students produced the following reasoning to justify that 37 – 17 > 37 – 18 : `37 is the same and –17 is greater than –18'. He was making use of a property of negative numbers, a topic that the students had briefly encountered in school. This argument was easily assimilated by many students and was the most frequent form of reasoning produced by them in their written responses to this type of problem.

Such exercises made the concept of term very salient for the students. They tried to apply the concept to other comparison problems, but not always successfully. In some problems the increase in one term was compensated by a decrease in the other, for example, 28 + 15 = 27 + 16 or 36 – 19 = 35 – 18. Such problems elicited a variety of reasons from the students. When the increase in one term was not compensated by a decrease in the other, or when the expressions had three terms each, students had more difficulty in writing reasons, although many were successful in judging which expression was greater.

In writing reasons, especially for expressions that were related in a simple manner, students often abbreviated their justification by referring to the terms not specifically, but positionally. For example, 'the first term is one less, the second term is one more', or 'one term is the same, the other is one less'. When combined with the high rate of correct judgements, such responses may reflect, at least in the case of some students, an increasing ease with such problems and a trend towards a stable perception of a structural pattern.

The concept of the product term was introduced to better understand the equivalence of expressions such as 43 + 68 + 32 × 35 and 35 × 32 + 43 + 68. Group C was able to apply this concept more successfully than the other groups. In the posttest for this question, students from Group C obtained 81% correct responses as against Group A – 48% and Group B – 40%. However, most students across all the groups were unable to judge the equality of 423–236+423+236+102 and 423+102+423, or the inequality of 423+236–423+236 and 423+236–(423+236). Groups A and C performed slightly better (40% and 24% respectively) on these questions than Group B (14%).

Symbolization

On the questions where students had to write algebraic expressions corresponding to given phrases, students from all the groups improved markedly. All the groups had received instruction on such questions and groups A and C learnt to symbolize both in arithmetic and in algebraic contexts. For the simple phrases such as 'three more than a' or 'six times a number t', all students of groups A and C wrote correct expressions, and group B had over 80% correct answers. For the phrase 'seven less than x', many students from groups A and B incorrectly wrote 7–x. However, all students of group C wrote the correct expression x–7. It turns out that while the word order in the English phrase follows the order in 7–x, the word order in the Marathi phrase follows the order in the correct expression.

Two questions taken from the SESM study were included in the pre- and post-tests (Booth 1984). In the first question, the students had to find the perimeter and area of a rectangle with sides m+3 and c. In the second, they had to find the perimeter of a star shaped figure with k sides each of length 5. These questions elicited near zero correct responses from all the groups. A simpler version of the second question, finding the perimeter for a star with 10 sides, each of length 4 elicited over 60% correct responses in all the groups. Clearly, knowledge of the concept of perimeter was not the obstacle. Further, in the guess-the-number game, most students chose a letter to represent the unknown and to write an expression, although only a few students could successfully manipulate the expression. Hence we find students able to symbolize using letters in the context of the guess-the-number game, but not in the perimeter and area questions.

Algebraic expressions

In the questions where students had to identify the terms in algebraic expression, groups A and C obtained over 90% correct responses, while group B had about 50% correct responses. In the questions on simplifying algebraic expressions, groups A and C (27% and 25% respectively) performed well below group B (46%). The total time spent by Group B on the simplification topic and on guess the number exercises which required simplification was about twice the time spent on these topics by Group A and C. However, on the questions where students had to identify expressions equivalent to a given algebraic expression, group A (63% correct) performed better than groups B (45%) and C (39%).

SUMMARY

The qualitative analysis of students' reasoning in comparing arithmetic expressions threw up some interesting results. There is positive evidence of the fact that students are able to perceive the structure of expressions from the widespread and ready use of the concept of term in justifying their comparisons and the near absence of justification based on computation. Some students referred to terms by their position rather than specifically, indicating a growing stability in the perception of a structural pattern.

We had hypothesized that learning the bridge concepts of term and equal expressions in meaningful arithmetic contexts prepares a base for a strong understanding of these concepts in algebra. The results from the teaching intervention are suggestive but not definitive. Groups A and C, which received instruction on arithmetic and algebra, did significantly better than Group B in writing an expression for a verbal phrase, and in identifying terms in algebraic expressions. Group A did significantly better than the other two groups in identifying equivalent expressions.

Students learnt to open brackets by using rules in the context of arithmetic or algebra or in both contexts. On these questions, all the groups improved their performance over the pre-test performance. Since group B did not receive any instruction in arithmetic, this is an instance where the students applied a rule learnt in the context of algebraic expressions to arithmetic expressions. It is not clear if any conceptual transfer is involved here since the rules could be applied purely on the basis of similarity in the form of the questions. Concepts are used in the formulation of rules and hence better conceptual understanding may lead to better learning and retention of the rules. The connection between concepts and rules in the context of manipulation of algebraic expressions needs further exploration.

Although the groups receiving instruction in arithmetic improved their performance on questions requiring the opening of brackets, we did not find direct evidence of its improving performance in algebraic manipulation. The most likely explanation for this is that the time spent on instruction on the latter topic was too short.

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AN INTRODUCTION TO THE PROFOUND POTENTIAL OF CONNECTED ALGEBRA ACTIVITIES: ISSUES OF REPRESENTATION, ENGAGEMENT AND PEDAGOGY[1]

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We present two vignettes of classroom episodes that exemplify new activity structures for introducing core algebra ideas such as linear functions, slope as rate and parametric variation within a new educational technology environment that combines two kinds of classroom technology affordances, one based in dynamic representation and the other based in connectivity. These descriptions of how mathematical and social structures interact in the classroom help account for significant algebra learning gains in recent SimCalc teaching experiments among 13-16 year old students.

A long-term goal of the SimCalc Project (Kaput, 1994) has been to exploit technology's capacity for interactive visualization tools and simulations linked to mathematical representations to provide an alternative to the algebraically based prerequisite structure of topics such as calculus to avoid the algebra bottleneck and democratize access to big mathematical ideas that are now inaccessible to the great majority of students due to the algebra barrier. But another shorter-term objective has emerged, partly as a result of the need to work within the existing structures and capacities of curricula and schools, and partly in response to the need to serve today's students, who cannot wait for long term strategies to take hold, no matter how promising.

Hence within the past five years, the SimCalc Project has developed strategies that use the interactive representational affordances of technology (visualization, linking representations to each other and to simulations, importing physical data into the mathematical realm in active ways, graphically editing piecewise-defined functions, etc.), to energize and experientially contextualize existing algebra courses, and to do so in ways that lay the base for more advanced mathematics, particularly calculus. Recently, we have been studying the profound potential of combining the representational innovations of the computational medium (Kaput & Roschelle, 1998) with the new connectivity affordances of increasingly robust and inexpensive hand-held devices in wireless networks (Roschelle & Pea, 2002) linked to larger computers (Kaput, 2002; Kaput & Hegedus, 2002).

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[1] This research was funded by a grant from the National Science Foundation (REC# 0087771). The opinions expressed here are those of the authors and do not necessarily represent those of the NSF.

EMERGING THEORETICAL COMMITMENTS

We have come to see classroom connectivity (CC) as a critical means to unleash the long-unrealized potential of computational media in education, because its potential impacts are direct and at the communicative heart of everyday classroom instruction – more so than internet connectivity. We are now beginning to build insight into how those new ingredients, in combination, may provide the concrete means by which that potential may be realized, because they may, in fact, help constitute the *first truly educational technologies*, intimately situated within the fundamental acts of active teaching and active learning. This embeddedness may indeed be more profound than we initially recognized, because these ingredients resonate deeply with broader views of learning as participation (Lave & Wenger, 1991) and no longer fit within a "learning in relation to a machine" (large or small, in the lab, classroom or even your hand) view of educational technology. Indeed, the paradigm is shifting towards one where the technology serves *not* primarily as a cognitive interaction medium for individuals, but rather as a much more pervasive medium in which teaching and learning are instantiated in the social space of the classroom (Cobb, 1994). We deliberately choose "instantiated" ahead of "situated" (Kirshner & Whitson, 1997) because we have repeatedly seen mathematical experience emerge from the distributed interactions enabled by the mobility and shareability of representations. The student experience of "being mathematical" becomes a joint experience, shared in the social space of the classroom in new ways as student constructions are aggregated in common representations – in ways reminiscent of, but distinct from those of a participatory simulation (Stroup, 2003; Wilensky & Stroup, 1999). This epistemological shift in the place of technology in classrooms is fundamental to our theoretical perspective.

CONTEXTUALIZATION OF VIGNETTES IN PRIOR SUCCESS STORIES

The empirical work behind this report investigated the impact of our constellation of technological, curricular and pedagogical innovations on student learning, *especially as measured by independent standard test items* on a pre/post-test basis. They include intense teaching experiments aimed at core algebra topics in middle and high-school classrooms in both Massachusetts and California. Results demonstrate comparably positive outcomes under substantially different instructional and technological conditions, somewhat different curricular targets, and different student demographics (Tatar, et al, in submission).

Pre/post-test measures (see http://www.simcalc.umassd.edu/) triangulated with observational video and field-note data in these and other instructional situations in undergraduate classrooms provide evidence of significant improvements in students' algebraic thinking as measured by students' performance. For example, in our after-school intervention (n=25), students performance was significantly better on post tests ($p<0.001$), with high effect sizes (Cohen's d=1.80sd) and strong gains occurring across disjoint populations (middle and high school) with relative statistical independence of prior knowledge, based on Hake's gain statistic (Hake, 1998) –

(<Post> – <Pre>)/(1 – <Pre>), reflecting strong impact of the intervention itself as exploiting very different kinds of prior knowledge in the two subpopulations (see Hegedus & Kaput, 2003; under review, for further details).

PARTICIPATORY AGGREGATION OF STUDENT CONSTRUCTIONS TO A COMMON PUBLIC DISPLAY

In a connected algebra environment the class is typically subdivided into numbered groups, where the size and number of groups fit both the given size of the class and the mathematical activity (ranging from the whole class to pairs). Groups are often, although not necessarily, defined "geographically" – students who are physically near each other. The students usually also "count-off" inside the group, so that each student then has a two-number identity that can then serve as "personal parameters," a Group Number and a Count-Off Number. Students then create mathematical objects – in the cases discussed here, linear Position vs. Time functions that drive animated screen objects. The functions depend in some critical way on students' respective personal parameters either on a hand-held device or on a computer. SimCalc MathWorlds runs on the TI-83Plus as a Flash Application (Calculator MathWorlds - CMW), and on desktop computers as a Java Application (Java MathWorlds - JMW). When using CMW or JMW (the scenario for this report), students' constructions are uploaded to the teacher where they are aggregated, organized and selectively displayed using JMW and discussed.

Staggered Start, Staggered Finish (Y=mX+b): The simplest case is to produce families of functions defined by a single parameter, such as the "b" in Y=2X+b, where b varies according to, say, Group Number. Then each student in a group produces the same position function (Y=2X+ Group Number) on his/her own device, so a given group's linear graphs all overlap (same Y-intercept and same slope = 2), and the different groups' graphs are parallel. When animated, the screen objects "representing" the members of a given group move alongside each other "as a group," while the different groups move at the same velocity but are offset by their initial positions (see Link 1 at http://www.simcalc.umassd.edu/PME04.htm). Here, the group provides mutual support, and it is of special interest that the superposition of graphs provides a strong visual realization of function equality.

By clicking on the overlapping graphs, we can bring different graphs of a group to the front. Further, the graphs and their corresponding objects ("Dots" in "Dots World") are color-coordinated, so the sequence of graph-colors that appear matches the sequence of colors of the objects of the given group. In addition, with the View Matrix (see Link 2 at http://www.simcalc.umassd.edu/PME04.htm), the teacher has virtually full control of which graphs and objects are displayed, and how they are organized or colored – e.g., by Group Number or by Count-Off Number.

With this simple example in mind, we present two vignettes that correspond to some of the learning gains demonstrated in our interventions.

Vignette 1: Varying M systematically – Slope as Rate both positive and negative

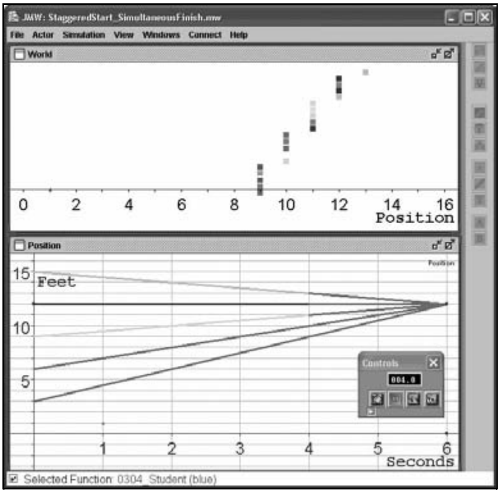

Figure 1: Simultaneous Finish

The Staggered Start – Simultaneous Finish activity is more complex than the former and requires the students to start at 3 times their Count-Off number but "end the race in a tie" with the object controlled by the target function Y=2X (so the target racer moved at 2 feet per second for 6 seconds and started at zero – see the bottom graph in Figure 1). Students now need to calculate how fast they have to go to end the race in a tie. And since they start at different positions, the slope of their graphs changes depending on where they start, which in turn depends on their personal Count-Off Number. Each group is limited to 5 people and while the group number does not affect their constructions it gives rise to a smaller, more manageable set of functions to discuss (see Link 3 at http://www.simcalc.umassd.edu/PME04.htm). Secondly, and more importantly, the Count-Off Numbers 4 and 5 give rise to two important slopes. The person with Count-Off Number 4 has a graph with constant slope, Y=0X+12, since he starts at 12 ft, which is the finish line, so he does not have to move! The person with Count-Off Number 5, starts beyond the finish line (15 ft) and so has to run backwards, thus forcing the student to calculate a negative slope. We have observed students in a variety of settings using various strategies including: numerical trial-and-error, inputting values into an X-coefficient field in the algebra window (a feature of the software for algebraic editing), slope-based analysis ("What slope to I need to connect my starting position to (6, 12)?"), velocity-based analysis ("How fast do I need to move get to 12 at 6 seconds?") and, by an implicit parametric-variation strategy, comparing what others had done in their group and averaging (i.e., observing that the slope of each person's graph changed by 0.5 as the Count-Off Number increases by 1).

Organizing and displaying student work is a strategic pedagogical decision, e.g., in focusing student attention on the underlying mathematical structure. An important question to ask before animating the motions for the whole class is "What will the race look like?"

The aggregate motion in the classroom display becomes a personal reality through the personal link with Count-Off Number. By aggregating and displaying the class work, students can observe how their personal construction fits into the "race" and allows them to note how people from other groups had constructed identical motions because of their Count-Off Number. In addition the shape of the graph and the parity of the slope for those who had to start past the finishing line (i.e. run backwards) was made more realistic and understandable in this motion-based scenario. In discussing different strategies and making these explicit in publicly examining outliers (incorrect answers) and natural outliers (motions for students with Count-Off Numbers 4 and 5) we have some evidence to why students began to improve on items of the pre-post test which involved slope analysis.

Vignette 2: Using both Group Number and Count-Off Number – creating "fans" to address the idea of slope-as-velocity

In this vignette we discuss an activity structure where we systematically increase the complexity of the variation by using the other part of the unique identifier – the group number. Students now have to create functions so that their object will travel at a ***velocity equal to their Count-Off Number*** but ***start at their Group Number***. We highlight here the move from a personal individual construction to a significant group structure to a class aggregation. Each group creates a family of functions, which is similar in shape ("a fan") to every other group but offset by starting position.

Students anticipate the visual form of the aggregation as is highlighted in the transcript, where the class was asked what the collection of graphs would look like.

J	It's gonna start at two, and it's gonna end at five, and… it's gonna look kinda like a fan. And, they're all going to start at the same place. <SNIP>
T	So, he's saying they're gonna start there, and then it's gonna kinda look like a fan?
J	Yeah they're gonna… {*spreads out fingers wide on one hand*} like that.
T	… You like that? {*hands up for more than half the class*}
	{Clapping and a few 'yeahs', as the results are displayed}

Here, John (J), indicates to the class physically with his hands (where his fingers resemble individual graphs) that the aggregation will resemble "fans" how these groups of functions will be displayed relative to each other. But although John knew where his graph was in the aggregate, he could not explain why in technical, slope terms. Here we highlight the social dynamic enabled by such a public event, which allowed two other students Alison (A) and Robert (R) to explain why John's graph (with slope 3) is the one highlighted by linking its slope to the velocity.

A	Go by velocity… however many… what number in a group you're in… how many increments he goes.
T	Okay. So, he's the third member of the group. So..
A	So he can go three times every second. Up three every second.
T	… How can we determine, Robert?

R See how far it goes… look between zero seconds and one second?

T You want to come show us?

R Okay. {Robert goes to the display and inscribes the first one-second segment of the graph with a 1-wide by 3-tall rectangle}

We observed a high level of engagement by the class and reaction to the animation of the aggregate, which resulted in a wide distribution of the actors by the end of the animation. This was an opportunity to highlight how some people finished at the same place and time but followed different motions, without delving into the algebraic detail of why this occurred. This activity intends to provide an experience embodying parametric variation across both the graphical representation (see figure 2) and the motion-based gestalt of the animated objects – just as the collection of graphs has a gestalt-shape captured in the fan metaphor, the collective motion has a gestalt (see (see Link 4 at http://www.simcalc.umassd.edu/PME04.htm) for a sense of it.

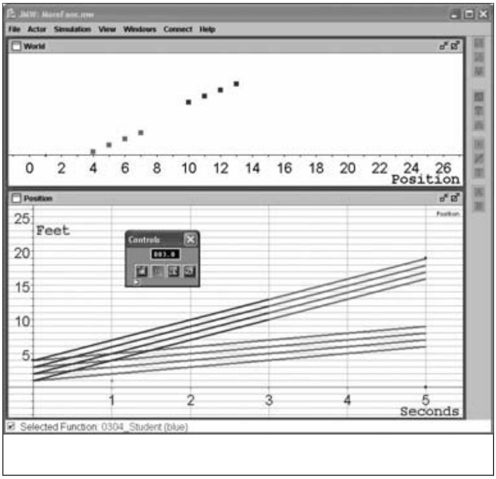

Figure 2: Making “fans” from two groups

We also highlighted and examined outliers, making a conscious decision in this case not to hide the names (or identifiers) of the students, although we note the issue of student anonymity in class (Scott, 1999). The software includes the ability for the teacher to hide names of selected functions if desired (identifying information appears in the lower left hand corner of the screen). Nevertheless, the class often took the initiative to determine who the outlier was, what the underlying error was, and then collectively correcting the mistake. In effect, the mathematical criteria of consistency comes to be socially embodied in class norms of “correctness” and coherence.

At this stage, differences in students’ work are based on inter-group variation (similar fans but off-set) and intra-group variation (the establishment of one fan where individual Count-Off Number varies slope). To further build meaning of how variation in their identifiers leads to variation in their corresponding graphs, we reversed the roles of Group Number and Count-Off Number: We asked students to repeat the task but now construct a motion where they travel at a ***velocity equal to your Group Number*** for 5 seconds, and ***start at your Count-Off Number***. Here students will produce a visually similar class aggregate, but their personal graph is now part of a fan constructed by members of other groups. In fact, their own group now constructs families of parallel lines, and, of course, their motions differ accordingly from the previous case.

CONCLUSION: ADDRESSING NEW PEDAGOGICAL DECISIONS

The connected SimCalc algebra classroom opens up a new learning environment for students, with increased intensity, structures and levels of participation. In presenting these brief vignettes outlining the student activity and decisions the teacher made during the post-aggregation phase of the activity (space prevents descriptions of within-group interactions), we have begun to describe how students begin to develop an understanding of one of the core ideas of high school algebra, slope-as-rate-of-change. Through such activities, students have both an individual "mathematical responsibility" to either their group or to the larger class via construction or interpretation of shared mathematical objects, as well as a vicarious participation in the joint construction upon aggregation. With careful pedagogical decision-making by the teacher students' attention is now moved along a trajectory from static, inert representations, to dynamic personally indexed constructions in the SimCalc environment on their own device, to parametrically defined aggregations of functions, organized and displayed for discussion in the public workspace.

Substantial teacher knowledge, a deep composite of content and pedagogy, is needed to facilitate movement along this trajectory and focus public mathematical dialogue on critical features of these visually shared objects to develop meaning. This is hinted at in the last illustration, where Count-Off and Group Numbers are interchanged. In effect, connectivity supports the pedagogical manipulation of student's focus of attention. But the teacher knowledge needed to take advantage of a connected classroom requires extended development.

How do your -		
Motion(s) Graphs Formula Tables	Look Different as Look the Same as	An Individual vs. the Group An Individual vs. the Class The Group vs. the Class

Table 1: Constructing pedagogical actions

Table 1 outlines a simple structure, which can guide the teacher's inquiry. Choosing one item from each column leads to a particular question that can be addressed to individual students, groups or the whole class. Successively using this outline might assist teachers in moving students along suitable learning trajectories in this environment and elevating mathematical attention.

In our later activities in the interventions, aggregation was used as a means for generalization and abstraction. As more work is conducted publicly, learning increasingly occurs in the social space to complement the individual device-interaction space. Encouraging students to make sense of their constructions publicly, annotating their graphs on the display space, and systematically highlighting parametric differences leads to more powerful understandings.

Having highlighted the benefits of social engagement in these activities, we also need to investigate in more detail the potential for negative social implications in some of these pedagogical actions and classroom decisions (e.g. issues of privacy, social embarrassment). We are confident, however, that by combining the two key ingredients, dynamic representations and connectivity technology, students can better understand fundamental, core algebra ideas by forming new, personal identity-relationships with the mathematical objects that they construct individually and collaboratively with their peers.

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THE TENSION BETWEEN TEACHER BELIEFS AND TEACHER PRACTICE: THE IMPACT OF THE WORK SETTING

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This paper presents part of an ongoing project on teachers' beliefs and practices in state schools and in privately owned exam preparation schools in Turkey. Extracts from an interview with a teacher who uses a technique that he disapproves of will be reported and discussed. The paper considers how the teacher reconciles his practices with his beliefs, drawing on both social psychology and socio-cultural perspectives.

INTRODUCTION

Research on teacher beliefs and goals is common in educational research, as is research on teacher practice, but pursuing the relationship between them has only recently been considered as an important issue of mathematics education. It has been established that teachers' beliefs about mathematics, teaching and learning have a significant influence on their instructional practices (Calderhead, 1996). In some cases, research has found that teachers' beliefs about mathematics, teaching and learning are consistent with classroom practice (Thompson, 1985). However, Thompson (1984) and Raymond (1997) have documented inconsistencies between professed beliefs and observed practices, with an implication that the teachers were unaware of a conflict between them. The assumption is usually that awareness of a difference between beliefs and practice would result in some attempt to change (Lerman, 2002). This paper reports a situation in which there is a tension between a teacher's beliefs[1] about how mathematics should be taught and his own classroom practice, and yet, despite showing an awareness of the conflict, the teacher is not trying to change.

THE RESEARCH

In this paper, we present a part of an ongoing project to investigate some teachers' beliefs about teaching and learning and their actual practices, in two different contexts in the Turkish education system for 17-18 year olds. Students of this age in Turkey are taught mathematics in two places. They attend state schools (SS), but at

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[1] Beliefs have been regarded as a "messy construct" (Pajares, 1992) - difficult to define as well as to elicit. We will therefore make use of Schoenfeld's (1998) concept of "professed beliefs" throughout the paper, so that whenever we refer to 'beliefs' this should be understood to mean professed beliefs, with no guarantee that they reflect genuine beliefs.

Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 2004 Vol 3 pp 137–144

weekends or in the evenings most of them also attend courses in privately owned schools (PC). The main objective of such private school courses is to prepare students for the university entrance examination (UEE), which is made up of multiple-choice questions. Private courses, rather than state schools, are the institutions where teaching for the multiple-choice tests is practiced. As one teacher put it [translated from the Turkish]:

> "The aim of mathematics teaching [in PC] is not to teach mathematics basically, but to prepare students for the examination they will take – to make them able to answer the questions that they will face in the examination in the most practical and easiest way. Our aim is not teaching mathematics deeply and with its theory. As an educator in private courses, our aim is to prepare them for the examination in a practical manner."

Teachers from each kind of institution (SS and PC) were interviewed using the hierarchical focusing technique (Tomlinson, 1989) and their lessons video recorded. The research reveals a widespread contrast in PC teachers between their beliefs about mathematics teaching, as expressed in interview, and their observed practices.

In this paper we will draw on the data obtained from a single PC teacher, a mathematics teacher with eleven years of experience of teaching in different PC. He has been chosen as the case study for this paper, because not only is he aware of the discrepancy between his beliefs and his practice, he is able to reconcile the two.

FINDINGS AND DISCUSSION

Extracts from the interview with the teacher are presented below. In the first, the teacher talks about a special problem solving method for multiple-choice questions, called 'numerical value technique' (NVT). Prior to the presentation of the actual transcript, we will clarify what is meant by NVT. Here is a typical UEE question:

For given $x < 3$ and $f(x) = x^2 - 6x - 2$ what is $f^{-1}(x)$?

$a)3 - \sqrt{x+11}$ $\quad b)3 + \sqrt{x+14}$ $\quad a)2 - \sqrt{x+14}$ $\quad a)2 - \sqrt{x+11}$

The essence of NVT lies in assigning a simple numerical value [like 0, 1 or 2] to each of the variables given in the root problem [lets say x = 2] and calculating the result with these values [f(2) = –10]. Then, the output values are assigned to the variables in each of the options and calculated. Since the options represent the inverse function, then for x = –10 we must get the result of 2. Only option (a) gives 2, so the correct choice is (a). If more than one option gives the required value, different numbers are given to the variables in the root expression, and the options are re-tested.

NVT enables students to solve problems in a very short time and usually without any recourse to the theoretical knowledge supposedly required by the question. The classroom observations as well as researcher's interactions with several PC teachers suggest that this method is commonly practised in PC as a method to reach the correct answer in a relatively easy way. As far as we are aware it is not used in SS. In the following excerpt the teacher describes his attitude to the use of this technique.

I: What do you think about using numerical values to solve problems?

T: Yes, this is a part our system. In terms of university preparation, preparation for university entrance examinations, this is part of our system…Using numerical values is of interest to them and they like it very much. 'Let's assign 1 to the value of 'a', and after that, lets give the options, lets put 1 for wherever you see 'a', what a simple thing, isn't it!' This is a part of our system, I mean, as a private course it is a part of us, we make use of it.

I: Do you mean it is one of the indispensables of private courses?

T: To me[2], look, sometimes you may not be able to remember the solution of a problem. Because the student may become nervous during the examination s/he may not be able to do things s/he can do normally. But if you approach them like 'you can solve it using numerical values' s/he can make use of a second method and s/he can possibly solve the problem in a practical manner with ease

I: Do you think it is a healthy method in terms of mathematics?

T: In terms of mathematics teaching it is not a healthy method. Because it keeps students away from formulas, it keeps them away from definitions. I mean without understanding the definition, without understanding the formulas, they want to solve problems. That's not healthy in terms of mathematics education.

Although NVT is taught efficiently and effectively, at the same time the teacher states that he regards NVT as not a healthy way to teach mathematics (lines 15-18). From his point of view, it keeps students "away from" a theoretical understanding of mathematics, a clear indication of disapproval of the technique that he teaches. It could be argued that the teacher's disapproval is not particularly deep-seated, and was perhaps a product of the question and the interview context rather than his true belief. This has to be acknowledged as a possibility, which the reader can consider by examining the excerpts given. However, the totality of the interaction with the teacher strongly suggests that the teacher did care about the issue. In the following excerpt, the teacher shows his disapproval in terms of the consequences of the context in which the teaching and learning takes place.

I: I came across situations where a student brings a problem to the teacher and asks questions like 'What should we do for this type of problem?' Do you think this mechanises mathematics? I mean, does what is taught in private courses mechanise mathematics?

T: Of course. I mean, whether we like it or not, we do it. There are some [question] forms and these forms should be learned. The aim here is to bring the students to such a level that they can solve these forms. Whether we like it or not, we have to mechanise a bit.

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[2]Considering the original interview recording, there is an unuttered but definite 'yes' at this point.

There is, it seems to us, a clear tension between beliefs and practice within his words, as well as the influence of the work setting, as shown by the expressions ‘whether we like it or not’ (lines 23; 25-26) and ‘we have to’ (line 26).

One of the crucial points in understanding this teacher, from our perspective, is that he is aware of the tension. If the teacher was unaware of the tension, as in the case reported by Raymond (1997), then one would not expect him to attempt to address it. Yet despite his awareness, he did not seem to be uneasy about the conflict.

I: Is it [NVT] characteristic of mathematics in private courses?

T: Whether one likes it or not because the characteristic of the University entrance examination is to deal with the practical side of mathematics[3]. In the exam it is not important the way you solve the problem, it is not important how the student solves the problem.

I: Let’s ignore state school mathematics or private course mathematics. Considering using numerical value to solve the problem, do you think it is an ideal way to teach mathematics?

T: It is not healthy. In my opinion, solving a problem using numerical value is only going for an easy[4] ride. But it perfectly fits with the private course approach. It attracts students’ attention. Students like it because students’ aim is solving the problem in any possible way, but it is not an ideal way to teach mathematics. From my perspective, it is going for the easy way, a kind of escape to an easy way.

Again the tension is prominent in lines 28, 35-36, and 38-40. The teacher does not feel comfortable with NVT. He claims that it is ‘going for an easy ride’ and it is not an ideal way to teach mathematics. However, he says ‘But it perfectly fits with the private course approach’, bringing his work setting into perspective.

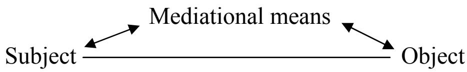
A description of this situation in socio-cultural terms can help to explain how this is possible. In the last few decades there has been a considerable shift in educational research from purely cognitive approaches to socio-cultural or cultural-historical studies in which the setting is considered as an important parameter to take into account. This change can be seen not only in educational research but also in the research of other social sciences (e.g. Chaiklin and Lave, 1993). Derived from writings of Vygotsky and his followers, the socio-cultural approach is based on the claim that “human action typically employs ‘mediational means’ such as tools and

[3] Here the term ‘practical’ refers to solving problems in a shortest time and quickest way without dealing with any theoretical aspect of the problem at all.

[4] The Turkish word ‘kolay’ is translated as ‘easy’ but this does not exactly correspond to the its meaning. The reader should take into account that ‘kolay’ clearly connotes disapproval in this context.

language, and that these mediational means shape the action in essential ways" (Wertsch, 1991, page 12). A "mediational means" stands between a subject (a person) and an object (a goal or purpose) as the practical way through which the object is achieved (see Figure 1).

Figure 1: an action triad

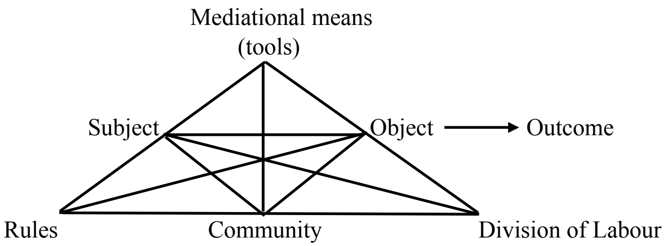


When a tool or other mediational means has been used frequently in a particular setting, the subject's actions are mediated by it, and the subject begins to "appropriate" the tool – that is to make it his / her own (Wertsch, 1998). Appropriation refers to the process though which a person becomes sufficiently familiar with a mediational means / tool that he or she is able to use it purposefully and flexibly in particular social environments (Grossman et al., 1999). The tools a person adopts and uses fundamentally affect his or her practice.

In the PC context the teacher's goal (object) is to make the students, as he puts it, "able to answer the questions that they will face in the examination". NVT is a 'mediational mean', or tool, towards the object of examination success. Furthermore, for this teacher (the subject), the technique has been 'appropriated', as suggested by lines 2-7, and this has been observed to have affected his practice fundamentally.

The importance of the work setting in a socio-cultural analysis is in putting the goal or purpose into a broader context. If the teacher did not work in a private school, it is likely he would have a different 'object', and would probably use different mediational means to attain it, since as Nikiforov (1990) observes, the goal constrains the options for action. The work setting is crucial to the conflict between the teacher's beliefs and his actions. In the activity theory of Engeström (1993, 1999), the 'triad' above is extended to include elements of the social context (see Figure 2).

Figure 2: an activity system



In the activity system of the PC work setting, NVT is regarded as a 'mediational mean' towards the goal of the activity partly because, as classroom observations suggest, it is a common technique used by PC teachers towards a shared goal (the outcome) of the PC community. The activity of the preparation of students for the examination in PC is therefore mediated by NVT. The expression ‘as a private course it is a part of us’ (lines 6,7) gives a clear signal of this. The teacher considered NVT to be indispensable for teaching in PC. One way of thinking of the teacher’s use of a technique that he disapproves of is that his goal-in-context has such an influence on him that he feels an obligation to use the generally deployed means to achieve it.

Socio-cultural theorists commonly refer to the ‘tensions’ or ‘contradictions’ in and between the components of an activity system as triggers for change (Engeström, 1993, 1999). Yet the tension between the beliefs and practice of this teacher do not seem to be bringing any pressure for change – any attempt, for example, to give up NVT or to make it more educative. In that respect, the socio-cultural perspective does not seem to have “action” to analyse. So, despite the invocations of Wertsch (1998) to “live in the middle” between the ‘reducing’ perspectives of psychology (when it considers the social context only in relation to its impact on the individual) and sociology (when it considers individuals only in relation to their impact on the social) it seems necessary to look to social psychology for an explanation of why the teacher’s inner cognitive conflict is not a motivation to change.

According to Eagly and Chaiken (1993), the dissonance theory of Festinger (1957) is the most widely accepted account of cognitive sources of motivation for change. This approach is based on the principle that disharmony among cognitive “elements” (people’s mental representations of their beliefs, attitudes, and attitudinally significant behaviours, decisions and commitments) motivates cognitive changes designed to restore harmony. Festinger (1957) suggests four such changes, which are presented here in a form pertinent to the present case:

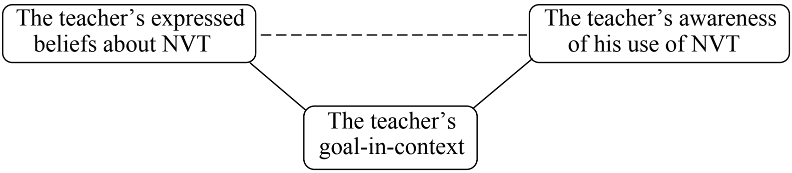
1. The teacher avoids thinking about the elements that conflict (specifically his belief that NVT is harmful), and so experiences disharmony only transiently;
2. The teacher tries to change his practice (e.g. stops using NVT);
3. The teacher changes his beliefs (e.g. comes to consider NVT less harmful);
4. The teacher incorporates into his thinking additional “consonant elements” which effectively ‘water down’ the tension between his beliefs and his practice, and thereby reduce the motivation to change.

The fourth of these seems to fit the current case. In Festinger’s (1957) analysis the use of NVT by a principled teacher might be seen as an example of “forced compliance”, which as his practical studies show, can be lived with on the basis of financial reward. The teacher has chosen to work for the private school, where the pay is good and the status is high, and accepts the consequences of doing so. However, this may be an example of focusing too much on the subject in isolation (Wertsch, 1998) and it does not adequately explain the teacher’s ‘appropriation’ of NVT, which is beyond mere compliance. A socio-cultural perspective would suggest that the additional “consonant elements” pertinent to the motivation of the teacher are provided by his participation in the “activity” of PC. The shared goals and collective action seem to be sufficient to dissipate the motivational effects of tension between beliefs and practice. This is evidenced by the mode of speaking of the teacher when referring to his disapproval of the action he takes, in that he makes statements of facts (how it is) rather than expressing values (how it should be). In Wagner’s (1987) terms, he uses the indicative mode, rather than the imperative mode.

CONCLUSION

The research presented in this paper highlights the significance and fundamental effects of ‘mediational means’. As observed throughout the teacher’s classroom activities and in the interview, his practice has been ‘mediated’ by the numerical value technique. Yet the teacher’s beliefs about NVT were negative. This had the potential to create a tension that might have motivated change, but did not. Careful consideration of the excerpts from interview has suggested that the teacher reduced the possible conflict by referring to the work context and, in particular, his goal as a teacher in that setting. Using Festinger’s (1957) vocabulary, he added a new “consonant element” to his thinking to avoid the conflict. Figure 3 schemetises this situation, where the dotted line indicates the potential tension / conflict.

Figure 3: A representation of the teacher’s inner conflict.



The study also documented that teachers’ goals, in particular when they are ‘imposed’ upon teachers (from the teachers’ participation in an activity system) are of such importance that it can lead to classroom practices that conflict with their beliefs. Although it is generally felt that “Beliefs have a strong shaping effect on behavior” (Schoenfeld, 1998, page 19), teachers’ beliefs about how mathematics should be taught can be overwhelmed by the goals in particular settings. In other words, goals can drive actions more than beliefs do.

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WHAT A SIMPLE TASK CAN SHOW: TEACHERS EXPLORE THE COMPLEXITY OF CHILDREN'S THINKING

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The Early Numeracy Project provided the opportunity for teachers to engage over a three year period in collaboratively developing resources to support early numeracy. Teachers field tested assessment items, and found one particular item to be particularly revealing about children's understanding of number. This paper discusses what it is about the task that captured teachers' interest. Task complexity, representational level, use of imagery and spatial reasoning, and the importance of disposition are described as possible ways to expand our lens beyond strategy use to more fully appreciate the complexity of children's thinking.

Research provides us with a variety of frameworks for considering children's development of progressively more sophisticated and complex understanding of number. Carpenter and Moser's (1984) 3 year longitudinal study of children's solution strategies for addition and subtraction, resulted in a framework for analyzing children's understanding of number, which provided the basis for Cognitively Guided Instruction (Carpenter & Fenemma, 1999). Steffe and Cobb's (1988) work on children's arithmetical meanings and strategies provided a framework that has been widely used, and is the basis for a number of subsequent projects related to children's mathematical thinking, such as the number framework developed by Wright (1998). The Early Numeracy Research Project (Clarke et al, 2001) built on some of this earlier work to provide sets of growth points for considering the development of children's mathematical ability. The British Columbia Early Numeracy Project [ENP] drew on these resources to shape an early numeracy assessment and follow up instructional resources.

The British Columbia Early Numeracy Project (Ministry of Education, 2003) was a three year collaborative initiative involving the B.C. Ministry of Education, University of British Columbia mathematics teacher educators, and teachers from six school districts in the province. The project's focus was to learn more about the ways to best assess and support the development of numeracy in the early grades. Project goals include involving teachers and researchers in: 1) the creation and use of performance-based tasks most appropriate for assessing numeracy in young learners; 2) the development and refinement of instructional strategies to support numeracy in school and at home; and 3) the development of reference standards on key assessment items that provide a portrait of young students' mathematical thinking. Sixteen project teachers extensively field-tested the ENP items while many teachers across the province field-tested items in their own classrooms as part of their school district's professional development initiatives on early numeracy. In the course of developing an early numeracy assessment with a focus on at-risk students, over 200 teachers field tested the assessment items. In professional development feedback sessions, teachers met to discuss the items, tried them with their students, then met again to analyze the results. Video tapes of children doing the items were used as the basis for these discussions, and were used in conjunction with a research based scheme to develop interpretation and scoring frames of reference (Kelleher, 1996; Nicol & Kelleher, 2003). Teachers were asked to comment on which ENP assessment items were most and least useful to them as teachers. In written and verbal feedback, 70% of the teachers ranked the Build and Change item above the 16 other assessment items. This frequent response prompted us to explore what it was about the task that teachers found so illuminating, and what aspects of mathematical activity were revealed by the task.

The “Build and Change” task involves students building a set of blocks, then changing the number as directed by the teacher, but first telling what they have to do to get the new number. Warm up examples are provided to ensure children understand the directions, and to ease them into predicting ahead of actually making the change. Directions are as follows:

Provide blocks or unifix cubes, and give students two warm-up examples:

“Show me 5 blocks. Now change it to 3 blocks.

What did you do?" (e.g., I had to take away two blocks.)

"This time tell me first…(fold your arms or sit on your hands)… how can you change your 3 blocks to 6 blocks? What will you need to do?"

You are working towards having the child tell you first what needs to be done. Provide another example if needed.

Examples for scoring:

a) Change 6 to 4

b) Change 4 to 8

c) Change 8 to 5

d) Change 5 to 12

Can the child predict what to do?

"Tell me first. What do you have to do to change your 6 blocks to 4 blocks?" You are looking to see whether and how the child says the set will need to be changed ahead of actually doing it.

Can the child solve the problem, and if so, how?

"Check to see if you are right."

Or, if there is no prediction, ask, "Show me how you can find out."

You are looking to see what the child does to actually make the change. You may need to ask, "How did you figure it out?"

The task generates a great deal of observable information and the teacher can ask questions for further information. For examples involving a decrease in the number (changing 6 to 4 and changing 8 to 5), it is possible to predict how to create the lesser number simply by visually analyzing the display of blocks. For examples where the number increases (changing 4 to 8 and changing 5 to 12), the child needs some way to construct the missing part, making these two examples even more challenging. Both addition and subtraction examples require an understanding of the question, the focus to attend to the given starting quantity, and the ability and inclination to respond verbally. If any of these were absent, the child was unable to complete the problems.

WHAT MAKES THE TASK SO INTRIGUING TO TEACHERS?

Using field notes collected during 15 professional development sessions and 140 written comment from teachers we analyzed what teachers thought of the Build and Change task and why they thought it was so intriguing.

Complexity of the task.

Many teachers who field-tested the Build and Change considered relatively simple yet were surprised by which students could and could not complete the task. Teachers found, for instance, that some children could not grasp the idea even after the two warm-up examples (in these cases the teacher would not continue). Some children, such as Josh (6 years, 1 month), successfully used the change idea in the scaffolded warm-up examples, but could not apply the idea when working independently without teacher help. Josh was successful with the warm-up examples but reverted to disregarding the starting sets. For example, in changing 6 to 4, he did not see how he could work with the existing 6 blocks, and instead said he needed to get 4 new blocks. He proceeded to build a set of four beside the starting group of 6, showing his four, and ignoring the 6 that remained. For changing 4 to 8, said he had to get rid of the four, moved them out of the way, and then brought back 8 new blocks. He was unable to build on to the given part for addition examples, and unable to see the part to remove in examples involving reducing the set. However, he was able to construct a requested set from scratch.

Variations in representation.

Teachers reported that they were surprised with how dependent many children were on sensory information to complete the task. They found that some children were unable to predict, and needed to actually move the blocks to make the changes. These children did not state ahead of time whether adding or removing blocks was required, even when asked that question. These children appeared to have no access to mental imagery for number, no personal strategies, and were unwilling to make an estimate. However, with the blocks and guided questioning, they were able to complete the changes. These variations in level of representation captured the interest of teachers.

Interviewers were interested in whether children who confidently completed the task using a direct modeling approach might be able to predict ahead of time what to do. Teachers, therefore, encouraged children to fold their arms beforehand to limit the sensory information. Mary (6 years, 4 months), for instance, in working on additive changing for the task, recounted the starting set then counted on blocks to reach the new total. When her interviewer asked her to fold her arms in an effort to see if she could predict how to change 8 to 5 before doing it, she complied but then proceeded to use her elbows to try to move the blocks. In subsequent examples she was able to imagine the change and describe it if it only involved a change of one or two.

Teachers were intrigued by the unique ways children supported their own reasoning. Some children appeared to be in the process of developing their own way of predicting changes by using a combination of manipulating materials, using visual aids, and attempting to access mental imagery. Adam (5 years, 4 months) in changing 6 to 4, easily analyzed the part/whole relationship and predicted that 2 needed to be removed. For the example of changing 4 to 8, Adam tried valiantly to "see" in his head the blocks that were needed. He was obviously trying to count on the needed blocks, but was unable to keep track of how many. He looked at the blocks, moved two, looked up and thought about it as he quietly counted, then added two more blocks and recounted. His actions showed he was beginning to mentally represent number, and had some imagery for counting on, but no tallying method to keep track of what he was adding other than using blocks to keep track of the change.

Use of strategies.

Estimating. Children's strategy choices were of great interest to teachers. They observed examples of estimating, counting on or back, or using known facts and grouping, and were intrigued to see the connection of representation level to strategy use. Sam (6 years, 3 months), did not have a way to predict the change accurately, but was willing to make an estimate, then check by making the change. Shane used an estimate for every example he was given, saying "Oh I'd say you need about five more." His estimates were usually close to the actual number needed, and sometimes were right on, but he had no way of checking other than actually moving the blocks. He required concrete models for all number work and did not appear to have access to mental representation of number. He used direct modeling and counting all for working out each example, never used counting on, and always counted from one to establish how many after a perceptual change. However, his use of estimates showed a ball park understanding of when to add or subtract, a grasp of the part whole relationships involved in the task, and the confidence to take a chance with his responses.

Counting. Many of the grade one children used counting on or counting back to make their predictions, and all of these examples were accompanied by a variety of kinesthetic, verbal, or visual cues. Some children used taps or fingers or head nods to tally the count, some used self talk such as "6 to 5 is one, 5 to 4 is 2, you need to take away two." Alex, when given "change 3 to 6" looked up at the clock and after a moment said "put three more". When asked how he knew, he said he "counts on the clock". His process of keeping track of the count involved very subtle body "nods" to accompany the count.

Grouping. Teachers were surprised by the children who didn't count by ones, but rather used known groupings to figure out the change. Chantelle, when asked to explain her reasoning for "change 3 to 6" said, "Remember our doubles? I did like…I pretended these were like these (made a column of 3) and I put three more in" (making 2 columns of 3, a 3x2 array with the blocks). She drew on her knowledge of the known combination 3+3=6 and the imagined arrangement of 3x2 to determine the change needed. Chris (6 years, 8 months) connected every example to familiar groupings, and carefully arranged the blocks into arrays of twos, threes, or fours. For changing 4 to 8, she said "put two more is six, and another two is eight". For changing 5 to 12, looking at the arrangement of 3 and 2 and using her hands to show how to add columns of three, said " make three, three, three, and three, four threes make 12". She then carefully added one block to the two to make a second set of three, and two more sets of three to get a 3x4 array. Her ability to see groupings and think in groupings rather than using counting by ones was apparent in all tasks in the assessment. She appeared to have a very strong visual spatial sense for quantity, and manipulated spatial groupings and mental groupings, with confidence.

DISCUSSION

Throughout the early numeracy project the participating teachers developed a keen interest in the importance of understanding children's thinking and how best to assess children's thinking. Teachers reported that their opportunities to try out items like the Build and Change task and then collaboratively discuss their results with colleagues was key to broadening their interpretive frameworks for making sense of student thinking. Listening to what other teachers found or could see in their students' work was an invaluable aspect of the project. Teachers also spoke about how seeing mathematical situations through the eyes of a child can be challenging. This is understandable. Once our adult familiarity with number concepts is established, and our skills become automatic, it is often difficult to reconstruct how a beginner might interpret and respond to number situations. Yet, teachers stated that opportunities to explore students' thinking was intriguing and fascinating.

Field-testing teachers were surprised that not all children used counting strategies to solve the Build and Change task, and were impressed with the range of strategies children used. They reported being particularly intrigued by the variations in sensory information that children seemed to depend on to support their thinking. Field notes and written feedback indicate that some teachers described these aspects of representational preference in terms of increasing competence, with more visual, auditory and tactile/kinesthetic involvement at early levels, and a gradual process of internalizing reasoning as children developed a greater understanding of number and a greater ability to work mentally.

Teachers were also intrigued by the number of children who appeared to rely on visual spatial reasoning in thinking about quantities. Many children would arrange the blocks into arrays and appeared to use the rows and columns as scaffolds to thinking about groupings and part/whole relationships. This reinforced for them the value of using graphic organizers such as ten frames and arrays for number so that children could capitalize on their spatial reasoning and subitizing capacity.

They recognized the importance of attention, organization, and the ability to keep track of more than one aspect of a problem. They were fascinated by the importance of mathematical disposition and how it related to success with the tasks, particularly perseverance, curiosity, and confidence. Teachers noted how these dispositional and procedural aspects of competence are ones that appeared to have an important impact on performance.

The Build and Change task challenged teachers' preconceived ideas about what mathematics young students can do and the strategies they might use. They reported that this task provided an efficient yet powerful portrait of their students' thinking. Counting has been accepted as children's primary means for making sense of number, and the majority of work on early number learning has attended to the development of counting approaches. However through trying the Build and Change (and other ENP tasks) teachers found that some children, particularly those with a strong visual spatial sense or poor auditory-verbal skills, preferred other pathways to making sense of number. In particular, it highlighted representational level as it relates to strategy use, visual spatial reasoning, and dispositional and procedural aspects of competence. These are all areas in need of closer scrutiny in order to enhance our ability to better understand the development of children's understanding of number. The Build and Change task captured the complexity of children's mathematical activity related to early number, and expanded teachers' thinking beyond a focus on counting to more fully appreciate children's ways of knowing about number.

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CONSTRUCTING KNOWLEDGE ABOUT THE BIFURCATION DIAGRAM: EPISTEMIC ACTIONS AND PARALLEL CONSTRUCTIONS

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The dynamically nested epistemic actions (RBC) model is used to describe the process of constructing knowledge about bifurcations of dynamic processes by a solitary learner. We observe a refinement of the three epistemic actions, Recognizing, Building-with and Constructing that have been identified in previous research based on the RBC model. We also observe that the Constructing actions are not linearly ordered but may go on in parallel. We observe the branching of a new construction from an ongoing construction. We use the term "branching" in order to describe this transition from a single construction to two parallel ones. In the paper we analyse why such branching occurs.

INTRODUCTION

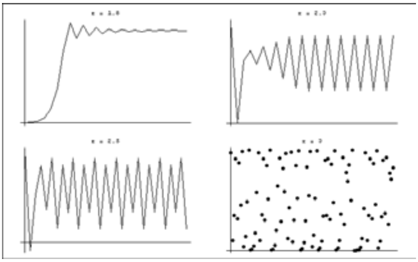
The dynamically nested epistemic actions model called the RBC model is described in Dreyfus, Hershkowitz & Schwarz (2001) and in Hershkowitz, Schwarz & Dreyfus (2001). Stehlikova (2003) demonstrated how it can be applied to introspective data and used for their interpretation. In the present study too, the RBC model is applied to introspective data. The learner in this study, the first author, Ivy, learned alone, with only books, the web and her interaction with Mathematica available as sources of external knowledge. Her detailed notes taken during the learning experience constituted the raw data for a written report she prepared some time after the experience. This report was prepared in collaboration between the two researchers. Ivy wrote what she considered to be a precise and detailed account of her thinking and acting at the time of the learning experience. The second author, Tommy, read this account and challenged the first author on every statement that seemed to reflect later additions, corrections or changes to what happened at the time of the learning experience. The report was then modified so as to answer these challenges. The final version of the report resulted from three rounds of such corrections. Independent research results (Nisbett &Wilson, 1977) show that people can produce accurate reports of their own cognitive activity if salient stimuli are provided. These salient stimuli were the field notes and the challenges.

The Mathematical theme

The learning process described in this article deals with bifurcations of dynamic processes, a complex topic, which occupied the learner during a period of approximately two weeks.

We observe the following iterative process: Given the quadratic function $f(x) = x+rx(1-x)$, where r is a real parameter, we look at the sequence of values $\{x_n\}$

produced from an initial value x_0, $0< x_0 <1$, by successive application of f, that is $x_{n+1} = f(x_n)$, for all $n\geq 0$. Independently of the choice of x_0, $0< x_0 <1$, for r =1.8 the process tends to the final state x = 1, which is a fixpoint of f, that is f(1) = 1.



However, for r = 2.3 the final state is a periodic oscillation between two values, a 2-period. For r = 2.5 the process approaches a 4-period and for r =3 it does not appear to become periodic at all (See for example Alligood, Sauer and Yorke, 1996, for more details). Ivy used Mathematica to confirm these phenomena numerically and graphically. (Figure 1).

Figure 1: Mathematica time series plots of the process for r=1.8 (top left), r=2.3 (top right), r=2.5 (bottom left) and r=3 (bottom right).

It was intuitively clear to Ivy that the rich and surprising pictures on the computer screen describe a mathematical reality. She felt that it was quite challenging to find the mathematical justification for the above phenomena. In this article, we analyze the epistemic actions in her construction of the transition from the 2-period to the 4-period.

THE STORY OF THE LEARNING EXPERIENCE

We divided the story of Ivy's learning experience, as reflected by the final version of the report into 16 episodes. In this section we present the first seven episodes in some detail and summarize the remaining ones. Each episode forms a cognitively coherent unit. For the purpose of analysis, each episode has been further divided into subunits, denoted by letters a, b, c, ...

With a view to investigating the transition from 2-period to 4-period, Ivy started in *episode 1* by focusing on r=2.3 (which is in the 2-period region) and confirming that the graphical, numerical and algebraic (quadratic equation) values of x for this r agree. She expected that in the region of the 4-period, there were analogous (though possibly more complicated) algebraic equations to be solved. In *episode 2,* she saw solving these equations as the problem to be tackled. More specifically, she realized that in the case of period 4, there are four values a, b, c and d such that f(a)=b, f(b)=c, f(c)=d and f(d)=a, whence $f^4(a) = f(f(f(f(a)))) = a$, $f^4(b) = b$, $f^4(c) = c$ and $f^4(d) = d$. She thus had to solve the equation $f^4(x) = x$. From a web resource she learned that the roots of $f^2(x) = x$ are also roots of $f^4(x) = x$ and that it was thus sufficient to solve the equation $\frac{f^4(x)-x}{f^2(x)-x} = 0$. This equation is a polynomial equation with parameter r. For simplicity, we will in the sequel denote it by p(x) = 0, or call it "the equation". In *episode 3*, Ivy attempted to solve the equation for a general value of r. This attempt failed and the failure, combined with the use of Mathematica, led her to a numerical way of thinking. Mathematica taught her that in the 4-period region, the equation is of order 12 and cannot be solved in general, but only for specific values of r. In the previous transition, from period 1 to period 2, the solutions were functions of r. In the present transition, this r-dependency became inaccessible since the equation was not solvable. Thus the numeric aspect took over. This led her, in *episode 4*, to focus on using other means to find the r at which the transition occurs.

4a	When I got stuck with this polynomial of order 12, I went back to the same website and there I learned that the transition from period 2 to period 4 could be found by setting the discriminant of p(x) equal to zero. There was no explanation why this solves the problem.

In the same web resource Ivy read that the discriminant D of a polynomial is defined as the product of the squares of the differences of the polynomial roots. With this definition, Ivy was not able to progress since she had failed to find the roots. However, in *episode 5*

5a	Searching further, I read in another web resource "up to some constant, the discriminant is the 'resultant' of a polynomial and its derivative. That is, the discriminant is the result of computing a certain determinant made from the polynomial coefficients".
5b	This was more promising since it contained a hint that the discriminant could be obtained without finding the roots first.

Ivy did not understand how the discriminant D could be obtained, but she was willing to exploit this alley anyway. Mathematica provided D, which was, after simplification, a polynomial of order 40 in r. Ivy needed the zeros of this polynomial. In *episode 6*, she found the one that was important, by asking Mathematica to factor D, yielding $r=\sqrt{6}$ as smallest real root.

Seeing that $r = \sqrt{6}$ is suitably located between 2.3 and 2.5, Ivy interpreted it as the transition point. On the basis of this numerical success, she began in *episode 7* to search for the mathematical reasons behind that success.

7	I was encouraged by the numerical appropriateness of $r=\sqrt{6}$ and
7a	I was optimistic that I now had the means to begin the analytical work of connecting between the requirement that D equals zero and the transition from period 2 to period 4.
7b	I felt that this undertaking would be helped if I would actually see the solutions of the equation in front of me. This had been my starting point: to solve $f^4(x) = x$.
7c	I thus decided to substitute $r=\sqrt{6}$ in the equation $p(x) = 0$ and to obtain the solutions.
7d	This was the only option I could think of, which would enable me to observe the structure of the solutions like in the case of the transition from period 1 to period 2.

7e	From the definition of D, I knew that when D equals 0, the equation has multiple roots.
7f	In the resulting of Mathematica output I obtained two pairs of double solutions, each listed twice. … I observed the other solutions. There were four pairs of complex numbers and their conjugates. I had no idea how to interpret the meaning of these complex solutions, not even the fact that they were not real.

Ivy was looking for real roots, multiple real roots, and the complex roots came as a surprise diverting her attention from the more relevant real double roots.

The apparition of the complex roots weakened the analogy with the case of the transition to the 2-period, where D≥0 implies real roots only. In another attempt to "save" the full analogy to the 1-period to 2-period transition, including the direct way of getting the real roots, Ivy refreshed (from a book) her knowledge about complex roots of polynomial equations, but she soon realized that this does not allow her to connect D=0 to the existence of real solutions.Numerically the results she had obtained were nice, but they were not satisfactory to her, since she had information on a single r only or on a discrete set of r-values in the best case. Algebraically, she was stuck. The impossibility of algebraic solution resurfaces now explicitly. At this stage of the learning experience, she turned to a graphical mode of thinking. A look at the bifurcation diagram led her to realize that the two (double) real x-values she had obtained for the 4-period at the transition point, were the same values as those of the 2- period as r approaches the transition point, i.e. that a bifurcation of the x-values occurs there. This emerging understanding came in terms of the x-values, together with their dynamic change as r varies. Now the questions were very clear: How are double real solutions, D=0, and period-transition connected? A previous search on the web helped to connect D to the coefficients not only of p but also of p', and to also connect multiple solutions of p=0 with solutions of p'=0. Exploiting the specific structure of p yielded an equation for the derivative that was known to Ivy to be characteristic for fixpoints changing stability. From here on progress was smooth and Ivy constructed her mathematical justification for finding the value of the parameter r corresponding to the transition point from the 2-period to the 4-period.

THE EPISTEMIC ACTIONS

In this section, we discuss Ivy's epistemic actions as we identified them in her report of the learning experience. With respect to Recognizing and Building-with actions, we only present modifications that have not been observed in earlier studies.

Less than Recognizing: R-

As mentioned earlier, Ivy learned alone, with books, the web and her interaction with Mathematica. In such a situation, she was repeatedly offered new information she had not explicitly asked for. In some cases (episodes 4a, 7f), she registered such information, and it later came to play a role in constructions of her knowledge structures. This new information was not a case of re-cognizing, nor did this registration on first encounter constitute a full cognizing of the information. On the other hand, the fact that Ivy was later able to make use of it indicates that some spark of an epistemic action did occur. We will denote such actions by R^- in order to indicate registration of information that, at least in the meantime, constitutes less than Recognizing. As an example of a R^- action, in episode 7f, Ivy had asked Mathematica to solve the polynomial equation. She expected real solutions only. Mathematica provided two pairs of double real solutions but there were also four pairs of complex conjugate numbers. Ivy didn't know how to interpret them but it influenced her further work.

A refinement of Building-with

When analysing Ivy's learning experience for B- actions (Building-with), we noticed a wealth of them. Every one of these B-actions contributes to some building but they do so in very different ways.

We discerned between B-actions of the problem solving type, the kind that had been observed in all previous studies using the RBC model, and B-actions organizing the problem space so as to make its further investigation possible. In fact, Ivy spent a considerable part of her time formulating and reformulating the questions she was asking and the problems she had to solve. This is an activity that is common for learners engaged in investigative activity. Such formulating of questions, tasks and intentions satisfies all criteria of epistemic B-actions: It does not, by itself, produce new mental structures but uses the available ones in order to organize and reorganize horizontally not only the knowledge one has, but also the knowledge one does not have yet and is looking for – the problem space. This distinction led to two quite distinct and different classes of B- actions. Hence we decided to use two different letters to denote them: P for B-actions of the Problem-solving type and V for B-actions supporting the inVestigation of the problem space. An example of a B-action of the V-type is 7a, in which Ivy formulates and thus clarifies to herself her intention to change the type of her thinking from empirical-numerical to analytic-algebraic and declares her aim to connect between D=0 and the transition point. She thus builds the cognitive fundament on which she can attempt further progress in the problem space. The verbalization of the algebraic connections she wants to establish, anticipates the process of generating a mathematical justification of the way in which she obtained the value $r=\sqrt{6}$ at the transition point. This motivation of achieving a justification drove the entire learning process.

Most of the Building-with actions of the V-type address the big issues Ivy dealt with and are thus intimately related to the constructing actions, which we will describe in the next subsection.

Constructions

The methodology we followed for identifying constructing actions closely followed the one that was used in Hershkowitz, Schwarz & Dreyfus (2001). Through detailed analysis of the report, we identified instances of vertical reorganization of knowledge structures, of added depth, and of integration of structures. Novelty of the resulting structure for the learner was used as a central criterion in the identification of constructions, and so was its verbal expression. The use of such verbal expression for further explanation was taken to be a definite sign of construction. In this section, we give a very short description of some of the constructions that form Ivy's overall construction C.

C_1 *Constructing the solutions of the polynomial equation:* We denote by C_1 the process of constructing the solution of the polynomial equation $p(x) = 0$ in order to find the 4-periodic points. The solution process is considered algebraically and numerically. Construction C_1 appears in episodes 2, 3 and 7 and in the later episodes.

C_2 *Constructing algebraic connections:* We denote by C_2 the process of constructing the algebraic connections between the existence of multiple (double) real solutions of the equation $p(x)=0$, the zeros of D and the transition point from period 2 to period 4. Construction C_2 appears in episodes 4-7 and in many later episodes. C_2 is different from the other constructions in the sense that it was very clear what the knowledge was that had to be constructed, but Ivy got stuck at several points in the process. The algebraic mode did not yield the results Ivy was looking for. As a consequence, the need for the other constructions arose, especially C_3.

C_3 *Constructing the link between the discriminant and the derivative:* We denote by C_3 the process of constructing the links between the derivative of a polynomial and the zeros of its discriminant. Construction C_3 appears in episode 5 and in the episodes toward the end of the story. The link between D=0 and the derivative first appeared in episode 5 but has been identified as the beginning of a construction only a posteriori, on the basis of the epistemic actions in the later episodes, where the role of the derivatives of p and related functions became central to Ivy's construction of knowledge.

C *Constructing the justification:* We denote by C the process of constructing the mathematical justification for finding the value of the parameter r corresponding to the transition point from period 2 to period 4. This process extends over all 16 episodes. The motivation of achieving a justification drove the entire learning process. This desire to generate a justification motivates C and thus indirectly motivates the other constructions, each of which is nested in and contributes to the overall construction C.

INTERACTING PARALLEL CONSTRUCTIONS

Ivy's construction of a justification for the transition from the 2-period to the 4-period constitutes a complex learning process. The complexity of the construction expresses itself in the fact it consists of several interweaved constructing actions that go on in parallel. In the previous section, these constructing actions were described separately. In the present section, we analyze the manner in which they interact with each other. The constructions are closely interrelated, and should therefore ideally develop in parallel. On the other hand, they are too substantial to go on simultaneously. As a consequence, the analysis of the learning experience produced interesting patterns by which constructions arise and interweave. These appear to be patterns that have not been observed in previous research. The following characteristic patterns have been observed: A new construction branches off from an ongoing construction, several constructions go on in parallel and then combine, one of several parallel constructions is interrupted and later resumed. Because of space limitations, we focus in this paper on branching. This phenomenon appears in episode 5 (C_3 branching off from C_2) and in episode 7 (C_1 branching off from C_2). Examples for combining and interrupting will be analyzed in a bigger paper in which we will describe in detail the later episodes of the learning experience.

Branching

In this subsection we describe the branching off of a new construction from an ongoing construction; such branching leads to a transition from a single construction to two parallel ones. We will provide an explanation why such branching occurs. We first deal with the branching point itself and then with the parallel development of the two constructions after branching.

Why C_1 branches off from C_2 at the beginning of episode 7?

C_1 has been interrupted at the end of episode 3 when the information provided by Mathematica was that the equation p(x) = 0 could be solved only numerically, and thus solving the equation would not yield the desired value of r at the transition point. At the end of episode 6, Ivy knew this value ($r = \sqrt{6}$). Thus this was an appropriate time for C_1 to be resumed. Moreover, C_2 couldn't go on in episode 7 without C_1, for the following reason: Ivy's aim was to connect D=0 and the transition point (7a). She used D=0 in order to obtain r. She had no idea how the discriminant was computed and at that stage, she wasn't ready to find out (which would have meant to resume C_3). She preferred to go back to the previous definition (4b) that connects the discriminant with the polynomial roots and to observe the structure of the solutions, and this was possible only with a return to C_1.

In both examples of branching, C_3 branching off from C_2 in episode 5, and in the present example of C_1 branching off from C_2 there is an ongoing construction, namely C_2, and another construction branching off from it. In the first case, the construction that branches off from the ongoing one is a new construction, C_3, and in the other case, the construction that branches off from the ongoing one consists of a return to a construction that had previously been active, namely C_1. But in both cases, the branching was essential for the ongoing construction to continue. The special character of C_2 contributes to the branching. As mentioned earlier, C_2 is different from the other constructions in the sense that it was very clear what the knowledge was that had to be constructed, but the algebraic mode did not yield the results Ivy was looking for. As a consequence, the need for the other constructions arose.

The parallel development of the two constructions after branching

Branching leads to two constructions being active simultaneously. This is a high demand on any learner. As a consequence, one of the constructions was interrupted at the end of the episodes that started with branching (C_3 at the end of episode 5 and C_1 in episode 7; both interruptions are caused by diverting attention to C_2). Nevertheless, the constructions develop in parallel during one episode after the branching point before the interruption. This may lead to the establishment of connections between the two knowledge structures being constructed. One of the two constructions provides the motivation for the other one. The new construction that branches off from the ongoing construction allows the influence of additional ideas to flow into the process. This positive influence will find its full expression only in the later episodes but it begins in the earlier stage in which the two constructions seem only to disturb each other.

We conclude with an observation about the relationship between the epistemic action R- and the parallel development of two constructions after branching. Sometimes new, unexpected R- information (for example, the information about the complex roots) obtained within one construction (C_1), at the moment when another construction (C_2), calls for the learner's attention, constitutes the immediate cause for the interruption of the first construction (C_1). However, at a later stage when the learner has assembled more knowledge, the same R- action can lead to the resumption of the interrupted construction, the resumption being based on the same information that was received without being requested.

In summary, we found, that in some contexts, such as a solitary learner dealing with an advanced mathematical topic, epistemic actions may be more varied and construction processes more intricate than observed in previous studies based on the dynamically nested RBC model of abstraction.

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EXPERIENCING RESEARCH PRACTICE IN PURE MATHEMATICS IN A TEACHER TRAINING CONTEXT

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This paper presents the early results of an experiment involving a class of elementary student teachers within the context of their mathematics preparation. The motivation of the exercise centred on giving them an experience with mathematical research at their own level and ascertaining its impact on their attitudes and beliefs. The students spent the first month working on open-ended geometrical topics. In the second month, working alone or in groups of up to four, they chose one or more of these topics then worked on a problem of their own design. The students spent the class time developing their ideas using strategies such as generating examples and non-examples, generalising, etc. Reference to books was not accepted as a research tool, but the instruction team monitored student progress and was available for questions.

INTRODUCTION

The paper examines the experience of mathematics research at their own level to "bring [elementary students teachers] to a point where they know the mathematician in themselves" (Gattegno, 1974, p. 79). Much work has already been done in this direction, under such topics as 'doing mathematics' (ATCDE, 1967; Banwell et Al, 1972; Wells, 1986; Brown & Walter, 1990; Schoenfeld, 1994), or 'investigations' (Banwell, Saunders & Tahta, 1972; Brown & Walter, 1990; Pòlya 1957). A review of writings about mathematics research practice (Hardy, 1940, Woodrow, 1985, Rota, 1981, Restivo et Al., 1993, Burton, 1999), however, shows that investigation and similar class work still contrasts with 'real' mathematics research. In other words, although investigative work in the classroom is a step towards modelling students' classroom experience on the experience of mathematicians, more work is required.

So "what does it mean to *do* mathematics, or to act mathematically?" (Schoenfeld, 1994, p. 55). Though it is often perceived that way by students, "Mathematics is not a contemplative, but a creative subject" (Hardy, 1940, p. 143). This sharp contrast between practice in pure mathematics and the sorts of activities one thinks of in relation to the traditional, or even the reform, mathematics classroom raises the question: How can classroom practice be modified to reconnect with practice in pure mathematics and what effect would that have on the students?

The topic of this paper is an attempt to answer just this question in the context of an elementary education programme. The experiment took place in a service course given by a mathematics department with a strong background in pure research. The course module is one of four mathematics courses required for an elementary education qualification, apart from the teaching methods component. It took place bi-weekly, and was given to 37 students, 33 of which were female. The majority of the students were Juniors (3^{rd} year) or Seniors (4^{th} year), with at least 4 graduate students completing prerequisites for a Masters' programme. Of the 35 students who completed the pre-course survey, 3 assessed themselves as having excellent mathematical ability, 15 as being competent, 14 as average, and the remaining 3 as either weak/poor. 25 of the same 35 acknowledged preferring algebra, 4 geometry, and 6 arithmetic. The mixture of self-determined ability and attitude in the class added another dimension to the experiment, particularly pertaining to the students' choice of topics for their projects. This resulted in a wide range of levels of mathematical thinking and creativity, analytical rigor, and others.

PEDAGOGICAL COURSE OUTLINE

The course was carefully designed around several didactic methods in order to optimise the classroom environment for supporting the anticipated learning approach. First, a significant part of the course curriculum, which is normally dealt with using the assigned textbook, was left to the end of the term in order to establish an unthreatening power relationship between students and instructors. The intention of this move was to give the students control of the knowledge, making their own experience the key, emphasising that the instructors *did not* have the answers, nor was there a textbook which does. Second, the project work was made the core of the course, not an extra, a 'Friday afternoon activity', by making it the main part of the grade. Third, the students were asked to collect all their work into a final portfolio and were allowed to resubmit previous assignments for additional points. Finally, the teaching style of the instructors and the contribution of the teaching assistant were carefully monitored and directed in order to restrict the possibilities of the instructors 'taking over' the student's project, or leading them to the answers.

September

The term was subdivided into three main parts. In September, the students were introduced to some mathematical topics and encouraged to develop ways of thinking mathematically emulating 'real' mathematics research as discussed in the introduction. In order to facilitate this development, a portion of the first class was spent in a whole class discussion of a mathematician's role and job. In addition, the course guideline placed special attention on the process, communication and outcomes of the explorations. Finally, the 'mini-projects' developed for September were designed as templates for investigations, in addition to providing the topics, mainly by keeping them as open-ended as possible. The students were given a chance to practice using mathematical research strategies. They were taught to think in terms of examples and counter examples, cases and constraints, patterns and systems of rules, conjectures, justifications and proofs. They were asked why rules they conjectured might always be true, or in which cases, and they learned to frame problems. The intent of this part was for the students to "do a non-mathematical activity and then reflect on it mathematically" (Morgan, 2003). In order for the students to focus on this mathematical thinking, the mathematical topics were approached in a very accessible manner. The topics included geometrical subject matter involving proper colourings[1] of regular triangular grids and simple polyhedra, symmetries of the same, the Euler characteristic, and others.

The approach entailed visual or manipulative activities using paper and colouring tools or modular connector sets such as Polydrons. The activities began with low levels of symbolic notation and terminology and ramped up to more sophisticated modes of communication, including use of graph theory to represent polyhedra, use of higher level counting techniques to verify colourings and symmetries, and use of precise language such as duality of configurations. Throughout this part of the course, the students were asked, as part of their homework assignments, to think of an 'interesting thing to look at next' in the context of the class work.

October

Following the training stage, the students were asked to pick topics for their projects, based on their ideas for 'interesting things to look at next'. They were expected to pose, and attempt to solve, their own mathematical problem, taken from one or more of the topics investigated earlier. It was made clear that their work should be "a step into the unknown. [...] The principal hope for an investigation [being] that totally unexpected things turn up, that different kinds of approaches to problems should appear as different pupils tackle different aspects of the problem in different ways." (Driver, 1988). Library research was therefore deemed unacceptable as mathematics research does not consist of looking up results in books and then presenting them. The students broke up into 19 groups, one of 4, three of 3, nine of 2 and six single people. Interestingly, every group of students came up with a topic they wanted to explore further. Though the instructors had prepared a list of 'fall back projects' for students who were not finding a starting point, these were unnecessary.

In the first week, the students submitted a proposal for which they received very restrained feedback, typically "would be very interesting mathematically", or "there is lots to explore here". Though in many cases, the proposed problem had to be reframed during the evolution of the project, this was declared acceptable, even normal in this context, as in 'real mathematical research'. This is typified in Schoenfeld's (1994, p. 59) example of the reformulation of the 'natural' definition of polyhedra: "New formulations replace old ones, with base assumptions (definitions and axioms) evolving as the data comes in." (Schoenfeld, 1994)

[1] A proper colouring is a way of colouring a tiling or polyhedron so that no two tiles or faces sharing an edge or side have the same colour.

In the following weeks, the students spent class time working on their projects, and the teaching team approached each group or individual in turn. This interaction was kept deliberately very hands-off, preserving the students' ownership of their projects. The guidance provided by the instructors was restricted to "validating [the students'] efforts" (Drake, 2003) and suggesting general strategies, preventing the closing off of any avenues of investigation. For example, Patrick (pseudonym), a student who was interested in examining the relationship between the number of faces of regular pyramids and their symmetrical proper colourings, was only considering pyramids made of regular polygons. This meant that he only had three cases to work with (the triangular, square and pentagonal pyramids, as the hexagonal case is degenerate). When it was suggested that he eliminate the regular polygon constraint, he was suddenly faced with an infinite class of polyhedra, giving him room to develop a conjecture involving the connection of primality and symmetry.

During October, the students were given the opportunity to work on mathematical problems of their own devising, at their own pace and at their own level, using methods of their choice. Though the pace, level and methods varied greatly, the class work in October was productive in most cases, and there were very few students who missed class, and in most cases, the students met and worked outside of class time. At the end of this part, the students were asked to hand in a 'first draft' of their project report. This document, which each student had to compose individually, including group members, contained three parts: First, the student described the initial motivation and topic chosen in September. In the second part, the student described "the project as it evolved, showing [their] work and thinking in full detail for each stage" (from guidelines). This part was expected to be the most extensive and was requested in order to emphasise to the student the importance given to the *process*, including direction changes and blind alleys. In order to help the students write up their work, they were given the following quote from the Gian-Carlo Rota's introduction to *The Mathematical Experience*: "A mathematician's work is mostly a tangle of guesswork, analogy, wishful thinking and frustration, and proof, far from being the core of discovery, is more often than not a way of making sure our minds are not playing tricks." (Rota, 1981)

Finally, the write up was to contain "a summary of [their] findings from section 2, referring to key examples and/or counter-examples. A precise statement of your claims and reasons why" (from guidelines). This section was considered to contain the results of their work, in contrast to the process of the previous section.

November and December

In November and December, regular classroom activities resumed, mainly for the purpose of course completion. The students were also given the opportunity to revise their project reports and everything, including homework re-submissions, was handed in at the end of term. Additionally, in the last few classes, the groups all gave an oral presentations of their projects.

STUDENT WORK

Many things can be said about the various projects. The level of enthusiasm and creativity achieved by the students, in particular, was unexpected. For example, one of the students began working on his project in the middle of September, not even waiting to hear the rest of the topics. Another group went way beyond the anticipated results, discovering a whole class of polyhedra, unknown to any of the instructors, with the property of self-duality. In order to accurately portray the work that was done in this experiment, both by the students and the instructors, this section will detail three projects. The description will place particular emphasis on the quality of the interaction between the students and the instructors. This is particularly important to illustrate the method of instruction, which is very difficult to pin point and relies heavily on teacher self-confidence and experience in mathematics research.

Project 1: relationships of properties

Carol and Samantha (pseudonyms) investigated proper colouring of polyhedra. They attempted to relate the number of edges of a polyhedron to the minimum number of colours required for a proper colouring. In order to investigate this problem, they began by generating 7 polyhedra with triangular, square and pentagonal faces, including a pentagonal dipyramid, a triangular prism, a cube with a pyramid on top, and a pentagonal anti-prism. They then generated tables of data, including descriptive tables containing the number of faces of each type for each polyhedron, and the number of faces, edges and vertices versus the minimum number of colours required.

They found that all their polyhedra could be coloured properly using either 3 or 4 colours. They also noticed that the pattern seemed to obey the following rules: “if the number of edges is divisible by 3, then the polyhedron needs 4 colors for proper coloring to occur” and vice versa. These rules are not mutually exclusive, and therefore their conjecture contains an internal flaw. This was not as dire as it could have been, however, as they disproved the validity of their conjecture using one of their existing shapes. The next conjecture they made concerned the number of faces at each vertex in relation to the number of colours necessary. Again, they found a counter-example in their collection. Though the students in this group did not arrive at a clear mathematical result, they made strong use of the general strategies that were developed in September. Interaction with the instructors was limited, for two main reasons. First, because of an “Artefact of the geography” (Drake, 2003): though the class was large, it was held in a very narrow room, and not all students were easily accessible. Second, they mostly kept to themselves and showed little interest in discussing their project during class. Suggestions were generally not followed. For example, the researcher suggested that the students concentrate on what forces the jump from 3 to 4 colours when constructing the polyhedron: “watch yourself colour it and focus on when it happens”. Interestingly, in the various write-ups and presentation, this group repeatedly emphasised their intention of working on a project that was relevant, in their view, to their future work in elementary education.

Project 2: mathematical creativity

Rob, Sandra and Jo's project was the one least connected to the September topics. They developed a mathematical system all of their own, involving a class of transformations, which they called manipulations, then explored the rules and constraints of the system. According to their definition, a manipulation is "either a change in length of a side or a change in angle measurement from a base shape." As their base shape of choice was the square, their manipulations netted them rectangles, parallelograms, etc. The subsequent exploration involved the determination of interdependence between the manipulations. In effect, can manipulation (a) occur without anything else changing, and if not, what does or can change. The types of problems they encountered early on concerned their counting and classifying system. If two opposite sides change length, does it count as one or two manipulations? They were also looking to find all possible resulting shapes, and finding their properties, including parallelism, symmetries, equality of length or angle. Though all this could easily be expressed in regular elementary school geometry terms, the significance of the project is in the students' 'stepping into the unknown' and in their attempt at formalising a system of rules they themselves had created and finding its structure.

Suggestions were made by the instructors throughout. An early example proposed that the students look at the 'manipulations' another way, for example as a displacement of a component, an edge or vertex, of the original square. Later, the course instructor suggested the students look for manipulations that can occur without others following, or which force others. Another suggestion involved the use of combinatorics to find all the possible combinations of manipulations, then eliminating the impossible ones. Two of these suggestions mainly centred on the *approach* to the question, the specific techniques used, rather than giving a redirection that would bring the question back to more conventional mathematics, or promote a solvable question. As for the one concerning interdependence of manipulations, it was made at a point when the group was looking for a direction to take, and therefore was timely and useful, and was in fact followed right up.

Project 3: analytical rigor

Although Patrick began his project with a very open field of study, things soon narrowed down to the relationship of symmetries of pyramids with regular polygonal bases to their proper colourings. He wanted to find out how many colours were necessary to colour a given pyramid, and how a resulting colouring impacted its symmetries. His first result showed that the base will always need to have a separate colour to all the other faces, since it touches them all. Following this, he found that pyramids with a base polygon having an even-number of edges will only need 3 colours, and that odd-numbered bases force a fourth colour. Examining the symmetries of the resulting pyramids, he found that reflection symmetries are only possible on even-numbered pyramids, as the odd ones have reflection planes passing through the edge between two sloping faces, which contradicts the proper colouring rule. He then continued his investigation by looking for symmetries in higher order pyramids. The results of this investigation showed that the factorisation of the number of sides of the pyramid plays an essential role in determining the symmetries of possible proper-coloured pyramids.

Though this project exemplifies a high level of mathematical rigor in the range of this class, it did not occur without consistent feedback on the part of the teaching team. The problem was narrowed down early on when the researcher suggested the student look at a class of simple cases, before the result was generalised to the student's initial choice. As mentioned in the course outline section, the field was opened up to include non-equilateral pyramids. Further along, a simple remark sent the student back to the work table: "Ah, but is this the only way to 4-colour this pyramid, or can you do it so some symmetry is conserved?" Though again the suggestions and feedback were designed not to close the question, but rather to facilitate a clearer view, Patrick made very intense use of them.

EARLY RESULTS AND CONCLUSION

The purpose of this experiment, from the research standpoint, concerns the students' potential change in view and attitude towards mathematics as well as the acquisition and development of general mathematical strategies. In order to evaluate this change, data were collected from all the participants. The students filled out a survey before and after the course, the discussion of the first class was recorded, the researcher observed and took notes in each class period, extensive interviews of the main instructor and teaching assistant were recorded, and everyone kept a journal.

Though the data have only just finished coming in, early observations can be made. In the journals, students expressed frustration in September and sometimes in October, though this was often temporary, and the following entry would express enthusiasm. In any case, this kind of frustration is quite common in mathematics research, and can even be a good sign. After all, as John Mason says in Thinking Mathematically, "Being stuck is an honourable state" (p. ix). In the after-course survey, the students were asked how they felt their project fit into their view of mathematics. 20 of the replies contained the word exploration or process. Terms used in other responses included creativity, innovative, and possibilities. When asked to rate which stage of the course they felt most useful for specific topics, 20 chose October as the most interesting. 20 chose October as the most instructive about teaching, while 11 found September to be most instructive. Though 23 students found that they learned most about mathematics in November (during regular book work), 15 found October the most instructive about mathematics. Finally, according to 34 students, their work in October emulated most closely the work of professional mathematician. In both pre- and post-course surveys, the students were asked to select three nouns that best describe mathematics as they view it. The following table shows the top three choices before and after the course, along with their score.

An exploration	Score before: 0	**Score after: 24**
Formulas	**Score before: 14**	Score after: 8
Numbers and Operations	**Score before: 20**	Score after: 5
Patterns and relations	Score before: 4	**Score after: 23**
Problem-solving	**Score before: 29**	**Score after: 23**

Table 1: Nouns most selected before and after, and their respective scores

Though these results only present a tentative view at best, the analysis promises to be interesting. In particular, the process section of the project hand-in, together with the journal entries, should prove enlightening.

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ENHANCING MATHEMATICAL LITERACY WITH THE USE OF METACOGNITIVE GUIDANCE IN FORUM DISCUSSION

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The purpose of the present study was to investigate the effects of forum discussion embedded within metacognitive guidance on mathematical literacy. In particular the study compares two learning environments: (a) Forum discussion with metacognitive guidance (FORUM+META); and (b) Forum discussion without metacognitive guidance (FORUM). Participants were 43 seventh-grade students (boys and girls) who practiced online problem solving in two classes. It was found that students who were exposed to FORUM+META discussion outperformed students that were not exposed to metacognitive guidance (FORUM discussion) on mathematical literacy. The effects were observed on various aspects of solving real-life tasks: (a) Understanding the task; (b) Using mathematical strategies; (c) Processing information; and (d) Using mathematical reasoning.

"Mathematical literacy is defined in PISA as the capacity to identify, understand and engage in mathematics as well as to make well-founded judgments about the role that mathematics plays in an individual's current and future life as a constructive, concerned and reflective citizen" (OECD/PISA, 2003, p. 20). The definition revolves around the wider uses of mathematics in people's lives and is not limited to mechanical operations. "Mathematical literacy" is used here to indicate the ability to put mathematical knowledge and skills to use rather than just mastering them within a school curriculum. To "engage in" mathematics covers not just simple calculation (such as deciding how much change to give someone in a shop) but also wider uses, including taking a point of view and appreciating things expressed numerically (such as having an opinion about a government's spending plan). Mathematical literacy is assessed by giving students tasks based on situations which represent the kind of problems encountered in real-life.

From a motivational perspective, such tasks are challenging tasks that are relevant to the students' world and daily life (OECD/PISA, 2003). Although real-life tasks are important, little is known at present on how to enhance students' ability to solve such

Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 2004 Vol 3 pp 169–176

tasks. The question: What characteristics should a learning environment have to facilitate the construction of students' mathematical literacy, merits further research.

E-learning is currently one of the popular environments of training and learning. In particular, participating in forum discussion has been touted as offering immense potential for improving the effectiveness of learning. Forum discussion allows asynchronous exchanges. It also permits one-to-one as well as one-to-many interactions. The learners are motivated, they are able to learn independently, and they can transfer and apply the knowledge to real–life situations (e.g., Deaudelin & Richer, 1999). However, this technology, as has been the case with prior technologies, raises the question about the pedagogical way of how to use that technology.

A review of pedagogical approaches shows that the majority put emphasize on the development of Self-Regulated approaches to learning (e.g., Butler & Winne, 1995; Mevarech & Kramarski, 1997; Kramarski & Mevarech, 2003; OECD/PISA, 2003). PISA describes Self Regulated Learning (SRL) as a style of activities for problem solving that includes: Evaluating goals, thinking of strategies and choosing the most appropriate strategy for solving the problem. The method of Mevarech & Kramarski, (1997), called **IMPROVE** emphasizes the importance of providing each student with the opportunity to construct mathematical meaning by involving him or her in self questioning that focus on: (a) *comprehending* the problem (e.g., "What is the problem all about"?); (b) constructing *connections* between previous and new knowledge (e.g., "What are the similarities/differences between the problem at hand the problems you have solved in the past? and why?"); (c) use of *strategies* appropriate for solving the problem (e.g., "What are the strategies/tactics/principles appropriate for solving the problem and why?"; and (d) *reflecting* on the processes and the solution (e.g., "What did I do wrong here?"; "Does the solution make sense?").

Generally speaking, researchers reported positive effects of metacognitive guidance on students' mathematical reasoning in cooperative learning (e.g., Schoenfeld, 1992; Mevarech & Kramarski, 1997; Kramarski & Mevarech, 2003) and in different computerized environments (e.g., Kramarski & Ritkof, 2002). Most of that studies examined the effects of metacognitive guidance on solving mathematical problems based on school curriculum and less on tasks based on situations which represent the kinds of problem encountered in real-life. Given these studies, the purpose of the present study was to investigate the effects of forum discussion embedded within metacognitive guidance on mathematical literacy. In particular the study compares two learning environments: (a) Forum discussion with metacognitive guidance (FORUM+META); and (b) Forum discussion without metacognitive guidance (FORUM). The effects will be observed on various aspects of solving a real-life task.

Method

Participants were 43 *seventh-grade* students (boys and girls, mean age 13.4) who practiced problem solving in two classes. One class (n=20) was exposed to forum discussion embedded with metacognitive guidance (FORUM+ META); and the other class (n=23) were exposed to forum discussion without metacognitive guidance (FORUM).

Treatments

Students in the two conditions practiced for six weeks (90 min. a week) of problem solving with the solution of three real-life tasks. The study was implemented in pairs. Students were encouraged to think about the task, explain it to each other, and approach it from different perspectives.

Since each treatment was composed of two components: The use of forum discussion with or without metacognitive guidance. We first describe each component separately, and then how the components were combined.

Forum discussion

The students that were exposed to forum discussion practiced problem solving of real-life tasks once a week in the computer lab (90 min).

The teacher did not interfere in the discussion, she encouraged the students to participate in the discussion, to send assignments each other, to reflect on the solutions and to submit questions regarding the solution process. The students were also encouraged to ask their friends for help when they encountered difficulties in understanding and correcting the solution, if needed. In addition, the students were asked to send the final solution to the teacher, in the forum or as an attachment file using word or excel.

Metacognitive guidance

The metacognitive guidance was based on the IMPROVE technique suggested by Mevarech & Kramarski (1997). The method utilized a series of four self-addressed metacognitive questions.

The *comprehension questions* were designed to prompt students to reflect on the problem/task before solving it. In addressing a comprehension question, students had to read the problem/task aloud, describe the task in their own words, and try to understand what the task/concepts mean. The *comprehension questions* included questions such as: “What is the problem/task all about?”; “What is the question”?; “What is the meaning of the mathematical concepts?”

The *connection questions* were designed to prompt students to focus on similarities and differences between the problem/task they work on and the problem/task or set of problems/tasks that they had already solved. For example: “How is this problem/task different from/similar to what you have already solved? Explain why”.

The *strategic questions* were designed to prompt students to consider which strategies are appropriate for solving the given problem/task and for what reasons. In addressing the strategic questions, students had to describe the *what* (e.g., What strategy/tactic/principle can be used in order to solve the problem/task?") the *why* (e.g., "Why is this strategy/tactic/principle most appropriate for solving the problem/task?") and *how* (e.g., " How can I organize the information to solve the problem/task; and "How can the suggested plan be carried out?").

The *reflection questions* were designed to prompt students to *reflect* on their understanding and feelings during the solution process (e.g., "What am I doing?"; "Does it make sense?"; "What difficulties/feelings I face in solving the task?"; "How can I verify the solution?"; "Can I use another approach for solving the task?").

FORUM+META discussion: Under this condition, students studied according to the IMPROVE method and FORUM discussion described above. They were encouraged to use the metacognitive questions during their discussion, in their written explanations when they solved the mathematical tasks and in their reflection on their friends' solution.

FORUM discussion: Under this condition, students studied the same as under the FORUM+META condition but they were not exposed to the metacognitive guidance.

Measures

Two measures were used in the present study to assess students' mathematical problem solving: (a) a pre-test that focused on students' mathematical knowledge prior to the beginning of the study: (b) a post-test that assessed students' mathematical ability to solve real-life task.

Pre–test of Mathematics Prior Knowledge: To control for possible differences prior to the beginning of the study, a 38-item pretest was administered to all students at the beginning of the school year. The test covered arithmetic knowledge taught prior to the beginning of the study in the following content: Whole numbers, fractions, decimals and percents. The test was based on multiple-choice items regarding basic factual knowledge and open-ended computation problems. In addition students were asked to explain their reasoning.

Scoring: For each item, students received a score of either 1 (correct answer) or 0 (incorrect answer), and a total score ranging from 0 to 38. The scores were translated to percents. The Kuder Richardson reliability coefficient was α =.87.

Mathematics Post-test: The Pizza Task- Your classmates organize a party. The school will provide the soft drinks, and you are asked to order the pizza. The class budget is NIS 85.00. Of course, you want to order as many pizzas as you can. Here are proposals of three local pizza restaurants and their prices. Compare the prices and suggest the cheapest offer to the class treasurer. Write a report to the class treasurer in which you justify your suggestion. The task is provided on Appendix A.

The Pizza Task has all the characteristics of "Mathematical literacy" that is needed to be engaged in solving such task. The situation is most familiar to junior high school students, the mathematical data is rich, and there is no ready-made algorithm for solving it. Quite often children at this age go to restaurants that offer pizzas having different prices, sizes, and supplements; and quite often they have to decide which kind of pizza is more consuming. The Pizza Task requires the use of a variety of sources of information (e.g., prices, size, number of supplements) and it has many different correct solutions. The solvers have to make computations, use different representations, and apply knowledge regarding geometry, fractions and ratio.

Scoring: Students' responses were scored on criteria based on the model of Cai, Lane & Jakabcsin (1996) for analyzing open-ended tasks: (a) Understanding the problem; (b) Processing information; (c) Using mathematical strategies; and; (d) Mathematical reasoning. Each criterion was scored between 0 (no response or incorrect response) to 4 (full correct response).

A full correct answer regards to referring to all the relevant data in each of the three proposals, making the calculations correctly, organizing of the information in a table, diagram, or an algebraic expression, making a correct suggestion based on the given information, and justifying the suggestion by explaining one's mathematical reasoning. In addition, the type of arguments that were provided in mathematical reasoning were analyzed on five criteria:

(1) Providing the final result; (2) Providing the computation process; (3) Using daily mathematical terms; (4) Using formal mathematical expressions and; (5) Using logic mathematical argument. Inter-judge reliability of categories was .89.

Results

Table 1 presents the mean scores, adjusted mean scores, and standard deviations on problem solving of the Pizza real-life task by time and treatment. Table 2 presents the frequencies and χ^2 test for providing arguments on the solution of the Pizza task by type of argumentation and treatment.

A one way ANCOVA and MANCOVA were carried out on the total score and on the various measures of the real-life task with the pretest scores used as a covariant.

Table 1: Mean Scores and Standard Deviations on the Pretest and on the Pizza Task, by Treatment

	FORUM+ META N=20	FORUM N=23	F
Pretest [a]: Prior knowledge			
M	70.38	65.55	F(1,42)=0.5
SD	23.04	20.60	
Posttest [b]: Pizza Task (total)			
M	3.52	1.87	F(1,41)= 31.19**
Adjusted M	3.44	1.86	
SD	0.30	1.04	
Understanding the Task [b]			
M	3.90	2.36	F(1, 41)=21.9**
Adjusted M	4.00	2.30	
SD	0.31	0.84	
Using Mathematical Strategies [b]			
M	3.60	2.39	F(1, 41)=29.8**
Adjusted M	3.67	2.33	
SD	0.50	0.50	
Processing Information [b]			
M	3.40	2.32	F(1, 41)=22.7*
Adjusted M	3.50	2.30	
SD	0.75	0.58	
Mathematical Reasoning [b]			
M	3.32	1.70	F(1, 41)= 27.2**
Adjusted M	3.42	1.59	
SD	0.65	0.70	

Note. [a] Range: 0-100, [b] Range: 0-4, *p< .001; ** p< .0001

Results (Table 1) indicated that prior to the beginning of the study no significant differences were found between the two treatments on their prior knowledge. But on the post-test it was found that students who were exposed to the FORUM+META discussion significantly outperformed their counterparts (FORUM discussion) in solving the real-life task on all the four criteria.

Further results (Table 2) indicated that students that participated in FORUM+META discussion used significantly more Logic-Mathematical arguments and formal mathematical expressions than students from the FORUM discussion (75%, 90%; 34.8%, 30.4%, respectively). Whereas, most of the FORUM discussion students based their reasoning on repeating their final result without explaining "why" they got it (65.2%; 30%, respectively for the FORUM vs. FORUM+META).

Table 2: Frequencies and χ^2 test for providing arguments on the solution of the Pizza task by type of argumentation and treatment

ARGUMENTATION	FORUM+ META N=20	FORUM N=23	χ^2
Providing the final result	6 (30 %)	15 (65.2 %)	5.31*
Providing the computation process	14 (70%)	7 (30.4%)	6.32*
Using daily mathematical terms	14 (70%)	14 (60.9%)	0.39
Using formal mathematical expressions	18 (90%)	7 (30.4%)	15.60***
Using logic – mathematical argument	15 (75%)	8 (34.8%)	6.96**

* p< .05; **p< .01; *** p< .001

Conclusions and Discussion

The findings showed that the metacognitive students were better able to solve a real-life task and communicate their reasoning. This is probably due to the fact that the metacognitive guidance embedded in FORUM discussion trained students to think about which strategies are appropriate for solving the task and why. By doing so, students suggested different kinds of representations, compared the strategies, and analyzed each strategy.

Our findings indicate that using FORUM discussion is not sufficient for enhancing mathematical literacy. There is a need to structure mathematical discussion, and that features of discussion such as giving reasons must be practiced and reinforced. This conclusion is in line with other studies that showed that asking students to answer *why* questions during the solution processes helped them to elaborate and retain information (Schoenfeld, 1992). These findings support other conclusions on the importance of integrating pedagogical uses with advanced technology, in particular metacognitive guidance (e.g., Deaudelin & Richer, 1999; Kramarski & Ritkof, 2002).

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APPENDIX A: The Pizza Task

TYPE OF PIZZA	PRICE PER PIZZA	DIAMETER	PRICE FOR SUPPLEMENTS
PIZZA BOOM			
PERSONAL PIZZA	3.50 N.I.S	15	4.00 N.I.S
SMALL	6.50 N.I.S	23	7.75 N.I.S
MEDIUM	9.50 N.I.S	30	11.00 N.I.S
LARGE	12.50 N.I.S	38	14.45 N.I.S
EXTRA LARGE	15.50 N.I.S	45	17.75 N.I.S
SUPER PIZZA			
SMALL	8.65 N.I.S	30	9.95 N.I.S
MEDIUM	9.65 N.I.S	35	10.95 N.I.S
LARGE	11.65 N.I.S	40	12.95 N.I.S
SPECIAL PIZZA			
SMALL	6.95 N.I.S	25	1 N.I.S
LARGE	9.95 N.I.S	35	1.25 N.I.S

THE IMPACT OF DEVELOPING TEACHER CONCEPTUAL KNOWLEDGE ON STUDENTS' KNOWLEDGE OF DIVISION

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This study investigated children's knowledge of division and its relationship to their teacher's conceptual understanding of division following Professional Development. A paper and pencil test was administered to 47 year 7 students and 2 teachers over 2 phases. Following the testing, six students and the teacher from each phase were interviewed. Results from this study indicate that most Phase 1 students rely on following a procedure with limited understanding. Their teacher displayed some conceptual understanding, however she too demonstrated a bias for procedural knowledge. This contrasts with Phase 2 teacher and students who demonstrated conceptual knowledge both in the test and the interviews.

INTRODUCTION

Division is required for many of the processes used in everyday situations, ranging from finding averages to determining rates and proportions involved with fuel use, cost and selling prices, and budgeting. It is also critical in problem solving and as part of the thinking underlying determination of area, volume and probabilities. A secure understanding of division, along with other forms of multiplicative thinking, is essential for work with fractions, ratios, algebra and further mathematics that marks the transition from the arithmetical thinking of the primary school to the more advanced thinking of the secondary curriculum and beyond (Booker 2003). Yet, many students experience considerable difficulties with division and it is often viewed as the most difficult of the four operations to learn by students and teachers.

A major source of these difficulties are the rote procedures that have often dominated its teaching and the language associated with them that provide little insight into the steps to be followed or their link to the underlying multiplication notions that allow division to be carried out (Booker et al 2004). Frequently, students have little understanding of the division concept in terms of sharing that provides a link to the multiplication facts needed at each step of the division process and are unable to read or interpret division statements or results (Greer 1995; Simon 1993).

Whether the teaching and learning of mathematics should focus on conceptual understanding or procedural competency has been the topic of research over many years (Ma 1999, Hiebert 1987). However conceptual understanding does not appear to be achieved by rote learning procedures in isolation, nor does simply knowing the steps necessarily indicate conceptual understanding of a mathematical process such as division (Silver 1987). Rather, the concepts and processes of mathematics need to grow hand-in-hand so that students can see connections between concepts and processes, interpret mathematics from multiple perspectives, be aware of the basic ideas fundamental to mathematics and possess an ability to reflect on previously learned concepts to give coherence to their knowledge (Ma 1999:122).

Ma's (1999) study compared conceptual understanding and procedural competency of experienced U.S. and Chinese primary teachers across a range of mathematical topics. She concluded that the major influence on students' learning of the fundamental ideas on which all future learning and problem solving depended was the strength of their teacher's conceptual understanding of mathematics and their capacity to use this to generate meaningful examples, explanations and processes.

In contrast, many students are expected to embrace division without the concept development stages that occurred with the other three operations because they have already had considerable experience with symbolic representations (Anghileri 1996; NCTM 2000). This has led to considerable difficulties in reading division where they are often unaware of the importance of order when using division expressions (Anghileri 1999) and in interpreting remainders in calculations (Silver et al 1993). Insufficiently developed understanding of the division concept also leads students to calculate unrealistic answers and then not question their results (Simon 1993).

An investigation of the division understanding of teachers and their students might then provide insights into the reasons for students' difficulties with division and its applications, similar to those Ma has reported for other mathematical topics.

THE STUDY

Forty-seven students and their teachers from two year 7 classes participated in a pencil and paper test containing 6 division questions made up of ten items and were asked to solve them without any imposed time limit. Following the testing 6students from each class and their teachers were interviewed on their test responses and their understanding of division. The two classes participated a year apart during which time professional development sessions were attended by most staff at the school.

Prior to the testing teachers used a process called *Divide Multiply Subtract bring down* commonly called *DMS bring down* to teach the division algorithm. Following Phase 1 professional development was conducted in the school where a focus on the development of conceptual understand by the use of games, and concrete representations of the sharing of division. This paper reports on the responses provided to 2 of these 6 questions aimed at identifying an ability to solve word problems through interpreting the problem, completing calculations involving internal zeros and interpreting their result to produce answers to the problems.

These questions are

Q5: When Movie World opened the Wild, Wild west ride, 6445 people went on the ride on the first day. If each wagon holds 7 people, how many full wagons could there have been?

Q6: A Birch Carroll and Coyle cinema needed 9238 packets of skittles to stack their shelves. If 4 packets are contained in each box how many boxes would need to be ordered?

RESULTS AND DISCUSSION

Student and teacher test results and interview responses are discussed in terms of the overall results and interview responses, error analysis of test responses, and the manner in which the remainder is interpreted to finally solve the problem.

Overall results

	Algorithm with recording	Algorithm without recording	Algorithm not recorded: answer only	No answer	Algorithm correct recording	Algorithm incorrect recording	% correct with calculation shown
Q5							
Phase 1 n=24	12	7	3	2	3	1	21%
Phase 2 n=23	20	1	2	-	16	1	80%
Q6							
Phase 1 n=24	12	7	2	3	6	Zero	31%
Phase 2 n=23	21	-	2	-	16	-	76%

Table 1: Comparison of Phase 1 and Phase 2 results

Table 1 shows the impact that professional development had on student results between Phase 1 and Phase 2. Phase 1 testing shows that only 21% of students were able to correctly complete the division algorithm for question 5 and 31%, for question 6. The year 7 students the following year (Phase 2) demonstrated far greater understanding, with 80% correctly calculating question 5 and 76%, question 6

Interview analysis

There was a noticeable shift towards conceptual explanations given by the six interviewed students in Phase 2 when compared to the interviewed responses for students in Phase 1. During the interviews the students where asked

> 'Please complete this division for me, 6)1845 and explain exactly what you are doing so that I can understand'.

Phase 1 students gave procedural descriptions. Some showed all their working and some chose to simply write in the quotient. Jimmy for example, wrote and said:

37 r3
6)1845

> 6 doesn't go into 1 so 6 into 18 goes 3 times. 6 doesn't go into 4 so 6 into 45 goes 7 with remainder 3. Easy.

Beth also exhibited considerable confusion although she remained very confident throughout the interview. Her working below is supported by the following description of her calculation.

307r2
6)1845

217

> Beth: Well 6 divided by 18 goes in 2 times with 6 left over. 64 divided by 6 equals 10 with 4 left over. 45 divided by 6 which is (counting on her fingers) 7 remainder 2.
>
> Interviewer: So what is your answer?

Beth: 217 (Beth replied as she wrote the number beside her calculation.)

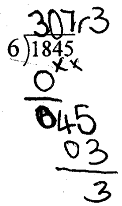
Interviewer: So is that a zero in there or ten?

Beth: You don't have to worry about the zero. The answer is just 217r2. (Pointing at her calculation)

Although Beth said *'6 divided by 18'* she calculated 18 divided by 6. She then went on to correctly say *'45 divided by 6.'* The ease with which she vacillates between the two descriptions of her calculation indicates her limited conceptual understanding of the division operation. She failed to recognise that the remaining 6 could have been shared with her initial calculation, that a 2digit number cannot sit in one place, that a zero marks a place and cannot simply be omitted and that 45 divided by 6 is not 7 remainder 2.

Leea correctly completed the algorithm, however her description is very procedural.

Leea: 6 into 1 can't go. 6 into 18, 3. Bring down the 4. 6 into 4 can't go so bring down the 5. 6 into 45, 7 remainder 3.

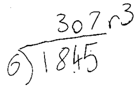


Leea has adhered to the *DMS bring down* process very strictly and has produced a correct answer. However this has not fostered a deeper understanding of the division operation nor the importance of place value.

The year 7 teacher from Phase 1 expressed concern during her interview that she did not have sufficient time to *'teach for understanding'* and that *'if the students just do as I tell them they will get it right.'* She said she liked structured teaching, which she classified as *'sit up and shut up.'* This style of teaching fits very well with the procedural explanation she gave for the division process:

> The students just have to learn to deal with one number at a time, apply the learned process of DMS bring down, and they will get it right every time.

Of the 6 students interviewed in Phase 2, Peter was the only student who chose to calculate his answer by writing the answer without any working:



He was also the only student to explain his calculation from a procedural perspective.

Peter: How many 6's can I put into one? I can't. How many 6's can I put into 18? 3. Write it up here. How many 6's can I put into 4? Can't, so I put down the zero.

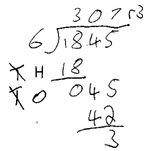
Interviewer: Why did you put a zero here? Can you explain?

Peter: You just do. Now, how many 6's can I put into 45? (counting quietly) 42, so 7 goes up here, remainder 3. The answer is 307r3.

Not only is Peter's explanation procedural, limited conceptual knowledge is evidenced by his inability to explain why the zero was placed in his answer.

Katie's work and explanation was typical of the remaining 5 interviewed students.

> Can I share 1 thousand among 6? No, so I rename as 18 hundreds. 18 shared among 6, which is 3. 3 times 6 equals 18. 18 minus 18. Then you bring the 4 down. Can I share out 4 tens among 3? I can't so I put zero here. Then I bring down the 5 and rename as 45 ones. 7, 6 times 7 equals 42 so 45 take away 42 is 3. I now have to rename to tenths. 6 into 30 tenths equals 5 with nothing left over.



Katie is aware of place value, that renaming is occurring and that division involves sharing. She is still using the language of 'bring down' as she renames however it could be argued that she understands exactly what she is doing and why.

Following the Professional Development, the Phase 2 year 7 teacher introduced her classroom to an approach to teaching mathematics where concrete materials were used and discussions encouraged to explain the concepts being developed. When asked how this approach had impacted on her teaching of division she replied:

> Once the kids get into the upper (primary) school they stop using concrete materials (for the operations) and the kids have a real reluctance to go back to using them. Maybe that is why DMS bring down was taught But with division being introduced in the upper school, and I have used games and other materials it is now OK to use concrete materials, in their eyes. I have found that by using concrete materials that kids get it, and I am talking about kids who haven't got it mathematically in the past. I think it is because we now do so much concrete stuff and they can all see it together and talk about it. You don't have to be able to work in the abstract to get it.

Error Analysis

When the errors made by the students are analysed further insight into the varying levels of student conceptual understanding is evident between Phase 1 and Phase2 students and their teachers. While the number of students who chose not to attempt the questions or to show any of their working has reduced from Phase 1 to Phase 2, the area of most change is in the process errors. Table 2 documents the student errors:

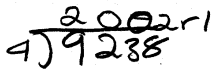
	Correct calculation	% Correct	Incorrect calculation	No attempt or no working	Internal zero error	Process error	Fact error	Misc errors
Q5								
Phase1 n=24	4	21%	15	5	5	9	4	1
Phase2 n=23	18	80%	4	1	3	-	1	-

Q6								
Phase1 n=24	6	31%	13	5	4	7	5	-
Phase2 n=23	16	76%	5	2	2	-	1	2

Table 2 Error Analysis

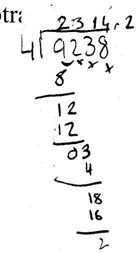
Phase 2 students did not make any errors that have been classified as process errors in contrast to the large numbers of process areas for questions 5 and 6 in Phase 1. However, internal zero and fact errors have continued.

In Phase 1 there were a variety degrees of process errors which provide good insight into the limited conceptual understanding these students possess. For example David wrote:



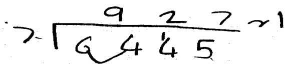
He has learned that when you cannot share a number in a place, you put a zero to indicate that no sharing has occurred. However, he has failed to share 230 and the remainder 1 is possibly the remainder from the sharing of 9 thousands.

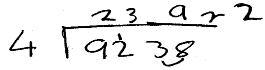
While Bella's work superficially appears to indicate substantially more understanding than the previous examples, on closer examination she demonstrates processing errors with both division and subtr



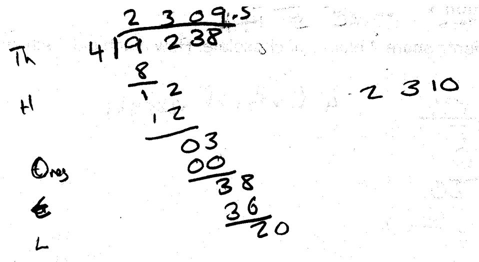
When Bella could not divide 3 tens between 4 she changed the divisor to 3 wrote in the quotient that 3 was shared once and that she had used all 4 tens. She then takes the 4 tens from 3 tens leaving her with one ten. Examples such as this indicate how superficial her understanding of the other operations may be, yet could be so very easily marked wrong without considerations to what aspect was incorrect particularly when her setting out looks like she has learned the process.

The results from both phases of this study indicate that a zero in the ones place in the quotient is more difficult to master than a zero in tens place. Phase 1 student, Daniel, has difficulty with internal zeros in both questions.





Divesh's work is a typical example of the 16 students in Phase 2 who correctly calculated the answers to questions 5 and 6. He clearly documents what place is being shared indicating place value knowledge and a thorough understanding of the concept of division. These children have had experience using games to demonstrate the sharing that is division, they then moved on to sharing MAB on a place value chart and then documenting the sharing in the form of the algorithm seen by the following example.



Interpretation of Remainder

The final aspect of these two questions to be analysed in this paper is the question of interpretation of the remainder to solve the problem.

	Correct calculation Remainder interpreted correctly	Difficulty Interpreting Remainder -calculation correct to this stage	No attempt to interpret remainder – calculation correct to this stage.	Difficulty Interpreting Remainder - calculation incorrect	No attempt to interpret remainder- Calculation incorrect
Q5					
Phase 1 n=24	0	1	3	1	14
Phase 2 n=23	8	1	8	1	3
Q6					
Phase 1 n=24	0	0	6	2	11
Phase 2 n=23	4	3	9	1	4

Table 3 Interpretation of Calculation

The example above shows that Divesh has interpreted the remainder in his calculation in order to arrive at his final answer of 2310 boxes. However, this was not commonplace as Table 3 shows.

No student in Phase 1 completed problems 5 and 6 successfully. There has been an improvement in students correctly completing both questions in Phase 2 but this remains at a low level, 47% for question 5 and 25% for question 6. When teacher responses are taken into consideration Phase I students can not be blamed for their low priority to interpret their calculation and solve the problem when their teacher does not have this aspect of problem solving as a priority and only interpreted one of the calculations herself.

Teacher Phase 1

920 + (5)
7) 6445

2309 + 2
4) 9238

2310 boxes requred

On the other hand, while the Phase 2 teacher gives the correct solution to both problems, she does not appear to have prioritised this to her students judging by their poor results on this aspect of problem solving.

CONCLUSION

The responses given by both students and teachers demonstrated considerable growth in conceptual understanding of division. Prior to PD both teachers taught the process known as *DMS bring down* when teaching the division operation. Following PD the Phase 2 teacher introduced games, use of concrete materials as representations and then mirrored these representations with documented working of the algorithm This allowed children to understand the sharing that occurred with greater respect for place value. The students did not have to learn a process they did not understand nor were they expected to work in the abstract. They were able to see exactly what was being shared and how this impacted on the final result. Phase 2 students outperformed Phase 1 students, their error rate was lower, their conceptual understanding of division was greater and they had began to consider interpreting their calculation to solve the problem.

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KINDS OF ARGUMENTS EMERGING WHILE EXPLORING IN A COMPUTERIZED ENVIRONMENT

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Emek Yezreel Academic College

In this paper there is a characterization of the arguments emerged while two 7th grade students were engaged in an investigation of several number theory concepts in a computerized environment. These emerging arguments were a result of the influence of the computerized environment together with the collaborative learning. Using the qualitative research methods, data is brought to show how the students constructed four kinds of mathematical arguments: basic, compound, elaborated and general masked as specific. These arguments that are example-based can serve as a basic knowledge for the developing of formal mathematical proofs.

Introduction

According to Webster's dictionary, '*argument*' is a discussion involving differing points of view; a process of reasoning. '*Argumentation*' is the process of developing or presenting an argument. More detailed definition was given by Wood (1999) who said that an argument is a discursive exchange among participants for the purpose of convincing others through the use of certain modes of thought. Argumentation is viewed as an interactive process of knowing how and when to participate in the exchange.

Researchers emphasize the positive influence of involving argumentation during the process of learning on the learners' learning quality (Okada & Simon, 1997; Chazan, 1993). According to Chi, Chiu & Lavancher (1994), the learner develops her/his ability to cope with complex problems; to be able to infer from a given data in case there are missing details and to be able to accomplish them. Furthermore, explanations given within the argumentation process might help the development of generalization skills (Crowley & Siegler, 1999).

In this paper there is a description of a case in which mathematical argumentation emerge and develop between 7th grade students working in an interactive computerized environment without a deliberate mentoring. The computerized environment has its influence on the characteristics of this argumentation which include mathematical regularities based on concrete examples of geometrical shapes (stars and regular polygons) and mathematical considerations relating to numbers' properties.

My aim in this paper is to characterize the different kinds of spontaneous arguments emerged in the geoboard environment and show how the argumentations' components are tightly related to this specific setting. To achieve this aim, I present an analysis of the students' discourse during two sessions (the fourth and the fifth out of ten) in which the students conceptualized two main concepts: the *n-star* and the *common denominators* concepts (for further details: Lavy and Leron, in press).

Methodology

The computerized environment under study consisted of MicroWorlds Project Builder (MWPB) –a Logo-based construction environment.

A group of ten 7th-grade students met several times after school hours in the school computer laboratory and explored the effects of the instruction *repeat n [jump k]* on geoboards of varying number of pegs. The instruction *repeat n [jump k]* results in the command *jump k* being executed n times in succession. Each choice of specific values for n and k results in a screen display of a regular polygon or a star with varying number of vertices (figure 1). The students were encouraged to look for mathematical patterns connecting the input numbers (n and k) and the shapes and the number of vertices of the resulting polygons or stars. These investigations led to the emergence in the students' discourse of concepts such as prime number, divisor and greatest common divisor (gcd). For detailed description of the computerized environment see Lavy and Leron (in press).

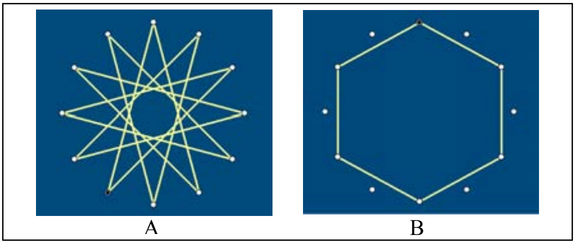

Figure 1: A – 12-star resulting from the command repeat 12 [jump 5]
B – 6-sides regular polygon resulting from the command repeat 6 [jump 2]

The students Noam and Jacob, whose discursive, and in particular, screen productions were captured by a video camera are in the focus of this paper. These students were selected because, more than their other colleagues, they tended to "think aloud" during their work. The major part of the research data is the verbalized discourse, which took part between the students during the exploration process. In addition, the research data included the screen pictures at every stage of the inquiry, the students' body language and every piece of written paper they produced.

The research data were analyzed by three tools: Inductive analysis, interactional analysis and scanning the students' discourse through the lens of Toulmin's terminology (1969). Inductive analysis (Goetz & Lecompte, 1984) is a method which integrates between scanning the data while looking for phenomenological categories, and successive refinement of them when confronted with the new events and interpretations. In keeping with this approach, there were no predetermined criteria or categories made. This kind of analysis helped in characterizing the different modes of working such as the mode of environment-dependent and the gradual shift towards concentrating mainly on mathematical concepts and their attributes. In addition, it helped to form the different logical 'blocks' relating to each one of the emerging argument in the discourse.

In order to be able to characterize the ways each one of the student expresses himself during the exploration process, a discursive tool of interactional analysis was applied (Sfard and Kieran, 1997). This analysis enabled the understanding of the contribution of each student to the argument construction.

Scanning the discourse through the lens of Toulmin's terminology enabled the characterization of the different emerged arguments. Toulmin in 'The Uses of Argument' (1969, pp. 85-113) proposed a useful model for analyzing and constructing arguments. The two most important elements of his model are *data*: facts serving as the basis for a claim; and the *claim*: a conclusion or generalization to establish or support. Additional components in the model are: *warrants* (or reasoning), which are general authorizing statements justifying the logical leap from data to claim and *backing,* which is the information that supports or offers a foundation for the warrant statement.

Results and Discussion

The exploration process of the relations between the logo instruction: *repeat n [jump k]* and the resulting shapes of stars and polygons for various values of *n* and *k* was accompanied by arguing, trying to justify and convince each other by the validity of one's claims. In this study, since the students were only in 7th grade, the emerging arguments included visual and/or intuitive justifications rather than formal ones.

The distinction between the different kinds of the emerging arguments was made according to the character of the data components and the reasoning used. For example, arguments which included screen images of a certain geoboard were classified as one kind of arguments while arguments which included claims received from previous arguments as data were classified as another kind of arguments. The following sections will include the identification of the four kinds of arguments found in the students' discourse and example to each one of them will be brought.

'Basic' argument

After checking few examples of jumps in 11-peg geoboard, Noam said: "There will be here only one polygon". In fact, Noam formulates the claim:" In an 11-peg geoboard there is only one polygon and the rest are stars"[4.52][1]. Noam's claim was a result of viewing few screen images which were the outcomes of different jumps in 11-peg geoboard. These picture screens were actually the data which was the basis for the arrived claim. The reasoning Noam gave for this claim was not verbal. It was based only on demonstrating additional examples of screen images on the same

[1] The number 4 stands for the fourth session of the investigation process and the number 52 stands for the line number of the discursive transcript in this session

geoboard. Since the validation of this claim was involved only by checking a finite number of cases, the students' reasoning for this claim was consisted of concrete examples of different jumps on the same geoboard. I termed this kind of arguments as *"basic argument"*. The level of generality of the claim in this argument is local. In this kind of argument the regularity relates to a *certain* geoboard and the data and reasoning consists of *additional concrete examples on the same geoboard only* – screen pictures of polygons or stars (figure 2).

Figure 2 includes a schematic description of a basic argument using the graphical description and terminology used by Toulmin (1969). The left rectangle refers to the argument's data, the right rectangle refers to the argument's claim, and the bottom rectangle refers to the reasoning used in this argument.

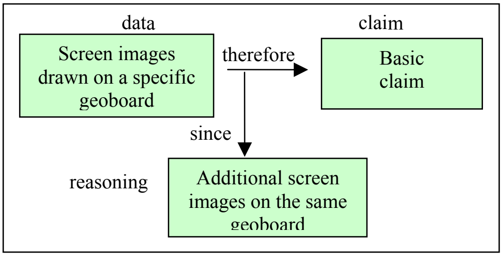

Figure 2: Schematic description of basic argument

Arguments of this kind were found at the first three sessions of the investigation activity. At this stage of the investigation, the students were not able to predict what will be the resulting shape (polygon or star) of a certain input (k) in a specific n-peg geoboard using the instruction *repeat n [jump k]*, and their claims were phrased only *after* testing some inputs of k in a specific geoboard.

'Compound' argument

From the third session of the investigation process, a different kind of arguments was found. Before terming the new argument, the raw data of an example to such an argument is presented:

> [4.73] **Noam**: you have already said that in geoboards with prime number of pegs there will be only one polygon and the rest will be stars.
>
> [4.74] **Jacob**: it is like this, but why.
>
> [4.75] **Noam**: because they are primes, 'cause they do not have any divisor. The divisors they have are one and themselves.

In this stage of the investigation, the students had already checked various geoboards with prime number of pegs. Before [4.73] was argued, they investigated 5-peg, 7-peg and 11-peg geoboards. The claim "In a geoboards with a prime number of pegs there will be one polygon and the rest will be stars" was actually phrased by Noam although it can be understood from the above excerpt as if Jacob have said it before. Jacob's reaction [4.74] also implies that this claim was not phrased by him. First Noam phrases the claim [4.73] and since Jacob was wandering why [4.74], he gives a vague justification: "because they are primes"[4.75]. Although Jacob did not say anything in return, looking at the video reveals that he was not satisfied with Noam's response. Being aware of Jacob's dissatisfaction of his answer, Noam tries to elaborate his justification by referring to properties of prime numbers (having only two devisors: one and the number itself). This is the first time in the discourse that mathematical considerations were given as justification in addition to relevant screen images. Yet, there were no explicit attempts made by Noam to connect between the mathematical considerations and the relevant screen images.

The above argument is an example of the second kind of the emerging arguments and is termed as "*compound Argument*". In this argument the mathematical claim relates to a group of geoboards sharing common attributes and the reasoning is composed of concrete examples and mathematical considerations related to number properties (figure 3).

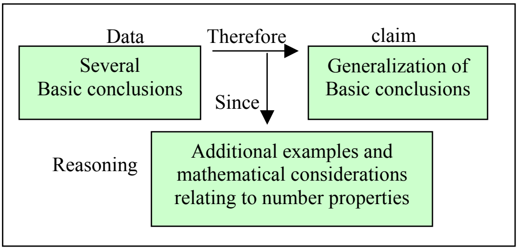

Figure 3: schematic description of a compound argument

The construction of compound argument was based on: basic conclusions constructed so far during the exploration process and controlled selection of screen images. The verbalized discourse of this stage of the investigation, shows that the students picked certain inputs for n (in this case, prime numbers) and checked only few inputs of jumps (k) for the instruction: *repeat n [jump k]* to verify their conjecture, while in the formulation of basic arguments they checked *all* the relevant inputs for k (from 1 to n-1). Connecting the mathematical properties of the related numbers to the resulting geometrical shape on the computer screen presumably reduced the need to check all the cases for each geoboard with prime number of pegs.

'Elaborated' argument

The third kind of argument was found during the fourth session. Before defining this kind of argument, first given the raw data of an example to this argument:

[4.179] **Noam**: I think I arrived to some regularity in stars of 24, of the… all the vertices. Here *jump 6*… here I got it [goes on typing]

[4.180] **Jacob**: Look, if you do a prime number, you get a star.

[4.181] **Noam**: No, not just a star, a star of 24…..

[4.240] **Jacob** (to the teacher): and you put in the *jump* an odd number, then I get a star. Why? Because it is not divisible….

[4.277] **Noam**: […] For me a star is only if it is not divisible….

[4.289] **Noam**: but look, in 24, but in a prime which is not divisible, it always comes out a star of 24. [pointing at the screen] look. Jump 7 is a star of 24.

At this stage of the investigation activity, Noam and Jacob had already discovered that in geoboard with prime number of pegs there is one polygon and the rest of shapes are stars. In the above excerpt they phrase the claim that when the jump size (k) in the command *repeat n [jump k]* is relatively prime to the geoboard's number of pegs (n) the resulting shape is an n-star. By *n-star* I mean a star of n vertices, where n is the number of pegs on the geoboard.

Until this stage of the inquiry, Noam and Jacob distinguished only between cases in which one can get a polygon and cases in which one can get a star. This is the first place in the discourse that they noticed that there are different "kinds" of stars. This was made possible after getting the screen image of the instruction *repeat 12 [jump 5]* in 12-peg geoboard. When Noam saw the 12-peg star resulting from the latter instruction, he moved back to 24-peg geoboard and only after verifying his conjecture, using Jacob's formulation, he putted it into words: " No, not just a star, a star of 24" [4.181].

The above example belongs to third kind of the emerging arguments which I termed as an "*elaborated argument*". This kind of argument is a refinement of compound argument. It is based on compound conclusion arrived earlier in the discourse, and additional examples which do not "fit" with the former conclusion. The argument's justification is based on concrete examples and mathematical considerations related to number properties (figure 4).

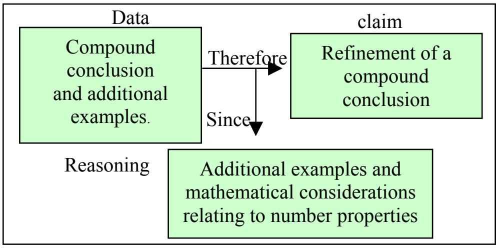

Figure 4: Schematic description of an elaborated argument

General argument masked as specific

By the end of the fourth and during the fifth session, the students were investigating for which inputs of n and k the instruction *repeat n [jump k]*, in an n-peg geoboard, will result in a star with less than n vertices. The regularity the pair was trying to uncover was that the number of vertices of the resulting shape equals *n / gcd(n,k)*. In fact, they did not formulate the above regularity; rather they used several local arguments verify it. First the raw data of two local arguments is given:

[5.5] **Noam** (with frustration): no only if you know [re-types the command] it will come out an 8-star because… [Counts aloud 8 vertices on the screen] look it comes out 9 [meaning: a jump of 9 in a 24-peg geoboard will result in an 8-star].

[5.6] **Jacob**: why?

[5.7] **Noam**: because actually 9… what is its common divisor with 24? …..

[5.45] **Noam**: ah, because 8, its common divisor with 20 is 4 and the common denominator is 5 [by this last part he apparently means that the "common denominator" of 4 and 5 is 20]…..

[5.55] **Noam**: ah, ok, ok, let's do 14 [changes to 14 peg-board, still thinking of a jump of 8] it's got a 7 star.

[5.56] **Jacob**: why? Because you divide by 2?

[5.57] **Noam**: no. because 2… because 8 is 2 times 4, and 7 is… what you need to multiply to get 14 is 2, right? [Noam and Jacob are looking at the 14 board on the screen].

[5.5-5.7] refers to the local claim: "a jump of 9 in a 24-peg geoboard results in a 8-star". This is the first time in the investigation that the students raise a claim without checking it first in the computer. This fact might imply that Noam had already a generalized assumption which came out in words as a concrete one.
[5.55- 5.57] concerns the example of *repeat 14 [jump 8]* (resulting in a 7-star), in which Noam finally gives a relatively clear procedural reasoning of how to calculate the number of vertices of the star. In fact, what he have said amounts to saying that $7 = 14/gcd\ (8,14)$. It can be said that the argument related to the case in which one can get a star with less than n vertices, was build from various concrete examples which implies on the general claim.

The above example belongs to fourth kind of emerged arguments which I termed as *"generalized argument masked as specific"*. This kind of argument is based on elaborated and compound conclusions arrived earlier in the discourse, and its justification is based on arguments which are specific examples of the generalized claim and mathematical considerations related to number properties (figure 5).

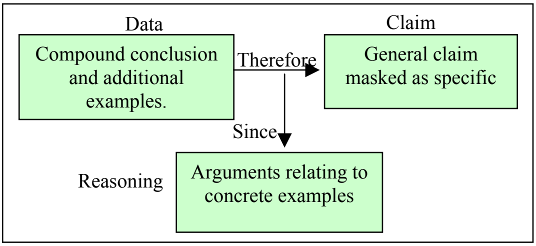

Figure 5: Schematic description of generalized argument masked as specific

Concluding remarks

The characteristics of the emerging arguments described above, imply on a tight connection to the computerized environment of the geoboard, namely data and reasoning rely on concrete examples of geometrical shapes (stars and regular polygons). However, from the second kind of argument and on, the reasoning used in these arguments include in addition to concrete examples of geometrical shapes, mathematical considerations relating to numbers' properties. The various interactions, namely student-student and students-computerized environment, enabled them to gradually develop the ability to see the connections between number properties and the related geometrical shapes. Each one of the students contributed to the mutual effort of building the mathematical argumentation.

The students' ability to argue during the investigation process was improved since there was no observed 'withdrawal' regarding the kind of the emerging arguments, namely there were not found in the students' discourse basic arguments after the students developed the ability to phrase compound or elaborated arguments. Finally, there is no doubt that this process of the students' collaborative effort to phrase arguments must be followed by a class discussion in which formal proofs will be built on the basis of them.

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USING SOLO TO ANALYSE GROUP RESPONSES

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The increased use of group work in teaching and learning has seen an increased need for knowledge about assessment of group work. This report considers exploratory research where the SOLO Taxonomy, previously used to analyse the quality of individual responses, is applied to group responses. The responses were created as part of an activity dealing with weather data and analysed for the quality of the description of variation. The research indicated that the hierarchy developed for coding the individual responses could also be used to code the group responses. Group responses were generally found to be of poorer quality than was suggested as possible by the responses of individuals within the group. Factors that may have contributed to the differing quality of the group responses are considered.

INTRODUCTION

The use of group work in teaching and learning is an important preparation for the collaborative expectations of project work to be undertaken in the 'real world'. As group work becomes more recognized as a legitimate form of learning (Edwards, 2002), and as assessment is an important aspect of teaching and learning, the assessment of group responses is becoming increasingly important. The Structure of the Observed Learning Outcome (SOLO) Taxonomy is being used increasingly to code individual responses: but what about responses submitted by groups? This paper reports on the analysis of group responses, part of a research project where students were required to respond both individually and then as a group.

GROUP ASSESSMENT

When assessing group work there are two major components to be considered, product and process, that is, what is produced and how it is produced. Assessment schemes related to group work often include either or both of these components (Devlin, 2002; Michaelson, 2003). While process is acknowledged as an important factor with group work, the focus in this report is assessing the group product. Various combinations of assessment of group and individual contributions to the group work can be created to produce final assessment grades to be allocated to individual students. Devlin (2002, pp. 7-12) outlines a variety of these combinations, one of which assesses the group product by allocating an overall group mark to each student in the group. To implement such an approach would necessitate assessing the group product and being confident that this was a reasonable representation of each group member's capabilities. However, care would need to be taken when assigning a group mark to individuals, as scores from group assessment are not always valid indicators of students' individual responses (Webb, 1993, p. 21). The question also needs to be raised as to whether group products can be assessed in a similar manner to individual products to give meaningful information about the group members.

CODING INDIVIDUAL RESPONSES USING THE SOLO TAXONOMY

Assessment of students based on the quality of their understanding and learning is a basic principle of Developmental-Based Assessment (DBA). This approach to assessment, with the mental structure of understanding as paramount, differs from outcomes-based assessment that focuses on what students are expected to know. Importantly, the developmental-based approach to assessment rests on the empirically established cognitive developmental SOLO Taxonomy model (Pegg, 2003, p. 238-239). This taxonomy consists of five modes of functioning, with levels of achievement identifiable within each of these modes (Biggs & Collis, 1991). Although these modes are similar to Piagetian stages, an important difference is that with the SOLO earlier modes are not replaced by subsequent modes and, in fact, often support growth in later modes. A series of levels have been identified within each mode. Three relevant levels are: *unistructural* (U) with focus on one aspect; *multistructural* (M) with focus on several unrelated aspects; and *relational* (R) with focus on several aspects in which inter-relationships are identified. These three levels form a cycle of growth that occurs in each mode and recurs in some modes, with each cycle being identified by the nature of the element on which it is based.

CODING GROUP RESPONSES USING THE SOLO TAXONOMY

SOLO is now widely used to code individual responses to tasks in mathematics (Pegg, 2003). Given the growing popularity of group work the question arises as to whether SOLO can also be used to code group responses. However, no reported research using SOLO to analyse group responses could be found. Given this lack of research, there are two possible approaches that could be taken when coding group responses. The first is to collect group responses to a task and then develop a hierarchy using SOLO as a framework to code the responses. The second is to code the group responses by using a SOLO-based hierarchy that has already been developed for individual responses to a similar task.

RESEARCH FOCUS

The reported research takes the latter approach, posing the question: can a hierarchy developed for individual responses be used to code group responses? Also, if such coding of the group responses is possible, then how useful is the information gained from such a coding of group responses? Especially important is the comparison of the level of group response to the levels of individual responses within the group.

METHODOLOGY

This was an exploratory study with research targeting students in Grades 7, 9 and 11 (aged 13 to 17) in a Australian secondary school undertaking a Weather Activity consisting of a Scenario based around choosing the most suitable month for a proposed Youth Festival. The stages in this activity, which spread over a number of lessons, included a rainfall segment, a teaching intervention and finally a temperature segment. In each segment the students were asked to describe individually the data for one particular month and then, as a group, compare their individual months to decide on the most suitable month. The teaching intervention involved the introduction of box-and-whisker plots to Grades 7 and 9 and a fuller statistics teaching-sequence for Grade 11. Students worked in self-selected groups of three to five. For more detailed description of the context of the research see Reading (under review). Although students were not specifically asked to describe the variation within the data, the key focus of the assessment of both individual and group responses was the description of variation that occurred within the response.

Detailed analysis of the individual responses is presented in Reading (under review), including discussion of the levels in the hierarchy developed, based on the SOLO framework, and sample responses from the various levels. Two cycles of U-M-R levels were identified representing increasing quality of response, the first cycle U1-M1-R1 based on qualitative descriptions of variation and the second cycle U2-M2-R2 based on quantitative descriptions. This hierarchy was used to code the group responses and then the performance of the groups was considered in relation to the performance of the individual members of the group.

RESULTS

The Reading (under review) description of variation hierarchy, developed from analysis of the individual responses, was found to be appropriate to code the group responses. Both researchers coded the responses independently and then coding levels were compared. Checking inter-coder reliability for the group responses, 17% of responses were initially coded at a different level by the two researchers but ensuing discussion was able to bring agreement on the level in all cases. This compared favourably with 13% initial coding disagreement for the individual responses. Coding for the individual responses and group responses are presented, by group, in Table 1 for the rainfall segment and in Table 2 for temperature. The grade of a group is indicated at the beginning of the group number, e.g., group 1104 is the fourth group in Grade 11. An 'nc' coding indicates that the response could not be coded because it contained no description of variation.

Previous analysis of the individual responses indicated that students often included an inference rather than just describing the data as required (Reading, under review). This resulted in the individual and group tasks both having an inference focus. Also, noticeable during the coding of the temperature group responses was the lack of use of statistical tools introduced in the intervention. This observation is similar to the trend observed when analysing the individual responses (Reading, under review).

Group	Student 1	Student 2	Student 3	Student 4	Student 5	Group
701	nc	U1	M2			M1
702	U1	U2	M1			U1
703	M2	U1	M1			M1
704	U2	M2	U1	nc		M2*
705	M1	M2	U1	U2		M1
706	M1	U1	M1	NA		U1
901	U1	U2	U1	U1		U1
902	R1	U1	M1	U2	nc	U1
903	U2	M1	M1			M2*
904	R1	M1	M1			U1
905	M1	nc	M1			R1*
906	M1	M2	R1			U1
907	M2	U1	M1	M1		U1
908	M1	U1	M1	M1		U1
1101	U2	nc	U1	M1		U1
1102	U1	U1	U2	U1		U1
1103	M1	M1	M1	M1		U1
1104	M1	U1	M1	R1		U1

* indicates group responses 'better than' or 'equal to' the best individual response within the group

Table 1: Rainfall segment response coding

Group	Student 1	Student 2	Student 3	Student 4	Student 5	Group
707	M2	M2	U1	nc		M2*
708	R2	M1	nc	U1		M2
709	M2	nc	U1	U2		U1
710	U2	nc	M2	nc		M2*
711	M2	U1	U1	U1	U2	U1
909	U2	U2	M2			U2
910	M2	M2	M2			U1
911	U2	M1	U1	U1		U2
912	U1	U2	U1			R1
913	M1	U2	U2			M1
914	U2	U2	M1	M2	U2	U1
915	M2	M1	M2	M2		M2*
1105	M2	R2	R2	M2		R1
1106	M2	U2	M2			M2*
1107	U2	U2	M1	M2	U2	nc
1108	U2	M2	U1	U1		M2*
1109	R1	M2	M2			U1

* indicates group responses 'better than' or 'equal to' the best individual response within the group

Table 2: Temperature segment response coding

Once coding of the group responses had been completed the level was compared to that of the highest level of individual response within the group, considering whether the group response was ‘better than’, ‘equal to’, or ‘poorer than’ the best individual response. The 21% (8/35) of group responses that were better or equal (see asterisks in Tables 1 and 2) comprise 27% (3/11) in Grade 7, 20% (3/15) in Grade 9 and 22% (2/9) in Grade 11, suggesting similar performance across the three grades. However, the better or equal group responses comprise 17% (3/18) for the rainfall segment compared to 29% (5/17) for temperature suggesting better performance. The number of groups is too small to test for statistical significance. This suggestion of better performance after the teaching-intervention is supported by a comparison between the number of individual students and groups giving second cycle responses. For rainfall 20% (13/66) of the individual responses and 17% (3/18) of the group responses were coded as second cycle. However, for temperature, 60% (39/65) of individual responses and 47% (8/17) of the group responses were in the second cycle.

Of the eight groups with equal or better responses only two gave results better than the best individual response. Both groups were from Grade 9 (rainfall), each comprising three students. Of the six groups, spread across the three grades, where the results were equal, allowing for individual responses that were unable to be coded, there were effectively three responses contributing to the group discussion. This suggests that the smaller groups produced the better quality responses. By comparison, the four largest groups, each of five students, all produced a U1 or nc response much weaker than the best individual response.

In an attempt to explain why some group responses were better than the best individual response within the group while others were not, researchers then considered the original wording of the responses, rather than just the coded level. Although space prevents detailing of such observations here, some instances are mentioned in the following discussion.

DISCUSSION

Having determined that the hierarchy developed for coding individual responses was useful for coding the group responses, the next issue to be addressed is the usefulness of the coded level of the group responses. What is the relationship between the quality of response produced by a group compared to the responses produced by individuals within the group? The results of this study support Webb’s findings that “scores from group assessment may not be valid indicators of many students’ individual competence” (1993, p. 21). However, whereas Webb’s results show that many of the group responses are of a higher level than that indicated by post-testing of individuals, group responses in the present study tend to be of poorer quality than those produced by the individuals involved.

When considering what might influence the quality of group responses, the following factors reported by other researchers as having improved the quality of group products are relevant. While, students have indicated the positive influence of putting ideas together with others, especially those well known to them (Edwards & Jones, 1999, pp. 284-285), researchers have identified as important, the need for students to be more experienced in group work (p. 286) and the need to receive instruction in working as a group (Edwards, 2002, p. 318). The current research acknowledged the positive influence of these factors by encouraging students, with previous group work experience, to work together in self-selected groups. However, the students did not have specific instruction in how to work as part of a group.

The results of the current study suggest that there were few group responses that were equal or better and the following factors may have influenced the quality of the response made by the group.

1. Homogeneity in the thinking of group members

It was noticeable that in some groups the individual responses were more homogenous, i.e., the students were responding at similar coded levels as well as using similar reasoning (based on the original wording of the responses). For both better groups (903 and 905), the students' individual responses displayed homogeneous thinking that may have enabled them to build on their ideas and to work through to a higher level. By contrast, inspection of the many heterogenous groups (e.g., 1109) indicates the production of a 'residue' response reflecting only what was common or agreeable to all individuals, i.e., the 'popular view', thus producing a poorer quality response.

2. Lack of recognition of the need to cite supporting statistics

It appears that poorer group responses resulted when students did not perceive a need to support discussions with statistics already cited in their individual responses. For example, in group 1105 all four individual responses were higher quality second cycle responses but the group response, although relational, contained no supporting figures and so was coded as first cycle. In another example, group 709 (with two stronger and two much weaker students) calculated the averages for the maximum temperatures for all four months to find the highest; but then apparently group members decided it was unnecessary to include these figures in their response, being satisfied with just describing what they had done.

3. Nature of the data

The better quality responses for temperature produced by individuals may have been due to the different ways that students view the two variables, with rainfall often treated as a dichotomous variable (rain/no rain), students thus perceiving less of a need to include quantitative support for their inference (Reading, under review). Analysis of group responses suggests that the nature of the data also influences the quality of the group response compared to responses of individuals involved. There are more of the better or equal group responses for temperature. A complicating factor when investigating such a notion is that the rainfall data was the focus in the segment prior to the teaching-intervention while temperature was the focus afterwards. However, as no group responses made any reference to the statistical tools introduced in the teaching-intervention it must be assumed that there was no direct influence on the quality of the group responses. The observed greater proportion of better group responses for the temperature segment is more likely attributable to the differing nature of the data than the teaching-intervention.

4. Group size

As detailed in the results, better responses were given by the groups with only three, or three effective, members. There is the possibility that with fewer contributing members these groups can more easily come to a consensus and perhaps improve the quality of response. However, with smaller group sizes it needs to be noted that that there is also more chance of homogeneity of thinking, and this has already been identified as contributing to a better quality group response.

It might also be considered that the differing purposes of the tasks for individuals (description) and groups (inference) could have an effect on the quality of response to each task. Although students may see the need to describe variation in their individual responses, they may not see the need to include such information for the inferences drawn in the group responses. However, any possible effect of this factor on the quality of the group response is less likely because many of the individual students made inferences as well as descriptions, before working together within the group.

There is clear evidence from this research that the group responses are not reflecting the best of the capabilities of the individuals within the group. Factors most likely to be causing such a trend include homogeneity in the thinking of the groups, the lack of recognition of the need to cite supporting statistics, the nature of the data and the group size. A previously noted limitation of this study, though, is that only the group product was considered, and not the process. However, the importance of process was noticeable. For example, an apparent lack of effort was evident in the non-serious attempt of the group 117 response that was not able to be coded, while the five individual responses in the group were all coded in the second cycle. This suggests that the stages of maturity of the student, or other social factors such as the effect of the dominant student, may be affecting the group responses. However, such factors should be considered as part of the process, the other important aspect of group work but not the focus of this report. Importantly, assessment of the process could also help to shed more light on the factors affecting the relatively poor quality of group responses in comparison to responses of the individuals involved.

IMPLICATIONS

This research has implications for educators as well as researchers. When group work is being assessed the factors identified above need to be taken into consideration and care must be taken with the relationship of that group assessment to individual assessment. Educators also need to remember that overall assessment of both groups and individuals should include aspects of the group process as well. The SOLO levels previously developed for individual responses have proven suitable for coding group responses but in situations for which a coding hierarchy has not yet been developed, researchers should consider using group responses to directly develop a hierarchy for coding based on the SOLO Taxonomy. Future research could also consider using the SOLO Taxonomy to develop levels to assess the quality of how individuals work together within a group, i.e., the group process.

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CONNECTIONS BETWEEN QUALITATIVE AND QUANTITATIVE THINKING ABOUT PROPORTION: THE CASE OF PAULINA

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Abstract

The case study presented in this report was part of assessing a teaching proposal on ratio and proportion. A group of sixth-grade students of elementary education in México participated in the implementation of the proposal. The girl of the case study was representative of those students in the group who had a lot of recourse to handling algorithms mechanically and whose elaborations made no sense at all, according to their answers to an initial questionnaire. A didactical program, developed in a problem-solving context for the research study, helped the girl widen her qualitative thinking and strengthen her quantitative thinking about proportion. Analyses of data collected from an initial questionnaire, the teaching process, a final questionnaire, and three interviews evidenced that enriching this girl's qualitative thinking about proportion allowed her widen quantitative relations and improve her handling of algorithms by providing the setting for meaningful applications.

Some theoretical antecedents of the investigation

Piaget and Inhelder (1978) pointed out, as a result of their experimental researches in education, that children acquire qualitative identity sooner than quantitative conservation. Thus, these authors made a distinction between qualitative comparisons and true quantification. According to Piaget and Inhelder (1972), the acquisition of the notion of proportion always starts in a qualitative and logical form before it becomes quantitatively structured. Piaget defined what is qualitative by using categories or classes of words. Our own interpretation of what is qualitative refers to what is based on linguistic recognitions by creating comparison categories such as big or small. Our interpretation is that what is qualitative consists of intuitive and empirical aspects as well, which are provided by our senses.

Piaget (1978) pointed out that the idea of order emerges during the transition from the qualitative to the quantitative realm, although the idea of quantity is not yet present. Piaget called these situations *intensive quantifications*. For us, this is what makes the transition from qualitative to quantitative thinking stand out.

On their part, Van den Brink and Streefland (1979) agreed with Piaget as to their research findings that qualitative aspects of thinking occur sooner than quantitative ones. However, Streefland usually had recourse to these findings in teaching contexts. In our approach for the designing of the didactical program as well as in the development of interviews for the case study of educational research we present in this report, we used that contribution by Streefland.

Research findings reported by Streefland (1984; 1985) emphasized that the early teaching of ratio and proportion topics must depart from qualitative levels of recognizing them. For that purpose, Streefland made use of didactical resources, which strengthen the development of perceptual patterns for supporting the corresponding processes of quantification. Streefland stated that qualitative reasoning evolves as the thinking of the child advances and he or she is capable of incorporating more elements for an analysis, which will allow him or her to consider different factors simultaneously.

Thus, since Piaget and Streefland took into account qualitative and quantitative thinking about proportion exhibited by their subjects under research, the rationale for our case study was strongly based on Piaget's and Streefland's findings. We based the didactical approach developed for our research on Streefland's realistic mathematics approach.

Hart (1988) and her collaborators had reported results of their research studies on proportional thinking as well. They found out that most students who participated as subjects in their researches considered that it was difficult to solve mathematics problems that involved proportion. However, Hart and her team analyzed collected data and evidenced that younger students as well as pupils in secondary school with less success had a certain sense of "what is seen right" or of "what seems to be a distortion." Hart designated the latter as a regulation from "common sense," which we recognized as intimately involved in "qualitative thinking." Moreover, Hart pointed out that the most advanced level of proportional thinking occurred in those subjects who had already constructed certain concepts.

We based the didactical context of our research on realistic mathematics education referred to by Streefland (1993). Realistic mathematics education has become a theory since reality is, in first instance, a source of information and the context for the application of teaching models, schemata, and notations—school productions that have an influence in social practice. This theory favors the development of research and practice of the teaching and learning of mathematics. Analogously, according to this realistic theory it is essential to link students' learning periods by resorting to the "strategy of change in perspective," which is characterized by the exchange of part of the information in the problem-situation being approached. Consequently, the possibilities for the reconstruction and production of problems become explicitly recognized by students, without losing their multifaceted conceptual richness.

Research problem

The case study we present in this report was part of a research study carried out for a doctoral dissertation (Ruiz Ledesma, 2002). Previously, other aspects and activities of that research have been presented and reported in various communications. The case study of our research is about a girl, Paulina, who solved ratio and proportion problems by having recourse to algorithms which made no sense and had no meaning at all.[1]

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1 According to Benveniste (1971), meaning is a "dictionary entry" and "a universal semantic category"; and sense is a semantic content, which is associated to particular constructions of language, it does not shape universal categories and usually keeps a close relation to specific modes of articulating them. Moreover, it is proper to emphasize that there is not a chronological sequence, or of precedence, in the development of sense and meaning. They are different semantic components, which complement each other.

We designed a teaching proposal embedded in this situation, with the aim of strengthening her establishing of solid connections between qualitative and quantitative thinking about proportion, so that she could improve her handling of algorithms by situating them into meaningful applications. The following question guided our research about Paulina's case.

Research question

Does the extensive handling of qualitative aspects of ratio and proportion allow the student to widen quantitative relationships of these concepts as well as to improve the handling of her algorithms?

Hypothesis

Enriching Paulina's qualitative thinking—by using integrated verbal categories, recognizing the compensations posed between these categories, and involving the corresponding empirical and perceptual data—favors the significance processes she has developed by using algorithms for solving ratio and proportion problems.

Methodology

The research process of the case study of Paulina included integrating results from analyses of data collected from (a) her answers to an initial questionnaire, (b) a teaching program designed under a constructivist-didactical approach, (c) a final questionnaire, and (d) interviews of "didactical nature." The interviews with Paulina were based on results presented by Valdemoros (1998). The research instruments were tested in a pilot study of a one-year school cycle and definitively implemented during a ten-month period of fieldwork. In this case-study report we present relevant examples of the use of the research instruments.

The initial questionnaire was applied to collect evidence of qualitative thinking about proportion. The tasks included in this questionnaire did not involve the use of quantities for their solution: it comprised comparison activities that allowed the student recognize similarity relationships between figures.

Figueras, Filloy, and Valdemoros (1987) defined *model* as a collection of teaching strategies which include meanings—of both technical and common languages—, didactical treatments, specific modes of representation, and their interrelations. In the didactical program, according to that definition, we designed several situations associated to "teaching models" so that Paulina could link her qualitative and quantitative thinking processes on proportion. We worked with those models at different stages of the research experiment, similarly to what Streefland (1993) pointed out in his realistic theory as to the "change strategy in perspective:" We created a model and tried to get the best out of it in the light of an idea, so that we could retake it and use it for another idea.

Twenty-nine students of sixth-grade of elementary education in México, who were eleven years old, solved the initial questionnaire. We chose Paulina for a case study because she was representative of those students who, in the initial questionnaire, had a lot of recourse to handling algorithms that made no sense and who simultaneously exhibited few elaborations in the qualitative context. Throughout the development of the teaching experience, Paulina exhibited enrichment of her qualitative thinking and, in spite of making a lot of progress in the numerical context, she did not abandon the qualitative context of proportionality. She achieved a close harmony of both contexts.

Analysis of Paulina's progress by comparing her answers in the initial and in the final questionnaires

The initial and final questionnaires were integrated by the same tasks, although their application had a different aim. The first questionnaire was applied for exploratory purposes, whereas the second one focused on evaluating the implementation of the teaching program. Eight months elapsed between the applications of both questionnaires: Thus, there was no influence of the first questionnaire on the students' answers to the second one.

In the initial questionnaire, Paulina exhibited a preference for using algorithms mechanically and very little work in the qualitative context. We observed that she almost did not use her common sense or visualization. From the thirteen tasks posed in the initial questionnaire, she solved nine of them correctly.

The first two tasks in the questionnaire were designed so that Paulina could give justifications of her answers by strongly resorting to qualitative appreciations and not taking into account explicit quantities associated to the given relationships of proportionality. We employed squared paper in the next three tasks of the questionnaire to favor a transition toward quantification. The remaining tasks in the questionnaire involved quantified situations of ratio and proportion. In these last tasks, we provided Paulina with certain numerical values and asked her for new values. In some of these tasks we used a table of numerical values as a mode of representation for the recognition of external and internal ratios. Now we present an analysis of two tasks Paulina answered incorrectly: task 1 and task 4.

In task 1, the drawing of a house was presented and the student was required to select the correct reduced sketching of the original drawing (see figure 1). Paulina selected a sketching that did not correspond to the original drawing and she argued that her choice resembled best the original drawing of a house. However, in the final questionnaire Paulina based her choice of the reduced drawing by having recourse to her intuition first and then by measuring each part of this drawing to obtain the ratios with corresponding magnitudes of the original drawing, although in her new explanation she mentioned again that "house C looks like Antonio's house" and added that "it is similar, that is, proportional" (see figures 1 and 2). Thus, we observed that, from the initial questionnaire to the final one, the expression "looks like" underwent a change of meaning for Paulina: she exhibited an understanding of the term "proportion" as the relationship of equivalence between two ratios (but she did not abandon her common sense, which was exploited throughout the teaching program). We can ascertain this based on other collected evidence: for instance, Paulina did not solve correctly task 4 in the initial questionnaire, but she did in the final one. It is important to make the explanations she elaborated stand out in this research study; they are included in figures 4 and 5.

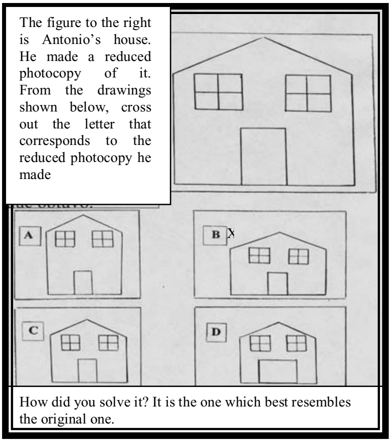

Figure 1. Task 1 of the initial questionnaire solved by Paulina.

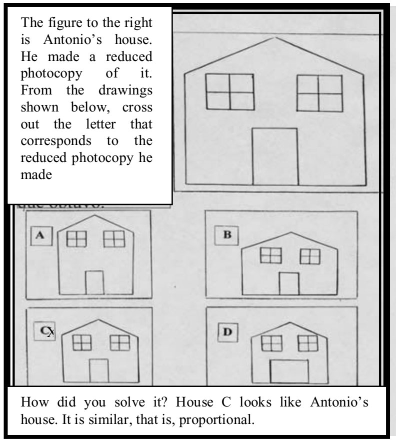

Figure 2. Task 1 of the final questionnaire solved by Paulina.

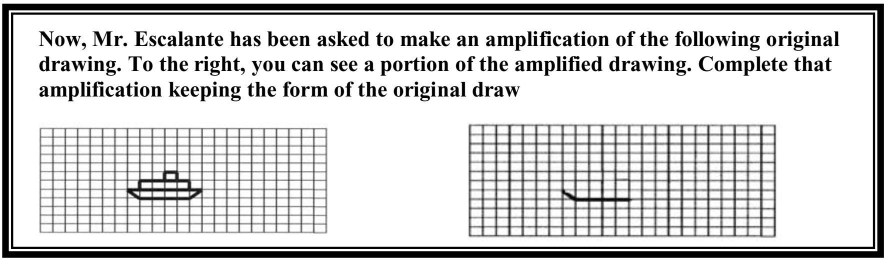

Figure 3. Task 4 in the initial and final questionnaires.

As shown in figure 4, Paulina completed the drawing but she did not notice that she had amplified it twice and not thrice. As seen from figure 5, Paulina showed the establishing of equivalence between two ratios that were obtained from comparing two corresponding magnitudes from the middle portion of the ship.

Analysis of Paulina's progress during the development of the teaching program

The solution of different tasks employed during the development of the teaching program, such as comparison activities, involved using quantities. These activities allowed Paulina recognize—by using very intuitive terms such as reduction and amplification—similarity relationships between figures and she could enrich her qualitative thinking. We worked with those notions by referring to concrete situations of the type of the experience of reproducing a drawing to scale and of the idea of using a photocopier.

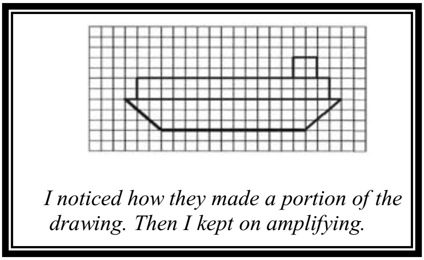

Figure 4. Task 4 solved by Paulina in the initial questionnaire.

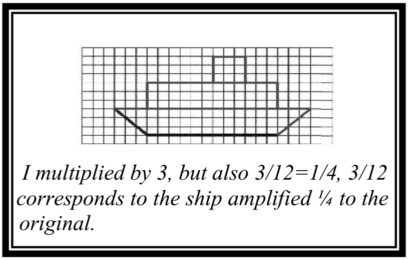

Figure 5. Task 4 solved by Paulina in the final questionnaire

The solution of different tasks employed during the development of the teaching program, such as comparison activities, involved using quantities. These activities allowed Paulina recognize—by using very intuitive terms such as reduction and amplification—similarity relationships between figures and she could enrich her qualitative thinking. We worked with those notions by referring to concrete situations of the type of the experience of reproducing a drawing to scale and of the idea of using a photocopier.

During the transition from qualitative to quantitative thinking, Paulina produced an ordering when comparing: she used the phrases “bigger than and smaller than”. This finding agrees with what Piaget (1978) pointed out. Later on, Paulina took measures to make comparisons. First, she compared different objects by placing one figure over another and then by using a measure instrument. In terms stated by Freudenthal (1983), the resources exhibited by Paulina at this development stage of her thinking are called “comparers.” After that, Paulina established relationships between magnitudes. She worked with natural numbers and employed fractions as well. Thus, at a very elementary level, she introduced herself to the field of rational numbers. The girl of this case study could designate a ratio as a relation between two magnitudes and a proportion as an equivalence relation between two ratios. This designation agrees with definitions given by Hart (1988).

When the working sessions ended, Paulina showed she had achieved a close relationship between her qualitative and quantitative thinking. This relationship implied the sense she made of her work in the numerical context, which was not revealed at the beginning of her work. Eventually, when the teaching experience ended and the final questionnaire was applied, Paulina’s meanings and quantification processes had been enriched. Now she could use a technical language in the designation context. She achieved a generalization stage in which new situations related to ratio and proportion were favored.

Analysis of Paulina's progress during the interviews

Paulina was interviewed in three different occasions, once a week, after the teaching program ended and the final questionnaire had been applied. The main purpose of the interviews was to asses the teaching program. The interviews consisted of asking Paulina to solve new tasks which aims were similar to those of the didactical program and of the questionnaires. Additionally, the development of the interviews gave feedback to Paulina.

With the first tasks we posed Paulina during the interviews, through her solution processes we could observe how she kept qualitative aspects to the light of having worked quantitative aspects, and how important it was for her to use visual images as well as her perception ability. Through the next tasks in the interviews, we also investigated how she handled numerical tables to recognize ratios and express these as fractions. During the interviews she exhibited her use of internal and external ratios, her transition from one symbolic system to another, and her posing of a situation where the use of proportions would be necessary to solve it.

This first interview was closely related to the Snow White and the seven dwarfs teaching model. Next, we show the development and analysis of that interview.

Paulina measured the length and the width of Snow White's wardrobe as well as the length and the width of each of the four drawings shown in the figure so that she could choose the required reduction. Once Paulina had chosen a wardrobe, she obtained the ratios between magnitudes of some of its parts and the corresponding parts of the original wardrobe. Now we show part of the interview with Paulina.

Interviewer: What did you base your choice of the dwarfs' wardrobe on?

Paulina: I took measures and found out that wardrobe B is proportional to Snow White's because all their ratios are equivalent. (Paulina pointed to what she had written, "12/8 = 6/4 =3/2.")

Interviewer: Will you please tell me how you obtained the ratios?

Paulina: By comparing measurements of Snow White's wardrobe with those of the dwarfs'. The numerator of each fraction measures certain part of Snow White's wardrobe: for instance, 12 is the length of the height, 8 is the length of the base, 3 is the length of one little window (she pointed to one of the drawings representing a decoration of the wardrobe), and 1.5 is the width of this little window. The denominators of the fractions are the measurements of the corresponding parts of the dwarfs' wardrobe. (The measurements Paulina mentioned are given in centimeters.)

Thus, Paulina established links to determine ratios based on taking measures. In another part of the same interview we could observe how she had recourse to her perception ability when she said, "Wardrobe A is too long, C is very wide, and D is very little. Although I did take measures, I noticed that those three wardrobes did not seem proportional to Snow White's."

Paulina exhibited that her handling of conceptual aspects was meaningful since she identified ratio as a relation and proportion as an equivalence relation between ratios. Moreover, we could notice that Paulina did not abandon the qualitative context, since she also used verbal categories and common sense to verify that her choice of the wardrobe was the right one. To this respect, she wrote that Snow White's wardrobe was equivalent to that of the dwarfs, and that as to their form they were equal although one was small and the other was big

Conclusions

Paulina exhibited a strong progress in relation to two important aspects:

1. The development of her qualitative thinking in relation to ratio and proportion.
2. The signification she gave to her using of algorithms.

During the processes of solving different tasks, Paulina exhibited how strong perceptual data became for her as well as how important it was for her to rely on her own experience. This is evidence about her achievements in the qualitative context of proportionality. The algorithmic work allowed us to explore the tacit recognition of the operators about which Paulina was thinking. These operators were natural numbers as well as fractions. The latter were used implicitly when multiplying certain value by a number and then dividing the result by another number, or vice versa, first dividing and then multiplying. In the context of what is now considered the construction of meanings, these—together with the processes of signification—were enriched. As to their designation, Paulina could eventually use the appropriate mathematical terms. Finally, she reached the point of constructing the concepts of ratio and proportion. This achievement was evidenced by the applications she made of those concepts in different contexts as well as by using their different modes of representation.

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TOWARDS HIGH QUALITY GEOMETRICAL TASKS: REFORMULATION OF A PROOF PROBLEM

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This paper analyses changes in the quality of a mathematical task as a result of its reformulation from a proof mode into an inquiry-based mode. The task is borrowed from the reformulation assignment given to the in-service teachers. The analysis of the task is based on implementation of the task with pre-service teachers. Through the lens of the task implementation I analyze task features that denote its qualities.

BACKGROUND

The quality of mathematical tasks

In their comprehensive analysis of mathematics lessons in US, Germany and Japan, Stigler & Hiebert (1999) pointed out the importance of kind of mathematics that is taught. ‘If the content is rich and challenging, it is more likely that the students have the opportunity to learn more mathematics and to learn it more deeply’ (p. 57). The researchers consider quality of school mathematics as function of *content elaboration*, *content coherence*, and *making connections* and state that the quality of mathematics at each lesson contributes to development of students’ mathematical understanding. Mathematical tasks that the teachers select as well as the settings in which the students are presented with the tasks determine quality of mathematical instruction.

In this paper task quality is defined by the set of four conditions (combined from Polya, 1981, Schoenfeld, 1985; Charles & Lester, 1982). First, the person who performs the task has to be *motivated* to find a solution; Second, the person has to have *no readily available procedures* for finding a solution; Third, the person has to makes an *attempt* and persists to reach a solution; Fourth, the task or a situation have several solving approaches. Obviously, these criteria are relative and subjective with respect to person’s problem-solving expertise in a particular field, i.e., a task, which is cognitively demanding for one person may be trivial (or, vice versa, unrewarding) for another.

Teachers’ role in teaching high quality mathematics

Inquiry dialog (Wells, 1999) may be seen as a way for increasing the quality of school mathematics. Inquiry tasks usually are challenging, cognitively demanding and allow highly motivated students' work. The students in such an environment conjecture, debate the conjectures, search for explanations and proofs and discuss their preferences regarding different ways of solution. Teachers’ awareness (Mason, 1998) of different solving approaches to a problem helps teachers act according to students ideas and be flexible in lesson orchestration (Leikin & Dinur, 2003). Although the importance of a dialogic learning environment is declared, and the teacher's flexibility in the classroom is widely discussed (Brousseau, 1997; Simon, 1997), such a classroom environment remains challenging and vague for teachers. No clear guidelines can be provided for each particular lesson, which is based on students' ideas and conjectures.

The quality of mathematics in any particular classroom depends on teachers' knowledge and beliefs. Teachers' previous experiences often are reflected in their skepticism with regard to changes in the nature and quality of mathematical tasks. Therefore pre-service teacher education is primarily intended to prepare prospective teachers to teach in ways different from those in which they learned as pupils, while in-service education is aimed at developing teachers' proficiency to teach in ways different from those in which they both learned and teach. In this paper I exemplify an implementation of a high quality task in teacher professional development. The analysis is focused on the quality of the task as reflected in problem posing and problem solving procedures in which the teachers were involved.

Supporting quality of mathematical tasks by inquiry in DGE

Nowadays it is rather natural to connect inquiry with computer-based learning environment in general and with dynamic geometry environment (DGE) in particular. This research developed through practice of in-service mathematics teacher education, which was based on the theoretical ideas about implementation of Dynamic Geometry in school mathematics (e.g., Chazan & Yerushalmy, 1998). Teachers' expertise is a crucial issue for an effective implementation of DGE, while one of the characteristics of such an expertise is teachers' ability to formulate powerful mathematics tasks for inquiry in DGE. Many studies explored the role of DGE in teaching-learning processes, namely in concept acquisition, geometrical constructions, proofs, and measurements (e.g., Mariotti, 2002; Jones, 2000). This paper is aimed to analyze changes in the quality of a mathematical task as a result of the adaptation of a proof task for the inquiry in DGE.

THE OBJECTIVES

The main objective of this paper is analysis of the changes in the quality of geometrical that emerge as the result of task adaptation from standard textbook proof task to an inquiry-based problem for work in DGE.

One of mathematics teachers' responses served a starting point for the analysis of the possible outcomes of task re-formulation, which is performed through the lens of problem-solving procedure exposed by 36 pre-service mathematics teachers (PMT) The analysis is based on my observations, written field notes, and videotaped mathematical performances of three groups of PMT.

THE TASK QUALITIES

Meeting the tasks

The task, which is in the focus of this paper, was raised in the in-service course "teaching an inquiry-based mathematics". In one of the assignments the teachers were asked "to choose a problem from a standard textbook and re-formulate it for an inquiry in DGE". The teachers performed the assignment individually and presented it to other teachers who participated in the course. They explained the re-formulation performed and described the classroom setting in which the re-formulated problem was implemented. One of the teachers (Anat) reformulated a task borrowed from a standard text-book (Goren, 1996):

The original task: *In the isosceles trapezoid ABCD the diagonals are perpendicular ($AC\perp BD$). Prove, that the altitude of the trapezoid equals to the mid-line joining the mid-points of the two sides of the trapezoid. Prompt: Built the altitude through O [the point of intersection of the diagonals]*

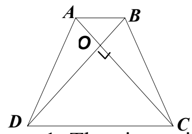
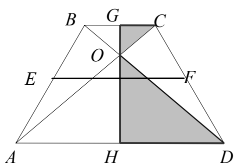


Figure 1: The picture in the textbook

The original task is a prove task, which, according to it placement in the textbook, clearly requires from the students application of the mid-line-of-a-trapezoid theorem. The drawing is presented and the prompt, which is given in the text of the task, simplify the solution and direct it towards one particular solving approach. An intended solution of the original tasks is depicted in Figure 2.



GH-height through *O*, then *OG* and *OH* are the medians, the altitudes and the bisectors in the triangles *BOC* and *AOD*.

Hence, triangles *AHD* and *AGH* are right isosceles triangles: $HD=HO$, $GC=GO$

Thus, $GH=GO+OH=GC+HD=½BC+½AD=EF$

Figure 2: **A solution of the original task**

Anat reformulated this task as follows:

Anat's problem: *Given an isosceles trapezoid with perpendicular diagonals. Compare the length of the altitude of the trapezoid and the length of the mid-line joining the mid-points of the two sides of the trapezoid?*

Anat's problem included requirement of comparing the two segments instead of proving their equality. The reformulation was based on the opening the task by "hiding" one of the properties of the given geometric figure. So, instead of proving the property, students had to find it, formulate it and prove it. However, this opening was very narrow and purpose directed. Students should, almost immediately, realize that the segments are equal. Additionally, Anat decided to provide her students with detailed guidelines for the construction of the figure in DGE since "usually students spend a lot of time on constructions". Her guidance included construction of a square and constructing the upper base of the trapezoid by joining the square diagonals inside the square. At this stage of learning the students in her class experienced in using dragging to perform geometrical explorations and the transformed tasks required this type of activity. Anat reported that students conjectured quickly that the segments are equal and proved their conjecture under her guidance.

Raising the task quality through teachers' discussion

Teachers' discussion on the task presented by Anat focused on the two main issues. First, teachers' differed in their opinions about the necessity of the *detailed guidelines for the construction*. Some of the teachers, as Anat, preferred providing students with the guidelines since "geometrical constrictions are not part of the curriculum", and "there is not enough time for this type of activity". The other group of teachers considered constructing a figure as an integral part of the inquiry tasks in DGE. They told they were happy with any opportunity to "teach geometrical constructions, as they develop students understanding of geometry and students' logical reasoning" Second, the teachers disagreed about the *level of openness of the question*. Some teachers found Anat's problem open enough and confirmed that they would "do the same". Other teachers tended "to open the task more" both by describing the figure as "an equilateral trapezoid with perpendicular diagonals" and by asking students to explore the figure properties.

One of the teachers, Maya, who was one of the group leaders, suggested a compromise:

Maya's problem: *In the isosceles trapezium ABCD the diagonals are perpendicular (AC⊥BD). Find possible relationships between two midlines of the trapezoid, which join mid-points of the opposite sides of the trapezoid.*

She stated that asking students to find all the properties of the given trapezoid is too vague. She suggested that "replacing the altitude in the original problem by the "second mead-line" makes the problem more elegant" and that changing the word "compare" by "find relationships" opens the question more since "it is clear that the altitude is perpendicular to the midline, however it is not obvious for the two mid-lines".

No consensus was obtained on each of these issues. The teachers liked Maya's problem "for themselves" however were uncertain regarding the task implementation with their students. Teachers' "craft knowledge" (in terms of Kennedy, 2002) embodied in their personal experiences with particular students' populations, and in their own (mostly limited) experiences with DGE, reflected in their intuitions about what is better for their students and how the tasks re-formulation may be better performed.

Challenging pre-service teachers with Maya's task: The setting

I explored the quality of *Maya's task* with pre-service mathematics teachers [PMT]. The task was probed with three groups of PMT of 12, 10, 12 teachers in each group. All the PMTs had BA in mathematics and were learning first year for the teaching certificate. At each session, that took place at the end of the year, the PMTs worked in DGE in pairs or in small groups of three. Overall 16 small groups (or pairs) were observed and interviewed collectively. I made field notes and reflective notes at each workshop. These notes were discussed with the PMTs during the consecutive meetings in order to confirm research suggestions regarding problem-solving procedures the PMTs encountered.

All the PMTs in three workshops were allowed using Geometrical Supposer (Schwartz, Yerushalmy & Shternberg, 2000) while solving the tasks. Interestingly, all of the PMTs made progress in similar sequences. All the small groups (pairs) of PMTs started with a "freehand drawing" (Chazan & Yerushalmy, 1998). Rather quickly (within 5-7 minutes) most of the participants started construction of the figure so that dragging would preserve the given properties of the trapezoid. Only 1 of 16 small groups of insisted on continuing "freehand drawing" with subsequent correction of the drawing.

When the construction was completed the students carried out measurement of different types and their conjectures were mainly based on the invariants observed while dragging. At the next stage the PMTs proved their conjectures. The next section presents the power of the mathematical tasks by addressing each one of these stages.

No ready-to use procedure: Freehand Drawing

All the PMTs started with drawing a trapezoid and then added diagonals to it. The ways in which the participants created an isosceles trapezoid were similar to those described by Chazan and Yerushalmy (1998). Isosceles trapezoid, which should be constructed in our investigation, had an additional property, i.e., perpendicular diagonals. Thus PMTs constructed the diagonals and “fixed” the angle between them by dragging the trapezoid vertexes. They found themselves “spoiling” the figure by further dragging. They tried to “fix” the figure again and again and sometimes were not able to obtain “exact properties of the figure”.

Obviously this inclination for freehand drawing was borrowed by PMTs from their experience in paper and pencil drawing. To verify this observation the PMTs in two (of three) groups were asked to draw an equilateral trapezoid with perpendicular diagonals on the paper a week after the activity took place. Of 22 participants only 3 started the drawing with diagonals even a week beforehand they discussed in details how to construct the figure. Three teachers started drawing with perpendicular diagonals since the "the drawing will be more precise this way".

Constructing the isosceles trapezoid with perpendicular diagonals in different ways

After realizing that freehand drawing does not allow exploring by dragging PMTs tried to perform "exact construction". This led them to the precise analysis of the properties of an isosceles trapezoid which diagonals are perpendicular. This analysis included thinking about necessary and sufficient conditions of the geometric figure.

At this stage of the work PMTs analyzed "which construction will allow dragging that preserves (a) the quadrilateral as a trapezoid and (b) perpendicularity of the diagonals.

Interestingly in each of the three groups of PMTs at least three different strategies for construction of the trapezoid with perpendicular diagonals were suggested. Figure 2 depicts two of these constructions.

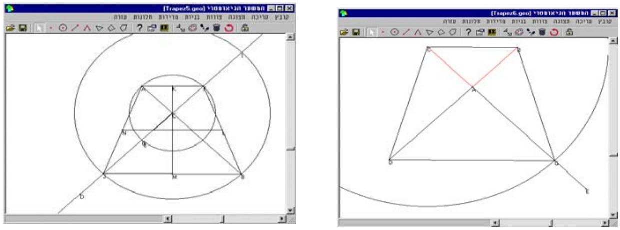


Figure 2: **Two different constructions**

Perpendicular diagonals that are congruent and are divided into two pairs of congruent segments by the intersection point served a sufficient condition for the construction of the given figure. The different constructions included construction of two pairs of congruent segments on the two perpendicular straight lines or completing a right isosceles triangle (the length constrains of the paper do not allow more detailed analysis of the constructions performed). Within each big strategy there were many variations and the teachers were always surprised by the amount of different ways in which different pairs constructed the trapezoid. Note here, that the construction that Anat suggested to her students was not produced by any of the experimental groups.

As mentioned above, only one (of 16) small groups of PMTs argued for sufficiency of freehand drawing. The students in this group were reluctant towards use of DG in teaching school geometry. At the construction stage they "braked and fixed" again and again their drawing and stated that "they may see the regularity". For them the midlines were always "almost perpendicular" and "almost equal" so they come to the conjecture as all other small groups that *in an isosceles trapezoid with perpendicular diagonals the midlines which join mid-points of the opposite sides of the trapezoid are equal each other and perpendicular to each other.*

Proving the conjecture in different ways

Interestingly in each of the 3 groups two different proofs for the conjecture were presented. One of the proofs was similar to one presented in Figure 2. The other proof was based on the construction that the PMTs performed.

Figure 3 depict computer screen in which the internal quadrilateral with vertexes in the midpoints of the given trapezoid is a square since the diagonals in the trapezoid are equal and perpendicular. The midlines that join mid-points of the opposite sides of the trapezoid are equal to each other and perpendicular to each other as the diagonals of the square. This construction-based proof (see Figure 3) usually was found as easier one, more elegant and convincing.

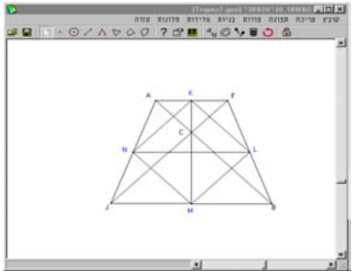


Figure 3: **The construction-based proof**

DISCUSSION AND CONCLUDING REMARKS

In this paper I tried to argue for the raising quality of mathematical tasks when adapting them to inquiry-based learning environment. The quality of the task was defined as depending on the four conditions. The first condition considers motivation for performing the task. As was shown in the paper inquiry problems that fit learners' level stimulate their motivation, like in the case of PMTs shifting from freehand drawing toward a systematic construction. It must be noted that construction procedure which is a part of the inquiry tasks on the one hand deepens analysis of the necessary and sufficient conditions of the given figure and on the other hand complicates the task performance. Based on this feeling of the complexity of the construction tasks many in-service teachers are inclined to provide their students with detailed guidance for the constriction. As it was shown, when performing the inquiry tasks, usually teachers had no readily available procedures for finding a solution. They had to make a certain attempt and persist to reach a solution. I tried to demonstrate that the inquiry tasks contrary to the original proof task had several solving approaches both at the stage of exploring the situation and at the stage of proving the conjectures. The Inquiry procedure seemed to be more connected and elaborated in all the three groups of PMTs. It should be noted that contrary to the Maya's task that is analyzed in the paper, Anat's task is not so distant from the textbook task and does not encompass the same qualities.

Zaslavsky, Chapman and Leikin (2003) suggested that a mathematical task is powerful if it involves dealing with uncertainty and doubt, engaging in multiple approaches to problem solving, identifying mathematical similarities and differences, developing a critical view of the use of educational technology, rethinking mathematics, and learning from students' thinking. The analysis performed in this paper explicitly addresses four of these characteristics. Additionally, one may see rethinking mathematics in teachers' reasoning about geometrical constructions as well as in their exploration of the midlines in quadrilaterals. As the paper presents my own learning from the teachers' thinking I assume that by using this task teachers may learn from students thinking.

Finally I would like to suggest one more condition that may be included into the list of conditions defining mathematical tasks of high quality, namely, the possibility to raise and discuss new mathematical question. I usually continue the mathematical discussion on Maya's task with a question: *Is it possible to inscribe a circle into an isosceles trapezoid with perpendicular diagonals*?

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MATHEMATICAL THINKING & HUMAN NATURE: CONSONANCE & CONFLICT

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Human nature had traditionally been the realm of novelists, philosophers, and theologicians, but has recently been studied by cognitive science, neuroscience, research on babies and on animals, anthropology, and evolutionary psychology. In this paper I will show—by surveying relevant research and by analyzing some mathematical "case studies"—how different parts of mathematical thinking can be either enabled or hindered by aspects of human nature. This novel theoretical framework can add an evolutionary and ecological level of interpretation to empirical findings of math education research, as well as illuminate some fundamental classroom issues.

A. INTRODUCTION

This paper deals with the relationship between mathematical thinking and human nature. I take from the young discipline of Evolutionary Psychology (EP) the scientific view of human nature as a collection of universal, reliably-developing, cognitive and behavioral abilities—such as walking on two feet, face recognition or the use of language—that are spontaneously acquired and effortlessly used by all people under normal development (Cosmides & Tooby 1992, 1997, 2000; Pinker 1999, 2002, Ridley, 2003). I also take from EP the evolutionary origins of human nature, hence the frequent mismatch between the ancient ecology to which it is adapted and the demands of modern civilization. To the extent that we do manage to learn many modern skills (such as writing or driving, or some math), this is because of our mind's ability to "co-opt" ancient cognitive mechanisms for new purposes (Bjorklund & Pellegrini, 2002; Geary, 2002). But this is easier for some skills than for others, and nowhere are these differences manifest more than in the learning of mathematics. The ease of learning in such cases is determined by the *accessibility* of the co-opted cognitive mechanisms. I emphasize that what is part of human nature need not be innate: we are not born walking or talking. What seems to be innate is the motivation and the ability to engage the species-typical physical and social environment in such a way that the required skill will develop (Geary, 2002). This is the ubiquitous mechanism that Ridley (2003) have called "Nature *via* Nurture".

These insights have tremendous implications for the theory and practice of Mathematics Education (ME), but to this date they have hardly been noticed by our community (but cf. Tall, 2001; Kaput & Shaffer, 2002). The goal of this paper is to launch an investigation (theoretical at this preliminary stage) of how the insights from EP may bear on the theory and practice of ME. Specifically, the goal is to investigate, in view of the above-mentioned mismatch, how different parts of mathematical thinking can be either enabled or hindered by specific aspects of human nature.

Consider, for example, the well-documented phenomenon that students tend to confuse between a theorem and its converse (e.g., Hazzan & Leron, 1996). As will be shown later, this phenomenon can now be understood as a clash between Mathematical Logic and the Logic of Social Exchange – a fundamental part of human nature.

In the rest of this document, I will describe my synthesis of the existing research on how mathematical thinking is enabled (Section B) or constrained (Section C) by human nature.

B. ORIGINS OF MATHEMATICAL THINKING[1]

In this section I consider the following (admittedly vague) question:

> *Is mathematical thinking a natural extension of common sense, or is it an altogether different kind of thinking?*

The possible answers to this question are of great interest and importance for both theoretical and practical reasons. Theoretically, this is an important special case of the general question of how our mind works. In practice, the answers to this question clearly have important educational implications. I take *common sense* to mean roughly the same as the cognitive part of human nature – the collection of abilities people are spontaneously and naturally "good at" (Cosmides & Tooby, 2000).

Recently, several books and research papers have appeared, which bear on this question, so that the possible answers, though still far from being conclusive, are less of a pure conjecture than they had previously been. The conclusions of the various researchers seem at first almost contradictory: Aspects of mathematical cognition are described as anything from being embodied to being based on general cognitive mechanisms to clashing head-on with what our mind has been "designed" to do by natural selection over millions of years.

However, these seeming contradictions all but fade away once we realize that "mathematics" (and with it "mathematical cognition") may mean different things to different people, sometimes even to the same person on different occasions. In fact, the main goal of this section is to show that all this multifaceted research by different people coming from different disciplines, may be neatly organized into a coherent scheme once we exercise a bit more care with our distinctions and terminology.

To this end, I will distinguish three levels of mathematics, *rudimentary arithmetic*, *informal mathematics* and *formal mathematics*, each with its own different cognitive mechanisms[2]. When interpreted within this framework, the research results show that while certain elements of mathematical thinking are innate and others are easily

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[1] More details on the topics of this section can be found in Leron (2003a).

[2] Strictly speaking, 'formal' and 'informal' ought to refer not to the mathematical subject matter itself but to its presentation, and may in fact describe two facets of the *same* piece of mathematics.

learned, certain more advanced (and, significantly, historically recent) aspects of mathematics—formal language, de-contextualization, abstraction and proof—may be in direct conflict with aspects of human nature.

Following is a brief survey of theses studies and the various suggested cognitive mechanism, organized according to my 3-level framework.

Level 1: Rudimentary arithmetic

Rudimentary arithmetic consists of the simple operations of subitizing, estimating, comparing, adding and subtracting, performed on very small collections of concrete objects. Research on infants and on animals, as well as brain research, indicates that some ability to do mathematics at this level is hard-wired in the brain and is processed by a 'number sense', just as colors are processed by a 'color sense'. Comprehensive syntheses of this research are Dehaene (1997) and Butterworth (1999). The innate character of this ability is evidenced by its existence in infants, its localization in the brain and its vulnerability to specific brain injuries. The likelihood of its evolutionary origins comes from its complex design features, the survival advantage it have likely conferred on our hunter-gatherer ancestors (e.g., in keeping count of possessions, and in estimating amount of food and number of enemies), and its existence in our non-human relatives (such as chimpanzees, rats and pigeons).

Level 2: Informal Mathematics

This is the kind of mathematics, familiar to every experienced teacher of advanced mathematics, which is presented to students in situations when mathematics in its most formal and rigorous form would be inappropriate. It may include topics from all mathematical areas and all age levels, but will consist mainly of "thought experiments" (Lakatos, 1978; Tall, 2001; Reiner & Leron, 2001), carried out with the help of figures, diagrams, analogies from everyday life, generic examples, and students' previous experience.

Some recent research, as well as classroom experience, indicates that informal mathematics *is* an extension of common sense, and is in fact being processed by the same mechanisms that make up our everyday cognition, such as imagery, natural language, thought experiment, social cognition and metaphor. From an evolutionary perspective, it is only to be expected that mathematical thinking has "co-opted" older and more general cognitive mechanisms, taking into account that mathematics in its modern sense has been around for only about 25 centuries – a mere eye blink in evolutionary terms (Bjorklund & Pellegrini, 2002; Geary, 2002).

Two recent books have presented elaborate theories to show how our ability to do mathematics is based on other (more basic and more ancient) mechanisms of human cognition. First, Lakoff & Núñez (2000) show (more convincingly in some places than in others) how mathematical cognition builds on the same mechanisms of our general linguistic and cognitive system; namely, they show how mathematical cognition is first rooted in our body via embodied metaphors, then extended to more abstract realms via "conceptual metaphors", i.e., inference-preserving mappings between a source domain and a target domain[3]. Secondly, Devlin (2000) gives a different account than Lakoff & Núñez, but again one showing how mathematical thinking has co-opted existing cognitive mechanisms. His claim is that the metaphorical "math gene"—our innate ability to learn and do mathematics—comes from the same source as our linguistic ability, namely our ability for "off-line thinking" (basically, performing thought experiments, whose outcome will often be valid in the external world). Devlin in addition gives a detailed evolutionary account of how all these abilities might have evolved.

Significantly for the thesis presented here, both theories mainly seek to explain the thinking processes involved in Level 2 mathematics, so that their conclusions need not apply to Level 3 mathematical thinking[4]. In fact, as I explain in the next section, they generally don't. Devlin's account, in particular, fits well situations in which people do mathematics by constructing mental structures and then navigate within those structures[5], but not situations where such structures are not available to the learner. For example, it is hard to imagine any "concrete" structure that will form an honest model of a uniformly continuous function or a compact topological space.

Level 3: Formal Mathematics

The term "formal mathematics" refers here not to the contents but to the form of advanced mathematical presentations in classroom lectures and in college-level textbooks, with their full apparatus of abstraction, formal language, de-contextualization, rigor and deduction. The fact that understanding formal mathematics is hard for most students is well-known, but my question goes farther: is it an extension (no matter how elaborate) of common sense or an altogether different kind of thinking? Put differently, is it a "biologically secondary ability" (Geary, 2002), or an altogether a new kind of thinking that ought perhaps to be termed "biologically tertiary ability"? This issue will be our focus in the next section. The mathematical case studies, as well as the persistent failure of many bright college students to master formal mathematics, suggest that the thinking involved in formal mathematics is *not* an extension of common sense; that it either can't find suitable abilities to co-opt, or it can even clash head-on with what for all people "comes naturally".

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[3] However, as Tall (2001) points out, we need also to take into account a process going in the reverse direction. Some of the results of the formal axiomatic theory (called "structure theorems") may feed back to develop more refined intuitions, or embodiments, of the concepts involved.

[4] The authors are not always explicit on the scope of mathematics they discuss, but see, e.g., "I am not talking about becoming a great mathematician or venturing into the heady heights of advanced mathematics. I am speaking solely about being able to cope with the mathematics found in most high school curricula." (Devlin, 2000, p. 271); and "Our enterprise here is to study everyday mathematical understanding of this automatic unconscious sort." (Lakoff & Núñez, 2000, p. 28).

[5] See in this connection his "mathematical house" metaphor on p. 125.

C. MATHEMATICAL THINKING VS. HUMAN NATURE: SOME MATHEMATICAL CASE STUDIES

The theoretical framework outlined above will now be applied to a sample of three mathematical "case studies" which, except for that framework, would have largely remained an unexplained paradox. Each of the case studies deals with a well-defined mathematical topic or task, which on the one hand seems rather simple and elementary, but on the other hand has been shown to cause serious difficulty for many people (i.e. most people fail on it).

Memorizing the Tables

Dehaene (1997) discusses the ample research evidence by psychologists, that many people find memorizing the multiplication table extremely hard: they take a long time to answer and they make many errors. There is also much evidence that all children demonstrate prodigious learning and memory capabilities in, say, learning the vocabulary and use of their mother tongue. How is it then that many have such difficulty remembering a few (less than 20 if you count carefully) multiplication facts? How can human memory, which in other contexts performs astonishing feats, fail on such a simple-looking task?

This example demonstrates in a nutshell—and elementary setting—the typical paradox that is at the core of this paper: People fail this task not because of a weakness in their mental apparatus, but because of its *strength*! Trouble is, what may have been adaptive in the ancient ecology of our stone-age ancestors, and is still adaptive today under similar conditions, may often be maladaptive in modern contexts. Returning to the present example, the particular strength of our memory that gets in the way of memorizing the tables is its insuppressible *associative* character: The *forms* of the number facts cannot be separated out and remain hopelessly entangled with each other. To quote Dehaene (1997): "Arithmetic facts are not arbitrary and independent of each other. On the contrary, they are closely intertwined and teeming with false regularities, misleading rhymes and confusing puns."

Mathematical Logic vs. the Logic of Social Exchange

Cosmides and Tooby (1992, 1997) have used the Wason card selection task, which tests people's understanding of "if P then Q" statements, to uncover what they refer to as people's evolved reasoning algorithms. They presented their subjects with many versions of the task, all having the same logical form "if P then Q", but varying widely in the contents of P and Q and in the background story. While the classical results of the Wason Task show that most people perform very poorly on it, Cosmides and Tooby demonstrated that their subjects performed relatively well on tasks involving conditions of *social exchange*. In social exchange situations, the individual receives some benefit and is expected to pay some cost. In the Wason experiment they are represented by statements of the form "if you get the benefit, then you pay the cost" (e.g., if you give me your watch, then I give you $20). A *cheater* is someone who takes the benefit but do not pay the cost. Cosmides and

Tooby explain that when the Wason task concerns social exchange, a correct answer amounts to detecting a cheater. Since subjects performed correctly and effortlessly in such situations, and since evolutionary theory clearly shows that cooperation cannot evolve if cheaters are not detected, Cosmides and Tooby have concluded that our mind contains evolved "cheater detection algorithms".

Significantly for mathematics education, Cosmides and Tooby have also tested their subjects on the "switched social contract" (mathematically, the converse statement "if Q then P"), in which the correct answer by the logic of social exchange is different from that of mathematical logic (Cosmides and Tooby, 1992, pp. 187-193). As predicted, their subjects overwhelmingly chose the former over the latter. When conflict arises, the logic of social exchange overrides mathematical logic. This theory adds a new level of support, prediction and explanation to the many findings (e.g. Hazzan & Leron, 1996) that students are prone to confusing between mathematical propositions and their converse. Again, they fail not because of a human cognitive weakness, but because of its *strength*: the ability to negotiate social exchange and to detect cheaters. Unfortunately for mathematics education, this otherwise adaptive ability, clashes with the requirements of modern mathematical thinking.

Do functions make a difference?

The phenomenon reported here came up in the context of research on learning computer science (specifically, functional programming), but has turned out to be really an observation on mathematical thinking. Interestingly, it is hard to see how this phenomenon could have been revealed through a purely mathematical task. The empirical research reported here is taken from Tamar Paz's (2003) dissertation, carried out under the supervision of this author. I only have space here for a very brief outline of this study. Cf. Leron (2003b) for more details.

In functional programming, functions are mainly viewed as a process, starting with an input value, performing some operations on it, and returning an output value. This image is nicely captured by the function machine metaphor. I have called this the *algebraic image* of functions, as opposed to the *analytic image*[6]. For example, the function `Rest L` takes a list `L` as input (for example, the value of `L` could be the 4-element list `[A B C D]`) and returns the list without its first element (in this example, `[B C D]`). Paz (2003) has found that many of the students' programming errors could be attributed to their assumption that the function actually changes the input variable (so that after the operation, L assumes the new value `[B C D]`). But in functional programming, as in its parent discipline mathematics, functions do *not* change their inputs, they merely map one value to another.

I propose to view this empirical finding as an example of the clash between the modern mathematical view of functions, and their origin in human nature. To do this, we need to look for the roots (mainly cognitive and developmental, but also

[6] We have two *concept images* of function, sharing one *concept definition* (Vinner & Tall, 1981).

historical) of the function concept – a synthesis of Freudenthal's (1983) "didactical phenomenology" and Geary's (2002) "biologically primary abilities": What in the child's natural experience during development, may have given rise to the basic intuitions on which the function concept is built? (Fruedenthal, 1983; Kleiner, 1989; Lakoff & Núñez, 2000.)

According to the algebraic image of functions, an operation is *acting on an object*. The agent performing the operation takes an object and does something to it. For example, a child playing with a toy may move it, squeeze it, or color it. The object before the action is the input and the object after the action is the output. The operation is thus transforming the input into the output. The proposed origin of the algebraic image of functions is the child's experience of acting on objects in the physical world. This is part of the basic mechanism by which the child comes to know the world around it, and is most likely part of what I have called universal human nature. Part of this mechanism is perceiving the world via objects, categories and operations on them, as described by Piaget, Rosch, and others. Inherent to this image is the experience that an operation *changes its input* – after all, that's why we engage in it in the first place: you move something to change its place, squeeze it to change its shape, color it to change its look.

But this is not what happens in modern mathematics or in functional programming. In the modern formalism of functions, nothing really changes! The function is a "mapping between two fixed sets" or even, in its most extreme form, a set of ordered pairs. As is the universal trend in modern mathematics, an algebraic formalism has been adopted that completely suppresses the images of process, time and change[7].

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[7] Professional mathematicians are able to maintain these images *despite* the formalism, but for novices the connection is hard to come by.

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THE MATHEMATICS PEDAGOGICAL VALUES DELIVERED BY AN ELEMENTARY TEACHER IN HER MATHEMATICS INSTRUCTION: ATTAINMENT OF HIGHER EDUCATION AND ACHIEVEMENT

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Abstract

This paper investigates the values implied in the mathematics instruction of an elementary school teacher in charge of fifth and sixth grade. The major research methods were classroom observation and interview. The research results discovered that the foremost mathematics pedagogical value of the teacher in this case study was "a good learning in mathematics makes easier for students to attain a higher education and for those who are able to attain a higher education can achieve a respectable social status and a successful life more easily." This value is an influence from one of Confucian dogma that it's extremely important for traditional Chinese intellectuals to pass the imperial examination and to earn a respectable social status. This paper also presents a brief discussion on the relationship between the values of mathematics instruction upheld by elementary teachers and the curriculum reform.

Introduction

The New Elementary Mathematics Curriculum (NEMC) in Taiwan was launched in 1996 and the primary educational goal of this reform was to "guide children to obtain mathematics knowledge from their daily life experiences and to develop the attitudes and abilities to apply mathematics effectively to solve problems they encounter in real life." Furthermore, it was to "nurture the attitude of respecting others through the practice of understanding and evaluating other students' solving process" and to "develop the consciousness to communicate, discuss, rationalize and criticize in mathematics language" (ME, 1993, p.91). The reform-oriented modes of teaching moved away from teacher-centered instruction toward pupil-centered learning and emphasized more group discussions than classroom lecturing. The Old Elementary Mathematics Curriculum (OEMC) emphasized on calculation skills but the NEMC focuses on problem solving ability.

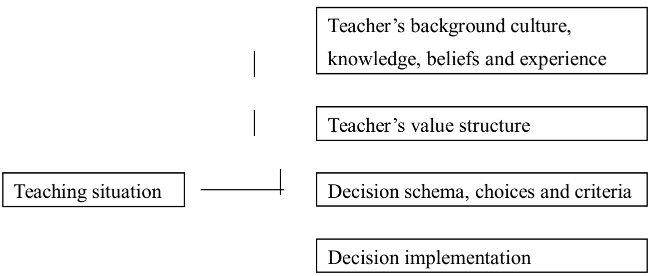
Starting in 2001, there has been another reform in mathematics curriculum at the elementary and junior high school level in Taiwan; it is named as "Mathematics Learning Area in Grade 1-9 Curriculum" (MLAGC). The curriculum goals of MLAGC preserve the ones of NEMC, which emphasize on the ability of problem solving, communicating and reasoning. However, one thing new and different about MLAGC is that it stresses on the cultivation of mathematics competencies, besides the acquirement of mathematics knowledge (ME, 2000).

The curriculum goals of both NEMC and MLAGC were influenced by the mathematics curriculum standards in the U.S. (NCTM, 1989, 2000). This paper addresses some of the socio-cultural issues related to the importation of curriculum and instruction from one country to another country with inherently different beliefs and values. For an instance, most people agree with the statement "mathematics is important". But what are some important goals of mathematics teaching? Is it the development of students' calculation skills? Or is it the ability of problem solving? Or is it the competence of communication and critical thinking? And if any one of them is the core of mathematics teaching, why is it so? Is it because "mathematics has importance for all learners who wish to get on in our society" or it is because "mathematics has importance for all learners in our society, because it helps them to understand and critique many of the structures of our society" (Bishop, 2001, p.234)? This research inquiry explores the personal values delivered by an elementary teacher in her mathematics teaching, especially the ones related to the aspect of "mathematics is important".

Theoretical Framework

Ernest (1991) indicated that every mathematics curriculum implies certain values and ideologies. Ideology is "an overall, value-rich philosophy or world-view, a broad inter-locking system of ideas and beliefs." (Ernest, 1991, p.111). Therefore, one can say that a mathematics curriculum reform signifies a change in values and ideologies.

Mathematics curriculum is realized by mathematics instruction and mathematics teaching carries implicit and explicit values (Bishop, 1988; Bishop, FitzSimons, Seah, & Clarkson, 2001; Chin, & Lin, 2001; Leu, & Wu, 2002; Swadener, & Soedjadi, 1988). Bishop (2001) proposed a diagram to illustrate how teacher's value structure affects mathematics instruction (i.e. decision implementation) (p.241).



According to the diagram, Bishop stated "the teacher's value structure monitors and mediates the on-going teaching situation, constructing options and choices together with criteria for evaluating them. The teacher thus is able to implement the decisions in a consistent manner." (Bishop, 2001, p.241) Consequently, mathematics teaching is not value-free.

In this study, the valuing theory (Raths, Harmin, & Simon, 1987) served as the foundation for exploring teachers' value-driven mathematical teaching. Values are any beliefs, attitudes, activities or feelings that satisfy the following three criteria: choosing, prizing and acting (Raths, et al., 1987). The criterion of choosing includes choosing freely, choosing from alternatives and choosing after thoughtful consideration of the consequences of each alternative. The criterion of prizing includes prizing, cherishing and affirming. The criterion of acting includes acting upon choices and repeating.

Methodology

In this case study, data were collected by various methods over a variety of schedules and topics including once-a-week, whole-unit and un-scheduled observations, and interviews to prevent the unjustified influence of any single method, mathematics topic or instructional event and to allow the triangulation of data and claims across multiple sources.

The research team met monthly with an external panel of three researchers with expertise in mathematics pedagogical values of Taiwanese secondary school teachers. Research data and interpretations were shared with the panel and discussed to improve data analyze, interpretations and research procedures.

The research subject of this case study is Ms. Lin, who has had nine years of teaching experiences in elementary school. She is currently teaching a class of twenty-seven fifth graders. Besides teaching mathematics, Ms. Lin was also responsible for Mandarin, Ethics, Health and some other subjects for her class. She was also responsible for monitoring and correcting her students' behaviors and conducts. At the time Ms. Lin was involved in the research, students from first to third grade in Taiwan were learning the NEMC while students from fourth to sixth grade were learning the OEMC.

This research is to investigate the values upheld by Ms. Lin about her mathematical teaching. According to Bishop's diagram on teacher's value structure (Bishop, 2001), researchers can retrace Ms. Lin's value structure through her mathematics instruction (i.e. decision implementation). Based on Raths et al's theory, researchers used classroom observations to notice repeated behavioral patterns during her mathematics lessons. The purpose of the interviews is to recognize the reasons why Ms. Lin developed these behavioral patterns and to formulate some value indicators, as well as to examine if the value indicators met the criteria of "choosing" and "prizing".

The time length for this research is one year. Ten lessons were observed and eighteen interviews were conducted. The topics covered in the ten lessons area of a circle, direct proportion, inverse proportion, scale on the map and the concept of probability.

Research Results

According to research data, it suggested that Ms. Lin's mathematics pedagogical value is that a good learning in mathematics makes easier for her students to advance for higher education (ex. college/university) and it's easier for a person with higher education to achieve a respectable social status and a successful life. This value of "one is to learn mathematics well as efforts to be successful" shares much similarity with one of the Confucian dogma "Those who excel in their study should become official."

How did Ms. Lin accomplish the mission of making her students learn mathematics well? The teaching behaviors and rationales of Ms. Lin' mathematics teaching can be analyzed from three perspectives: before class (preview), during the class (classroom instruction) and after class (review and assessment). The reasons provided by Ms. Lin were put in quotation mark.

From the perspective of previewing before class: Ms. Lin requested her students to preview and the purpose of preview is so that "students can have a better understanding when I explain the solving strategy provided in the textbook." Another purpose of preview is "…to invite those students who preview to take the role of a teacher and let them explain the solution in front of the class." Ms. Lin's criteria for choosing the presenters were "those who are confident in expressing his/her thoughts and who can express ideas/concepts clearly." Ms. Lin would not pick those students who she believed were incapable of delivering a clear explanation because she was afraid that "they would delay the schedule and/or ruin the atmosphere in the class". From her above statement, it is evident that Ms. Lin didn't value the cultivation of students' ability of communicating and reasoning as one of the goals in learning mathematics because she regarded this kind of practice as a possible destructor to the progress of a lesson. One may wonder if the progress of a lesson is such a major concern to Ms. Lin, then why didn't she teach the whole lesson by herself so that she could have a complete control? That's because Ms. Lin found out that students were more open and daring to ask questions during students' presentations, compared to her lecturing.

From the perspective of in-class instruction: Ms. Lin would stick to the teaching plan she prepared in advance and present the content of the textbook in a systematical order. Ms. Lin stated, "I'm really afraid not following the teaching plan I prepared in advance. I'm afraid that I would forget to say something important or lose control of the teaching procedures and/or the flow of instruction. I don't want this kind of situation to happen because it is as if I take away some of my students' rights to learn". What is the reason behind Ms. Lin's apprehension about ruining her students' right? That's because the questions on the mathematics exams are similar to the questions on the textbook and Ms. Lin had the concern that her students would be at a disadvantage if she had not clearly and effectively taught the solutions provided in the textbook. And why is it that questions on the mathematics exams are similar to the questions on the textbook? That's because there is only one unified version of midterm/final for all the students on the same grade (even though each class has a different mathematics teacher, usually their homeroom teacher) and there is a possibility that students' performance would suffer if the test-maker designed questions unlike the ones on the textbook. If the situation is to happen, then parents may accuse teachers for not teaching well enough and teachers would in turn blame the test-maker.

Ms. Lin strictly demanded her students to be attentive during the math class. She said, "I cannot stand even one sentence from my students during the class time. Students absolutely cannot talk. If one student talks for the first time, I would call up his/her name and give him/her a serious warning. If he/she does it the second time, then sometimes I would reprove. Or if the weather is not bad, I would ask him/her to stand on the hall as a punishment. The reason I'm doing this is that I believe students cannot understand because they don't listen carefully. If they concentrate to my lecture, then they can understand."

From the perspective of reviewing: After a lesson is complete, Ms. Lin would ask students to do four or five math problems during the morning hours and she would also photocopy the questions on the exams from previous years as in-class practice or homework. Ms. Lin distributed a math review sheet almost daily during the first semester and she reduced it to two sheets per week during the second semester. However, she hardly handed out any review sheet for other subjects. Ms. Lin explained, "Mathematics cannot be learned in a day or two. For all the subjects studied at the elementary school level, mathematics requires the most time and effort to learn and practice". She stated, "I made great emphasis to my students that they need practice more. With more practice, it comes faster calculation. Consequently one can learn more in a given time." She asserted, "Repeated practice is important for mathematics learning, besides understanding the concepts. In order to demonstrate one's understanding and calculation skills promptly during a test, he/she needs to practice over and over again". Ms. Lin further pointed out, "There are students in my class who can do every problem right if they have time but they can only finish two third of the test questions within the test time and they get only two third of the grades. There are always time limits for the entrance exams of senior high school and college in Taiwan. It's such a great pity if one cannot finish all the test questions he/she can solve correctly within the test time constraint. Therefore I encourage my students not only to understand the concepts but also to speed up their answering time. That's why I ask my students to practice repeatedly".

In addition to the regular review, Ms. Lin would spend about a week before the midterm/final for some extensive review. She stated, "There are several teachers who do not review for their students. They regard review as students' own responsibilities. I cannot dismiss the job of reviewing for my students. I take students' grades as my own responsibility. Since I am a teacher for fifth and sixth grade, I should prepare my students ready for junior high school so that they will perform well. I cannot say I don't care if my students get bad grades at the junior high school, even though they are no longer my students. The feeling is especially intense when I hear some graduates complain about their elementary teachers not teaching them well enough. If I can be more responsible and watchful about their academic performance, they may have a better and easier learning time at junior high school.". From this quote, it shows that Ms. Lin cared about not only her students' grades but also other people's judgment about her. There are some researches suggested that Chinese teachers care more about other's judgment about them than the western teachers (Biggs & Watkins, 2001).

From the perspective of test assessment: Ms. Lin would have two paper-and-pencil tests for each mathematic unit, which is about a week of class time. These tests were usually similar questions from textbook or workbook with different figures. Sometimes same questions or questions from previous test with a slight change of figures would be tested repeatedly.

Ms. Lin's criteria for grading were strict. If a student did a word problem with correct solving procedure and calculation but wrote down the answer with wrong unit, she would deduct all points. Ms. Lin stated, "My teacher told me that if there is any little mistake on a question of the entrance exam for senior high school and college, then a whole question would be marked wrong and one doesn't get any credit for it. I want my students to be familiar with this kind of grading policy used in the entrance exam." (However, this is only some impression Ms. Lin had while she was a student. It doesn't match with the reality or current situation.) In addition, Ms. Lin didn't allow students to use calculators on any math tests. She explained, "Calculators are not allowed for the entrance exam in Taiwan. I want to train my students to master calculation skills and well adapt the test system in Taiwan."

Sometimes Ms. Lin would use the class time of Ethics or Health to teach mathematics. Ms. Lin expected her students to learn mathematics well. She stated, "The notion that I convey to my students is that the fact that they live in Taiwan means that they have to compete intensively for the entrance of higher education. Mathematics and English are the two most decisive subjects. And what an elementary teacher can do is to consolidate students' mathematics". (During the research, elementary students didn't have to learn English.)

Discussion

The teaching behaviors of Ms. Lin's mathematics instruction included designing a well-organized teaching plan in advance, lecturing systematically about solution in the textbook, demanding students to be fully concentrated during the class time, providing many review sheets for students after a lesson is complete, giving tests regularly and reviewing constantly. The purpose of all these work is to strengthen students' capability in taking mathematics exams. These teaching behaviors are common in Taiwan and they are similar to the teaching behaviors founded for the Chinese society in Mainland China and Hong Kong, which are concentrated listening, vicarious learning, careful planning, timed questioning and associated activity (Biggs & Watkins, 2001). All these three places have an exam-dominated education system.

Ms. Lin's mathematical teaching behaviors can be explained from her values on mathematics instruction, that is, "One who can learn mathematics well have a better chance to attain a higher education and it's easier for those who get a higher education to achieve a respected social status and a successful life." The urge to learn mathematics well stems from the Chinese tradition that intellectuals should study hard for the imperial exam and once they pass the exam then they can become a government official and this in turn is a glory and honor to her family name and ancestors (Huang, 1994).

The reform on the mathematics curriculum at the elementary and junior high level during the recent ten years in Taiwan is basically an implantation of western culture. During the process of reforming, teachers, parents and the mass population experience a regression in students' calculation skills. Nevertheless, the values emphasize in NEMC and MLAGC, such as "the cultivation of the consciousness of communicating, discussing, reasoning and criticizing", cannot be assessed easily by paper-and-pencil tests. Consequently, teachers, parents and the mass population cannot realize the advantages of NEMC and MLAGC. Furthermore, most of the entrance exams in Taiwan are paper-and-pencil tests. Due to the disapproval from teachers, parents and the mass population about NEMC and MLAGC and the discord/clash between the tradition of exam-dominated education system and the newly-introduced mathematics pedagogical values, there has been a growing opposition and resistance about the reform. As a response to this public resentment, the Ministry of Education in Taiwan announced a revised version of MLAGC in 2003. This latest version is similar to OEMC, which emphasizes calculation skills and advocates lecturing (ME, 2003). With this kind of inconsistency, the elementary mathematics curriculum is chaotic at the present time.

Values are implied in mathematics itself, mathematics curriculum and mathematics instruction. The reformers should thoroughly investigate the prevalent mathematical pedagogical values upheld by mathematics teachers and carefully evaluate whether the innovated values can be accepted by most of the mathematics teachers and the general public and whether they are compatible with the established education system and society structure. If the existing values and the new values disagree or even conflict with each other, then some pertinent policies and effective measures should be planned and implemented to ensure a smooth transformation. Some possible means may be inculcating the innovation ideas to the mathematics teacher education programs and/or reforming the entrance exams accordingly. It is only when teachers (parents and mass population as well) accept the new values willingly and actively can a curriculum reform become successful.

Acknowledgements: Thanks are given to Ms. Lin, who was kind and open in sharing her ideas with us. We thank our assistant Ms. I.H. Hong, who did a great job on the audio-video tape transcriptions. We are grateful to NSC for funding the project (NSC 89-2511-S-152-001 and NSC 89-2511-S-152-019). The opinions expressed do not necessarily reflect the views of NSC.

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VISUAL REASONING IN COMPUTATIONAL ENVIORNMENT: A CASE OF GRAPH SKETCHING

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Abstract
This paper reports the case of a form six (grade 12) Hong Kong student's exploration of graph sketching in a computational environment. In particular, the student summarized his discovery in the form of two empirical laws. The student was interviewed and the interviewed data were used to map out a possible path of his visual reasoning. Critical features that enable visual reasoning in graphing software are discussed.

Introduction

Visualization is gaining important status in mathematics and mathematics education (Arcavi, 2003; Hershkowitz, Arcavi & Bruckheimer, 2001). With the rapid development of ICT, attempt to use virtual visualization as a mean for learners to construct mathematical knowledge and meanings has been an active research agenda (see for examples, Leung & Lopez-Real, 2002; Noss, Healy & Hoyles, 1997). In Chan (2001), a study was conducted to investigate the impact of graphing software on the learning of curve sketching in a form six (grade 12) class in Hong Kong. Graphmatica and Winplot were chosen as the graphing environments that students could use during the exploration tasks. In particular, one of the tasks was to ask students to investigate (using the computer only) the shapes of the family of graphs that has the symbolic form

$$y = (x+a)^m (x+b)^n (x+c)^p$$

where m, n, p are positive integers and a, b, c are real numbers. A student, Kelvin, came up with two empirical laws that could facilitate the sketching of these graphs without resorting to rigorous mathematical analysis. He claimed that this theorizing was a consequence of what he saw on the computer screen. Furthermore, he was able to apply his laws to sketch the graphs of the form

$$y = (x+a)^{\frac{m}{n}} (x+b)^{\frac{p}{q}} (x+c)^{\frac{r}{s}}.$$

Kelvin was interviewed in depth twice after he presented his findings to the class. In this paper, we will try to interpret on Kelvin's thinking patterns in his exploration and to sieve out possible elements that might contribute to visual reasoning in a computational graphing environment.

The Exploration Task

First, we outline roughly the tasks in the study that we are interested in. It consists of a sequence of explorations using Graphmatica or Winplot.

Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 2004 Vol 3 pp 233–240

A1. Investigate the family of graphs of the form $y = x^n$ where n is a positive integer.

A2. Investigate the family of graphs of the form $y = (x+a)^n$ where n is a positive integer and a is any real number.

A3. Investigate the family of graphs of the form $y = (x+a)^m(x+b)^n(x+c)^p$ where m, n, p are positive integers and a, b, c are real numbers.

B1. Investigate the family of graphs of the form $y = x^{\frac{m}{n}}$ where m and n are positive integers.

B2. Investigate the family of graphs of the form $y = (x+a)^{\frac{m}{n}}(x+b)^{\frac{p}{q}}(x+c)^{\frac{r}{s}}$ where m, n, p, q, r, s are positive integers and a, b, c are real numbers.

There were subtasks for each item and students were asked to work in pairs.

Kelvin's Path of Visual Reasoning

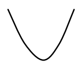
Tasks A1 and A2

Kelvin had learnt how to sketch the graphs $y = x^2$ and $y = x^3$ before this exploration, however he didn't know what $y = x^n$ looked like in general. The computer helped him to discover the general shape of $y = x^n$.

Excerpt from interview 2: (I: the interviewer; K: Kelvin)

I: What did you learn in the tasks concerning $y = x^n$?

K: When the index is odd, the shape looks like: 

When the index is even, the shape looks like: 

I: Do you mean that you don't know this beforehand and you learnt it from the computer during the exploration tasks?

K: Yes, but I knew the basic graph $y = x^2$, $y = x^3$ in form four but I do not know the graphs of $y = x^4$, $y = x^5$ before.

At this stage, it appears that when Kelvin saw the shapes of $y = x^n$ on the computer screen for different values of n, he was able to make a conjecture on the general shape of $y = x^n$. His prior knowledge on $y = x^2$ and $y = x^3$ played a pivoting role in this process of generalization since these were the two graphs that Kelvin had an analytical understanding on, and this knowledge on the polarity of the index of x seemed to *visually agree* with the general case. It's interesting to note that Kelvin ascribed the word "basic" to the graphs of the two specific cases. It seemed that the visual experience of $y = x^n$ was integrating with what Kelvin had already known. A process of meaning exchange between symbolic representation and computer generated visual image was taken place: the polarities of the index of x became shapes.

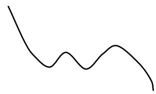
Task A3

When Kelvin was asked to present his findings for exploration Task A3, he proposed two laws, law of combination and law of continuity, which he claimed would facilitate the sketching of the graph of $y=(x+a)^m(x+b)^n(x+c)^p$ in general. A summary of these two laws is as follow:

Law of Combination for $y=(x+a)^m(x+b)^n(x+c)^p$

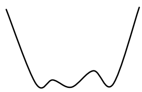
(A) (i) When the sum of the indices of the factors is odd, then the overall shape of the curve would look like:

Positive leading coefficient



Negative leading coefficient

(ii) When the sum of indices of factors is even, then the overall shape of the curve would look like an U (∪) or an inverted U (∩) with some 'mountains' and 'valleys':



Positive leading coefficient

Negative leading coefficient

(B) The factors of the function are considered independently. The graphs of the factors are drawn individually first, then they are combined by reflecting (when necessary) both ends of the graphs about x-axis, with reference to the signs of y when x◊$\pm\infty$.

Law of Continuity

During the combination of separate curves of the factors, continuity of curve should hold.

During the interview, Kelvin indicated that part (A) of the law of combination was inspired by his discovery in Tasks A1 and A2.

Excerpt from interview 2: (I: the interviewer; K: Kelvin)

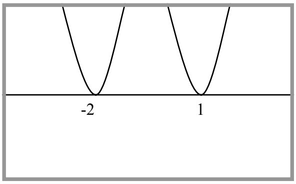
I: Kelvin, how do you think out the Law A(1) and A(2)?
K: This originates from a lesson earlier but I forget which lesson. Is it the first lesson, I am not sure?
I: What is the focus of that lesson?
K: The lesson we studied the $y = x^2$, $y = x^3$ and mm…
I: Is it the lesson we studied $y = x^n$?
K: Yes.

For Part (B) of the law of combination, Kelvin credited its formulation as a consequence of his interaction with the computer and his interpretation of the computer's visual feedback. Here's how he described the process of the discovery.

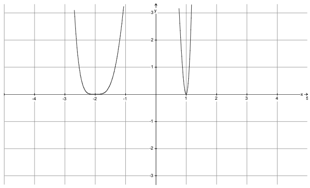
Excerpt from interview 2: (I: the interviewer; K: Kelvin)

I: There is another question I like to ask. How did you think of the method of combining the graphs of all separate factors to get the overall graph?

K: This is owing to the display of curve (*on the computer screen*) during the lesson. For instance, for the graph of $y = (x-1)^2(x+2)^4$, the software shows (*he made the following sketch*):

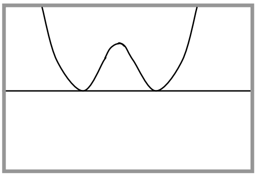


Kelvin's sketch of what he saw in computer



Actual image from the software

K: But, after applying function of 'zoom out', the separated curves are 'joined together' like the graph below.



Kelvin's sketch of what he saw on the computer of the graph after adjustment of the y-scale

K: So, I thought that I have to draw the separate curves of the factors first and then joined them together, and reflections of separate curves about x-axis may be needed.

The immediacy of visually displaying the scaling effect (via the zoom function) that the graphing software could provide for Kelvin was the key to open the door to this deep speculation. Kelvin was experiencing visually a process of discovering global features (the whole) via local properties (the parts), in particular, of how (prompted by the visual feedbacks) to paste together local characteristics to reveal a "generalized whole". More interesting, the *inability* of the graphing software to show the whole continuous graph at the first instance when Kelvin put in the equation was probably the most significant moment in the whole discovery process. Kelvin *saw* the graphs (separated from each other) that he was acquainted with in Task A. This laid the foundation for his two laws. Then the 'zoom out' function of the software that "joined together" the "separated curves", as described by Kelvin, gave Kelvin a visual fact (the continuity of the graph) that he needed to accept and to cope with. This opened up a cognitive space in which he had to simultaneously fuse together all the visual information and his prior knowledge to formulate an explanation, even laws. This was how Kelvin explained why he needed the law of continuity.

Excerpt from interview 2: (I: the interviewer; K: Kelvin)

K: In form six, we were taught the concept of continuity and differentiability. These curves should be continuous.

I: Yes, but what stimulates you, does the computer help you in any way?

K: After drawing the curves of separate curves. They have to be continuous. Therefore, the curves are smooth and continuous. And it is observed that some of the graphs of factors are reflected about x-axis to maintain the characteristic of continuity. Therefore, I try to isolate this important feature in additional to the law of combination.

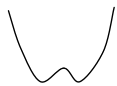
I: Do you know why these curves are continuous?

K: Naturally equations of these forms should be continuous … this is my guess.

The zoom function acted as a channel to a global view of variation, allowing Kelvin to experience contrast, separation, generalization and fusion. It is interesting to compare the zoom function in a graphing software with the drag function in a dynamic geometry environment, both of them serve as a kind of interactive amplification tool for the learner to see global behavior via variation (see Leung, 2003). Kelvin's explanation on how to paste two separated curves together was quite intriguing.

Excerpt from interview 1: (I: the interviewer; K: Kelvin)

K: (Described with drawing) In cases when the two intercepts of two curves are closed, the shape results in a small 'valley' (*he probably meant 'low hill'*) like this:

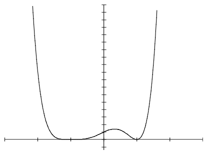


But when the two intercepts are separated far away, say +100 and –100, y value of the joining curve ($\cap$) between this two basic curves becomes very large.

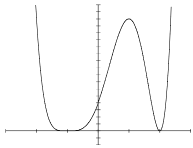
I: Yes. Do you know the reason?

K: I think that when the factors are considered independently, the curves themselves have their own shapes and sizes. But, when they are combined, they have to suit each other. For instance, in order for these two curves to be combined, their joining parts have to be diminished a bit. Suppose the two shapes are identical, it is unreasonable to squeeze the jointed part. They do not work like this. So, the part will be cut off.

The following graphs were probably typical of what Kelvin saw on the computer screen that led him to this line of reasoning.



$y=(x-1)^2(x+1)^4$



$y=(x-2)^2(x+1)^4$

It is worth noting that Kelvin used phrases like "the curves themselves have their own shapes and sizes" and "they have to suit each other" to describe the combination process of the two graphs. The visual variation resulting from symbolic variation somehow endowed "personalities" to the graphs and Kelvin reacted to them by "feeling".

Excerpt from interview 1: (I: the interviewer; K: Kelvin)

I: Your reasons are full of 'feeling'!
K: Yes, my explanation is full of art. The explanation of Fan is of logical.
I: What do you mean by 'art'?
K: … See the shapes of graphs so as to determine how to draw. But I seldom … . This differs from approach adopted by Fan who looks (considers) numbers and he did series of process of 'arithmetic' and then draw graph. I draw graphs directly and then observe (find features). So, he is logical while I am of art.

(Fan was another student who approached these tasks in an analytical way)

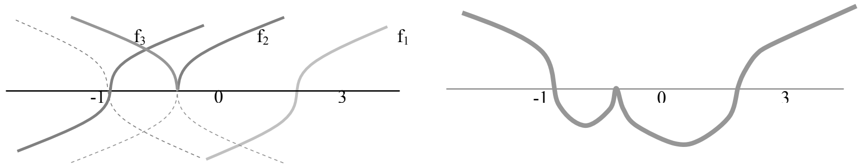
Very seldom do we explicitly relate doing mathematics with doing art. However, in this graph sketching exploration via a computational environment, Kelvin 'explained' Mathematics like an artist interpreting a piece of visual artwork. "See the shapes of graphs so as to determine how to draw." This is in contrast with the usual way of graph sketching, that is, determine how to draw (via rigorous analysis), then see the shapes. The graphing software situated Kelvin in a medium in which he could "see" mathematics in ways that enabled him to "feel" mathematics. Could this element of "feeling" via "seeing" (whatever is its meaning at this point) be a relevant factor that can contribute to visual reasoning?

Tasks B1 and B2

After Task A and the formulation of the two laws, Kelvin went ahead to attempt Task B. He had no difficulty in Task B1 to discover the basic shapes of $y=x^{\frac{m}{n}}$. To sketch the graph of $y=(x+a)^{\frac{m}{n}}(x+b)^{\frac{p}{q}}(x+c)^{\frac{r}{s}}$, Kelvin used the two laws he developed and illustrated the sketching procedure with a specific example.

Excerpt from interview 1: (I: the interviewer; K: Kelvin)

K: To answer this logically, let use an example to explain. (Kelvin drew the graphs on paper.)



Discrete curves drawn by Kelvin A continuous curve drawn by Kelvin

$$y = (x+1)^{\frac{1}{3}} x^{\frac{2}{3}} (x-3)^{\frac{1}{3}}$$

K: The latest theory is that for $x > 3$, the graph follows the curve with x-intercept 3 (f_1). In the middle, graph follows the middle one (f_2) while on the left graph follows the left one (f_3). In order to make the curve continuous, the basic graphs have to be reflected about x-axis.

I: Why?

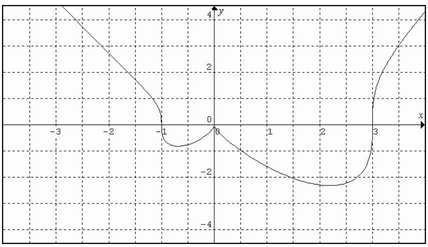
K: This is my observation. And I haven't work out any proof.

I: You haven't got any reason but you think that this is correct. Are you sure that this method is correct? Are you sure?

K: Observation only …

I: Okay.

The actual graph of $y = (x+1)^{\frac{1}{3}} x^{\frac{2}{3}} (x-3)^{\frac{1}{3}}$ looks like:



We can see the striking resemblance between Kelvin's sketch and the actual graph. Kelvin was able to apply his laws in a different situation. This may imply that these two laws were rich in mathematical content even though Kelvin hadn't "work out any proof" and his reasoning relied very much on intuition and observation. In this respect, as far as exploring graph sketching using the computer only, Kelvin's discovery has to be an exemplary one.

Final Remark

We have tried to trace a possible path on which Kelvin developed his two empirical laws for graph sketching using graphing software environment. The zooming (re-scaling) function of this environment seems to play a pivotal role in the visual reasoning process. Local scales give partial pictures that can evoke prior knowledge while global scales reveal the whole that might create cognitive conflicts which need to be resolved, hence the drive to make conjecture and to theorize. The ease of transition between local images and the global images of a graph via the zoom function turns the awareness of the learner to focus on the critical features that determine the continuity (or discontinuity) between local properties and global properties. These variable visual images can further stimulate a learner's intuitive feeling about mathematical behaviours that might contribute significantly in the visual reasoning process. Kelvin's two laws were heuristic and they are operationally quite effective. Mathematical analysis needs to be performed to verify and to fine-tune them. Nevertheless, these laws were indigenous to the graphing computational environment and they were obtained by systematic visual observation via scaling. There are at least two critical features in graphing software that are conducive to visual reasoning. The first one is its ability to instantly let the learner sees the graph of practically any equation. In this sense, there is no need to sketch the graph, rather, the question now becomes why certain graph looks certain way. Secondly, the amplification of the ability to vary visual images in different modalities (in our case, re-scaling) allows a learner to see invariant properties (e.g., Kelvin's two laws) of the "generic graph". These two features are the catalysts in a visual reasoning process. We hope this paper serves as a convincing example for them.

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ELEMENTARY SCHOOL STUDENTS' USE OF MATHEMATICALLY-BASED AND PRACTICALLY-BASED EXPLANATIONS: THE CASE OF MULTIPLICATION

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This paper focuses on elementary school students' use of mathematically-based (MB) and practically-based (PB) explanations. The mathematical context used in this study is multiplication. Two issues are discussed. The first issue is a comparison between MB and PB explanations used by students before they are formally introduced to multiplication in school as opposed to the explanations they use afterward. The second issue is a comparison of the types of explanations used for multiplication without zero as opposed to explanations used for multiplication with zero. Results show that more students use MB explanations than PB explanations. However, when multiplying with zero, many students use another type of justification (ie. rule-based explanation).

It is a long held belief that when elementary school children seek to describe their mathematical thinking or explore mathematical concepts they will use tangible items to manipulate or relate these concepts to real life contexts (e.g., Cramer & Henry, 2002; Fischbein, 1987; National Council of Teachers of Mathematics [NCTM], 1989; NCTM, 2000). This goes along with Piagetian theory which places students of this age at the concrete operational stage. Yet, according to the Standards for School Mathematics (NCTM, 2000), by the "middle and high grades, explanations should become more mathematically rigorous" (p. 61). Fischbein (1987) agreed and took this one step further, "One has to start, as early as possible, preparing the child for understanding the formal meaning and the formal content of the concepts taught" (p. 208). Is it possible to introduce elementary school students to formal mathematics if they are so reliant on concrete examples? Perhaps elementary school students are too young for rigorous explanations but not too young for explanations that are less formal but nevertheless rely solely on mathematical notions. This study focuses on the types of explanations that elementary school students use. By focusing on the types of explanations used we reexamine the premise that elementary school students need explanations that rely on tangible items or real life stories and investigate the possibility of introducing explanations that rely solely on mathematical notions in these grades.

Explanations have been classified in many different ways throughout the years. This study investigates mathematically-based (MB) explanations and practically-based (PB) explanations. MB explanations employ only mathematical notions. PB explanations use daily contexts and/or manipulatives to "give meaning" to mathematical expressions (Koren, in press). This classification distinguishes between explanations that are based solely on mathematical notions but are not necessarily rigorous, and complete, formal explanations. Formal explanations are usually referred to at the high school and undergraduate level. The term PB explanation was coined to include any explanation that does not rely solely on mathematical notions.

Much research has been done relating to the use of PB explanations in the elementary school mathematics classroom. Many of these studies (e.g., Koirala, 1999; Nyabanyaba, 1999; Szendrei, 1996; Wu, 1999) found that each type of PB explanation has its own set of pitfalls which need to be avoided or remedied by the teacher. Other studies have investigated the use of mathematical explanations in the elementary school that do not rely on manipulatives or real life stories (eg., Ball & Bass, 2000; Lampert, 1990). However, these studies took place in inquiry-based classrooms where mathematical discourse was encouraged. The current study investigates how students who study in more traditional classrooms explain multiplication without zero and with zero.

METHODOLOGY

Subjects

Subjects in this study were divided into younger and older students. The first group consisted of twenty second graders from three different schools who had not yet been exposed to multiplication in class. The second group consisted of ninety-one third, fifth, and sixth grade students from four different schools who already had experience with multiplication in school.

Instruments

Because young children express themselves better orally it was decided to interview the second grade students as opposed to using questionnaires. During the interview, students were asked to solve and give their own explanations for multiplication problems without and with zero. Older students were asked to fill out questionnaires. Both instruments included the following multiplication problems:

$3 \times 2 =$

$2 \times 3 =$

$3 \times 0 =$

$0 \times 3 =$

The first two problems sought to establish how multiplication without zero was solved and explained and if the subject used the commutative property of multiplication as an explanation. The second two problems allowed us to investigate how subjects solved and explained multiplication of a non-zero number by zero. Specifically, these two questions investigated if the types of explanations used by the subject for multiplication by non-zero numbers differed from the explanations used for multiplication with zero. Furthermore, these two questions allowed us to investigate if the subject differentiated between 3x0 and 0x3. The problem 3x0 fits well into the definition of multiplication as repeated addition when the multiplier is a positive integer and indicates the number of times 0 is to be added to itself. However, when the multiplier is non-positive, as in the case of 0x3, difficulties may arise. Therefore, it was of particular interest to investigate how subjects would explain this problem and how prevalent the use of the commutative property would be in this case.

Procedure

As stated before, second graders were each interviewed individually. Every interview was audio taped. Older students filled out questionnaires in their classroom with their class teacher and the researcher present. Students were told to solve each problem and provide explanations that they might use if they were asked to explain their solution to a friend in class who did not know the answer. Students worked individually without consulting the teacher, the researcher, or other students.

RESULTS

This section discusses the results of the interviews and questionnaires. First, we discuss the various explanations and how they were categorized into MB and PB explanations. Explanations for multiplication without zero and with zero were categorized in a similar matter and examples are given for each. We then present separately the distribution of the types of explanations used for multiplication without zero and with zero and discuss these results. Finally, we look at the differences between explanations used by second grade students who had not yet been introduced to multiplication in class and third, fifth, and sixth grade students who already had experience with multiplication in school.

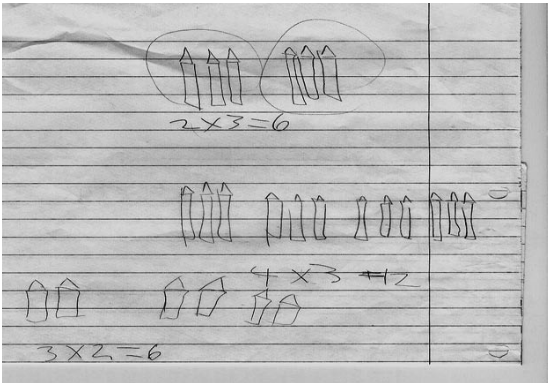
Categorization of explanations

MB explanations

As stated above, MB explanations employ only mathematical notions. In this category we included explanations that did not rely on the use of pictures, concrete objects, or stories. Many explanations were based on the definition of multiplication as repeated addition. As one fifth grader stated, "Multiplication is pretty simple. It's like quick addition". This type of explanation usually involved representing the multiplicand (the second number) the number of times that is indicated by the multiplier (the first number) and then successively adding these numbers. An example of this is the following explanation for 3x2 given by a second grader, "2 and another 2 is 4 and another 2 is 6". Similarly, when explaining 3x0, one fifth grader wrote, "0+0+0=0". Also in this category were explanations that relied on sequencing, such as one second grader's explanation, “I just counted by 2's, … 2, 4, 6”. There were also students who based their explanation on the word “times”, such as, “This one is like saying 3 two times, which would be six”. A similar explanation was given by a second grader for 0x3, "You don't have to write any times 3". Finally, explanations that were based on the commutative property of multiplication were considered MB explanations. Included in this category were explanations that explicitly used this property calling it by its proper name as well as explanations that stated that the order of the factors is irrelevant to the solution. As one sixth grader wrote, "2x3=6. It is the exact same as before, (3x2). In multiplication just like addition it does not matter which order the numbers are".

PB explanations

PB explanations were defined above as explanations that use daily contexts and/or manipulatives to "give meaning" to mathematical expressions. Most explanations in this category included pictures and stories that the students used to "give meaning" to the multiplication task. The following is an illustration of a second grader's PB explanation of why 2x3=6 and 3x2=6:



This student originally answered that 2x3 equals 12. When explaining her solution, she drew 2 sets with 3 pencils in each. The student realized her mistake and then wanted to figure out how many sets of 3 pencils she would need in order to have 12 pencils. This led her to draw 4 sets of 3 pencils and write "4x3=12". Finally, she drew 3 sets of 2 pencils to illustrate why 3x2=6.

One fifth grader drew 3 circles with 2 x's in each as an explanation for 2x3=6. When explaining 3x0 he drew 3 empty circles and wrote "3 groups of nothing". Another fifth grader wrote a story, "You have three guests and each of your guests wants two pancakes. How many do you have to make?" When explaining multiplication with zero, a different fifth grader wrote, "You have 3 ice creams but you don't eat any. How many did you eat? (0)". One sixth grader used a picture accompanied by a story. He drew 3 large circles with 2 dots in each and wrote underneath, “There are 3 cages and 2 animals are in each cage. Now count all the animals that are in the cages”.

Rule-based (RB) explanations

RB explanations have not been mentioned earlier because they were not the focus of this study. However, when explaining multiplication with zero many students sited the rule that every number times zero must equal zero. This explanation was not categorized as either MB or PB, and was therefore given its own category. For an explanation to be categorized as RB it had to be clear that the student was generalizing his explanation to all multiplication examples with zero as a factor. An example of this can be seen from the third grader who wrote, “Everything that you multiply by zero is equal to zero”.

Other

Included in this category are students who did not offer any explanation at all for their answers and students who offered explanations that were not clearly MB, PB, or RB.

Multiplication without zero

In this section we discuss the students’ explanations for 3x2=6 and 2x3=6. It should be noted that most students used the same type of explanation for both examples. In other words, most students who used a MB explanation for 3x2 used a MB explanation for 2x3 and likewise for PB explanations. That being said, three students (about 3%) gave different types of explanations for the two different examples. A few students gave both MB and PB explanations for the same task. Results are summed up in Table 1. Percentages are based on the number of students in each grade.

Table 1: Distribution of the types of explanations per grade for multiplication without zero (in %)

Task	3 x 2				2 x 3			
Grade	2 n = 20	3 n = 35	5 n = 32	6 n = 24	2 n = 20	3 n = 35	5 n = 32	6 n = 24
MB	85	86	56	63	85	89	72	71
PB	10	-	19	29	10	-	19	21
MB & PB	5	-	22	-	5	-	6	-
Other	-	14	3	8	-	11	3	8

Results show that in every grade, students are more likely to use MB explanations than PB explanations. One might have expected that younger children are more likely to base their explanations on their life experiences while older, more mathematically experienced students, would choose MB explanations. However, this was not the case in this study.

Multiplication with zero

Two new issues arose in students' explanations for multiplication with zero that were not present in the examples without zero. First, not all second graders knew the correct results of multiplication with zero. Second, many older students offered rule-based explanations without explaining the rule. In this section we will first discuss the solutions given by younger and older students and then present the types of explanations used for multiplication with zero.

Although all second graders interviewed knew multiplication without zero, 15% incorrectly solved 3x0 and 40% incorrectly solved 0x3 (see Table 2). It should be noted that many children changed their minds several times and only their final answers are considered. It is very interesting to note that all (except for one) of the younger students, who answered incorrectly, claimed that 3 times 0 (or 0 times 3) equals 3 and not all students who answered one question correctly answered the second correctly. These results show that although students may know multiplication without zero, it does not necessarily follow that they will know multiplication with zero. Possibly, this is because children's first experiences are with the set of natural numbers. Students, especially younger ones, often relate zero to "nothing" or "emptiness". At times, this can lead to an incorrect solution.

Table 2: Distribution of the types of explanations per grade for multiplication with zero (in %)*

Task	3 x 0				0 x 3			
Grade	2 n = 20	3 n = 35	5 n = 32	6 n = 24	2 n = 20	3 n = 35	5 n = 32	6 n = 24
MB	80 (70)	40 (33)	31 (31)	21 (21)	75 (35)	57 (46)	37 (37)	21 (21)
PB	5 (5)	-	16 (16)	16 (16)	10 (10)	-	13 (13)	17 (17)
RB	-	29 (29)	34 (34)	46 (46)	-	23 (23)	31 (31)	37 (37)
MB &PB	5 (-)	-	-	-	-	-	-	-
Other	10 (10)	31 (21)	19 (16)	17 (17)	15 (15)	20 (14)	19 (13)	25 (25)

* Percentages of correct solutions are given in parentheses.

One of the second grade students who claimed that 3x0=3 explained, "Because 3 times 0, you don't add anything so it stays the same number". Another second grader was also confused by the zero:

Interviewer: What about 3 times 0?

Student: It's 3. (This is an automatic response.)

Interviewer: Tell me, why do you think it's 3?

Student: Cause it's 0, so it's nothing.

Previously, this student had drawn a picture using sets of tally marks to illustrate why 3x2=6 and 2x3=6. In light of this drawing, the interviewer asked the student if he could draw a picture for 3x0. He drew 3 tally marks. Then, to illustrate 0x3, he drew a big empty circle and said, "It's just nothing".

Other students answered correctly that 3x0=0 but were still confused as to whether zero should be considered a number or not. This is illustrated by the following exchange with a second grader who had clearly used repeated addition when multiplying 3 by 2 but found multiplying by 0 quite different:

Interviewer: And what is 3 times 0?

Student: 0.

Interviewer: Why?

Student: Because…it doesn't have a number. If you had one, then it could be different. Because you can't do 3 *times* 0. It's still 0.

Interviewer: Why can you do 3x2 but you can't do 3x0?

Student: Because 0 is a number but it's… it's nothing. It's nothing.

Among older students, only 6 third graders (17% of the third graders) answered incorrectly that 3x0=3 and 0x3=3. One fifth grader did not answer at all. Overall, 92% of the older students knew that multiplication with zero always results in zero. This is a great increase over the second graders and shows that almost all students who learn multiplication in class knew that multiplying with zero results in zero.

From Table 2 we see that there are similarities and differences in the results for 3x0 and 0x3. MB explanations are used more often than PB explanations for both tasks. However, RB explanations were used less for 0x3 than for 3x0 as many students used the commutative property, a MB explanation, for the second task. Although one might think that older children, aware of the commutative property, would always have a ready explanation for 0x3, this was not always the case. In fact, a comparison of the tasks among sixth graders shows an increase in the use of "other" explanations for 0x3 than for 3x0. Perhaps this is due to the multiplier being a non-positive integer.

Second graders did not use RB explanations at all. These explanations were only used by students who had been introduced to multiplication in school. However, among the older group of students, it should be noted that for both tasks, the use of RB explanations rises from grade to grade. This does not imply that students are unaware of what lies behind the rule they cited. However, the question remains as to what might have caused this dramatic increase. Is this the only explanation presented in class? If not, then why do so many students recite this rule and not a different explanation?

When comparing the results of multiplication without zero to that of multiplication with zero, we see a decrease in the use of both MB and PB explanations for multiplication with zero, most likely due to the use of RB explanations. Significantly, there are more "other" explanations for multiplication tasks with zero than for without zero. This may be the result of students' difficulties, regardless of age, incorporating zero into the number system.

Discussion

Although the focus of this paper was on MB and PB explanations, the use of RB explanations cannot be ignored. Before students have formal learning they are not exposed to any rules. They try to give meaning to the mathematics, either by connecting it to life experiences or by basing it on already known mathematical concepts. After being introduced to multiplication in class, students give up using

MB explanations in favor of more RB explanations. Is this a trend that we want to encourage? One of our goals as mathematics educators is to help our students move from PB explanations to MB explanations. In the beginning of this paper we asked if it is possible to introduce more formal mathematics to young children. Knowing that the move to formal mathematics may be difficult, we should examine the possibility of introducing more MB explanations to elementary school students.

This study shows that even young students are capable of using explanations that rely solely on mathematical notions. Is this true only for multiplication tasks? We need to examine students' use of MB explanations in other mathematical contexts as well. We also need to investigate how these findings may be used in practice by teachers in the classroom and, in line with Fischbein's (1987) recommendation, investigate how MB explanations may be used to prepare students for the formal content of mathematics.

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MATHEMATICAL DISCOVERY[1]: HADAMARD RESURECTED

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In 1943 Jacques Hadamard gave a series of lectures on mathematical invention at the École Libre des Hautes Etudes in New York City. These talks were subsequently published as The Psychology of Mathematical Invention in the Mathematical Field (Hadamard,1945). In this article I present a study that mirrors the work of Hadamard. Results both confirm and extend the work of Hadamard on the inventive process. In addition, the results also speak to the larger context of 'doing' and learning mathematics.

What is the genesis of mathematical creation? What mechanisms govern the act of mathematical discovery? This "is a problem which should intensely interest the psychologist. It is the activity in which the human mind seems to take the least from the outside world, in which it acts or seems to act only of itself and on itself" (Poincaré, 1952, p. 46). It should also intensely interest the mathematics educator, for it is through mathematical discovery that we see the essence of what it means to 'do' and learn mathematics. In this article I explore the topic of mathematical discovery along two fronts, the first of which is a brief synopsis of the history of work in this area. This is then followed by a glimpse at a study designed to elicit from prominent mathematicians ideas and thoughts on their own encounters with the phenomenon of mathematical discovery.

HISTORICAL BACKGROUND

In 1908 Henri Poincaré (1854–1912) gave a presentation to the French Psychological Society in Paris entitled 'Mathematical Creation'. This presentation, as well as the essay it spawned, stands to this day as one of the most insightful, and thorough treatments of the topic of mathematical invention. In particular, the anecdote of Poincaré's own discovery of Fuschian function transformations stands as the most famous contemporary account of mathematical creation.

> Just at this time, I left Caen, where I was living, to go on a geological excursion under the auspices of the School of Mines. The incident of the travel made me forget my mathematical work. Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step, the idea came to me, without anything in my former thoughts seeming to have paved the way for it, that the transformations I had used to define the Fuschian functions were identical with those of non-Euclidean geometry. I did not verify the idea; I should not have had the time, as, upon taking my seat in the omnibus, I went on with the conversation already

[1] It should be noted that, although arguments can be made for the distinction between *invention* and *discovery,* Hadamard (1945) made no distinction. As such, for the purposes of this article the two terms will be used similarily and interchangeably.

> commenced, but I felt a perfect certainty. On my return to Caen, for conscience' sake, I verified the results at my leisure. (Poincaré, 1952, p. 53)

Inspired by this presentation, Jacques Hadamard (1865-1963) began his own empirical investigation into mathematical invention. He based this investigation on a 30 question survey authored by psychologists Édouard Claparède and Théodore Flournoy which was published in the pages of L'Enseignement Mathematique in 1902 and 1904. Although impressed with their intensions, Hadamard was critical of Claparède and Flournoy's work. He felt that the two psychologists had failed to adequately treat the topic of mathematical invention on two fronts; the first was the lack of comprehensive treatment of certain topics and the second was the lack of prominence on the part of the respondents. In particular, Hadamard felt that as exhaustive as the survey appeared to be, it failed to ask some key questions – the most important of which was with regard to the reason for failures in the creation of mathematics. This seemingly innocuous oversight, however, led directly to what he termed "the most important criticism which can be formulated against such inquiries" (1945, p.10). This also lead to Hadamard's second, and perhaps more damning, criticism. He felt that only "first-rate men would dare to speak of" (p.10) such failures, and so, Hadamard retooled the survey and gave it to friends of his for consideration – mathematicians like Henri Poincaré and Albert Einstein, to name a few – whose prominence were beyond reproach. The results of this seminal work culminated in a series of lectures on mathematical invention at the École Libre des Hautes Etudes in New York City in 1943. These talks were subsequently published as *The Psychology of Mathematical Invention in the Mathematical Field* (Hadamard,1945).

Hadamard's treatment of the subject of invention at the crossroads of mathematics and psychology was an entertaining, and sometimes humorous, look at the eccentric nature of mathematicians and their ritualistic practices. His work is an extensive exploration and extended argument for the existence of unconscious mental processes. To summarize, Hadamard took the ideas that Poincaré had posed and, borrowing a conceptual framework for the characterization of the creative process in general, turned them into a stage theory. This theory still stands as the most viable and reasonable description of the process of mathematical invention. In what follows I present this theory, referenced not only to Hadamard and Poincaré, but also to some of the many researchers who's work has informed and verified different aspects of the theory.

MATHEMATICAL INVENTION

The phenomenon of mathematical invention, although marked by sudden illumination, consists of four separate stages stretched out over time, of which illumination is but one part. These stages are initiation, incubation, illumination, and verification (Hadamard, 1945). The first of these stages, the *initiation* phase, consists of deliberate and conscious work. This would constitute a person's voluntary, and seemingly fruitless, engagement with a problem and be characterized by an attempt to solve the problem by trolling through a repertoire of past experiences (Bruner, 1964; Schön, 1987). This is an important part of the inventive process because it creates the tension of unresolved effort that sets up the conditions necessary for the ensuing emotional release at the moment of illumination (Barnes, 2000; Davis & Hersh, 1980; Feynman, 1999; Hadamard, 1945; Poincaré, 1952; Rota, 1997).

Following the initiation stage the solver, unable to come to a solution stops working on the problem at a conscious level (Dewey, 1933) and begins to work on it at an unconscious level (Hadamard, 1945; Poincaré, 1952). This is referred to as the *incubation* stage of the inventive process and it is inextricably linked to the conscious and intentional effort that precedes it.

> There is another remark to be made about the conditions of this unconscious work: it is possible, and of a certainty it is only fruitful, if it is on the one hand preceded and on the other hand followed by a period of conscious work. These sudden inspirations never happen except after some days of voluntary effort which has appeared absolutely fruitless and whence nothing good seems to have come … (Poincaré, 1952, p. 56)

After the period of incubation a rapid coming to mind of a solution, referred to as *illumination*, may occurs. This is accompanied by a feeling of certainty (Poincaré, 1952) and positive emotions (Barnes, 2000; Burton 1999; Rota, 1997). With regards to the phenomenon of illumination, it is clear that this phase is the manifestation of a bridging that occurs between the unconscious mind and the conscious mind (Poincaré, 1952), a coming to (conscious) mind of an idea or solution. However, what brings the idea forward to consciousness is unclear. There are theories on aesthetic qualities of the idea (Sinclair, 2002; Poincaré, 1952), effective surprise/shock of recognition (Bruner, 1964), fluency of processing (Whittlesea and Williams, 2001), or breaking functional fixedness (Ashcraft, 1989) only one of which will expand on here.

Poincaré proposed that ideas that were stimulated during initiation remained stimulated during incubation. However, freed from the constraints of conscious thought and deliberate calculation, these ideas would begin to come together in rapid and random unions so that "their mutual impacts may produce new combinations" (Poincaré, 1952, p. 61). These new combinations, or ideas, would then be evaluated for viability using an aesthetic sieve (Sinclair, 2002), which allowed through to the conscious mind only the "right combinations" (Poincaré, 1952, p. 62). It is important to note, however, that good or aesthetic does not necessarily mean correct. As such, correctness is evaluated during the fourth and final stage – *verification*.

HADAMARD RESURECTED

As mentioned, my interest in this topic has not been limited to its historical roots. I have also engaged in a number of studies pertaining to the topic (c.f. Liljedahl, 2002). The latest of these studies can best be described as a resurrection of Jacques Hadamard's work. That is, a portion of his original questionnaire was used to elicit from contemporary mathematicians ideas and thoughts on their own encounters with the phenomenon of mathematical discovery[2]. In what follows I present a summary of this study.

Hadamard's original questionnaire contained 33 questions pertaining to everything from personal habits, to family history, to meteorological conditions during times of work (Hadamard, 1945). From this extensive and exhaustive list of questions the five that most directly related to the phenomena of mathematical discovery and creation were selected. They are:

1. Would you say that your principle discoveries have been the result of deliberate endeavour in a definite direction, or have they arisen, so to speak, spontaneously? Have you a specific anecdote of a moment of insight/inspiration/illumination that would demonstrate this? [Hadamard # 9]
2. How much of mathematical creation do you attribute to chance, insight, inspiration, or illumination? Have you come to rely on this in any way? [Hadamard# 7]
3. Could you comment on the differences in the manner in which you work when you are trying to assimilate the results of others (learning mathematics) as compared to when you are indulging in personal research (creating mathematics)? [Hadamard # 4]
4. Have your methods of learning and creating mathematics changed since you were a student? How so? [Hadamard # 16]
5. Among your greatest works have you ever attempted to discern the origin of the ideas that lead you to your discoveries? Could you comment on the creative processes that lead you to your discoveries? [Hadamard # 6]

These questions along with a covering letter were then sent to 150 prominent mathematicians (see below) in the form of an email.

Hadamard set excellence in the field of mathematics as a criterion for participation in his study. In keeping with Hadamard's standards, excellence in the field of mathematics was also chosen as the primary criterion for participation in this study. As such, recipients of the survey were selected based on their achievements in their field as recognized by their being honored with prestigious prizes or membership in noteworthy societies. In particular, the 150 recipients were chosen from the published lists of the Fields Medalists, the Nevanlinna Prize winners, as well as the membership list of the American Society of Arts & Sciences. The 25 recipients who responded to the survey, in whole or in part, have come to be referred to as the 'participants' in this study.

The responses were initially sorted according to the survey question they were most closely addressing. In addition, a second sorting of the data was done according to trends that emerged in the participants' responses, regardless of which question they

[2] I would like to acknowledge the contribution made by Peter Borwein with respect to the collection of data for this study.

were in response to. This was a much more intensive and involved sorting of the data in an iterative process of identifying themes, coding for themes, identifying more themes, recoding for the new themes, and so on. In the end, there were 12 themes that emerged from the data, each of which can be attributed to one of the four stages presented in the previous section. Some of these themes serve to confirm the work of Hadamard while others serve to extend it. Given the limited space available here, in what follows I provide a *very brief summary* of four of the themes that extend our understanding of various stages of the inventive process.

The De-Emphasis of Details

A particularly strong theme that emerged from this study was the role that detail does NOT play in the incubation phase. Many of the participants mentioned how difficult it is to learn mathematics by attending to the details, and how much easier it is if the details are de-emphasized.

Stephen: Understanding others is often a painful process until one suddenly goes beyond the details and sees whole what's going on.

Mark: There is not much difference. More precisely, I seldom study or learn mathematics in detail.

In some cases, this also manifested itself as a strategy for problem solving and research.

Carl: Get the basics of the problem firmly and thoroughly into the head. After that, an hour or two each day of thinking on it is all that's needed for progress. [..] For that reason, I started some 20 years ago to ask students (and colleagues) wanting to tell me some piece of mathematics to tell me directly, perhaps with some gestures, but certainly without the aid of a blackboard. While that can be challenging, it will, if successful, put the problem more firmly and cleanly into the head, hence increases the chances for understanding. I am also now more aware of the fact that explaining problem and progress to someone else is beneficial; I am guessing that it forces one to have the problem more clearly and cleanly in one's head.

In presenting his strategy for getting " *the basics of the problem firmly and thoroughly into*" his head, Carl has come up with a strategy that de-emphasizes the details by forcing the transmission of the problem through a medium wherein details are impossible. That is, talking de-emphasizes details and, as a result, will "*put the problem more firmly and cleanly into the head, hence increases the chances for understanding*."

The Role of Talking

Further to this previous theme, it is also clear from the mathematicians' responses that, while working in the initiation phase, they have a much higher regard for transmission of mathematical knowledge through talking than through reading. This is best summarized in the comments of Jerry and George.

Jerry: I assimilate the work of others best through personal contact and being able to question them directly. [..] In this question and answer mode, I often get good ideas too. In this sense, the two modes are almost indistinguishable.

George: I get most of my real mathematical input live, from (good) lectures or one-on-one discussions. I think most mathematicians do. I look at papers only after I have had some overall idea of a problem and then I do not look at details.

Considering these last two themes (de-emphasizing of details and the role of talking) it becomes clear that the painstakingly rigorous fashion in which mathematical knowledge is written, both in journals and in text-books, as well as the detailed fashion of over-engineered curriculums stand in stark contrast to the methods by which mathematicians claim they best come to learn new mathematics.

Deriving and Re-Creation

A third theme that resonates with the two previous themes, while again speaking to the initiation phase of the inventive process, is the importance of deriving mathematics for oneself. In part, this theme reflects the personal practices of assimilating the work of others through active re-creation of the mathematics rather than passive reading of it.

Debby: I need to work out examples, specific instances of the new ideas to feel that I have any real understanding. Anything one creates oneself is much more immediate and real and so harder to forget.

Mark: When there is a need to fully assimilate something, I must redo everything in my own way.

Dick: I've forgotten what Hadamard had to say on this, but for me there's no difference, - in order to 'understand', I have to (re)create. To be sure, it's much easier to follow someone else's footsteps, i.e., it is much easier to prove a result one knows to be true than one that one merely guesses to be true.

However, the theme also manifested itself as advice to young mathematicians about how to best approach doing mathematics in the expansive response from Peter.

Peter: If you have an idea, develop it on your own for, say, two months, and only then check whether the results are known. The reasons are: (1) If you try to check earlier, you won't recognize your idea in the disguise under which it appears in the literature. (2) If you read the literature too carefully beforehand, you will be diverted into the train of thought of the other author and stop exactly where he ran into an obstacle. This happened to a friend of mine, who started three Ph.D. theses in totally unrelated fields, before he finished one.

Again, this stands in stark contrast to the conventional methods by which mathematical knowledge is often conveyed.

The Contribution of Chance

Up until now, the themes that have been presented all pertain to the initiation phase of the inventive process. Indeed, seven of the 12 emergent themes deal with this stage. The theme presented here, however, concerns the illumination stage. From the responses provided by the participants it became clear that, for them, *chance* plays a very large role in illumination and insight. There are two types of chance, *intrinsic chance* and *extrinsic chance*. Intrinsic chance deals with the luck of coming up with an answer, of having the right combination of ideas join within your mind to produce a new understanding. This was discussed by Hadamard (1945) as well as by a host of others under the name of "*the chance hypothesis*" and is nicely demonstrated in Dan's comment.

Dan: And relevant ideas do pop up in your mind when you are taking a shower, and can pop up as well even when you are sleeping, (many of these ideas turn out not to work very well) or even when you are driving. Thus while you can turn the problem over in your mind in all ways you can think of, try to use all the methods you can recall or discover to attack it, there is really no standard approach that will solve it for you. At some stage, if you are lucky, the right combination occurs to you, and you are able to check it and use it to put an argument together.

Extrinsic chance, on the other hand, deals with the luck associated with a chance reading of an article, a chance encounter, or a some other chance encounter with a piece of mathematical knowledge, any of which contribute to the eventual resolution of the problem that one is working on. This is best demonstrated by the words of Mark and Carl.

Mark: Do I experience feelings of illumination? Rarely, except in connection with chance, whose offerings I treasure. In my wandering life between concrete fields and problems, chance is continually important in two ways. A chance reading or encounter has often brought an awareness of existing mathematical tools that were new to me and allowed me to return to old problems I was previously obliged to leave aside. In other cases, a chance encounter suggested that old tools could have new uses that helped them expand.

Carl: But chance is a major aspect: what papers one happens to have read, what discussions one happens to have struck up, what ideas one's students are struck by (never mind the very basic chance process of insemination that produced this particular mathematician).

Once again, the idea that mathematical discovery often relies on the fleeting and unpredictable occurrences of chance encounters is starkly contradictory to the image projected by mathematics as a field reliant on logic and deductive reasoning. Extrinsic chance, in particular, is an element that has been largely ignored in the literature.

CONCLUSIONS

Mathematical discovery and invention are aspects of 'doing' mathematics that have long been accepted as standing outside of the theories of "logical forms" (Dewey, 1938, p.103). That is, they rely on the extra-logical processes of insight and illumination as opposed to the logical process of deductive and inductive reasoning. This study confirms this understanding as well as adds to this cohort of extra-logical processes the role of chance. In addition, this study also comments on the initiation phase of the inventive process in showing that it is best facilitated through the de-emphasizes of details and transmission of knowledge through talking and (re)creation. As such, this study also contributes to our understanding of the larger contexts of 'doing' and learning mathematics.

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SUPPORTING TEACHERS ON DESIGNING PROBLEM-POSING TASKS AS A TOOL OF ASSESSMENT TO UNDERSTAND STUDENTS' MATHEMATICAL LEARNING

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The study was designed to support teachers on designing problem-posing tasks to understand students' mathematical learning. Seven classroom teachers and the researcher collaboratively set up a school-based team participating in an assessment project that assists teachers in implementing assessment integral to instruction. Four categories that the teachers generated problem-posing tasks included number sentences, pictorial representations or drawings, mathematics languages, and students' solutions collected in a class. Insight into teachers' understanding of students' learning was identified through students' responses to the problem-posing tasks. The tasks that students engaged in generating problems were a useful tool for assessing students' understanding and then informing teachers making further instructions.

Key words: assessment tasks, problem posing, assessment integral to instruction.

INTRODUCTION

The reformed curriculum suggested that every instructional activity is an assessment opportunity for teachers and a learning opportunity for students (NCTM, 2000). The movement emphasized classroom assessment in gathering information on which teachers can inform their further instruction (NCTM, 1995). Assessment integral to instruction contributes significantly to all students' mathematics learning. The new vision of assessment suggested that knowing how these assessment processes take place should become a focus of teacher education programs.

Problem-posing task referred to in the study was that the task teachers designed requires students to generate one or more word problems. The professional standards suggested that teachers could use task selection and analysis as foci for thinking about instruction and assessment. According to De Lange (1995), a task that is open for students' process and solution is a way of stimulating students' high quality thinking. Training teachers in designing and using assessment tasks has also been proposed as a means of improving the quality of assessments (Clarke, 1996). However, the design of open-ended tasks is a complex and challenging work for the teachers who are used to the traditional test. Thus, the tasks involving in the study were considered as an informal way of assessing what and how individual student learned from everyday lesson. Thus, the preparation of the tasks involving in this study was not prior to instruction; rather, teachers generated them from the activities in which students engaged in everyday lesson. The mathematics contents covered in the textbooks were a dimension of the assessment framework of the study.

The reformed curriculum calls for an increased emphasis on teachers' responsibility for the quality of the tasks in which students engaged. The high quality of tasks should help students clarifying thinking and developing deeper understanding through the process of formulating problem, communicating, and reasoning (MET, 2000). Thus, these cognitive processes were the other dimension of the assessment framework. The tasks teachers designed in the study were to assess students' problem posing, communicating, and reasoning. Due to the limitation of space, this paper is primarily concerned with the problem-posing tasks.

Problem-posing is recognized as an important component in the nature of mathematical thinking (Kilpatric, 1987). More recently, there is an increased emphasis on giving students opportunities with problem posing in mathematics classroom (English & Hoalford, 1995; Stoyanova, 1998). These research has shown that instructional activities as having students generate problems as a means of improving ability of problem solving and their attitude toward mathematics (Winograd, 1991). Nevertheless, such reform requires first a commitment to creating an environment in which problem posing is a natural process of mathematics learning. Second, it requires teachers figure out the strategies for helping students posing meaningful and enticing problems. Thus, there is a need to support teachers with a collaborative team whose students engage in problem-posing activities. This can only be achieved by establishing an assessment team who support mutually by providing them with dialogues on critical assessment issues related to instruction.

Problem-posing involves generating new problems and reformatting a given problems (Silver, 1994). Generating new problems is not on the solution but on creating a new problem. The quality of problems in which students generated depends on the given tasks (Leung & Silver, 1997). Research on problem posing has increased attention to the effect of problem posing on students' mathematical ability and the effect of task formats on problem posing (Leung & Silver, 1997). Such problem-posing tasks that situations were presented in a story form were created by researcher rather than by classroom teachers. Moreover, there is a little research on teachers' responsibility for the variety and the quality of the problem-posing tasks. The way in which teachers explored to create tasks for students generating problems from a contrived situation was investigated in the present study.

THE ASSESSMENT PRACTICES IN MATHEMATICS CLASSROOM PROJECT

The Assessment Practices in Mathematics Classroom (APMC) project funded by the National Science Council was designed to develop a teacher program in which supports teachers on practicing assessment integral to instruction. An aim of the project was to assist teachers to recognize how students develop their learning with understanding, and how this can be supported through the program. To reach the aim, encouraging teachers used mathematical journal as an informal way of gathering the information about students' thinking processes, strategies, and their developing mathematical understanding to assess individual entire learning process by writing about mathematics. Assessment tasks as the prompts of mathematical journal that was served to establish a better means of communication among students, parents, and teachers about mathematics leaning taking place in classrooms.

Supporting teachers on generating mathematical tasks was the kernel part of the APMC project, the concerns included that: 1) supports a method of assessment that allows students to demonstrate their strengths; 2) stimulates students to make connections between mathematical ideas; 3) promotes high quality of problem-posing, communicating, and justifying one's way of thinking; 4) generates creative tasks that do not separate mathematical processes from mathematical concepts; 5) generates the tasks for assessing what and how students learned from a lesson. To generate the high quality of the tasks from everyday lesson, the tasks covered in each journal including one or two problems were reasonable.

The rationale of the APMC project was a social constructivist's view of learning. Learning is viewed as the product of social interaction in a professional community (Vygotsky, 1978). Therefore, activities related to generating assessment tasks were structured to ensure that knowledge was actively developed by the teachers, not imposed by the researcher. The generation of assessment tasks as the part of practices of assessment integrated into instruction was initiated from teachers' everyday instruction and modified by professional dialogues. Thus, the teachers frequently observed, discussed, and reflected to the quality of assessment tasks altogether.

The study reported in this paper was focused on how a school-based professional development program supported teachers on designing problem-posing tasks as a tool of assessing students' mathematical understanding. Two research questions require to be answered: What supports did teachers need when they created assessment tasks initiated from instruction? What kinds of problem-posing tasks did teachers generate from everyday instruction for assessing students' understanding?

METHOD

To achieve the goal of the study, a school-based assessment team consisting of the researcher and seven teachers was set up to discuss the assessment issues which occurred in a classroom by comparing to others'. Because the same mathematical content lent itself to a focus and similar pedagogical issues addressed drew attention from each teacher, leading to in-depth discussions, the seven teachers were recruited from two successive grade levels. The second- and third-grade classrooms were the primary contexts for teachers learning to generate tasks. Participations in the regular weekly meetings were the other primary context for the teachers improving the quality of the tasks. The three second-grade teachers were P2, Q2, and R2; four third-grade teachers were A3, B3, C3, and D3. Their teaching experiences were ranged from 3 to 16 years. The role of the researcher was not to provide ready-made tasks for the teachers to use, but to create the opportunities for teachers sitting together to design creative assessment tasks for students.

The teachers had little knowledge of assessment integral to instruction, so that classroom observation was used as a means of increasing their awareness of generating tasks initiated from the lessons observed altogether. The teachers had routine weekly meetings lasting for three hours. The meetings gave teachers to share creative tasks mutually and to rethink if the tasks gathered information of students' in-depth understanding. At the very beginning of the study, I encouraged them to create at least an assessment task each week integral to their teaching. The assessment tasks were encouraged to be open-ended questions contrived by the teachers as part of homework that students completed after school. Moreover, they required bring students' responses to the tasks for others to analyze. Besides, each teacher required reporting in the meeting what they learned from the tasks they administrated and what information they gathered from students' responses to the tasks.

Data was gathered through classroom observations, problem-posing tasks, regular weekly meetings, and students' responses to the tasks. The routine weekly meetings were audio-recoded and the lessons were video-recoded. The audio- and video-tapes were transcribed to be faithful as possible to the teachers' exact words. Students' responses to the tasks were transcribed as possible to students' exact words to produce readable English.

RESULTS AND DISCUSSION

The tasks that students generated problems from a contrived situation were categorized according to the following four elements: 1) number sentences; 2) pictorial representations; 3) mathematical languages; 4) students' various solutions collected in everyday lesson. Each of these elements is addressed below and includes the problems that students generated and the supports of assessment incorporating into instruction that the assessment team provided.

Category I: Giving a number sentence to create word problems

Designing a creative task that students generate word problems is a new experience for the teachers. A typical problem-posing task the teachers designed was given by a number sentence. The typical task was occasionally covered in the current textbooks. This was the only category that the teachers designed in the very early of the study. The task given by a number sentence was bounded to the mathematics contents that the teachers taught.

It is found that the task dealing with problems generated from a number sentence helped teachers perceived the difficulties students encountered. The teacher A3 reported that either the multiplicand 1 or 0 is particular difficulty for her to explain to students. Hence, she conducted the following Task 1 to examine if students recognized the meaning of multiplicand 1. Task 1 and three students' responses to the task are displayed as follows.

Task1: If you were a teacher, how would you give your students a problem situation represented by 1x 5 = ( )? Write it down in words." (A3, 10/12/2000).

Wu: There are five third-grade classes in Din-Pu School. There are clocks in each class. How many are clocks there altogether?

Hwei: There are freezers in each house. How many are freezers in the five houses?

Sue: A cow produces a bucket of milk. How many buckets do 5 cows produce totally?

After analyzing students' responses, she realized 11 of 35 students still having the difficulty with understanding the multiplicand 1. Of the responses, students were not able to distinguish the difference between the multiplicand 1 and integer 1. "A group of 5" [1] referred to in textbooks was expressed by 5x1=( ). Wu and Hwei had the same misunderstanding with the meaning the multiplicand 1. Wu understood incorrectly "1" for 1x5=( ) as "Each class has clocks" instead of "Each class has a clock".

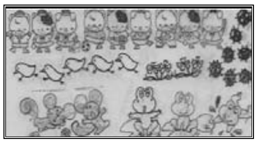
One of the teachers, B3 recommended A3 bring one of the two improper problems to classroom to ask students to repair it. Next day, A3 acted as though she needed help and then asked students, "Is it wrong? *[There are five third-grade classes in Din-Pu School. There are clocks in each class. How many are clocks there altogether?]*. Could you help me to repair it so that it can be solved?". As observed, the majority of students devoted to repairing the improper problem.

It shows that the task allowed A3 to gain insights into the way students constructed mathematical understanding. The improper problems that students generated were served as the indicator of their unclear understanding. These improper problems made profitable when asking students to repair them and informing teachers' decision-making. Thus, correcting misconceptions or repairing the improper problems that students responded to the tasks became a common activity for the teachers at the very beginning of each class.

Category II: Giving a picture or drawing to formulate word problems

With the exception of the task by a given number sentence, it is hard for the teachers to create a new task without supports. The researcher, as a learning partner of the teachers, shared the possible tasks referred to the literature of problem posing to the school-based assessment team. Interestingly, the teacher P2 created a new task by a given picture. P2 with more than six years of teaching in second grade perceived that the beginners of learning multiplication should understand clearly on the concept of multiplication. She conducted the Task 2 to assess if her students understood the meaning of multiplication.

Task 2: This is a picture about the princess and 7 dwarfs.
If you were a teacher, what word problems would like to formulate? (P2, 10/29/2000)



The problems Horng created as follows.

1. There are 2 mice. Each mouse has 2 legs. How many legs are totally?
2. There are 4 flowers and each has 5 leaves. How many leaves are there?
3. There are 7 bugs. Each bug has 6 legs. How many legs are the bugs totally?
4. There are 7 dwarfs. Each dwarf has 2 eyes. How many eyes are there totally?

The day after the lesson, the teacher P2 brought the bug problem Horng provided

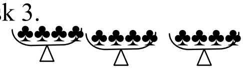
[1] The textbook of my country dealing with "a set of 5 apples" as "one five" "five times one" "5x1=( )" is not consistent with those of other countries. Students do not learn the multiplication until they are in the second grade.

into the classroom to ask students to solve it. During the class, Horng persistently kept an eye on seeing if his problem was solved. After grading, 8 out of 32 students still hardly made the distinction between 6x7=( ) and 7x6= ( ) (Observation, 10/30/2000). Later on, R2 reacted to the lesson in the weekly meeting followed immediately the observation and said:

> "Students did not look the problem carefully to identify which is the size of each unit or the number of unit; as a consequence, they misused the two expressions. This confusion occurred in my class as well (R2, Meeting, 10/31/2000)".

This confusion became as a focus of the professional dialogues. B3 shared her last year experience of teaching second-graders and suggested that it is necessary to explain "6x7=( )" meaning "6 is the size of each unit and 7 is the number of the unit".

We noticed that the teacher P2 got the supports from the others and also supported to others of the professional team. Again, at the very beginning of the third day of the lesson, P2 made a remediation for her students to clarify the meaning of "6 sets of 7 meaning 7 times 6 and can be expressed as 7x6=( )". From P2 sharing the task, R2 got an insight and adapted it into as the Task 3.

Task 3: Using the figure to generate a word problem. 

Category III: Giving a mathematical language to formulate word problems

The task 4 that P2 conducted was to assess if her students understood the of language "6 sets of 5", '5 times 6" and connecting to 5x6=( )".

Task 4: (1) Draw a picture and create a word problem for "6 sets of 5".
(2) Draw a picture and create a word problem for "5 sets of 6". (P2, 11/02/2000)

P2 brought students' responses to the task 4 to a weekly meeting. After analyzing students' responses, we found that more than 80% of her students understood well the distinction between "6 sets of 5" and "5 sets of 6" whatever it was displayed in a picture or a word problem. Only 2 out of the 30 students still were confused with the distinction between these two terms. Based on the classroom observation and professional dialogues in the weekly meetings, the teacher D3 with only one year of experiencing second grade reflected to her last year teaching and stated that

> "I finally realized why my students had the difficulty with using a number sentence to represent a multiplication problem. It is resulted from my neglect of the significance of understanding the meaning of multiplication (P2, meeting, 11/06/2000) ".

Category IV: Displaying students' solutions to formulate word problems

In the midterm of the first semester, we observed a lesson related to two-digit subtraction without regrouping. As observed, the teacher Q2 and students engaged in discussing three students' solutions of the problem *"Tom has 39 dollars. He needs 15 dollars to buy a sandwich. How much money does he have now?. Using mathematical expressions represents your solution"*. The three solutions were:

Yee: 39-25=()	*Mei: 39-25=()*	*Tai: 39-25=()*
30-20=10	*39-20=19*	*29-25=4*
9-5=4	*19- 5= 14*	*10+4=14*
10+4=14		

Q2 asked the three students to come to the front of the classroom to explain their solutions and justify their thinking to others. We, as observers, were surprised with students articulating their own thought so clearly. However, C3 suspected if those students who were silent in the class understood what the classroom happened. Interestingly, in the homework of the day, Q2 generated the following Task 5 including two of the solutions displayed in the classroom to examine if students understood the discussions. The task Q2 conducted was initiated from her everyday teaching.

Task 5: We solved a word problem in today lesson. The solutions were given by Yee and Mei as follows. Would you please to write possible problems they solved?

Yee:	Mei:
39-25=( )	*39-25=( )*
30-20=10	*39-20=19*
9-5=4	*19- 5= 14*
10+4=14	

Students' responses to the Task 5 were sorted into three categories. The first is the problem was the same as the one solved in the classroom. The second was the new problems students created correctly. The third category shows parental interventions with students' work but it is unreasonable. For instance, *"Dad gave Joe 30 chocolates. Joe ate 20 of them. Mom gave him 9 more chocolates. Joe ate 5 more. How many chocolates does Joe have now?" and "Dad gave Eddy 39 chocolates. Eddy ate 20 of them. A day later, he ate 5 more. How many chocolates does Eddy have now?".*

Because students' unreasonable work with parental intervention was not acceptable by the teacher, several parents therefore learned from this task about the role of working with students' assignments to be as a supporter instead of a provider.

The task 5 indicates that the teachers were available with the tasks generated from everyday teaching. The teachers participating in the study found that classroom discourses on mathematical ideas became a resource of conducting such kind of task. Through the entire year, generating assessment tasks incorporating into everyday classroom teaching became as part of the mathematical instruction. This is a natural source and never terminates to initiate assessment tasks.

DISCUSSION

This study supported the teachers on creating an environment in which problem posing is a natural process of mathematics learning. The supports included the strategies of helping them in designing problem-posing tasks integral to instruction and the variety of tasks that situations require students formulate problem. As a result, the supports contributed the teachers to optimizing the quality of assessment and instruction, and thereby optimized the learning of the students.

For teachers, the problem-posing tasks allowed them to gain insight into the way students constructing mathematical understanding and served to be a useful assessment tool. As an assessing tool, the tasks incorporating into everyday instruction, decisions about task appropriateness were often related to students' communication of their thinking, or the students' problem-solving strategies displayed in classroom. The mathematics concepts to be taught at a grade level became as an elementary element of designing assessment tasks integrated into instruction. Other decisions concerning the appropriateness of a task were relevant with teaching events students encountered in everyday lesson.

The assessment task integral to instruction referred to in the study was characterized by the tasks conducted by the teachers collaborating with the researcher. The tasks created by the teachers and modified by the assessment team are more likely to improve the quality of the tasks. The result supports Clarke's claim (Clarke, 1996). The intervention of the researcher contributed more to the theoretical perspectives of problem posing, while the involvement of the teachers devoted to their classroom assessment practice. Comparing to the assessment tasks generated by individual, sharing multiple perspectives of appropriateness of task in a school-based assessment team was likely to achieve the purpose of task and the variety of task.

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PROSPECTIVE ELEMENTARY SCHOOL TEACHERS' SOLUTION STRATEGIES AND REASONING FOR A MISSING VALUE PROPORTION TASK

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This paper adds to the research studies on prospective elementary school teachers' mathematics knowledge with findings on proportional reasoning. In addition to analyzing prospective elementary school teachers' solution strategies for solving a missing value proportion task, it also examines the nature of the drawings and writing they used to explain their strategies.

Previous studies revealed that both prospective and practicing U.S. elementary school teachers possessed limited knowledge of mathematics, including the mathematics they taught (Ball, 1990; Ma, 1999; Post, Harel, Behr & Lesh, 1991; Simon, 1993, Simon & Blume, 1994). There is a growing consensus that the development of the "profound understanding of fundamental mathematics" (Ma, 1991) takes a long time and should be considered as a part of professional continuum that begins with the mathematics courses specifically designed for prospective teachers.

Even though conventional wisdom suggests that the mathematics knowledge possessed by the teachers plays a major role in shaping their instructional practices, there has been very little evidence that support a strong link between the number of mathematics courses teachers took and their student achievement (National Research Council [NRC], 2001). This result suggests the need to pay careful attention to both the content and instructional approach of these courses.

To address this issue, the American Mathematical Society and the Mathematical Society of American jointly put forth recommendations for the mathematical preparation of teachers in a book titled "The Mathematical Education of Teachers" (Conference Board of the Mathematical Sciences [CBMS], 2001). Furthermore, the book's authors point out the need for more detailed descriptions of prospective teachers' mathematical thinking, including their alternative conceptions, problem solving approaches and reasoning. Such information is needed to enable mathematics faculty who teach mathematics courses for prospective teachers to facilitate their students' development of mathematical proficiency more effectively. The current paper helps to meet this need by describing prospective elementary school teacher's reasoning and justification within the context of a missing value proportion task.

Proportional reasoning plays an important role in school mathematics (NCTM, 2000). When solving ratio, rate and proportion problems, students draw upon their concepts of multiplication and division of whole and rational numbers which are all part of the multiplicative conceptual field that is fundamental to upper elementary and middle

Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 2004 Vol 3 pp 265–272

school mathematics (Thompson & Saldanha, 2003).

Prior research studies on proportional reasoning have identified k-12 students' errors and difficulties, as well as the kinds of informal strategies they use in solving proportion tasks. Kaput & West (1994) identified three informal reasoning strategies commonly observed in solving missing value proportional tasks: 1) coordinated build-up/build down processes, 2) abbreviated build-up/build down processes, and 3) a unit factor approach. They suggested that the informal approaches were based on relationship that emerged naturally from given situations. The strategies defined in these previous studies might not translate to problem-solving methods used by adults. This paper seeks to address the following questions: Will prospective elementary school teachers be able to come up with reasoning strategies similar to those identified by the previous studies when they were encouraged to do so? What types of strategies will they develop? These are questions that the current paper seeks to address.

Furthermore, this paper analyzes the drawings and writings prospective elementary school teachers used to explain their solution strategies. Many researchers studied children's mathematical thinking have observed the spontaneous use of pictures or diagrams by students encountering problematic situations embedded in rich contexts. However, Diezmann and English (2001) also noted several difficulties students have in using pictures or diagrams. They point out the need for teachers to actively facilitate the development of "diagram literacy" among their students, which suggests that college mathematics courses should provide prospective elementary school teachers with opportunities to develop such literacy themselves. Another major question this paper seeks to address is: Given a missing value proportion task embedded in rich context, what types of drawings will prospective elementary school teachers develop to explain and justify the meaning behind their computation steps?

METHODOLOGY

The data reported in this paper is part of a research project that aimed at identifying the types of solution strategies, mathematical reasoning and justification prospective elementary school teachers developed as they participated in a specially-designed mathematics course focusing on numbers and operations. In this course, students were provided with opportunities to develop multiple methods to solve problems, to explain their approaches and reasoning to other students, worked toward understanding reasoning different from their own, to evaluate the efficiency of different strategies, and to make connections between different representations.

The concepts of ratio and proportion were discussed at the middle of the course through comparative ratio problems and a distance-time-speed problem, after instructional units on meanings and models of multiplication and division, and the concepts of factors, LCM and GCD. The technique of cross-and-multiply was strongly discouraged because students tend to carry out this method as a rote procedure that requires little proportional reasoning (Kaput & West, 1994).

Following the unit on ratio and proportion, students worked with problems on fractions, decimals and percents. The connections between these concepts and proportionality were highlighted continually throughout the units.

This paper focused on the analyses of thirty-six written solutions to one proportional task from students' final exam papers. The statement of the problem, which we call *Painting Walls*, is as follows:

> *If it took Jane 3/4 hour to paint a wall that was 12 ft by 12 ft, how long will it take to paint another wall that is 15 ft by 16 ft?*
>
> *a) Draw pictures to illustrate the situation in the problem.*
>
> *b) Explain how you can use your pictures to solve this problem.*

At a very basic level, *Painting Walls* can be classified as a missing value proportion. However, the fraction quantities, time conversion, and length and area measurements added layers of complexity for students.–Although students used the area model frequently throughout the semester to convey the meaning of multiplication and as a setting to explore LCM and GCD, it was not used when discussing proportionality in this course. Also, students had not worked on a missing value proportion task involving fractions. Therefore, *Painting Walls* had the potential to assess prospective elementary school students' ability to reason and to make connections among various mathematics topics they had discussed throughout the course.

The analysis of the data was guided initially by the following questions.

1. What types of solution strategies and reasoning did prospective elementary school teachers use to solve this particular missing-value proportional problem?

2. What kinds of drawings did prospective elementary school teachers use to justify their computation steps? Were they able to provide drawings and written explanations that supported their computation steps? If not, what seemed to be the difficulties?

Strategies with conceptual structures similar to what Kaput and West described were identified first. New categories and subcategories were formed to account for additional variations. Next, each strategy was classified based on whether or not the drawings supported the computation steps or not. Finally, descriptions were created to illustrate different types of drawings and to highlight the descrepencies between the computation steps and the drawings/written explanations.

SUMMARY OF RESULTS

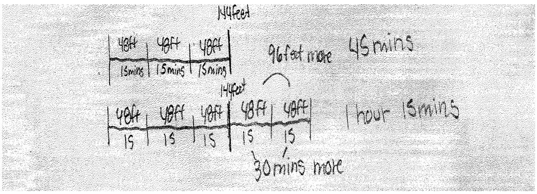
In all, twenty-two students (61%) solved the problem correctly and nine additional students (25%) had a valid strategy to solve the problem but mistakenly stated the additional time instead of the total time or made other minor computation errors. From the subset of thirty-one correct or partially correct papers, six different solution strategies were identified (Table 1).

Strategy 1a: Equivalent ratio/multiplicative factor approach (Using 144 sq. ft. to 45 min. as unit ratio)	4 (13%)

Strategy 1b: Equivalent ratio/multiplicative factor approach (Using 48 sq. ft. to 15 min. as unit ratio)	11(35%)
Strategy 1c: Equivalent ratio/multiplicative factor approach (Using 12 sq. ft. to 3.75 min. as unit ratio)	3(10%)
Strategy 2: Per min. approach	8(26%)
Strategy 3: Per sq. ft. approach	2(6%)
Strategy 4: Ratio table approach	3(10%)

Table 1: Percentage of students used each strategy

Conceptually, strategy 1a, 1b and 1c are all similar to what Kaput and West (1994) identified as "abbreviated build-up/build-down process." Yet, there were three different "unit ratios" students used to build-up. The following is one example of student drawing and reasoning using Strategy 1b.



We know that each of the walls have 144 feet. The second has 144 +96 =240 feet. We divide 144 by 3 cause it takes 3/4 of an hour to paint. So it takes 15 mins for every 48 ft. So 96 feet more on the second wall is 48+48 = 96. So 30 more min. was spent on the second wall.

Figure 1. An example of Strategy 1b.

This student first reasoned proportionally to figure out the unit ratio 48 sq. ft. for 15 min. Then she figured out the amount of time needed for painting the extra area, the 96 sq. ft., was two of 48 sq. ft. Thus, she would need two 15 min. more to paint the second wall.

Students who used Strategy 2 (Per Min.) and Strategy 3 (Per Sq. Ft.) approaches formed a new rate quantity, either 3.2 ft per min. or 0. 3125 min. per ft. West and Kaput considered these two strategies conceptually similar and called them both "unit factor approach." However, to justify them with pictures required totally different drawings. For example, a 12 by 12 grid embeded in a 15 by 16 grid could only be used to justify the Per Sq. Ft. approach but not the Per Min. approach. Yet three students who used the Per Min. approach drew this picture and referred to each 1by1 cell as one min.

The ratio table approach was used by three students and was well-grounded in the context and the numerical relationship of the problem. All three students created drawing similar to the following one (Figure 2). These students subdivided the extra area into three regions: 4 by 12, 3 by 12 and 3 by 4. Then they used the proportional reasoning to figure out the amount of time needed for smaller regions based on "45 min. for 12 by 12" For example, one student reasoned that 4 by 12 would require 15 min. because 4 is 1/3 of 12 (and 15 is 1/3 of 45). She then noticed that 3 by 12 and 3 by 4 would add up to the total area of 48 sq. ft., the same as the 4 by 12. Thus another 15 min. will be needed to paint the remaining region. The students' use of several proportionalities involving fraction relationships and their flexibly switching between the linear and area measurement made this a unique and rich approach worth noting.

Figure 2. An example of a drawing accompanying the Ratio Table approach.

Even though a relatively high percentage of the students were able to come up with valid reasoning strategies to find the numerical answers to the *Painting Wall* task, only sixteen students were able to draw meaningful pictures to explain their computation. The following table shows the percent of students within each strategy who were able to justify with their reasoning with pictures successfully.

Strategy 1a	Strategy 1b	Strategy 1c	Strategy 2	Strategy 3	Strategy 4
0%	82%	67%	13%	50%	100%

Table 2: Percentages of successful pictorial justification within each strategy

The most common drawing that failed to explain the computational steps looked like the one in Figure 3. Two separate rectangles were drawn with relevant information noted but with no additional mark on the two rectangles to help connect the problem context to the computation steps. This kind of picture illustrated the initial static status of the given problem, but did not show how this particular image could facilitate the conceptual process needed to solve the problem.

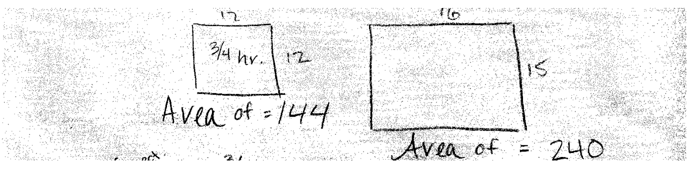


Figure 3. The most common unsuccessful drawing.

Finally, among incorrect solutions, the most common error showed the inability to coordinate the linear and area measures as noted by Simon and Blume (1994). The following is such an example (Figrue 4). Instead of conceiving 12 by 12 multliplicatively, this student came to the conclusion that 4 by 4 would be 1/3 as large as 12 by 12, thus requiring 1/3 the amount of time, or 1/4 hour. She then reasoned that 15 by 16 was very close to 16 by 16, so it would require approximately one hour to paint a 15 by 16 wall.

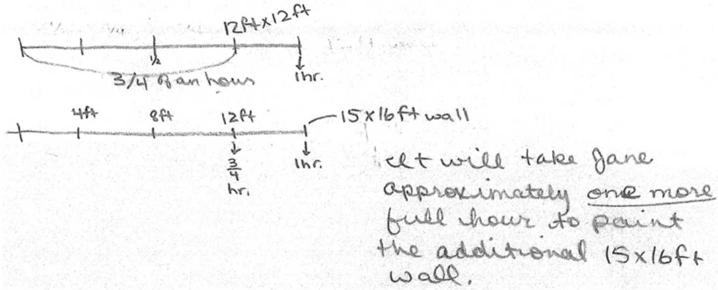


Figure 4. An incorrect solution.

SUMMARY OF CONCLUSIONS

As indicated at the beginning of this paper, instructors of prospective elementary school teachers could benefit from having information about the conceptions and strategies these students generally employ when solving mathematics problems. The current study identified six different solution methods prospective elementary school teachers used to solve a missing value proportion task. Three of these methods (using 144 sq. ft. to 45 min. as the unit ratio, Per Min. and Per Sq. Ft. approaches) have been identified by other researchers in prior studies with k-12 students, while the other three were more unique because of their tie to the given numerical relationship and the area context.

Even though a high percentage of the prospective elementary school teachers could solve the missing value problem with approaches other than the traditional cross-multiply method, a much lower percentage of these student could draw appropriate pictures to explain the meaning behind their computations. Many students drew static pictures to illustrate the given problem situation and did not use those pictures to explain the computation steps. Others attempted to use their pictures but were unable to use them successfully. Interestingly, students who used the three previouslyidentified methods had the most difficulty in drawing pictures and providing written explanations to justify the connection between the computation steps and pictures.

Why did students have difficulty in drawing pictures to explain these particular methods? Results from this and previous studies suggested two possible explanations.

First, many prospective elementary school teachers might have used these methods because they recognized the given task as a typical missing value task. Having identified the problem-type, they then used a method with which they had prior success in solving other missing value tasks without thinking more deeply about the meaning of the method itself in the given context. As Thompson and Thompson (1994, 1996) noted, having strong conceptual understanding of a particular mathematics concept alone might not be sufficient for a teacher to help his or her students understand that particular concept. Considering the level of mathematical abstraction at the elementary school levels, it's vitally important to provide opportunities for prospective elementary teachers to ground their computations in some image-rich context so that they will be better equipped to facilitate the conceptual development of their own students.

Second, many students decided to draw rectangles to illustrate the problem context. But as illustrated by Figure 2, it was not an easy task to figure out the answer to this question strictly based on the rectangular drawings with the given dimensions of the walls. It was also very challenging to explain certain approaches such as Per Min. approach using rectangular drawings. Some students recognized these difficulties and chose to use line models to represent the "quantities (area vs. time)" of the problem, rather than the physical dimensions (length by width vs. time) and used linear models similar to Figure 1 to support their computations. There is a clear need for prospective elementary school teachers to continue developing their "diagram literacy," including knowing when to use what types of drawings (Diezmann & English, 2001.)

Providing future elementary school teachers with mathematics tasks that are rich in context and encouraging them to develop drawings and representations to convey the meaning of their solution methods to other students appeared to have great potential to help them ground their mathematical reasoning. The results of this study also point to the need to discuss linear and area concepts earlier in the mathematics courses for prospective elementary school teachers, rather than in the second course as typically done in the U.S.. The hope of the mathematics education community is to place reasoning and justification at the central focus of mathematics instruction (Ball & Bass, 2003.) To support this goal, more studies are needed to investigate how various mathematics topics can be reasoned and justified by prospective elementary school teachers, and also to determine how mathematics courses can be specifically designed to enhance the reasoning and justifying abilities of future teachers.

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THE INTRODUCTION OF CALCULUS IN 12TH GRADE: THE ROLE OF ARTEFACTS

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The paper concerns the analysis of the role of artefacts and instruments in approaching calculus by graphic-symbolic calculator at high school level. We focus on an element of the introduction of calculus: the global/local game. We discus the hypothesis that the zoom-controls of calculator support the production of gestures and metaphors that foster the shift from a global to a local point of view. The analysis of protocols confirms that the exploration of several functions through the zooming process was supported by gestures and language. They appeared during the zooming process and in pupils' answers to the tests when the calculator was no more available and when the task concerned mathematical objects.

INTRODUCTION

This paper refers to our research about the introduction of calculus at secondary school level in a graphic-symbolic calculator environment (Maschietto, 2002). In particular, we focused on this introduction in terms of a transition between mathematical fields and we tackled it in terms of a reconstruction of relationships among mathematical objects. At the beginning of the teaching of calculus at high school level, pupils meet functions as algebraic and geometrical objects mainly under two points of view, the global point of view and the pointwise point of view. A global point of view on functions involves two aspects. The first aspect consists of considering functions as entities defined by a formula and/or a graphic representation. The second aspect concerns global properties of functions. A pointwise point of view consists of considering a function by the values taken on one or several chosen points, belonging to its domain. The beginning of the study of calculus is characterised by the addition of the local point of view. This forces to pay attention, for example, to a specific value and to the points near to it or, in the graphic setting, to what happens around a chosen point. Then within the objects characterising the transition and emerging from researches in mathematics education, we chose to study the articulation between a global point of view and a local point of view about functions, which we called the 'global/local game'. For analysing this game, we developed a didactical engineering (Brousseau, 1997), where we considered some cognitive research (above all Longo, 1998; Lakoff and Núñez, 2000) that studies the

1 Research funded by the MIUR and the Università di Modena e Reggio Emilia (COFIN03 n. 2003011072)

conceptualisation in mathematics (Longo is particularly interested in cognitive analysis of the foundations of mathematics, while Lakoff and Núñez are mostly interested in the analysis in terms of embodied cognition and metaphors). The situations are proposed in a graphic-symbolic calculator environment (TI-89, TI-92).

In this paper, we pay attention to the role of artefacts and instruments in approaching the global/local game under a Vygotskijan perspective.

THEORETICAL FRAMEWORK

With respect to artefacts, Vygotskij distinguishes between the function of mediation of *technical tools* and that of *psychological tools* (or *signs* or *tools of semiotic mediation*), discusses on their relation and offers a list of examples:

> language, various systems for counting, mnemonic techniques, algebraic symbol systems, works of art, writing, schemes, diagrams, maps, and mechanical drawings, all sorts of conventional signs and so on (Vygotskij 1974).

In the process of internalisation, technical tools (e.g. a concrete instrument to be handled in problem solving, graphic-symbolic calculator) become psychological tools (e.g. signs), starting in this way the cultural process:

> The use of signs leads humans to a specific structure of behaviour that breaks away from biological development and creates new forms of a culturally-based psychological process. (Vygotsky, 1987).

Elsewhere, the same author emphasizes the need to combine three different contributions in the process of internalisation:

> Children solve practical problems helping themselves with language, eyes and hands. This unit of perception, language and action, which in the end produces interiorization of the visual field, constitutes the central theme for all types of analyses regarding the origin of exclusively human forms of behaviour (Vygotskij, 1987).

This internal visual field is a part of the student's internal context where to carry on mental experiments, also supporting the production of mathematical reasoning. This emphasis on the body (eyes, hands and action) is consistent with the quoted position of Lakoff & Núñez (2000) according to which mathematical ideas are, to a large extent, grounded in sensory-motor experience.

An important contribution to the analysis of the process of internalisation, when artefacts are into play, is given by Vérillon and Rabardel (1995). In their studies, drawing on cognitive ergonomics they suggest the possibility that the use of an artefact causes in the subject the activation of schemes of use that transform it into an instrument. In particular, Artigue and al. (Artigue, 2002) have exploited these studies in order to develop an instrumental approach to both mathematical software and graphic-symbolic calculators in mathematics education.

A further elaboration of this idea is given, among others, by Bartolini Bussi, Mariotti & Ferri (in press), who integrate the cognitive analysis with the didactical analysis of the teacher's role. In a teaching experiment on artefacts taken from the history of perspective drawing (i. e. one of the roots of modern projective geometry), they validate two research hypotheses that concern: (1) the relationship between the intrinsic polysemy of any artefact and the expected - desired polyphony of classroom activity, through the essential role played by the teacher; (2) the relationship between the physical features of some artefacts and the bodily activity involved. They guess that similar hypotheses may be validated in other fields of experiences, related to new technologies (e.g. microworlds). In this paper, we aim at reformulating and discussing the second hypothesis with respect to graphic-symbolic calculator environment.

HYPOTHESIS AND METHODOLOGY

The second hypothesis above highlights a link between experienced events (and dynamic features of situations for pupils) and processes of conceptualisation in mathematics. Drawing on these researches, we focus on the link about the global/local game and we precise this hypothesis of embodiment as it follows.

The Hypothesis of Embodiment. In suitable activity with GS calculators, the Zoom-controls (ZoomIn and ZoomBox ZoomOut, ZoomStd) support the production of gestures and metaphors that foster the shift from a global to a local point of view, and produce a specific language which is maintained also later when the calculator is no more available (e.g. in paper and pencil environment) and when the task concerns mathematics objects.

We mean that it is possible to introduce some kind of manipulation (that is dynamics) in the treatment of graphic representations of functions through the zoom-controls. Then, the transformations of the graphic representation of a function through the use of zoom-controls and the experience of perceptive phenomena of local linearity (that can be formulate like that "when a curve is enlarged around a point of differentiability, it becomes locally linear") which these transformations may give <u>rise to the formulation</u> of a specific language, the production of gestures and the construction of metaphors from pupils. These elements might be exploited in processes of construction of mathematics objects such as tangent line and linear approximation of a curve at a chosen point (local linearity leads to mathematical analytic concepts such as the concept of limit and derivative).

We developed a "didactical engineering" to introduce the global/local game through the notion of local linearity. It was structured according to the following path: from identifying the graphic phenomenon of local linearity ("micro-straightness") during the first session to its mathematics formulation during the second and the third session. The first session is based on the use of the artefact: through the zoom-controls, pupils are lead to explorations of graphic representations of functions. In the second session, calculator (an instrument according to Rabardel and al.-1995- because of the activation of schemes of use) is used in order to discuss about mathematical character of micro-straightness (This part is not analysed in this paper).

Each session was organised in two parts: group activities first and then collective work orchestrated by the teacher. The collective work was not strictly conceived as a mathematical discussion according to Bartolini Bussi (1996), nevertheless it was managed in order to favour the sharing of results of the explorations among pupils and teacher and to let gestures and metaphors to emerge (according to Boero & al., 2001). The didactical engineering (3 sessions) was implemented in three Italian classrooms (fourth year of a 'Liceo Scientifico' with 17-18 year old students) in May 2000 (exp_A) and May 2001 (exp_B and exp_C). We analyse some excerpts of protocols in the following paragraph.

THE ANALYSIS OF PROTOCOLS

We present three excerpts: the first excerpt shows some gestures that were associated to a strategy of exploration and connected to zoom-controls in the group activities of the first session, the second excerpt concerns the collective work of the first session, the third excerpt concerns some answer to tests we gave to pupils at the end of each experiment.

The first session – The zooming process

In group activities, pupils had to explore six functions around chosen points, some of these functions are differentiable everywhere and others have singular points. At the beginning the exploration is guided, then the students are asked to use the zoom control more freely. In the text of the worksheet, we impose the shift from the calculator to paper and pencil in order to represent some screen.

In the transcript below (taken from exp_A), we have described the gestures used by pupils. We have underlined the words uttered at the same time of gestures and we have described them below. However, it is necessary to consider that the duration of gesture was not limited to the pronunciation of the associated word.

Students: PM, RE, GA. Researcher: C

58 PM: We always consider the same <u>point</u> (G58a), at the beginning we see how the <u>graph changes</u> (G58b) making <u>ZoomIn</u> (G58c), then from the standard window we verify how the curve changes making <u>ZoomOut</u> (G58d) this time [he said to C] we have a <u>bigger curve</u> (G58e) as a graph.

<u>G58a</u>. The point is showed by two fingers touching each other in the air (thumb and forefinger). PM indicated the point on the screen of his calculator and kept this position with the other fingers close to each others. <u>G58b</u>. The palm of the right hand was considered as the plane of the graphic representation. <u>G58c</u>. The joint fingers (right hand) shifting towards the palm (left hand), which was facing downward, indicated that the ZoomIn control was being used here. This gesture was made twice. <u>G58d</u>. His fingers started as joint from the palm of the hand and they separated in a movement upward (his hands was opening). <u>G58e</u>. The curve is traced in the air, with the same trend of the curve displayed by the calculator at this moment.

59 C: Do you think that "studying around a point" means…

60 PM: But, is it wrong?… because if we see the standard graph, zoom standard (G60a), as the element of reference, we take a point (G60b) and then we make ZoomIn (G60c) and then we make ZoomOut (G60d), we can better understand how it [*the curve*] develops in a detailed way and then however in…

G60a. His fingertips (thumb and forefingers of the two hands) made now a little rectangle. G60b. The point was indicated by two fingers close to each other, but not touching each other. G60c. This is accompanied by a repeated movement downward. G60d. The return to zoom standard to make a ZoomOut is accompanied again by the construction of the little rectangle.

The ZoomIn control is used in order to see some of the characteristics of the curve in a detailed way and is associated to a movement downward meaning an "entrance into the curve". The ZoomOut control, that is used to obtain a bigger curve and to better study its characteristics, is associated to a movement upward meaning an "exit from the curve", that corresponds to a moving away from the curve. PM also creates a space in front of him for controlling these processes (the standard window of the calculator becomes a little rectangle that is constructed by his fingers (G60a) under his eyes). This excerpt shows how PM enriches his use of artefact with gestures that seem to generalise those which are connected to zoom-controls. They seem also to act as a mediator between the global point of view and the local one. PM's gestures lead to interpret the particular as downward and the general as upward.

In general, our analysis showed that pupils have different behaviours according to their role (pupil using calculator and pupil drawing curves in paper and pencil environment) in the group activities. For instance, RE's gestures are different from PM's ones: since she had to draw the curve appearing on the screen, she tries to appropriate the trend of curve following it by her finger. In the same way, the language, that is associated to explorations and to the characterisation of the local linearity, changes: it is rather dynamic for pupils making the explorations, it is rather static for pupils drawing graphic representations in paper and pencil environment.

The analysis of this excerpt suggests that the zoom-controls introduce a kind of dynamics in manipulating graphic representations. Then it confirms our hypothesis.

The first session – At the beginning of the process of mathematisation

The collective work focuses on the linear invariant, obtained by explorations (for instance, figure 1): teacher solicits pupils to attribute a linguistic expression to the graphic sign displayed by the calculator at the end of several explorations. The analysis of the three experiments shows that the attribution of this term is very delicate and not evident at all. Communication is possible because pupils find a shared context in the common experience of the explorations of graphic representations of function.

Figure 1: standard window [-10,10]x[-10,10] and micro-straightness

In the collective discussion of exp_A, pupils agreed upon the expression “zoomata lineare”[2], which highlights the idea of the process of enlargement and the use of instruments. It pays less attention to the local property. In the collective discussion of exp_C, the phase of definition had been very rich. At the beginning, pupils proposed some terms (such as “segmentizzazione” or “segmentizzata”[2] where the root of the word “segment” can be recognized. These terms are different from the previous one (exp_A), because they stress the result of the enlargements rather than the process.

In the second session, the defined expression is used by teacher as a tool of semiotic mediation, because it is given as a sign to be interpreted. The task (“Determine the equation of the line which is displayed by the calculator”) leads to remake an exploration and starts the process of mathematisation.

Test – Some use of graphic sign and linguistic expressions

The following excerpts show how pupils recall both linguistic expressions and signs in different way in their answers to the questions of the test. They also show some personal adaptation of shared terms.

The first excerpt (cf. figure 2) presents CF’s answer to the question of giving an example of a micro-straightness curve (“zoomata lineare”) at a point. In the text of test, the use of linguistic expression evokes both the zooming process and its result. Two graphic signs point out the process: (1) a standard graphic representation of a curve with a rectangle and the point P inside, (2) two arrows between the two free-hand drawings. The rectangle is an evidence of localisation and it refers to the use of calculator (perhaps ZoomBox). Because of the absence of a formula to insert into the calculator, this is a mental experiment. The presence of two arrows pays attention to the shift to a local point of view and to the linear invariant.

Instead of exp_A, the evaluation of exp_B is based on the comparison between two tests about tangent line to a curve at a point: a test which is given before the beginning of didactical engineering and a similar test which is given at the end. In figure 3, we point out the linguistic expression “to make a zoomata lineare” in order to justify the existence of the tangent line at the chosen point. This indicates that the pupil associates a precise mathematical meaning to it. As in the previous example, he can not use calculator for the zooming process, which is therefore only evoked.

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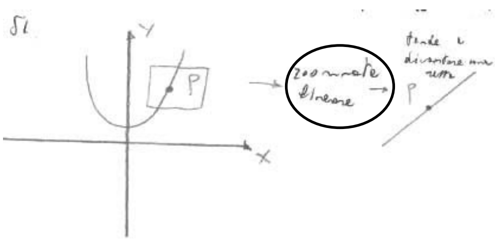
[2] Italian expression; a possible translation “linear zooming”.



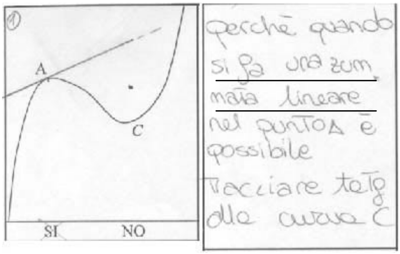
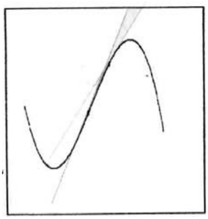


Figure 2: pupil CF (exp_A) Figure 3: pupil BE (exp_B)

In this example (cf. figure 4), SM has changed the linguistic expression "micro-linear" into a reflexive verb ("si microlinearizza" in Italian words), which indicates a potential transformation of the drawn curve into a line around the chosen points. This answer suggests that zoom-controls have been internalised and they have became a tool of verification. Also in this case, the calculator is not available.



Si, in tutti e due i punti, perche' nei dintorni di entrambi la curva si microlinearizza.

Yes, in all the two points [*the tangent line can be drawn*], because the curve becomes micro-linear on the neighbourhoods of both [*points*].

Figure 4: pupil SM (exp_C)

CONCLUDING REMARKS

In our research, the introduction of calculus is conceived in terms of global/local game and it focuses on the local linearity. In this paper, we have discussed the Hypothesis of Embodiment, basing our analysis of artefacts on a Vygotskijan theoretical framework. We have studied the mediation of zoom-controls of graphic-symbolic calculator in approaching local linearity. In particular, we have studied how the use of zoom-controls supports the production of gestures and metaphors that foster the shift from a global to a local point of view and that produce a specific language, which is maintained also later when the calculator is no more available. The analysis of protocols confirms our hypothesis. In fact, the exploration of several functions through the zooming process was supported by gestures and language. They appeared not only during the zooming process, but also in pupils' answers to test when the calculator was no more available and when the task concerned a mathematical object such as tangent line at a curve. They were revealed in this particular context, but they had their roots in deeper relationships with space and movement. Gestures seemed to play a particular role in the collective discussions when the process of mathematisation began and pupils did not yet have analytical instruments and associated technique to answer to the questions.

In this paper, we have started to discuss a reformulation of the Hypothesis of Embodiment and its validation in this technological environment. For space constraints, we have not yet discussed the first hypothesis stated by Bartolini Bussi, Mariotti & Ferri, i. e. the Hypothesis of Polysemy. Our future research, on the one hand, will carry on the analysis suggested by the Hypothesis of Embodiment, and, on the other hand will proceed to the reformulation and the discussion of the Hypothesis of Polysemy, that is also expected to be suitable for the environment of graphic-symbolic calculators.

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THE CRITICAL ROLE OF INSTITUTIONAL CONTEXT IN TEACHER DEVELOPMENT

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In this paper we document the importance of institutional context in both constraining and enabling the work of mathematics teachers. We build from our current and ongoing collaborative efforts with middle-grades mathematics teachers to provide an analytic approach and resulting analysis that clarifies the critical role of institutional context in teacher development. The analysis delineates the communities of practice whose enterprises are concerned with how mathematics is taught and learned in the district and the importance of their interconnections (cf. Wenger, 1998). Our approach can best be viewed as a tool designed to support transformative educational change as iterative processes of continual improvement in mathematics education.

INTRODUCTION

In our ongoing collaborative efforts with teachers, we have noted the importance of the institutional setting in both constraining and enabling the work of teachers and school leaders. As part of our research efforts, we have therefore developed an analytical approach for situating mathematics teachers' instructional practices in the context of the institutional settings of the schools and school districts within which they work (Cobb, McClain, et al., 2003). The approach treats instructional leadership and teaching as distributed activities (cf. Spillane, Halverson, et al., 2001) and involves delineating the communities of practice (cf. Wenger, 1998) within a school or school district whose enterprises are concerned with how mathematics is taught and learned. As will become apparent when we document the various communities of practice within the district and the interconnections between them, teachers and leaders constitute significant aspects of the environment for each other (cf. McDermott, 1976). The members of each community therefore afford and constrain the practices developed by members of other communities. It is in this sense that we will speak of the practices of each community being partially constituted by the institutional setting in which its members act and interact.

We illustrate the analytic approach by focusing on one urban school district in which we have collaborated with a group of middle-grades (students age 12-15) mathematics teachers for the past three and a half years[1]. Our goal in working with the teachers has been to support their eventual development of instructional practices in which they place students' reasoning at the center of their instructional decision making. In the envisioned forms of instructional practice to which the collaboration aims, students' interpretations and solutions are viewed as resources on which the teachers can capitalize to achieve their instructional agenda. Instructional materials would then serve not as blueprints for instruction but as resources that teachers adapt to the context of their classroom as informed by conjectures about both students' reasoning and the means of supporting its development. Furthermore, implementation of text resources would become a process of conjecture-driven adaptation rather than one of fidelity of reproduction. However, the complex and demanding nature of instructional practices of this type indicate the importance of social resources such as those afforded by the development of a professional teaching community (Gamoran et al., 2003). When situated within such a community, the process of instructional improvement then becomes a collaborative, problem-solving activity in which teachers generate knowledge about both students' mathematical reasoning and the process of supporting its development (Franke et al., 2001). The development of the professional teaching community was therefore a concurrent goal of our research.

In the following sections of this paper, we begin by providing an overview of the setting. We then discuss the theoretical framework and methodology we use for analyzing a school or district as a configuration of communities of practice. Against this background, we present an analysis of both the relevant communities of practice and the interconnections between them. In doing so, we clarify the critical role of school and district leaders in mediating the state- and federally-mandated high-stakes accountability program and claim that these were not solo accomplishments, but were instead partially constituted by the institutional setting in which they were developed and refined.

SETTING

The school district, which we call Washington Park, is located in a large city in the southwest United States and serves over 5,000 students, 42% of whom are minority students. There is high turnover in student enrollment. As an example, the student turnover rate during the 2001-2002 academic year at one of the three middle schools was 29% and the English Language Learner population doubled during a two-week period. A high-stakes accountability testing program was in place when we began collaborating with teachers in the district. In this program, students are tested in mathematics at each grade level on a nationally norm-referenced test. The results of these assessments are disseminated widely in the local media, and school and district leaders are held accountable for student performance. The district is of interest because school and district leaders have responded to the tests *not* by attempting to regulate teachers' instructional practices, but by giving teachers access to material resources and by supporting their development of social and personal resources. As an example, the district adopted a National Science Foundation [NSF] funded middle-grades textbook series and received a NSF implementation grant. In addition, the district routinely hired mathematics educators to conduct professional development sessions with the mathematics teachers.

THEORETICAL FRAMEWORK

The theoretical approach that we have taken when conducting the analysis builds from the work of Wenger (1998) to involve identifying the communities of practice within the school and district whose missions or enterprises are concerned with the teaching and learning of mathematics. In doing so, we take Wenger's three interrelated dimensions that serve to characterize a community of practice as constructs for our analysis: 1) a joint enterprise, 2) mutual relationships, and 3) a well-honed repertoire of ways of reasoning with tools and artifacts. These constructs guided our analysis as we worked to identify the significant communities of practice. We then built from Wenger to characterize the interconnections between these communities by focusing on 1) boundary encounters, 2) brokers, and 3) boundary objects. The approach grew out of pragmatic concerns for clarifying the critical role that teachers and school leaders play in mediating high-stakes accountability testing since an emphasis on high-stakes tests often stands in direct conflict with instruction focused on students' deep understanding of significant mathematical concepts.

Methodologically, we used what Spillane (2000) refers to as a snowballing strategy and Talbert and McLaughlin (1999) term a bottom-up strategy to delineate the communities of practice within the Washington Park district whose missions or enterprises are concerned with the teaching and learning of mathematics. The first step in this process involved conducting audio-recorded semi-structured interviews with the collaborating teachers to identify people within the district who influenced their classroom instructional practices in some significant way. The issues addressed in these interviews included the professional development activities in which the teachers have participated, their understanding of the district's policies for mathematics instruction, the people to whom they are accountable, the influence of high-stakes test scores on their instructional practices, their informal professional networks, and the official sources of assistance on which they can draw. In order to corroborate these interview data, we also administered a survey that addressed these same issues to all middle-grades mathematics teachers in the Washington Park district. The second step in this bottom-up or snowballing process involved interviewing the people identified in the teacher interviews and surveys in order to understand both their agendas as they relate to mathematics instruction and the means by which they attempt to achieve those agendas. We then continued this process as we identified additional people in this second round of interviews who actively attempt to influence how mathematics is taught in the district. This comprehensive data corpus allowed for the longitudinal analysis of the emergence of the communities of practice by testing and refining conjectures against the data in a systematic manner as described by Cobb and Whitenack (1996) which is consistent with Glaser and Strauss' (1967) constant comparative method.

RESULTS OF ANALYSIS

Communities of Practice

As we analyzed these data, the communities of practice that we identified, in addition to the *professional teaching community* that emerged from our collaboration, were the district-wide *mathematics leadership community* and the *school leadership community* in each of the three schools in which the teachers work. The core members of the *mathematics leadership community* were three *mathematics teacher leaders* [MTL's] based in each of the three middle schools who receive 50% release time from teaching to lead the district's instructional improvement effort in mathematics. A number of teachers were also members of this community but had more peripheral roles. The MTL's were, for their part, full members of the professional teaching community and participated in all sessions.

In addition to the semi-structured interviews conducted with the core members (e.g. the three MTL's), the data generated to document the activities of the mathematics leadership community include a series of follow-up interviews, scheduled monthly meetings, frequent informal discussions, and an ongoing email exchange with the MTL's as well as observations of professional development sessions that the MTL's conducted in the district. These data consistently indicate that the MTL's viewed themselves as members of a broader community of mathematics education reformers and had a relatively deep understanding of and a commitment to the general intent of reform proposals for mathematics teaching and learning. The data also consistently indicate that the joint enterprise of this community was to improve the mathematics understanding of all students by assisting teachers in developing a relatively deep understanding of both the mathematical ideas addressed in the reform textbook series and the ways in which students' reasoning might evolve as they complete instructional activities. The MTL's assumed that fidelity to the curriculum correlated strongly with high test scores.

The *school leadership community* in each of the three middle schools consisted of the principal and the assistant principal. In addition, the mathematics teacher leader and one or more teachers in each school were peripheral members. We have relied on semi-structured interviews conducted with the school leaders to document the activities of these communities, and have triangulated these interviews with the collaborating teachers' descriptions of the settings of their work. These data document that the joint enterprise of each of the school leadership communities was to support mathematics teachers' efforts to improve the quality of mathematics teaching and learning in the district while remaining vigilant about student test scores on high-stakes tests. The interviews indicate that the school leaders, like the MTL's, viewed fidelity to the curriculum as evidence of effective instructional practice. They pursued their agenda for mathematics teaching and learning by providing resources, arranging schedules to facilitate collaboration, and modifying observation forms so that they supported reflection rather than assessment.

Interconnections between Communities of Practice

To this point, we have documented that the practices of the professional teaching community, mathematics leadership community, and school leadership communities were in broad alignment. However, we have not explained either how this alignment was sustained or how the practices of the mathematics leaders and school leaders related to and influenced teachers' instructional practices. To address these issues, we have to take the analysis one step further by delineating the interconnections between the various communities that we have identified. In doing so, we distinguish between three types of interconnections: 1) boundary encounters, 2) brokers, and 3) boundary objects.

The first type of interconnection arises when teachers' or leaders' routine participation in the practices of their community involves *boundary encounters* in which they engage with members of another community. As an illustration, analysis of data indicates that boundary encounters occurred in the Washington Park district when both mathematics leaders and school leaders conducted classroom observations. Additional boundary encounters included grade-level meetings that the MTL's conducted with teachers, and regularly scheduled meetings between the school leaders and the mathematics teacher leader in each school. The mathematics teacher leader's institutionalized role as authority with expertise in the teaching and learning of mathematics was readily apparent in these meetings.

The second type of interconnection that we documented concerns the activities of *brokers* who were at least peripheral members of two or more communities of practice. Brokers can bridge between the activities of different communities by facilitating the translation, coordination, and alignment of perspectives and meanings (Wenger, 1998). Their role can therefore be important in developing alignment between the enterprises of different communities of practice. In the Washington Park district, the MTL's were the most visible brokers. As we have noted, they were not only members of the mathematics leadership community, but were also core members of the professional teaching community and peripheral members of the school leadership community. In this pivotal role as brokers between their own and the other communities, the MTL's had at least partial access to the practices of both the professional teaching community and the school leadership community. This in turn enabled them to provide the school leaders and teachers with access to the practices of each other's communities.

The third type of interconnection between the communities of practice involves the use of a common, boundary object by members of two or more communities as a routine part of their activities. Analysis of data clarifies that in the Washington Park district, boundary objects include the curriculum materials, the State Standards, and reports of students' test scores. As Wenger (1998) notes, boundary objects are based on what he terms reification[2] rather than participation. Wenger defines reification as "the process of giving form to our experience by producing objects that congeal this experience into 'thingness'" (p. 58). He argues that in creating reifications, "we project our meanings into the world and then we perceive them as existing in the world, as having a reality of their own" (p. 58). However, as he goes on to emphasize, reifications cannot capture the richness of lived experience precisely because they are frozen into a concrete form such as a text. As a consequence, although a reifying object is a relatively transparent carrier of meaning for members of the community in which it was created, there is the very real possibility that these objects will be used differently and come to have different meanings when they are incorporated into the practices of other communities. Even when this occurs, common boundary objects that are used differently in different communities can nonetheless enable the members of these communities to coordinate their activities. Consequently, as Star and Griesemer (1989) demonstrate, successful coordination does not require that members of different communities achieve consensus. Boundary objects do not therefore carry meanings across boundaries but instead constitute focal points around which interconnections between communities emerge.

DISCUSSION

The analysis we have presented demonstrates that the critical role of individual school leaders were not solo accomplishments but were instead partially constituted by the institutional setting in which they worked. We have also seen that in meeting regularly with mathematics teacher leaders in their school, they had the opportunity to negotiate their interpretations of the reform instructional materials with a person who was constituted in the district as a content expert. These and other aspects of the institutional setting in which the school leaders worked both afforded and constrained their development of leadership practices that involved supporting teachers' learning by giving them access to resources and by engaging in the discourse of *educational reform* rather than of *high-stakes testing* when they interacted with them (cf. Confrey, Bell, & Carrejo, 2001). In a very real sense, what it meant to be a school leader in the Washington Park district was partially constituted by the institutional setting in which they developed and refined their practices. Consistent with the distributed perspective on mathematics teaching, the analytical approach also characterizes individual teachers' instructional practices as situated and as partially constituted by the institutional setting in which they work. The analysis is therefore significant because it provides a case where high-stakes accountability testing did *not* delimit opportunities for teachers to develop instructional practices that focus on significant mathematical ideas and that aim to support students' development of relatively sophisticated mathematical understandings.

A number of investigations document that teachers' instructional practices are profoundly influenced by the institutional constraints that they attempt to satisfy, the formal and informal sources of assistance on which they draw, and the materials and resources that they use in their classroom practice (Ball, 1996; Brown, Stein, & Forman, 1996; Feiman-Nemser & Remillard, 1996; Nelson, 1999; Senger, 1999; Stein & Brown, 1997). The findings of these studies indicate the need to take account of the institutional setting in which teachers develop and refine their instructional practices. It is only when we do so that we can adequately explain both our success in supporting the teachers' development of increasingly sophisticated instructional practices and the district's success as assessed by student performance on high-stakes tests.

The potential value of such an approach is that it can support teacher development efforts by enabling researchers and teacher educators to monitor the institutional settings of the sites in which they are working on an ongoing basis. In this regard, the analytic approach can best be viewed as a tool that is designed to support transformative educational change as iterative processes of continual improvement in mathematics education. Analyses of the topology of communities of practice and their interconnections can provide guidance for reform efforts that aim to transform rather than merely augment currently institutionalized instructional and leadership practices.

Notes

1. This analysis was supported by the National Science Foundation under grant Nos. REC-0231037 and REC-0135062, and by Office of Educational Research and Improvement under grant No. R305A60007.
2. Reification as Wenger (1998) defines it should not be confused with Sfard's (1994) use of this same term.

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STUDENTS' PERCEPTIONS OF FACTORS CONTRIBUTING TO SUCCESSFUL PARTICIPATION IN MATHEMATICS

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This is a report of a project investigating students' perceptions of the extent to which their own efforts influence their achievement at mathematics and their life opportunities. We conducted two hour interviews with over 50 students, as well as collecting other data. Even students who were confident, successful and persistent exhibited short term goals. It also seems that classroom culture may be an important determinant of under participation in schooling.

INTRODUCTION

This is the report of an investigation of some possible causes of the malaise that characterises much of the experience of school for many students between the ages of 10 and 15 (termed the middle years) in Australia, a period that coincides with the transition from primary school (the first seven years) to secondary school. The key focus of the research was the students' perceptions of the extent to which their own efforts contribute to their success in, and enjoyment of, school. We see such research as critical because under participation in schooling is connected to missed life opportunities, high drop out rates, and reduced employment prospects, the economic cost of which has been estimated at over $2.5 billion to Australia (King, 1999).

Researchers have noted a decline in school engagement of young adolescents as compared with their engagement in primary school (Hill, Holmes-Smith, & Rowe, 1993), increased truancy, and greater incidence of disruptive behaviour, alienation and isolation (Australian Curriculum Studies Association, 1996). Hill et al. (1993), for example, reported that, in the middle years, there is a noticeable arrest in the progression of learning observed through the primary years.

The alienation seems to be most acute in the case of disadvantaged students. Lokan, Greenwood and Cresswell (2001), for example, argued that recent reforms have failed to address the obvious disadvantage of particular groups of students, and have not resulted in significant gains in engagement, especially in the middle years of schooling. Hill et al. (1993) noted that the bottom decile seems not to progress academically beyond Year 4.

Hill, Mackay, Russell and Zbar (2001) summarised a range of initiatives to address participation of students in schooling in the middle years. Predominantly the projects they summarised sought to address the decline in the level of students' engagement and liking of school, to promote a sense of identity and self esteem, and to develop in students the confidence to foster autonomous learners. Our focus is on the perspectives of the students to seek to identify the causes of the difficulties they

Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 2004 Vol 3 pp 289–296

experience. Without a clear understanding of these factors, structural or teacher professional development initiatives are unlikely to be successful.

Anticipated sources of pupil alienation

We believe that identifying the nature of pupil alienation and possible avenues to solution rest in coming to understand the perceptions or beliefs that students have about themselves and the opportunities that schooling offers. In particular, we examined perceptions that students' in the middle years have of their capacity to influence their own achievement.

The research framework is based on work by Dweck (2000) who identified two views of intelligence which she saw as being fundamental to understanding the way that people view themselves. One is a fixed view of intelligence entitled *entity theory* in which people believe that their intelligence is predetermined at birth and remains fixed through life. Dweck suggested that students who believe in the entity view require easy successes to maintain motivation, and see challenges as threats. The alternate perspective is where students see intelligence as malleable or *incremental* and they can change their intelligence and/or achievement by manipulating factors over which they have some control. Students with such incremental beliefs often choose to sacrifice opportunities to look smart in favour of learning something new.

Directly connected to these views of intelligence are the ways that people describe their own goals. Dweck suggested that some people have *performance* related goals, and rely for success on tasks that offer limited challenge. When experiencing difficulties, such people lose confidence in themselves, tend to denigrate their own intelligence, exhibit plunging expectations, develop negative approaches, have lower persistence, and deteriorating performance. Such students particularly seek positive judgments from others and avoid negative ones.

There are others, according to Dweck, who have *mastery* oriented goals who tend to have a hardy response to failure and remain focused on mastering skills and knowledge even when experiencing challenge. Mastery oriented people do not blame others for threats, do not see failure as an indictment on themselves, rather they hold learning goals which are to increase their competence when confronted with difficulty. Confidence in their own ability and success are not needed to build mastery oriented objectives.

Dweck argued that an *entity* view of intelligence leads students to focus mainly on *performance* goals whereas the *incremental* theory allows students to focus on *mastery* oriented goals.

It is interesting to consider the implications of this for teaching. Students who believe in the entity theory of intelligence could be a direct result of significant adults such as parents and teachers who tended to exaggerate the positives and protect them from negative information. Dweck claimed that, by their actions, some teachers teach students that they are entitled to a life of easy low effort successes, and argued that this is a recipe for anger, bitterness and self doubt. Dweck suggested that some teachers respond to students experiencing difficulty by providing easier tasks, the net effect of which is to create a climate in which challenges are feared rather than addressed.

Dweck (2000) argued that teachers can teach specific behaviours such as decoding tasks, perseverance, seeing difficulties as opportunities, and learning from mistakes. This emphasis is directly compatible with quite separate research strands on self fulfilling prophecy (e.g., Brophy, 1983), and motivation (e.g., Middleton, 1995).

Our key research questions were:

- To what extent does the students' orientation to mastery or performance relate to their confidence and achievement?

- What are students' perceptions of the extent to which their own effort contributes to their success at school?

SOURCES OF DATA

Data were collected from one year 8 class in each of four schools in a regional Australian city. The data sought students' responses to questions and tasks relating to both English and Mathematics. The surveys were administered to, and interviews conducted with, over 50 students.

The interviews took the form of a teaching conversation. Two sets of six hierarchical tasks on a similar topic were constructed in both English and Mathematics, ranging from very easy to very difficult. In the case of Mathematics, we posed a set of six tasks on the area of figures ranging from counting squares to a sophisticated task requiring interpretation of a scaled drawing. For each task the interviewer posed the task, sought the student's explanation of their strategy and their perception of whether they were correct. If correct, the interviewer instructed the student to attempt the next task. The intention was that eventually nearly all students would confront the challenge of a task which was difficult for them. The students were asked how they felt about the challenge they experienced, and the type of support they needed to solve the problem. We also sought students' responses to a vignette about advice they might give to one of their peers who was a potentially high achiever who deliberately does not try.

The survey included items from three instruments adapted from Dweck (2000), asking students to rate their self confidence and achievement, their persistence, their perception of the value of schooling, and what constitutes successful learning. These data were supplemented by their teachers' rating of their achievement and effort in Mathematics and English. Only the results related to Mathematics are presented here.

RESULTS

During the interview, students were asked up to six questions requiring the calculation of area, stopping when the student responded incorrectly. The first question was a trivial task requiring students to count squares which all students could do. The second task was also simple but required students to count half squares as well. All but one did this successfully. The third task asked students to draw a shape, in which the prompt suggested using half squares. Four more students were unable to do this.

The next task asked the students to calculate the area of the shape in Figure 1. Three more were unable to do this. While the task is slightly easier than the curriculum for these year 8 students would suggest, it is nevertheless a reasonable challenge, and that 38 out of the 46 students could do this suggests that their mathematics progress is at least satisfactory.

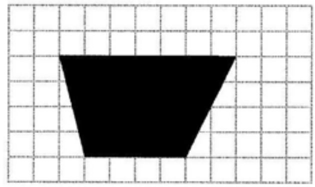

Figure 1: The 4th area question.

As an aside, the observers noted that many students tried to apply a rule for task 4 even though they had been successful on the previous tasks without using a rule.

The fifth task was slightly more demanding, as shown in figure 2, and this was at the level expected by curriculum for this level. That over one quarter of the students responded correctly suggests that these students, at least are progressing well at their mathematics. There was a sixth, much harder question that was completed by four students.

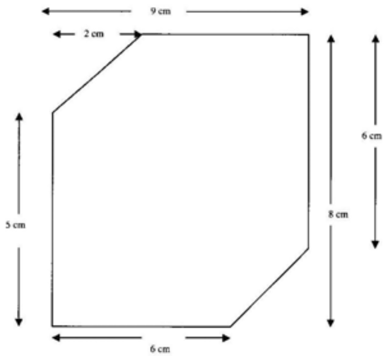

Figure 2: The 5th area question.

On the survey, the students were surprisingly confident in their own capacity to learn mathematics. Table 1 presents results from selected items from the survey, on which the students rated their responses on a six point scale, including strongly agree, agree, mostly agree, with similar options for disagreeing.

Table 1: Student self confidence (n = 46) (%)

	Strongly agree	Overall agree
I feel confident that I can learn most maths topics	38	94
I can learn anything in maths if I put my mind to it	41	94
If I find the work hard, I know that if I keep trying I can do it	37	91

About one third were very confident, most are confident, and they see a link between achievement and effort. Further responses related to effort are presented in Tables 2 and 3.

Table 2: Student self rating (%) of effort - positive (n = 46)

	Strongly agree	Overall agree
My friends say that I keep trying when maths gets hard	15	83

Table 3: Student self rating (%) of effort – negative (n = 46)

	Strongly disagree	Overall disagree
You are either good at maths or not. You cannot get better by trying	56	82
If I can't do the work in maths I give up	31	96

In both positive and negative forms, these students see themselves as persistent, and further confirm a link in their minds between effort and achievement. It is interesting to compare their self perceptions with those of their teachers who were asked to rate the students on their estimation of these students' effort and achievement. The ratings, choosing options from "poor" to "great" are presented in Table 4.

Table 4: Teachers' ratings (%) of student achievement and effort (n = 46)

	Great	Average or better
Achievement	29	82
Effort	46	90

In other words, the teachers also rate the students predominantly as achievers who try hard. Based on these student self ratings and the teacher ratings, it could be assumed that the students are progressing well and try hard.

It is interesting to compare the responses of the 12 students who demonstrated higher achievement by completing question 5 as presented in Figure 2. About half of these students strongly agreed with the propositions in Table 1, with the rest agreeing. Only 3 of these strongly agreed that their friends would say they keep trying when it gets hard. While 8 of this achieving group strongly disagreed with the proposition that you are either good at maths or not, and you cannot get better by trying, there were 2 who agreed. The teachers rated 8 out of these 12 students' achievement as great, and for 10 they rated their effort as great. In both items the rest were rated as good achievement. Perhaps these achieving students' self ratings are below what might be expected.

There were 8 students who did not reach or complete question 4, and these could be considered as below the expected level. Only 1 student did not agree with the propositions in Table 1, and 2 students strongly agreed with them. Only one disagreed that their friends would say they keep trying when it gets hard, and none agreed with the proposition that you are either good at maths or not, and you cannot get better by trying. Only one agreed with the proposition "If I can't do the work in maths I give up". The teachers rated 3 of these students as good, and 5 of them as having good or great effort. Perhaps these students' self ratings are optimistic.

Overall, both the high achieving and low achieving students are confident in their ability, they feel they try hard, and they see achievement as connected to effort. We had not anticipated this result, and suspect that working in schools on either aspects is not likely to address whatever are the causes of the apparent threats to participation in the middle years.

To gain some insights into what the students overall considered success in mathematics, they were invited to give an open response to the prompt

> I know when I am doing well in maths.

While there were many responses, 24 of the responses were categorized as "Getting correct answers and completing the work"; 23 as "Seeking teacher praise and good marks"; and 11 as "Emphasising learning and understanding". Some students had responses scored in more than one category. We would rate only the 11 students giving the third category of response as clearly mastery oriented. We rate the 23 students giving the second category of responses as clearly performance oriented, and we infer that responses in the first category are indicative of a performance orientation. Interestingly all of the 12 higher achieving students gave performance oriented responses, although focusing on what the good students would do. Only one of the low achieving students mentioned understanding. In other words, all but one of the students we rate as mastery were neither in the high or low achieving group.

To gain some sense of the importance the students attribute to mathematics, they were invited to give an open response to the prompt:

> What are the advantages of being good at maths.

About half of the responses were related to getting a better job or assisting them in their life generally. We take these responses to indicate an acceptance of the value of mathematics, and the worth in learning it. The other half of the responses were school and mark (grade) oriented. This may be evidence that substantial numbers of students have limited perception of the value of mathematics, and see it only as a school oriented task. Such perceptions would be vulnerable to external threat. Interestingly 11 out of the 12 higher achieving students gave job or life oriented responses whereas only 3 of the 8 low achieving students did so. Twenty-two students indicated that they persevered when the task got difficult and nearly all of these responses were related to getting the right answer.

The students were posed with a scenario of a friend who was good at maths but does not try. When asked to explain why this might be, in open response format, nearly half of the responses were either that their friend was trying to be popular or that they were scared of being bullied. Interestingly this finding was even more marked for the corresponding English prompt. This is perhaps the key finding in this study and has significant implications for the culture of schools, and the value for learning communicated by our society.

DISCUSSION AND CONCLUSION

In interpreting these results it should be noted that the process of seeking ethics agreements across grades and schools meant that only students who returned the forms were included. This may have had a biasing effect. Nevertheless there was a spread of achievement evident and so the results are informative at face value.

Overall, these students were surprisingly confident in their own ability, they perceived themselves as trying hard, they saw these as linked, and they achieved up to expectations on the mathematics tasks. The teachers' ratings generally confirmed the student self ratings, although it was noted that the weaker students seemed rule oriented in a counterproductive way. The students seemed very aware of the importance of effort. It seems that the schooling of the students in this study has developed an awareness of the importance of effort, and of metacognitive awareness of their approaches to problems. The students overall seemed aware that some students underachieved through lack of effort.

We suspected that students would give up when posed difficult tasks and this would provide the prompt for our discussions. However, in both the English and Mathematics tasks all students persevered for the whole time. It should be noted that the situation was artificial in that an adult observer was with the students individually for all the time, and this does not reflect a classroom situation. Nevertheless it does show that all of these students were willing to persevere under these conditions. Perhaps teachers could seek to simulate such conditions with difficult students at times.

Inferences from some responses suggested that generally the students have a performance orientation, not only to mathematics but also to effort. It confirms the Dweck conjecture that orientation to mastery or performance is not connected to confidence or achievement. Teachers will not address the students' participation in schooling solely by improving confidence, achievement or even awareness of the connection between effort and achievement (although these are obviously desirable).

Many of the responses, that we interpreted as evidence of a performance orientation, may be related to short term goals. In other words, the students saw pleasing the teacher, getting questions correct, getting the work completed, and scoring well on tests as the desirable goals. Students may benefit if teachers direct attention explicitly to the longer term goals of deep understanding, linking new knowledge to previous knowledge, linking new knowledge to its usefulness and application, and generally focusing on the mastery of the content rather than performance to please the teacher or parents, or even their own self esteem through any competitive advantage.

About half of the students connected success at mathematics to life opportunities. Interestingly nearly all the better students saw this connection, and few of the weaker students did. Teachers could well find ways to make connections between the content and its long term value. This is also connected to the purpose of schooling.

In an open item, nearly half of responses to a prompt seeking explanations for under participation suggested that either students deliberatively do not try in order to comply with a particular classroom culture or avoid the perception of trying due to threats of sanctions by peers. Perhaps this is a key finding. The students seem to have the necessary self confidence and appreciation of the contribution of effort and persistence, but may under contribute due to characteristics of the classroom culture. Teachers and schools could well address this issue as a priority and seek strategies for addressing it.

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THE COGNITIVE - MOTIVATIONAL PROFILES OF STUDENTS DEALING WITH DECIMAL NUMBERS AND FRACTIONS

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This pilot study determined some typical cognitive-motivational profiles of Finnish lower secondary students dealing with fractions and decimal numbers. Forty seven students from grades 7-9 participated in a number concept test, where also the motivational aspects, such as self-efficacy, certainty and tolerance were measured. Four distinctively different profiles were found where the cognitive aspect of task sensitivity and the motivational aspect of tolerance were crucial. The results suggest that if students' cognitive distance to the task demands is too wide, the cognitive conflict is passed unnoticed. In addition moderate sensitivity combined with high estimation of self-efficacy and low tolerance seems to be restrictive to a more radical change and deeper understanding of the concepts.

INTRODUCTION

Conceptual change refers to a situation, where learners' prior knowledge is incompatible with the notion of the new conceptualization and where learners are prone to have systematic errors or misconceptions suggesting that prior knowledge interferes with the acquisition of the new concept. In mathematics, this kind of situation is typical when the students are struggling to learn the concept of rational numbers while their prior thinking of numbers is based on natural numbers (Merenluoto & Lehtinen, 2002; Merenluoto, 2003). In several empirical studies it has been found that humans have an innate cognitive mechanism related to numeral reasoning principles (e.g. Gallistel, & Gelman, 1992; Starkey, 1992) which is based on the discrete nature of objects and strengthened in everyday experiences and linguistic operations (Wittgenstein, 1969) of counting. Later these prior concepts of discreteness are strengthened in order to teach and learn the notion of natural numbers. Because these concepts seem self-evident, self-justifiable or self-explanatory, they easily lead to overconfidence (Fischbein, 1987). As such they might act as an obstacle for conceptual change or lead to mistakes and misunderstandings on more advanced domains of numbers.

The advanced properties of rational numbers as a compact set of numbers are not explicitly taught at the lower levels of mathematics education. These properties are, however, embedded in the representations, rules of operations, and of order for these numbers which are essentially different compared to respective rules of natural numbers. Thus, every extension of the number concept demands new rules to be learned for operations and the use of a new kind of logic often leading to many different, but systematic problems and misconceptions in mathematics learning (c.f. The multiplier effect, see Verschaffel, De Corte & Van Coillie, 1988).

In several empirical studies it has been found that the misconceptions resulting from problems with prior knowledge of numbers seem to be exceptionally resistant to teaching attempts and that the students a prone to have difficulties with decimal numbers and fractions (e.g. De Corte & Verschaffel 1996; Verschaffel, De Corte & Borghart, 1997). In fact the problem in the school context is that the students are not very well aware of their prior conceptions and are prone to create models, where the prior knowledge is inconsistently combined with the new thinking. In this kind of situation we can speak about a problem of conceptual change. It is possible to claim that in process of a radical change in the thinking of numbers the students are forced to tolerance the ambiguity which comes from newly learned operations and characteristics of numbers while they do not yet fully understand the concepts (c.f. Sorrentino, Bobocel, Gitta & Olson, 1988; Stark, Mandl, Gruber & Renkl, 2002). In fact it is possible that coping with a new complex conceptual system is possible only if the learner has sufficient metacognitive skills to deal with conflicting notions such as when the same number is possible to present in infinite many representations (like fractions) or dealing with the infinity (Merenluoto, 2003).

Motivation seems to be related to the conceptual change in a very complex way. For certain, interest and the feeling of self-efficacy (e. g. Ford, 1992; Schoenfeld, 1987) are the fundamental aspects of high tolerance of ambiguity. However, high self-efficacy and certainty also seems to increase learners' tendency to pass the possible cognitive conflict unnoticed (e.g. Merenluoto & Lehtinen, 2002).

The research on conceptual change (e.g. Carey, 1985; Chi, Slotta & DeLeeuw, 1994; Duit, 1999; Hatano & Inagaki, 1998; Karmiloff-Smith, 1995; Vosniadou, 1994; 1999) has this far mainly dealt with cognitive factors, but especially during the last few years several researchers have agreed that these processes can not be explained in mere cognitive terms, but also motivational aspects (e.g. Pintrich, Marx, & Boyle, 1993; Linnenbrink & Pintrich, 2002), should be considered.

The aim of this pilot study is participate to this discussion and to analyse the relations of cognitive and motivational factors in students dealing with decimal numbers and fractions.

METHOD

The participants in the pilot study were students on grades 7-9 at Finnish comprehensive school, grade 7 (n = 15), grade 8 (n = 17) and grade 9 (n = 15), the percentage of girls was 40 %, 18 % and 67 % respectively. All the students had the same teacher.

In the beginning of *the procedure* the students were asked to fill in a questionnaire on own estimation of how much they had understood of the mathematics taught at school, their self-efficacy and tolerance with difficult problems in mathematics. The teacher was also asked to estimate the same variables for each of the students. *Phase 2.* The students were given a two-paged rational number concept test with 26 tasks testing sorting of numbers, identification of different representations of numbers (see Table 1), the density of numbers on the number line, and basic calculations. They were also asked to estimate their certainty on the answers with a 5-point-likert-scale, from 1 (a wild guess) to 5 (as sure as I know that 1 + 1 = 2) and to pick the most difficult and easy problem on each page. All the variables were calculated as percentages of maximum.

Students' *achievement level* in mathematics was estimated by the teacher on the scale from 5 to 10. Besides analyzing the answers to the tasks qualitatively the *test scores* (representing also the students' cognitive sensitivity to the tasks) were also scored from 0 to 1, where zero was given, if the answer was incorrect. The reliability of respective *certainty scores* was: alpha .799.

Tolerance of ambiguity was measured using two components: 1) *Estimated tolerance* was measured with the teacher's and students' answers to statement with a 5-point-likert-scale "If the task feels difficult I/the student do/does not do it", alpha .619; 2) *Test tolerance* was measured as the number of tasks done (score 1 per each task) and as the quality and thoroughness of explanations in the tasks (score 0-2 per each task).

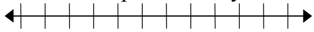
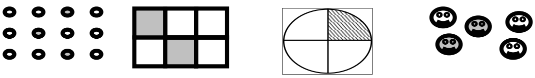
Students' *self efficacy in mathematics* was measured with a 5-point-likert-scale with statements such as "I am good in solving problems", "I'm doing well in mathematics at school", "I like difficult problems, then I can struggle to solve them", five items alpha .832. *Experience of understanding* the students were given a rectangle (1 cm x 10 cm) and asked to color as large part as they estimated to have well understood about the mathematics they had been faced this far at school. The colored portion was measured as percentages.

RESULTS

The results refer to major problems with rational numbers (Table 1). According to the results the students had a high tendency to a mistaken transfer from natural numbers to the domain of rational numbers, such as giving an answer of "one" when asked, how many decimal numbers there are between numbers 0.50 and 0.52. The mean score for all the students was 52 per cent (SD 21) and the mean of certainty estimations was 64 per cent (SD 15). Between the grades there were no statistical differences in test scores or certainty scores. Instead, the statistical differences were related to the achievement levels of the students (high, average, low), the task scores, $F(2, 44) = 18.5$, $p = .000$; $\eta = .452$, and certainty scores, $F(2, 44) = 6.66$, $p = .003$; $\eta =$

.232. The difference was due to the difference between the high and low achieving students (Scheffe, p < .001).

Table 1
Examples of the tasks used in the test with the frequency and percentage of correct answers

	Number of correct answers
1. How many decimal numbers there are between 0.50 and 0.52 on the number line?	3 (6 %)
2. Which decimal number is the next after 0.60?	
3. Mother bought half a kilo of grapes costing 2.10 euros per kg. How much is her change from 10 euros?	29 (62 %)
4. Sort following fractions from the smallest to the largest: $\frac{1}{2}$ $\frac{5}{6}$ $\frac{7}{8}$ $\frac{2}{100}$ $\frac{6}{4}$	23 (49 %)
5. Sam had run one eighth of his route and there was 3.5 km left. How long is the route?	17 (36 %)
6. Which number is represented by ‘x’?  0 x 4/9	13 (28 %)
7. 1/2 x 1/8 =	16 (34 %)
8. 4 : 1/3 =	12 (26 %)
9. The identification task:  3/12 1/4 0.333… 1/3 2/8 2/5 0.25 2/6 Draw a line to connect the picture to the number presenting the coloured portion of it. (It is possible that there are several numbers for the same picture.)	The mean of correct connections: 6.7 (56 %)

Typical to all the students was that they had significantly more difficulties in dealing with fractions than in dealing with decimal numbers, paired samples difference in test scores, t = 9.25; p =.000, and in certainty scores, t = 11.90; p =.000.

The only gender related difference was in students' self-efficacy estimations, F (1, 46) = 5.38, p = .025; η = .107, and in the estimations of understanding of school mathematics, F (1, 46) = 13.84; p = .001; η = .239, the estimations of the boys were higher than the same for the girls.

Table 2 shows the high correlations between the cognitive and motivational variables measured in the test. Achievement level in mathematics had the highest correlations to test score, self-efficacy, understanding of mathematics, and estimated tolerance, but lower correlations to the measured test tolerance and certainty scores. In addition, the correlation of the achievement level in mathematics to the test tolerance was higher for the girls (.599) than for the boys (.320). The students had a clear tendency to over estimate their certainty and tolerance with difficult tasks.

Table 2
Correlations between the cognitive and motivational factors measured in the test

	1.	2.	3.	4.	5.	6.
1. Achievement level in mathematics estimated by the teacher						
2. Estimated understanding of school mathematics	.669**					
3. Estimated self-efficacy	.719**	.778**				
4. Estimated test tolerance	.711**	.684**	.624**			
5. Test tolerance	.395*	.279	.338*	.308*		
6. Test score	.710**	.498**	.474**	.528**	.430**	
7. Certainty score	.480**	.543**	.423**	.410**	.288	.541**

** p< .01; *p< .05

To analyze the typical cognitive - motivational profiles for the students a cluster analysis was used on the variables in Table 2. Four distinctive different profiles were found.

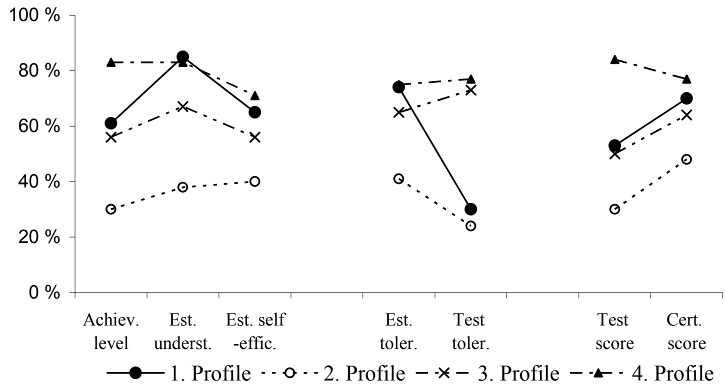


Figure 2. Typical profiles of the students in the cognitive-motivational variables measured in the test

The *first profile* was the most typical for the boys in the study (sixteen students, fourteen boys and two girls). Typical to this profile was significantly high estimations of their own understanding, tolerance with difficult tasks, and certainty compared to their average achievement level, low test tolerance, and moderate test scores. Their clear over estimation of certainty (see Figure 2) suggests some kind of illusion of understanding. A typical student on this profile was Carl (name changed). He was an average student in mathematics and liked mathematics. He also explained that he is good in mathematics, but did not want difficult problems. But then again he found mathematics discouraging. Thus he gave conflicting answers. His test score was 38 per cent of the maximum, but his certainty estimations optimistically 84 per cent. He gave correct answers in sorting the decimal numbers, in the word problems (Table 1) and in the calculations with decimal numbers but had problems in all the tasks where fractions were used.

The students on the *second profile* (seven boys and four girls) were low achievers in mathematics. In sorting the decimal numbers, many of them used a rule: "the number is small if it has many decimals" and they typically sorted the fractions by the nominators or denominators. In the identification task (Table 1) they found only the four obvious connections with high certainty and chose this task to the easiest task on the page. These results refer to a low cognitive sensitivity to the demands of the tasks thus suggesting a wide cognitive distance to the concepts of rational numbers.

The *third profile*, where eleven students were classified on (seven boys and four girls) was especially characterized with high tolerance where there were no statistical difference between the estimated tolerance and test tolerance as it was the case in the profiles one and two. But like in previous profiles, they over estimated their certainty in the tasks. These were high and average achievers in mathematics. A typical student on this profile was Eric (name changed) who liked mathematics and difficult problems. He had some intuition of several possible answers to the question of the next number after the number 0.60, giving the answers 0.61 and 0.60001. He found eight of the 12 correct connections in the identification task, but failed in all other tasks where fractions were used.

To the *fourth (high) profile* were classified eight students (two boys and six girls). These were high achieving students in mathematics, with the highest test scores and sensitive certainty scores (no statistical difference to the scores). This was the only profile where the teachers estimation of students' self-efficacy in mathematics was significantly higher than their own, t(repeated measures) = -3.88; $p = .005$. Only one student in this profile had problems in sorting the fractions. Seventh grader Ann (name changed) was one of the three students in the whole group, who's answer to the questions pertaining to the density of numbers on the number line (like tasks 2-3, Table 1) suggested deeper understanding of the rational numbers. She was a high achiever in mathematics who estimated that she had understood everything taught at school and liked mathematics. She answered that she is good in mathematics and that mathematics is easy and useful, but needs work. She gave correct answers to the majority of the tasks. Her answers to the most difficult tasks referred to a quite high sensitivity to fundamental feature of density of rational numbers, when she explained that between number 0.50 and 0.52 there exist two or more numbers "because they can be tenths, hundreds, etc." And when asked about the next number after .60, she explained: “may be my answer is wrong, but in the decimal numbers there can be very many numbers after the comma. Thus, I can not answer to that question”. This was the best answer to this question in the whole group.

DISCUSSION

The results of the test refer to major problems with decimal numbers and especially with handling the fractions. The answers of even the best students suggested mostly operational level of understanding indicating to an enrichment kind of learning (c.f. Vosniadou, 1999). Only in the answers of a few students there were some indications of deeper level of thinking suggesting a preliminary state of conceptual change. The identification task and the tasks of sorting the numbers were the best indicators of the quality of conceptual understanding in the test (see also Sowder, 1992).

The high correlations between the cognitive and motivational factors refer to the importance of considering the motivational aspects in the research on conceptual change. But they also refer to the complex interaction of these variables in learning where more research is needed. The students' achievement level in mathematics had a significant relation to the high sensitivity to the cognitive demands and high tolerance of ambiguity that seems to be optional for the conceptual change. However, similar to our earlier empirical results from the upper secondary level (Merenluoto & Lehtinen, 2002) also in this data from lower secondary levels of mathematical education, the moderate operational understanding of the concepts has a tendency to prevent the students' from noticing the cognitive conflict. The results also confirm that in the attempts to teach for conceptual change it is crucial to consider the cognitive distance between students’ prior knowledge and the new phenomenon to be learned.

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THE NUMBER LINE AS A REPRESENTATION OF DECIMAL NUMBERS: A RESEARCH WITH SIXTH GRADE STUDENTS

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Based on Janvier's (1987) model on representations, in this paper we examine 12 year old students' understanding of the concept of decimal numbers. For this reason the study was conducted with the use of three kinds of tests related to decimal numbers. These three tests involved recognition, representation and translation tasks. In particular, the idea of the number line as a geometrical model is being discussed in respect to representations and translation between different representations. The application of the implicative statistical method demonstrated a compartmentalization of the different tasks and this signifies that there is a lack of coordination between recognition, representation and translation in decimal numbers.

INTRODUCTION

One of the aims of mathematics instruction is to achieve the understanding of mathematical concepts through the development of rich and well organized cognitive representations (Goldin, 1998; NCTM, 2000; DeWindt-King, & Goldin, 2003). In this study the term representation is interpreted as the tool used for representing mathematical ideas such as tables, equations and graphs (Confrey & Smith, 1991).

The first aim of this study is to investigate 12 year old students' understanding of addition of decimal numbers. Second, to explore the way these students understand and utilize the representations of decimal numbers as different concepts – modularization – or as different expressions of the same concept. The third aim of the study is to investigate which modes of representation of decimal numbers are the most difficult ones. Finally, the study is dealing with the differences in students' achievements concerning the issue addressed by the different exercises – concept and addition of decimals.

THEORETICAL BACKGROUND

Decimal numbers, representations and number line

According to Janvier (1987) the understanding of a mathematical concept passes through at least 3 stages:

1. The recognition of the mathematical concept among different representations
2. The flexible manipulation within a system of representation
3. The translation from one system of representation to another

In particular the ability to translate from one system of representation of a concept to another is of great importance, concerning mathematical problem solving and the learning of mathematics in general.

The concept of decimal numbers is included in mathematics curricula and it is considered to be of great significance especially due to its application and use in every day life. The specific concept can be linked to the concept of fractions, since decimals can be considered as the parts of the whole, a whole that has been divided into 10, 100, 1000, or some other number of parts that is in a power of 10. This is the reason why instruction should not approach decimals as an isolated mathematical concept. Decimal concepts need to be related to a variety of fraction ideas and to place value. Furthermore, teaching decimals in a comprehensive way should give students the opportunity to flexibly, utilize and link a variety of representations concerning the decimal numbers: linear models, such as the number line, manipulatives, surface, symbols and currency (Thomson & Walker, 1996).

Research findings often disagree in regard to the importance of number line as a didactical model in general and as a means of representing integers and rational numbers (Ernest, 1985; Behr, Lesh, Post, & Silver, 1987; Lesh, Post, & Behr, 1987; Raftopoulos, 2002; Gagatsis, Shiakalli, & Panaoura, 2003; Michaelidou & Gagatsis, 2003). For example, Thomson and Walker (2000), underline the fact that the knowledge of how to use the number line as a means to represent the decimals is quite necessary since it contributes to the development of concepts not only related to the identification and comparison of decimals but to the ability to perform operations as well.

Sometimes the disagreement of the researchers on the role of the number line is due to the fact that number line is not a common or standard representation but a geometrical model. In fact a directed straight line in Euclidean Geometry can serve as a geometrical model for addition, subtraction, multiplication and division of rational numbers as well as for constructing irrationals. Operations on real numbers are represented as operations on segments on the line. A straight line with a scale on it belongs to a mixed type of representations. On the one hand, it functions as a new comprehensive geometrical model with the rational numbers corresponding not only to directed segments and operators on them but also to a set of distinct points on the line. On the other hand, the scale can be used as a means of arithmetization. Points on the line can be numbered in such a way that differences of numbers measure distances of corresponding points. In this way, every geometric operation – like segment addition – can be translated into an arithmetic operation and carried out algorithmically (Gagatsis, et al., 2003).

Two ideas have been seminal to this study: The first is Janvier’s classification on the understanding of a mathematical concept and the second is the idea of number line as a geometrical model. Thus we propose the following four research questions:

(a) Are 12-year old students able to recognize decimal numbers in different representation systems, verbal, symbolic and number line?

(b) Can 12-year old students flexibly manipulate the concept of decimals within a given representational system?

(c) Can 12 year old students accurately translate decimal concepts from one system of representation to another? And are some translations easier than others?

(d) What are the relationships among children's responses to tasks of recognition, representation and translation?

METHOD

A hundred and twenty 12-year-old students from three primary schools in Limassol, Cyprus participated in this study.

The study had three distinct phases. In phase 1, Test A was administrated to all 120 students. Test A, which had a multiple choice format aimed to examine students' ability to recognize and represent the concept of decimal numbers in a variety of different representations – line segment, number line and rectangular surface (Fig.1).

For example: Circle those diagrams where the shaded part presents 0.9

Line segment Number Line Rectangular Surface

Figure 1

Test B, included tasks dealing with addition of decimal numbers. Some of these items were presented exclusively in symbolic mode (i.e 0.04 +0.52 =). Some tasks were presented on a number line and aimed to examine students' ability to translate from the number line to the symbolic expression and vice versa (Fig.2).

Show on the number line the following mathematical expression: 0.3+ 0.5=

0 1

Figure 2

Test C included four word problems which involved decimal numbers. Sixty students were asked to solve the problems in any way they wished. The remaining sixty students were instructed to solve the problems using the number line.

An athlete run a distance of 0.3 km and stopped to have some rest. Afterwards he ran a distance of 0.4 km. What is the total distance that the athlete had run?

The research design and the selection of the items are similar to the ones used in previous studies which examined the use of number line as a means for representing mathematical concepts (Michaelidou & Gagatsis, 2003; Shiakalli & Gagatsis, 2003).

Variables

Test A: Recognition Item. AL: Recognition on Number Line Item. ASUR: Recognition on Rectangular Surface Item. AS: Recognition on Line Segment Item.

Test B: Addition of Decimals Item, B-number (i.e B1): Addition of Decimals in Symbolic Mode Item. BL: Addition of Decimals Involving Translation from the Number Line to the Symbolic Expression. BS: Addition of Decimals Involving Translation from the Symbolic Expression to the Number Line.

RESULTS

We considered that – beyond frequencies of success (given by SPSS) – one appropriate statistical method to be used was the implicative statistical analysis of Regis Gras. This statistical method allows the examination not only of the difficulty level of the questions in the 3 questionnaires but also the relations between students' responses to the tasks in the 3 questionnaires. To this end we have used the statistical package CHIC (Gras, Peter, Briand, & Philippé, 1997). One of the diagrams produced by this method is the similarity diagram which represents groups of variables which are based on the similarity of students' responses to these variables.

The Understanding of the Concept of Decimal Numbers

The data suggest that students perform better in items including the rectangular surface concerning the recognition items (Table 1). Concerning the tasks involving the representation of decimals, students perform better in tasks where the given representation is the rectangular surface (Table 2). As far as the translation mode is concerned students perform better in translating from the number line to the symbolic expression in comparison to the translation from the symbolic expression to the number line (Table 3).

Success Percentage Concerning the Recognition of Decimals –Test A	
Representation	Percentage (%)
Line Segment	43
Number Line	46,7
Rectangular Surface	63,6

Table 1: Recognition of Decimals

Success Percentage Concerning the Representation of Decimals –Test A	
Representation	Percentage (%)
Number Line	83,5
Rectangular Surface	98,3

Table 2: Representation of Decimals

Success Percentage Concerning the Translation Ability –Test B	
Translation Mode	Percentage (%)
Number Line – Symbolic Expression	68.6
Symbolic Expression – Number Line	53.7

Table 3: Translation ability

Finally there was no significant different between the control and the experimental group (a>0.05) despite the fact that the experimental group was instructed to use the number line whilst the control group was instructed to solve the problems without any restriction regarding the solving procedure.

Relations between tasks-Modularization

The similarity diagram showed a formation of four groups of tasks (Figure 1). The first group (1L, A3SUR, AR6L, B4) involved tasks that examined students' ability to recognize and represent decimals. The second group (A1S, AR4SUR, B13S, B14L, B1, B6L, B9S, B12L, B10L, B11S) was comprised by tasks that explored students' translation ability concerning the addition of decimals. The third group (A2L, A2SUR, AR4L, AR5L) mainly consisted of tasks including number line as a means of representing decimals. Finally, the last group (AR6SUR, B5, B7S, B8L, B2, B3, B15S) consisted tasks involving addition of decimals. According to the above results the tasks of both Test A and B are modularized according to the kind the task - translation, recognition, representation, addition.

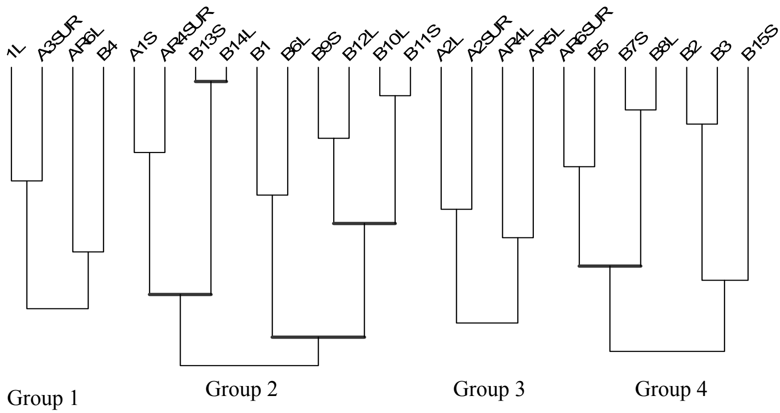

Figure 1: Similarity Diagram of Tasks included in Test A and Test B which examines Modularization

DISCUSSION

Understanding the Concept of Decimal Numbers

Low percentages of success in recognition items (Table 1) and translation items (Table 2) show that the representations of decimal numbers are not sufficiently developed and do not consist a unified whole. High percentages of success in representing the concept of decimals using a number line, a line segment and especially the rectangular surface can be attributed to instruction which usually gives great emphasis to isolated representations such as the rectangular surface. In this way the modularization of knowledge of decimal numbers is reinforced.

Moreover, the sub concept of part whole of rational numbers seems to dominate children's knowledge concerning rational numbers since students treat the number line which is infinite as a line segment from which they have to select a part. Consequently, students perform better in representation tasks since they were rectangular surfaces and number lines including the space from 0 to 1. On the contrary, concerning recognition items, students face difficulties especially when then number lines include spaces beyond 1.

Students' Performance in Different Modes of Translation

Students face difficulties in both kinds of translation tasks: from the number line to the symbolic expression and vice versa. Translation from number line to the symbolic expression seems to be easier than translation from the symbolic expression to the number line. This might be attributed to the fact that students are more familiar with the use of ruler, which is a linear representation, as a tool for measuring length. Consequently, they are more familiar with the translation from the number line to the symbolic expression.

Difference in performance concerning the two translation modes indicates that students deal with the two modes as if they are different concepts and not two different modes of the same concept. According to Janvier (1987) a translation involves two modes of representation: the source and the target (i.e number line Symbolic expression). To achieve directly and correctly a given translation, one has to transform the source "target-wise" or to look at it from a "target point of view". There are certain implications for instruction since the above comment suggests teaching strategies should emphasize to this two way procedure (Janvier, 1987).

Students' performance in Different Tasks

Students perform well in representation tasks but they face difficulties while dealing with the recognition tasks and the addition tasks. This difference in performance maybe due to the fact that among the recognition tasks were perceptual distractors, that is tasks with misleading information i.e number lines including spaces greater than the space from 0 to 1. This fact along with the fact that the sub concept of the rational number as a part whole is dominant in students' thought led students to the wrong selections. They treated number lines not as continuous models where a point obtains meaning if the position of two other points is determined, but as line segments from which they had to select a part. So students considered as correct representations, number lines which included spaces beyond the 0 to 1 space and they were representing numbers greater than the given decimal number.

Finally, students faced difficulties in tasks of addition of decimal numbers. This result indicates that addition tasks and especially the addition tasks that included translation procedures presupposed the ability of performing the operation of addition, the ability to represent the adders and the operation on a number line and the ability to manipulate and interpret the information represented by the number line and its features. Consequently, addition tasks might have caused a great cognitive load to the students dealing with them.

The performance of students varies according to the value of the digits that consist the adders. Tasks including adders which exclusively consist of tenths or hundredths seem to be easier than tasks that include adders which consist of both tenths and hundredths.

Number line and Problem Solving

According to the results there were no statistically significant differences between the means of success of students that solved the problems with no restriction to the solution procedure and the students that were asked to use the number line in order to solve the same problems.

Of course the fact that there are no significant differences among the two groups might equally indicate that the number line must have caused more difficulties to the some of the students of the experimental group. It is highly possible that the necessity to manipulate and interpret the number line and its features have been added to the cognitive load of the students and led them to erroneous approaches concerning the problems. In fact the number line is not a simple representation but a geometrical model (Gagatsis et al., 2003) which propose a continuous interchange between geometric and symbolic representation.

Compartmentalization or Modularization of Tasks

According to the results, the tasks of Test A and Test B form isolated groups based on the kind of issue addressed in each task. To be more specific, there are groups mainly consisting of recognition and representation tasks – tasks of Test A. In addition to this, there are groups in which the majority is translation tasks –tasks of Test B. Finally, there is modularization even among the addition tasks according to the number of the digits of the adders –tenths, hundredths. The modularization occurred underlines the fact that students have not developed a unified cognitive structure concerning the concept of decimals since their ideas about decimals seem to appear partial and isolated. More specifically the application of the implicative statistical method demonstrated a compartmentalization of the different tasks and this signifies that there is a lack of coordination between recognition, representation and translation in decimal numbers.

Concerning the last hypothesis, and the observed phenomenon of modularization of tasks, a suggestion should be that future research should look into organizing and conduction intervention programs. These intervention programs should deal with number line as a means of representing decimal numbers. In this way future studies will have the opportunity to investigate possible differences between control groups – students which are not involved in the program – and experimental groups –students involved in programs which focus on the manipulation and interpretation of number line as a means of representing decimal numbers.

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PRESERVICE TEACHERS CONCEPTIONS OF MATHEMATICS-BASED SOFTWARE

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Preservice teachers do not feel prepared to teach mathematics using technology (Smith & Shotsberger, 2001). To address this issue, we have developed a tool-based categorization of mathematics software and used it to instruct preservice teachers. Changes in thinking that occurred as a result of the course are analyzed primarily using repertory grid techniques supported by heuristic questions, reflections and Internet communication group correspondence.

INTRODUCTION

Technology has tremendous potential for enhancing mathematics instruction; it can be used to strengthen student learning and to assist in developing mathematical concepts. Technology can enrich student learning in the areas of richer curricula, enhanced pedagogies and more effective organizational structures (Dede, 2000). However, technology has not reached its potential in preservice teacher instruction; newly graduated teachers often do not have the experience to use computers in the classroom or knowledge about available software (Gunter, 2001).

In a recent study conducted by Smith and Shotsberger (2001), most preservice teachers identified technology as important in mathematics education to assist in the development of concepts but were uncomfortable discussing the specific uses of technology for instruction due to lack of knowledge. Many preservice teachers feel that they are not prepared to teach using technology after they graduate (Carlson & Gooden, 1999). The question that then develops is what kinds of experiences preservice teachers should have with regards to the integration of technology and mathematics.

An effective way to prepare preservice teachers to use technology in mathematics is to prepare them to utilize technology for student use as a tool. A tool can be defined as a cultural artifact that "…predisposes our mind to perceive the world through the 'lens' of the capability of that tool," making it easier or harder to perform certain activities (Brouwer, 1997, p. 190). For example, to solve multi-step algebraic equations, a pencil is a tool that is beneficial in assisting with solving the equation. Use of the pencil allows for the problem to progress, providing a record and visualization of the process. Some technological tools in mathematics are computer programs and software, calculators, and languages (like Logo) (Connell, 1998).

Lajoie (1993) describes the many benefits of using the computer as a tool for instruction in an educational setting. First, technological tools help to support the cognitive processes by reducing the memory load of a student and by encouraging awareness of the problem-solving process. Second, the tools can share the cognitive load by reducing the time that students have to spend on computation. Third, the tools allow students to engage in mathematics that would otherwise be out of their reach, stretching the students' opportunities. And fourth, the technology tools support hypothesis testing by allowing students to easily test conjectures.

We have developed five general categories of software examined with the idea of tool-based use in the mathematics classroom. All of these categories can be used as part of a complete mathematics curriculum with each type of software highlighting a different type of learning.

Review and Practice Tool

When software is used for reinforcement of previously learned material, the software falls into this category. The software is simply used to drill the student on a specific area of mathematics. The student does the same type of mathematics problems in a repeated manner. No new conceptual material is introduced. With review and practice software, the program controls materials, tasks and feedback in a highly directed manner.

Review and practice software is usually designed to be used individually with little teacher intervention. Students solve problems, asking for assistance from peers or the teacher when questions occur. An example of software in this category is *Pre-Algebra Math Blaster Mystery*. This program is designed to reinforce skills already learned in pre-algebra. There is an emphasis on computation, estimation, proportions, ratios and percents.

General Tool

When software is designed for use across a variety of mathematical domains, the software falls under this category. General software is designed for many different applications, the teacher must examine the area of mathematics that the software will be used in and develop lessons that promote the type of learning he or she will focus on. General software can be used for a wide range of grade levels and mathematical subjects.

Geometer's Sketchpad is an example of software designed for general use. It is a dynamic geometry computer program that has gained respect for its potential at assisting students with the possibility of testing conjectures, emphasizing critical thinking and problem solving, through active manipulation of graphical objects and procedural scripts.

Specific Tool

Software designed to emphasize learning in a particular area of mathematics is an example of specific software. The focus with specific software is the learning of a distinct mathematical topic, such as fractions, reflections, order of operations, etc. This differs from the review and practice category in that the focus is on learning new content, not reviewing a specific mathematical concept. *TesselMania* is an example of specific software.

TesselMania, while not a heavily researched application, has promise as a technology that emphasizes transformational geometry, specifically concepts of rotation, translation, and glide reflection. Because it is complex, including combinations of these concepts in the tessellation process, the program can support critical thinking and deep conceptual understanding.

Environments Tool

Software used as an environments tool integrates different types of learning in a variety of subject areas. The software provides an environment that is not normally possible in the classroom, and students make investigations into the given setting. Environments software provide a virtual place for students to guide their mathematical learning, taking students to a new place without requiring them to leave the classroom. Allowing student investigations into problem solving based on mathematical inquiry, sometimes the software is designed for cooperative investigation. The teacher does not present the software to the student rather, the teacher acts as a facilitator, assisting the students as requested and posing inquisitive questions or comments to keep students on track or to clarify.

The Jasper Project (Cognition and Technology Group at Vanderbilt, 1997) is an example of software that can be utilized as an environments tool. The Jasper Project is designed for use with students in grades 5 and up; there are a total of twelve scenarios utilizing real world examples, with an emphasis on either statistics and probability, distance/rate/time, geometry or algebra (CTGV, 1997). The videos include problem-solving environments that promote mathematical thinking through the scaffolding design of the software (Nicaise, 1997).

Communication Tool

Communication software is software that is designed for sharing information between students and another party or parties including the instructor, other teachers, students or professionals (in education or outside of the field). The idea is to increase understanding of mathematical concepts and ability to articulate mathematical arguments and concepts through discourse.

Groupware, videoconferencing, chats, electronic bulletin boards, e-mail and listserves are examples of software that has been developed for use as a communication tool (Jonassen, Howland, Moore & Marra, 1999). Students have an opportunity to go back and look at discussions that took place previously, and reformulate their thoughts if necessary. Groupware, videoconferencing, chats, electronic bulletin boards, e-mail and listserves are examples of software that have been developed for use as a communication tool (Jonassen, Howland, Moore & Marra, 1999).

Tool-based software has been an area of focus in mathematics education for some time. However, there has not been research in the 5 major categories of tools described in this paper. The focus of this paper is to examine changes in thinking, particularly the understanding of the affordances and constraints embodied by the 5 classes of software, by preservice teachers. We expected that examination of these affordances and constraints through experience with, and analysis of, exemplars of the 5 classes of tools would have an affect on the thinking of preservice teachers. In short, this research attempts to identify *how* their thinking will alter as a result of this exposure to be more specific regarding the fit of tool to task, and broader in conceptualization of the kinds of software appropriate for integration into mathematics instruction.

METHOD

Two case studies were investigated: one a preservice mathematics teacher focusing on secondary education, the other a preservice teacher focusing on primary education. Utilizing the methods of Personal Constructs Theory (Kelly, 1955), pre-repertory grids and post-repertory grids were administered to each preservice teacher, classroom observations and transcripts were recorded, and heuristic questions were administered and analyzed in threaded electronic conversations.

Students were enrolled in an upper-division course for preservice teachers designed as an introduction to mathematics-based software. The students met once a week for a 6 weeks, 2 hours per session. Examples from one class of software (see above) was presented each week, with the exception of the communication software which was used throughout the course. After each software experience, students participated in a communication group and completed heuristic evaluations of the applications in an online threaded discussion (Squires, 1997). In each of these discussions, we seeded the conversations by asking students to make distinctions among the different applications they had experienced, regarding the constraints and affordances each offered to mathematics instruction.

To assess change in conceptualization of software tools, we analyzed pre- and post-repertory grids, which asked participants to compare different software they had experienced, and to proffer concepts that distinguished one from another in teaching mathematics. By examining change in both depth of constructs and their organization cognitively, it is possible to determine quantitatively, changes in teachers' conceptualizations due to the intervention of the course. Ward's Method of cluster analysis was applied to constructs in the repertory grids to determine inter-construct distances. Inverse Scree tests were then applied to the agglomeration schedule of different cluster solutions to determine the number of significant versus error clusters (Lathrop & Williams, 1987). Transcripts of observations, reflection responses, communication group discussions and heuristic responses, were used to contextualize and describe the point in which changes in thinking took place in the course. These analyses assisted in the creation of an overall model of how thinking changed in regards the use of technology in the mathematics classroom for the two preservice teachers studied.

RESULTS

Analysis of repertory grids indicated that following instruction, teachers' understanding had broadened to include more defining features, and became more organized with respect to the 5 general classes of mathematics tools presented above. In addition, the ways in which examples of software were categorized changed in organization, indicating that teachers were using a pedagogical lens by the end of instruction. We provide a brief discussion of one participant's constructs organization as an illustration:

Software Categorization

This preservice teacher, Andrea, classified the software she had had experience with in her own instruction into 2 distinct categories at the beginning of the course (Figure 1). She divided productivity software such as PowerPoint and word processing programs into a category and the rest into an "other" category, those softwares that were not solely productivity-oriented. The similarities and distinctions of the non-

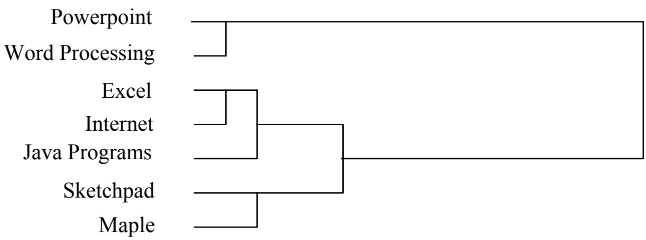

Figure 1: Dendogram of Software Categories, Pre-Instruction

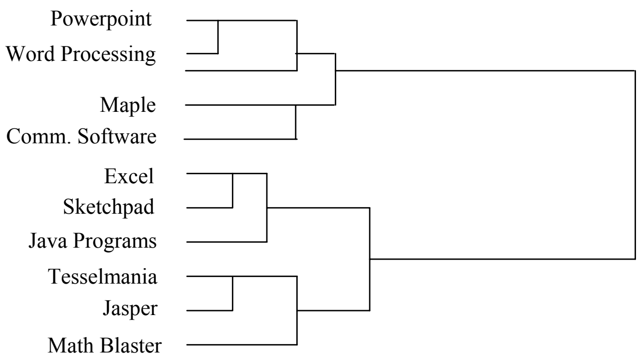

Figure 2: Dendogram of Software Categories, Post-Instruction

presentation types of software are not clearly apparent to the student in this model.

After the completion of the course, the student has made clear distinctions between the types of software mirroring the similarities they have in supporting student learning (Figure 2). She maintains a category that relates to teacher-directed presentation types of activities. However, the second category is more oriented toward tool-based use of software to aid the general mathematical growth of students through the use of open-ended software exploration. The third category pertains to software used to assist the student in a specific mathematical area; it includes software that is supportive of review and practice techniques along with environments and specific softwares.

Constructs Development

After just a brief exposure to the software, this student was able to develop more defined clusters, increasing her constructs from 10 to 27 (Table 1). The constructs also more general in the beginning, referring to what the software can do (for example, add, subtract, make along list of numbers in seconds; you can solve equations and do different kinds of graphs). After the course, the constructs focused more on the support the software offers to student learning and mathematical development (for example, students can explore and discover results; makes students really think about the problem and the answer) including the use of software as a tool to aid in student learning (general tool used to learn many topics).

The elementary teacher had similar results. Her first comparison of software types included two categories: a review and practice category and a category of "others" also. After the course, she developed 4 categories: review and practice, communication software, a general category and a specific category. Her constructs increased from 13 to 35. She created 3 distinct clusters in the beginning, and after the course she still had 3 clusters, expanding the clusters with more defined and developed constructs.

DISCUSSION

Previous research indicates that the majority of preservice teachers believe that technology is important for its own sake (Quinn, 1998). However, university preparation in the field of technology use in the classroom is inadequate (Gunter, 2001; Smith and Shotsberger, 2001). When they graduate, preservice teachers are not ready to teach mathematics with technology, nor are they aware of the possibilities in learning the software can support.

Preservice teachers are ready to learn about the use of technology for use in the mathematics classroom. Exposure to the 5 tool-based categories along with follow up analysis in the form of heuristic questions, communication questions and reflections offer preservice teachers the opportunity to develop a stronger foundation of mathematics-based software knowledge. This base provides the preservice teacher

Clusters	**Pre-Instruction Cluster Constructs**	**Post-Instruction Cluster Constructs**
Cluster 1- Focuses on Algorithms and Topics	More specific topic; You learn how to enter a particular problem in the program; You can solve equations and do different kinds of graphs; Students find it interesting because they get to do something	Gives the results and shows you the steps; Can be used to add, subtract, make a long list of numbers in seconds; You can solve equations and do different kinds of graphs; It has solutions; You learn how to enter a particular problem in the program; Students don't learn much (negative); Used for solving one problem
Cluster 2- Emphasis on Discovery through Activities	Can discover properties by doing activities on the program; There is a picture that you can play with to learn and discover something; Students don't learn much (negative)	Can discover properties by doing activities on the program; Students can explore and discover results; Challenges students; It is done in groups and students can learn from each other; Students can learn from a tutorial that lets students interact with the software; Makes students really think about the problem and answer; Students think at higher levels; Provides different data representations so you have a better chance of reaching all the students; More specific topic; It is a specific tool to help students learn; Students find it interesting because they get to do something; Allows students to be creative; There is a picture that you can play with to learn and discover something; Has more options for students to learn; Gives you a chance to express not only what you learned but if you learned it right; You can reflect upon what you have learned
Cluster 3- Relates to Options within the Software	Provides different data representations so you have a better chance of reaching all of the students; Has more options for students to learn; Can be used to add, subtract, make a long list of numbers in seconds	You can use it for different subjects; There are different ways to approach the problem; General tool used to learn many topics; Graphics and sound overpowers the learning (negative)

Table 1: Cluster Membership of Mathematics Software Constructs

with new knowledge regarding the specific features of software that enable students to learn mathematics, and the fit of those features to particular goals of classroom instruction.

The findings of this study suggest that exposure of the 5 categories of mathematics-based software can lead to positive conceptual change. The thoughts of the preservice teachers became more developed and comprehensive after experiencing and reflecting on the affordances and constraints of tool-based mathematics software. This suggests that with time to experience and reflect, preservice teachers can alter their thoughts concerning the categories and features of software for use in the mathematics classroom.

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HELPING CHILDREN TO MODEL PROPORTIONALLY IN GROUP ARGUMENTATION: OVERCOMING THE 'CONSTANT SUM' ERROR

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We examine eight cases of argumentation in relation to a proportional reasoning task - the 'Paint' task - in which the 'constant sum' strategy was a significant factor. Our analysis of argument follows Toulmin's (1958) approach and in the discourse we trace factors which seem to facilitate changes in argument. We find that the arguments of 'constant sum strategists' develop in the presence of (i) a small group of children with conflicting strategies, (ii) a teacher-researcher who draws attention to the context of the problem as a resource for backing warrants, and (iii) a pictorial tool which can facilitate informal communication.

INTRODUCTION

It has long been 'known' that children's strategies and errors can be used as a starting point for effective mathematics teaching. Bell, Swan, Onslow, Pratt and Purdy (1985) suggest that 'conceptual diagnostic tests' can help teachers become aware of their pupils' strategies. Bell et al. (1985) suggest that after significant errors are identified, they can be resolved through 'conflict discussion'. Williams and Ryan (2001) also complemented research on diagnostic tools with research into arguments in discussion of small groups of pupils.

Accordingly, in this study, we complemented our previous research on diagnostic assessment of children's proportional reasoning (Misailidou and Williams, 2003) with analyses of their argumentation while working on selected ratio tasks. A 'small group collective argumentation' approach was designed and the data were analysed drawing primarily on discourse analysis and sociocultural theories of learning (Gee, 1996). Within this framework the significance of dialogic argument in supporting the group's knowledge-building was recognized (Yackel, Cobb and Wood, 1991 and particularly for proportional reasoning, Pesci, 1998) and also the importance of tools (Hershkowitz and Schwarz, 1999) and the critical role of the teacher in coordinating all the above (Williams and Ryan, 2001).

In this paper we report results on the development of arguments from eight groups working collaboratively on an item called 'Paint'. This item comes from a diagnostic test that we have constructed (Misailidou and Williams, 2003) and it had been found to produce a large number of interesting errors, including the rarely reported 'constant sum' error. A pictorial representation of the item was used as a tool for facilitating discussion because previous analysis had shown it to make an unusually significant impact on the item difficulty (Misailidou and Williams, 2003b).

METHODOLOGY

The approach taken is multiple case study methodology (Yin, 2003) of discourses, in which each discourse is seen as an opportunity for each individual to develop arguments. We analyse each individual's discourse as a sequence of more or less complete arguments, that is, a series of statements relating data, conclusions, warrants and possibly backings: we describe this as their 'discursive path' (see next section for the definition of these terms).

We formed groups of pupils, whose responses to the 'Paint' item (see below) had been different on our previously administered and analysed diagnostic test, thus engineering conflict. Each group consisted of three pupils and was involved in a researcher-guided discussion. The children were set the task of persuading each other by clear explanation and reasonable argument of their answer. The researcher, adopting the teacher's role, established rules for the children's argument in order to facilitate participation in discussion. Moreover, she tried to ensure that the arguments for the errors were clearly voiced and justified. Finally, the children were asked to summarise what they had learnt.

The 'Paint' item was adopted from Tourniaire (1986):

> Sue and Jenny want to paint together.
>
> They want to use each exactly the same colour.
>
> Sue uses 3 cans of yellow paint and 6 cans of red paint.
>
> Jenny uses 7 cans of yellow paint.
>
> How much red paint does Jenny need?

A pictorial representation of the item (shown in Figure 1 together with drawings from the pupils and the researcher) was used as a tool for facilitating discussion.

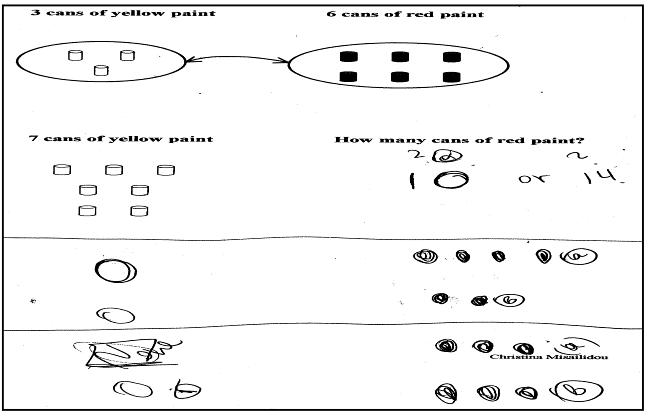

Figure 1: The 'pictures-sheet' used in the group discussions

The ‘Paint’ item provoked the highest frequency of the ‘constant sum’ (Mellar, 1987) error which is the incorrect answer ‘2’. A pupil applying the ‘constant sum’ strategy thinks that the sum of Sue’s cans should be equal to the sum of Jenny’s cans: 3+6=9 therefore 7+2=9. Pesci (1998) also reported the occurrence of this error in ‘colour mixing’ problems but generally it has not been given much attention in the literature. We believe that this error is important because of its very high frequency in certain tasks and because of the fact that a lot of pupils provide ‘adequate’ explanations for a ‘constant sum’ answer (i.e. they justify it by recourse to the context of the problem). Thus in this paper we will focus on the pupils that made this error in test conditions and then participated in the group discussions on ‘Paint’.

DATA ANALYSIS

The pupils’ arguments in each group discussion were recorded and analysed using Toulmin’s (1958) method. The utterances that made up the arguments were classified as ‘data’, ‘conclusions’, ‘warrants’ and ‘backings’. Data are the facts that are requested as a foundation for the conclusion. Warrants are the utterances which demonstrate that ‘taking the data as a starting point, the step to the…conclusion is an appropriate and legitimate one’ (Toulmin 1958, p. 98). Finally, backings are defined as the assurances that strengthen the authority of the warrants. The children's arguments were schematically represented using Toulmin's categories and then coded in order to assess changes in the provided arguments for adopted strategies.

Cobb’s distinction (2002) of children’s discourse about mathematics as ‘calculational’ and ‘conceptual’ was applied and refined to the context of proportional reasoning. Four categories were found relevant to classify pupils’ explanations in order to examine the development of their reasoning:

1. Numerical explanation: The pupil explains a numerical performance for finding an answer without justifying it contextually.

2. ‘Adequate’ explanation but non-multiplicative conceptualisation of the task: The explanation is defined as ‘adequate’ when the pupil connects numerical performances to contextual data. The pupil’s conceptualisation of the task context is defined as ‘non-multiplicative’ when it prohibits the construction of proportional relations.

3. ‘Adequate’ explanation and pre-multiplicative conceptualisation: The pupils have conceptualised the task ‘pre-multiplicatively’ when they can think relationally about quantities (for example more red paint than yellow is needed for a dark shade of orange), but not yet proportionally. (This is a slight variation of Resnick and Singer’s (1993) ‘protoquantitative relational reasoning’)

4. ‘Adequate’ explanation as a result of a multiplicative conceptualisation.

Pupils are not considered to *reason* proportionally if they just give an answer which is the result of a multiplicative calculational process, but rather when, additionally, their argumentation indicates multiplicative backing: i.e. we require evidence in the actual discourse and do not infer reasoning which is not stated. Hence, Category 4 is considered to be an indication of multiplicative reasoning. Category 3 is considered equally important since it is hypothesized that it can lead to Category 4 with appropriate teaching interventions. (Resnick and Singer, 1993)

'Discursive paths' for each of the pupils that took part in the group discussions were composed and studied. We define 'discursive path' as the evolution of the pupil's argumentation in the discussion. By combining discursive paths across group discussions we generalise patterns of changing arguments that we think of as 'changes of mind' (really they are changes of talk).

RESULTS

We present results from the eight groups that discussed the item 'Paint'. An example of a typical group discussion is given first and then the findings from all the groups are summarised. Our focus is the discursive paths of the group members that provided the answer '2' to the item in previous testing.

Example of a typical group discussion

Jane, Ann and Arpita (11 years old) were selected from the same class to form a discussion group because in previous testing they had provided three different responses to the 'Paint' item: '2' ('constant sum' strategy), '10' ('additive' strategy: 3+3=6 so 7+3=10) and '14' ('multiplicative' strategy: 2x3=6 so 2x7=14). The discussion, with the guidance of one of the authors, lasted 30 minutes and was audio taped. We will focus on Jane's discursive path through the discussion.

Initially, all the pupils recalled their response to the test and were invited to present an argument for it. Jane explained her answer '2'as: '6 add 3 is 9. That's how many paint cans Sue's got. And 7 add 2 equals 9. So, they should have exactly the same amount of paint. So, Jenny needs 2 cans of red paint.'

Her argumentation is represented using Toulmin's terminology in Figure 2. It is also labelled according to the categories mentioned above.

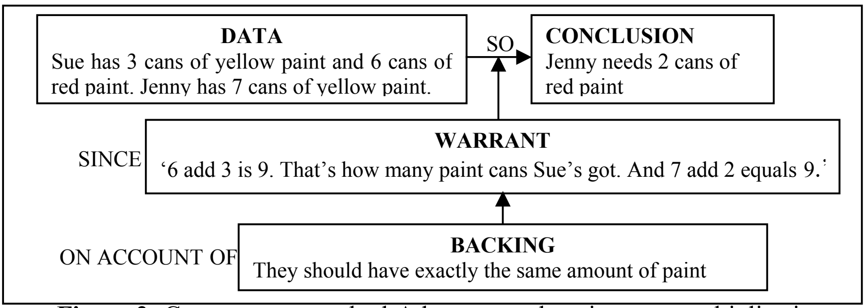

Figure 2: Constant sum method-Adequate explanation, non-multiplicative

After each pupil had presented their method, the researcher asked for reflection on all the methods: Jane insisted on her 'constant sum' method and influenced Arpita to adopt it. Then, in order to sustain and enrich the discussion, the researcher distributed the 'pictures-sheet' that is shown in Figure 1 as a tool that would facilitate discussion. From then on, she demonstrated throughout the discussion the use of the 'pictures-sheet': verbally (she referred, whenever possible, to the 'pictures-sheet') with her gestures (she pointed to it) and with her actions (she drew on it).

With the help of the 'pictures-sheet' a part of the discussion that was coded as 'exploration of the context' was initiated. This part was completed in three steps:

Step 1: The group realized that there is an essential condition that affects the answer to the problem. After the researcher's prompt, Ann discovered the significant sentence: 'They want to use exactly the same colour'.

Step 2: Prompted by the researcher, the group searched the meaning of the above condition: They established the difference between the 'same colour' and the 'same amount of cans'.

Step 3A: The pupils focused on essential contextual elements such as the resulting shades by using different answers to the problem. At this point Ann realized that the answer '2' was not reasonable:

> 'It's not exactly the same. *[She points with her pencil on the 'pictures-sheet' at Sue's cans]* Sue has more red paint on the first one *[points at Sue's red cans on the 'pictures-sheet']*…and then there's more yellow on that one *[she points with her pencil on the 'pictures-sheet' on Jenny's cans]*.

Step3B: The pupils clarified that the resulting (from the mixing of paints) colour was orange.

Following the 'exploration of the context' Jane changed her mind and declared that '2' was not an appropriate answer. Her argumentation is presented in Figure 3.

DATA
Sue has 3 cans of yellow paint and 6 cans of red paint. Jenny has 7 cans of yellow paint.

SO

CONCLUSION
Jenny should not use 2 cans of red paint.

SINCE

WARRANT
[If Jenny uses 2 cans of red paint] 'It would be darker there. *[pointing at her pictures sheet]* Sue's would be darker.'

ON ACCOUNT OF

BACKING
She's got more red paint *[than yellow. Jane points at Sue's red cans and then at Sue's yellow cans on the 'pictures-sheet']*.

Figure 3: Rejection of 'constant sum'-Adequate explanation, pre-multiplicative

It is hypothesized that the essential elements that made Jane change her mind were:

1. The introduction of the 'pictures-sheet' and its use by the group and the researcher,

2. The 'exploration of the context' part of the discussion and

3. Ann's argument about rejecting the answer '2'

After all of them had rejected the answer '2', they seemed confused whether to accept '10' or '14'. So the researcher introduced a new 'extreme case' to the problem by drawing on the 'pictures-sheet':

> 'Let me try…another person, Maria. *[Draws 3 cans of red paint on her 'pictures-sheet']*. She has 3 cans of red paint. OK? With this method *[the multiplicative method]*…how much yellow paint does Maria need?'

This is an extreme case because if the additive (instead of the multiplicative) method is used, the resulting answer is '0', so the resulting colour is red instead of orange. Thus the additive method does not work for this case.

After the group had found the answers for both methods and drawn them on the 'pictures-sheet' it seemed natural for them to reject the additive method and adopt the multiplicative one after offering adequate explanations for their choice. Jane's argumentation for finally accepting the answer '14' (the 'third stage' of her discursive path) is not of interest for this paper. We just provide her backing, which indicates a multiplicative conceptualisation of the task at the end of the discussion: 'With doubling…we will make exactly the same shade of orange [for every case].'

In summary, Jane's participation in the discussion group seem to have facilitated her way towards proportional reasoning as indicated by her 'gradually multiplicative' argumentation: 'Constant sum' method and non-multiplicative conceptualisation → Rejection of 'constant sum' method and pre-multiplicative conceptualisation → Multiplicative method and multiplicative conceptualisation.

It is hypothesized that the critical factors that lead to her 'change of mind' were: *(a)* the group's (and hers) gradually focused discourse on the context of the problem *(b)* the enrichment of her (and the group's) means of communication by incorporating the 'pictures-sheet' in her discourse (e.g. Figure 3) and by drawing on it and *(c)* the researcher's introduction of an 'extreme case' to the task.

Combination of all the group discussions

The 'Paint' item was discussed by eight groups, each consisting of three pupils (aged 11 or 12). Eight of these pupils (one in each group) made a 'constant sum' error when tested (prior to the discussions) in the 'Paint' item but all of them 'changed their mind' during the discussions as indicated by their discourse. By combining the basic stages from the discursive paths of each of these eight pupils, a general pattern emerged presented in Table 1.

The general pattern in Table 1 is consistent with Jane's 'three-stage' discursive path and highlights the two basic factors that provide a 'remedy' for the 'constant sum' method: *(a)* the exploration of the context and *(b)* the various uses of the 'pictures-sheet'. Furthermore, a third factor that conflicts with 'additivity' and influences the pupils towards a multiplicative approach is: *(c)* the introduction of an 'extreme case' by the researcher.

Stage	Decision	Explanations	Change in discourse followed:	**Task: PAINT**
1	'2' (8 cases)	Adequate, non-multiplicative (7 cases) Numerical, non-multiplicative (1 case)		**Changing from the Constant Sum Error to an 'adequate strategy'**
2	Reject '2' (8 cases)	Adequate, pre-multiplicative (8 cases)	1. (8 cases) The introduction of the 'pictures-sheet' by the researcher and its use, in various ways by the whole group, and then 2. (6 cases) The 'exploration of the context' part of the discussion, and then 3. (3 cases) Argumentation from a peer about rejecting the answer '2'	
3	Reject'10' and adopt '14' (5 cases)	Adequate - multiplicative (5 cases)	1. (5 cases) The introduction by the researcher with the help of the 'pictures-sheet' of an extreme case to the task, and then 2. (3 cases) Argumentation from a peer	**Conversion towards a multiplicative method**

Table 1: Changing from the Constant Sum Error to an 'adequate strategy'.

CONCLUSION

We conclude that pupils who use the 'constant sum' method for solving certain ratio tasks could learn to reason more adequately through discussion in small 'conflict' groups of children: i.e. groups with conflicting initial responses on the given tasks. Learning, (conceived here as a short term, and perhaps temporary, change of argument or talk in discussion) however, can occur under conditions:

1. The context of the task must be challenging to provoke pupils to negotiate the task contextually first and then seek a solution. It is hypothesized that with this kind of 'contextual discussion' pupils can learn to conceptualise a task multiplicatively or, in other words, to 'model proportionally'. The context of our 'Paint' problem (that provokes various interpretations of the 'way paint is used') has proven to be exceptionally challenging and effective.

2. A 'tool' should be provided that *(a)* makes the context of the task more prominent *(b)* provides a shared means of communication for pupils and *(c)* facilitates their expressions and arguments. This last element is particularly important for young children who do not yet posses adequate mathematical terminology to communicate their thoughts and strategies. The pictorial representation shown in Figure 1 meets these requirements. (Additionally, it affords the crucial argument about (0, 3) that gives an unacceptable red colour instead of orange and brings an apparently successful 'change of mind' from additive to multiplicative arguments.)

Acknowledgement

We gratefully acknowledge the financial support of the Economic and Social Research Council (ESRC), Award Number R42200034284

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ABSTRACTION IN MATHEMATICS AND MATHEMATICS LEARNING

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It is claimed that, since mathematics is essentially a self-contained system, mathematical objects may best be described as abstract-apart*. On the other hand, fundamental mathematical ideas are closely related to the real world and their learning involves empirical concepts. These concepts may be called* abstract-general *because they embody general properties of the real world. A discussion of the relationship between abstract-apart objects and abstract-general concepts leads to the conclusion that a key component in learning about fundamental mathematical objects is the* formalisation *of empirical concepts. A model of the relationship between mathematics and mathematics learning is presented which also includes more advanced mathematical objects.*

This paper was largely stimulated by the Research Forum on abstraction held at the 26th international conference of PME. In the following, a notation like [F105] will indicate page 105 of the Forum report (Boero et al., 2002).

At the Forum, Gray and Tall; Schwarz, Hershkowitz, and Dreyfus; and Gravemeier presented three theories of abstraction, and Sierpinska and Boero reacted. Our analysis indicates that two different contexts for abstraction were discussed at the Forum: abstraction in mathematics and abstraction in mathematics learning. However, the Forum did not include a further meaning of abstraction which we believe is important in the learning of mathematics: The formation of concepts by empirical abstraction from physical and social experience. We shall argue that fundamental mathematical ideas are formalisations of such concepts.

The aim of this paper is to contrast abstraction in mathematics with empirical abstraction in mathematics learning. In particular, we want to clarify "the relation between mathematical objects [and] thinking processes" (Boero, [F138]).

ABSTRACTION IN MATHEMATICS

What does it mean to say that mathematics is "abstract"?

Mathematics is a self-contained system separated from the physical and social world:

- Mathematics uses everyday words, but their meaning is defined precisely in relation to other mathematical terms and not by their everyday meaning. Even the syntax of mathematical argument is different from the syntax of everyday language and is again quite precisely defined.

- Mathematics contains objects which are unique to itself. For example, although everyday language occasionally uses symbols like x and P, objects like x^0 and $\sqrt{(-1)}$ are unknown outside mathematics.
- A large part of mathematics consists of rules for operating on mathematical objects and relationships. Sierpinska calls these "the rules of the game" [F132]. It is important that students learn to manipulate symbols using these rules and no others.

We claim that the essence of abstraction in mathematics is that mathematics is self-contained: An abstract mathematical object takes its meaning only from the system within which it is defined. Certainly abstraction in mathematics—at all levels—includes ignoring certain features and highlighting others, as Sierpinska [F130] emphasises. But it is crucial that the new objects be related to each other in a consistent system which can be operated on without reference to their previous meaning. Thus, self-containment is paramount.

Historically, mathematics has seen an increasing use of axiomatics, especially over the last two centuries. For example, numbers were initially mathematical objects based on the empirical idea of quantity. Then mathematicians such as Dedekind and Peano reconceptualised numbers in axiom systems which were independent of the idea of quantity. Euclid, Hilbert, and others performed a similar task for geometry. But, as Kleiner (1991) states, "whereas Euclid's axioms are idealizations of a concrete physical reality … in the modern view axioms are … simply assumptions about the relations among the undefined terms of the axiomatic system" (p. 303). In other words, mathematics has become increasingly independent of experience, therefore more self-contained and hence more abstract.

To emphasise the special meaning of abstraction in mathematics, we shall say that mathematical objects are *abstract-apart.* Their meanings are defined within the world of mathematics, and they exist quite apart from any external reference.

So why is mathematics so useful?

Mathematics is used in predicting and controlling real objects and events, from calculating a shopping bill to sending rockets to Mars. How can an abstract-apart science be so practically useful?

One aspect of the usefulness of mathematics is the facility with which calculations can be made: You do not need to exchange coins to calculate your shopping bill, and you can simulate a rocket journey without ever firing one. Increasingly powerful mathematical theories (not to mention the computer) have led to steady gains in efficiency and reliability.

But calculational facility would be useless if the results did not predict reality. Predictions are successful to the extent that mathematics models appropriate aspects of reality, and whether they are appropriate can be validated by experience. In fact, one can go further and claim that the mathematics we know today has been developed (in preference to any other that might be imaginable) because it does model significant aspects of reality faithfully. As Devlin (1994) puts it:

> How is it that the axiomatic method has been so successful in this way? The answer is, in large part, because the axioms do indeed capture meaningful and correct patterns. … There is nothing to prevent anyone from writing down some arbitrary list of postulates and proceeding to prove theorems from them. But the chance of those theorems having any practical application [is] slim indeed. (pp. 54-55)

Many fundamental mathematical objects (especially the more elementary ones, such as numbers and their operations) clearly model reality. Later developments (such as combinatorics and differential equations) are built on these fundamental ideas and so also reflect reality—even if indirectly. Hence all mathematics has some link back to reality.

EMPIRICAL ABSTRACTION IN MATHEMATICS LEARNING

Learning fundamental mathematical ideas

Students learn about many fundamental, abstract mathematical objects in school. In this section, we discuss the meaning of abstraction in this learning context. We begin by looking at some examples.

Addition. Between the ages of 3 and 6, most children learn that a given set of objects contains a fixed number of objects. A little later, they realise that two sets can be combined and that the number of objects in the combined set can be determined from the number of objects in each set—a procedure which later becomes the operation of addition. Students learn these fundamental arithmetical ideas from counting experiences: They find that repeatedly counting a given set of objects always gives the same number, no matter how often it is done and in which order. As they recognise more and more patterns, counting a combined set is gradually replaced by "counting on" and eventually the use of "number facts" (Steffe, von Glasersfeld, Richards, & Cobb, 1983).

Angles. There is good evidence that, at the beginning of elementary school, students have already formed classes of angle situations such as corners, slopes, and turns (Mitchelmore, 1997). To acquire a general concept of angle, students need to see the similarities between them and identify their essential common features (two lines meeting at a point, with some significance to their angular deviation). Even secondary students find it difficult to identify angles in slopes and turns, where one or both arms of the angle have to be imagined or remembered (Mitchelmore & White, 2000).

Rate of change. The most fundamental idea in calculus is *rate of change*, leading to differentiation. A major reform movement over the last decade or so has been concerned with making this idea more meaningful by initially exploring a range of realistic rate of change situations. In this way, students build up an intuitive idea of rate of change before studying the topic abstractly. A leading US college textbook (Hughes-Hallett et al., 1994) devotes a whole introductory chapter to exploring realistic situations, and in Australia similar materials have been published for high school calculus students (Barnes, 1992).

Characteristics of empirical abstraction

The above examples show how fundamental mathematical ideas are based on the investigation of real world situations and the identification of their key common features. Hence, a characteristic of the learning of fundamental mathematical ideas is *similarity recognition.* The similarity is not in terms of superficial appearances but in underlying structure—for example, in counting, space, and relationships. To get below the surface often requires a new viewpoint, as when a student imposes imaginary initial and final lines on a turning object in order to obtain an angle.

There is a leap forward when students recognise such a similarity: As students relate together situations which were previously conceived as disconnected, they become able to do things they were not able to do before. More than that, they form new ideas (such as addition, angle, and rate of change) and are incapable of reverting to their previous state of innocence. In a sense, these new ideas *embody* the similarities recognised. Of course, single ideas rarely evolve in isolation; for example, the idea of angle is inextricable linked to ideas such as point, line, parallel, intersection and measurement which can also be traced to similarities students recognise in their environment.

This process of similarity recognition followed by embodiment of the similarity in a new idea is an *empirical abstraction* process. It is well described by Skemp (1986):

> *Abstracting* is an activity by which we become aware of similarities ... among our experiences. *Classifying* means collecting together our experiences on the basis of these similarities. An *abstraction* is some kind of lasting change, the result of abstracting, which enables us to recognise new experiences as having the similarities of an already formed class. ... To distinguish between abstracting as an activity and abstraction as its end-product, we shall ... call the latter a *concept.* (p. 21, italics in original)

Thus number, addition, angle and rate of change are all empirical concepts, and they take their place in students' learning alongside other empirical concepts such as colour, friend, and fairness.

Piaget (1977) made a distinction between abstraction on the basis of superficial characteristics of physical objects (*abstraction à partir de l'objet*) and abstraction on the basis of relationships perceived when the learner manipulates these objects (*abstraction à partir de l'action*). But both are based on the child's physical and social experience, and in both similarity recognition is essential. In using the term *empirical abstraction* to cover both cases, we are making the distinction between abstraction on the basis of experience and what we shall call *theoretical abstraction* (see below).

EMPIRICAL ABSTRACTION AND MATHEMATICAL ABSTRACTION

From empirical concept to mathematical object

When students learn a fundamental mathematical idea in the way described above, three things happen: They learn an empirical concept, they learn about a mathematical object, and they learn about the relationship between the empirical concept and the mathematical object. Empirical concepts are often rather fuzzy and difficult to define. For example, the empirical concept of circle is that of a perfectly round object—but "perfect roundness" can only be defined by showing examples. A circle becomes a mathematical object only when it is defined as the locus of points equidistant from a fixed point: It is then clearly defined in terms of other mathematical objects. However, for this definition to be meaningful, an individual must see that the locus of points equidistant from a fixed point gives a perfectly round object and vice versa.

We have already referred to mathematical objects as abstract-apart. To emphasise the distinction between abstraction in mathematics and mathematics learning, we shall call empirical concepts *abstract-general*: Each concept embodies that which is general to the objects from which the similarity is abstracted.

Gravemeier also focuses on how "formal mathematics grows out of the mathematical activity of the students" [F125], calling the process *emergent modelling.* The Realistic Mathematics Education movement, to which Gravemeier belongs, has previously called it *vertical mathematisation* (Treffers, 1987). We prefer to call this process *formalisation*, since its main purpose is to select abstract-apart relationships which capture the *form* of an abstract-general concept. (So "formal mathematics" is the study of mathematical forms.) For example, the locus definition of a mathematical circle precisely expresses the perfect roundness of an empirical circle.

Linking mathematical objects to empirical concepts

There is strong evidence that many student difficulties in learning mathematics can be traced to the fact that, when they learned about an abstract-apart mathematical object, they made no link to the corresponding abstract-general concept (Mitchelmore & White, 1995). Consider again the previous three examples.

Addition. Many young students experience difficulty learning elementary arithmetic. One explanation is that they do not understand the empirical meaning of the operations: Symbols such as + and × are learned apart from the abstract-general concepts of addition and multiplication on which they are based. Early number research (Steffe et al., 1983; Wright, 1994) has led to projects such as *Count Me In Too* which have closely linked early arithmetic to students' counting experiences, with a measurable improvement in learning (Mitchelmore & White, 2003).

Angle. Many student difficulties with angles arise because the angle diagram is abstract-apart. Williams (2003) gives a particularly extreme example: Her case-study secondary school student successfully made a generalisation about the angle sum of a polygon, but he could not identify the angles of the triangles into which he had divided the polygon. In fact, it is quite possible to teach an abstract-general concept of angle as early as Grade 3, as White & Mitchelmore (2003) have shown.

Calculus. Calculus instruction based on abstract-apart differentiation leads to a *manipulation focus* (White & Mitchelmore, 1996). Students do not see symbols as representing anything, so they cannot use the manipulative techniques they have learned to solve contextual problems. Their concept of differentiation has been truly decontextualised and therefore impoverished, instead of being abstract-general and rich (Van Oers, 2001).

The preceding discussion emphasises the value of making a clear distinction between empirical concepts and mathematical objects.

MORE ADVANCED MATHEMATICS LEARNING

The learning of fundamental mathematical ideas is only one component of learning mathematics: More advanced ideas need to be developed out of the fundamental ideas. Some of these ideas (such as square roots) can be readily linked back to abstract-general concepts; others (such as a zero exponent) seem to have no counterpart in normal experience. In addition, students need to learn to operate within an abstract-apart system—an aspect of mathematics learning which takes on increasing significance in university mathematics as the links to experience become thinner and thinner. But even professional mathematicians use empirical concepts as an aid to intuition (Boero, [F137]).

The formation of new ideas within mathematics is well described by the Schwarz-Hershkowitz-Dreyfus *Nested RBC Model of Abstraction*. They define abstraction as "an activity of vertically reorganizing previously constructed mathematics into a new mathematical structure" [F121]. New mathematical objects are constructed by "the establishment of connections, such as inventing a mathematical generalization, proof, or a new strategy of solving a problem" [F121]. This abstraction process is quite different from empirical abstraction, and is best described as *theoretical abstraction.* Sierpinska's ignoring/highlighting process is another example of theoretical abstraction.

Gray & Tall's idea of a *procept*—"the amalgam of three components: a *process* which produces a mathematical *object,* and a *symbol* which is used to represent either process or object" [F117]—also clarifies the development of ideas within mathematics. The construction of a procept seems to us, however, to be more akin to formalisation than abstraction.

Historically, some more advanced mathematical objects have been constructed by a process similar to empirical abstraction. An example is group theory:

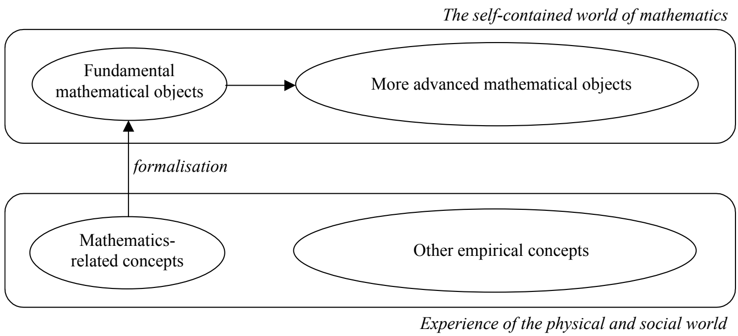
> The abstract concept of a group arose from different sources. Thus polynomial theory gave rise to groups of permutations, number theory to groups of numbers and of "forms" … and geometry and analysis to groups of transformations. Common features of these

> concrete examples of groups began to be noted, and this resulted in the emergence of the abstract concept of a group in the last decades of the 19th century. (Kleiner, 1991, p. 302)

Other examples are rings, fields and vector spaces. Our arguments above would suggest that the learning of such mathematics would be most effective if it were based on a process of similarity recognition followed by formalisation.

SUMMARY AND CONCLUSION

The term *abstraction* has different meanings in relation to mathematics and the learning of mathematics. Previous abstraction theorists have tended to focus on the process of developing ideas within mathematics. In this paper, we have tried to redress the balance by exploring the role of empirical abstraction in the formation of fundamental mathematical ideas. This is a crucial process, since many fundamental, abstract-apart mathematical objects need to be linked to abstract-general empirical concepts if their learning is to be meaningful.



Grossly over-simplified, we see the whole picture as follows:

In practice, the formation of mathematics-related empirical concepts and their formalisation into mathematical objects may occur simultaneously—especially in school learning. Also, more advanced mathematical objects may be linked directly to empirical concepts and not only indirectly via fundamental objects.

Like Boero [F138], we believe that “we are still far from a comprehensive theoretical answer to the challenge of mathematical abstraction in mathematics education”. A clear response to this challenge would be of great value to researchers and teachers alike. Examining and differentiating the different forms of abstraction involved in learning mathematics constitute one step along the path to this goal.

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REFLECTIVE SYMMETRY IN CONSTRUCTION AND PROVING

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The aim of this research is to advance understanding of how mathematical knowledge functions in the proving in geometry. We focus on the rules whose mobilization is due rather to the mathematical knowledge at stake than to the proof. We observed students who are asked to solve construction problem and proving problem. The problems require to mobilize the "same" rule from the theoretical point of view. The first result is that even if the students can construct correctly a symmetric and are conscious of the necessity of perpendicular and equal distance, it doesn't mean that they can use the rule appropriate in proving.

INTRODUCTION

What relationship does it exists between the knowledge about a specific mathematical notion and the nature of proof? There would be a strong relationship, as many researches report the role of proof for mathematical understanding (cf. Hanna, 2000). Our research interest is, in a sense, inversive: try to make clear not the role of proof for knowledge but the role of knowledge for proving. In this paper we focus on the state of knowledge in geometrical construction and proving and propose an explicitation for their relationship or differences.

We have selected a precise domain in geometry, the reflective symmetry[1]. The reason for this choice is first that many researches have been done on symmetry, especially for the construction and recognition problems in the paper-pencil environment as well as computer-based (Küchemann, 1981; Grenier, 1989; Bell, 1993; Hoyles & Healy, 1997; etc.). It will facilitate the analysis of students' knowing of reflection. The second reason is that the reflection is rarely involved in proving problems in the class. This means that the students of our observation should rely on and mobilize the knowledge acquired in non-proving context.

THEORETICAL FRAMEWORK AND RESEARCH METHOD

The structure of proof or reasoning can be expressed by the triad: given statement, rule of inference, and conclusion (Duval, 1991; Figure 1). As the rule of inference connects two statements, it can be expressed in the form of an implication "If A then B".

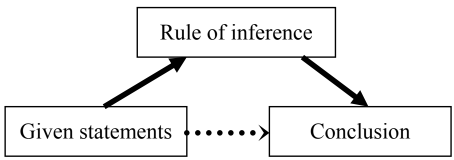


Figure 1 Structure of reasoning by Duval

So, a proof will be the combination of some triads like in a graph. Of course, some of these triads – the obvious ones in the eyes of the readers – might not be written in the proof. A model developed by Toulmin (1958) to analyze argumentation also has a similar structure. Referring to this structure, we consider that the knowledge involved in reasoning comes within the rule of inference, because statements cannot be connected without it.

We also consider the hypothesis that people do not have rules as a long and huge list, in other words, rules are also based on something which support them. Toulmin (1958) call it "backing". In the words of cK¢ model developed by Balacheff and his research group (Balacheff & Gaudin, 2003; Balacheff, 2000; etc.), this is a conception or instantiated knowledge. We use, as a tool of analysis, the cK¢ model which gives us certain points of view for analyzing not only the proof but also the students' products and behavior from an epistemic point of view. In this model a conception is characterized by the quadruplet (P, R, L, Σ): the set of problems on which the considered knowledge is efficient, the set of operators involved in the problem solving activity, the system of representation used (i.e. semiotic system), and a control structure. We pick up two aspects from this description of a conception: operators which can correspond to the rule of inference in the proof structure and control structure which is behind the decision to use operators and validates them. We call in this paper "rule" what can be expressed as "if A then B" which would be mobilized as a rule of inference or operator.

The questions concerning the rules have been studied, with the term "conditionality of the statements", by the Italian research group (Boero et al., 1996; 1999; etc.). They analyzed the process of its generation, and its link to proving process. Our study would be situated to a more basic problematic with the notion of symmetry, that is, diagnosis of students' state of knowledge from the rule point of view. The questions posed are:

- What rules may exist for the symmetry?
- Which rules may be mobilized in the construction and proof problems? Do they have the same nature?

To reply to these questions, we take the following process. First, we give a theoretical analysis of rules involved in the use of symmetry. Second, we analyze the construction process of symmetric to identify rules which may be mobilized. Based on these results, we propose a proving problem which requires the same rule as construction, and organize an observation experiment for the 9th grade students. We present the data collected in the observation and a case study with specific students.

ANALYSIS OF RULES FOR THE REFLECTIVE SYMMETRY

We analyze the possible rules generated for the symmetry. From this analysis we construct a framework for analyzing the data obtained in the observation. Three types of rules concerning symmetry can be formalized as a form of implication with respect to the relationship between symmetry and geometric properties or objects.

Type 1. (symmetry $\Rightarrow$ property) rule connecting symmetry to an geometric property:

R_1: If $P_2Q_2 = \mathrm{Sym}(P_1Q_2, d)$ then $P_1Q_1 = P_2Q_2$[2]

R_2: If $P_2 = \mathrm{Sym}(P_1, d)$, then $P_1P_2 \perp d$

R_3: If $P_2 = \mathrm{Sym}(P_1, d)$, then $P_1M = MP_2$ ($M \in d$);

Type 2. (properties $\Rightarrow$ symmetry) rule connecting some geometrical properties to symmetry. This is a rule which characterizes the symmetry:

R_4: If $P_1P_2 \perp d$ and $P_1M = MP_2$, then $P_2 = Sym(P_1, d)$ ($M = d \cap P_1P_2$)

R_5: If $P_1M = MP_2$ and $P_1N = NP_2$, then $P_2 = Sym(P_1, d)$ ($M \neq N \in d$);

Type 3. (symmetry $\Rightarrow$ symmetry) rule expressing symmetrical relationship:

R_6: If $P_2 = Sym(P_1, d)$ and $P_4 = Sym(P_3, d)$, then $P_2P_4 = Sym(P_1P_3, d)$.

R_7: If $P_2P_4 = Sym(P_1P_3, d)$, then $P_2 = Sym(P_1, d)$ and $P_4 = Sym(P_3, d)$.

The above list of rules is not exhaustive at all. These are not always of the form "if A then B" when being mobilized. But we try to identify them in the resolution process and students' statement collected in the protocol. For example, when one says a property of the symmetry such as "two symmetric segments have the same length", we understand that the rule "R_1: If $P_2Q_2 = Sym(P_1Q_2, d)$ then $P_1Q_1 = P_2Q_2$" is used.

CONSTRUCTION FROM RULE POINT OF VIEW

We analyze here which type of rules are mobilized in the correct construction process of symmetry. This analysis makes clear the relationship between construction and proof from the rule point of view. In fact, we will be able to make a proving problem with a rule needed in the construction.

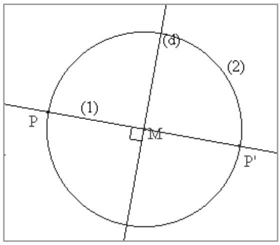


Figure 2

The usual construction for a symmetric point would have the procedures like Figures 2 and 3. These two procedures can usually be found in the school textbook. We analyse these cases. The first one is to draw first (1) the perpendicular line PM to the axis *d* through a given point P, and (2) take the distance PM to the "other side". For the perpendicular line, the triangular or compass and for the equal distance, the compass or ruler with graduation can be used as instruments. The operators mobilized in this construction are the actual construction procedures, (1) and (2), which give a point P' on the paper. They would be based on "R_4: If $P_1P_2 \perp d$ and $P_1M = MP_2$, then $P_2 = Sym(P_1, d)$". In fact, mathematically, the perpendicular line PM is drawn and the point P' which is at the equal distance as PM is taken, since these two geometrical properties give a symmetric point.

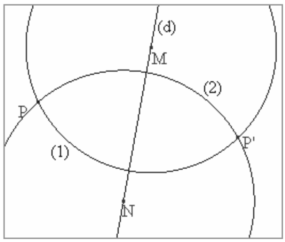


Figure 3

The second procedure (Figure 3) is to draw two circles whose centers are on the axis, and the intersection P' is symmetric of P. It is based on "R_5: If $P_1M = MP_2$ and $P_1N = NP_2$, then $P_2 = Sym(P_1, d)$". In fact, two circles are drawn to get a set of equidistant points from two points on the axis, and two conditions are enough to determine one point on the plane, so two equal distances gives a symmetric point.

From this analysis, we can find that the type of rules needed for the construction is type 2 (properties $\Rightarrow$ symmetry) which characterizes a symmetric point. We also find that in the construction, the rule is not operational but predicative. In fact, it's not rule but actual action that draws a symmetric on the paper. In the words of cK¢ model, they are used not as an operator but as a control. So, two aspects are possible for a same symmetric "element".

PROBLEMS IN THE OBSERVATION

We organized an observation to see how students solve construction and proving problems which require a same rule, type 2, R_4. We present here two of four problems given in the observation.

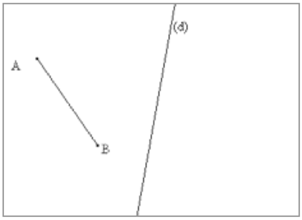
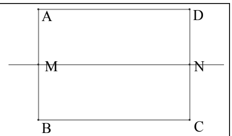


Figure 4

Problem 1 is usual one. We ask to draw the symmetric segment of AB with any usual instruments (ruler with graduation or not, compass, triangular, protractor, etc.). Any instruments can be used, because we didn't want to give restriction for constructing other geometric objects than symmetric. What we want to know is the geometric properties mobilized for the construction of symmetry, but not the techniques on the instrument, nor the mobilized properties for the construction of perpendicular line or equal distance. For the given figure, we took into account some didactical variables identified by Grenier (1987), such as the slope of axis, the direction of segment with respect to the axis, etc., in order to appear the incorrect procedure. As demonstrated by the analysis presented in the previous section, to construct the symmetric point of A, one of type 2 rules should be used. And it's same for B. After drawing the symmetric points A' and B', it needs another rule which allows to draw A'B' as a symmetric segment of AB. It would be a type 3 rule "R_6: If P_2 = Sym (P_1, d) and P_4 = Sym (P_3, d), then P_2P_4 = Sym (P_1P_3, d)".

Problem 2 is to recognize two symmetrical segments and to prove it. Since this is a proving problem for the characterization of symmetry, the type of rules which should be mobilized is type 2, as in the problem 1. In fact, to prove that the segments AD and BC are symmetric, it is needed at first that extremities of segments are symmetric, such as "AM = MB and AB ⊥ MN, so A and B are symmetric". After the proof for extremities, the proof of the symmetry of segments requires the R_6 rule.

Problem 2: *The quadrilateral ABCD is a rectangle. Let M, N, be the midpoints of opposite sides AB et DC. Are the segments AD et BC symmetric with respect to the line MN? Reply with "yes", "no", or "not always", and prove your answer.*



We have observed 25 students of 9th grade (aged almost 14 years) in France: we had 11 pairs, and only one group of three who are asked to work together and give one answer. This observation method aims at expliciting students' conception as far as possible. The observers of each pair noted students' behaviors.

In France, proof learning begins progressively from explanation or justification at the 6th grade and is taught mainly in relation to deductive reasoning at the 8th grade in geometry. The notion of symmetry is introduced at 6th grade with construction. After 6th grade, it is often used as a tool to analyze other geometric objects. The reflection is not often used in the proving context.

THE DATA FROM THE OBSERVATION

We give at first the data of students' answers. For the problem 1, most pairs give a correct construction with a triangular, compass, ruler with graduation or not. In the table below, the pairs using triangular drew perpendicular line to the axis *d*. The compass is used for the equal distance, while the pair 10 uses ruler with graduation for it. Only the pair 2 gives an incorrect construction. The line drawn is not perceptively perpendicular to the axis *d* and the instrument used does not have perpendicular. For the problem 2, all pairs give a correct answer "Yes", two segments are symmetry. But the pair who gives a correct proof is only the pair 9.

As we analyzed it in the previous section, the rules which should be used are same. It seems that if the students use the triangular and the compass for the construction of symmetric point, they use the type 2 rule R_4. However, our data shows that most students who gave a correct construction do not give a correct proof, even if the same rule is needed. The questions posed now are "What is the difference?", "Why can't they give a correct proof?", and especially concerning the type 2 rule "Do they really have the type 2 rules?". We try to answer these questions.

Pair	Problem 1: instruments	Problem 2
1	Triangular, compass	Yes
2	ruler, compass	Yes
3	Triangular, compass	Yes
4	Triangular, compass	Yes
5	ruler, compass	Yes
6	Triangular, compass	Yes
7	Triangular, compass	Yes
8	Triangular, compass	Yes
9	Triangular, compass	Yes
10	Triangular, ruler with graduation	Yes
11	ruler, compass	Yes
12	Triangular, compass	Yes

A CASE STUDY

We are going to analyze the products and protocols of the first pair, Delphine & Baptiste (1), from the rule point of view. We identify the rules mobilized in the protocol for the construction and proving problems, and analyze the related controls.

Construction for the problem 1 by Delphine & Baptiste (1)

Their construction would be accepted by most teachers at the secondary school (Figure 5). They draw the perpendicular line with triangular and take equal distance with compass for A and B separately, and connect two points A' and B'. Then, they mark some codes for equal distance and right angle which allow us to diagnose that they are conscious of the perpendicular attached to triangular and the equal distance to compass, and their necessities for the symmetry. It's also explicit in the protocol.

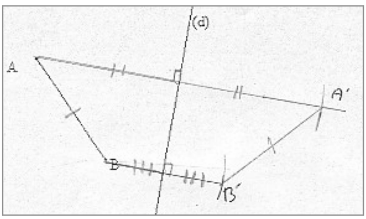


Figure 5 Construction by pair 1

11. Baptiste: this one, you have to make it symmetrical.

12. Delphine: yah. You draw a perpendicular segment to the line d, passing through B. Then with your compass,

(…)

24. D: then, you make the same thing for … there, it's not perpend … you are doing ... what are you doing!?
25. B: what? I'm drawing
26. D: no! It should be a right angle, there.
27. B: it's perpendicular.
28. D: it should be perpendicular.
29. B: hummmm,
30. D: it's not perpendicular. Do you know why it should be perpendicular?

(…)

36. D: you see, in fact, it should be the same segment, when one sees in front of the mirror.
37. B: hum.
38. D: It makes us an inverse. In fact, it's inverse. So, it should be the same slope with respect to the line d.
39. B: OK, I see.
40. D: to be the same slope, well, it should be perpendi …

In the protocol, it seems that Baptiste is not conscious of the necessity of perpendicular. But Delphine clearly states it and "right angle" [26; 28; 30]. After [30], Delphine tries to explain its necessity, but her explanation is not clear [36; 38; 40]. It relies on some examples given by the perceptive definition "mirror effect" [36]. In the words of cK¢ model, the necessity of perpendicular is validated by a perceptive control "mirror effect". We also consider that the construction with perpendicular is one justification for the necessity of perpendicular, such as "symmetric segments which satisfy the perceptive control could be constructed with perpendicular, so it is necessary for the symmetry".

Well, which rule do they mobilize? As the perpendicular and equal distance are used, can we interpret them as the type 2 rule "R_4: If $P_1P_2 \perp d$ and $P_1M = MP_2$, then $P_2 = \mathrm{Sym}(P_1, d)$" is mobilized? From the construction, it seems that they do. But, we will find from the problem 2 that the answer is rather negative than positive.

Proof for the problem 2 by Delphine & Baptiste (1)

For the problem 2, they construct a proof in which two following rules can be identified:

R_{DB1}: "if $P_1P_2 \perp P_2P_3$, $P_4P_3 \perp P_2P_3$ and $P_1P_2 = P_4P_3$, then $P_1P_4 // P_2P_3$"

R_{DB2}: "if $Q_1Q_2 // d // Q_3Q_4$ and $Q_1Q_2 = Q_3Q_4$, then $Q_1Q_2 = \mathrm{Sym}(Q_3Q_4, d)$"

The first rule R_{DB1} appears twice in the first eight lines. It is not correct rigorously, because P_1P_4 and P_2P_3 are not parallel if P_1 and P_4 are at the different side each other with respect to the line P_2P_3. The second rule R_{DB2} is identified in the last sentence (last four lines). This is not also correct.

> Yes
> (AM) ⊥ (MN)
> (DN) ⊥ (MN)
> [AM] = [DN]
> thus (AD) // (MN)
> (BM) ⊥ (MN)
> (NC) ⊥ (MN)
> [BM] = [NC]
> thus (BC) // (MN)
> As the segments [BC] and [AD] are parallel to (MN) and they are the same length, then they are symmetric with respect to the line (MN).

As they use the perpendicular "AM ⊥ MN" as a hypothesis (while this property is not stated in the problem statement) and the equal distance "AM = MB" in the proof, we can find that they have enough properties for characterizing symmetry. But they don't mobilize the rule "R_4: If $P_1P_2 \perp d$ and $P_1M = MP_2$, then $P_2 = \text{Sym}(P_1, d)$".

62. D: For me, I think "Yes". I explain you why. Because you see there, there is a right angle with respect to this line.
63. B: they are equal.
64. D: yes, and a right angle, here, they are parallel.

(…)

68. D: yeah, in fact, you see, MA is equal to MB. You see?
69. B: yes, yes, yes, I understand it.
70. D: so, as they are both the perpendicular segments

(…)

76. D: and the line AB is perpendicular to the line MN.
77. B: hum.
78. D: and the line BC is perpendicular to the line MN. They are both perpendicular to the same line, and MA is equal to MB. So AD is parallel to MN.
79. B: hum. Yes, yes, I understand.
80. D: On the contrary, how shall we write? I don't know.

In the protocol, it's clear that Delphine notices perpendicular or right angle [62; 64; 70; 76] and equal distance [68; 78] and the right angle is identified as an important criterion for symmetry [62]. But she doesn't notice the rule R_4. At the end of proving process, the rule R_{DB2} is mobilized without discussion [117-118]. This time, parallel and equal are stated together by Baptiste.

117. B: parallel and equal. It's symmetric.
118. D: that's right. I leave you to write down a little.

DISCUSSION

In the previous sections, we presented the data obtained in the observation and analysis from rule point of view. The first result is that most students who gave a correct construction cannot give a correct proof, even if the same rule is required. Moreover, although they are conscious of the properties necessary for proving, it's not same as using a correct rule. In the case of Delphine & Baptiste (1), even if they can construct correctly and are conscious of the necessity of perpendicular for symmetry, it doesn't mean that the appropriate type 2 rule (R_4 in the problem 2) can be use. Thus, the construction with geometric instruments is not enough to acquire and mobilize rules characterizing symmetry.

Why don't Delphine & Baptiste (1) mobilize R_4 in proving? Don't they have it? We consider that they don't have the rule R_4. One of the reasons is that the perpendicular and equal distance are mobilized separately in both problems, while they should be together to be a rule and utilizable in the proof. In fact, in the construction (problem 1), Delphine doesn't talk about the necessity of perpendicular and equal distance together and never gives statement characterizing symmetry. For example, her statement "it should be perpendicular" often appeared [26; 28; 30] is not based on the type 2 rule but on the type 1 rule "R_2: If $P_2 = \text{Sym}(P_1, d)$, then $P_1P_2 \perp d$". She also states "it should be the same segment" [36], "it should be the same slope" [38]. These statements have the same nature as for the perpendicular [26; 28; 30], because these are properties of symmetry and imply the type 1 rules, not the type 2. In the proving (problem 2), as we have already mentioned, the right angle is an important criterion for symmetry for Delphine. But it's not also together with equal distance. In the proving process, while they found many geometrical properties – perpendicular or right angle [62], same length [63], parallel [64], same distance [68; 78] –, "parallel" and "same distance" which allow us to diagnosis the rule R_{DB2} are qualified for proving at the end. Therefore, we understood that the students do not have the appropriate type 2 rule R_4.

We should also consider the role of rule in two problems. As we have found in the theoretical analysis, the rule for the construction is mobilized in a predicative form, while it should be operational in proving, even though the same aspect of symmetry is dealt. So, the construction without explicit rule, but with some separated geometric properties would be accepted in the school context, while proving requires the formalized and operational rule.

NOTE

1. In this paper, when we say "symmetry", it means reflective symmetry.
2. "P_2Q_2 = Sym (P_1Q_1, d)" means, in this paper, that two segments P_1Q_1 and P_2Q_2 are symmetric with respect to a line d. The order of P_1Q_1 and P_2Q_2 are indifferent, while it would be different from the cognitive point of view. And we use this notation also for the point.

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STUDENTS' IMPROPER PROPORTIONAL REASONING: THE CASE OF AREA AND VOLUME OF RECTANGULAR FIGURES

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In this paper, we investigate the predominance of the linear model in 12-13 year old Cypriot students, while solving non-proportional word problems involving area and volume of rectangular figures. Using three different kinds of tests related to the context of the word problems presented we attempt to identify a differentiation in students' responses. The results reveal students' tendency to apply proportional reasoning in problem situations in which this kind of reasoning is not suited. This tendency appears to decrease in the second phase of the study when the context of the problems was changed. The data also suggest that students are not able to detect the common non-linear character of area and volume tasks and therefore, deal with them in a different way.

INTRODUCTION

The rule of three, or in modern terminology the linear function, has been an important mathematical tool of explaining and mastering phenomena in different fields of human activity (Freudenthal, 1973; De Bock, Verschaffel, & Janssens, 1998). This suggests that proportionality, ratio preservation and linearity are universal models, a view that is reinforced by their frequent use. Moreover, the importance and significance of proportionality, as a mathematical tool, is also determined by the fact that its use dispenses one from rethinking a situation. Such a dispensation is usually gladly accepted.

The basic linguistic structure for problems involving proportionality includes i) four quantities *(a, b, c, d),* of which, in most cases, three are known and one unknown, and ii) an implication that the same relationship links *a* with *b* and *c* with *d*. In the case of true proportionality, this relationship is a fixed ratio (Behr, Harel, Post, & Lesh, 1992). However, if a problem matches this general structure, the tendency to evoke direct proportionality can be extremely strong even if this is not the case (Verschaffel, Greer & De Corte, 2000).

As Freudenthal (1983) points out, linearity is such a suggestive property of relations that one is readily seduced to dealing with each numerical relation as though it were linear. This phenomenon can be found even in traditional word problems where mathematical procedures, such as the rule of three, tended to be applied to problem situations without consideration of the realistic constraints (Verschaffel et al., 2000).

Students' tendency to apply proportional reasoning in problem situations for which it is not suited is partially caused by characteristics of the problem formulation, with which pupils learned to associate proportional reasoning throughout their school life. Multiplicative structures, notably those that on a superficial reading may create an illusion of proportionality, provide examples of inappropriate invocation of proportionality, as a result of an unconscious reaction to linguistic form (Greer, 1997). Thus, both at the level of individual students, and throughout history, there is a non-reflective link built up between the mathematical structure of proportional relationships and a stereotyped linguistic formulation (Verschaffel et al., 2000).

Freudenthal (1983) focuses on the appropriateness' of the linear relation as a phenomenal tool of description and indicates that there are cases in which this primitive phenomenology fails. One of these cases, which will be the focus of the present study, is the case of the non-linear behavior of area and volume under linear multiplication.

As De Bock, Verschaffel, & Janssens (2002b) point out, students' former real life practices with enlarging and reducing operations do not necessarily make them aware of the different growth rates of lengths, areas and volumes. Therefore, students strongly tend to see the relations between length and area or between length and volume as linear instead of quadratic and cubic. As a consequence, they apply the linear scale factor instead of its square or cube to determine the area or volume of an enlarged or reduced figure. This tendency is in line with the Intuitive Rule Theory (Stavy & Tirosh, 2000) which suggests that a change in a quantity A causes the same change in a quantity B.

Understanding that multiplication of lengths by d, of areas by d^2 and of volumes by d^3 is highly associated with the geometrical multiplication by d, is mathematically so fundamental, that, phenomenologically and didactically it should be put first and foremost (Freudenthal, 1983). Students should be able to distinguish that for instance, volume is proportional to length only when width and height are held constant; and similarly to width (or height) only when the other two variables are held constant. It is conceptually important and essential for students to understand the difference between the product of two variables in double proportion tasks, and the product of one variable by a constant in simple proportion problems (Vergnaud, 1997). Students have to break the pattern of linearity and become aware of the multi-dimensional impact of increase.

In recent years, there has been a considerable effort from researchers (De Bock et al., 1998; De Bock, Van Dooren, Janssens, & Verschaffel, 2002a; De Bock et al., 2002b; De Bock, Verschaffel, Janssens, Van Dooren, & Claes, 2003; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2003) to examine and overcome students' tendency to deal with non-proportional tasks concerning area as if they were proportional. In particular, De Bock et al., (1998) revealed an alarmingly strong tendency among 12-13 year old students to apply proportional reasoning in problem situations concerning area for which it was not suited. However, even the use of a number of different experimental scaffolds like the increase of the authenticity of the problem context (De Bock et al., 2003) and the use of metacognitive and visual scaffolds (De Bock et al., 2002b), did not yield the expected result. Only the rephrasing of the usual missing value problems into comparison problems proved to substantially help many students to overcome the “illusion of linearity” (De Bock et al., 2002b).

The actual processes and the mechanisms used by students while solving non-proportional problems were unraveled by means of interviews. It appears that not only is the "illusion of linearity" responsible for inappropriate proportional responses, but also intuitive reasoning, shortcomings in geometrical knowledge, and inadequate habits and beliefs about solving word problems (De Bock et al., 2002a; De Bock, Van Dooren, Janssens, & Verschaffel, 2001). All these aspects appear to favor a superficial and deficient mathematical modeling process, which leads to the unwarranted application of linearity.

From the literature review it becomes evident that the “illusion of linearity” is not a result of a particular experimental setting. In contrast, it is a recurrent phenomenon that seems to be quite universal and resistant to a variety of forms of support aimed at overcoming it (De Bock et al., 2003). Proportions appear to be deeply rooted in students’ intuitive knowledge and are used in a spontaneous and even unconscious way, which makes the linear approach quite natural, unquestionable and to a certain extend inaccessible for introspection or reflection (De Bock et al., 2002a). Therefore, as Verschaffel et al. (2000) illustrate, it takes a radical conceptual shift to move from the uncritical application of this simple and neat mathematical formula to the modeling perspective that takes into account the reality of the situation being described.

In order to achieve this radical conceptual shift in the present study, we asked students to confront two different experimental settings with a common aim; to investigate the predominance of the linear model in 12-13 year old Cypriot students and to create a cognitive conflict in order to differentiate students’ behavior while solving non-proportional word problems involving area and volume of rectangular figures.

METHODS

Participants in the study were 307 students of grades 7 and 8 of 6 different gymnasiums of Cyprus. Specifically, the sample of the study consisted of 163 students of grade 7 (12 year olds) and 144 students of grade 8 (13 year olds).

The study was completed in three different phases in each of which a different written test was administered. All three tests were administered separately in the span of 15 days for about 30 minutes each. The first test (Test A) was administered to all 307 students of both grades and consisted of 5 different word problems, two of which concerned volume (pr.1, 4), two area (pr. 3, 5) and the fifth one length (pr.2). All problems were in a multiplicative comparison form where the area or volume of the rectangular figure was given as well as the relation that connected it with the area or volume of the new figure, respectively. The purpose of this test was to examine the extend to which Cypriot students apply proportional reasoning while solving non-proportional word problems involving area and volume of rectangular figures.

The second test (Test B) was administered to only 157 of the students that participated in Test A and consisted of the same 5 word problems of the first test, but with a differentiation in the amount of data given for solving each problem. In particular, in each problem in addition to the area or volume of the rectangular figure and the relation that connected it with the area or volume of the new figure, respectively, the dimensions of the first rectangular figure were also given. The purpose of this test was to examine whether the inclusion of the dimensions of the figures would influence students in such a degree as to execute the multiplicative comparison first with the dimensions of each figure and then find out the area or volume of the new figure, instead of applying direct proportionality between length and area or volume respectively.

The third test (Test C) was administered to the remaining 150 students that were not administered Test B and consisted of the same 5 word problems of Test A, but with a different presentation. In particular, each item of the test was accompanied by two alternative answers of two fictitious peers. One expressed the dominant misconception that the area and volume are directly proportional to length whereas the other expressed the correct answer. Students were then asked to find the solution strategy each peer used to find the answer given and then to choose the correct reasoning justifying their choice. Purpose of this scaffold was to create a cognitive conflict in students' minds in order to question the appropriateness of the direct proportionality between length and area and length and volume.

It is worth mentioning that all three tests included the formulas for finding the area and volume of rectangular figures. As for the grading of the tests, each correct answer was assigned the score 1, each wrong answer the score 0, whereas in the cases that the mathematical expression for the problem was correct but not the answer, the score 0.5 was given.

For the analysis and processing of the data collected the statistical package of SPSS was used as well as an implicative statistical analysis by using the computer software CHIC (Bodin, Coutourier, & Gras, 2000). The statistical package CHIC produces three diagrams: (a) the similarity diagram which represents groups of variables which are based on the similarity of students' responses, (b) the implication graph which shows implications $A \Rightarrow B$. This means that success in question A implies success in question B and (c) the hierarchical tree which shows the implication between sets of variables. In this study we use only the similarity diagram.

RESULTS

The analysis of the data collected revealed the tendency of 12-13 year old Cypriot students to apply proportional reasoning in problem situations, concerning area and volume of rectangular figures, for which it was not suited. From Table 1, one can detect the great difference in students' achievement at the non-linear tasks of area and volume in relation to the linear tasks of length. This difference, even though more prominent in Test A, is statistically significant in all three tests (t_A=40,9, p=0,00, tB=15,29, p=0,00, tC=21,92, p=0,00). Students' achievement at the non-proportional tasks of area and volume is also differentiated, at a statistically significant level, in the experimental settings of Tests B and C. More specifically, students have greater success while dealing with the non-linear tasks of Tests B and C compared with the respective tasks of Test A. As far as the difference in students' achievement at the non-linear tasks of area and volume is concerned, this is statistically significant and in favour of the area tasks, only in Test A (t =-2,99, p=0,003) and in none of the other two tests. Therefore, it seems that both interventions assisted the diminution of this difference.

	Problem	Test A	Test B	Test C	Test	t	p
Area	3	7%	32%	19%	A-B	-7,38	0,00*
					A-C	-3,51	0,01*
	5	13%	34%	24%	A-B	-4,86	0,00*
					A-C	-3,07	0,00*
Volume	1	6%	30%	19%	A-B	-6,78	0,00*
					A-C	-4,17	0,00*
	4	7%	31%	21%	A-B	-6,43	0,00*
					A-C	-4,51	0,00*
Lenght	2	90%	90%	93%	A-B	-0,43	0,67
					A-C	-1,32	0,19

*p< 0, 01

Table 1: Students' percentages at Tests A, B and C

Figure 1 illustrates the similarity diagram of all variables (tasks) of Tests A and B. Students' responses to the tasks are responsible for the formation of three clusters (i.e., groups of variables) of similarity. The two first groups consist of the same tasks (1, 3, 4 & 5), which represent the non-linear problems of area and volume of Tests A and B, respectively. The third group consists of the linear problems of Tests A and B, something quite natural since both tasks are the same.

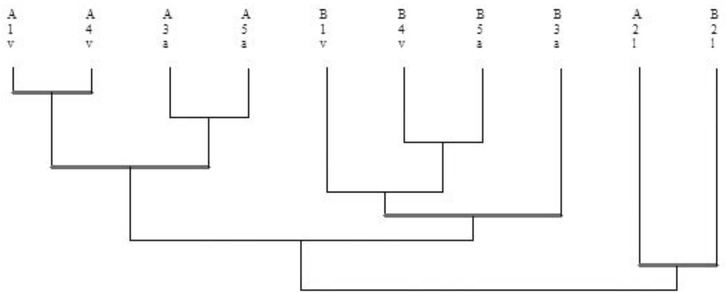

Figure 1: Similarity diagram of the variables of Tests A and B

Note: Similarities presented with bold lines are important at significant level 99%.

The first similarity cluster is formed by two distinct sub-groups of tasks that correspond to the problems of volume (1 & 4) and area (3 & 5) of Test A. The fact that the area and volume tasks are separated in Test A indicates that, prior to the intervention, students deal with these problems in a different way, without taking into consideration their common non-linear character.

The above tendency does not seem to apply to the non-proportional tasks of Test B, that constitute the second similarity cluster, since at this intervention all the problems of area and volume are mingled together. Consequently, students seem to realize that the tasks that are asked to deal with are not just problems of different mathematical content, but also problems that are characterised by the same phenomenon. That is, they realise the common non-linear character of the tasks despite the differentiation of the dimension number of the rectangular figures.

In Figure 2 all the similarity relations of the tasks of Tests' A and C are illustrated. As in Figure 1, three distinct similarity groups are formed. The first two groups consist of the same tasks (1, 3, 4 & 5), which represent the non-linear problems of area and volume of Tests A and C respectively, whereas the third group consists of the linear problems of Tests A and C.

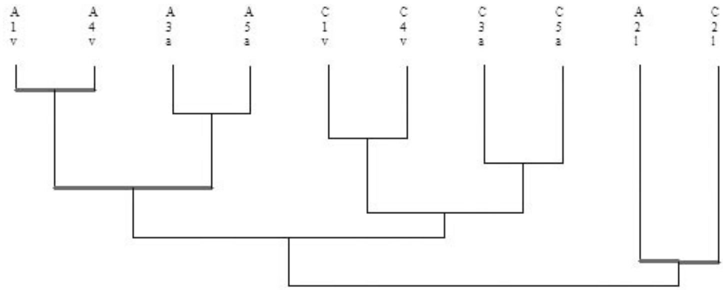

Figure 2: Similarity diagram of the variables of Tests A and C

In this figure, in contrast with the previous one, the first and the second similarity cluster, are formed by two distinct sub-groups of tasks that correspond to the non-proportional problems of volume (1 & 4) and area (3 & 5) of Tests A and C, respectively. The existence of this task separation indicates that students, prior and after the intervention, do not take into consideration the common non-linear character of the tasks and deal with them in a different way.

DISCUSSION

The results of our study reveal the great discrepancy in students' performance while dealing with linear and non-linear geometrical tasks. The existence of this difference was due to the mathematical errors students' made, while dealing with the non-linear tasks of area and volume, because of their tendency to see the linear function everywhere (Gagatsis & Kyriakides, 2000). One explanation could be that students, especially in Test A, failed to realise the multidimensional increase of the rectangular figure's size, since proportional reasoning appeared to be deeply rooted in students' intuitive knowledge. It could also be asserted that students' weak performance in non-proportional items may be merely the result of a superficial reading of the tasks that spontaneously connected them with a stereotype problem formulation, which is linked with proportional reasoning (De Bock et al., 2002b). Therefore, it seems that students matched the area and volume tasks with the primitive linear model, since the words double, triple etc., that were used in the problems, triggered the operation of a linear multiplication. Consequently, the operation of linear multiplication was applied to the two numbers embedded in the problem text and the result of the calculation was found, without referring back to the problem text to check for reasonableness (Greer, 1997). Another explanation could be their tendency to respond in line with the intuitive rule Same A – Same B, according to which the same change that occurs to the length will also occur in the area and volume.

Students' quite natural and spontaneous use of proportional reasoning appears to be questioned when Tests B and C were administrated. In particular, the cognitive conflict that both experimental settings promote seems to impel an examination of the appropriateness of the linear model for the solution of all tasks. Therefore, both interventions yield significant positive effects on students' performance on non-proportional items. However, these effects, as with other experimental manipulations, were disappointingly small and thus did not suffice to make the illusion of linearity disappear (De Bock et al., 2003).

Students' performance on non-proportional tasks was differentiated, depending on whether the task concerned the area or the volume of a rectangular figure, only at Test A. The experimental design of Test B succeeded in making students realize the common non-linear character of the area and volume tasks, despite the differentiation of the dimension number of the rectangular figures. The main reason for this is the fact that in Test B, all the students that overcame the illusion of linearity used the dimensions of the figures, which were given in the problems, in order to find the area or the volume of the figure. In particular, they first performed the multiplicative comparison with the dimensions of the figures and then used the formula for finding either the area or the volume of the figure. It must be noted that this was students' favoured method in most non-proportional tasks of all three tests, even if the dimensions of the figures were not included in Tests A and C. However, students in Tests A and C treated volume and area tasks differently, without understanding their uniformity.

In a subsequent research, the cognitive factors that prevent students from realising the common non-linear character of area and volume tasks can be investigated. Moreover, the results of a more systematic didactical intervention, concerning the non-proportional nature of geometrical tasks, can be evaluated with regard to their sufficiency to modify Cyprus's mathematics curriculum, so that this illusion of linearity be diminished.

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ABSTRACTION AND CONSOLIDATION

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What is involved in consolidating a new mathematical abstraction? This paper examines the work of one student who was working on a task designed to consolidate two recently constructed absolute function abstractions. The study adopts an activity-theoretic model of 'abstraction in context'. Selected protocol data are presented. The initial state of the abstractions and changes that were observed during the consolidation process are discussed. Features of consolidation noted are: reconstruction of the abstractions, increased resistance to challenges, developing a language for the abstractions and greater flexibility.

ABSTRACTION AND CONSOLIDATION

Abstraction and consolidation are important issues in mathematics education but there are differing interpretations as to what these constructs are and involve. An empiricist view of abstraction is that it involves generalisations arising from the recognition of commonalities isolated in a large number of specific instances (see Ohlsson & Lehtinen, 1997 for a critique of this view). An alternative position recognises the importance of the social and contextual factors in abstraction (see van Oers, 2001). This paper adopts an activity-theoretic model of 'abstraction in context' proposed by Hershkowitz, Schwarz & Dreyfus (2001) which we outline shortly. Consolidation is often associated with skill acquisition (Meichenbaum & Biemiller, 1998) and the retention of information (McGaugh, 2000; Spear, 1978). There is little research on consolidating abstractions. We review the research in this area after outlining the Hershkowitz et al. (2001) model of abstraction.

Hershkowitz et al. (2001) view abstraction as an activity of vertically reorganising previously constructed mathematical knowledge into a new mathematical structure ('structure' is their generic term for structures, methods, strategies and concepts – we employ this term in describing their work but avoid it in describing our own work). The new abstraction is the product of three epistemic actions: recognizing, building-with and constructing. Their theory posits that the genesis of an abstraction passes through three stages: (a) the need for a new structure; (b) the construction of a new abstract entity where recognizing and building-with already existing structures are nested dialectically and (c) the consolidation of the abstract entity facilitating one's recognizing it with increased ease and building-with it in further activities. The authors assume *a priori* that recognition of new structures in further activities will consolidate these structures and consequently students will progressively be able to recognise and build-with them with increasing ease. Hershkowitz et al. (2001) and a companion paper (Dreyfus, Hershkowitz & Schwarz, 2001), however, are primarily concerned with the process aspects of abstraction (stages a and b) rather than the outcome and the consolidation of an abstraction.

Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 2004 Vol 3 pp 353–360

Three other papers consider consolidation with respect to this theory of abstraction. Tabach, Hershkowitz & Schwarz (2001) and Tabach & Hershkowitz (2002) examine the construction of knowledge and its consolidation. They touch on the importance and necessity of the consolidation after the construction of knowledge but do not analyse the process of consolidation. A major contribution, in our opinion, to understanding the process of consolidation comes from Dreyfus & Tsamir (2001). They analyse the protocol data of one student and conclude that consolidation is a long-term process in which an abstraction becomes so familiar that it is available to the student in a flexible manner. They identify three modes of thinking that take place in the course of consolidation: building-with, reflecting on the building-with, and reflecting. They claim that building-with actions are the most direct and elementary means of consolidation. They characterise the consolidation of an abstraction with the constructs: immediacy, self-evidence, confidence, flexibility and awareness.

Although this work is important further investigations are in order as their claims are based on one case. Our work aims to further understand and characterise the process of consolidating an abstraction. We provide, below, an overview of our study, present protocol excerpts of 17-year-old boy working on the consolidation task and conclude with a discussion of issues raised from the protocol data.

THE STUDY AND TASKS

The work presented in this paper is part of a larger study examining aspects of human interaction and the RBC model of abstraction. The wider study employed an explanatory multiple case study methodology (Yin, 1998). 20 Turkish grade 10 students (16-18 years of age), seven pairs and six individuals, worked on four absolute value of a linear function tasks over four consecutive days. These students were selected from a larger sample on the basis that they could complete the tasks but they had not encountered the mathematical content of the tasks. Four pairs and three individuals were scaffolded by the interviewer (the second author) in their work. All interviews were transcribed. Initial analysis of the protocols was similar to the protocol analysis described in Hershkowitz et al. (2001).

The objects of the first, second and fourth task were to construct a method to draw the graphs of, respectively, $|f(x)|$, $f(|x|)$ and $|f(|x|)|$, by using the graph of $f(x)$. The third task was designed to consolidate the abstraction of the first and second tasks. This paper reports on a protocol generated in the third task, which had five questions. In the first question, the function $f(x)=2x-2$ is given and students were asked to draw the graphs of $|f(x)|$ and $f(|x|)$. The second question asked students to describe how to obtain the graphs of $|f(x)|$ and $f(|x|)$ from $f(x)=ax+b$. In the third and fourth questions an hypothetical situation was depicted where three fictitious students (Aylin, Cem, and Arzu) made claims about how to obtain the graph of, respectively, $|f(x)|$ and $f(|x|)$ from a given graph of $f(x)$. We present the claims in the appropriate part of the protocols. All of these claims were plausible but not quite correct. The students were expected to evaluate each claim. In the fifth question students were presented with six graphs and asked to identify whether they could be the graphs of $|f(x)|$ and/or $f(|x|)$.

PROTOCOL DATA

We present excerpts from Tugay's protocol for the third task. Tugay (not his real name) was one of the three students who worked alone with the interviewer scaffolding the work. All three of these students successfully achieved the desired abstractions of the first two tasks. In selecting protocol excerpts we attempt to present an overview of the work and include all excerpts referred to in the discussion section. To present data and to discuss issues in depth, in the space available, we focus on the consolidation of $f(|x|)$, though as the third task concerned both $|f(x)|$ and $f(|x|)$ we mention $|f(x)|$ at times. 'I' refers to the interviewer and 'T' refers to Tugay. Each complete utterance was given a new line number. We interleave protocol excerpts with our comments.

Tugay reads question 1. The interviewer asks how he will obtain these graphs.

4T: As far as I remember, I can either first draw the graph of $f(x)$ and then take the symmetries accordingly or I can substitute different values of x and then draw the graphs.

Tugay describes the symmetry for $|f(x)|$ and proceeds to $f(|x|)$.

10T: For the graph of $f(|x|)$, we draw a line parallel to y-axis through the intersection point of $f(x)$ and y-axis. Then we take the symmetry of a ray by flipping up and down accordingly. But I am not too sure how! Maybe according to given graph, I guess.
11I: You told me what you remember. How you could draw these graphs now?
12T: Umm, I think it'd better if I substitute different values of x and then draw the graphs because I feel more secure in that way. Maybe afterwards I can use what I developed before.

Tugay draws the graphs by substituting and moves on to the second question.

44T: I think I use the first question for this. Let's see… but I need to draw the graph of $f(x)$ first so that I can see what happens…

Tugay graphs $f(x)$, describes how to obtain the graph of $|f(x)|$ and moves on to $f(|x|)$.

52T: …after x becomes negative… first we find the intersection point of $f(x)$ and y-axis. Then we draw a line which passes through this point and is parallel to the y-axis. I'd better explain with this graph. We draw the line of y=-2 because it is the intersection of $f(x)$ and y-axis. Then this part [of $f(x)$] under this line will be flipped up to obtain the graph of $f(|x|)$.
53I: You mean you take the symmetry in the line of y=-2?
54T: Hmm yes, symmetry but the line… the symmetry line changes according to $f(x)$.

Between the 56T and 107I Tugay worked on the third question about the graph of $|f(x)|$. We move on to his work on the fourth question which concerns the graph of $f(|x|)$. He reads the first statement (by Aylin who claims that "To obtain the graph of $f(|x|)$, one needs to take the symmetry of the negative $f(x)$ values in the x-axis because this function includes absolute value which makes every negative $f(x)$ values positive and positive values exist only over the x-axis"). The interviewer suggests a graph.

110T: I wonder if a graph of $f(|x|)$ can take negative values, I mean under the x-axis… it could be… let's draw a random graph. In order to draw the graph, we should draw a line passing through the intersection of y-axis and $f(x)$… and then take the symmetry in that line.

He draws a graph of $f(|x|)$ and focuses on negative values of x and concludes that Aylin is wrong. He moves on to the second statement in this question (by Cem who claims that "There is no difference between the graphs of $f(x)$ and $f(|x|)$ for the positive x values but we cannot say anything about the difference for the negative x values, which depends on the equation of $f(x)$").

124T: Are both graphs the same for the positive x values? Both graphs appear to be the same… for the positive x values… 'There is no difference'… yes… there is no difference…
125I: Do you think he is right?
126T: There seems no difference for now… we have to consider the whole theory to come to a decision… umm, negative x values for the graph of $f(|x|)$… I think we can say something about the graph of $f(|x|)$ for the negative x values… when the x values are negative then we take this part symmetry in a line which parallels to the y-axis… but the difference between $f(x)$ and $f(|x|)$?… Well, the difference is evident… while $f(x)$ is linear, the graph of $f(|x|)$ is something like the shape of V…but one arm of V is the symmetry at the negative x values…

Between 127 & 133 the interviewer challenges Tugay's reasoning.

134T: Well, first of all, I remember that for the positive x values the graph of $f(x)$ remains absolutely the same, well I don't know if I can say 'absolutely'. But for the negative values of x, it was enough to take the symmetry. In fact we made use of analytic geometry for the solutions so…but I am not sure if what I am saying is definite…I am confused…

The interviewer suggests that they return to the first question and examine the graphs. Tugay notes that $f(|x|)$ has the same value for points $\pm n$. The interviewer asks why.

154T: I think due to the absolute value sign, I mean it is outside of the x and that means… regardless of the sign of the values of x, they will have the same value of y.
155I: What does this tell us about the symmetry?
156T: So it tells us perhaps that all of the graphs of $f(|x|)$ are symmetric in the y-axis. In fact I remember that I told something about it on the second task but I did not realise today.
157I: Perhaps? When you say perhaps I feel…
158T: I need to look at once again…[he looks at earlier graphs] yes, all of the graphs must definitely be symmetric in the y-axis because different values of x with different signs must have the same value of y, which is why it must be symmetric in the y-axis.

Tugay restates his confidence in this formulation.

162T: It is evident that we can say that there is no difference between the graphs of $f(x)$ and $f(|x|)$ for positive x values. At the same time we can surely say that the part corresponding to the negative x values must be the symmetry of the ray which is on the right side of the y-axis. So Cem is wrong. We can say how to obtain the graph even without an equation.

They move on to the third statement in this question (by Arzu who claims that "To obtain the graph of $f(|x|)$, one must not change the part of the graph of $f(x)$ at negative x-values and simultaneously the symmetry of this part must be taken in the y-axis").

164T: No, it is not so… I mean when we take the symmetry of the graph of $f(x)$ at positive values of x in the y-axis we obtain the $f(|x|)$…

They discuss this and the interviewer challenges and asks Tugay for a justification.

172T: Because… every value will be positive in the absolute value…it does not matter for positive values whether they are in the absolute value sign or not because it is positive anyway so it does not change. However, the negative values differ if they are in the absolute value I mean when they are in the absolute value sign then they change, they alter into positive and thus result changes…so when one substitutes, for example –2 for x in the $f(x)$, then one would obtain a different result from the result of $f(|x|)$ when one substitutes –2… because |-2| is a positive value and this is –2 in the $f(x)$. So they are totally different

173I: So, for the positive values in the $f(|x|)$…

174T: Let me put it another way. In $f(|x|)$ when we substitute positive values we obtain a result which is the same as $f(x)$. However, if we substitute negative values in $f(|x|)$ we get different results from that of $f(x)$ when the same negative values are substituted in the $f(x)$.

175I: Which shows that…

176T: That proves that the graph of $f(x)$ at the positive x-values is exactly the same graph as the graph of $f(|x|)$. But as the negative x values change in the $f(|x|)$ so does the graph of $f(x)$ when transformed into the graph of $f(|x|)$… I think I made my point, right?

177I: Yes, but say how we can obtain he graph of $f(|x|)$ from the graph of $f(x)$ once again.

178T: Well in fact we can obtain the graph of $f(|x|)$ in two different ways. The first one is that …we can draw a parallel line to the y-axis through the intersection point of $f(x)$ and y-axis. And then for the negative x-values we can take the symmetry of that part of graph in this line. Secondly, well we can take… umm the graph of $f(x)$ at the positive x values remains the same; I mean we can take the symmetry of this part in the y-axis and cancel the part of $f(x)$ at the negative x-values… and so this is $f(|x|)$… yes definitely so.

DISCUSSION

We discuss the initial state of the abstraction of $f(|x|)$ and the changes that were observed during the consolidation process. We also briefly relate our findings to the model proposed by Dreyfus & Tsamir (2001).

The initial state of the new abstractions

Tugay's new abstractions did not appear to be firmly consolidated when he started the third task in that he was not confident in their validity. He was, for example, able (10T) to describe how to obtain the graph of $f(|x|)$ from the graph of $f(x)$, but his comment "I'm not too sure" suggests that he was not certain about this construction. He also expressed feelings of insecurity (12T) with regard to this construction as a means of obtaining the graph of $f(|x|)$. His comment "if I substitute" suggests that he is uncertain about the validity of the abstractions constructed in the first and second tasks. A hesitancy in defending formulated abstractions for a considerable period after their constructions was common in all the protocols of students who made these abstractions. In Tugay's case we can see his uncertainty reappearing as the interviewer probes different aspects of the graphs. In 134T, for example, he states that he was not sure if his symmetry argument for negative values of x were correct and stated "I am confused".

In the early parts of the protocols of the third task students made extensive use of specific examples and these examples were used as a basis for formulating their ideas. Only students who consolidated the abstractions in this task went beyond specific examples and then only in the latter parts of the protocols. This is not surprising but it does draw attention to an apparent need to ground the new abstractions in concrete examples. In Tugay's case he states, (44T), "I need to draw the graph of $f(x)$ first so that I can see what happens". He did not talk about the relationship between the graphs of $f(x)$ and $f(|x|)$ until he had drawn them (44T - 52T). The points and lines he constructed were cojoined with demonstrative adjectives in his discourse: "this point … this line" (52T) – he appeared to be unable to formulate his constructions in general mathematical terms free from specific lines and points. We take this, and the uncertainty noted above as evidence that the new abstractions are fragile and need to be consolidated.

Changes in the course of consolidation

As Tugay worked on the third task he appeared to consolidate his abstractions from the first two tasks. We focus on: reconstruction of the abstraction, increased resistance to challenges, developing a language for the abstraction and greater flexibility.

It appeared to us that Tugay reconstructed his abstractions of $|f(x)|$ and $f(|x|)$ in the initial stages of the third task. Reconstruction is a process in which abstractions are derived from past constructions, i.e. the abstractions are not simply recalled. In 44T-54T, we see Tugay combining and manipulating various bits of information about absolute values, symmetries and graphs. This process continues throughout the third task. For example, later in the protocol (172T) we see his justification that the graph of $f(|x|)$ is the same as the graph of $f(x)$ for positive values of x by combining bits of information and actively reorganising them. We do not equate reconstruction with consolidation but reconstruction appears to be an important part of consolidation.

The fragility of new abstractions, we believe, makes students reluctant to use them to counter challenges. In the course of consolidation, however, students begin to resist challenges by establishing interconnections between the new abstractions and established mathematical knowledge and by reasoning with these abstractions.

Tugay established interconnections between the graph of $f(|x|)$, absolute values, symmetry and linear functions (154T, 158T, 172T & 174T). In 172T, for example, Tugay explains why $f(x)$ and $f(|x|)$ are the same for positive values of x by establishing connections between the graph of $f(|x|)$, absolute values and $f(x)$. Shortly after (176T) a change in the tone of his assertions can be noted, "that proves … I think I made my point". This change to a confident tone continues from 176T, e.g. compare "yes definitely so" (178T) with "I don't know if I can say absolutely" (134T). This aspect of this abstraction, that $f(x)$ and $f(|x|)$ are the same for positive values of x, appears to be fully consolidated as he used this to confidently elaborate how to obtain the graph of $f(|x|)$ (174, 176, & 178). We posit that Tugay's initial insecurity in his claims about his new abstractions partly stem from the fact that the interconnections between the new abstractions and existing knowledge were not sufficiently well established. The more connections students make between the new abstractions and existing knowledge, the more meaningful and accessible the new abstractions become, and students become more confident and resistant to challenges.

Apart from the confidence of students' language as the abstractions of the first and second tasks are consolidated in the third task, there was a qualitative shift in the clarity and precision of their language in the course of the third task. It seems that language development (to describe new abstractions) has a dialectical relationship with the consolidation of the abstraction. For example, in Tugay's protocol, his description in 52T of the graph of $f(|x|)$ lacks precision and is slightly ambiguous whereas in 178T his mathematical language is precise and unambiguous. A lack of precision in the initial part of the third task is not surprising, but language development during the task is significant with regard to consolidation in that the language of the new abstractions needs time to develop.

Students' use of examples is closely related to this development in their language of the abstraction. Prior to consolidation students appear to need concrete examples to formulate their thoughts but after consolidation they appear to use examples to demonstrate assertions. Tugay, for example, in 52T, articulates his thoughts by referring to specific properties of graphs. In 172T, however, he uses examples to clarify, to convince the interviewer, and to justify his assertions.

The use of specific examples to articulate thoughts suggests to us that the new abstractions are somewhat inflexible. When Tugay, for example, was asked to give an account of the graphs of $f(|x|)$ (52T) he appears to begin stating a general rule, "after x becomes negative", but later he focuses on a specific graph. Later in this protocol (174T), however, he quickly provides an alternative way to view $f(|x|)$ – and does so without recourse to a specific example. The phrase "let me put it another way" along with the confident and precise way he states this other way, suggest to us that he has consolidated this abstraction and is using it flexibly.

Comments on Dreyfus & Tsamir's consolidation model

We deliberately chose not to employ Dreyfus & Tsamir's (2001) constructs in this analysis, to avoid a narrow line of enquiry in a new area of research. Their paper, however, is an important one and it is useful to make some comparative comments.

They isolate three distinct modes of thinking in the consolidation process: building-with, reflecting on the building-with and reflecting. Our analysis of Tugay's protocol shows that building-with was the dominant mode of thinking throughout the third task. He occasionally reflected on building-with, for example, when he says in 134T, "it was enough to take the symmetry. In fact we made use of analytic geometry". We did not, however, note what Dreyfus & Tsamir call 'reflection', "an impressive reflection on a wide range of mathematical and psychological issues" (ibid., p.27). We do not think it is an essential part of consolidating an abstraction.

Dreyfus & Tsamir claim that consolidation occurs both in using new abstractions and while reflecting on them. Our data supports this. In the early stages of the third task Tugay reconstructed his new abstractions and later developed convincing arguments to defend his claims. During the reconstruction he used the abstractions. When challenged he developed convincing arguments where he both used and reflected on the new abstractions. This helped him to establish interconnections between his established mathematical knowledge and the new abstractions.

Dreyfus & Tsamir put forward five psychological and/or cognitive constructs associated with the progressive consolidation of an abstraction: immediacy, self-evidence, confidence, flexibility and awareness. Our data broadly supports this. We have already discussed confidence and flexibility. Regarding immediacy, there are clear indications in Tugay's protocol that this develops during consolidation. At the beginning of the third task (12T) Tugay was rather slow in describing how to draw the graph of $f(|x|)$ and somewhat hesitant in evaluating the initial two propositions in the fourth question (110T, 124T, & 126T). However, towards the end of the task he was quickly describing (162T & 178T) ways to obtain the graph of $f(|x|)$ and reacted to the third proposition in the fourth question (164T) almost immediately after reading it. Regarding self-evidence and awareness, these appear to be natural consequences of this consolidation process and it may not be always possible to refer to particular utterances to exemplify.

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CRITICAL SENSE IN INTERPRETATIONS OF MEDIA GRAPHS[1]

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This paper reports on a study that investigated the interpretation of media graphs developed by student teachers. The analyses of interviews indicate that cognitive, affective and contextual, aspects might constitute important components of the interpretation of graph such as those that we can find in print media. Particularly we discuss Critical Sense in graphing which is a skill that people can use to read the data, building an interpretation that balances these different aspects. The discussion of results might contribute to an understanding of these aspects, and the development of strategies, which help teachers think about the teaching and learning of graphing in ways that will support the development of Critical Sense.

INTRODUCTION

The interpretation of graphs might be conceptualised as a process by which people can establish relationships within data and infer information when reading graphs. In this paper, we discuss the idea of Critical Sense in graphing as a skill to analyse data and its interrelations rather than simply accepting the initial impression given by the graph (Monteiro and Ainley, 2003).

The term 'critical' identifies an important philosophical aspect, which is related to social, political, economical and affective issues involved in the educational process (e.g. Freire, 1972, 2003). Researchers have also related this critical perspective to mathematical education (e.g. Skovsmose, 1994). Our approach converges on general ideas developed in those studies. However, we are especially interested in the study of Critical Sense in graphing as an important skill related to the role of citizens, who need to be able to look critically at statistics presented by different sources.

In particular, Ainley (2000) states that the increasingly widespread use of graphs of many kinds in adverting and the news media for communication seems to be based on an assumption that graphs are transparent in communicating their meanings. The transparency is conceptualised as being inherent in the graph itself. In contrast to this perspective, Ainley (2000) argues that a graph may be considered as transparent for a particular user if it is both visible to reading, and invisible in giving access to features of the phenomenon it represents.

Ainley (2000) emphasises that the transparency of graphs emerges from the process of using them in a specific context. For example, when readers engage in a meaningful situation of interpretation they can use graphs to imagine ways of travelling through a symbolic space where events and narratives unfold (Carraher, Schliemann, and Nemirovsky, 1995).

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[1] This research is supported by CNPq – Conselho Nacional do Desenvolvimento Científico e Tecnológico (Brazil).

We believe that the interpretation of graphs is a complex process in which different elements are involved. In particular, we suggest three aspects, which are related to the idea of Critical Sense: cognitive, affective, and contextual. Certainly, this classification does not cover the entire range of components, but it seems helpful in our investigation of the problem.

Critical Sense in graphing is linked with the cognitive processes, which are associated with "formal" knowledge, such as that linked with the structure of graphs: framework, specifiers, labels and background (Friel, Curcio, and Bright, 2001). The cognitive factor might also be associated with the critical role of citizens when reading graphical representations. We need to be aware of and criticise graphs that might contain technical errors in their presentation, or may be technically correct but display irrelevant or misleading content.

To be critical is not just an intellectual or rational action, but also has an affective component. Even though professionals with high levels of schooling, and technical knowledge in graphing, utilise aspects from their beliefs and desires, when they are interpreting graphs such as those which we can find in print media (Monteiro, 2002).

The graph is not itself an isolated or neutral construct, it displays a content which might be related to a wider range of topics. The term context might be related to that content but also it might be associated with the situation in which the interpretation of graphs is developed. Gal (2002) suggests two main kinds of contexts: 'enquiry' and 'reader'. In *enquiry* contexts people act as 'data producers' and usually have to interpret their own data and report their findings (e.g. researchers and statisticians). R*eader* contexts emerge in everyday situations in which people see and interpret graphs (e.g. graphs are published in different types of periodicals). R*eader* contexts demand a certain level of *statistical literacy* in which readers can interpret, critically evaluate, and comment on statistical information, arguments, and messages.

Cognitive, affective, and contextual aspects are interrelated during the interpretation of graphs. For example, cognitive aspects should also encompass informal knowledge, such as that related to intuitions, which might be associated with beliefs and other kind of affective elements. Therefore, we can use the notion of *inclusive separation* (Da Rocha Falcão et al., 2003) to remark the fragile borders between these aspects.

We see Critical Sense in graphing as a skill in which people balance the influence between these three elements. The teaching and learning of graphing in ways which encourage the development of Critical Sense is a challenge for teachers who need to guide the pedagogical setting to situations in which statistically relevant aspects are discussed (Monteiro and Ainley, 2003). In the next sections we discuss part of a study which explores Critical Sense in graphing in student teachers, as a way of helping us, and them, think about teaching and learning graphing in ways that will support the development of Critical Sense.

EXPLORING CRITICAL SENSE IN GRAPHING

118 primary school student teachers took part in our study. A group was formed from 64 second-year students from an undergraduate education course. They were following three different specialisms: Mathematics, Science and English. Another group of participants was composed of 54 education post-graduate (PGCE) students. The study had three main stages. Initially, we gave a questionnaire to all student teachers just before they took a data handling section in a curriculum methods course in primary school mathematics. The questionnaire consisted of items that examined their familiarity with a *reader* contexts and their background in mathematics and statistics. It comprised two tasks based on print media graphs. Secondly, we made observations of the activities developed during the data handling section. Finally, a few months after the first and second stages, we interviewed some volunteers. However, because of lack of space, in this paper we focus only on the data related to the interviews about one of the graphs.

The interview included the same graph given on the questionnaires (see Figure 1), which was chosen for three main reasons. Firstly, we anticipated that the topics associated with the graph were related to the interests of the students, most of them living and studying in or near Warwickshire. Secondly, the graphs seem to have accessible levels of complexity of mathematical relationships and concepts. Basically, the graphs present absolute and rational numbers, and percentages. Thirdly, we choose a graph which might be straightforward to interpret. However, it was hard to find a media graph which we classify as totally "clear" and "easy" for interpretation. Therefore, we recognized even though in this straightforward graph, the way in which the target has been represented might mislead the participants.

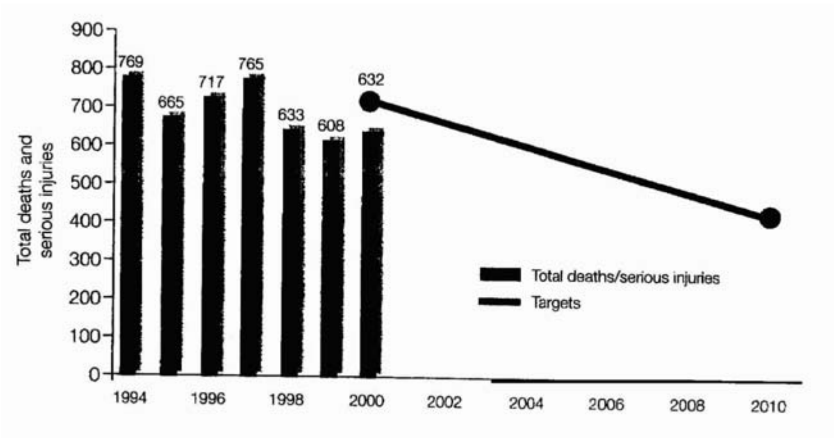

Figure 1: graph reprinted from *Quality of life in Warwickshire*, 2001, pp. 93-94.

The interviews served to investigate the typology of Curcio (1987), who proposed a multiple choice test composed of three kinds of questions for students interpreting traditional school graphs: *reading the data, reading between the data,* and *reading beyond the data.* In our study, the utilisation of Curcio's typology was not a kind of "replication". The context of interview, the graph topic chosen, and the formulation of questions (see below) seemed to propose a different approach for the process of interpretation.

Reading the data questions: What is the total of number of deaths and serious injury per year? What is the lowest actual death and serious injury rate?

Reading between the data questions: Between 1994-1995, and 1997-1998, there was a decline in the number of deaths and serious injuries. Which period represents the greatest decline? Which years represent the highest and lowest number of deaths and serious injuries?

Reading beyond the data questions: What is your prediction for death rate and serious injury in 2001? If the targets for 2000-2010 were met, what do you think the pattern might be for 2010-2020?

The interviews also were learning opportunities for student teachers to think about their own interpretation of graphs. They could also reanalyse their previous questionnaire's answers, and discuss their expectation of teaching in graphing.

STUDENT TEACHERS INTERPRETING MEDIA GRAPHS

Generally, the questions involving "reading the data" and "reading between the data" demanded direct answers. However, these questions provided an opportunity to the participants for carrying out an initial exploration of the data. On the other hand the "read beyond the data" questions generated a wider and deeper exploration of data displayed on the graph. We present some examples of interpretations produced when we asked the "read beyond the data" questions, which might help our discussion.

The first exchange comes from the interview with Betty, a 41 year-old PGCE student with a degree in English. She was trying to identify any trend related to the increases and decreases during the period between 1994 and 2000 to answer the question.

R – "What is your prediction for death rate and serious injury in 2001?"

B - In 2001… Well I don't know. It's… <u>there doesn't seem to be a trend</u>. It gone from 765 to 633… it's dropped down again, but then its gone up to 632 [year 2000], and here I'm presuming that this was the rate it wasn't the set target. <u>I'm presuming but either way even if you look at the graph, it has gone up</u>…

Betty was looking carefully at the figures presented, and realised that the graph presents the 2000 rate as target starting point. She seemed concerned about the graph's structure, which is clearly associated with cognitive aspects. On the next part she continued describing technical aspects involved in the graph, but she interpreted the graph based on her personal experience.

B - …But throughout the whole period there hasn't been a set trend it dropped down [1994-1995], it's gone up [1995-1996], it's gone up [1996-1997], dropped down [1997-1998], dropped down [1998-1999], it's gone up [1999-2000]. If you based it on that… My husband is a currency trader; so all day is very boring he looks graphs all day. And he follows trends. That's how he buys and sells currency depending on trends. So He would look at this graph, he would say "are well the trend on it is to fluctuate" and he would draw lines in, and than he would say: "ok, dropped down, went up twice, dropped down twice, it's gone up". Now he would plot a line to see, and he would go back over many years, he would look for the trend to see it follow a certain pattern. It is actually quite interesting. So he would want to look at more than this, he would probably say: "well maybe it would rise a little bit". But, Again for me I don't have any information. I don't know what they're doing. Are they… you know… advertising more, trying to educate people, making people to wear seat belts, and things. It's hard to predict from these figures what's going to happen.

The confrontation between the description of currency trader's strategies of analysis and the engaged context of interpretation seemed to be a justification for the limits of a efficient application of technical procedures. Then, she started to conjecture about the social context in which the data might be related. When encouraged to specify a prediction, she recognized that it was difficult but she made a prediction.

B - If I have to then, I would say it would rise slightly. Well if was 632 in 2000, maybe in 2001, if it rose there, maybe 640, something like that. But it's a guess. And I don't have enough information to be able to make a trending [moving with her hand like a curves of a graph, going up and down]… a trending estimate of it.

It seems that her reasonable answer was based on the articulation of cognitive, affective and contextual aspects about data presented on the graph.

In the interview with Teresa, a 19 year-old second-year student taking Maths specialism, we can also observe similar arrangement of the aspects involved in the interpretation. Facing a question that did not have "exact answer", she initially tried to observe any tendency from the data displayed. Suddenly she realised that possible tendency could not be the only factor that can predict the answer. Then, she tried to make suppositions based on her opinion about the possibilities of change the data.

R – What's your prediction for death rate and serious injury in 2001?

T – Hum… Well… From this it looked like it kind of … went down and than went up, went down and then starting to go up… So it might got a little bit more… But than it depends on a kind of what's being changed. Maybe… Like whether they've done anything in particular to try and reduce road accidents so they just make a prediction… I don't know… Yes, say… say 665. I mean that will be … that kind of match the graph slightly… so we're going to go lots a little bit like this…

After she gave the answer considering the social factors, she "came back" to the graph to justify that her interpretation would be coherent with the trend presented. However, on the following part of the interview, she demonstrated her uncomfortable feeling about representation of the target.

T - The target… I don’t know. The targets don’t really mean anything; I mean you can put the target to go down to zero… So if it is about there…then that is about 665 if the targets right. I mean they gone up a bit from here, haven’t they? [Year 2000] Perhaps, they will stay the same. (…) I don’t know. I mean I wouldn’t… It’s difficult, because always you have a bit suspicious about where it came from…then You know perhaps all the speed limits have been changed now in 2000 and what … certainly bought all the speed limits down, put road humps and things in all dangerous the roads and… So… Yeah.

The same kind of sceptical approach was observed when she was answering the next “read beyond the data” question. She made a distinction between what the graph represented, and what would be a “realistic” answer based on her own analysis of the data.

R – How about if the targets… from 2000 to 2010 were met. What do you think the pattern might be from 2010 for 2020?”

T - If they were met! Wow! That will be good. From 2010 to 2020… I think it will probably level out… hum… because always is going to be some… It might be down a little bit more… But… [Measuring with fingers on the graph]. It is not going be down the same steepness as because we will never get nobody dieing unfortunately. So I think it will go down perhaps a little bit and level out. I think. Yeah.

R - Do you guess a number?

T – By 2020, perhaps… I don’t know… Perhaps 300 even, maybe, yeah.

R – Do you want to comment this…?

T – Hum… I mean… Yeah… Deaths and serious injuries… there was a serious injury… and you can always class a serious injury as the same every time? Yeah… and… I suppose Warwickshire does that mean all of roads in Warwickshire? Are all the motorways included or… that kind of thing? The target I don’t really… I mean… They… It looks like they are taking a line… near perhaps I don’t know… But I don’t think it sounds realistic… and than… Well, I suppose unless they… If they had a bit writing to say what, and how they going to make this happen. There its not going to happen by magic, is it? So, yeah… (…) This is always interesting in the way… I know you can’t really compare them. But they’re put next to each other on a kind of “compare these” kind of way. But, you can’t really compare them because its… I don’t know whether the proportion is going down or not… Don’t know.

Teresa definitely did not believe in the trend represented on the graph, and analysed the structure of the graph criticising the implicit intention of showing the relationship between actual rates and targets displayed as rectilinear decrease.

The following exchange is from the interview with Hillary, 35 years old PGCE student with degree in Music. She seemed to want to believe in the trend, but it also seems that she did not find a strong argument to base her answer on.

R – “If the target for 2000-2010”… there is a target there… “What do you think the pattern would be from 2010 for 2020?”

H – 20… All right… hum… I think provided that technology doesn't take over people's well being… Than… <u>I think the pattern should decline</u>. But there are so many other things that might influence that pattern, like population rates … and… It is difficult to say… it is really difficult… <u>it is hard question that</u>… But I think… <u>I think it would be a decline</u>. <u>I think there always be a decline, because it is such important issue</u>… And then… There obviously… it always has been history of some kind of decline. <u>But obviously things come along the way that interrupt the flow</u>… obviously here [pointing to 1997 figure on the graph] there is… more deaths on the roads. There are reasons… <u>Well, I don't know. It is hard to say whether its death and injuries</u>. (…) But obviously that was addressed, because there was a big drop there [1997-98]. So, I think there always a kind of picture of a decline, or an attempt for a decline. With something as serious you know… as this issue.

Hillary alternates, looking at the patterns shown on the graph, the context in which the road accident occurs, and expressing her desire to see safer roads with lower levels of accidents. She seems to be reluctant to face up to the complexity of the question. However, when she was encouraged to try to specify a prediction, Hillary managed to "guess" an answer that seems to be based on the graph, but considering aspects such as the "hope" that was implicitly present on the interpretation.

R – If could say a rate as well?

H – Rate? … Do you want that I say what I think that death and injury rate might be…? Right. <u>So if it's starting at 500 which its obviously that's what they're hoping</u>… I don't think its actually going to hit the bottom. I think there is always be deaths and serious injury on the road. I don't think you ever avoid that happening, but it might be… For instance, a target… realistic might be straight from 500 to… say 300… Yeah, it seems a realistic target.

After Hillary answered the questions, the interviewer invited her to reanalyse the answers produced months before. It was an opportunity in which she compared both situations of reading of the graph. But, it was a moment in which she could make explicit a factor, which might be meaningful for her interpretation: she was actually involved in an accident.

H – I have been involved in an accident myself … It wasn't a particular serious accident. But, …I can perhaps relate to this statistics more …I think. I can actually see what it's telling me.

We can infer from analyses that Hillary's motivations and wishes played a prominent role in her interpretation. The fact she cares about the road accidents, and that she was actually involved in one of them was an essential meaning of the graph for Hillary. For example, she was trying to see what she wished, even though criticising and recognising the limits of her interpretation.

CONCLUSIONS

Our analyses of interviews indicated that the way in which we asked about predictions from the data helped the students in building an interpretation that involved aspects, which were objectively absent. It seemed that as they worked through the interview the graph became more transparent: they looked *at* the graph, but they also looked *through* the graph. They seemed to be aware that technical knowledge about the interpretation was not enough to answer the questions. They needed to use other resources such as opinions and feelings about the data, and knowledge of the context. The participants could travel through *symbolic space* (Carraher et al. 1995), which emerged from the interpretation. However, on the other hand, they recognised that they needed to balance the different elements, which played roles in their interpretation, which indicated the use of what we call Critical Sense.

Thus we see Critical Sense as offering a different perspective on the statistical literacy needed for the interpretation of media graphs from that presented by Curcio (1987), whose category of *reading beyond the data* seems to focus essentially on cognitive skills. Our concern with the teaching and learning of graphing leads us to question whether traditional pedagogic contexts offer opportunities for students to engage in the kind of activity in which Critical Sense may develop. A further focus of our research will be to compare characteristics of traditional pedagogic contexts with the experiences of student teachers during our interviews.

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BETWEEN AFFECT AND COGNITION: PROVING AT UNIVERSITY LEVEL

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In this paper we report on a case study of a university student (third year of Mathematics course). She was engaged in proving a statement of elementary number theory. We asked her to write the thoughts that accompanied her solving process. She was collaborative and her protocol is suitable to study the interrelation between affect and cognition. We seize in her performance a poor control of the solving process intertwined with emotions and beliefs on the mathematical activity.

INTRODUCTION AND BACKGROUND

It is widely recognized that in mathematical activity "'purely cognitive' behavior is extremely rare" (Schoenfeld, 1983, p.330). Already in the first half of XX century mathematicians such as Henri Poincaré and Jacques Hadamard reflected on the nature of mathematical activity and singled out aspects that nowadays could be ascribed to the affective domain[1]. Nevertheless, traditionally research on students' performances has concentrated primarily on cognition, less on affect, and still less on interactions between them. This way of looking at students' behavior engaged in mathematical tasks has shown its limits. According to Goldin (2002, p.60) "When individuals are doing mathematics, the affective system is not merely auxiliary to cognition – it is central". Leron and Hazzan (1997, p.266) claim that:

> In many cases, cognitive factors may be just the tip of the iceberg. Rather, forces such as the need to make sense, trying to hang on to something familiar, the pressure to produce something (anything!) to fulfill the expectations of the instructors – in short, trying to cope – are dominant.

Recent research, see (Araújo et al., 2003), (Gómez-Chacón, 2000), (Schlöglmann, 2003), addresses to analyze the interrelation of affective and cognitive factors. Following this stream of research in this paper we report on a case that seems to us suitable to integrate affective and cognitive domains. It concerns a university Mathematics student engaged in the proof of the following statement (compare with Euclid, VII, 28):

> Prove that the sum of two relatively prime numbers is prime with each of the addends
> (Two natural numbers are relatively prime if their only common divisor is 1)[2]

Our analysis of the way she faces the proof is in line with the observation made by DeBellis and Goldin (1997, p.211), who think it is essential to focus on "subtle

[1] It is well known that there is a problem of terminology, see (Leder, Pehkonen & Törner, 2002). We adopt the view of McLeod (1992), who claims that "the affective domain refers to a wide range of beliefs, feelings, and moods that are generally regarded as going beyond the domain of cognition. H. A. Simon (1982), in discussing the terminology used to describe the affective domain, suggests that we use affect as a more general term; other terms (for example, beliefs, attitudes, and emotions) [are] used as more specific descriptors of subsets of the affective domain." (p. 576)

[2] The definition of relatively prime numbers was included in the statement to prevent difficulties linked to ignoring it.

emotions, such as puzzlement, curiosity, frustration, or confidence, inherent in solving mathematical problems." Through a meticulous scrutiny of the student's protocol we look for these "subtle emotions", which contribute to shape the student's process of proving.

Our working hypothesis is that in proving a statement a student follows two intersecting pathways: the cognitive, and the affective. The cognitive pathway encompasses steps such as reading the text of the statement, understanding it, designing the plan and developing it through different proving techniques. Our statement, which encompasses just arithmetic and algebraic concepts, could orient students to expect that the proof may be carried out by developing familiar routines of elementary arithmetic and algebra, that is to say through "sequential procedures in which each mathematical action cues the next", see (Barnard & Tall, 1997, p.42). But this 'automatic-like' style is not enough for proving our statement. The proving process is more complex and students have to control and to link together cognitive units, since "mathematical proof requires the synthesis of several cognitive links to derive a new *synthetic* connection" (*ibidem*, p. 42). It is also required to activate anticipatory thinking, see (Boero, 2001). The impossibility of proving only automatically makes essential to mobilize other resources useful to go on, among them to review the different methods of proof (by counterexamples, by contradiction, by induction…) and to choose the most suitable to the purpose. The statement proposed could have been proved rather easily by contradiction: we will see that one of the reasons of difficulty for our student resides in having started the proof without a preliminary reflection on methods of proving, which could have oriented her toward this method.

The cognitive pathway towards the final proof presents stops, dead ends, impasses, steps forwards. The causes of these diversions reside only partially in the domain of cognition; they are also in the domain of affects. Thus, beside the cognitive pathway, we have to consider the affective pathway, which is described by DeBellis and Goldin (1997, p.211) as "a sequence of (local) states of feelings, possibly quite complex, that interact with cognitive representational configurations". The focus of our work will be on the modalities of this interaction.

A main point of our analysis will be the role of beliefs. According to (Schoenfeld, 1983; 1987; 1992) beliefs appear to be an overriding factor in students' performances, more rooted than emotions and attitudes. Beliefs may concern the mathematical activity or the person. Beliefs about self are closely related to notions of metacognition, self-regulation and self-awareness. Among the beliefs about self McLeod (1992) mentions confidence in learning mathematics (e.g. a belief about one's competence in mathematics). This author also takes into account causal attributions (what subjects perceive as cause of their success and failure). They are categorized along three main dimensions: locus (internal vs external), stability (stable vs unstable), controllability (controllable vs uncontrollable) of the causal agent.

We also take into account the categorization of proof schemes given by Harel and Sowder (1998). These schemes are grouped in three main classes: external (ritual, authoritarian, symbolic), empirical (inductive, perceptual), analytical (transformational, axiomatic). In the following we will be concerned with ritual proof scheme (which manifests itself in the behavior of judging mathematical arguments only on the basis of their surface appearance) and symbolic scheme (proof is carried out using symbols without reference to their meaning).

METHODOLOGY

In our study we consider a student attending the final year of the university course in Mathematics. She attended all basic courses (algebra, geometry, analysis…) and some advanced courses in mathematics. Her curriculum encompasses one course of mathematics education, in which our experiment was carried out. In this course students are regularly engaged in activities of proving, developed as follows:

- a problem is given
- the students are aware that the problem is at their grasp
- the students are asked to write the solving process and contemporarily to record the thoughts that accompany their work
- the students work out the problems, solving them individually
- the individual protocols produced are analyzed by all students.

The goal of these activities is not the proof by itself, nor marks are given to the performances. Instead, the students are asked to focus on what they think and do when proving. They are allowed to use pseudonyms (usually they do it). Emotions and feelings are not explicitly mentioned as required information, nor they have been mentioned in the mathematics education course. In order to avoid the influence of time in performances, see (Walen & Williams, 2002), it is given to the students as much time as they need. The case study we refer to (student "Fiore") is set in this context: she worked with a will, was very collaborative, and provided us with rich information. The whole experiment is reported in (Morselli, 2002).

In our analysis Fiore's protocol is split into its component steps, signed by us with numbers (bold font). In this paper the sentences (translated by us) are typed (italic font) in a shape as similar as we can to that in her protocol (signs, symbols, layout are kept). In our comments we will attempt to understand the factors that shaped the student's performance rather than judging it from an expert's perspective.

FIORE'S PROTOCOL AND OUR ANALYSIS

1. *Help! I'm not familiar with prime numbers! ←(but it does not matter)*

Fiore's first reaction is emotional ("*Help!*"): she expresses panic. This panic may be seen as a consequence of her low self-confidence. In the very moment she approaches the statement of the problem she is already judging it out of her grasp. Since she takes for granted that she will have difficulties in solving the problem, she is looking for these difficulties rather than attempting to actually understand the problem. Her concern is to check whether she has the knowledge connected to each word, thus she isolates each word loosing the general meaning of the statement. The result is that her first reading is superficial and not goal oriented. Polya (1945) has pointed out the importance of an efficient reading of the text as the starting point in mathematical performances. The first passage in the protocol brings to the fore affective aspects as a cause of non efficient reading of the text.

Fiore's initial focus is on the word "prime", while the adverb "relatively" is neglected. Her reaction is negative ("*I'm not familiar with prime numbers!*"): she compares what she thinks is needed to solve the problem and what she feels to know: her conclusion is that she is not adequate to the task. We may say that the affective factor (low self-confidence) turns to be a cognitive factor as it influences the way she reads the text. On the other hand her superficial reading enhances her feeling of inadequacy to the task.

The student's behavior reminds us a widespread belief, see (Schoenfeld, 1992), that doing mathematics requires to memorize rather than to understand. Possibly Fiore is not aware of holding it, but we feel that this belief accompanied her along all the solving process, both in the choice of strategies and in the reactions to difficulties. After this first phase of panic she comes back to the text and reads it more carefully. The critic word "prime" is no more frightening, because she caught that the real clue is "relatively prime" and not "prime" alone. The new clue is more comfortable, since being relatively prime refers to a concept (divisibility) that has been treated from early school times and for which she met specific algorithms, while the concept of prime number keeps a degree of mystery and uncontrollability (no way to elementarily decide whether a number is prime or not, etc.). Fiore expresses relief ("*but it doesn't matter*") and writes:

2. *I do a few numerical checks:*
2 and 7 2+7=9. Are 9 and 2 relatively prime? Yes.
Are 9 and 7 relatively prime? Yes

Fiore uses this numerical example just to get in touch with the problem, but she does not see anything in it because she confines herself to one example (which, in addition, is trivial because is based on two prime numbers). Fiore is not really committed because she doesn't rely on exploration; she already has the idea of using algebra as a tool for working on the problem, as shown in the following step:

3. *I stop. It's fine. I'm already thinking of the way of representing two relatively prime numbers through the algebraic language. I have to think for a while.*

Again we find a behavior which brings to the fore Fiore's beliefs about mathematical activity. On one hand the explorative aspect (here numerical exploration) is completely neglected in her way of conceiving mathematical activity, on the other hand she trusts in algebra as a powerful tool to obtain proofs in an automatic-like way. The consequence is that she privileges syntactic aspects of reasoning (manipulation of symbols) rather than semantic ones and appears to be not concerned with meaning. Fiore offers us a telling example of behavior whose aspects refer both to ritual and symbolic proof schemes, see (Harel & Sowder, 1998).

Now the critical point is how to represent two relatively prime numbers, as evidenced by Fiore's sentence "*I have to think for a while*." The cognitive difficulty in finding a representation pushes Fiore to look back to her memories on the subject, but the problem is that she has not an efficient strategy; she goes on trying to recall facts and definitions at random.

The following steps will show an additional problem, i.e. Fiore's low mastering of algebra. She tries to recall the concept of factorization, but she can not master it in an operative way, thus she has to resort to numeric examples.

> **4.** I *have n. m cannot be a multiple of n.* [She calls the two numbers m and n]
> *Do I have to use the factorization of a number...?* ← *natural! But how?*
>
> *2=2·1 and 7=2·3+1 (no, 7 is prime!)*
> *3=3·1 and 8=2·2·2*

At this point Fiore's goal has shifted from the main task to the task of recalling the factorization of numbers. She tries to apply the algorithm of factorization to 7 that is a prime number, and this shows that she doesn't associate meaning to what she is doing. We note again that prime numbers are a source of confusion.

> **5.** *6 = 2·3 and it* [the second number m] *cannot be a multiple of 2, it cannot be a multiple of 3…. The first number fitting the requirement is 5, which is prime*

The conclusion of this part of Fiore's reasoning is:

> **6.** *Help! I cannot do it, I still do not see anything. The deepest darkness*

We see here an example of

> the interplay between affective states and the heuristic or strategic decisions taken by students during problem solving [….]. For example, feelings of frustration while doing mathematics may encode (i.e., represent) the fact that a certain strategy has led down a succession of "blind alleys", and (ideally), these feelings may evoke a change of approach. (Goldin, 2002, p.61)

It is remarkable the use of the metaphor based on "*see*" and "*darkness*", which is very efficient for transmitting Fiore's panic when her expectation of immediate feedback and clues is not realized. Again it emerges her view of mathematical activity as an automatic activity, which does not require personal elaboration and creativity. She expects the solving process to be linear, say made up of a sequence of stages that follow each other in a linear and continuous way. When she chooses a path she expects to arrive at the end, that is to say she does not contemplate the possibility of dead ends and failures. When she meets a situation that requires a time of reflection she feels lost ("*The deepest darkness*"). Schoenfeld (1992) points out that one of the typical students' beliefs is that "Students who have understood the mathematics they have studied will be able to solve any assigned problem in five minutes or less" (p.359). Fiore shows the persistency of this belief even at university level: she is a mathematics student, thus, in principle, she is not hostile to this discipline, but this does not prevent her from keeping this misleading belief about mathematical activity. Her underestimation of the role of reflection in the solving process may come from her very poor experience in actively doing mathematics during her school and university career. She has always seen mathematical facts presented as finished products, where the complex process that has brought to the solution is hidden. This is the reason why she lives every step of her solving path that has not an immediate consequence (because it is wrong or because it needs some deepening) as a failure and not as a physiological component of the process.

After this moment of discouragement Fiore goes back to a numerical example:

7. *3 and 14=7·2*

Once again the example is trivial and she does not reflect on it (see the further step). This step makes even clearer her view of exploration through numbers: it is a kind of *rite*, which is applied without considering it an actually efficient strategy. The use of numerical examples is simply a way of restarting the process of solution. This restart constantly marks the moments when she feels the failure. After she writes:

8. *n m such that*
$n = p_1^{r1}p_2^{r2}p_3^{r3}$

(yes, in general a natural number is a product like this, it is a product of powers of prime numbers)

$m = p_4^{r4}\, p^{r5}$
$n+m = p_1^{r1}\, p_2^{r2}\, p_3^{r3} + p_4^{r4}\, p_5^{r5}$

This representation could hint that if $n+m$ is not prime with n there is a factor, say p_1^{r1}, which is common to $n+m$ and n. It is easy to see that this factor is also a factor of m. This contradicts the hypothesis that n and m are relatively prime. But Fiore, faced with this representation, can not see this development. We hypothesize that her blindness is due to the fact that, after having reached her goal of factorizing, she expects to go on in an almost automatic way. We stress again that this expectation intertwines with the absence of a useful strategy (proof by contradiction) and of critic control of her proving process.

9. *But in algebra* [she means university course of algebra]*, indeed, we have worked a lot on these things! May I have forgotten everything?*
[But does this mean I have not understood what I studied???
What a troubling question! I hope it is matter of memory]

Fiore ascribes the cause of her failure to internal and personal reasons, such as lack of remembering and understanding (internal causal attribution) and not to the lack of a strategy in proving. Fiore focuses on memory because she has the stable belief about mathematical activity as an activity relying on applying rules and algorithms. This, together with the belief about self, generates the internal causal attribution ("*May I* *have forgotten everything?*”). The attribution concerning remembering is rather stable: in the initial step **1** Fiore refers to familiarity that is linked to the past experience, in the last step **9** again she recalls forgetting/remembering. There is a further internal causal attribution linked to understanding (“*But does this mean I have not understood what I studied*???”). This latter causal attribution apparently is less stable or at least, Fiore, being aware of its gravity, attempts to reject it: as a matter of fact she ends by expressing the hope that only remembering is the problem (“*I hope it is matter of memory.*”) Fiore’s belief about remembering is not unusual among university students. Burton (1999, p.31) reports that “68% of the students in the ‘most-able’ class, that is presumably and possibly the future mathematicians, prioritized memory over thought.”

We could say that Fiore is always oriented to reproduce rather than to create, she lacks of mathematical creativity. Maslow (1962) considers self-actualizing creativity, which does not come from a particular talent (“genius”), but from personality. He claims that while ordinary peoples feel uncomfortable with unknown and uncertain situations, creative peoples live such situations as pleasant challenges. Fiore makes explicit her anguish about past (remembering, understanding) and feels unsafe and uncomfortable toward future (the beginning of each step.)

FINAL REMARKS

Our initial assumption on the intertwined nature of cognitive and affective factors has led us to scrutinize through two different lenses (cognitive and affective) Fiore’s protocol.

Through cognitive lens	**Through affective lens**
• poor exploration and production of examples	• low self confidence
• low mastering of algebra	• internal causal attribution
• scant anticipatory thinking	• view of mathematical activity
• no reflection on proving strategies	• low creativity

Only one type of lenses would have deprived us of important details and given an unfaithful picture of the situation, while the two different lenses allow to see causal links among affective and cognitive components of the student’s behavior. From the cognitive point of view it clearly emerges that the discriminating element in filling or not filling the task is the mastering of proving strategies. This finding was quite obvious and predictable, but becomes more interesting if we ask ourselves why Fiore, who, in theory should be ‘expert’, does not master proving strategies. To answer we resort to the domain of affect. In our analysis we have stressed the moments where the beliefs about mathematical activity may have influenced the outputs. As Schoenfeld (1987, p.34) claims, “Beliefs have to do with your mathematical *weltanshauung* or world view. The idea is that your sense of what mathematics is all about will determine how you approach mathematical problems”. In addition to this fact, Fiore shows a low capability to manage her personal emotional responses and their interactions with cognition. Emotions about emotional states and emotions about cognitive states, which are such an important component of her affective pathway, reveal themselves as a burden in her cognitive pathway.

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AN ASPECT OF MATHEMATICAL UNDERSTANDING: THE NOTION OF "CONNECTED KNOWING"

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Much advice about teaching for understanding implies that teacher should help children to develop connections between aspects of their experience, knowledge, and skills. This paper outlines points from the literature about different types of connections and describes relevant points from four case study teachers said and did.

TYPES OF CONNECTED KNOWING

Few authors claim to have studied teaching mathematics for understanding *per se*. One exception is Knapp et al. (1995), who studied experienced teachers in 15 schools while gathering longitudinal data via standardised tests of mathematical understanding. The features of meaning-oriented instruction identified are:

- (a) broadening the range of mathematical content studied to give children a sense of the breadth of mathematics and its applications;
- (b) emphasising connections between mathematical ideas;
- (c) exploring the mathematics that is embedded in rich, "real life" situations;
- (d) encouraging students to find multiple solutions and focusing students' attention on links between the solution processes used; and
- (e) creating multiple representations of ideas (e.g., drawings and physical objects).

Every one of these characteristics of teaching mathematics for understanding involves connections, but each implies different types of "connected knowing".

Overall, the research literature available to Australian teachers, many professional development programs, and curriculum advice provided by state and national government bodies suggest that the development of connections in mathematics education is important. Thus how teachers interpret such suggestions, as well as whether and how they facilitate the development of a range of types of connections, are important research questions. My analysis of many relevant articles (see Mousley, 2003) demonstrates that literature on the development of mathematical understanding has a focus on connected knowing, but authors tend not to distinguish between possible meanings of this term. The three most common interpretations were: (a) connections that learners make between new information and existing understandings; (b) relationships between different mathematical ideas and representations; and (c) links that teachers and children make between school concepts and the mathematical aspects of other everyday contexts. There are other possible meanings, such as personal connections that individuals make with educational content (see for example Ocean, 1998), but reference to these is not common.

Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 2004 Vol 3 pp 377–384

Over the past five years, I have carried out research into how teachers describe and develop students' mathematical understanding. The remainder of this paper reports on my research into ways that four of the teachers, all from one school in a rural area in Victoria, Australia, attended to the development of "connected knowing".

THE RESEARCH CONTEXT AND METHODS

Case study methods were used. About 4 weeks were spent in each of 4 classrooms. David and Susan were the teachers of the Year 6 (age 1112) classes; Tracey and Frea taught the Year 2 (age 78) classes. Teachers were interviewed several times, and their mathematics lessons were videotaped. The resulting audiotaped and videotaped data were analysed to find examples of what the teachers believed and did in relation to the development of their pupils' mathematical understanding. This paper focuses on a small sub-set of the data. The full report of this (Mousley, 2003) is descriptive, with close reference and electronic links to extensive multimedia appendices.

Making connections between new information and current knowledge

This first interpretation of connected knowing—the most commonly-used—was popularised by Piaget, who portrayed understanding as constructed, developed and organised as a result of cognitive interaction between an sensory experience and existing schema. According to Piaget (1926) and others (e.g., Steffe, Cobb & von Glasersfeld, 1988), sensory input is filtered, arranged, and stored in complex networks of concepts, rules, and strategies. These shape cognition as new experiences are either assimilated into existing schemata or accommodated by a change in their structure. Understanding mathematics involves developing a harmonious network of information that may include images, relationships, errors, hypotheses, anticipations, inferences, inconsistencies, gaps, feelings, rules, and generalisations (O'Brien, 1989).

Many researchers have drawn on this constructivist approach. For example, Hiebert and Lefevre (1986) wrote, "Perhaps 'understanding' is the term used most often to describe the state of knowledge when new mathematical information is connected appropriately to existing knowledge" (p. 4). Similarly, Nickerson (1985) claimed that teaching for understanding involves "the connecting of … relatively newly acquired information to what is known, the weaving of bits of new knowledge into the integrated whole" (p. 234). Pound (1999) proposed that any developmentally appropriate curriculum starts with children's ideas: "the starting point must be the child's current understanding—our efforts must go into helping each child to make the connections which will promote idiosyncratic personal understanding" (p. 51).

Many useful ideas are underpinned by this concept. The notion of "learning trajectories" (e.g., Simon, 1993; Steffe & Ambrosio, 1995), for instance, is based on the idea that new knowledge will further develop what has just been learned. Cobb & McClain (1999) referred to an "instructional sequence [that] takes the form of a conjectured learning trajectory that culminates with the mathematical ideas that constitute our overall instructional intent" (p. 24). Hiebert et al. (1997) used the term "residue", involving the understandings children gain from teaching being used as a basis for further planning, but Sfard and Lincheveski (1994) noted the difficulty of predicting when any sequence of activity will become connected and hence more generally meaningful:

> It often happens that three or four steps in instruction add little to the child's understanding of arithmetic, and, then, with the fifth step, something clicks. The turning points at which a general principle becomes clear to the child cannot be set in advance by the curriculum. (pp. 101–102)

In Australia, advice to mathematics teachers typically includes the point that new ideas need to be connected with what is known:

> We can only take from any situation the parts that … can be linked to some existing ideas we have. … Some learning can readily be accommodated within existing conceptual structures. Other learning requires a relatively simple extension or adjustment of ideas we already have. (Australian Education Council, p. 18)

In my research, videotaped lessons showed that the students' knowledge was drawn on frequently to introduce lessons but very infrequently at other times unless students made spontaneous contributions such as "I remember when we did easier ones last year" (Boy, Grade 2). Early in lessons, there were many teachers' requests for students to recall previous in-school experiences.

> Cast your mind back to last week, … Do you remember how amazed we all were that a baby sperm whale, when it is born, is longer than this room? Do you remember how we estimated metres? Think back. Have a think about how long a metre was. (David)

All 4 teachers encouraged children to "fold back" (Kieren & Pirie, 1991), but when analysing videotapes of such interactions I noted that teachers (probably keen to get the lesson underway) generally chose only higher-performing students to talk about concepts from the previous lesson. Further, all of the teachers commented spontaneously that children do not always remember or think to apply what has been taught in previous lessons.

> They looked at rulers last week. They found things that were about 30cm long. They should have understood 40cm would be longer than a ruler. … With all my years of teaching, it is still surprising … They know something … but they don't remember that when they need it. (Tracey)

Planning for teaching of sequential ideas was also seen as difficult as the result of the range of understandings that children have and their abilities to make connections quickly:

> Some of the children like [Boy, Grade 6] are one step ahead of the class … But some of the others [are not]. I spend time planning the best way to step them through, but then they just rely on me to teach them [sequentially] … the better you structure the learning, the less effort they make. (David)

Making connections between mathematical concepts

The next most commonly-mentioned interpretation of connected knowing, in relation to the development of mathematical understanding, involved students understanding relationships between varied mathematical objects—including links between specific facts, ideas, representations, processes, propositions, etc., as well as links across these categories. For example, Hiebert and Lefevre (1986) wrote that:

> Conceptual knowledge is characterised most clearly as knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network. (pp. 3–4)

Hiebert, Carpenter, and colleagues asserted that learning with understanding involves development of an internal network of representations, with the strength of a child's understanding being determined by the number and strength of its connections. They point out that in the development of understanding, communication and reflection are vital:

> Communication works together with reflection to produce new relationships and connections. Students who reflect on what they do and communicate with others about it are in the best position to build useful connections in mathematics. (Hiebert et al., 1997, p. 6)

Skemp's theorising of the concept of "relational understanding", where pedagogy is aimed at the construction of relational cognitive schemata, is relevant here.
A possessor of such schemata can, in principle, produce an unlimited number of plans for getting from any starting point within a schema to any finishing point (Skemp, 1976, p. 23). Byers and Herscovics (1977) distinguished further between "relational understanding", or the ability to deduce particular procedures through a recognition of mathematical relationships, and "formal understanding" that involved the ability to connect mathematical forms of notation with the appropriate ideas and to combine these into sequences of mathematical reasoning. Focusing on teaching for understanding of decimals, Hart (1989) provided some practical examples of the networking of mathematical ideas. In her discussion of the results of a test that focused on place value, she pointed out that:

> It is above all clear that learning of whole numbers … involved internalising a whole chain of relationships and connections, some with the place-structure itself (e.g., 0.9 is equivalent to 0.90) and some linking to other concepts like those of fractions (e.g., the notion of one hundredth and its relationship to one tenth), some visual correspondences and some connecting to applications in the 'real' world. (p. 64)

When asked about ways that they developed children's understanding of mathematics, both of the case study Year 6 teachers volunteered that they made a point of linking operations. When asked for an example, David said "Take place value: it is not only the basis of numerals and the number system but also measurement, decimals, percentages, and so on.” Susan said that she had been to an extended professional development course for primary teachers of mathematics several years previously, and “One thing that was really emphasised was constructivism and ways to help children build knowledge networks where maths ideas are all linked together in meaningful ways”.

However, in the many Grade 6 lessons observed over their 8-week period, emphasis on connections between mathematical forms or operations was rarely explicit. The exception was when percentages were taught and both teachers emphasised the different ways that fractions can be expressed. Generally, though, such understandings needed to be gleaned from school experience. In fact, when asked about whether she emphasises connections between the four processes, Susan acknowledged, “I don’t emphasise them much. Probably only the best of the children are able to make all the necessary connections”.

One of the Year 2 teachers, Frea, thought that making connections between operations had the potential to confuse her pupils. Without prompting, she gave an example.

> Well, I know that multiplication is like adding. [She gave example of repeated addition]. The children might realise that. I am sure some of them do—not many though. No, I think they do not think of it as addition. … I want them to see multiplying as repeated groups, not as repeated addition. That might confuse them. It’s incidental.

I noted several instances where the use of context-based cross-curriculum linkages resulted in cross-mathematics linkages. An example of this was David’s lesson where children made “Flying Machines” from scrap materials, measured the length of flights, averaged the results for each machine, and graphed then wrote about the outcomes. While David saw this as a Science-Mathematics-Language integrated activity, the lesson clearly helped the students to develop connected understandings within mathematics as well as across subject areas.

Making connections to everyday experience

The third common interpretation involves bi-directional connections between the “real world” and school mathematics. In one way, it involves children coming to understand ways that mathematical knowledge and skills can be applied in everyday contexts. Alternatively, children draw on their everyday experiences to understand school mathematics ideas as well as the need for specific mathematical processes.

The aim of facilitating links between mathematics concepts and everyday experiences underpinned the *Connected Mathematics Project*. Here, Ben-Chaim, Fey, Fitzgerald, Benedetto, and Miller (1997) concluded that taking a problem-based approach that draws on children’s knowledge of everyday contexts (see, for example, Zawojewski & Hoover, 1996) could help students to construct effective networks of knowledge, understanding, and skills.

While the idea of basing learning activities on children's out-of-school experiences seems logical, it necessitates "multiplied and more intimate" contacts between teachers and students, and thus "more, rather than less, guidance by others" (Dewey, 1938, p. 8). It is also important to note that some connections that seem obvious to adults prove difficult for children to make. Thomas (2000), for instance, showed that while many children grouped objects in tens to make them easier to count (about 20% in Grade 2 and 60% in Grade 6), only one (Grade 1) child of 132 observed and interviewed was able to explain why it made sense to do that. What most adults take for granted proved to be difficult for the young children to grasp, as it involves not only understanding the structure of the number system and being able to group and count by tens, but also the second form of connected knowing above—understanding of the way that this knowledge and this skill are related.

Again, advice was readily available to available to the case study teachers. For instance, their course advice document, referring to the link between everyday experience and mathematics pedagogy, states that:

> If this connection is not made, students may see school knowledge as different from, and not relevant to, their out-of-school life. … meaningful activities will encourage the learner to explore their understanding of the world and mathematics. At all levels teachers should draw upon students' prior everyday activities to ensure abstract ideas are linked to the familiar. (Victorian Directorate of School Education, 1995, p. 10)

All four of the teachers mentioned this strategy, without prompting and early in the interviews. Susan explained:

> I have always held the philosophy that maths has to be real, that if it can't be applied practically, and made use of in everyday life, its value for most people is questionable. If it is not made real, it is not made understandable. … That's particularly true when I am introducing new ideas. If they can see the use for it, and make the links with real world interests and events, then they will take an interest. But if they can't they just won't make the effort to comprehend.

In the videotaped lessons, however, it was noticeable that there was more time spent in making connections with children's everyday experience when a new topic were introduced than at other times. Such appeals to real contexts were also generally short-lived. Typically, and especially in the teaching and learning of Number, teachers moved quickly from discussion of children's experience to exercises involving abstract examples and work.

All 4 teachers talked about the difficulty of finding appropriate contexts for abstract ideas and operations, but Susan articulated the need for children to be able to refer back to simple understandings of practical contexts as they meet more difficult operations. For example, regarding dividing fractions by inverting the divisor and then multiplying, Susan said:

> When you cut half a cake into quarters you don't need to calculate how many pieces you

> have or how big each piece is. But if you do it and discuss the result, at least the kids will have met the idea in a simple form, so they will have some comprehension of not how it works so much but the fact that it does work—and that it can be shown to work.

No overt discussions about how particular mathematics are likely to be applied in everyday contexts in the children's later lives were observed, other than one comment by Tracey: "You have really achieved something, haven't you? You understand it, so you can use it all the time now at home and in school". However, implicit links were made in many of the lessons observed, particularly through the use of word problems, discussions about everyday events (e.g., percentage being used to describe the fat content of foods and in calculation of sale prices) in lesson introductions as well as in sets of application problems used in latter parts of many lessons. Susan described her pragmatics approach: "I think it is not important to have everything useful in their own lives today, so long as they get a sense that maths is sensible and will be useful in all sorts of ways in the future".

CONCLUSION

It is clear that the making of connections is important activity for both teachers and learners in classrooms where teaching is aimed at building mathematical understanding. Articulation of the types of connections that could be made is useful, as categories can be used as a basis for research into teachers' beliefs and actions as well as for pre-service and in-service professional development.

When discussing how they develop children's mathematical understanding, the primary teachers participating in my research referred spontaneously to the three different types of connections that are referred to most commonly in relevant literature and in their formal curriculum documents, but in practice their development of "connected knowing" could have been stronger, more frequent and more consistent. Whether their level of activity in this respect is common in other classrooms, schools and countries is a point for further research, and there is also potential to research how other teachers, texts, curriculum structures, and teacher educators facilitate the development of these three, and other, forms of connections.

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ALGEBRAIC AND GEOMETRIC APPROACH IN FUNCTION PROBLEM SOLVING

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This study explores students' algebraic and geometric approach in solving tasks in functions and the relation of these approaches with complex geometric problem solving. Data were obtained from 95 sophomore pre-service teachers, enrolled in a basic algebra course. Implicative statistical analysis was performed to evaluate the relation between students' approach and their ability to solve geometric problems. Results provided support for students' intention to use the algebraic approach to solve simple function tasks. Students who were able to use geometric approach had better results in solving complex geometric problems. Implications of findings for teaching functions are discussed.

INTRODUCTION AND THEORETICAL FRAMEWORK

There is an increasing recognition that functions are among the most important unifying ideas in mathematics. Functions form the single most important idea in all mathematics, in terms of understanding the subject as well as for using it for exploring other topics in mathematics (Romberg, Carpenter, & Fennema, 1993, p.i). The understanding of functions does not appear to be easy, given the diversity of representations associated with this concept, and the difficulties presented in the processes of articulating the appropriate systems of representation involved in problem solving (Yamada, 2000). Although it is complex and difficult, attaining a deep and rich understanding of the concept of function is crucial for success in mathematics. Yet, despite its importance, numerous studies suggest that many students, including pre-service teachers (Even, 1993), demonstrate an instrumental understanding of the concept of function (Sfard, 1992).

In this study the term representations is interpreted as the tools used for representing mathematical ideas such as tables, equations and graphs. Concern has been growing about the role of representations in teaching and learning mathematics. The NCTM's Principles and Standards for School Mathematics (2000) document includes a new process standard that addresses representations (p. 67), while the status of function representations has been elevated even further. The use of different modes of representations and connections between them represents an initial point in mathematics education at which students use one symbolic system to expand and understand another (Leinhardt, Zaslavsky, & Stein, 1990, p. 2).

Several researchers addressed the importance of connections between the different modes of representations in functions and in solving problems (Moschkovich, Schoenfeld, & Arcavi, 1993; Yamada, 2000). Students' handling of different representations permits ways of constructing mental images of concepts in functions.

Moreover, the connections between the different modes of representations influence mathematical learning and strengthen students' ability in using mathematical concepts of functions (Romberg, Fennema, & Carpenter, 1993, p.iii). Knuth (2000), indicated that students have difficulties in making the connections between different representations of functions (formulas, graphs, diagrams, and word descriptions), in interpreting graphs and manipulating symbols related to functions.

The theoretical perspective used in the present study is related to a dimension of the framework developed by Moschkovich, Schoenfeld, & Arcavi (1993). According to this dimension, there are two fundamentally different perspectives from which a function is viewed. The *process perspective* and the *object perspective* have been described. From the *process perspective,* a function is perceived of as linking x and y values: For each value of $x,$ the function has a corresponding y value. Students who view functions under this perspective could substitute a value for x into an equation and calculate the resulting value for y or could find pairs of values for x and y to draw a graph. In contrast, from the *object perspective,* a function or relation and any of its representations are thought of as entities—for example, algebraically as members of parameterized classes, or in the plane as graphs that are thought of as being 'picked up whole' and rotated or translated (Moschkovich et al., 1993). Students who view functions under this perspective could recognize that equations of lines with the form $y = 3x + b$ are parallel or could draw these lines without calculations if they have already drawn one line or they can fill a table of values for two functions (e.g., $f(x) = 2x$, $g(x) = 2x + 2$) using the relationship between them (e.g., $g(x) = f(x) + 2$) (Knuth, 2000). The algebraic approach is relatively more effective in making salient the nature of the function as a process while the geometric approach is relatively more effective in making salient the nature of function as an object (Yerushalmy & Schwartz, 1993).

Sfard (1992) has argued that the ability of seeing a function as an object is indispensable for deep understanding of mathematics. Furthermore, developing competency with functions means moving towards the object perspective and graphical representation (Moschkovich et al., 1993). Students being able to profitably employ object perspective can achieve a deep and coherent understanding in functions. We believe that this point to point approach, that is the algebraic approach, gives students only a mere and local image of the concept of function. On the contrary, the geometric approach gives students a global approach of the concept of functions, so that students, who can manipulate and use it, will perform better in solving complex geometric problems.

The purpose in this study is to contribute to the mathematics educational research community's understanding of the algebraic and geometric approach students develop and use in solving function tasks and to examine which approach is more correlated with students' ability in solving complex geometric problems.

METHOD

The analysis was based on data collected from 95 sophomore pre-service teachers enrolled in a basic algebra course during the spring semester in 2003. The subjects were for the most part students of high academic performance admitted to the University of Cyprus on the basis of competitive examination scores. Nevertheless there are big differences among them concerning their mathematical abilities.

The instruments used in this study were two tests. The first consisted of four tasks, implementing simple tasks with functions. In each task, there were two linear or quadratic functions. Both functions were in algebraic form and one of them was also in graphical representation. There was always a relation between the two functions (e.g. $f(x) = x^2$, $g(x) = x^2+2$). Students were asked to interpret graphically the second function. The second test consisted of two problems. The first problem consisted of textual information about a tank containing an initial amount of petrol and a tank car filling the tank with petrol. Students were asked to use the information to draw the graphs of the two linear functions and to find when the amounts of petrol in the tank and in the car would be equal. The second problem consisted of a function in a general form $f(x) = \mathrm{a}x^2+\mathrm{b}x+\mathrm{c}$. Numbers a, b and c were real numbers and the $f(x)$ was equal to 4 when x=2 and $f(x)$ was equal to -6 when x=7. Students were asked to find how many real solutions the equation $\mathrm{a}x^2+\mathrm{b}x+\mathrm{c}$ had and explain their answer. The tests were administered to students by researchers in a 60 minute session during algebra course.

The results concerning students' answers to the above tasks and problems were codified in three ways: (a) "A" was used to represent "algebraic approach – function as a process" to tasks and problems. (b) "G" stands for students who adopted a "geometric approach – function as an entity". (c) The symbol "W" was used for coding wrong answers. A solution was coded as "algebraic" if students did not use the information provided by the graph of the first function and they proceeded constructing the graph of the second function by finding pairs of values for x and y. On the contrary, a solution was coded as "graphical" if students observed and used the relation between the two functions (e.g. $g(x) = f(x) +2$) in constructing the graph of $g(x)$. This paper is focused on the first two types of responses. Moreover, implicative analysis (Gras, Peter, Briand & Philippé, 1997) was used in order to identify the relations among the possible responses of students in the tasks and problems. Therefore, twelve different variables representing the algebraic and geometric approaches emerged. More specifically, the following symbols were used to represent the solutions, involved in the study: (a) Symbols "T1A", "T2A", "T3A" and "T4A" represent a correct algebraic approach to the tasks and "P1A" to the first problem (second problem could not be solved algebraically). (b) Similar, symbols "T1G", "T2G", "T3G" and "T4G" represent a correct geometric approach to the tasks and "P1G" and "P2G", correct graphical solutions to the problems respectively.

For the analysis and processing of the data collected, implicative statistical analysis was conducted using the statistical software CHIC. A similarity diagram and a hierarchical tree were therefore produced. The similarity diagram represents groups of variables, which are based on the similarity of students' responses to these variables. The hierarchical tree shows the implications (A◊B) between sets of variables. This means that success in A implies success in B (Gras, et al., 1997).

RESULTS

The main purpose of the present study was to examine the mode of approach students use in solving simple tasks in functions and to test which approach is more correlated with solving complex geometric problems. Most of the students correctly solved the tasks involved linear functions (T1 and T2). Their achievement radically reduced in tasks involved quadratic functions (T3 and T4) and especially in solving complex geometric problems, only 27.4% and 11.6% of the 95 subjects correctly provided appropriate solutions (Table 1). More than 60% of the students that provided a correct solution chose an algebraic approach, even in situations in which a geometric approach seemed easier and more efficient than the algebraic. Furthermore, in the second problem, most of the students failed to recognize or suggest a graphical solution as an option at all.

	Tasks and Problems					
	T1**	T2	T3	T4	P1	P2
Geometric approach with correct answer	24.2*	23.2	19	21	16	11.6
Algebraic approach with correct answer	59	56.8	22.1	24.3	11.4	0
Incorrect Answer	16.8	20	58.9	54.7	72.6	88.4

Table 1: Students' responses to tasks and problems.

* Numbers represent percentages.

* T1 – T4: Tasks in functions. P1 – P2: Geometric problems.

Students' correct responses to the tasks and problems can be classified according to the approach they used and they are presented in the similarity diagram in Figure 1. More specifically, two clusters (i.e., groups of variables) can be distinctively identified. The first cluster consists of the variables "T1A", "T2A", "T4A" and "T3A" which represent the use of the algebraic approach (process perspective). The second cluster consists of the variables "T1G", "T2G", "P1G", "P2G", "P1A", "T3G" and "T4G" and refers to the use of the geometric approach (object perspective) and solving geometric problems. The emergence of the two clusters is in line with the assumption of the study and reveals that students tend to solve tasks and problems in functions using the same approach, even though in tasks that a different approach is more suitable.

It can also be observed from the similarity diagram that the second cluster includes the variables correspond to the solution of the complex geometric problems with the variables representing the geometric approach.

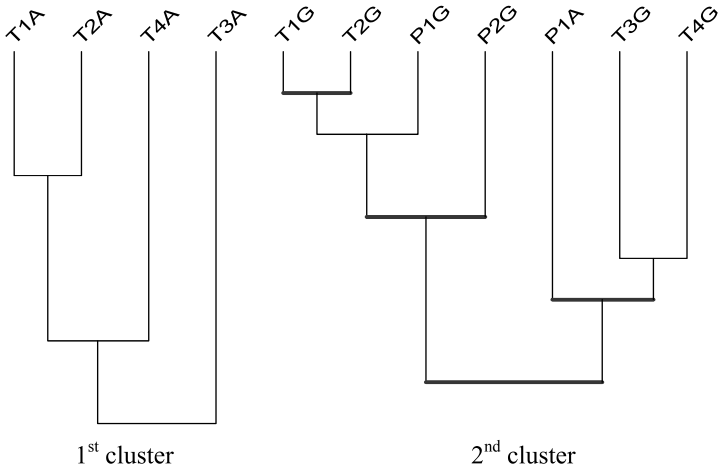

Figure 1: Similarity diagram of the variables

Note: Similarities presented with bold lines are important at significant level 99%

More specifically, students' geometric approach to simple tasks in functions is closely related with their effectiveness in solving complex geometric problems. This close connection may indicate that students, who can use effectively graphical representations, are able to observe the connections and relations in geometric problems, and are more capable in problem solving. It is also important to acknowledge that almost all of the similarities in the second cluster are statistically significant at level 99% and this refers to the geometric approach and the complex geometric problem solving.

Significant implicative relations between the variables can be observed in the hierarchical tree, which is illustrated in Figure 2. First, three groups of implicative relationships can be identified. The first group and the third group of implicative relations refer to variables concerning the use of the geometric approach – object perspective and variables concerning solution to the geometric problems. The second group provides support to the existence of a link among variables concerning the use of algebraic solution-process perspective. This finding is in line with the findings emerged from the similarity diagram. The formation of these groups of links indicates once again the consistency that characterizes students' provided solutions towards the function tasks and problems. Second, the implicative relationship (P2G, P1G, T2G, and T1G) indicates that students, who solved the second problem by applying the correct graphical solution, have implied the application of the object perspective – graphical representation for the other problem and the four tasks. An explanation is that, students who have a solid and coherent understanding of functions can identify relations and links in complex geometric problems and thus can make the necessary connections between pairs of equations and their graphs, and easily apply geometric approach in solving simple tasks in functions.

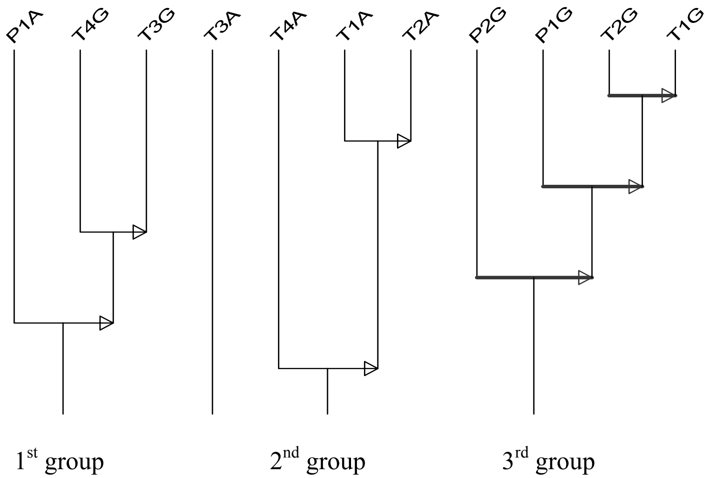

Figure 2: Hierarchical tree illustrating implicative relations among the variables

Note: The implicative relationships in bold colour are significant at a level of 99%

DISCUSSION

It is important to know whether pre-service teachers are flexible in using algebraic and geometric representations in function problems. Although problems used in this study are some of those taught at school, subjects had difficulties, especially when they needed to implement geometric approach. Many students have not mastered even the fundamentals of the geometric approach in the domain of functions. Students' understanding is limited to the use of algebraic representations and approach, while the use of graphical representations is fundamental in solving geometric problems.

The most important finding of the present study is that for the group of pre-service teachers two distinct sub-groups are formatted with consistency: the algebraic and the geometric approach group. The majority of students' work with functions is restricted to the domain of algebraic approach and this process is followed with consistency in all tasks. As a consequence, few students develop ability to flexibly employ and select graphical representations, thus geometric approach. The present study is in line with the results of previous studies indicating that students can not use effectively the geometric approach, which engenders within the object perspective (Knuth, 2000). The fact that most of the students chose an algebraic approach (process perspective) and also demonstrated consistency in their selection of the algebraic approach even in tasks and problems in which the geometric approach (object perspective) seemed more efficient or that they failed to suggest a graphical approach at all, is particularly distressful considering that the students participated in the study thought to be representative of our best students.

Moreover, an important finding is the relation between the graphical approach and geometric problem solving. This finding is consistent with the results of previous studies (Knuth, 2000; Moschkovich et al., 1993), indicating that geometric approach enables students to manipulate functions as an entity, and thus students are capable to find the connections and relations between the different representations involved in problems. The data presented here suggest that students who have a coherent understanding of the concept of functions (geometric approach) can easily understand the relationships between symbolic and graphic representations in problems and are able to provide successful solutions. Moreover, data provided support that there is a close relationship between the use of a geometric approach in functions and better understanding of equations, graphs and functions in general.

Researchers have suggested that the difficulties students have on tasks and problems in functions that require the implementation and use of graphical representations may be due to the fact that students can not focus in the information a graph provides. Thus, students may be unaware that the graphical representation offers a means for determining a solution (Moschkovich et al., 1993). Moreover, in some cases the graphical representations create cognitive difficulties that limit students' ability to make connections between the algebraic and the graphical representations (Gagatsis, Elia, & Kyriakides, 2003). One factor that has a significant influence on students' preference in the algebraic solution is probably the curricular and instructional emphasis dominated by a focus on algebraic representations and their manipulation (Dugdale, 1993). In their textbooks, students are usually asked to construct graphs from given equations using pairs of values, while solving geometric problems follows the same procedure (Leinhardt, et al., 1990). As a result, graphical representations are seemed as unconnected to the corresponding algebraic representations and students fail to make the necessary connections between them and furthermore to effectively employ geometric approach and graphical representations (Yerushalmy & Schwartz, 1993).

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CHILDREN'S DEVELOPMENT OF STRUCTURE IN EARLY MATHEMATICS

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In a descriptive study of 103 Grade 1 children, including 16 longitudinal case studies, we investigated the use of mathematical structure across a range of mathematical tasks. Children's structural development was inferred from their imagistic representations, supported by analysis of problem-solving behavior during videotaped task-based clinical interviews. We describe with examples how internal representational systems change through four stages of structural development.

In this paper we describe an analysis of mathematical structure present in children's representations as they solve five mathematical tasks during clinical interviews: time, *triangular pattern, area, length* and *picture graph*. Goldin (2002) emphasises that individual representational configurations, whether external or internal, cannot be understood in isolation. Rather they occur within representational systems. These are not mere collections of representations, but have complex structures that in practice may be ambiguously defined or context-dependent. In this study we build upon previous analyses (De Windt-King & Goldin, 2001; Goldin & Passantino, 1996; Mulligan, 2002; Mulligan, Mitchelmore, Outhred & Russell, 1997; Thomas, Mulligan & Goldin, 2002), and extend the mathematical content domains with the aim of making as explicit as possible the bases for children's structural development.

BACKGROUND

In a longitudinal study of Grade 2 to 3 children's intuitive models for multiplication and division, Mulligan and Mitchelmore (1997) found that the intuitive model employed to solve a particular word problem did not necessarily reflect any specific problem feature but rather *the mathematical structure that the student was able to impose on it*. Students acquired increasingly sophisticated strategies based on an equal-groups structure and their calculation procedures reflected this structure.

Another 2-year longitudinal study of Grade 2 to 3 children's representations focused on how they imposed structure, or lack thereof, on numerical situations (Mulligan et al., 1997). Low achievers were more likely to produce poorly organised, pictorial and iconic representations lacking in structure. These children lacked flexibility in their thinking; they were barely able to replicate models of groups, arrays, or patterns. Low achievement was attributed to children's dependence on unitary counting and weak visualisation skills. A follow-up study of 24 of these children tracked to Grade 5 indicated that low achievers consistently lacked mathematical structure; pictorial and iconic representations dominated responses (Mulligan, 2002). High achievers, however, used abstract notational representations with well-developed structures from the outset in Grade 2.

Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 2004 Vol 3 pp 393–400

Further studies investigating children's mental images of number concepts have identified structural elements such as grouping, regrouping, partitioning and patterning found within the recordings of the numbers 1 to 100 (Thomas et al., 2002). Mathematically gifted children's representations showed recognisable mathematical structure and dynamic imagery, whereas low achieving children's representations showed no signs of underlying structure, and use of static imagery.

Our interest in mathematical structure has also drawn upon research on imagery and spatial visualisation (Battista, 1999). Imagery appears to be a central influence in structural development of mathematical ideas (Pirie & Martin, 2000). Battista (1999) defines *spatial structuring* as:

> the mental operation of constructing an organization or form for an object or set of objects. It determines the object's nature, shape, or composition by identifying its spatial components, relating and combining these components, and establishing interrelationships between components and the new object. (p. 418)

Children's spatial structuring has been highlighted in studies of two and three-dimensional situations such as arrays of squares in rectangles, and cubes in rectangular boxes (Battista, 1999; Battista, Clements, Arnoff, Battista, & Borrow, 1998). Most children learn to construct the row-by-column structure of rectangular arrays and also acquire the equal-groups structure required for counting rows and layers in multiples (Outhred & Mitchelmore, 2000; Reynolds & Wheatley, 1996).

The following conclusions are drawn as common findings from these studies:

- The more that the child's internal representational system has developed structurally, the more coherent, well-organised, and stable in its structural aspects will be their external representations, and the more mathematically competent the child will be.
- A child's external imagery reflects the structural features of his or her internal representations, and this provides a view of the child's conceptual understanding.
- There are common structural features evident in children's representations across mathematical content domains; spatial structuring is a critical feature.

THE PURPOSE OF THIS STUDY

Our analyses of structural development has not documented early stages sufficiently, and has relied on specific number tasks. We therefore posed the research question:

- Do young children (children aged from 5 years 6 months) develop and use structure consistently across different mathematical content domains and contexts?

METHOD

The sample comprised 103 Grade 1 children, 55 girls and 48 boys, ranging from 5.5 to 6.7 years of age who were administered task-based interviews. This broadly representative sample was drawn from nine state schools in Sydney. A sub-sample of sixteen children representing extremes in mathematical ability was selected for in-depth case study over two years on the basis of initial interview data.

Thirty tasks were developed to explore common elements of children's use of mathematical and spatial structure within number, measurement, space and data. Each task required children to use or represent elements of mathematical structure such as equal groups or units, spatial structure such as rows or columns, or numerical and geometrical patterns. Number tasks included subitizing, counting in multiples, fractions and partitioning, combinations and sharing. Space and data tasks included a triangular pattern, visualising and filling a box, and completing a picture graph. Measurement tasks investigated units of length, area, volume, mass and time. This paper reports an analysis of children's responses to the *time, triangular pattern, area, length,* and *picture graph* tasks (see Table 1).

Table 1: Selected interview tasks

Time: clockface	Draw a clock and show 8 o'clock on it.
Triangular pattern	(Show flash card with triangular pattern of six dots.) Draw from memory exactly what you saw.
Area: unitizing	(Show 3x4 rectangle with squares drawn along two adjacent sides.). Finish drawing the squares, exactly like these, to cover all of this shape.
Length: ruler	(Show drawing of long thin rectangle.) Imagine this strip is a ruler. Draw as many things as you can remember about a ruler.
Picture graph	(Show table of data.) There are 7 dogs, 5 cats, 3 birds. Finish drawing the graph to show the number of animals.

Children were asked to visualise, then draw and explain their mental images. They were given plenty of time and opportunity to provide alternate solutions and reproduce or modify their drawings. Operational definitions and a refined coding system were formulated from the range of responses elicited in pilot interviews and in accordance with coding employed in previous studies. A coding system was complied from analysis of videotapes with an inter-rater reliability of 94%.

Children's representations were considered with respect to three dimensions: (i) the type of component signs in the representation (pictorial, iconic or notational); (ii) the stage of structural development evidenced by features such as equal units; and (iii) the static or dynamic nature of the image represented. Additionally, we considered (iv) the verbal explanations describing the representation.

RESULTS

Children's responses indicated four broad stages of structural development:

1. In an initial *pre-structural* stage, representations lacked any evidence of mathematical or spatial structure; most examples showed idiosyncratic features.

2. This is followed by an *emergent* inventive-semiotic stage where representations show some elements of structure and in which characters or configurations are first given meaning in relation to previously constructed representations.
3. The next stage shows evidence of *partial structure*: Some aspects of mathematical notation and/or spatial features such as grids or arrays are found.
4. The following stage is a *stage of structural development*, where the representations clearly integrate mathematical and spatial structural features.

In order to illustrate levels of structural development, we discuss representative examples below. (Codes such as C1, T2 refer to individual drawings, not to stages.) We focus on the child's imagistic representations and how these conform to mathematical aspects such as the correct numerical quantity, use of formal notation, uniformity in size of units, and spatial organization of units.

Pre-structural stage of development

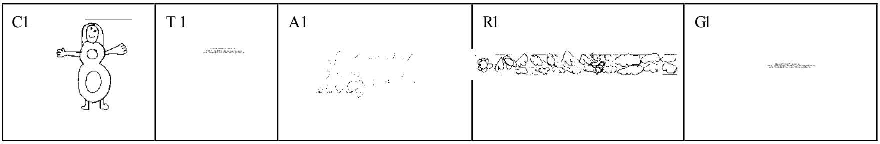

Figure 1: Stage of pre-structural development

Figure 1 shows representations that reflect some features of the mathematical tasks, such as a dog in G1, but there is no evidence of mathematical or spatial structure. Idiosyncratic features dominate, for example, depiction of a clock (C1) as a girl - although there is an attempt to represent the numeral 8. In the triangular pattern (T1), the row of circles bears no relationship to the triangular shape, numerical pattern, or quantity of the circles in the stimulus picture.

Emergent, inventive-semiotic stage of development

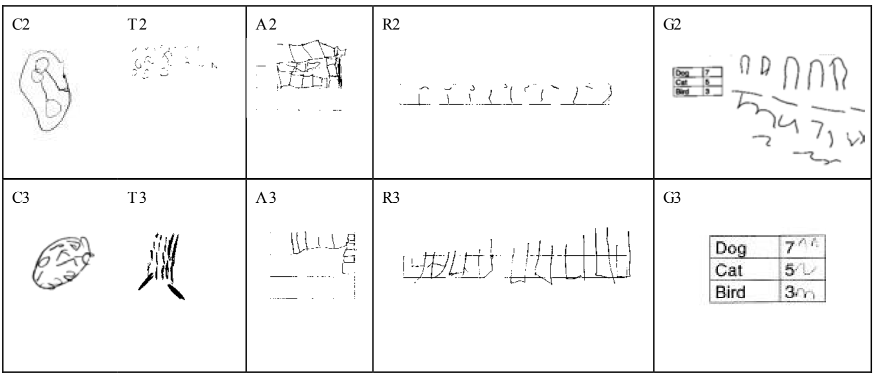

Figure 2: Stage of emergent structural development

Figure 2 shows evidence of emergent, inventive-semiotic representations. These include iconic, pictorial, and notational signs with some structural feature but focus on the aspect of the task representation most significant to the child. Figure T2 shows squiggles that are unrelated to the triangular pattern of dots but there is at least a representation of the correct quantity of circles. Figure T3 shows a triangular form drawn as a ‘Christmas tree’ and an attempt to draw the pattern as vertical rows of five dots. There is little awareness of the structure or number of items in the pattern; there is some indication of spatial structure with equally spaced marks. We found evidence of meaning assigned to idiosyncratic characters where attempts are made to use icons (dashes, dots, lines, objects), and symbols to represent numeral quantity, equal units or ‘spatial structure’.

Stage of partial structural development

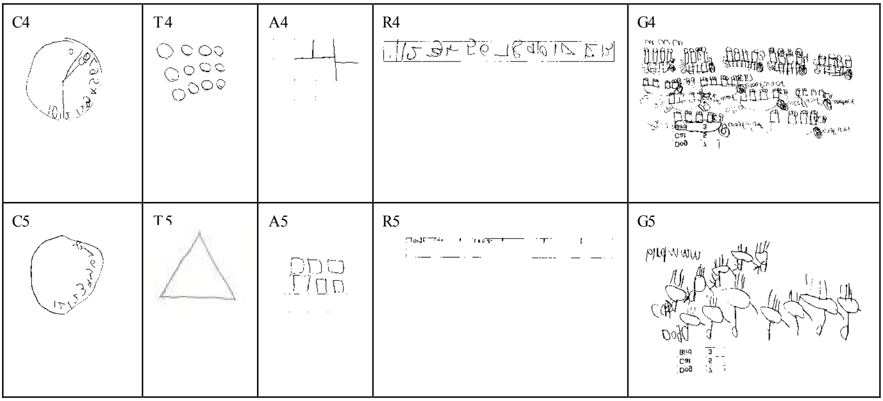

Figure 3: Stage of partial structural development

At this stage children depict partial structure in their representations, with at least one structural element; recordings were more organised but independent of idiosyncratic features. For example, drawings G4 and G5 show correct quantities, but there is no attempt to draw rows or align items with the grid provided. Representations of animals dominate, but these images are directly related to the task content. Representations for the area task show an awareness of structure, consistent with the findings of Outhred & Mitchelmore (2000). Drawings A4 and A5 show partial structure as a correct number and size of squares in a border pattern, and correct number and alignment of individual squares respectively. These examples do not, however, show any indication that rows and columns are coordinated – although the equal groups structure of multiplication may be emerging.

Stage of structural development

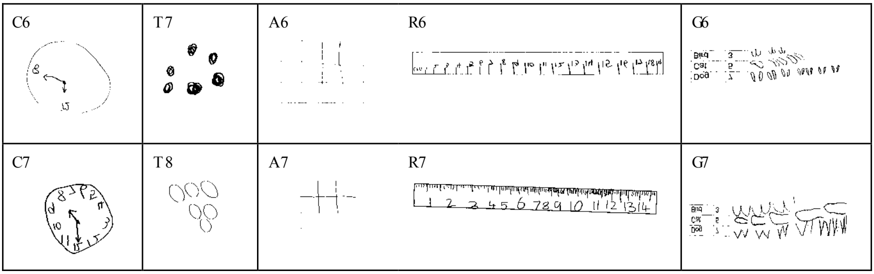

Figure 4: Stage of structural development

Responses showing well-developed features of mathematical and spatial structure are depicted in all the examples in Figure 4. These show clear evidence of an internal system that uses common elements of mathematical structure (e.g. the use of equal sized units in the clock face, area and ruler examples; the integration of spatial and numerical pattern in the triangular pattern task; and one-to-one matching and horizontal alignment in the graphs task). In the ruler task, structural features such as equal-size units were often symbolised as centimetres with subdivision, indicating conceptual understanding of linear measurement prior to formal instruction (Bragg & Outhred, 2000; Nuhrenborger, 2002).

The above results show that children's responses to the tasks can be reliably coded for the presence of structural features to obtain a developmental sequence. We questioned whether this structural development is consistent across tasks. Responses to all 30 tasks were coded for all 103 children and the matrix examined for patterns. It was found that the children could be unambiguously sorted into four groups. In Group 1, 95% or more of their 30 responses fell in the pre-structural stage. In Group 2, 80% or more of the responses fell in the stage of emergent structural development, and in Group 3, 80% or more of the responses fell in the stage of partial structural development. In Group 4, 95% or more of the responses fell in the stage of structural development. These results indicate an astonishing consistency of structural representation across tasks.

Case studies of low achieving children indicated that the key difficulty was not necessarily the comprehension of tasks or their ability to draw or to count, but the child's lack of perception of structure. All the low achieving children in fact fell into Group 1. Conversely, the high achieving children all fell into Group 4 and readily expressed the mathematical structure in all or almost all of the tasks.

DISCUSSION

This study has produced two extremely important findings:

- Young children's perception and representation of mathematical structure generalises across a wide variety of mathematical tasks.
- Early school mathematics achievement is strongly correlated with the child's development and perception of mathematical structure.

The first result supports what many have said about the importance of structure in mathematical understanding (Gray, Pitta & Tall, 2000). This study, however, advances our understanding by showing that stages of structural development can be measured and described across a wide variety of tasks.

We must clearly ask why it is that some young children do not develop structure in their representations of critical mathematical concepts. Teaching young children to focus on structure in early mathematics may require professionals to pay more attention to children's representations and to help them focus more clearly on the mathematical structure inherent in the various situations encountered. In individual studies, a structural approach has already shown benefits in terms of children's understanding of length and area. Consistent application of a structural approach (e.g. assisting children to visualise and record simple spatial patterns accurately) could lead to much broader improvements in children's mathematical understanding. We plan to undertake longitudinal investigations (using multiple case studies) to track the structural development of low and high achieving children from school entry through to primary level. We aim to identify classroom influences that tend to promote or impede the development of structure in children's representations.

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ON THE FRAGILE, YET CRUCIAL, RELATIONSHIP BETWEEN MATHEMATICIANS AND RESEARCHERS IN MATHEMATICS EDUCATION

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The relationship between mathematicians and researchers in mathematics education has often been fragile. Yet it is crucial. We conducted a series of themed Focus Group interviews with mathematicians from six UK universities. Pre-distributed samples of mathematical problems, typical written student responses, observation protocols, interview transcripts and outlines of relevant bibliography were used to trigger an exploration of pedagogical issues. Here we elaborate the theme "the relationship, and its potential, between mathematicians and researchers in mathematics education" that emerged from the data analysis. We do so by presenting the participants' views on this relationship in terms of: ***obstacles****,* ***desired characteristics*** *and* ***potential benefits****.*

INTRODUCTION

The relationship between theory and practice in mathematics education is often fraught with suspicion (Sierpinska & Kilpatrick 1998), even hostility. This applies across the educational spectrum – for example within the primary and secondary sectors where policy makers often suffer criticism that their decisions are rarely and marginally informed by research in mathematics education [Brown in (Sierpinska & Kilpatrick 1998)]. Nowhere however is this more evident than within the tertiary sector (Ralston 2003). Recent developments in the world of university mathematics, such as the changing enrollment and profile of the student intake (Holton 2001), have resulted in a need for mathematics departments to rethink curricula, e.g. (Kahn and Hoyles 1997), and pedagogical practices, e.g. (McCallum 2003). In doing so a rapprochement between the worlds of mathematics and mathematics education research has become vital [Artigue in (Sierpinska & Kilpatrick 1998)].

Here we elaborate this issue through drawing on the views of university mathematicians participating in a study currently in progress in the UK. For an outline of the methodology of the study see the ENDNOTE. Participants were twenty mathematicians, pure and applied, with teaching experience ranging from a few years to several decades, all but one male and of varying rank. In six out of the eighty *Stories* which formed the analytical units of the study (see ENDNOTE) the participants expressed views on what we present here grouped as: *obstacles* and *desired characteristics* of the relationship between mathematicians and researchers in mathematics education; and, *potential benefits* for mathematicians engaged as educational co-researchers.

OBSTACLES

The participants in the study generally acknowledged that, despite its importance, the relationship between mathematicians and researchers in mathematics education is weaker than it ought to and could be. Across the six *Stories* there was extensive discussion on what these weaknesses are. Issues of *trust*, *access*, *priority*, *communicability, applicability* and *subtlety* dominated this discussion.

Participants often stated that we, the team of researchers in mathematics education conducting this study, were the 'first ones' they 'ever talked to'. While appreciating the fact that our mathematical background allowed elaborate examination of learning and teaching issues that are specific to university mathematics (see BENEFITS), they also expressed concern that, if that was not the case, they would have found it difficult and rather unproductive to participate in 'content-free' pedagogical discussion.

Asked about where in the mathematics literature a mathematician is likely to find educational articles, the *Notices* of the American Mathematical Society were mentioned as was the *Mathematical Gazette* in the UK. However, participants admitted that they would 'never come across' papers such as the ones published in the *PME Proceedings* or in mathematics education journals. At the heart of the problem seems to be the fact that the two worlds do not meet very much: both mathematicians and researchers in mathematics education need to publish in and read journals in their own areas (e.g. for the purpose of research-assessment exercises) and there is precious little time for reading each other's journals. 'We are more likely to read pedagogically thoughtful books than journals (e.g. Polyà)', as one participant put it. 'In its bulk' the mathematics community does not 'look to this type of research as a source of knowledge or ideas about mathematics teaching. It just doesn't… whether it should or not is a different matter', he suggested. He then concluded: it is still the case that the image of a mathematics department that pays a lot of attention and contributes to research in mathematics education could suffer as this appears as a digression from its main research agenda.

Issues of access and priority notwithstanding, participants highlighted a deeper difficulty in establishing and maintaining communication between the two worlds. Many mathematics departments are committed to the idea that 'research driven tertiary education is the only and best way to teach mathematics', to the 'terribly arrogant' idea that 'we are the holders of the knowledge of how tertiary mathematics should be taught' and that 'universities should not be giving [degrees] to people if they don't have active research faculty'. Under these circumstances rethinking the profile of a mathematics graduate as someone who may also have pedagogical faculty (namely could become a mathematics teacher or a researcher in mathematics education) then becomes extremely difficult – as does an appreciation of what engagement with research in mathematics education may have to offer in terms of university-level pedagogical practice.

On the other hand most participants reserved weary suspicion towards epistemological debates within the educational community which were seen as moving the target from offering to mathematicians something that is 'somehow connecting with what I [am] doing in a lecture theatre or in a seminar group' to 'egotistical nonsense' about how, for example, individual educational researchers use particular terminology. As one participant put it: in order to 'pay attention', 'first I would want them to kind of not be arguing about what they are meaning'. This comment reflects a more profound epistemological gap between the two worlds: for these mathematicians concept definitions within mathematics are stable – stability having been achieved through arduous negotiation and evolution over the years – and so should concept definitions be within educational research. Whether this is desirable or even feasible (Schoenfeld 1994) within the community of educational researchers is a matter of debate. It seems however an issue that could be resolved were the two worlds to meet more fruitfully.

Furthermore, according to some participants, the 'noble cause' of 'involving both communities' is not always best served by a presentation of educational findings that is often seen as 'almost indecipherable'. Additional evidence to these statements comes from the following fact: across the eleven Cycles of Data Collection, the brief literature reviews on each of the six *Themes* (see ENDNOTE) accompanying the Datasets that participants were invited to consider prior to the interviews *never* became part of the conversation spontaneously – unlike the samples of data (from students' writing, interviews etc) that proved highly effective triggers for discussion. 'Indecipherability' seemed to be the main reason while some participants suggested that some educational 'jargon' is inevitable if analysis is to do justice to the subtlety of data. Sophistication in the data analysis was seen as a *sine qua non* for the participants: in the few occasions where, following probing by the interviewees, participants discussed, for example, models of mathematical understanding, they appeared to be variably impressed. On a proactive note, one participant suggested that small glossaries could facilitate the exchange amongst interlocutors from different disciplines. Trying to understand writing in mathematics education is actually a 'fun' exercise: 'in the same way in which you can read a political essay and so on and you can take a certain pleasure just from that'.

Even further, we see the issue of building a common language as not separate from building a 'mutual agenda' (Barbara Jaworski, personal communication) of research – all stages of it, from inception through to execution and dissemination. Or, as she put it, for example regarding data analysis: 'I [wonder] to what extent the mathematicians contribute to the analysis or consider it when it is written'. One way of addressing the issue of communicability of findings is ensuring that the audience these findings aim at addressing has ownership and participation in the process that brought these findings to existence in the first place. Other characteristics of research in mathematics education seen by the participants in our study as desirable are presented in the next section.

DESIRES

Participants often proposed that dissemination of educational research findings should be done more systematically in mathematics departments (for example through seminars, workshops etc.). Tertiary level teacher training, at least in the UK currently in the form of courses on higher education practice for newly appointed lecturers (http://www.ilt.ac.uk), was proposed as ‘the kind of places in which this kind of research should be disseminated’. However some participants suggested that often these courses are bogged down to ‘epitomising the worst aspects of professional education’ by being ‘content-less’. ‘If I had gone to one of those things’, observed one experienced lecturer, ‘and someone had said something about the ε-δ definition of continuity or the definition of a group or a specific […] thing in chemistry or had talked in these terms about it, I would have been fully attentive and I would have got something from it. So… it is not that the audience is not there’, because ‘the problem [of effective teaching, of recruitment etc.] is there’. While not underestimating the value of interdisciplinary discussion of pedagogy, a younger lecturer amongst the participants observed that, in the face of the challenge of a new lecturing job, where the pressures to prove one’s worth in the substantive field (see also OBSTACLES), ‘content-less’ training courses can engender impatience and be seen as a waste of time.

Beyond ways in which to improve the dissemination of educational research findings within the mathematical community, discussions amongst the participants focused primarily on characteristics of research in mathematics education which would enhance for them the appeal of *collaboration and engagement*. In these discussions issues of methodology (e.g. selecting focus of the research, methods for collecting and analysing data, language) dominated.

In terms of selecting foci for educational research, participants expressed a strong interest in studies which focus on the teaching and learning of specific concepts or topics. Cross-topical studies of mathematical learning were also mentioned. One participant outlined an example: ‘a longitudinal study of a single student’, one that would focus on ‘what in fact are the conceptual hurdles one must overcome’. While acknowledging interviewers’ comments that student-centred studies are already in existence, the participants stressed that mathematicians are not always aware of these studies (returning to the issues raised in OBSTACLES) and, furthermore, would hugely appreciate a consideration in these studies of their own experiences and views. Or as one participant put it: we ‘all have an intuitive feeling on what goes or what doesn’t go. You look at these students, you look at their faces, you know that they are lost’.

In terms of building a common language to discuss the teaching and learning of mathematics at university level, the participants often emphasised that a sufficiently strong mathematical background on the part of researchers in mathematics education helps alleviate some deeply rooted suspicion on the side of mathematicians. Or as one participant strongly put it: ‘…to be honest if people want to find open doors in a mathematics department they need to be able to talk to mathematicians about mathematics, and, if they can’t, maybe they are in the wrong business.’ Another participant followed this comment with a recommendation for 'friendliness' towards the community of mathematics educators. Significantly he also raised the issue of the epistemological gap (see OBSTACLES) between the two communities and made a plea to the mathematical community for a less 'absolutist' spirit when engaging in pedagogical discussion.

Despite reservations towards educational research findings that do not always directly prescribe effective practice (see OBSTACLES), participants in this study expressed a preference for a type of research that 'brings questions, not answers to the table'. The latter, one participant observed, is 'far less interesting'. Drawing on specific examples from the Datasets used in this study, the participants often expressed a preference for a methodology that allows naturalistic data to be examined in an open forum where the researchers in mathematics education, given that they are the apparent educational experts in the discussion, do not assume a position of authority but engage in a more equitable and collaborative exploration of pedagogical issues – a characteristic of this study that was emphatically appreciated by the participants (see BENEFITS).

In addition to a preference for open, naturalistic methodologies, some participants made statements about how this openness needs to be combined with attention to detail and sensitivity to the complexity of pedagogical issues. In particular some participants expressed reservation towards studies which attempt to explain mathematical behaviour in terms of broad variables, thus being exposed to 'the danger of smudging … different things together'. And 'smudging together the effect of certain conditioning, perceivable conceptual things, cognitive things, just smudging all together under the same banner' would be worrying, one participant concluded and stated a preference for a highly focused, 'clinical' approach.

But, we wondered, behind some of these broad variables, e.g. gender, do not often lie controversial issues (such as the issue of gender representation in mathematics faculties in the UK)? For example, in the context of this study one could claim that, by apparently focusing in the Datasets on mathematically-specific issues on learning, by maintaining 'that there is something relatively safe in the kind of questions that we are asking' and by 'not asking for acting upon the problems' (our words during the last group interview) we have stayed clear of controversy in the fear of risking the mathematicians' participation. As a 'safer route' into 'how people involved in a mathematics course are thinking' this adoption of a minimally interfering observer's role, prior to attempting to effect change, was seen by the participants as an appropriately subtle method. One pinned this idea as follows: '… we have been talking about things covered in several different years with as little as possible change on how we view them. […]. If we were sort of on the fly during the process, changing how we presented certain concepts what we would be saying would change very rapidly and we would be somehow observing something, we would be participating in it and […] not to the benefit of understanding what is going on. […] it would be like trying to hit a moving target.' This insight into pedagogical issues was described by participants as the major benefit from collaborative engagement with educational research. In the following section we elaborate their comments on this issue.

BENEFITS

Participants evaluated their experience of the study and often these evaluations led to more general statements about potential benefits from engagement as educational co-researchers. 'There are things I will teach differently. There are things that I feel like I understand better of mathematics students than I did', said one. In particular the research process helped the realisation to emerge how 'one should be liaising with the other lecturers' in order to discuss 'what things we are doing that confuse [students]'. The questioning, 'all guards down', exploratory and non-prescriptive spirit of these discussions coupled with the 'refreshing discipline' in the specificity of questions being asked in the Datasets were also appreciated (see also DESIRES). As one participant put it, 'this is a platform by which we start to rationalise'.

The naturalistic character of the data ('these are real people struggling with a real difficulty, in real time') used as trigger for discussion was often praised. Its character almost steered the conversation away from the hectic frame of mind within which the lecturers' encounter with the masses of student writing usually takes place: there was no need to determine marks or distinguish between 'right' and 'wrong' answers. 'It is really a confirmation that […] the quantitative story about learning is devoid of meaning' suggested one participant. '… a tick box of the average understanding of the concept of group among the students is a figure that has absolutely no meaning at all in it, in comparison with the individual detailed discourse of this student'. And while this may vary across students 'there are similar hurdles and I think that defining the hurdles is important'.

The challenge within the 'rationalisation' mentioned above was also seen as 'a perfect role reversal': 'this idea of having *lecturers* [as opposed to students] thinking hard about what they are understanding when we write […] on the board'. 'I think now I don't have any more answers than when I started [the study] but certainly I don't take things for granted anymore, from colleagues or from students. I think I am much more open-minded on what might be going on inside other people's brains', one participant said of the 'fascinating' increase in pedagogical awareness that participating in the study resulted in. 'The intellectual challenge for me', he continued referring to his teaching 'is […] converting what I know into communicating that to students' and this is 'no routine task'. 'I think we should take an interest in this because obviously we are spending a lot of effort on things that aren't working. The routine milling that we do since the middle ages probably won't produce students who have a broad view of mathematics as an exciting intellectual challenge. I think we can do this much better and I think we can start by mathematicians talking to each other about their own prejudices […] before having a go at innocent students!'. This is 'starting at the right end' if we are to consider alterations of practice, he concludes. The participants often suggested ways in which their teaching practice was affected by the opportunity for a closer look at students' thinking and ways in which students and mathematics teachers across the educational levels can benefit from such reflective processes (we elaborate these elsewhere, for update information: http://www.uea.ac.uk/~m011).

CONCLUSION

The participants in the study acknowledged that, despite its importance, the relationship between mathematicians and researchers in mathematics education is weaker than it ought to and could be. *Trust*, *access*, *priority*, *communicability*, *applicability* and *subtlety* were the dominant **obstacles** they identified. Beyond ways in which to improve the dissemination of educational research findings within the mathematical community (for example through seminars, workshops, tertiary level teacher training etc.) participants often focused on **desired characteristics** of research in mathematics education that would enhance the likelihood of their engagement. These primarily included issues of *methodology* (e.g. selecting focus of the research, methods for collecting and analysing data, language). Finally, with regard to potential **benefits**, the participants valued the opportunity to appreciate the *qualitative paradigm* exemplified by the study, expressed a preference for *in-depth* studies of teaching and learning and elaborated the *gains in pedagogical insight*, both in terms of awareness and in terms of specific modifications in their practice. With regard to the latter, and once the current phase of data analysis is complete, we envisage an Action Research study, where a selection of John Mason's 'tactics' (2002) will be employed and evaluated by practitioners in the context of lectures, seminar groups and assessment of students' written work.

ENDNOTE

This 15-month, LTSN-funded (http://www.ltsn.ac.uk) study engages groups of mathematicians from six institutions in the UK as educational co-researchers (Wagner 1997). There were 11 Cycles of data collection, six with five mathematicians from the University of East Anglia (Cycles 1-6), where the authors work, and five from elsewhere (Cycles 1X-5X). Six Data Sets were produced for each of Cycles 1-6 on the themes *Formal Mathematical Reasoning I: Students' Perceptions of **Proof** and Its Necessity; Mathematical Objects I: the Concept of **Limit** Across Mathematical Contexts; Mediating Mathematical Meaning: **Symbols and Graphs**; Mathematical Objects II: the Concept of **Function** Across Mathematical Topics; Formal Mathematical Reasoning II: Students' Enactment of **Proving Techniques** and Construction of Mathematical Arguments;* and, *A **Meta-Cycle**: Collaborative Generation of Research Findings in Mathematics Education*. The Datasets for Cycles 1-5 were used also for Cycles 1X-5X. Each Dataset consisted of: a short literature review and bibliography; samples of student data (e.g.: students' written work, interview transcripts, observation protocols) collected in the course of the authors' previous studies (http://www.uea.ac.uk/~m011); and, a short list of issues to consider. Participants were asked to study the Dataset in preparation for a Focus Group Interview - see Madriz (2001) and Nardi & Iannone (2003) for a rationale for using this tool for data collection. Interviews were digitally recorded. The interviews from Cycles 1-6 were fully transcribed (the data from Cycles 1X-5X were used as supportive material in the analytical process). Each interview was about 200 minutes long and generated a *Verbatim Transcript* of about 30,000 words. In the spirit of Data Grounded Theory (Glaser & Strauss 1967) eighty *Episodes*, self-contained extracts of the conversation with a particular focus, emerged from a preliminary scrutiny of the transcripts and were transformed into *Stories*. These are narrative accounts in which we summarise content, occasionally quoting the interviewees verbatim, and highlight conceptual significance. The eighty *Stories* were grouped in terms of the following five *Categories*: *students' attempts to adopt the 'genre speech' of university mathematics* (Bakhtin 1986); *pedagogical insight: tutors as initiators in 'genre speech'*; *the impact of school mathematics on students' perceptions and attitudes*; *one's own mathematical thinking and the culture of professional mathematics*; and, ***the relationship, and its potential, between mathematicians and mathematics educators*** (25, 25, 4, 20 and 6 *Stories* respectively). Here we focus on the last *Category*.

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STUDENTS PREFERENCE OF NON-ALGEBRAIC REPRESENTATIONS IN MATHEMATICAL COMMUNICATION

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This research study deals with the modes of representation that ninth-graders choose in order to communicate their problem solving paths and justifications, and the relation between these modes of representations and achievement level. The findings are based on analysis of 350 answers to problems that demanded communication of reasoning, explanations, and justifications. The results indicate that only a few students, who are very high achievers, choose to communicate via algebraic representations, even after two extensive years of learning algebra. These results might be related to difficulties students have with the abstraction of algebra and the way algebra is taught in school – an issue that should be considered by curriculum developers and teachers.

INTRODUCTION AND THEORETICAL BACKGROUND

Over the past decade the emphasis in the math classroom is shifting from routine procedures towards developing mathematical thinking, reasoning, and communicating. The more the students' mathematical language develops, the better they can reason (NCTM, 1989, 2000). The importance of mathematical communication, using mathematical symbols - in writing, and in 'ordinary language', lies in bridging between articulation and reflection, and serves as a means to participating in the mathematical community (Fried & Amit, 2003; Morgan, 1994).

Mathematical communication is strongly related to problem solving and reasoning. The process of successful problem solving is dependant on the following: *problem representation skills* which include constructing and using mathematical representations in words, graphs, tables and equations, *solving* and *symbol manipulation* (Brenner et al., 1997). Students can communicate their explanations for a mathematical strategy or solution in a variety of ways: symbolically (numerical and/or algebraic symbols), verbally, diagrammatically, graphically, or by tables of data (Shield & Galbraith, 1998). The significance of presentation in problem solving is that it reveals the ways in which students process the problem, and as a matter of fact, the modes of representations are the external reflection of the thinking and solution processes (Cai, Magone, Wang & Lane, 1996). In this study we observe the ways students communicate their justification for solution paths, and how do they choose to represent them.

Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 2004 Vol 3 pp 409–416

METHODOLOGY

The aims of this study are to examine the modes of representation that ninth-graders choose in order to communicate their problem solving paths and justifications, and to investigate the relation between the modes of representations and achievement level.

Settings and instruments

The population of this study comprised 164 ninth-grade students (83 male and 81 female) who participated in a regional test. Five multi-ability classes, each from a different school were selected to compose the sample (The number of students from each class: 37, 37, 27, 34, 29). They came from similar socio-economic backgrounds; most of them low to middle class. All students had a similar mathematical background, because their studies followed the national curriculum.

The research instrument was compromised of three problems taken from a regional test in mathematics. The test was consisted of 46 problems referring to varying areas within the ninth grade curriculum. About half of the items were multiple-choice, and the others were constructed response, short answer and a few were open-ended problems (to different degrees of "openness").

Three problems, in which students had to communicate their explanations and justifications, were chosen as the research instrument.

The first problem dealt with optimization, and required that the students choose between two telephone companies and justify their preference. Data on the cost of monthly charges and per-minute costs were provided. Such a problem had never appeared in their math textbooks and therefore was considered to be a non-routine problem.

The second problem dealt with rate of change: water drained from a pool at a given constant rate. The students were asked whether the pool would be completely empty after one hour had elapsed. This type of problem appears frequently in textbooks but in the test they had to justify their conclusions.

The subject of the third problem was the relation between the area and circumference of a rectangle. The problem was formulated in two parts: a multiple-choice question with three choices, and then justification of their choice. This is a non-routine problem and does not appear in textbooks.

Data Collection and Analysis

Data was collected from students' test booklets. The data were analyzed qualitatively to identify the mode of representation, and then the qualitative results were quantified. (Note: neither the teachers nor the students knew at the time the test was taken that some of the tests would be researched later on. This fact increased the authenticity of the answers.)

Level of Achievement

The whole test score, excluding the three instrument items, defines the level of achievement in this research. Scores are on a scale of 0-100. The tests were scored by the researchers, according to strict guidelines set by the Ministry of Education.

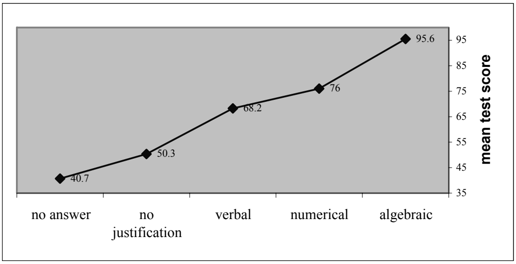
Modes of representation

In order to find how students communicate, a qualitative analysis was implemented. Students' explanations and justifications were sorted according to the following representation modes: *algebraic* - explanation was represented by an equation or a function; *numerical* – explanation was represented arithmetically, including computations and manipulation; *verbal* - justification was written in words, or *diagrammatically and graphically* - explanation was represented by diagrams, graphs, or other pictorial illustration. This analysis was based on Shield & Galbraith (1998) modes of representations, but modified after a pilot, in such a way that symbolic representation was separated into two categories: numerical and algebraic. For each problem the mean test scores of the students who had chosen to communicate in a specific representation was calculated. This obtained a mean score per representation per problem.

Findings

Observation of how students chose to communicate indicated that the vast majority preferred verbal and numerical modes, and a minority preferred an algebraic mode. From the total of 350 justified answers, 153 answers (44%) were represented in a verbal mode, 131 (37%) answers in a numerical mode, and 39 (11%) answers were represented in an algebraic mode. In the area-perimeter problem, 26 answers were in a diagrammatical mode. Since this problem was perceived as a geometry problem – the use of a diagrammatical mode was natural. In the other problems, only in one case was there use of a graphical mode.

Examination of the relations between representation and achievement indicates that the students who choose algebraic representations were high-achievers, and their mean test scores were 95%-97% (pending each problem, see below). Those who chose numerical representations had mean scores of 76%-85%, and those who chose verbal representations had mean scores of 68%-75%. Students who did not attempt to justify their answers or did not answer the problems at all were low-achievers in the entire test (scored less then 45%). These results were valid for all three problems. The following graphs represent the relation between achievement level (test scores) and the choice of representation mode for each problem. Enclosed is also the distribution of representation mode within each problem.



Number of Answers:

Algebraic: 18

Numerical: 76

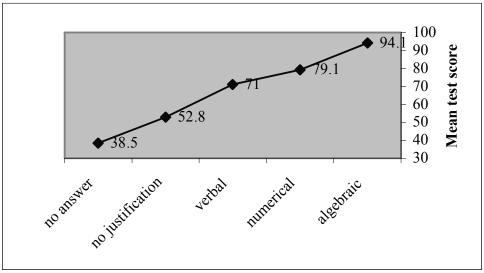
Verbal: 25

Graphical: 1 (excluded from the diagram)

No justification: 7

No Answer: 37

Fig 1: Optimization problem: Achievement related to representation modes



Number of Answers:

Algebraic: 15

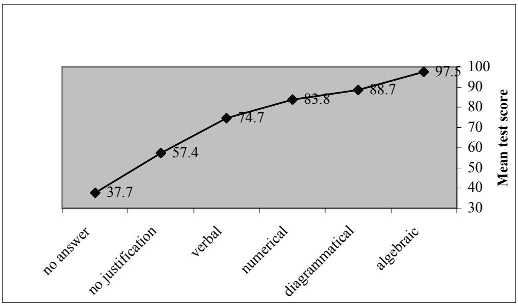
Numerical: 50

Verbal: 57

No justification: 10

No Answer: 32

Fig 2: Rate of change problem: Achievement related to representation modes



Number of Answers:

Algebraic: 6

Numerical: 5

Verbal: 71

Diagrammatical: 26

No justification: 22

No Answer: 34

Fig 3: Area-perimeter problem: Achievement related to representation modes

DISCUSSION

Reform in mathematics education emphasizes written communication as a tool to develop and deepen mathematical thinking and reasoning. In this paper, we report on the modes of representation students choose in the process of communicating their problem solving paths and justifications, as appeared to three mathematical items in a regional test. Such communication can be represented in a variety of ways: mathematical symbols (numerical or algebraic), graphs, verbal, diagrams, and sketches (Shield & Galbraith, 1998).

A major finding in this research is that only a few students choose to communicate their solution paths in an algebraic mode. There are other studies reporting students' preference to justify and explain mathematical solutions in a verbal mode (Cai et al., 1996), or to solve problems presented verbally in non-algebraic methods (Nathan & Koedinger, 2000). However, this outcome is quite surprising, bearing in mind that the students in the current study have experienced almost 2 years of algebra studies. Focusing on the results, it is evident that the students, who choose algebraic representations, are students who achieved high scores in the test.

Why is this so? One explanation is that students just find that communicating in an algebraic mode is difficult for them (even after two years of studying algebra!). The use of numbers generalized by letter designations is an abstraction that raises difficulties for students (Hembree, 1992). Some students even fail to construct a meaning for the ideas of algebra or to connect them with pre-algebraic ideas. In order to use the language of algebra, students need to get used to a more different and more abstract mode of thinking than that used in arithmetic, and students tend to retreat to a more "solid ground" such as numbers or words (Herscovics & Linchevski, 1994; Lee & Wheeler, 1989). The same phenomenon was found by Hazzan (1999). She found with undergraduate students, that coping mentally and cognitively with new mathematical styles of representation and new concepts, leads the students to adopt mental strategies that involve reducing the level of abstraction. This type of regression has also been reported for students that had been introduced to algebra but who had the tendency to stay in the arithmetic modes (Lee & Wheeler, 1989), or with calculus students (Amit & Vinner, 1990.) When dealing specifically with argumentation, it was found that algebraic arguments are even harder to follow, and students prefer arguments presented in words (Healy & Hoyles, 2000). All the above indicates that using algebraic means in order to justify and explain problem solving procedures is really hard for students. Therefore it makes sense that only high achievers dare or are willing to choose this mode representation.

Another explanation is related to the way algebra is taught in middle school. According to the national syllabus, the first two years of algebra are mainly devoted to solving equations and systems of equations, plotting graphs, and solving problems with one or two unknowns. When students have (seldomly) to justify their outcomes, they do it by substituting a number in an equation or by using everyday language.

Shield & Galbraith (1998) found that the writing products of students are constrained by the mathematical presentations to which they have become accustomed. If students do not experience whatsoever the use of algebra for argumentation, then only the “talented and brave” dare to do so, as the results of this research indicates.

CONCLUSIONS and IMPLICATIONS

We do not underestimate the importance of the use of verbal or numeric representation in mathematical communication, especially in local one-time situations. However, we cannot overestimate the importance of an algebraic representation as a powerful, general, global, and comprehensive communication mean. Such means should not be the estate of the high achievers only; rather it should be accessible to every student. The current research indicates that this is not the case. In order to achieve this goal, the use of algebraic representation should be integrated into the teaching of algebra from the first stage, and students should gain experience in using algebra for argumentation and justification. If such a feat is implemented in schools, communication will be a real service to mathematics education rather then a lip service.

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LEARNING TO SEE IN MATHEMATICS CLASSROOMS

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This paper reports a study that involved teacher education students investigating teaching practice by collecting and analyzing video clips of their own mathematics teaching. One student's case is used to portray what prospective teachers attend to in the process of filming/editing and describing/analyzing their teaching practice. Shifting from noticing what to film in retrospect to noticing and filming what was happening in the moment was key for this student as she began to see her own practice as site for learning. This paper points to the power of opportunities for teacher education students to not only learn to analyze video of other's teaching but to also analyze their own..

Prospective teachers spend a great deal of time observing in real and video recorded classrooms all throughout their teacher preparation programs. Learning to see beyond the surface and mundane features of classroom practice, however, requires more than simply watching. The pitfalls of learning to teach in classroom settings have long been explored and reported (e.g., Feiman-Nemser & Buchmann, 1989; Britzman, 1991). On the one hand, the practice setting provides powerful and memorable experiential learning. On the other hand, the familiarity of classrooms serves to confirm more than to challenge prospective teachers' limited conceptions and beliefs about teaching and learning. Prospective teachers' prior experiences and beliefs about what it means to teach and learn in classrooms, accumulated over many years of being in classrooms as students, play an important part whenever prospective teachers are presented with scenarios of classroom practice whether it be in actual, videotaped, or textual forms. Thus using the classroom setting as a context for learning to teach, therefore, is hardly unproblematic.

The research literature reports on the numerous benefits and potential for the use of video tapes and video-recordings as means to promote and broaden prospective teachers' understanding of teaching (e.g., Lampert & Ball, 1998; Sullivan & Mousley, 1996). Yet, the use of video technology in teacher preparation programs raises important questions about its role and potential. It brings to the fore the need to examine alternatives to simply showing students pieces of classroom video in our teacher education courses. In this study the authors explored a strategy for using video technology in teacher education courses that involves students in filming, editing, and sharing video excerpts of their teaching as a medium for individual and collective analysis of practice. The research we report here investigated the following questions: (a) What do prospective teachers attend to when filming/editing, and when describing/analyzing classroom practice? (b) How do these experiences influence prospective teachers' understanding of classroom practice?

THEORETICAL CONSIDERATIONS

Following Ball and Cohen (1999) we argue for a broad view of teaching practice to include all that is associated with teaching. Opportunities to stand back and reflect on practice, to inquire with others, to prepare for teaching, to analyze curriculum texts, are all aspects of practice. Our role as teacher educators, then, is to consider ways in which we might engage our students in authentic aspects of practice so that they may learn to use teaching practice as a source of inquiry and continued professional learning. Ball and Cohen (1999) refer to this as developing a stance of inquiry in terms of learning both in and from teaching practices. Sherin (2001) refers to this as developing a professional vision of classroom events.

Sherin's research (2001) with teachers analyzing their own classroom videos is one strategy that shows promise for supporting the development of teachers' professional vision or way of making sense of classroom activities. Sherin argues that watching and analyzing videos of their own teaching enabled teachers a different kind of opportunity to learn from practice. Without the pressures to respond to student comments, consider next pedagogical moves, or address immediate management issues, teachers can focus their attention on understanding and interpreting what happened rather than on how they might or should respond in the moment. Unlike real-time practice which requires teachers to be simultaneously aware of and consider many issues and pedagogical alternatives, analyzing video provides teachers opportunities to narrow their attention to a particular issue, activity, or student. Video allows teachers to be observers in their own classrooms; it is a different kind of analysis that enables teachers to *distance themselves*, to use Jaworski's (1994) term, from the immediacy of practice and to shift their professional vision to areas that might not be noticed while teaching.

Mason (2002) conceptualizes this professional vision as being "sensitised to notice things" (p.1). Professional development, according to Mason is about developing the sensitivity to notice. Expert teachers, for instance, notice and see aspects of classroom practice in ways that beginning teachers do not. Mason offers the idea of noticing as an intentional stance– "a collection of practices both for living in, and hence learning from, experience, and for informing future practice" (p. 30). The discipline of noticing is about making the effort to notice particular things, for instance, in classroom teaching or research, and to be able to notice these things when needed or "when it would be useful to have noticed [them] (and not merely later, in retrospect)" (p. 31). And so learning to notice does not just involve noticing aspects of teaching that before went un-noticed but also includes the sensitivity and inclination to be aware.

CONTEXT OF THE STUDY

The context for this study was a Problem-Based Learning [PBL] cohort in a 12-month elementary education program for post-baccalaureate students at a large university in British Columbia, Canada. The PBL cohort is an option for students entering the teacher education program and uses the principles of problem-based learning and case-based teaching. Students, with the guidance of their tutor, discuss, frame, and research various issues of teaching and learning that they identify in a given written case. After a 13-week teaching practicum, PBL students are required to produce their own written case of teaching and learning as a final assignment in the program. All students are invited to use their own teaching as a context for their final case project.

Ten PBL students volunteered to participate in a study that provided support and access to digital technologies in order for them to film and analyze their own teaching as a context for their final case report. Participants were mainly middle class Caucasians with a small number from Chinese, Indo-Canadian, and Japanese backgrounds. During their practicum participants met regularly (about every two week) as a large group or as smaller groups in their respective schools to discuss their teaching and interesting incidents that could frame the context of their final case report. Students were given access to laptop computers, digital cameras, and technical support. With this technology students could document their teaching, capture pupil thinking, interview teachers or parents and edit these clips in order to collectively help each other interpret and make sense of teaching practice. Follwing their practicum students had three weeks to produce and respond to a case of teaching and learning. Whereas other students in the PBL program produced written cases, these students produced a digital case in the form of a web page that stemmed from the student's particular inquiry question and consisted of edited video data linked with analysis.

Data collection included video recordings of six student group meetings held during the practicum, transcripts of two interviews held with each participant–before and after they completed their digital case, students' completed digital cases, and video recordings of students working on their cases. Data were analyzed for the kinds of questions participants explored in their digital cases, what they chose to video tape and attend to, and their claims to how this process influenced their teaching. This paper reports on a case study (Stake, 1995) conducted with one of the participants. In deciding which participant to focus on we used Patton's (1990) concept of intensity sampling and Stake's (1995) idea of instrumental case study. From this perspective examples or cases are chosen as "intense" in that they are "excellent or rich examples of the phenomenon of interest, but not unusual cases" (Patton, p. 182). We chose to focus on Janine's story, not because her story represents that of others, although aspects of it do, but because it provides an interesting example through which to explore the nature and complexity of prospective teachers' noticing when given opportunities to document and analyze their practice using video technologies.

RESULTS AND DISCUSSION: JANINE'S STORY

Janine entered the teacher education program with an undergraduate degree in Sociology and Women's studies. She chose the PBL teacher education option over other available options because she thought it would provide her with opportunities to investigate issues of teaching and learning at a more in-depth and critical level than, according to Janine would typically be expected in other options. She came with a strong background in school math and was placed in advanced math school groups but stated that she hated math: "I was good at it but I hated it. In math there was just no room for creativity." Janine's practicum teaching was in an elementary school with a high English as a second language population; the school's population consisted of students from at least 20 different nationalities. Janine was assigned a Grade 1/2 combined class and during her 13-week practicum was expected to teach up to 80% of the curriculum.

Attending to 'what to attend to'.

In the early stages of her practicum Janine expressed some initial frustration in trying to decide what she should point the camera at. With a Grade 1/2 class, Janine felt that nothing very interesting would be captured, particularly with children at this young age level who, Janine said "were just learning to communicate their ideas." For more than a month she did little or no filming of her students or her teaching. As she explained:

> I just can't seem to find the time. Sometimes I think of an interesting episode to video but it is always in retrospect that I think I should have video-taped it because it's hard to predict what interesting thing is going to happen.

Although during her campus course work Janine did participate in viewing and analyzing video clips of young children articulating their mathematical thinking she does not connect this activity with possibilities for her own students. Furthermore the act of filming for Janine seemed to be conceptualized more as a representation of a finished product than as a tool for investigation. That other prospective teachers in her school were filming and sharing their clips did not help Janine consider what she could do with her young students. Janine did notice interesting aspects of her class that could be filmed, however, she noticed these in retrospect.

Shifting attention.

At a midpoint in the practicum it was suggested to Janine and her peers that they select a math problem that they could give to students across grade levels. It was hoped that by having prospective teachers ask the same problem to a range of pupils they would have a shared context through which to discuss their clips. Janine and her peers (who taught Grade 7, Grade 5, and Grade 4) chose the Checkerboard problem that asked students to determine the number of squares in an 8 by 8 square checkerboard. Janine adapted the problem for her Grade 1/2 students and made the checkerboard a 4 by 4 rather than an 8 by 8 board. In order to accommodate both posing the problem to pupils and filming their response, Janine decided to select certain pupils and film them working on the problem outside of the regular class. It was at a group meeting where Janine shared her clips of students working on this problem with the other prospective teachers that a change in what she was attending to can be seen. During this meeting she excitedly showed what her students were doing, the various ways in which they approached the problem, and how they spoke about it. In talking about the problem with her peers she stated:

> I would actually like to give this same problem to different kids, and after seeing how everyone else did it, it would give me some more ideas on how I could approach the problem. Because I struggled with wanting to give them enough information that they could go on and have enough success with it, but without giving them too much information that my guidance was over bearing on their thoughts.

Janine began to see value in using the camera to capture student thinking and although she actively provided commentary on her students' different responses to the task, her interest for further investigation focused on exploring pedagogical approaches to the problem. Interestingly, the video clips raised a question for her regarding the answer to the problem and how she might introduce students to the answer. She commented:

> I would have liked to have had an answer for them, but I don't know the answer…. I would have liked to have given them the answer and then had them work backwards. I didn't know the answer and still don't know the answer. I think I do, but I don't know if it is right.

Janine's comments provide evidence of her shift in attention to being concerned about what to film, or noticing in retrospect what might be interesting to film, to considering the value of students' thinking. This seemed to focus her attention on ideas for pedagogical approaches to the problem more so than ideas for thinking further about students and the mathematics involved in solving the problem. She does notice that her not knowing the answer to the problem limited her pedagogical response to students but at this time she does not see the need to act upon but this noticing. Nonetheless, this excerpt also points to the comfort Janine felt in being able to share what it was she didn't know.

Noticing what the same students do in different contexts.

A few weeks later Janine completed a critical thinking science unit on sound with her students and filmed the students' working on it. She noticed that her students were, as she said "able to successfully consider taught elements through a critical lens and put their knowledge to work in a critical activity – designing a better human ear." Through this lesson and through filming/editing her videos she noticed that students were willing to try and test their ideas in the context of a science lesson and how different this was for the same students working on a math problem. In reflecting on this Janine commented:

> In science through the challenge of designing the best model ear, they were willing to take risks and had no problem with being wrong or trying something else and really working through the problem. But when I give a math problem that they couldn't find the answer to right away, if getting the answer involved more than one step, they would shut down. It was fascinating.

With this comparison between how students responded in various subjects Janine tentatively defined her case project to focus on students' mathematical dispositions and how she might help students take risks, persevere and solve difficult mathematical problems. It was suggested in a group meeting, when Janine posed her question, that she focus on understanding the nature of the issue, that is students' mathematical dispositions, before she explored ways of improving it. With this, Janine set out to interview students by posing various problems to them and then asking them to describe the characteristics of a good mathematics student or to think of metaphors to describe mathematics. She interviewed parents, teachers, and her own family members. No longer focused only on the ways in which she should engage her students mathematically she sought to explore aspects of why students engaged in mathematics problems differently from how they engaged in science problems and how they saw themselves as mathematical thinkers. Evidence of this shift in attention is also illustrated in Janine's comments made to her peers during one of the prospective teachers' last group meetings before returning to the university campus to work on their cases. Referring to her efforts to makes sense of her peer's questions she stated: "I just this intellectual stuff … I really feel now that I can step back and see my teaching from a different standpoint."

Transformed views of teaching

Over the next three weeks Janine worked with her peers on her case project producing a digital case of linked video clips and analysis around the question of "What is mathematical disposition and how might it be fostered?" In interviews following the development of her case project Janine spoke about the project as something that totally occupied her thoughts. She mentioned that through the response to her questions through viewing and analyzing her video clips, she developed a heightened sense of awareness.

> I started carrying a little note pad around with me wherever I'd go because I'd be thinking about it all the time…–at the beach, on the bus, or driving at a stop light… it is really hard to put into words but this whole project was life altering, not only for my teaching but for me and for how I view things. My understanding is so much deeper.

When asked to comment on what prompted her to shift from her initial skepticism of using her classroom as a context for analyzing her teaching to the value she now sees, Janine commented that being able to frame and research her own question was key. She continued:

> Once I started interviewing students it was really interesting–like what's going on with these kid's thinking. .. But initially I really thought of it as a chore and unrealistic.. There is so much you need to think about in the practicum that I thought I couldn't do this [project] as well … But that's not how I see it now. .. This whole project has made me think about how to learn from teaching in ways that I didn't see before.

In exploring this further, Janine made links between the question she pursed for her project and her own dispositions toward mathematics and learning. For Janine, framing a question, analyzing her students' thinking, and considering implications for her practice involved her in taking risks that she normally didn't do. Her interest in understanding her students' risk taking (or lack of it) in math led her to become aware of and examine her own risk taking (and lack of it) in her learning and teaching. This awareness for Janine was transformative.

CONCLUSION

Janine's story is an example of how one prospective teacher through collecting and analyzing video clips of her teaching developed a stance of inquiry that stemmed from her efforts to make sense of aspects of her own teaching practice. She developed Jaworski's (1994) distancing lens of practice and to some extent Sherin's (2001) notion of professional vision by moving outside the immediacy of teaching to take a non-judgmental position as a researcher observer. She moved from not seeing what she could learn from her students to attending to their mathematical dispositions, to noticing parallels between her own learning and that of her students.

Mason (2002) refers to the act of noticing-marking-recording as an initial aspect of disciplined noticing that moves retrospective noticing into noticing in the moment so that we present ourselves with choices for how we could act. Noticing-marking-recording involves the act of noticing retrospectively and marking it, or as Smith (2003) suggests, "naming" it, so that we can increase our awareness and sensitize ourselves to what we would like to attend to. The act of marking or naming what is noticed in teaching practice is an important aspect of owning and authoring the inquiry involved in learning from teaching. In the case of Janine opportunities to film and analyze video clips of her students working in different domains prompted her to notice, mark, and re-mark the differences in students' scientific and mathematical risk taking. This led her to notice and attend to her own learning and the opportunities risk taking provided her. We suggest that for beginning teachers using video to help them notice-mark-record their practice is key to helping them shift their attention. We can only speculate how this noticing will be used for Janine in her practice, nevertheless this study provides insight into how prospective teachers can engage in a meaningful investigation of their own practice.

What seemed to prompt Janine's inquiry and noticing was an opportunity to investigate her own question related to her own teaching, and to do so not only while she was teaching but once her practicum was finished. Although before her practicum Janine had been exposed to and engaged in the analysis of videos of classroom teaching she did not see the value of these for her own teaching while she was teaching. Prospective teachers' past experiences as school students are their main resource when interpreting what happens in classrooms. These extensive experiences make it a challenge for them to explore alternative interpretations for the teachers' and the students' words and actions. This raises questions for us as teacher educators in the use and implications of the use of videos during coursework. The use of video provides opportunities for the analysis of real practice. It offers shared opportunities to examine practice–to replay, rethink, and inquire into possibilities for teaching. Yet, the results of our study suggest that prospective teachers need not only opportunities to analyze and learn to analyze the practices of more experienced teachers through video, but also need opportunities to study their own practice through collecting and analyzing video of their teaching. The results of our study prompt us to explore alternative ways of using videos of practice in teacher education to help our students broaden their interpretive frameworks and see mathematics teaching, their own and others, as opportunities for learning.

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STUDENTS' WAYS OF INTERPRETING ASPECTS OF CHANCE EMBEDDED IN A DICE GAME

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In this paper seventh-grade pupils' ways of handling aspects of probability have been investigated. The aspects in question were embedded in a dice game, based on the total of two dice. Four different set-ups of dice were included in the situation in which they were up to explore optimal strategies for winning the game. How children understand concepts is regarded from the perspective of how the pupils' understanding varies with their interpretation of the situation, in which the concepts are embedded. Empirical data have been analyzed with intentional analysis, a method by which we regard pupils' act as intentional. The results show approaches of extremes and of a number model, as consequences of how the pupils process and bring to the fore information in the situation.

BACKGROUND

Two research perspectives are seen in the area of chance encounters. First there is the psychology/cognitive perspective including the works by Kahneman and Tversky, quoted and developed in Gilovich *et al*. (2002), with focus on analyzing patterns in order to identify misconceptions and judgmental heuristics. The second perspective is that of mathematicians and mathematics educators, with focus more on learning probability from a mathematical point of view (Shaughnessy, 1992).

The results of numerous psychological studies are reflected in the research of mathematics educators, in that the psychologists have provided a theoretical framework in considering judgmental heuristics. An extensive mapping based on the psychologist approach, and from related studies of misconceptions, can be identified in the literature, including *representativeness and availability* (Gilovich *et al.*, 2002), the *outcome approach* (Shaughnessy, 1992) as well as the *equiprobability bias* (Lecoutre, 1992).

The bias in focus here is equiprobability. Regarding the total of two dice this notion implies that all sums are equally likely to appear. Based on her results Lecoutre (1992) argues that this bias mainly stems from a conceptualization of the random experiment as being only a matter of chance. Confronting pupils with the total of two dice in a computer-based environment Pratt (2000) also identifies responses in accordance with equiprobability. He concludes that the participants base their decisions in random experiments on different types of resources, external as well as internal. Regarding equiprobability the four local internal recourses *unpredictability, irregularity, unsteerability* and *fairness* are mainly in play.

In a study of factors affecting probabilistic judgments Fischbein *et al*. (1991) argue that there seems to be no natural intuition regarding the order of the two dice. This implies that children are not aware of all possible combinations when they are

Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 2004 Vol 3 pp 425–432

comparing different totals of the dice. In connection with this Keren (1984) also found evidence that it is important to identify the sample space decisions are based on, "the knowledge of the sample space used by the students is crucial for understanding their responses" (*ibid.*, pp. 127).

Synthesizing the discussion so far, results indicate that pupils' responses in situations of uncertainty are affected by what resources they make available, and how they choose to make use of them, in a given situation. Their responses can be seen as a question of how they process and bring to the fore information for their judgments. This, I believe, challenges the view of misconceptions and, particularly, the equiprobability bias. The problem of chance encounters is rather a question of how pupils interpret and organise the situation as a whole. In the theoretical considerations I will discuss such processes from a constructivist perspective, in terms of contextualization and differentiation.

THEORETICAL CONSIDERATIONS

Regarding personal resources, Fischbein (1975) argues that intuitions play a prominent role. He distinguishes between primary intuitions as cognitive acquisitions, derived from individual experiences, without systematic instruction, and secondary intuitions as formed by education and linked to formal knowledge. Resources in general and intuitions in particular bear a strong likeness to what is commonly called *alternative frameworks* (Driver, 1981). Such frameworks, as intuitions established in every-day life, are in the same way related to teaching objects as primary intuitions are related to secondary intuitions. Considering learning from such an experience-based standpoint, the constructivist tradition usually regards this as a process in which naive, alternative, conceptions are abandoned in favour of more scientifically based knowledge. But since a constructivist approach to learning also presupposes the two basic principles continuity and functionality such a learning model is difficult to accept. First of all, and in relation to continuity: How can new formal knowledge be constructed on the basis of naive conceptions, if the two forms of knowledge are inconsistent with each other? The large mapping of different kinds of misconceptions in probability emphasizes this problem. Another problem is that knowledge does not always seem to be stable between similar situations. These issues illustrate the problem of transfer that the constructivist perspective struggles with, and which is emphasized by a model of learning as a process of abandoning naive, alternative notions in favour of more scientifically based knowledge.

As the constructivist perspective focuses on the individual construction of a learning object, criticism has also been raised against its low priority of the situated interaction between individual and environmental aspects (Säljö, 2000).

Discussing conceptual change, Caravita and Halldén (1994) argue that a more appropriate way to conceptualize learning, in accordance with a constructivist approach, would be to describe it in terms of thinking strategies, such as an expanded repertoire of them as well as a refined organization of and between them; "Learning is then a process of decentering, in the Piagetian sense, rather than the acquisition of more embracing logical or conceptual systems replacing earlier less potent ones" (pp. 106).

Based on this reasoning, learning can be looked upon as a problem of differentiating between contexts for interpretations. But in accordance with a constructivist view, context here refers to students' personal constructions. If we let the *conceptual* context denote personal constructions of concepts embedded in a study situation, as well as the *situational* context denotes interpretations of the setting in which learning occurs, and the *cultural* context refers to constructions of discursive rules and patterns of behavior in the society, we can talk about students' ways of appropriating new conceptions as a problem of *contextualization* (Halldén, 1999; Wistedt & Brattström, in press).

Halldén (1999) stresses that these different kinds of contexts are in play simultaneously as we are trying to solve a task. Depending on how we interpret the situation, by focusing certain aspects, they get different priorities in the contextualization process. In studies of learning conceptual structures are of certain interest, why the conceptual context is in focus in analyzing learning situations.

Such a meaning-making process is more in tune with the principles of continuity and functionality; old ideas are combined (and recombined) with other old ideas and new ideas, with respect to personal interpretations of a phenomenon or event.

In line with the contextualization approach I will in this paper describe seventh-grade pupils' ways of handling aspects of probability as a problem of their different ways of contextualizing tasks that bring forward such aspects.

METHOD

By confronting students, during a game situation, with a mathematical content not presented to them before in school I was hoping to create a situation in which a variety of contextualizations would appear. The content in focus was probability, a subject that in terms of conceptualization has shown to be interesting in relation to mathematics as well as to every-day life.

Eight participants were divided into four groups, with two students in each group. In the group discussions, which were tape-recorded and fully transcribed, they were up to explore optimal strategies for winning a dice game, based on the sum of two dice. The dice were designed to bring to the fore several aspects of probability and simultaneously give the students the opportunity of encountering small differences in mathematical structure between different situations. Each team had a board with areas marked 1-12. They also got a set of markers, which they were asked to distribute as they liked among the 12 areas. In the moment of play, which was videotaped, two teams played against each other. Here they took turns on rolling the dice. If one team or both of them had at least one marker in the area, which was marked with the sum of the dice, they removed exactly one marker from this area.

The team who first removed all its markers from the board won. The situation included four different set-ups of dice presented to the students in the following order.

1. The yellow setting – Here the faces were marked with one and two eyes, distributed as (111 222) and (111 222). The set-up was aimed to model the well-known experiment of throwing two coins.
2. The red setting – Included two different dice, each with a distribution of two outcomes among the faces as (222 444) and (333 555). This design implied an interaction in which the order of the dice didn't have to be taken into account.
3. The blue setting – Similar to the yellow with the difference that there were now four sides marked one and two sides marked two, that is (1111 22) and (1111 22).
4. The white setting – Similar to the red setting but now with the distribution shifted towards the lower numbers as (2222 44) and (3333 55).

A central purpose with the third and fourth settings was to stimulate the process of contextualization, with respect to combinations and proportionality.

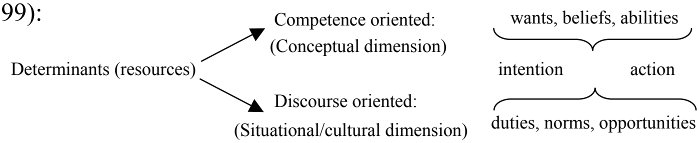
In the analysis I followed the principles of intentional analysis (von Wright, 1971; Halldén, 1999). This means, in order to understand a sequence of activities, that behaviour has been regarded as intentional. The intention in question gives meaning to the behaviour. One way of structuring such an intentional explanation is, according to von Wright, in the form of a practical syllogism, of which the following can serve as an illustration:

P1 A person P intends to bring about x.
P2 P believes that to bring about x would require the fulfillment of y.
C Thus: P does y.

What you see as an observer is the conclusion, i.e. a person P doing y. By ascribing P the intention x we can find reasons for P doing y, under those circumstances that are implicated in premise 2. In terms of von Wright (1971):

> Behaviour gets its intentional character from being seen by the agent himself or by an outside observer in a wider perspective, from being set in a context of aims and cognitions. This is what happens when we construe a practical inference to match it, as premises match a given conclusion. (pp. 115)

Premise 2 should be seen as a mental stage, connecting the intention x with the verbal or non-verbal behaviour in C. von Wright describes this relation in terms of internal and external determinants. Based on principles for this an educational approach has been worked out, which is related to the previously referred perspective of contextualization, in order to explain the relations in the practical syllogism (Halldén, 1999):



RESULTS AND ANALYSIS

Rounds 1 and 2 – A question of finding possible outcomes

In the group activities, there seems to be no doubt that the major intention for the participants is to win the game. During the two first play-rounds they interpret this issue as a matter of differentiating possible outcomes from impossible ones. They focus possible sums, regarding the dice in question. However, what does not seem to be in focus in such a contextualization is the various ways sums can be represented. Regarding the first design this was not surprising as I had reasons to believe that the pupils would not be able to take into account the order of the dice, i.e. the difference between the outcomes (1, 2) and (2, 1). But, still focusing only on possible sums, none of the groups reflect on the distinct different ways 2+5 and 4+3 either, representing the sum of seven in the second setting. Actually, two of the groups, using an approach of extremes, ended up with a sample space in the second setting as {5, 6, 7, 8, 9}, including the impossible outcomes 6 and 8. The approach of extremes was a strategy in which they started by identifying the smallest and largest values five and nine and after that ascribing possible outcome to all sums within these extremes. This approach even more implies that the pupils are only focusing resulting sums, with little or no considerations regarding underlying processes generating the sums.

In order to understand the pupils' ways of ascribing probabilities to the sums, in terms of distributing markers within the identified sample space, I argue that we have to keep in mind their interpretation of the situation; that the experiment is a question of finding possible sums. Since their action, regarding distribution of markers, in great details are similar during the two first settings the first round may serve as an illustration. Referring to the sample space {2, 3, 4}, Sabina in group A suggests a solution to the distribution of markers as:

Sabina: How many on each? 24 divided by 3…it will be 8, doesn't it?

In group D we identify a similar approach. After that Lars and Petra also have identified the same sample space {2, 3, 4} Petra moves on with:

Petra: We are going to have 24 [referring to the number of markers]…it will be 8 on each. If we have 8 on each there will be 3 times 8, which is 24.

These responses could be explained in terms of Lecoutre: The pupils just consider the situation as a matter of chance and therefore place equiprobable. I could agree with that. But since it seems to be that they just bring to the fore, and base their judgments on, the single outcomes 2, 3, and 4, such responses seem to be reasonable and not clear biases. By this I mean that an equiprobability response, based on an idea that everything is just a matter of chance, may be determined by the reason that they only have made available, that is, are aware of three different outcomes. As no further information is processed by the pupils it seems reasonable for them to conclude that each outcome is equally likely to appear, a notion which is in line with the classical definition of probability. This implies that the pupils in some sense may connect sample space with probability for events within this sample space.

From the utterances above, there are also reasons to believe that their behaviour has been determined by conceptions concerning norms and expectations according to school mathematics. They find a solution to the task presented by application of tools that are relevant for the culture of education; that is, the numbers 3, 8, and 24, included and the computational devices of multiplication and division (cf. Säljö, 1991). Since the Swedish expression "delat med", used above by Sabina, could be interpreted as either "divided by" or "distributed among" it could also be argued that an imprecise language in the activity plays a crucial role as well.

Rounds 3 and 4 – Contextualizing in terms of a number model

In a similar way as the first two rounds were related by a common contextualization the activities were in accordance between the last two settings as well. Now focus was shifted from the resulting sums towards a more detailed exploration of the situation. But for such an approach to happen the participants had to be aware of the differences in design between the first and the third setting of the dice. Two of the four groups did not recognize this by themselves. Instead they contextualized in the same manner as earlier and therefore again respond in terms of equiprobability. Being aware of that, the observer made a choice to intervene in the situation, in that he made them aware of the differences between the designs.

Observer: Are these the same as the yellow dice?

Tom: Yes, they are the same.

Observer: Okay. So you don't see any differences between the blue dice and the yellow dice?

Tom: Wait! There is some difference between the twos.

Louise: There are fewer twos.

Observer: Will that matter?

Tom: Yes, we shall place little more on two… and not so many on four. Look, there are many, many ones and not so many twos!

Tom's answer on the first question emphasizes that the sums have been and still are in focus at that particular moment. However, being stimulated regarding differences in design the last utterance implies that he has made the information, offered by the observer, explicit for himself. Hence we can assume that all groups, from this point on, are aware of the situational determinant, regarding contrast between designs.

Overall, this awareness affects the pupils' ways of interpreting the aspect of chance in the situation in similar ways as is evident in Tom's last utterance above. In group C – ending up with a linear model of 16 markers on two, 12 on three and 8 on four in the third round – this was approach by:

Sabina: There are more ones then twos, so it is twice as big chance…that it will be two then it will be…

Peter: Four!

This type of reasoning is in several aspects in accordance with what Lecoutre (1992) calls the number model and Pratt (2000) relates to as the (structural) fairness resource. This means that the pupils now, in a more detailed way, focusing structural features underlying the random process, by interpret the situation as a question of mirroring the structure of individual dice in the structure of the total. Regarding such a contextualization, which in turn emphasizes that they take in consideration the number of ways they can represent each sum, the pupils again make use of extremes. However, this time the approach of extremes is more directed towards smallest and largest chance of the sums. Starting by focusing the most likely sum to appear, this strategy was exhibited during the fourth setting by Tom in group A as:

Tom: We should have many fives. Look here, I got 4 of twos and you got 4 of threes.

In group C Sabina emphasizes this further, in that she concludes:

Sabina: It should be most at five since there are most of threes and of twos… and least on nine, since there are few high numbers.

With respect to such approach the sums three and seven respectively were only reflected upon as being in between the two identified extremes. That means that the pupils do not take into account either the order of the dice or the distinct different ways of representing the sum of seven in the fourth setting.

CONCLUDING DISCUSSION

The aim of the paper has been to describe how chance encounters can be viewed as a problem of contextualizing. By ascribing intentions to sequences of activities we could find reasons for such meaning-making processes, under circumstances that may be described in terms of determinants.

In the results above I have argued that two main contextualizations appeared during the activities. During the two first set-ups it became evident that the pupils interpreted the game as a question of finding possible outcomes. I have argued for how such an approach, affected by situational and cultural conditions, restricted their strategy of distributing markers in that they only were aware of three single outcomes. Considering the last two settings, the pupils interpreted the situation as a question of mirroring the structure of individual dice in the structure of the total. Regarding such an interpretation, which in turn emphasizes that they reflect upon the number of ways they can represent each sum, an approach of extremes was used. An approach of extremes was used in the two former rounds as well, as a device for ascribing possible sums. However, in the two latter settings the approach was more directed towards smallest and largest chance of the sums.

Even if it could be argue that learning has taken place, in that the pupils deviate from equiprobability, the results indicate as well the crucial importance for pupils to make appropriate contextualizations. What is obvious is that neither of their interpretations activates a more systematical approach regarding possible outcomes. By this I mean that the sample space in focus, the sample space which they are aware of, is of great importance for their responses. They seem to have a natural intuition regarding proportionality, but as their contextualisations does not stimulate them to bring to the fore all representations of each sum, they base their decisions on a limited amount of information. Thus, to explore chance encounters, I claim that it is of crucial importance to take into account how the pupils interpret a phenomenon, an event or a situation as a whole.

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THE IMPACT OF STATE-WIDE NUMERACY TESTING ON THE TEACHING OF MATHEMATICS IN PRIMARY SCHOOLS

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This paper reports on teachers' views of the effects of compulsory numeracy testing in Years 3, 5, and 7 in Queensland schools. Teachers were surveyed on (i) the validity and worth of the tests, (ii) the impact the tests had on their teaching of mathematics, and (iii) how they were using the results of the tests. Although the results reveal a great diversity of beliefs and practices among teachers, attitudes are very negative. The tests have not greatly influenced teaching practices and the results of the tests are not being used to any great extent to inform planning apart from identifying gaps in the schools' mathematics programs.

INTRODUCTION

In Queensland (Australia) primary schools have experienced an increased emphasis on numeracy (and literacy) skills since the mid 1990s. A review of the school curriculum (Wiltshire, McMeniman, & Tolhurst, 1994) lead to the introduction of the *Year 2 Diagnostic Net* and *Year 6 Test* in schools in the mid 1990s (Queensland Schools Curriculum Council, 1996). Although the Year 6 Test was discontinued in 1997 (making way for the proposed Year 3, 5 & 7 Tests), the Year 2 Net continues to be used. It has been received well by primary teachers and has had a positive impact on their teaching of mathematics (Nisbet & Warren, 1999).

Further, at a national level, performance-based assessment and reporting was promulgated in the mid 1990s (Australian Education Council, 1994a), and all states were given individual responsibility for its implementation. Consequently in Queensland, Student Performance Standards were introduced but faced teacher opposition and so was unsuccessful, despite the fact that substantial funds were provided for professional development projects (Nisbet, Dole & Warren, 1997).

In 1997, a National Literacy and Numeracy Plan was adopted in all states to (i) identify students at risk, (ii) conduct intervention programs, (iii) assess all students against national benchmarks, and (iv) introduce a national numeracy reporting system (Department of Education, Training & Youth Affairs, 2000). Consequently, annual compulsory state-wide testing was introduced for students in Years 3, 5 and 7 in 1998. In August each year, all students in Years 3, 5 and 7 in Queensland schools sit for tests in numeracy (and literacy). The tests are devised by the Queensland Studies Authority (QSA) and are distributed to all government schools.

In Queensland, a broad interpretation of *numeracy* is assumed, embracing the perspectives offered by Willis (1998) that numeracy (i) includes concepts, skills and processes in mathematics, (ii) is described in terms of everyday situations in which mathematics is embedded, and (iii) implies that students can choose and use mathematical skills as part of their strategic repertoire. Hence the Queensland tests

Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 2004 Vol 3 pp 433–440

cover number, measurement, geometry, chance and data, and test skills of calculation (written, mental & calculator methods), and real-world problem solving.

A review of the Year 3, 5 & 7 testing program (Queensland School Curriculum Council, 1999) identified potential benefits and concerns related to such state-wide testing. The suggested benefits for teachers include the identification of students' strengths and weaknesses, data to inform planning and teaching, the provision of results for various groups (boys, girls, students of non-English speaking backgrounds, & indigenous students), and identifying teachers' professional development needs. Issues of concern include narrowing the curriculum, a tendency to teach to the test, having assessment items not based on the classroom program, and the potential for misuse of results (e.g. the publication of 'league tables' of 'good' and 'bad' schools).

The reports sent to schools after the annual tests contain extensive information on the results of the tests for the school including: results for each test item and each section (number, space, measurement & data) for each year-level, for each subgroup (boys, girls, NESB, & indigenous students), and for each student, with comparisons with the state averages. Further, all incorrect answers are recorded for each item for each student, and items for which the school scored 15% above and 15% below the state average are listed. With such information supplied, teachers and administrators are in a position to identify strengths and weaknesses of the school's program, compare their results with those of other schools, and take what they may consider to be appropriate action.

The nature and extent of the action taken by schools naturally varies across the state, and some of this information has been gathered by QSCC (later QSA) in surveys of participating schools. For example the survey undertaken in relation to the reports about the 2001 tests indicated that schools would make extensive use of the information in the reports. For instance, 80% of schools indicated that they would use the data for diagnosis of individual students' needs, and 78% indicated they would use the data to inform school programming.

However, it is not known whether these intentions reflect the opinion of class teachers (and not just the principal) and whether the schools and their teachers actually put the test results to such uses. Evidence gathered in a pilot study suggests that although schools may have good intentions, they don't actually get around to using the results. The current study was designed to determine the extent to which schools analyse and use the test data and teachers' views of the Year 3, 5 & 7 tests.

The adoption of the Year 3, 5 & & Numeracy Tests has been yet another change that primary teachers in Queensland have had to cope with in recent times. Much of the literature on teacher change and professional development acknowledges the importance of teacher beliefs as well as teacher knowledge in the cycle of professional growth. For instance, the importance of teachers' knowledge and beliefs in the cycle of professional growth was confirmed by Kyriakides (1996) who found that the failure of a mathematics curriculum change in a centralized system was due to the fact that teachers' perceptions of mathematics were inadequately considered at the adoption and implementation stages. Similarly, Philippou and Christou (1996) noted that if new ideas are to find their way into mathematics classrooms, it is imperative that change agents have a deeper understanding of classroom teachers' views, beliefs, conceptions and practices. Their study found that although teachers may be aware of and accept contemporary ideas (in their case about assessment), there can be a distance between their knowledge and intentions on the one hand, and their actual practice on the other hand.

The traditional model of implementing curriculum innovation assumes that teacher change is a simple linear process: staff development activities lead to changes in teachers' knowledge, beliefs and attitudes, which, in turn, lead to changes in classroom teaching practices, the outcome of which is improved student learning outcomes (Clarke & Peter, 1993). Later models of teacher change recognise that teacher change is a long term process (Fullan, 1982) and that the most significant changes in teacher attitudes and beliefs occur *after* teachers begin implementing a new practice successfully and can see changes in student learning (Guskey, 1985). The professional development models of Clarke (1988) and Clarke and Peter (1993) are refinements of the Guskey model which recognise the on-going and cyclical nature of professional development (focussing knowledge, attitudes & beliefs) and teacher change.

Such models can help explain why some educational innovations are successful, and others not. The introduction of the Year 2 Diagnostic Net was successful because teachers saw positive outcomes for pupils and they valued the Net's overall effect (Nisbet & Warren, 1999). However the introduction of Student Performance Standards in mathematics was a failure because teachers did not believe that the extra work entailed in performance-based assessment and reporting was worthwhile. Further, they received little support for the move (Nisbet, Dole & Warren, 1997).

The primary aim of the current study was to investigate (a) teachers' attitudes to and beliefs about the Year 3, 5 & 7 tests (agreement with tests, & their validity & purposes), (b) how schools and teachers use the test results (identifying students with difficulties & gaps in the curriculum), (c) the impact of the tests on teachers' practices (preparation for the test, influence on content & method), and (d) the responses of teachers and pupils to the tests. A secondary aim was to determine the effect of school location, school size, and level of teaching on the above attitudes, beliefs and practices.

METHODOLOGY

This study was conducted by survey method. A questionnaire was constructed containing items about teachers' attitudes, beliefs and practices relating to the Year 3, 5 & 7 Tests (as described above), plus items relating to the teachers' grade level, teaching experience, school location and school size, and an item for 'any other comments'. The results of a pilot study of 34 teachers in city and rural schools conducted in the previous months (Nisbet, 2003) were used to revise and expand the questionnaire items. A five-point Likert scale (from 1 = 'disagree strongly' to 5 = 'agree strongly') was provided for responses, and teachers were also invited to comment in selected items. A sample of 56 primary schools representative of size, disadvantaged-schools index and geographical location across Queensland was selected[1] and a total of 500 questionnaires were sent to the schools (having estimated the number of teachers in each school from the data on pupil numbers). Although the response rate was small (24.2%), the sample was representative of teachers' year level and position (Year 1 to Year 7, principal, deputy, & mathematics coordinator), teaching experience (from 1 year to 40 years), geographical location (Brisbane i.e. capital city, provincial city, rural & remote), and school size (categories from <20 pupils to >400 pupils).

RESULTS

The results are presented in six sections – (i) teachers' attitudes to and beliefs about the Year 3, 5 & 7 tests (ii) how schools and teachers use the results of the tests, (iii) the impact of the tests on teachers' practices, (iv) the responses of teachers and pupils to the tests, (v) the effect of school location and size, teaching level and teaching experience on the above attitudes, beliefs and practices, and (vi) other comments.

Teachers' attitudes to and beliefs about the Year 3, 5 & 7 tests

Opinion is divided on agreement with tests in principle, however the overall view of the tests and their purposes is quite negative. Although more teachers agree with the tests (47.1%) than disagree (33.8%), almost one in five (18.2%) are undecided. A minority of teachers agree that the tests are a good way of ensuring accountability (17.3%), a good way of comparing their school with other schools (34.7%), or a good way of comparing their school with the state (38%). Further, the majority of teachers (60.3%) think that the tests do nothing to assist pupils' learning.

Teachers think that the tests have little validity. Only 25.7% believe that the results of the tests give an accurate indication of the pupils' numeracy ability. Fewer teachers (12.4%) believe that the results of the tests give an accurate indication of the quality of the school's numeracy program. Fewer still (5.8%) believe that the results give an accurate indication of the teacher's ability to teach mathematics.

The uses that schools and teachers make of the results of the tests

It appears that schools make use of the results in some ways but not others. For example, 67.5% of teachers report that their school analyses the results to identify topics causing difficulties. Further, 59.2% believe that their school analyses the results to identify gaps in content taught, and 66.2% believe that the school analyses the results to identify pupils experiencing difficulties. However, only 36.7% of teachers report that the school uses the results of the numeracy tests to notify parents about the school's overall performance. Similarly, only 38.4% of teachers report that the school informs the community about the school's overall performance, and 7.5% of teachers report that the school obtained expert advice on analysing the results.

At the personal teacher level, the data appear more negative compared to those above in regard to the school's use of the test results. For instance, only 40.5% of teachers report using the results to identify individual students who are having difficulties. Fewer teachers give students feedback on their strengths (38.9%) and weaknesses (27.2%), or use the results to encourage pupils (36.4%). Only 19.9% of teachers report using the results to judge how well the class is progressing, and only 21.5% use the results to plan their teaching. These low figures may be due to the fact that 77.7% of teachers believe that the results arrive too late in the year to be of use.

The impact of the tests on teachers' practices

The level of impact of the tests on teachers' practices varies greatly depending on the domain of practice. The majority of teachers (91.7%) report showing pupils how to fill in the answers before the day of the test (e.g. colouring response bubbles, writing numbers in boxes) and giving pupils a practice test before the day (89.2%). However, very few report that the tests have influenced what they teach in mathematics lessons (31.7%), how they teach mathematics (26%), or how they assess it (20%).

The responses of pupils to the tests

It appears, according to teachers, that the numeracy tests are a negative experience for pupils. Only 20.6% of teachers report that their pupils cope with the tests, and the majority of teachers (61.2%) report that their pupils become anxious with the tests.

The effects of geographical location, school size, level of teaching and experience

The effects of geographic location, size, and level of teaching were investigated by conducting chi-square tests on cross-tabulations of the substantive items with categories of location, size and teaching level.

<u>School location</u> had a significant effect on eight (out of 29) items: Brisbane schools use the test results to identify gaps in the curriculum more than other schools (χ^2 = 23.7, p = .02). Pupils in Brisbane schools are more likely than other pupils to be anxious with the tests (χ^2 = 41.8, p = .0001). Teachers in Brisbane schools agree less than other teachers with the whole idea of the tests (χ^2 = 31.5, p = .008) and the idea that the tests ensure accountability (χ^2 = 23.8, p = .02). Rural schools are more likely to use the results to identify topics causing difficulties (χ^2 = 33.6, p = .001) and are more likely to report the test results to the community (χ^2 = 21.6, p = .04). Rural teachers more than other teachers are influenced by the tests in regard to how they teach mathematics (χ^2 = 27.6, p = .006) and how they assess it (χ^2 = 32.2, p = .001).

<u>School size</u> had a significant effect on only four items: Teachers in large schools (>400 pupils) agree less with the idea that the tests ensure accountability than teachers in other areas (χ^2 = 47.01, p = .001). Teachers in large schools are influenced less by the tests in regard to what mathematics they teach (χ^2 = 44.4, p = .001), how they teach it (χ^2 = 47.8, p = .001), and assess it (χ^2 = 51.5, p = .001).

<u>Level of teaching</u> had a significant effect on only five items. For instance, Year 3 teachers used the test results to identify pupils experiencing difficulties less than

teachers of other Year levels ($\chi^2 = 10.1$, $p = .04$). Teachers who thought their pupils cope with the tests more than pupils in other grades were Year 6 teachers ($\chi^2 = 9.7$, $p = .045$), and Year 7 teachers ($\chi^2 = 13.6$, $p = .01$). Year 6 teachers agreed with the idea of testing more than other teachers ($\chi^2 = 11.4$, $p = .045$). Lastly, principals believed more than the other teachers that the tests influenced what mathematics was taught in the school ($\chi^2 = 9.8$, $p = .04$).

Years of teaching experience correlated significantly with only two items – positively with use of the test results to identify pupils experiencing difficulties ($r = .24$, $p = .01$), and negatively with use of the test results to plan teaching ($r = -.19$, $p = .04$).

Other comments

Issues raised by teachers were mostly negative and related to the tests themselves (problems with the language of the tests, the ability of students especially those in Year 3 to read the tests, ambiguity and formatting of the tests), the way the tests are administered (time of year & lack of consistency across classes & schools), curriculum issues (a perceived mismatch between the tests & the syllabus, unsuitability to multi-age classrooms), the effect on pupils (emotional stress & reinforcement of negative self image, especially in Year 3), effect on teachers (stress), and philosophical concerns (unfairness of some schools using the actual test as a practice test beforehand).

DISCUSSION

It is clear from the survey data that teachers have not embraced the Year 3, 5 & 7 Numeracy Tests to any great extent, nor have the tests had much impact on their teaching. Less than half of the teachers agree with the tests in principle, and the majority think that the tests do little to assist learning or ensure accountability. The data also reveal some inconsistencies in the teachers' responses, especially on the use made of the test results. Although the majority report that the school uses the results to identify topics causing difficulties and gaps in the curriculum, only 21.5% report that they personally use the results to inform their planning, and only 19.9% use the results to judge how well the class is progressing. Similarly, two-thirds of the teachers believe that the school analyses the results to identify pupils experiencing difficulties, but less than 40% give pupils feedback on their strengths or weaknesses.

There are three possible reasons for these inconsistencies. Firstly, the great majority believe that the results arrive at the school too late in the year to be of any use to the school or the individual teachers. The results arrive within two or three weeks of the end of the school year, a time when the Year 7 students are getting ready to transfer to high school, and the pupils in Years 3 and 5 are soon to move up to Years 4 and 6 respectively. At the start of the following year, their new teachers have had a holiday break and are then preparing for their new classes. Last year's test results are not high on the agenda. The issue of timing of the tests and the release of results should be seriously considered by the testing authority (QSA) with a view to giving schools more opportunity to take advantage of the data available in the schools' test results.

Teachers’ attitudes to the tests may change for the better if the results arrive earlier, giving schools ample opportunity to analyse and act upon the results.

Secondly, it is clear that not many schools seek expert advice on how to analyse the test results and what action to take in the light of the analysis. There is also an inconsistency between the teachers’ responses and the schools’ intentions which are signalled in the reports sent to the testing authority. Intentions don’t seem to match the reality afterwards. Although the test-result data sent to schools are comprehensive, schools may need expert assistance in analysing and interpreting the data, and secondly in working out what changes need to be made to the school work program, classroom pedagogy, and teaching/learning support.

Thirdly, the low degree of usage of the test results may be symptomatic of the teachers’ negative attitudes and beliefs as expressed by a lack of support for the tests and a belief that the tests have little validity. This situation confirms the importance of beliefs and attitudes in teacher change (Guskey, 1985; Clarke & Peter, 1993).

In terms of the Clark and Peter (1993) model of professional growth, practical issues such as lateness of the reports and lack of expertise or support for analysing and interpreting the test data are probably limiting the amount of activity in the *Domain of Practice* which in turn is limiting teachers seeing valued outcomes (*Domain of Inference*) and influencing teachers knowledge and beliefs (*Personal Domain*). Thus the cycle of professional growth is severely impeded. It appears that the potential for enhancing numeracy outcomes and improving practice has not yet been fully tapped.

It seems that teachers perceive the Year 3, 5 & 7 Numeracy Tests quite differently to the Year 2 Net, which is seen in a very positive light (Nisbet & Warren, 1999). There is obvious concern expressed by most teachers about their students coping with the numeracy tests and the levels of anxiety caused by the tests, especially in Year 3. Such concerns about testing children in Year 3 give reason to investigate (a) the validity and appropriateness of putting children of a young age through such an experience, and (b) the alternative of extending the Diagnostic Net into Year 3.

The fact that the effects of school location, school size, teaching level, and years of experience were evident in only a few items implies that the problems and issues revealed in the data are system wide, and not confined to small pockets of schooling. Teachers’ negative beliefs and attitudes to the numeracy tests are proving to be a barrier to their acceptance and limiting efforts to taking advantage of the tests to improve students’ numeracy outcomes and attitudes. To turn this around, teachers need to see something positive in relation to the improvement of pupils’ performance, and this in turn needs an ‘external source of information, stimulus or support’ (Clarke & Peter, 1993). This could take the form of an initial trial of a numeracy-enhancement project involving specific professional development for teachers and school administrators, publication of the benefits obtained through the trial, and a subsequent extension of the project into other schools and districts.

[1] Staff from Australian Council for Educational Research (ACER) provided assistance with the sample design and selected the sample of schools. The ACER sampling frame is compiled annually from data provided by the Commonwealth and each State and Territory education system.

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"THE POETRY OF THE UNIVERSE": NEW MATHEMATICS TEACHERS' METAPHORIC MEANING-MAKING

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Contemporary metaphor theory here provides a framework for an initial exploration of metaphoric language used by pre-service mathematics teachers to describe mathematical knowledge and learning. Taking metaphoric expressions used by student teachers in early course assignments Il begin to consider the affordances and constraints that arise from meanings of mathematics structured by these metaphors and will consider how these might relate to student teachers' beliefs and impact upon their emerging practice.

INTRODUCTION

Studies in (mathematics) teacher education have questioned the extent to which programmes of initial teacher education can challenge mathematics teachers' dispositions (Brown & Borko, 1992; Grouws & Schultz, 1996). Social and educational backgrounds have a considerable influence on the emerging teaching dispositions and practice of new teachers (Noyes, 2003) and it is important for student teachers and those involved in their professional development to acknowledge and critique this tendency. In order to expose these influences students are required to submit, in the first week of the PGCE secondary mathematics course, an assignment exploring their 'starting' position as teachers. They are expected to write about their experiences of learning and using mathematics - both at school and in their lives more generally, about teaching and about the nature of mathematics itself.

The way that mathematics is taught varies depending upon location, both geographically and culturally; at an international level (for example Stigler & Hiebert, 1999), but also at a national level. Children have different experiences of growing up, of schooling, and of learning mathematics. Consequently, a cohort of forty to fifty student mathematics teachers will have a diverse range of mathematical histories, these being dependent on schooling and the wider social background. The mathematics teacher belief literature reports the complex relationship between beliefs about mathematics and teaching practices. Thompson's (1992) summary points out that professed beliefs are not always an accurate reflection or indicator of classroom practice but considerable evidence associates beliefs and practices. She commences her work with Hersh's assertion that:

> One's conception of what mathematics is affects one's conception of how it should be presented. One's manner of presenting it is an indication of what one believes to be most essential in it...The issue, then, is not, What is the best way to teach? but, What is mathematics really all about? (Hersh 1986, p13, cited in Thompson, 1992, p127)

Within the aforementioned student' assignments, focused in part on the nature of mathematics, students describe their mathematical histories and beliefs with a great variety of metaphoric expressions. This paper will try to draw some of these together under a few overarching root metaphors. Following contemporary metaphor theory, I contend that the metaphoric expressions used to describe mathematical knowledge and practice relate to individual's personal beliefs about mathematics and thereby probably relate in some way to pedagogic practice. However, this 'think piece' does not seek to validate the claims that it makes but rather proposes further avenues for inquiry.

With this fundamental interrelation between metaphoric systems and knowledge I want to explore the affordances and constraints of the metaphors most commonly found in the students' work. Consider, for example, two of the students' metaphors: if *mathematics is a toolkit* (reflecting an instrumentalist view of mathematics), what meaning of mathematics can and can't be constructed with it, and compare this with where one can or cannot get to if *mathematics is a journey*. The following discussion is necessarily brief but does outline some issues arising from such a tropological analysis. At the same time I acknowledge that these metaphors are not mutually exclusive nor do I currently have sufficient data to categorise students or relate their metaphoric expressions to other psycho-social factors or teaching practice.

METAPHOR

Over forty years ago Black (1962, cited in Ortony, 1993) presented an *interaction* view of metaphor in critique of the then prevailing *comparison* view of metaphor. This theoretical shift has since been followed by further developments towards a 'contemporary metaphor theory'. During that period of development Elliot (1984) asserted that education metaphors' "incompleteness makes them flexible instruments for communication, but they lack depth"(p.39). However, this (metaphorically framed) perspective failed to acknowledge the theoretical developments that were taking place at the time. The *depth* that Elliot considered to be missing has in fact been shown to be so fundamental and taken for granted that it often goes unnoticed. More than that, such deeply ingrained metaphoric language is a means of transferring meaning from one context to another. Consider the example *education as market*. The language of markets and economics has seeped into education discourses throughout the world (Apple, 2000; Gewirtz, Ball, & Bowe, 1995). Through the use of such a conceptual metaphoric schema perceptions of the meaning of the education system are changed. This as an example of a generative metaphor (Schön, 1993)one which has power to actually change the way people think about, and consequently act, concerning education. Schön's examples from social policy urge the use of alternative metaphoric frames, which themselves might lead to frame conflict and subsequent restructuring. The same intention is very apparent in Thomson and Comber's (2003) *reframing* of the discourse around disadvantage, and in Sfard's (1998) warning against relying solely on one metaphor to conceptualise learning.

Conflicting metaphoric frames could be utilised in a similar way to challenge (new) teachers' already well-established beliefs.

Lakoff and Johnson's (1980) early work in this field demonstrated how language employs metaphorical constructs to build meaning on the basis of prior conceptual understandings. Through a series of detailed studies of English' root metaphors they demonstrated how all meaning making sits upon a foundation of metaphor. That pioneering work was supported by Reddy's (1993) exploration of the *language as conduit* metaphor, with its critique of the damage caused by such an all pervasive, unproblematised metaphoric language system. Lakoff's (1993) later work reports a thorough account of metaphoric systems in everyday thought and language. Metaphors (i.e. the root metaphors, for example *life is a journey)* are not primarily linguistic tools but are conceptual mappings. He describes three main types of metaphor image schemas: "containers, paths and force-images" (p. 228) and it will be useful to see how these relate to what I consider to be the root metaphors in the students' narratives concerning mathematics.

A METAPHORIC FRAMEWORK

The range of metaphoric statements used by student teachers is extensive but themed under what I have already described as the key or root metaphors. I take each of these root metaphors and explore them in some detail, using the students' own scripts as exemplification. I have brought together these metaphorical descriptions under four headings: mathematics as structure, language, toolkit and journey. The metaphors of, and associated with, mathematics as toolkit and as language are predominant in the student narratives.

Mathematics as language

That mathematics is considered to be a language is evident in a large number of students' scripts. Moreover, as a language it is considered to have properties of internal logic and is useful for describing and communicating. Indeed, with regards the latter the students variously consider mathematics to be "the international language":

> What is most important to me is how it is universally understood by all across the globe. Mathematics is…a language in its own right. (PC)

> For me Mathematics is the poetry of the universe… To put Mathematics in context with other disciplines: maths is the poetry of everything, or the language with which everything communicates with everything and how it relates to everything. (SE)

There is a strong sense of the pre-existence of this language, and when described as *the language of the universe* it appears to have a self-existence, embodied in nature itself. This positivistic pantheism, elevating mathematical language to an almost divinic status (e.g. "God is mathematics" (IR)) mirrors Lerman's (1990) description of mathematics as "the last bastion of absolutism":

> Whether people like it or not maths is everywhere. (AB)

> Mathematics can exist without real life situations, maybe even without people to invent it. It certainly is argued that mathematics is discovered like a truth about the universe, rather than invented like a language … (maths is) a mystical world of its own. (JG)

This self-existent quality of mathematics means that it can act almost like a teacher itself:

> Mathematics has made me a logical thinker. (BC)
>
> Mathematics teaches clear and logical thinking. (SY)

So for many of these students mathematics is not simply a pre-existent language but it seems to be able to speak for itself. They explain that this language is useful for describing the world and is a means of interpreting the world around us. However, there is still the sense in this usage that the world can be interpreted using mathematics because mathematics is itself 'the language of the universe', and so the process of coming to know mathematics is akin to 'tuning in' to the voice of nature.

As "the last bastion of absolutism", such positivistic assertions concerning mathematics could lead to a sense of superiority. If it is *the* language of the universe; the ultimate means of description and understanding, then those who can understand and speak this language are in a privileged position. If it is merely another humanly constructed language of description and communication then it has different relative importance in the field of knowledge. Furthermore, if it is a language to be learned, then how should it be learned, through use or through study of its grammatical structure?

Mathematics as toolkit

The idea that mathematics is a toolkit, or set of skills, is the predominant discourse of student' conceptions of what mathematics is. Mathematics has utility!

> It is perfectly reasonable to view mathematics as a toolkit, a bag of rules, methods and conventions that we can use to model, interpret or change the world around us. (SA)

As well as being useful these tools can be "powerful" (DP) which reflects the measure of status afforded to mathematics in the previous section. In order to be able to use these tools correctly children are described as needing to practice their use. There is a standard set of *tools* that are introduced to children and then they need to develop expertise in a) tool use and b) tool selection. Children who achieve technical mastery with this tool set will be able to move on to use more dangerous tools, or perhaps specialise their *trade* with a particular subset of the mathematical toolkit. The student teachers do not explain how these tools came to be or whether children can invent new tools to do a better job.

This metaphor is used much more in the context of solving problems, as an odd-job tool kit, rather than the way in which an artisan might create something new with tools. So mathematics is described by these students as being the tools themselves rather than the artefacts created by the use of the tools.

These instrumentalist perspectives betray an attitude to mathematics that is necessarily useful and has to be relevant to real life problems. Very many applicants to the PGCE course find the idea that maths *is useful for everyday life* a difficult concept to defend, critique, or convince children of. For many student teachers this metaphor indicates that teaching (children how to use) mathematics is an apprenticeship where they, as the expert tool user, will pass on their trade secrets to children, jealously guarding the traditional ways.

Mathematics as journey

This root metaphor, which might more accurately be ‘learning mathematics is a journey’, is not as common in the students’ descriptions of mathematics as the previous two metaphors but it tries to explain the process of mathematical learning. There is some overlap with the structure metaphor (below), particularly with regards the fallibilist (Ernest, 1991; Lerman, 1990) conception of historically constructed mathematical knowledge structures. Here, journey is a process that includes places and timings, obstacles and short cuts, dead ends and all too often, going around in circles. Some learners are understood to progress more quickly than others, some get stuck or are hindered, and others never get through to reach the desired destination. Many of the teachers’ personal stories contain references to getting ahead, or going faster, on this journey and relate this to their emerging liking for the subject during childhood. For the majority of new teachers the route is pre-planned as in this example:

> Mathematics is a journey, understanding one step leads to another, and each step relies on the existence of the previous one. This journey, for me, is what Mathematics is all about…How far each individual chooses to travel is up to him or her, but all must take the same initial steps. At a later stage the path begins to split as different areas of mathematics become more defined but again the first steps must have been taken for travel to continue…The purpose of mathematics as a school subject is to demonstrate to pupils the significance of all of these paths and to invite them on the journey. It is then our job to make the journey exciting and eventful and each destination a memorable achievement. (CF)

So what is important here is not how one travels or which route one takes but “how far” individuals travel. The advent of the UK National Curriculum in the late eighties, and more recently the Framework for Teaching Mathematics, has made the official route much clearer. These official documents mark the way for journeying school mathematicians. Many of our student teachers completed their secondary mathematics education with the influence of such maps and route plans. In this respect there is some conflict between their tutors’ perspectives, who were much more mathematical explorers and pioneers. When the student teachers complete their school placements they discover that oftentimes tutors’ tales of mathematical explorations are a far cry from the well-worn track that they now so often have to follow.

A related image is that of climbing a ladder; children learning mathematics go up the hierarchical ladder rungs only moving up to the next rung when they are firmly positioned on the previous one. To go down to a previous mathematical idea is negative and acceptable attainment at the end of compulsory schooling is rung X. Such a conception of the hierarchical nature of mathematical knowledge is of course the language of National Curriculum levels, a fiction themselves. However, as part of a generative metaphoric structure they have created meaning and are part of a frame that needs continued critique and restructuring.

Mathematics as structure

Again, like the above metaphor, the *mathematics as structure* metaphor is not seen in its root form but many related metaphoric expressions can be seen in the students' work. Mathematics is described as a building; a network; a framework; as branched; as a fabric, and so on. Some of the references are concerned with the mathematical ontology itself whilst others are more concerned with the epistemology of coming to know.

> Mathematics is a building block. (EB)
>
> Mathematics is taught because it underpins a great many other subjects that are taught at school and beyond. (JP)

In addition students describe mathematical tools as keys, keys that presumably open up this building of knowledge. The students' ascription of priority and significance can be seen again here in the notions of foundations or underpinning. The claim that mathematics 'underpins all of life' is fantastical as all of life is not compatible with a foundation that mathematics could provide. This echoes the 'maths is *useful* for everyday life' myth that seems to permeate student teachers discourse.

Similar notions are envisaged in the idea of mathematics as a framework, it is something upon which a more substantial construction can be built. The difference in this image is the reduction of hierarchy and the increased inter-linking. It differs from the notion of network in that a framework is something that is built upon or extended/expanded whereas a network describes a more self-contained structure:

> The fact that mathematics is a vast network of ideas to explore remains one of my favourite aspects of the subject and I found the teachers who used exploration and discovery to bring the subject alive were among the best I was taught by. (VN)

This usage was unusual (as was the idea of exploration). Interestingly, VN is perhaps the most highly qualified student on the course and her initial ideas about teaching mathematics show how, at least in theory, her beliefs seen through this metaphor relate both to her experience, personal philosophy and notions about good teaching. The image in this network of mathematical knowledge is that the links are as valuable as the nodes, i.e. knowledge is as much contained in the linkages of the network as it is in the separate ideas.

One more thing to include within this section, although it could perhaps be situated within the *mathematics as language* section, is the notion of the aesthetic value of mathematics. Many students describe mathematical structures as beautiful, which is arguably an image is not associated with a *mathematics as toolkit* metaphor:

> And its nature is elegant and beautiful. (RO)
>
> However my interest in mathematics is not only because of its applications and its power, I think it has a beauty and fascination of its own. (CS)

These two quotes reflect the self-existence of mathematics described above.

CONCLUDING COMMENTS

Of the four groups of metaphoric expressions discussed above, Lakoff's three image schemas can be seen: the *structure* metaphor is a 'container', whilst the *journey* is a 'path' and 'force' is represented in the metaphor of *toolkit*. The first metaphor I presented was that of language and I would suggest that this fits more with the former and latter of Lakoff's categories. Reddy's analysis of language as a conduit is similar to the notion of mathematical language as a container of mathematical concepts. Also language has power in the same way that the students' consider mathematics to have power or pre-eminence.

Why these metaphors? Why not other path metaphors? For example, why does the notion of journey appear more commonly that that of exploration for these UK pre-service mathematics teachers? Why is the *mathematics as toolkit* metaphor so predominant, and to what extent is that subject to cultural variation? As I have shown within each of these root metaphors there are a range of metaphoric expressions and different meaning arise from their various uses, for example *journey* or *exploration*, *network* or *foundation*. Conceptual shifts within these metaphoric spaces might lead to changes in belief and an associated self-critique of student teachers' developing classroom practice.

That mathematics is seen to be a toolkit fits with teacher belief research, where an instrumentalist view of mathematics and mathematics teaching is acknowledged to be prevalent amongst new trainees. Platonist absolutist beliefs can also be seen threaded through these metaphorical tropes. However, a number of questions arise from this brief examination. How have these metaphoric systems developed and can they be changed? If, through the use of alternate generative metaphors, teacher educators can create conflict situations that might shift meanings of mathematics, how might this affect the conceptual positioning of new teachers and their subsequent teaching? How strong are official discourses in the promulgation of the current metaphors and can they be countered (NC levels are such an example)?

Liston and Zeichner (1991) urge teacher educators to engage new trainees in critical reflection of their beliefs and the social conditions in which they are developing as teachers. A critical examination of the metaphoric construction of their beliefs, as theorised by the contemporary theory of metaphor, provides another means of challenging teachers during their early socialisation into (mathematics) teaching.

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WHAT STUDNETS DO WHEN HEARING OHTERS EXPLAINING

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The purpose of this paper is to investigate what students do when hearing others' explanations. For this purpose, we set two video-cameras in a classroom, one of which recorded only the target student throughout the lesson and one of which recorded the teacher and other students explaining at the blackboard. The learning processes of two elementary school students will be analyzed. The analysis will demonstrate that the students induced their own learning by putting what they had noticed into new contexts activated by the information which they had selected from the others' explanations.

INTRODUCTION

It is widely recognized today that discussions in classrooms are important for learning mathematics. To help students participate in such discussions effectively and, as a consequence, construct mathematical knowledge, many studies have investigated discussions in whole-class or small-group settings and deepened our understanding of social aspects of mathematics classes. For example, Yackel (1995) analyzed the discussions in the inquiry mathematics classrooms and illuminated some factors which can influence students' explanations or constitute a situation for explanation. Webb et al. (2002) analyzed small-group works and pointed out that help-seekers needed to ask appropriate questions or requests to elicit good explanations.

In Japan, mathematics lessons, especially introductions of new ideas, include the time of discussing solution methods after students work individually or with their neighbors (see Stigler & Hiebert, 1999, p. 79). In this discussion, some students explain their ideas to the class. Even though they explain their ideas and some of other students ask questions about their explanations, many of the classmates spend most of the time hearing others explaining or asking. If so, we should pay attention to students who hear others explaining, as well as those who explain their ideas.

While the importance of social aspects has gotten our attention, some researchers seem to direct our attention to individual students who learn mathematics in social settings like classrooms. For example, Waschescio's (1998) indication, which was made after critically reviewing some researches about social constructivism in mathematics education, seems to imply that we should pay more attention to manners of learning of individual students. Anthony (1996) and Nagashima (1998) analyzed learning processes of individual students in ordinary secondary mathematics lessons and illuminated the different learning goals they had.

Following such indications, it may be valuable to investigate what individual students do or experience when they hear others explaining in mathematics classrooms. The purpose of this paper is to analyze the learning processes of students, focusing on the phase where they hear others' explanations, in order to gain insights into what kinds of learning can occur in such phases.

GATHERING DATA

In the following sections, we will analyze the learning processes of two students: One is a fifth grade male student, Shingo, and another is a sixth grade female student, Shizue. They attended different elementary schools in Japan. To investigate what students do when they hear others' explanations, in observing each lesson, we set a video camera so that it could record only the targeted student throughout the lesson. We sometimes operated the camera to zoom in and record how the student worked on his/her worksheet. Another video camera was set at the back of the classroom. It recorded the teacher and other students explaining at the blackboard. We made the transcripts from the videotaped records. They included what the target students did and what happened in the classroom (e.g. what kinds of explanation were presented by other students) at that time. These transcripts and the videotaped records are the data used in the following analyses. All the names mentioned in this paper are pseudonyms.

EPISODE1: SHINGO'S LEARNING

Shingo's behaviors during working individually

In his 5th grade class, the topic was to expressing the quotient of two whole numbers with fractions. The teacher posed the following task to the class: "We want to divide 2ℓ of milk among three people. How many liters of milk can one person get?"

Shingo wrote "2÷3" and started its long division algorithm. As it continued endlessly, he changed "2÷3" into "3÷2" after looking at the neighbor's notebook. At this moment, in responding to other students' question, the teacher announced that "3÷2" did not fit the situation of the task. He returned to "2÷3." The teacher interrupted and initiated the whole-class discussion. When some students mentioned that 2 was not divisible by 3, Shingo implemented a long division of 2÷3 and rounded off its quotient to obtain 0.67 and 0.7. He also calculated 0.67×3 and 0.7×3. After that, some students mentioned the rounded-off answer 0.7 and the class confirmed that it was an approximate value. The teacher encouraged the student to devise the way of expressing the quantity accurately.

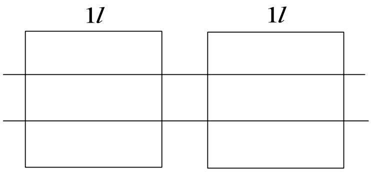

Fig. 1

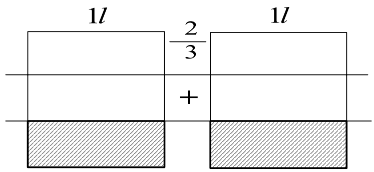

Fig. 2

The teacher drew two squares on the blackboard and added two horizontal lines crossing these squares. When he said, “I’ve divided it into three parts. Do you understand?” Shingo nodded slightly. The teacher handed the students worksheets which included the diagram shown in Fig. 1. He asked them to paint the area representing the quantity for one person, express it with a fraction, and write the reason for using that fraction. Shingo painted the lowest two rectangles (Fig. 2: There were not “2/3” and “+” at this moment). Hearing someone saying “two thirds,” Shingo wrote down “2/3” on his worksheet. But he erased this immediately.

After that, Shingo wrote his explanation on the worksheet: “When dividing 2ℓ among three people, a share for one person is 1/3. And two pieces of that.” He looked at his worksheet for about 30 seconds and modified and extended this explanation as follows: “When dividing 1ℓ among three people, a share for one person is 1/3. If changing it to 2ℓ, I can paint two pieces for one person. So, I can paint one piece for 1ℓ.” During Shingo was writing the last sentence, the teacher initiated the whole-class discussion again. He worked on his worksheet for about 15 minutes, aside from 6 minutes for the intervening whole-class discussion.

What did he do when hearing others’ explanations

The whole-class discussion started with the explanation of two students who thought the quantity for one person to be 2/6. They told that it was 2/6 because two squares in Fig. 1 were divided into six equal pieces and two pieces of them were a share for one person. Shingo looked at his worksheet as he listened to the explanation of the second student. Following these students’ explanation, the teacher said, “The expression 2÷3 becomes two sixth, doesn’t it?” Shingo nodded a few times.

At this moment, another student, Masato, raised his hand and insisted that the quantity for one person was two thirds. He went to the blackboard and explained his idea as follows: (i) Dividing the left square, which representing 1ℓ, into three pieces; (ii) Adding 1/3 from them and another 1/3 from the right square; (iii) Since adding them led to 2/3, 2÷3 became 2/3. The teacher repeated his explanation with Masato. When the teacher confirmed with Masato whether two 1/3’s were added, Shingo wrote down “+” and “2/3” between two squares in his worksheet (see Fig. 2). Some students uttered that they could understand the both ideas. The teacher told to the class that they thought there were two answers. Shingo put his head on one side.

Four students expressed their opinions: (Ken’ya) since two squares are separated, the answer is 2/6; (Ikumi) the answer must be one of 2/6 and 2/3, but the opinion will be divided; (Yuko) 2/3 resembles the original expression 2÷3; (Masato) as 2/6 means that 6 people share the milk, 2/3 may be better. When Masato expressed his opinion, Shingo told to the neighbor girl as follows: “It is 2/3. ‘Cause, speaking in terms of fractions, 1/3, 2/3…it is not 2/6.” After that, the teacher asked the class whether there were two answers to 2÷3. While one student spoke loud that he was not sure, Shingo told to the neighbor girl as follows: “But 1/3 and 1/3 does not become 2/6.”

When the teacher encouraged the class to resolve this question by themselves, Shingo raised his hand and told to the class as follows: "Adding 1/3 and 1/3 usually becomes 2/3. The denominator does not change in such addition." Another boy, Sou, mentioned referring to the worksheet used in the previous lesson that 2/6 was not equivalent to 2/3 and it was less than 2/3. The teacher confirmed with the class that 2/6 was equivalent to 1/3 and 3/9. During this discussion, Shingo looked at the teacher and speaking children, but he wrote or spoke nothing. Finally, Ikumi said that she could understand both ideas and the teacher announced to keep thinking about this issue in the next lesson.

Discussion about Shingo's learning

Even at the end of individual workings, he did not write how many liters of milk one could get. He only wrote that two pieces should be painted for one person. In the first half of the whole-class discussion, he nodded a few times when the teacher mentioned 2/6 as the answer. Shingo did not know $2÷3=2/3$ at all when the whole-class discussion began. In the second half of the whole-class discussion, Shingo raised his hand and set out his idea which supported the opinion that the answer was 2/3. When the teacher made an announcement about the next lesson, Shingo said to the neighbor girl, "It is two thirds, since the denominator does not change." Shingo had arrived at the conviction that $2÷3$ becomes 2/3 through the whole-class discussion of this lesson.

Because Shingo spontaneously wrote the answer "2/3" on his worksheet when hearing Masato's first explanation, this explanation can be considered critical for Shingo to understand $2÷3=2/3$. In fact, when the teacher mentioned the existence of two answers immediately after this explanation, Shingo put his head on one side. When other students expressed their various opinions, Shingo told the neighbor girl a few times that the answer was not 2/6 but 2/3. His behavior changed before and after the Masato's first explanation.

In repeating the Masato's explanation, the teacher wrote "1/3" in two painted pieces of the diagram on the blackboard. But Shingo did not write these "1/3" on his worksheet. On the other hand, he wrote down the "+" sign on his worksheet before the teacher used this sign to write "1/3+1/3=2/3." In the Masato's explanation, the idea of addition was most important for Shingo.

He might attend to this idea because it could bridge the gap between what he had done during his individual working and the goal of expressing the quantity with a fraction. Through his individual working, Shingo had realized that when dividing 1ℓ of milk among three people, the quantity for one person became 1/3. He had also found that when dividing 2ℓ, the quantity for one person could be expressed by two out of six pieces. The Masato's explanation made it possible for Shingo to put these findings in the context of addition of fractions. Viewing them in this context, Shingo could integrate his findings to achieve the above goal. Furthermore, in this context, he could bring forward another reason in favor of the answer 2/3: "The denominator does not change in such addition." No students mentioned this reason before he presented it.

Before hearing the Masato's explanation, Shingo might view these two pieces in the context of the number of equally-divided pieces. This is a reason why he nodded a few times when the teacher mentioned the answer 2/6, although he noticed that each piece represented 1/3. What he knew about the problem situation was basically the same before and after the Masato's explanation. Through his peer's explanation, however, Shingo could put it in another context and arrive at a certain conviction.

EPISODE 2: SHIZUE'S LEARNING

Shizue's behaviors during working individually

In her sixth-grade class, the topic was an introduction of division of fractions. The students worked on the following task using the worksheet (Fig. 3): "3/4$d\ell$ of paint is needed to paint 2/5m^2 of wall. How many square-meters of wall can be painted using 1$d\ell$ of paint?" Before working individually, the class reflected the preceding lesson and made sure that the expression for this task was 2/5÷3/4. Since they had not yet learnt how to calculate it, the teacher asked the students to devise and find out the area.

Shizue painted 3/4 of the 2/5 m^2 wall (Fig. 4) and wrote "2/5÷3/4=8/15". She erased this "8/15" and the painted part, and added two bars as shown in Fig. 5. Shizue moved her pencil in the air over "2/5÷3/4" for a while and wrote "8/15" again. She painted the part expressing 2/5 m^2 wall (i.e. below two rows in the square). She seemed to get in a bind. Then she made "the reduction" of "2" and "4" and changed the answer to 2/15.

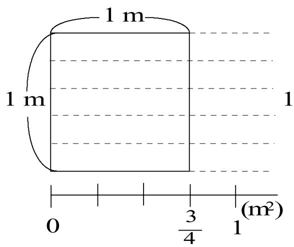

Fig. 3

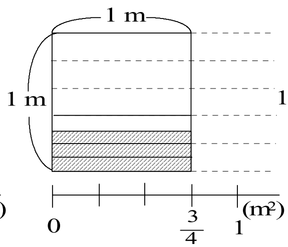

Fig. 4

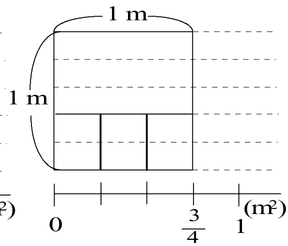

Fig. 5

When the teacher handed her a hint card, she erased this expression. This card included the diagram shown in Fig. 6 and its hint was to mark the area which could be painted by 1$d\ell$ of paint. She added "1/4" and "2/4" to the number line and counted 6 small rectangles in Fig. 5. After that, Shizue drew lines enclosing that area (i.e. a larger rectangle) in the worksheet (Fig. 7). She quickly counted 8 small rectangles in this area one by one. Then she wrote the expression "3/4÷2/5= /2" with "the reduction" of "4" and "2." However, she erased this expression at once.

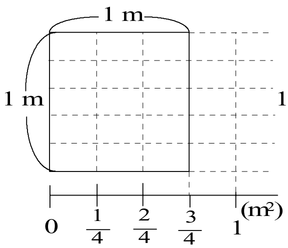 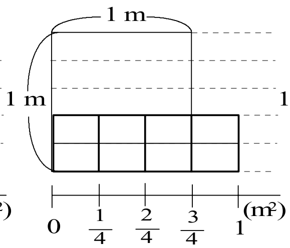 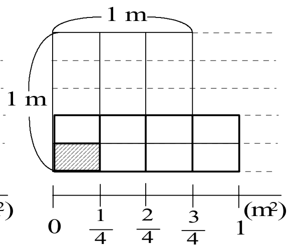

Fig. 6 Fig. 7 Fig. 8

The teacher handed her another hint card. It asked how many square-meters the leftmost two rectangles were (i.e. the area which can be painted by 1/4 $d\ell$) and asked how many pieces of such part constituted the area painted by 1$d\ell$ of paint. Shizue wrote "1/4" as the answer to the first question and "4/4=1" as the answer to the second question. After that, the teacher called on one of her classmates, Nana, to share her idea with the class. Shizue's individual working lasted about 23 minutes.

What Shizue did during hearing others' explanations

The teacher copied Nana's diagram (Fig. 8) on the blackboard. Because Nana hesitated to explain her idea by herself, the teacher explained it to the class using that diagram and reading her worksheet. Nana's idea was as follows: (i) In Fig. 8, the area which can be painted by 1$d\ell$ of paint is the big rectangle enclosed by bold lines; (ii) Draw vertical lines at 1/4 and 2/4; (iii) There are 15 small rectangles in the 1m^2 wall represented by a square; (iv) Since the area painted by 1$d\ell$ of paint is consisted of 8 small rectangles, it has an area of 8/15 m^2.

After the teacher explained the step (i), Shizue wrote "2/5÷4/4=2/20" ("2" of 2/5 was cancelled by the denominator "4" of 4/4). However, when the teacher explained the step (ii), Shizue erased this expression and looked at the blackboard again. The teacher proceeded to the step (iii) and counted 15 small rectangles one by one. When he counted the 13th rectangle, Shizue looked at her worksheet and added two lines to her diagram (Fig. 7) to change it into one like Fig. 8. Then she counted 15 small rectangles one by one. She also pointed the two rightmost rectangles in the bold lines. After she heard the teacher's explanation of the step (iv), Shizue wrote "2/5÷3/4=8/15" on her worksheet. Though another student began to explain her idea, which was different from Nana's one, Shizue erased this expression and attempted to write down the step (iii) with her own words. After several such attempts, Shizue wrote down on her worksheet what the teacher wrote on the blackboard. When she finished writing it down, the third student explained his idea based on a number line. She wrote nothing on her worksheet concerning the ideas of the second and third students.

Discussion about Shizue's learning

According to the post-interview, Shizue obtained "2/5÷3/4=8/15" by multiplying numerators and denominators crossly. She might learn this method outside school. Although she implemented this calculation at the beginning of her individual working, she did not seem to be certain of how to use this method. She modified the above expression into one with the "reduction" and changed its answer into 2/15. She erased this modified version soon when receiving the first hint card. However, after hearing the Nana's idea, Shizue returned to the answer "8/15" and the expression "2/5÷3/4=8/15" and never wrote other answers. Hearing the Nana's idea helped Shizue arrive at the conviction that 2/5÷3/4 becomes 8/15.

In the Nana's idea, the step (iii), 15 small rectangles in the 1m^2 wall, attracted Shizue most strongly. When the teacher and Nana copied her diagram on the blackboard, Shizue looked at them and did not work on her worksheet although her last diagram (Fig. 7) was slightly different from Nana's (Fig. 8). She started to write new expression when the teacher explained the step (i). But this expression, 2/5÷4/4=2/20, was not related to the Nana's idea, because "4/4" in it was the number she wrote during her individual working and the way of calculation was the same as her previous way. Even when the teacher mentioned the two vertical lines, she did not react to it. This is consistent with the fact that she did not extend vertical lines when receiving the hint card, whose diagram (Fig. 6) had dotted vertical lines crossing the square. What Shizue directed her attention to was the information that there were 15 rectangles in 1m^2.

The reason why Shizue reacted to this information may be that it could bridge the gap between her cross-multiplying method and the diagram in her worksheet. She had found that there were 8 rectangles in the area painted by 1$d\ell$. Nana's information about 15 rectangles could complement this finding and validate her initial answer "8/15." Drawing vertical lines crossing the square might make sense to Shizue as far as it produced 15 rectangles in the square. In other words, it was her findings about the situation that made Shizue react to certain information of the other's explanation.

It should be noted here that Shizue had drawn the vertical lines before receiving the first hint card (Fig. 5). She had also recognized the small rectangle as the unit for measuring the area painted by 1$d\ell$ of paint. However, Shizue treated these vertical lines and the small rectangle within the larger rectangle in Fig. 7. This can be supported by the fact that she answered "1/4" and "4/4=1" to the questions of the second hint card. The explanation of Nana's idea made it possible for Shizue to place them in a new context, the square representing the 1m^2 wall. In this sense, the Nana's idea was not a brand new one, but it placed Shizue's idea in a new context. Putting her idea in such a new context was, however, critical for linking her calculation method and the findings she noticed during the individual working period.

CONCLUSIONS

The learning processes of the two students have some common characteristics. First, the students selectively picked up the information from others' explanations. They seemed to select it for bridging the gaps they might feel during their individual workings. Second, using the selected information, the students put what they had known before into the contexts which were different from ones they had adopted during their individual workings. While the students had noticed the basic elements of the ideas before they heard others' explanations, the selected information enabled them to put those elements in new contexts. And putting them in such contexts resolved the gaps the students felt before that. That is, what the students did when hearing others explaining was to induce their own learning by putting what they had noticed into new contexts activated by the information which they had selected from the others' explanations.

This observation concerning what the students did when hearing others explaining, implies other important points. First, elementary school students who seem to merely hear others' explanations can learn mathematics actively. Second, what students do during their individual workings is a critical factor for such active learning, because it partly constitutes the above-mentioned gaps (*cf*. Nunokawa, 2001) and because such learning can occur by putting it in new contexts. If we remember that the students mentioned in this paper selected the information which they thought could resolve the gaps they felt, it can be said that what students do during their individual workings is also critical for selecting required information from others' explanations. While the information selected from others' explanations brought in new contexts, what the students had done before directed what to be selected and how to use it.

Observing the students' behaviors and learning throughout the lessons, we can say that what students do during their individual workings plays a central role for their learning when hearing others explaining. If we follow this discussion, it may be important for us to encourage students to: (i) try to make sense of problem situations as far as possible so that they can have something to be put in new contexts; (ii) be aware of kinds of gaps between what they are doing, on the one hand, and goals to be achieved or what they know, on the other hand. Such encouragement may generate fruitful time of students hearing others' explanations.

Acknowledgement

This research was partially supported by Grant-in-Aid for Scientific Research (No. 14580186), Ministry of Education, Culture, Sports, Science, and Technology.

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CHANGES IN TEACHERS' PRACTICES WHILE ASSUMING NEW LEADERSHIP ROLES

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This study explores the notion that as teachers develop their leadership skills their classroom practices gradually reflect a reform orientation. Within a systemic change project, an 18-month leadership institute was designed to nurture teachers' professional growth and to develop the skills and knowledge needed to plan and present mathematics reform recommendations to colleagues. Changes in teachers' practices were investigated using a case-study design by analyzing the questions asked, the cognitive demand of the instructional tasks posed, and the questions asked on formal assessments. Findings from this study suggest that teachers' practices change when they examine their own teaching practices.

Despite calls for reform in school mathematics by The National Council of Teachers of Mathematics (NCTM, 1980, 1989, 2000) classroom practices in United States during the last century have shown little change (Cuban, 1993; Stigler & Hieber, 1999). To support mathematics reform, NCTM (1991) described a need for mathematics teacher leaders who would become specialists "positioned between classroom teachers and administration" (p. 375) to assist with the improvement of mathematics education. These teachers would be responsible for building the content and pedagogical knowledge of colleagues, refocusing conversations from activities to an analysis of practice, and arranging collaborative investigations about student thinking.

Studies that examine teacher leadership are built upon the belief that, as teachers enhance their leadership skills and emerge as leaders, their practice also evolves (e.g.,Louckes-Horsely et al., 1998; Swanson, 2000). It is assumed that as teachers develop their leadership qualities, they gradually assimilate a reform orientation and their practice changes. The current study extends the research base on teacher leadership and change by examining this assumption. More specifically, this study asked the following question: How do teachers' classroom practices, as evidenced by their classroom discourse, instructional tasks posed, and assessments, change while participating in a leadership institute designed to promote professional growth?

THEORETICAL ORIENTATION

The research question focused my attention on the psychological processes of teacher change within a mathematics classroom. Symbolic interactionism facilitates the interpretation of the interactions between people as indications of personally constructed meaning (Becker & McCall, 1990; Denzin, 1992). Communication is thought to be a symbolic process that consists of an ensemble of social practices (including language, intonation, gestures, and written symbolic representations) that portray an individual's private construction of knowledge. Thus, an individual's interactions can be analyzed and interpreted to indicate this constructed knowledge

Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 2004 Vol 3 pp 457–464

(Becker & McCall). The character and new uses of verbal language by an individual during social interactions may indicate their assimilation of new ideas (Kumpulainen & Mutanmen, 2000). Kumpulainen & Mutanmen found that teachers' classroom discourse changed when their practice portrayed reform recommendations. In this study, I interpreted the symbolic interactions between teachers and their students to indicate the teachers' understanding of reform mathematics instruction. I analyzed the classroom discourse and the instructional tasks posed by the teacher to discern how teachers supported her students' development of mathematical ideas.

METHOD

Ethnography permits a researcher to examine and depict a social reality constructed through the actions of people (Gubrium & Holstein, 2000). Recently, ethnography was adopted to different disciplines and new forms of ethnography have emerged (Tedlock, 2000). A modified ethnographic methodology permits a researcher to select data sources, create collection methods, and report findings that respect the participants' perspectives and answer new research questions. In this study, I used a modified ethnographic methodology to develop a close relationship with three case-study teachers in their everyday lives to investigate the realm of meaning that they created as portrayed through their interactions with students.

Context

The Primary Mathematics Education Project (PRIME), a systemic change project, was a cooperative mathematics professional development venture between Illinois State University and a large, urban mid-western school district. A teacher leadership institute was created during the second year of the project to meet the concerns of teachers and project leaders for continued professional development after the project's conclusion. Ten teachers from different elementary schools joined the PRIME Leadership Institute and taught first through fourth grade.

The PRIME Leadership Institute was designed to nurture reflection and develop teachers' leadership skills. Teachers were involved in a lesson study during the first third of the 18-month leadership institute to help them deepen their content and pedagogical knowledge. Next, teachers created a list to describe high quality professional development and used these guidelines to plan and present reform mathematics recommendations to colleagues. The remaining leadership meetings explored how teacher leaders can support change in the social and political structure of schools.

Participants

Three teachers participating in the Leadership Institute were selected as case-studies and represented different levels of prior leadership activity within their school community. This research report describes the changes in the teaching practice of one case-study teacher, Ms. Edelweiss. Ms. Edelweiss was a third grade teacher who reported no leadership responsibilities in her school before joining the Leadership Institute (leadership application, June, 2001) and seldom spoke during faculty meetings (principal interview, May 3, 2002). During the 18-month Leadership Institute, Ms. Edelweiss began a mathematics study group at her school, planned and presented three professional development sessions for her school district, and made two presentations at regional NCTM conferences.

I assumed the role of both a participant and researcher in this study. I answered Ms. Edelweiss's questions about mathematics content and reform pedagogy. During classroom observations I made field notes, asked students to explain their solutions, and sometimes posed a new question to extend the task.

Data Sources

Three data sources were used to investigate the changes in Ms. Edelweiss's teaching practice: monthly classroom observations, field notes, and formal assessments (graded assignments). Twelve classroom observations over an 18-month period of time were made of Ms. Edelweiss with a pre-lesson and post-lesson interview using a semi-structured interview protocol. The classroom observations were audiotaped and transcribed for analysis. Ms. Edelweiss collected the formal assessments that she utilized during three time periods: fall 2001, spring 2002, and fall 2002.

Analysis

The computer software, winMAX (Kuckartz, 1998), was used to manage the data, to code transcriptions, and to sort coded segments into categories. The classroom discourse was analyzed using constant comparative methods (Merriam, 1998) to determine the types of questions the teachers asked (Driscoll, 1999), to determine the cognitive demand of the posed task and instances when the cognitive demand of the task changed during instruction (Stein, Smith, Henningsen, & Silver, 2000). Three time-ordered conceptual matrices were constructed to collapse these data and compare for patterns of change (Miles & Huberman, 1994). A conceptual matrix was constructed for each of three assessment collection periods to display the types of questions asked by Ms. Edelweiss and the cognitive demand of them (Stein et al). Changes in the percentages of the questions with high cognitive demand were interpreted as evidence of change.

My field notes included student solution strategies and questions that I wanted to ask Ms. Edelweiss during the post-lesson interviews. These notes were analyzed for evidence of change. The classroom interactions and changes in the types of questions that Ms. Edelweiss asked on assessments were interpreted to indicate a different conception of what it means to teach and learn mathematics. Member checking (Merriam, 1998) was utilized to check my analysis and interpretations of classroom practices.

RESULTS

Ms. Edelweiss represents a teacher who assumed new leadership responsibilities while participating in a Leadership Institute designed to develop reflection, presentation knowledge, and an understanding of the change process. During the 18-month institute, she changed three aspects of her teaching practice: use of questions, level of the cognitive demand of tasks, and perceptions about teaching and learning. The following two excerpts provide an illustrative example of these changes in practice.

Initially, Ms. Edelweiss used students' prior experiences to develop the context for an instructional task. She asked students to describe their family's garden and drew a square on the board (observation, October 16, 2001). Then she asked how much fencing would be needed to protect it. One student suggested that a side could be 4 feet and Ms. Edelweiss asked other students in the class to define the length of the square's other sides. Then she asked, "What's the perimeter?"

1: Devon	Sixteen feet.
2: Ms. E	Are you sure?
3: Devon	Sixteen feet.
4: Ms. E	Are you sure?
5: Devon	Yup.
6: Ms. E	Randy, what's the perimeter of the square?
7: Randy	Eighteen feet.
8: Ms. E	Are you sure?
9: Randy	Yes.
10: Ms. E	Mitch.
11: Mitch	Sixteen feet.
12: Ms. E	Okay, we'll go with sixteen feet. Does anyone see an
13:	arithmetic problem here?
14: Mitch	Eight plus eight.
15: Ms. E	What did you do? (Pause 8 seconds)
16: Mitch	Added those two sides (pointed at the opposite sides) and the
17:	other two.
18: Ms. E	Were you thinking of adding fours? (Wrote 4 + 4 + 4 + 4 on the
19:	board. Mitch nodded.). Were you thinking of adding the sides
20:	together? (Mitch nodded.)

During this lesson, Ms. Edelweiss maintained tight control of the classroom interactions by initiating questions and calling on individuals to respond. Ms. Edelweiss deviated slightly from a traditional discourse pattern (teacher initiates the interaction-student responds-teacher evaluates the response) when she solicited several possible solutions (lines 6, 10). She asked her students if they were sure (lines 2, 4, 8) but did not pursue the question. Ms. Edelweiss validated Mitch's response and then asked Mitch to state a procedure for finding the perimeter of a square (line 12). After he stated his strategy, she asked a leading question (line 18) and used his response to summarize the procedure that she expected them to use (line 19-20). In doing so, reduced the cognitive demand for the succeeding questions on the prepared worksheet.

Initially, Ms. Edelweiss asked me content and pedagogical questions. During the post-lesson interview, Ms. Edelweiss commented, "They understand perimeter but when we do area, they get them confused. What should I do?" (October 16, 2001). She considered the impact of curricular materials on student learning stating, "Well, this [textbook lesson] is so concrete that it basically tells you the answer… Reform curriculum there's more thinking and it's easier to ask questions… With the textbook there's just one way to think about it [mathematics]." While Ms. Edelweiss recognized the support reform curriculum provided to develop students' mathematical thinking, she continued to use the traditional textbook for the majority of her lessons. When I asked her about this choice she responded that the fourth grade teachers expected her students to know the material in the textbook. I interpreted these responses and actions to indicate a conception of teaching and learning that students learn mathematics through repeated practice after being shown a procedure to follow.

Ms. Edelweiss participated in a lesson study during leadership meetings (September 2001-February 2002) to promote her own professional growth. Her group investigated how students make change during a monetary transaction. In January, they considered how the type of questions asked influenced both the information they gained about students' mathematical thinking and students' opportunity to learn (field notes, January 15, 2002). After this discussion, Ms. Edelweiss changed the kinds of questions that she asked and maintained the cognitive demand of the task. These changes of practice were portrayed on February 8, 2002, when students explained how they shared three brownies between four people. In the following illustrative example, Lizzie drew three brownies on her paper. Two of the brownies were cut in half and the third brownie was divided into fourths. Each person was given two pieces, a half and a fourth.

Ms. E: Okay, four people three brownies. How much do they all get?
Lizzie: Three fourths.
Ms. E: Okay, would you explain this one to me.
Lizzie: One half plus one fourth (pointed at the half piece and then the fourth).
Ms. E: Okay, write it down. (Pause 3 sec) equals (pause 10 sec). One half is the same as how many fourths?
Lizzie: Uhmmm.. Two fourths.
Ms. E: Why don't you write two fourths right there? (Pause) Plus one fourth.
Lizzie: Equals three fourths.
Ms. E: Now what can we do so that the next time you have two different denominators?
Lizzie: Uhm. I could just divide them all into fourths and none of them in halves.
Ms. E: Do you need to divide them into fourths as you're working the problem or divide them into fourths when you just trying to find your final answer?
Lizzie: Divide them into fourths as I'm doing the problem, it makes it easier to add them up and see what it is.
Ms. E: Okay.
Lizzie: Yeah, make them all fourths and then divide them up. Then you just see how much each gets. (She drew three brownies, divided them, and drew lines to four people.
Ms. E: But this way, you started with 1/2 plus 1/4 and you changed…
Lizzie: To two fourths.
Ms. E: So if you add 1/2 and 1/4, what did you do?
Lizzie: You just make them all fourths and add them up. You don't have to cut everything up, you just have to do that for one person.

Ms. Edelweiss asked Lizzie to reflect on her solution and explain her reasoning (lines 1, 3). Lizzie was told to record her solution using an equation and then Ms. Edelweiss prompted her to think about the relationship between a half and a fourth (line 6-7). Recognizing that Lizzie added two fractions of different size pieces, Ms. Edelweiss directed Lizzie to use the standard algorithm by representing one half as two fourths (line 9-10). She then focused Lizzie's attention on using common denominators to add fractions (line 12). Lizzie considered an alternative method of sharing the brownies by dividing all of them into fourths (line 14). Lizzie then proceeded to draw a model representing this strategy (line 23). Ms. Edelweiss redirected Lizzie's attention back to her original representation (lines 25, 27) and Lizzie made a connection between the two strategies (lines 28-29). Without a suggested strategy, this task represented a problem with high cognitive demand. Ms. Edelweiss maintained a task with high cognitive demand by providing Lizzie an opportunity to construct her own models for dividing brownies as she explored the concept of fractional pieces and operation of division.

Reflecting on the role of questions, Ms. Edelweiss reported that she asked questions out of her own curiosity, expounding, "If I don't ask them questions, I won't know where they are or what they are thinking" (interview, March 21, 2002). Questions were no longer posed to find out if they knew an answer but to reveal their thinking and assess their understanding. Her actions indicated a new conception of teaching and learning. She asked questions to discover students' mathematical understanding and then used those insights to make instructional decisions to develop their thinking.

DISCUSSION AND IMPLICATIONS

The changes in Ms. Edelweiss's teaching practice suggest a psychological change in her conceptions of teaching and learning. Initially, she asked questions that led students to describe a procedure that was used to solve subsequent problems and thus reduced the cognitive demand of the task. After participating in a lesson study, questions were used to help students reflect on their solution strategy, consider the relationship between mathematical models, and make a connection between the standard algorithm and alternative strategies. Thus, she maintained or elevated the cognitive demand of the task. Her perceptions about teaching and learning also changed after participating in a lesson study. Ms. Edelweiss ask students' to find out their understanding and used this knowledge to plan instruction that developed her students' mathematical ideas instead of teaching established procedures to solve problems.

These changes in practice occurred when she deepened her own pedagogical and content knowledge while participating in a lesson study. Changes in the other two-case study teachers also occurred during the same time period, suggesting that the three case-study teachers developed a new interpretation of reform mathematics recommendations while engaged in conversations about teaching and learning (Olson, 2003).

Ms. Edelweiss maintained these changes in practice during the remaining eleven months on the leadership institute but did not exhibit any new changes of practice. Assuming leadership roles that included informal discussions about mathematics reform with colleagues, planning and presenting professional development, and discussing the change process did not lead to additional changes in her practice. While Loucks-

Horsely, Hewson, Love, and Stiles (1998) suggest that teachers' practices will change as they develop leadership, this study indicates that teachers' practices do not necessarily change to reflect a reform orientation while they develop their leadership skills. I found that the three case-study teachers gradually assimilated a reform orientation using analytical frameworks that helped them to examine their practice from new perspectives.

Swanson (2000) and Snell and Swanson (2000) found that teachers developed their leadership over a long period of time while they gradually developed practices that reflect reform recommendations. On the contrary, study indicates that teachers can change their practices and develop leadership to support mathematics reform in a relatively short period of time. Ms. Edelweiss began the study with little influence in her school community. While participating in the 18-month Leadership Institute, Ms. Edelweiss demonstrated leadership growth in her school. She organized a mathematics study group and orchestrated professional development for her colleagues. Additional research is needed to determine whether teachers who gain leadership over a short period of time are able to maintain their influence and a reform orientation in their practice after professional support is reduced.

The work described in this report was funded in part by the National Science Foundation, grant number ESI-9911754. Opinions expressed are those of the author and not necessarily those of the Foundation.

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STUDENTS' STRUCTURING OF RECTANGULAR ARRAYS

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This paper presents the results of a study of the structural development of young students' drawings of arrays, and in particular, the significance of using lines instead of drawing individual squares. Students' array drawings were classified on basis of numerical properties, and perceived structural similarities that reflected the spatial properties of arrays. The relationship between these two aspects was investigated and a sequence for the development of array structure is postulated.

INTRODUCTION

The rectangular array model is important for mathematics learning because of its use to model multiplication, to represent fractions and as the basis for the area formula. Although array models are used to show multiplicative relationships, students may not see structural similarities of discrete arrays and arrays as a grid of contiguous squares, thus they may not connect an array of squares with multiplication.

Fundamental understandings of rectangular array structure would appear to be that the region must be covered by a number of congruent units without overlap or leaving gaps, and that a covering of units can be represented by an array in which rows (and columns) are aligned parallel to the sides of the rectangle, with equal numbers of units in each. The most efficient way of drawing an array is to draw equally-spaced lines parallel to the sides of the rectangle, constructing equal rows and columns. However, many young students cannot do this (Outhred & Mitchelmore, 1992). In this paper we make inferences as to how students' understandings of array structure progress from a collection of individual units to (perpendicular) intersecting sets of parallel lines.

The action of physically covering a rectangle with unit squares suggests a counting process whereas the array model is used to exemplify multiplication. To link the array model to multiplication, students need to perceive first that the rows are equal and correspond to equivalent groups. In theory, such a perception equates to a repeated addition model. The second perception, that the array is a composite of composites, equates to a multiplicative model. Steffe (1992) believes that students' recognition and production of composite units are key understandings in learning about multiplication. However, students may not fully understand the relationship between multiplication and addition (Mulligan & Mitchelmore, 1997) and may persist in counting.

Only gradually do students learn that the number of units in a rectangular array can be calculated from the number of units in each row and column (Battista, Clements, Arnoff, Battista, & Borrow, 1998). These authors classified Grade 2 students' counting methods into levels of increasing sophistication. At the lowest level, students counted in a disorganised manner. Then there was what Battista et al. call a paradigm shift to treating the array in terms of rows. Some students were unsure of how to find the number of rows, while others were able to find the number of rows when the number of squares in the orthogonal direction was given but estimated this number otherwise. By contrast, at the highest strategy level students immediately used the numbers of units in each row and column to find the total by multiplication or repeated addition.

In area measurement, emphasis on area as covering encourages counting (Hirstein, Lamb & Osborne, 1978; Outhred & Mitchelmore, 2000). Students may not see congruence as crucial to measurement; they may perceive individual pieces resulting from partitioning regions as counting units rather than fractional portions of a referent whole (Hirstein et al., 1978; Mack, 2001). Students who count units are also unlikely to link area measurement to multiplication, which is fundamental to understanding the area formula. Use of concrete materials also encourages counting and does nothing to promote multiplicative structure. The materials themselves can obscure structural features of unit coverings (Doig & Cheeseman, 1995; Dickson, 1989) and obviate the need for students to structure arrays.

Student's drawings of arrays

How do students develop mental representations of array structure by abstraction from physical or pictorial models? There is evidence that students' drawings of array models can reveal their mental representations (Besur & Eliot; 1993). We shall assume that students' difficulties in representing arrays are a consequence of limited conceptions of array structure, rather than of inadequate drawing skills.

Several researchers (Battista et al., 1998; Outhred & Mitchelmore, 2000) have emphasized the relationship between array structure and either counting or measurement. Neither study focused on development of array representation, nor interpretation of students' drawn constructions in terms of their understanding. No systematic study detailing the structural development of students' array drawings, and in particular, the significance of using lines instead of individual squares, appears to have been reported in the literature.

METHODOLOGY

A large sample of 115 students, with approximately equal numbers of boys and girls from a range of cultural groups, was randomly selected from forty grade 1 to 4 classes (aged 6 to 9 years) in four schools in a medium-socioeconomic area of a large city. Individual interviews with the students were conducted early in the school year.

The sequence of drawing, counting, and measuring tasks involved representing arrays of units given different perceptual cues, calculating the numbers of elements in arrays, and constructing arrays of the correct dimensions when no perceptual cues were given. These particular skills focus on linking the unit (in this case, a square), iteration of this unit to cover a rectangular figure, and the lengths of the sides of the figure. Information concerning the strategies that students used to solve array-based tasks was inferred from students' strategies as they drew. In this paper only a subset of the tasks will be included. These tasks are summarised in Figure 1.

The drawing items (D1, D2, and D3) all required the students to draw arrays of units, but did not require measurement skills. Responses to Task D1 should indicate students' perceptions of the essential features of an array because no drawing cues were given and to copy a figure students require some knowledge of its properties. Tasks D2 and D3 were presented to elucidate students' abilities to construct arrays given different cues, which required that students imagine increasingly more of each array in order to draw it. The responses to the tasks provide information about the skills involved in representing arrays and the order in which these skills are learnt.

Task	Unit	Requirements
D1	Cardboard tile 4cm square	Cover a 12cm x 16cm rectangle (enclosed by a raised border) with 4 cm cardboard unit squares, work out how many units, and draw the squares.
D2	Drawing of a 1cm square	Draw array given units along two adjacent sides of a 4 cm x 6 cm rectangle.
D3	Drawing of a 1cm square	Draw array given marks to indicate the units on each side of a 5 cm x 8 cm rectangle.

Figure 1 **The array drawing tasks (D1, D2, D3)**

RESULTS AND DISCUSSION

The students' drawings were sorted in two ways, based on analysis of the drawings, supplemented by the interview notes. First, the drawings were classified on the numerical properties of arrays, and second, on the basis of perceived structural similarities that reflected the spatial properties of arrays. The numerical classification was based whether students drew equal rows (columns) and whether the dimensions corresponded to the array that had been indicated. The spatial classification was based on covering the region without leaving gaps and the degree of abstraction shown in the drawings, that is whether students drew individual squares or lines. All three tasks showed the same three levels for numerical properties and five levels for spatial properties. However, students often produced drawings at different levels for different tasks, so the levels are not a classification of students. The three final numerical levels are shown in Figure 2.

Level 1. Unequal rows (columns): There may be an incorrect number of columns with an unequal number of units in each (1a) or a correct number of rows with an unequal number of units in each row (1b).

Level 2. Equal rows (columns)—incorrect dimensions: Rows and/or columns have an equal, but incorrect, number of units. There may be an incorrect number of rows with an equal, but incorrect, number of units in each (2a) or a correct number of rows with an equal, but incorrect, number of units in each (2b).

Level 3. Numerically correct array: Rows and columns have an equal and correct numbers of units. However, the array is not always spatially sophisticated.

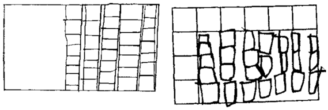
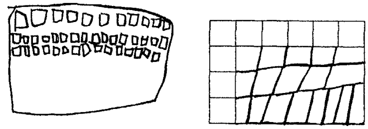
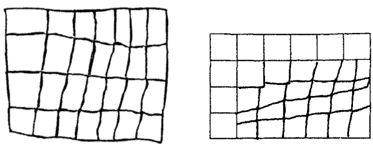
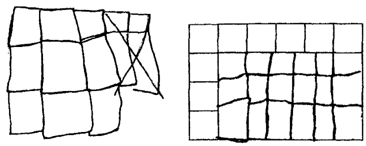
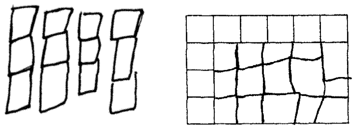
1(a) incorrect number of columns, each with unequal numbers of units 	**1(b)** correct number of rows with an unequal number of units in each. 
2(a) incorrect number of rows with an equal, but incorrect, number of units in each row 	**2(b)** correct number of rows with an equal, but incorrect, number of units in each row 
3 correct number of columns, equal and correct number of units in each 	

Figure 2 **Examples of each numerical level for Tasks D1 and D2**

Spatial structuring levels

The most important skill in representing an array, partitioning into rows and columns, seems to be based on an understanding of a fundamental property of rectangular arrays: the elements of an array are collinear in two directions. The five level classification of spatial structure (see Figure 3 for examples from Task D3) describes students' increasing level of knowledge of array structure from Level 1 to Level 5.

Level 1 Incomplete covering: The units do not cover the whole rectangle. They are drawn individually and may be: (a) unorganised elements; or (b) arranged in one dimension but not connected.

Level 2 Primitive covering: An attempt is made to align units (drawn individually) in two dimensions. Units cover the rectangle without overlap but their organisation is unsystematic.

Level 3 Array covering—Individual units: Units are drawn individually, are approximately equal in size, and are aligned both vertically and horizontally. Drawings show correct structure—equal numbers of approximately rectangular units in each row and column. The array is not constructed by iterating rows.

Level 4 Array covering—Some lines: Students realise that units in rows (or columns) can be connected and use some lines to draw the array.

Level 5 Array covering—All lines: The array is drawn as two (perpendicular) sets of parallel lines. Row iteration is therefore fully exploited.

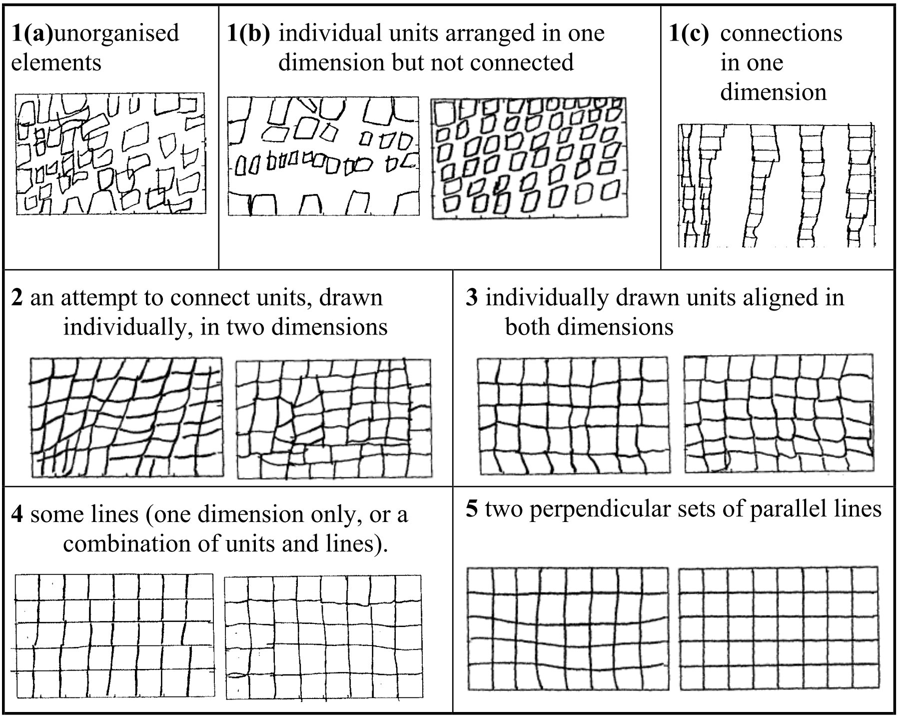


Figure 3 **Examples of each spatial level for Task D3**

The above sequence is developmental in the sense that each level is more sophisticated than the previous ones and the levels show a clear grade progression. This is not to say that students necessarily progress through each level in turn. At Level 1, no discernible strategy is used to cover the rectangle. Young students frequently draw individual units with large gaps between them but as they realise the importance of alignment, their drawings increase in regularity and the row/column structure becomes correspondingly apparent. As student knowledge increases the units become connected first in one, then in two dimensions (that is, students gradually seem to understand the importance of covering the region). Until students attempt to join the units in two dimensions, the rows and columns are not usually aligned. The strategies used to construct coverings at Levels 1, 2 and 3 might be termed local rather than global (Battista & Clements, 1996). The students focus on parts of the structure—for example, iterating rows or joining adjacent squares—but they have no global scheme for coordinating an array.

Level 4 indicates the emergence of a coordinated scheme for showing units as composites in one or two dimensions. There are various transition stages between drawing individual units and an array. For instance, lines may be drawn across the width of the rectangle to indicate rows with units in each row marked off individually, or some individual units (usually the top row and the left column) is drawn as a guide to drawing the array (see Figure 3, Level 4). The most abstract method of drawing an array is as two (perpendicular) sets of parallel lines (Level 5), because this method is furthest removed from the physical action of covering a rectangle with individual units. By Level 5 students appear to have internalized the row and column structure.

The relationship between numerical and spatial levels

The relationship between numerical and spatial levels for Task D3 showed that few students (7%) drew a numerically correct arrangement without using some lines (Levels 4 or 5). The converse was also true, all students' drawings classified as spatial Levels 1 and 2 were numerically incorrect. The distribution of numerical and spatial levels with grade for Task D3 is quite different from Task D1 where, in most of the drawings that did not show a systematic array covering (Levels 1 and 2), rows and columns usually contained unequal numbers of units. Nevertheless, quite a large proportion (37%) of these drawings were numerically correct. Once students began to use lines to draw the array (Levels 4 and 5), they always drew equal numbers of units in each row but,21% of students did not show the correct number of units in each row. However, Task D1, in which numerical structure had to be deduced, would have been far more difficult if the model had had equivalent dimensions to Task D3 (5x8).

CONCLUSION

In summary, the results of this study show students' drawings of rectangular arrays develop between Grades 1 and 4 from single squares to an accurate array with a concomitant understanding of alignment and composite units. Analysis of students' drawings indicated that representing an array of units using two perpendicular sets of parallel lines is more difficult than might be expected, indicating that the structure of a square tessellation is not obvious to students but must be learned.

In initial representations of arrays, many Grade 1 students did not see the importance of joining the units so that there were no gaps, and drew units individually. As they attempted to align squares, their drawings became increasingly regular and the structure became correspondingly apparent. Until students began to join the units in two dimensions, they did not usually align rows and columns. Before drawing arrays using only lines, some students drew lines across the width of the rectangle to indicate rows and marked off the units in each row individually while others drew some individual units (usually the top row and the left column) as a guide to drawing the array. By Grade 4 most students had learnt that the physical action of covering a rectangular area with units was equivalent to an abstract representation using lines.

For some students the lines shown in an array may be only a visual feature unrelated to numerical structure. However, drawing lines in one dimension appeared to be a precursor to recognising rows as composite units. Such recognition helped students to perceive that squares could be constructed by joining lines in the other direction, and hence realise the two-dimensional structure of an array. The comparison between numerical and spatial structure across the three tasks shows that drawing correctly aligned units is necessary, but not sufficient for correct numerical structure when drawing arrays of large dimensions (as in Task D3). Although it might be argued that the relationship is a consequence of the strong correlation between knowledge of array structure and age, students in Grades 2, 3, and 4 solved measurement tasks when the units were not indicated (Outhred & Mitchelmore, 2000), so the more salient discriminator would appear to be array structure. Moreover, for the indicated grid task, numerical level was a stronger predictor of spatial level than grade. The results of this study, combined with a teaching experiment (see Outhred, 1993) suggest the following sequence (see Figure 4) for the development of array structure.

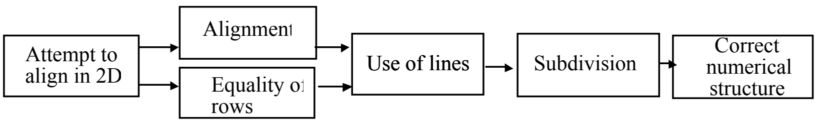


Figure 4 **The hypothesised development of array structure**

Understanding of array structure (as demonstrated by ability to complete an indicated grid) has been shown to be a prerequisite for students to progress from array-based activities with concrete or pictorial support to more abstract tasks, involving multiplication and measurement. Only students who drew an array using at least some lines successfully solved a measurement task in which students had to construct an array of the correct dimensions (5x6) by accurate estimation or by measuring the side lengths of the rectangle with a ruler (Outhred & Mitchelmore, 2000). The results of the measurement tasks reported in the above study reinforced the significance of the formation of an iterable row as the foundation of an understanding of array structure. In addition, an understanding of subdivision was found to be crucial when cues to the array structure were not given. Students have to clearly identify the significance of the relationship between the size of the unit and the dimensions of the rectangle. Although it may seem self-evident to adults that the number of units in the array must depend on the measurements of the sides, it was clearly not obvious to students. Thus, teaching about array structure must include activities that provide students with experience of partitioning a length into equal parts. Subdividing a rectangular region into equal parts depends on students being able to partition a length into a required number of parts, as well as knowing that an array can be represented using lines.

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CONSTRUCTION OF MATHEMATICAL DEFINITIONS: AN EPISTEMOLOGICAL AND DIDACTICAL STUDY

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*The definition-construction process is central to mathematics. The aim of this paper is to propose a few **S**ituations of **D**efinition-**C**onstruction (called SDC) and to study them. Our main objectives are to describe the definition-construction process and to design SDC for classroom. A SDC on "discrete straight line" and its mathematical and didactical analysis (with students' productions) will be presented too.*

INTRODUCTION

This paper would like to show that "definition" is not only a meta-mathematical term. Actually, mathematical definitions can be approached from two different standpoints. The first one consists in taking for granted that definition is not a problem and that the definitions provide mathematical concepts: "le premier piège est de croire facile à acquérir ce qui est simple à énoncer" (Kahane-1999-p12) [1]. When we are constructing a concept, a dialectical process involving both the construction of the definitions and the construction of the concept is at work: "A definitional procedure is a procedure of concept formation" (Lakatos-1961-p54). We shall start from those two points of view to introduce our research topic: *definition-construction*, in other words: what type of defining situations are designed and analysed in mathematics education research? Does the analysis of those situations give specific results concerning concept-formation? Can they really help towards the assimilation of mathematical concepts? Which theoretical background can we use for the analysis of such situations?

EXISTING RESEARCH ON DEFINING ACTIVITIES

Some researchers in mathematics education stress the need for a learner to be an apprentice-mathematician. Freudenthal (1973), in particular, tackles the case of mathematical definitions, which don't have to be considered as arbitrary rules by pupils. To illustrate his point of view, he gives the example of the classification of quadrilaterals in geometry, and thus underlines the nature of the exploration of several properties of different quadrilaterals; his theoretical approach rests on the Van Hiele levels. Freudenthal specifies two different types of defining activities: *descriptive (a posteriori) defining* and *constructive (a priori) defining*. There are a systematisation of existing knowledge and a production of new knowledge. This kind of defining activity is visited again by De Villiers (1998) who underlines that students are active learners in such situations. In this connection, Vinner emphasizes the importance of constructing definitions: "the ability to construct a formal definition is for us a possible indication of deep understanding" (Vinner-1991-p79). Within his theoretical framework, Vinner suggests to expose a flaw in the students' *concept image* of a mathematical concept, in order to induce students to enter into a process of reconstruction of the *concept definition*. Can we imagine other kinds of situations involving definition-construction? This is precisely what Borasi proposes, insisting on mathematical inquiry, and more specifically on the role of mathematical definitions. She proposes three instructional heuristics for the design of defining activities: the in-depth analysis of a list of incorrect definitions of a given concept; the use of definitions in specific mathematical problems and proofs, the exploration of what happens when a familiar definition is interpreted in a different context (Borasi – 1992 – p155), and she underlines the difficulties in building defining activities in which unfamiliar concepts are at stake (she uses the notion of "à la Lakatos"). Duchet (1997) proposes such problem-situations, called *research situations*, inspired by ongoing mathematical research, in which a definition-construction process may appear: but Duchet's analysis is not specifically turned to **S**ituations of **D**efinition **C**onstruction (called SDC).

Let us underline the major objectives of these previous authors (and others) in order to point out their common denominator i.e. the definition-construction.

Type of situations (*definition-construction*)	Mathematical concept	Mathematical field
Classification (starting from representations, examples and counterexamples)	Quadrilaterals (Freudenthal/De Villiers) Convexity (Fletcher)	Geometry
Redefining		
- starting from representations, ex/cex (familiar concept)	- Function / triangles (Vinner)	Analysis /Geometry
- starting from a list of incorrect definitions (familiar concept)	- Circle (Borasi)	Geometry
- redefining in a different context	- Taxicab metric (Borasi) /Square on a sphere (Duchet)	Geometry
- extending definitions	- Exponentiation beyond the whole numbers	Algebra
Problem-situation	- Generator, minimality (Displacements on a regular grid map) (Duchet) - Polygon (Borasi)	Geometry-algebra Geometry

Table 1: summary of didactical problematics on *definition-construction*

From this table, different features appear: first, the nature of mathematical concept at stake in situations of *definition-construction* seems to be specific, because almost all of those concepts come from the geometry or combinatorial geometry. Second, the predominance of classification and redefining situations: it is perhaps directly link to the real difficulty in designing a "problem-situation" whose resolution involves a *definition-construction*. Thirdly, there is no common theoretical background for the global analysis of SDC. In fact, the existing theoretical backgrounds used (*concept image*, *concept definition* -Vinner- & **T**heory of **D**idactical **S**ituations-Brousseau) are useful because they help us to grasp some key elements for the design (TDS) and the analysis of students' processes (*concept image*), so we have no choice but to model mathematical definition-construction process (present specifically in Lakatos's work). Thus, the mathematical and didactical study of processes of definition-construction both involves a description of these processes in mathematics and a typology of SDC including classification, redefining situations among others. Our challenge consists in building theoretical tools, efficient for the characterization and the analysis of processes of definition-construction, in modelling the dialectic between concept-formation, definition-construction and proof (cf. Lakatos). The didactical stakes lie in the fact that the SDC constitute a real challenge for concept construction, and for the evolution of students' conceptions about definitions.

THEORETICAL FRAMEWORK & AIMS

We need to consider the concept of "definition" through its main features i.e. language (a definition is a specific discourse), axiomatic (a definition is inscribed in a mathematical theory) and heuristic (a definition-construction process, which is heuristic, leads to concept formation). The main references we chose for describing conceptions about "definition" are Aristotle (language), Popper (axiomatic) and Lakatos (heuristic). We chose the cK¢ model (Balacheff-2003) in order to describe these three conceptions of the notion of definition, because it allows a recognition of definition-construction process, and thus, it brings elements for analyzing mathematical concept formation. Balacheff presents a *conception* as an "instantiation" of a subject's knowledge by a situation and stresses that conceptions and problems are dual identities. Starting from a psychological presentation of a concept (referent, signified and signifies – Vergnaud, 1991), the cK¢ model calls *conception* (C) a quadruplet (P, R, L, Σ) in which:

- P is a set of *problems*: this is the sphere of practice of C;
- R is a set of *operators* (to solve a problem 'means' to modify it with a sequence of operators);
- L is a *representation system* (it allows the explanation of the elements of P and R)
- Σ is a *control structure* (in control structure, there are strategic knowledge and meta-knowledge specific to a given class of mathematical problems).

We would like to underline that validation is a key aspect of conceptualisation (Vergnaud introduces the notion of *theorem-in-action*). That the reason why the cK¢ model proposes a control structure: a clear identification of a control structure and the related operators (indeed, a meta-level with respect to action) allows the conceptualisation process to occur through a complex interaction with action and representation. We will concentrate our attention in this paper on Lakatos's conception. For the description of the others conceptions, see Ouvrier-Buffet (2003).

The Lakatosian conception (key elements)

Starting from an "intriguing relation" discovered for some polyhedra [2], Lakatos's dissertation (1961) tries to test it in different ways and hence, three main focuses of interest appear: definition-construction, concept-formation and proof. This work deals with three viewpoints relating to the mathematical concept of definition: the linguistic (it is specific to the Aristotelian conception), the axiomatic (gets rid of by Lakatos; this standpoint is described in the Popperian conception) and the heuristic. The latter is the particularity and thus the interest of the Lakatosian approach. Three notions are present in this heuristic approach of definitions: *naive* definition, *zero-definition*, *proof-generated definition*. Each one has a specific role and a place in the concept-formation. A *zero-definition* is a tentative definition emerging at the beginning of the research process. It may evolve into a *proof-generated definition* or just disappear. It is brought about by proof and stands out as the most important notion in Lakatos's view: the product of *proof-generated definition* is directly linked to the type of SDC (i.e. problem-situation, according to Lakatos). Logically, the zero-concepts may be naïve, but Lakatos concentrates his attention on the expansion of zero-concepts; according to him, this expansion is not possible from naïve concepts. Hence, we have "stages" in a definition process, but how can the evolution of the definition-construction incorporate them and describe the operators and controls between *zero-definition* and *proof-generated definition* for instance?

The operators in fact are specifically related to the proof and the heuristic perspective in which the definitional procedure is inscribed. The generation of examples and counter-examples, in a refutative view, is certainly the most important operator, adding to the functions ascribed to a definition. These functions implicate specific stages in the definitional process: for instance, the functions of communication and denomination generate *zero-definitions*, and the catalysis of the proof brings *proof-generated definitions*. The main control structure refers to the proof i.e. the validity of the studied proof. The other is directly linked to the lack of counter-examples for refutation, but how can we stop the refutation-process? Lakatos informs us that we may stop the expansion of concepts, where it stops being a fertilizer to become a total weedkiller and underlines that a scientific research starts and ends with problems. Hence, this excerpt testifies to the implicit operators and controls existing in a definition-construction process.

Typology of SDC

Our theoretical work, by modelling conceptions of definition, brings a typology of SDC: we distinguish three types of SDC, called *Classification*, *Mathematisation/Modelling*, *Problem-situation*. The first one includes Fletcher's and Freudenthal's proposals, with mathematical objects accessible by theirs representations. The characterisation of the second is initiated by its name; let us give an example of such a situation we have not given yet: "define a mathematical object, which can represent the set of plants" (i.e. the elements of the whole vegetable kingdom) [3]. The third one is called *Problem-Situation* with reference to Lakatos's situation; it includes *research-situations* (Duchet-1994&1997). According to Lakatos, starting from a vague idea of a mathematical concept (such as Euler's formula) can be enough for marking the beginning of a definitional procedure (Lakatos-1961,p69).

PRESENTATION OF A SDC – METHODOLOGY

We have experimented a SDC (*Classification*) with the mathematical concept "tree" before. We have given an analysis of the students' definition-construction processes by *zero-definitions* in Ouvrier-Buffet (2002).

We choose here the mathematical object "discrete straight line", for two main reasons. First, it is of the core of current problem in present mathematical research. This object is accessible by its representation, it is non-institutionalised, and thus allows a re-problematisation of the axiomatic problematics (it cannot be done with Euclidian geometry because the latter is too institutionalised). Second, it permits two types of SDC: the situation *Classification* starts from about ten representations of discrete lines (which are or resemble discrete straight lines). The text of the *Problem-situation* is: "draw discrete triangles and explain your construction". This is problematically close to Lakatos's *Problem-situation* because it involves a search for objects, mathematically still unknown but being dependent on a referent (real straight line). Our objectives are to analyse students' definition-construction process, to explore the influence of the possible explicit demand of definition on the latter and to determine the feasibility of different types of SDC with a common object.

Hence the methodology we chose: three groups of three or two students (from the first university year, scientific section but not especially from the mathematical sections) have taken part in this experimentation (2 or 3 hours required). There were videotape recorded; an observer was present for recalling the instructions (if necessary). Two situations were conceived with two different starting points: the first one (groupA) did not include an explicit request of definition and referred to an axiomatic problematic; neither examples nor counter-examples of discrete straight lines were given to students. The second one (groupB) is a *Classification*-situation, starting from examples and counter-examples of discrete straight lines non-identified as such, and it includes an explicit request of definition. We will use the description of the Lakatosian conception (via cK¢) in order to present the potentiality of the chosen mathematical object for SDC, on the one hand, and to analyse students' definitional processes on the other.

Presentation of the mathematical object "discrete straight line"

This presentation will allow both a mathematical explanation of this concept and its potentialities from the point of view of definition-construction (we will present this aspect with the notion of *zero-definitions* and the possible evolution of them).

To consider a discrete straight line (on a regular grid map, while colouring pixels) can generate a reference to real straight line and thus a use of properties of the last in order to define the same object in a discrete context. We call these problematics "real straight line". If one draws a real straight line, and chooses some pixels crossed by it for forming a discrete straight line, the criteria of choice are requested (see figure 1 below). Thus, different *zero-definitions* are conceivable (the words in bold type mark the orientation of the evolution of the *zero-definitions*: see Zdef1,2,3).

The second problematics is called "regularity": it consists in researching a regularity in the sequence of stages of pixels (see figure 2 below: how to modify a sequence in order to obtain a better regularity?). This problematic brings us two potentially evolutionary *zero-definitions*: Zdef4,5.

There is a third more complicated approach, called "axiomatic". It consists in questioning the mathematical object "discrete straight line", in connection with our knowledge of Euclidian geometry, and thus, for instance, studying the intersection of two discrete straight lines, the number of discrete straight lines by two given pixels etc. This approach is more difficult.

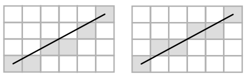
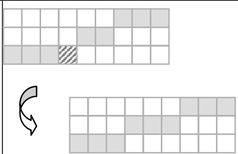
Problematics	Figures	*Zero-definitions*
"Real straight line" Function of def.: to build the object	 … or … ? Figure 1	Zdef1: set of the pixels crossed by a real line. Zdef2: set of the pixels "**the nearest"** of a real line. Zdef3: set of the pixels "**inside"** a band
"Regularity" Function of def.: to recognize, to build the object	 Figure 2	Zdef4: sequence of stages of pixels with specific **properties**. Zdef5: sequence of pixels' stages with a uniform repartition, **non-improved** from the regularity viewpoint.

Table 2: two problematics and the *zero-definitions* of "discrete straight line"

To characterize the pixels "the nearest" of a real line, searching a property relating to the sequence of stages (called chaincode string), leads to a theorem. This approach to the discretization of a straight line by checking linearity conditions is directly related to number theoretical issues in the approximation of real numbers by rational numbers. These linearity conditions can be checked incrementally, leading to a decomposition of arbitrary strings into straight substrings (cf. Wu-1982).

RESULTS

Two problematics were tackled by students: they bring several *zero-definitions*.

Group	*Zero-definitions* produced	Operators, controls	Final Statement
Group A (*Problem-*	- Zdef 1,2,3 (abandon because problem of the	Perceptive controls	*Arithmetical rule* involving slope

situation)	criteria of choice) - Zdef5 (search of regularity by modification of stages)		
Group B (*Classific ation*)	- Zdef1 (abandon because of the non appropriate use of a ruler for a mathematical definition) - Zdef4	Counter-examples, linguistic and logical operators; Perceptive controls	*Repetition of a sequence, in the same direction. The difference between two stages does not exceed 1.*

Table 3: students' statements and their evolution

For these two groups, the first approach consists in using the "real straight line", and thus, some aspects of the *concept image* of straight line appear among what follows: perceptive regularity, slope and infinity of the points. This *concept image* is here insufficient in view of the difficulty of the discrete straight line concept, but still present for students' perceptive controls. The final statements produced by students are very close to actual mathematical definitions. It is noteworthy that these two groups abandon the "real straight line" approach (because of the problem of the choice criteria or the use of a ruler) for the benefit of the "regularity" approach. The students change their point of view relating to the mathematical object and thus abandon the external referent "real straight line". In that way, they define really the "discrete straight line", now fully considered as a mathematical concept.

We can formulate a hypothesis about the influence of the explicit demand of definition on the process of definition-construction, in particular on the evolution of *zero-definitions*: groupB mobilizes more operators taking part in the definition-process than groupA. In this case, the explicit request of definition seems to be profitable for the definition-construction process because it seems to favour a connection progress between different definitions and mobilizes specific operators (linguistic and logical) which contribute to the reflexivity on definition and questions about the presence of new counter-examples (refutation-process). The lack of any form of control concerning the function of definition is conspicuous. Clearly, there has been no simultaneous treatment of the two functions of definition (i.e. drawing a discrete straight line and recognizing it).

CONCLUDING REMARKS

Different markers attest a students' process of definition-construction: the presence of *zero-definition(s)* underlines its beginning, and the mathematical treatment of these potential definitions consists in studying the lacks of these "working definitions", analysing the different implications between them. All this process involves specific Lakatosian operators such as testing a definition with a research of refutation by a counter-example, but also Aristotelian operators, linguistically and logically orientated, such as searching a minimal, non-redundant definition (that implies a reflexivity on the definition as a statement but also as a characterisation of a concept). This points to great potentialities of SDC for revising students' conceptions on the notion of "definition", but also for the exploration and the understanding of mathematical concepts above all. To explore and capitalize on these potentialities implies the designing of a set of SDC, with different kinds of SDC, including *Classification* and *Problem-situation*. We have to analyse more precisely the *Problem-situation* phenomenon, as it is presented by Lakatos (in connection with proof): that is what the study of *research-situations* (Duchet) seems to be promising.

NOTES

1. We do not delude ourselves into thinking that what can be easily expounded can be easily assimilated.

2. Euler's formula: V-E+F=2, where V is the number of vertices, E the number of edges and F the number of faces. Notice that Euler had defined the concepts of vertex and edge.

3. It can be the mathematical concept of "tree".

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MATHEMATICAL ABSTRACTION THROUGH SCAFFOLDING

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This paper examines the role of scaffolding in the process of abstraction. An activity-theoretic approach to abstraction in context is taken. This examination is carried out with reference to verbal protocols of two 17 year-old students working together on a task connected to sketching the graph of $|f(|x|)|$. Examination of the data suggests that abstraction is a difficult activity that can sometimes be beyond students' unassisted efforts, in which case supportive intervention of a scaffolder through several means of assistance is observed to help the students achieve the abstraction.

The issue of abstraction has attracted the attention of many educators (Dienes, 1963; Piaget, 1970; Skemp, 1986). Purely cognitive views see abstraction as ascending from 'concrete' to 'abstract', e.g. "the extraction of what is common to a number of different situations" (Dienes, 1963, p.57). Criticism of this view of abstraction comes from an epistemological point of view which recognises that contextual and social factors are crucial to knowledge acquisition (see van Oers, 2001). Empirical studies of abstraction in context are a relatively recent phenomenon and include Noss and Hoyles' (1996) concept of situated abstraction and Hershkowitz, Schwarz and Dreyfus' (2001) activity-theoretic model of abstraction. This paper works within the Hershkowitz et al.'s model.

Hershkowitz et al.'s (2001) model was inspired by Davydov's (1990) epistemological theory pointing a dialectical connection between abstract and concrete. They provide an operational definition of abstraction as "*an activity of vertically reorganising previously constructed mathematics into a new mathematical structure*." The new structure is the product of three epistemic actions: recognising, building-with, and constructing ('RBC theory of abstraction' hereafter). Recognising a familiar structure occurs when a student realises that a structure is inherent in a given mathematical situation. Building-with consists of combining existing artefacts in order to meet a goal. Constructing consists of assembling knowledge artefacts to produce a new structure. These actions are dynamically nested in such a way that building-with includes recognising, and constructing includes both recognising and building-with.

In empirical studies of abstraction researchers claim to gain insight into students' abstraction processes via the help of a knowledgeable agent e.g. the researcher/ interviewer. Hershkowitz et al. (2001), for example, note that the interviewer in their study aims to induce the student to reflect on what she is doing so that she might progress beyond the point that she would have reached without the interviewer. Others argue that the successful completion of an abstraction process is contingent upon providing the student with 'hinting' (Ohlsson and Regan, 2001) and 'shifting the focus of activities' (van Oers, 2001). All three sets of researchers effectively argue that abstraction is not an easy process which may be beyond the learners' unassisted efforts. This has links with the theoretical concept of 'scaffolding'.

Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 2004 Vol 3 pp 481–488

Scaffolding may be defined as any kind of systematic guidance given to a learner to develop and achieve his/her fullest potential, which is beyond his/her actual present ability (Chi, Siler, Jeong, Yamauchi and Hausmann, 2001). The key element of scaffolding is 'the sensitive, supportive intervention of a teacher in the progress of a learner who is actively involved in some specific task, but who is not quite able to manage the task alone" (Mercer, 1995, p.74). Scott (1997) develops the idea of 'sensitive intervention' arguing that throughout scaffolding, in interacting with the learner, the tutor is aware of and responsive to existing modes of and any changes in a learner's thinking, and thus has the opportunity to support the development of the learning goal. Scott breaks the teacher's responsiveness down into three elements: *monitoring* – monitor present performance of the learner, *analysing* – analyse the nature of any differences between present performance and performance required by the learning goal, *assisting* – respond with an appropriate intervention to address differences in performance. When the learner makes progress towards the learning goal, the level of assistance is decreased and responsibility may be handed over to the learner (Bruner, 1983).

Despite the implications of empirical investigations of abstractions for scaffolding, published research to date has not addressed the link between scaffolding and abstraction. This paper examines the role of scaffolding in the process of abstraction within the framework of the RBC theory. The examination is carried out with regard to two 17-year-old girls working together within a scaffolded situation to construct a new mathematical structure. The paper provides a brief description of the girls' joint work and the scaffolder's intervention. It then discusses how scaffolding functions and relates this to the achievement of a mathematical abstraction.

BACKGROUND

The study: The study presented in this paper is part of a larger study focusing on the role that scaffolding and students' interaction play in the formation of mathematical abstraction within the framework of the RBC theory. For the study, data were collected from the students working on four tasks connected with sketching the graphs of absolute values of linear functions. The students were selected on two criteria: (1) they had the prerequisite knowledge needed to embark on the tasks; (2) they were not to be acquainted with the intended abstractions. In order to select a sample that met the criteria, a diagnostic test was prepared and applied in Turkey to 134 students aged 16-18. 20 students were selected and organised so that 14 worked in pairs and six worked alone. Four pairs of students and three individuals worked within a scaffolded environment, the rest without. Verbal protocols were audio-recorded.

The task: Four tasks were prepared and applied on four successive days. The overall aim of the first, second and fourth task was to construct a method to draw the graphs of, respectively, $|f(x)|$, $f(|x|)$ and $|f(|x|)|$ by using the graph of $f(x)$. The organisational structure of these three tasks was identical apart from the mathematical objects i.e. $|f(x)|$, $f(|x|)$ and $|f(|x|)|$. The third task was prepared to consolidate the first and second tasks. This paper reports on protocols generated in the fourth task, which had five questions. In the first question the students were asked to draw the graph of $|f(|x|)|=|(|x|-4)|$ and to comment on any patterns in the graph. In the second question, they were asked if they saw any relationship between the graph of $|f(|x|)|=|(|x|-4)|$ and the graph of $f(x)=x–4$. In the third question, the graph of $f(x)=x+3$ was given and the students were asked if they could draw the graph of $|f(|x|)|=|(|x|+3)|$ by using the given graph as an aid. In the fourth question, four linear graphs, without equations, were given and the students were asked to obtain the graph of $|f(|x|)|$ for each one. In the fifth question, students were asked to explain how to draw the graph of $|f(|x|)|$ by drawing on the graph of $f(x)$. By the end of the task, the students were expected to construct a method to sketch the graph of $|f(|x|)|$ by using the graph of $f(x)$, which will be referred to as 'the structure of $|f(|x|)|$' in the paper.

THE DATA

Part of the two girls' verbal protocol on the fourth task, parsed into episodes, is presented in this section. Please note that some comments have been inserted in square brackets to assist the reader to follow the interaction amongst the participants.

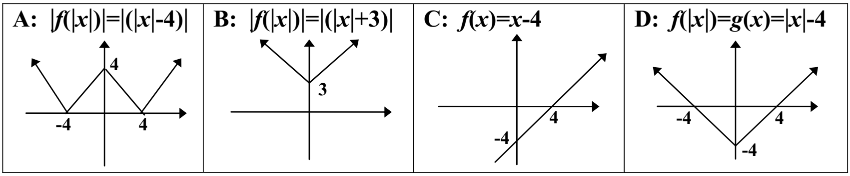

Figure 1: The graphs obtained by the students.

Episode 1:

At the beginning of the task the interviewer did not intervene, in order to observe how far they could go without his assistance. The students obtained the graphs of $|f(|x|)|$ for the first and third questions (see figure 1A and 1B) correctly by substituting into the given equations for different values of x. After considering these graphs together, as can be seen in the excerpt below, they stated that they were unable to find a general method.

141H: *I don't think we can ever understand how to use f(x) to draw the graph of |f(|x|)|…*

142S: *The first graph [figure 1A] was something like W-shaped… but this graph [figure 1B] is V-shaped…*

143H: *They are totally different! How can we speak in a general way… even this question made things worse… rather than helping us.*

144S: *We'd better stick to substituting… we can answer the next question by substituting.*

This excerpt clearly shows that the structure of $|f(|x|)|$ is beyond the students unassisted collaborative efforts. At this point the interviewer intervened and suggested that the students return to the first question. Between utterances 149 and 164 (not shown), the interviewer helped the students recognise what they know about the graphs of $|f(x)|$ and $f(|x|)$.

Episode 2:

165I: *Ok, if you pay a closer attention to the equation… I mean look at the expression itself, |f(|x|)|, it is a combination of these two [of |f(x)| and f(|x|)]. Do you see that?*

166H: *Yes, that's right. We already mentioned about this at the beginning…*

167S: *Yeah, this [|f(|x|)|] is a combination of f(|x|) and |f(x)|… for example [the graph of] |f(x)| never goes under the x-axis…*

168I: *Ok, let's think about it and consider what you know. How can we use our knowledge to obtain this graph [of |f(|x|)|]?*

169S: *Look, it makes sense… I mean [the graph of] |f(x)| doesn't pass under the x-axis as [values of] y is always positive and also the graph of f(|x|) is symmetric in the y-axis… so the graph of |f(|x|)| doesn't take negative value and symmetric in the y-axis.*

170H: *Yeah, it makes sense now… look, if |f(|x|)| is a combination of f(|x|) and |f(x)|, we can think about it like a computation with parenthesis…*

171I: *Computation with parenthesis?*

172H: *I mean for example when we are doing computations with some parentheses like… let's say for example… err ... (7-(4+2)), then we follow a certain order…*

173S: *Right, I understood what you mean… we need to first deal with the parenthesis inside of the expression, is that what you mean?*

174H: *Yeah, I think it is somehow similar in here, I can sense it but I can't clarify…*

175S: *I know what you mean but how could we determine the parentheses in here?*

176I: *You both made an excellent point. OK, let's think about it together. In the expression |f(|x|)|, can we think about the absolute value sign at the outside of the whole expression as if a larger parenthesis, which includes another one just inside?*

In this episode, the interviewer stresses that $|f(|x|)|$ could be seen as a combination of $f(|x|)$ and $|f(x)|$. This prompts S to recognise some properties of the graphs of $f(|x|)$ and $|f(x)|$ in relation to the graphs of $|f(|x|)|$ (167 and 169). In 168, the interviewer asks the students to think about how to obtain the graph of $|f(|x|)|$ by making use of what they already know and thus sets a subgoal to develop a strategy about how to obtain the graph of $|f(|x|)|$. In 170, H recognises the order of the operational priority of computations including parentheses and proposes that $|f(|x|)|$ may be treated the same way. In 171, the interviewer probes H to understand her intention. Based on H's interaction with S, the interviewer has an opportunity to monitor and analyse their performance. The students, between 172 and 175, are developing an appropriate strategy but they are not sure if their approach is reasonable or how they might determine the 'parentheses' in the expression of $|f(|x|)|$. In 176 the interviewer intervenes to keep the students in pursuit of the subgoal and gives positive feedback indicating that their approach is reasonable. He further accentuates how the absolute value signs in the expression $|f(|x|)|$ might be used in the similar way to parentheses.

Episode 3:

177H: *Aha, I got it… I know what we will do.*

178I: *Could you please tell us?*

179H: *We can consider f(|x|) as if it was the smaller parenthesis!*

180I: *Smaller parenthesis?*

181H: *I mean it should be the first thing that we need to deal with.*

182S: *Yeah, I agree… I think we should begin with the graph of f(|x|) and first draw it.*
183H: *But what next?*
184S: *Then we can use the absolute value at the outside… in the similar way of doing computations.*
185H: *But we will be drawing graphs? Can we really do this?*
186S: *I am not too sure if we can… but it sounds plausible…*
187I: *What you are doing here is not computation of course… but you are making an analogy, I mean you are making some certain logical assumptions based on your earlier experiences… and I see no problem with that… let's draw the graph by considering what we've just talked about and then decide if it will work or not?*

In this episode, the interviewer's help in 176 prompts H to start to build a plan as to how to execute the strategy by using what they recognised in the second episode. The interviewer here asks some probing questions (178 and 180) to gain insights into the students' explanations in order to monitor how the given assistance in 176 is taken up. The interviewer later observes the students and analyses their performance on the basis of their interaction. The students put forward that they could first draw the graph of $f(|x|)$ and then consider the absolute value sign at the outside of the expression of $|f(|x|)|$. Yet, they are not sure if they can do so or if this approach works. In 187, the interviewer gives the students positive feedback and explains why their approach is reasonable. He also assures them that he does not see any problem with their approach. After that he sets another subgoal to the students and asks them to draw the graph of $|f(|x|)|$ by considering what they have just discussed.

Episode 4:

188H: *What are we doing now?*
189S: *We will draw first the graph of f(|x|).*
190H: *Ok let's draw the graph now… [They draw the graph of f(|x|) (see figure 1D) by using the graph of f(x) (see figure 1C)].*
191I: *Alright, you drew the graph of f(|x|). But this is not what we expected to find, is it?*
192S: *No… we will now draw |f(|x|)|.*
193H: *Do you know how? Well, the next step is not too clear to me!*
194I: *Ok, just to make your job a bit easier, let's rename f(|x|) as g(x). So what you need to find turns into [he stops]…*
195S: *|g(x)|*
196H: *Aha! I can see it now…*
197I: *What is it*?
198H: *That means we will draw the absolute value of this graph… I mean we need to take the absolute value of this graph… oh it is so clear now, do you understand?*
199S: *Of course, but renaming the expression helped me see it clearly now…*
200I: *Ok, let's think about it now, how can we apply absolute value to this graph?*
201S: *|g(x)| never takes negative values… I mean it never passes under the x-axis.*
202H: *We will be taking the symmetry of the rays [she refers to the line segments (see figure 1D)] under the x-axis.*
203S: *Yes.*
204H: *Ok then, let's draw it now. We are now drawing the graph of |f(|x|)|.*

205S: *We were taking the symmetry of this part [the line segment on the fourth quadrant (see figure 1D)] in the x-axis… and we should also take the symmetry of that part as well [the line segment on the third quadrant (see figure 1D)]… according to x-axis.*

In this episode, the students satisfy the subgoal set by the interviewer, to draw the graph of $|f(|x|)|$, through two steps: (1) by drawing the graph of $f(|x|)$ and then (2) by drawing the absolute value graph of $f(|x|)$. Regarding the first step, the students recognise the structure of $f(|x|)$ that they constructed in the second task and use it to draw the graph of $f(|x|)$ (190). They have, however, some difficulties in seeing the second step (193). The interviewer realises this and assists them by renaming the expression of $f(|x|)$ as $g(x)$. He then invites the students to discuss how to apply the structure of $|f(x)|$, which they constructed in the first task, to the graph of $f(|x|)$ (200). In doing so the interviewer sets a sub-subgoal to the students, which is to draw the absolute value graph of $g(x)$. The interviewer seems to break down the subgoal, which was set at the end of the third episode, into further sub-subgoals. Satisfaction of these goals requires the students to reorganise their earlier constructions of $|f(x)|$ and $f(|x|)$ to draw the graph of $|f(|x|)|$. To do so, they recognise the structures of $|f(x)|$ and $f(|x|)$, appeal to the features of these two structures, and apply it to build the intended graph (e.g., 201, 202, 203, 204, and 205). In doing so, the students construct a method to obtain the graph of $|f(|x|)|$ from the graph of $f(x)$.

Episode 5:

Between the utterance of 206-227 (not shown) the students draw the graph of $|f(|x|)|$ for the third question by first drawing the graph of $f(|x|)=g(x)=|x|+3$ and then drawing the graph of $|g(x)|=|f(|x|)|$. They obtain the same graph of $|f(|x|)|$ as they obtained previously by substituting.

228I: *Right let's go on to the fourth question…what will you do in this question?*
229H: *We will draw the graphs with the same method again.*
230S: *Yes*
231H: *[They talk about a graph of f(x) given in the 4^{th} question] Ok… now… first of all…*
232S: *The graph of f(x) at the positive values of x will remain the same*
233H: *First we obtain the graph of f(|x|)…*
234S: *Yes [they are drawing the graph of f(|x|)]*
235H: *Now we will draw its absolute value graph.*
236S: *That means we will take the symmetry in the x-axis.*
237H: *All of the parts over the x-axis remain as they are and…*
238S: *The parts under the x-axis will be cancelled and their symmetries will be taken in the x-axis [they drew the graph of |f(|x|)| successfully].*

In this episode, the students are expected to draw the graph of $|f(|x|)|$ for each of the given graphs of $f(x)$ in the fourth question. As can be seen from the students' interaction, they are able to regulate themselves and proceed without any help from the interviewer. The interviewer is still monitoring and analysing the students' performance but he does not feel the need to intervene. Thus in this episode the interviewer hands responsibility over to the students. After the students manage to reach to the new structure of $|f(|x|)|$, they become relatively self-regulated. Having drawn the graphs asked in the fourth question correctly, the students go on to the fifth question where they are asked to give a method to obtain the graph of $|f(|x|)|$. As can be seen from the excerpt below, the students have constructed the structure of $|f(|x|)|$.

243S: *First of all, by making use of the graph of f(x) we obtain the graph of f(|x|)=g(x) and then obtain |g(x)|…*

244H: *To do this, first when drawing f(|x|), part of f(x) at the positive [values of] x remains unchanged… umm then this part is taken symmetry in the y-axis and err and also part of f(x) at the negative [values of]* x *is cancelled. After that, we apply absolute value to this graph, and for this… umm… negative values of y are taken symmetry in the x-axis and thus we obtain the graph of |f(|x|)|.*

DISCUSSION

Construction is central to the RBC theory of abstraction in that without it abstraction cannot be claimed to take place. It requires students to combine and reorganise already constructed mathematical structures so as to create a new one. The protocols suggest that when the construction of a new structure is beyond the students' unassisted efforts, supportive and sensitive intervention of a scaffolder to direct the students' actions and attentions, and thus regulate their work and effort, is likely to induce them to make progress towards abstraction. The presented data reveal three major facets of scaffolding in the process of abstraction which are discussed below.

1. Based on the monitoring and analysing, the scaffolder assists the students through several means: The scaffolder's acts here appear to consist mainly in a continuous cycle around three elements: monitoring, analysing, and assisting. He constantly monitored the learners' performance when they took action in response to the given assistance and also when they were interacting with each other. This monitoring helped him analyse the learners' situation to determine the difference between their existing level and the intended level of performance. Based on the analysis he decided on the type of (and adjusted the amount of) assistance. It should be noted that there are many types of assistance in scaffolding such as explaining, questioning, feedback, and hinting (see Chi et al., 2001; Tharp and Gallimore, 1988). However, the scaffolder's selection of the type of assistance was completely subjective and dependent on his perception and interpretation of the situation. We thus posit that the type of assistance is not the central element to successful scaffolding as long as it is provided at the right time and results in progress on the part of the students towards the intended abstraction.

2. The scaffolder regulates the students by organising the main goal of the activity into subgoals and sub-subgoals: The scaffolder had the vision of the target goal of the activity and expected performance. This helped him to regulate the students towards the achievement of the main goal of the activity by setting new subgoals in such a way that attainment of each subgoal moved the students closer to the construction of a new structure. In order to get the students to attend to a subgoal, he even broke it down into sub-subgoals. Pre-determination of these subgoals and sub-subgoals is not possible. Quite the contrary, they emerge as the interaction amongst the participants evolves. Therefore the structure of a subgoal was negotiated in the interaction itself and required the scaffolder to make dynamical adjustment of it depending on his monitoring and analysing. In order to achieve these subgoals, the students at times needed to recognise a structure(s) constructed earlier (recognising); at other times to use these recognised structures to satisfy the subgoal(s) (building-with); and still other times to assemble and reorganise previously constructed knowledge artefacts to produce a new one (constructing). It seems that goals that students have, or are given, determine the nature of and initiate a series of epistemic actions that are required to attain the goal. For example, in the presented data, when the goal was to draw the graph of $f(|x|)$ the students needed to recognise and use the structure of $f(|x|)$ that they had constructed earlier. However, when the goal was to sketch the graph of $|f(|x|)|$, they needed to reorganise the structure of $|f(x)|$ and integrate it with the structure of $f(|x|)$ (see Episode 4).

3. The scaffolder steadily reduces the amount of assistance and gradually hands the responsibility over to the students as they make progress towards the main goal of the activity: As a result of scaffolder's monitoring and analysing, when he felt that the learners could proceed through the task without needing assistance from him, he gradually reduced the amount of help to the level of none and handed the responsibility over completely to the learners. This was the case after the students have constructed the structure of $|f(|x|)|$ (see Episode 5). The complete handover of the responsibility is thus likely to indicate that the students have become acquainted with a new structure and that the main goal of the activity, abstraction, is attained.

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THE USE OF DIAGRAMS IN SOLVING NON ROUTINE PROBLEMS

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This paper explores the role of diagrams in a specific problem solving process. Two types of tests were administered to 194, 12 year old students, each of which consisted of six non-routine problems that could be solved with the use of a diagram. In Test A students were asked to respond to the problems in any way they whished whereas in Test B problems were accompanied by diagrams and students were asked to solve these problems with the use of the specific diagrams presented. The results revealed that there was no statistical significant difference between the two tests. The result also revealed that it was not the same group of students that were successful in the two tests.

INTRODUCTION

The mathematics education community has espoused the importance of developing children's problem solving skills (for example, National Council of Teachers of Mathematics, 2000; Shoenfeld, 1992). In the same vein, research in mathematics education discusses the importance of using multiple representations in the problem solving process (Lesh, Behr, & Post, 1987; English, 1996). Markmann (1999) interprets the term "representation" as the concept that includes the represented world, a representing world, a set of rules that map elements of the represented world to elements of the representing world, and a process that uses the information in the representing world. Diagrams are considered as one kind of such representations (Novick & Hurley, 2001). The represented world in this case is a description of a problem to be solved, while the representing world contains the spatial diagrams as an abstract form, along with their applicability conditions. Specifically, a diagram is a visual representation that displays information in a spatial layout (Diezmann & English, 2001). In problem solving a diagram can serve to represent the structure of a problem. Diagrams are considered structural representations, in which the surface details are not important and this is their main characteristic and differentiation from pictures and drawings (Veriki, 2002). Diagrams typically rely on conventions to depict both the components of the situation being represented and their organization. These conventions must be learned and understood before the diagrams can be understood and successfully used (Diezmann & English, 2001).

According to a number of researchers the ability to use diagrams is a powerful tool of mathematical thinking and problem solving, because they are used to simplify complex situations, they concretize abstract concepts, and they substitute easier perceptual inferences for more computationally intensive search processes and sentential deductive inferences (Novick & Hurley, 2001; Diezmann & English, 2001;

Kidman, 2002).This paper discusses a part of a larger study which aims to investigate the impact of diagrams in solving non-routine problems.

THEORETICAL BACKGROUND

Networks, Matrices and Hierarchies

The efficient use of a diagram depends on its suitability for a given situation. The appropriateness of a diagram depends on how well it represents the structure of the problem. Novic and Hurley (2001) propose three general–purpose diagrams that suit a range of problem situations, networks, matrices and hierarhies. These diagrams highlight structural commonalities across situations that are superficially very different. These diagrams are especially useful in elementary non-routine problems (Booth & Thomas, 2000; Diezmann & English, 2001). Networks consist of sets of nodes with one or more lines emanating from each node that link the nodes together. Networks do not have any predefined formal structure (Fig.1). Matrices use two dimensions to represent the relationships between two sets of information. A cell in the matrix denotes the intersection of value i on one variable and value j on the other variable (Fig.2). Hierarchies comprise diverging or converging paths among a series of points. A single node gives rise to at least two other nodes (Novick & Hurley, 2001; Diezmann & English, 2001) (Fig.3).

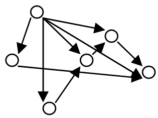
Figure 1: Networks

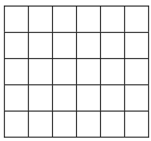
Figure 2: Matrices

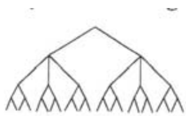
Figure 3: Hierarchies

In Networks the nodes specify values along a single variable. Any node may be linked to any other node and all of them have identical status. In Matrices the rows and columns specify values along two distinct variables. Values on the same dimension may not be linked. All the rows have identical status, as do all of the columns. In Hierarchies the nodes at a given level have identical status, but the nodes at different levels differ in status. While in Networks the links between the nodes may be unidirectional, in Matrices the links between the nodes are non directional, while in Hierarchies the links are directional (Novick & Hurley, 2001; Diezmann & English, 2001). Networks and Hierarchies may show paths connecting subsets of nodes, while matrices do not show subsets.

Similar diagrams are often used in solving non routine problems. Non routine problems are the problems that do not involve routine computations but the application of a certain strategy, in this case a diagram, is required in order to solve the problem (English, 1996). Non routine problems are considered more complicated and difficult than routine problems in which only the application of routine computations is involved in their solution (Shoenfeld, 1992). Research concerning the efficiency of the use of diagrams in solving non routine problems is often contradicting and thus inconclusive (Diezmann, & English, 2001; English, 1996; Booth & Thomas, 2000). Thus, the present study purports to throw some light to the contribution of the three diagrams in the solution of non-routine problems. More specifically, the study investigated two questions:

(a) Does the *presence* of diagrams in non-routine problems increase students' ability in solving them? and

(b) How does the presence of diagrams in non-routine problems affect students' reponses?

(c) Does the *use* of networks, hierarchies and matrices (diagrams) facilitate students in solving non-routine problems?

(d) Does the use of diagrams facilitate all students?

METHOD

To examine the impact of diagrams in the problem solving process we constructed two tests. Test A consisted of six problems. The problems used in the study were chosen based on three criteria. First, the problems were taken from previous pieces of research, which showed that children used diagrams similar to the ones that we intended to use in this study in order to solve them (Diezmann & English, 2001; Booth & Thomas, 2000). Second, students were familiar to similar problems since they appear in their mathematics textbooks, and third, the teachers' guide book suggests that these problems should be solved using similar diagrams with the ones we indebted to use in the study.

The first problem inquired about the distance between four trees. Specifically, students had to find out the distance between two trees while the distance of the other trees was given. In the second problem students had to find the step that the cleaner was standing when he started cleaning the windows. Problems 1 and 2 could both be solved with the use of Network diagrams, because the problems had identical status items (trees and steps) that may be linked with each other. In both problems it was important to show paths connecting the items.

The third problem concerned combinations $A \times B \times C$. Children were asked to find out all the possible combinations of two cards (Christmas or Easter cards), their colour (green or yellow) and their ribbon's colour (red or blue). The fourth problem asked children to find out the wining team among four teams. Problems 3 and 4 could be solved using the Hierarchy diagram because they involved items that were distinguished according to different levels. In addition, the links between the items were directional and only one route existed between any two items.

The fifth problem asked children to match each of the four friends with their own favourite kind of books according to some given information. The sixth problem was a combination problem $A \times B$ (four flavours of ice cream and three types of cones). These two problems could be solved using the Matrices diagram because both of them had two sets of items. The items within a set could not be combined amongst themselves (for example the 3 different cones). All possible combinations of the items across sets should be considered (cones and flavours).

Test A was administered to 194 Cypriot students in grade 6. Students were asked to solve the six problems (a1-a6) in any way they wished. Test B contained isomorphic problems, this means problems that had the same structure as those in the first test and was administered to students a week after the administration of Test A. Each problem in Test B (b1-b6) was accompanied by a diagram (bgi-bg6) that students were asked to use in order to solve the problems.

For the analysis and processing of the data collected, statistical analysis was conducted by using the computer software SPSS and CHIC (Bodin, Coutourier, & Gras, 2000). A t-test analysis was produced to examine the difference between students' achievement in the two tests. The statistical package CHIC produces three diagrams. The similarity diagram which represents groups of variables which are based on the similarity of students' responses to these variables. The implication graph which shows implications $A \Rightarrow B$. This means that success in question A implies success in question B. Finally the hierarchical tree which shows the implication between sets of variables. In this study we use only the first two.

RESULTS

In regard to the first question of this study the mean score of Test A was 14,36 (20) and the mean score of Test B was 14,55(20). The t-test analysis revealed that there was no statistical significant difference between students' achievement in the two tests ($M\alpha$-$M\beta$=-0,196, df=193, p=0,636).

In regard to the second question of the study, according to the similarity diagram (figure 4), students' responses to the problems of the two tests are separated except from the problem b1. The emergence of these two clusters of variables (a1,a2,a3,a5,a6) and (b2,b6,b3,b4,b5) show that students perceive the two tests as completely different tasks failing to realise that the two tests included isomorphic problems. The similarity diagram (Figure 4) shows that students have not realized the structural resemblance of the problems in the two tests. With the presence of the diagrams in Test B they encountered the problems as completely different tasks. It is not clear why b1 joined cluster a. It may be conjectured that the high level of difficulty of problems a1 and b1 caused this similarity between them.

The similarity diagram also shows statistically significant similarity at level 99% (Note: Similarities presented with bold lines are important at significant level 99%) only between the problems in test A. Particularly, there is statistically significant similarity between the variables a1-a3. The similarity between students' responses in problems a1 and a2 may be due to their similar structure. There is also a statistically significant similarity between the group of variables a1-a3, and the group of variables a5-a6. This can be explained in certain extend to their similar structure (a5 and a6, a1 and a2) and the level of difficulty of these non-routine problems. As it was pointed out earlier on, these problems are considered difficult and therefore are usually solved

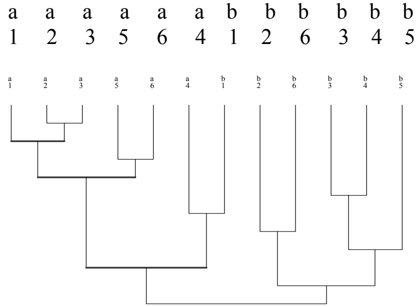

Figure 4: Similarity diagram of the responses to the problems in the two tests

by mathematically able students. On the contrary in Test B there does not appear to be any statistically significant similarity at level 99% between any of the problems, suggesting that the presence of each diagram in the problems had a determinative role to students' responses. One explanation that may be given is that the appearance of the diagrams was causing different behaviour to students resulting to different responses. Whereas in Test A success in the problems depended solemnly on students mathematical abilities, in Test B students also had to interpret and use the diagrams efficiently. This may have caused the lack of any similarity between students' responses in Test B.

The implication graph in figure 5 shows significant implicative relations between the problems in Test A. Specifically, it suggests that success in a1 implies success in a3. Therefore in Test A, a1 was the most difficult and a4 the easiest problem. While in Test A there is a hierarchy of difficulty, shown in the implication graph in figure 5, this hierarchy does not exist in Test B. This implies that with the presence of the diagram some problems became easier for some students, while other problems became more difficult. In addition, the implication graph shows that students who solved correctly a problem in Test A did not necessarily solved correctly the isomorphic problem in the Test B and reversely.

In regard to the third question whether the use of the diagrams facilitates students ability in solving non-routine problems, figure 6 gives some interesting information. The implication graph shows that students who used the networks in b1 and b2 also used the hierarchy in b3 and matrices in b6. However, this use does not imply the successful solution of the foremost mentioned problems.

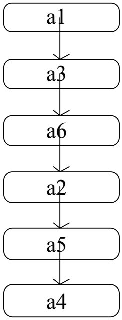

Figure 5: implication graph illustrating relations among the twelve variables, a1-a6 and b1-b6.

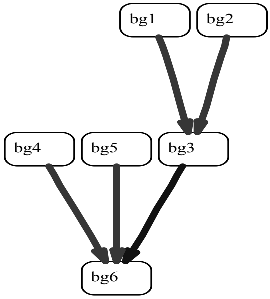

Figure 6: implication graph illustrating relations among the use of the diagrams bg1-bg6 and the correct answers to problems b1-b6.

In regard to the fourth question, Table 1 shows in detail the students' responses to the two tests.

Wrong Responses in Test A		Responses in Test B		Correct Responses in Test A		Responses in Test B	
A1	W:**112 (58%)**			A1	C: **82 (42%)**		
		B1	W: **84 (75%)**			B1	W: **41 (50%)**
		B1	C: **28 (25%)**			B1	C: **41 (50%)**
A2	W: **77 (40%)**			A2	C: **117 (60%)**		
		B2	W: **43 (56%)**			B2	W: **37 (32%)**
		B2	C: **34 (44%)**			B2	C: **80 (38%)**
A3	W: **97 (50%)**			A3	C: **97 (50%)**		
		B3	W: **57 (59%)**			B3	W: **33 (34%)**
		B3	C: **40 (41%)**			B3	C: **64 (66%)**
A4	W: **22 (11%)**			A4	C: **172(89%)**		
		B4	W: **12 (55%)**			B4	W: **23 (13%)**
		B4	C: **10 (45%)**			B4	C: **149 (87%)**
A5	W: **62 (32%)**			A5	C: **132(68%)**		
		B5	W: **27 (44%)**			B5	W: **41 (31%)**
		B5	C: **35 (66%)**			B5	C: **91 (69%)**
A6	W: **75 (39%)**			A6	C: **119 (61%)**		
		B6	W: **33 (44%)**			B6	W: **15 (13%)**
		B6	C: **42 (66%)**			B6	C: **104 (87%)**

Table 1: Pupils' responses to problems a1-a6 and their responses to the isomorphic problems b1-b6. (W: Wrong responses, C: Correct responses)

The important aspect of the Table 1 is that although these problems are considered difficult for primary school level and usually they are expected to be solved only by mathematically able students, the presence of the diagram in Test B has been very helpful for some students. However these students were not the same as those who solved the isomorphic problem in Test A. This is evident by the fact that a number of students that have solved the problem correctly in Test A they were not able to solve the isomorphic problem in Test B. Some students seem to have failed to see the representing structure in the diagram of the problem, while for other students the diagram proved to be very helpful.

DISCUSSION

The present study is within the framework of the ongoing discussion about the role of mathematical representations, and more specifically of diagrams, in the problem solving process. The results of the study suggest that the presence of the diagrams in Test B did not increase students' ability in solving the non routine problems. This is exemplified by many students' failure to "see through" the diagram the structure of the problem even though similar diagrams are used for the solution of these kinds of problems in their classroom.

The presence of each diagram had a determinative role in students' responses to each problem. While students' responses to Test A depended solemnly on students' mathematical ability, in Test B students also had to interpret and use the diagrams efficiently. It appears that not all students were able to do so. Success or failure to use the diagrams correctly might be due to a number of reasons. First, students' inability to interpret the diagram correctly and second, some students' lack of experience of solving problems with the presence of diagrams.

The results of the study show that the efficient use of a diagram did not imply the successful solution of a problem and reversely the successful solution of a problem did not imply the efficient use of the accompanied diagram. It can be argued that students perceived the problems with the accompanied diagrams as different tasks failing to perceive each diagram as an additional aid for the solution of the problems.

The results of the study show that the diagrams make the problems easier for some students while they make the problems more difficult for some other. For this reason, students who solved the problems in Test B were not the same students who solved the problems in Test A. It can be claimed that pupils with different visuo-spatial abilities (Booth & Thomas, 2000) responded differently in the tests.

Diagrammatic literacy is an essential component of students' mathematical development (NCTM, 2000). In order to use diagrams efficiently, students must develop the ability to translate the word problem into a diagrammatic representation and the ability to interpret a diagram in terms to a given word problem (Novick & Hurley, 2001; Diezmann & English, 2001). However, what is clearly suggested in this study is that this is a skill that needs to be developed and is not inherent to all students. A specific diagram does not have the same impact on all students. It is likely that the diagrams presented may be incompatible to some students' mental representations. Therefore, it is very important for students to be engaged in multiple representations in the problem solving process. Development of students' diagrammatic literacy may only be achieved though carefully designed instructional activities. These may turn diagrams into an effective tool for thinking.

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DESIGNING FOR LOCAL AND GLOBAL MEANINGS OF RANDOMNESS

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This research aims to study the ways in which 'local' events of randomness, based on experiencing the outcome of individual events, can be developed into 'global' understandings that focus on an aggregated view of probability (e.g. probability of an event). The findings reported in the paper are part of a broader study that adopted a strategy of iterative design, in which a computer-game was developed alongside the gathering of evidence for children's use of the game. In response to a range of tasks, 5 and 8 year-old children manipulated the sample space in ways that generated corresponding outcomes in the game. The findings illustrated how the game provided new tools of experience that afforded the construction of novel meanings for randomness.

INTRODUCTION

Research within the constructionist framework has shown how technology can empower children in the use of stochastics. Wilensky (1997) based his work on the conjecture that both the learner's own sense-making and the cognitive researchers' investigations of this sense-making are best advanced by having the learner build computational models of probabilistic phenomena. He shows that through building computational models, learners can come to make sense of core concepts in probability such as normal distribution. Learners are supported in building and developing their mathematical intuitions and, through this construction process, mathematical objects are seen to be more concrete as learning progresses. Pratt's (2000) study also showed that by using a computational system, children managed to make sense of local and global probabilistic meanings, referring to *local* meanings of randomness as those based on experiencing the outcome of individual events, and *global* meanings that focused on an aggregated view of probability. Local meanings in probability are characterised by the fact that the next outcome is unpredictable and there is irregularity. Global meanings have the following characteristics: the proportion of outcomes for each possibility is predictable, the proportion will stabilise with an increasing number of results (large numbers) and there is control through manipulation of the sample space. Pratt describes the development of a computer-based domain within which children (aged between 10 and 11 years) manipulate stochastic 'gadgets', representing everyday objects such as a die, a coin, a lottery and a set of playing cards. Individual learners were put in situations where they could express their beliefs in symbolic (programming) form and articulate the beliefs that they held, and construct and reconstruct them in the light of their experiences.

The aim of this paper is to describe the design of a game constructed simultaneously to afford expressive power to children aged 5 to 8 in the domain of probability, and to provide a *window* (Noss & Hoyles, 1996) on that thinking as it is developed. It presents the ways in which local meanings of randomness, based on experiencing the outcome of individual events, can be developed into global understandings that focus on an aggregated view of probability (e.g. probability of an event). The broader study also assesses whether and how the explicit linking of local and global meanings via a rule-based system, assists in effecting this evolution. However, in this paper, we will focus on design and on ways in which our design-criteria facilitated children's expressions of their ideas of randomness.

THE MAJOR DESIGN PRINCIPLES OF THE GAME

There were three major principles that informed the design of the game, as are described in the following paragraphs.

The manipulable sample space (and distribution): A 'lottery machine' represented an "executable sample space" or distribution in the game. The 'lottery machine' was a visible manipulable engine for the generation of random events and with it the children could directly manipulate the outcome of the game. The direct manipulation and linked connections provided by the software allowed children to set in motion the mechanism to trigger an event, and be able to link the execution of that event with an outcome on the screen.

The spatial representation of sample space: The presentation of the lottery machine in the game was geometrical/spatial, whereas in previous work (for example Konold, 1989) the sample space was either hidden (i.e. not available for inspection or manipulation) or represented only in quantitative form (by using only numerical quantities). The lottery machine contained balls of different colours, which made it possible for children to carry out as many events as they liked without being obliged to think about numbers (as they would have to using dice, coins etc.). Moreover, in the final iteration, the children also had the opportunity to change the probability of an event by changing the size of the balls and their arrangement.

The existence of local and global events in the game and a visible link between them: The game gave the children the opportunity simultaneously to see on screen the local and global representation of an event in their sample space. A *local* event refers to the trial-by-trial variation and the *global* to the aggregate view of each single trial[1]. In practice, local events might be used by children to make sense of short-term behaviour of random phenomena, while global events are associated with long-term behaviour. The game made visible the link between the short-term and long-term behaviour. Thus, whilst individual outcome could be seen as a single trial in a

[1] Many studies (for example Pratt, 2000; Ben-Zvi and Arcavi, 2001; Konold and Pollatsek, 2002; Rubin, 2002) have shown the importance of linking local and global understandings with an aggregate view, and how complex students may find this process.

stochastic experiment, the totality of these outcomes gave an aggregated view of the long-term probability of the total events.

A DESCRIPTION OF THE GAME AND ITS DIFFERENT PARTS

An example of a lottery machine can be seen in Figure 1.

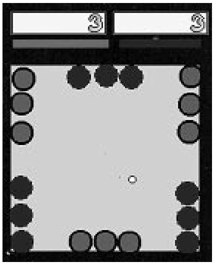

Figure 1: A lottery machine with its scorers showing the local events of the game

The lottery machine here is represented by the large square. In it, a small white ball bounces and collides continually with a set of static *blue* and *red* balls. Children could change and manipulate a number of aspects in order to construct their own sample space: the number, the size, and the position of the balls in the lottery machine, and they could also create new objects with their own rules. As programmed initially, collisions with the *blue* balls (light grey balls in the figures) added one point to the blue score and moved the 'space kid' one step down the screen. In this way the lottery machine controlled the movement of the space kid (see Figure 2).

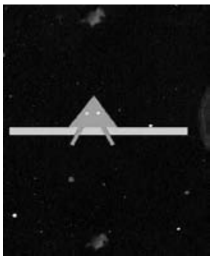

Figure 2: The space kid and the planets that represent the result of the game

An important design criterion was to choose a tool that worked with rules to afford children the opportunity to understand how the elements of the game are connected and moreover, to link local and global events. Figure 3 shows how the features of Pathways[2] provided the link between the lottery machine and the space kid.

[2] This research was funded by the European Union as part of the *Playground Project*, Grant No. 29329. See http://www.ioe.ac.uk/playground/

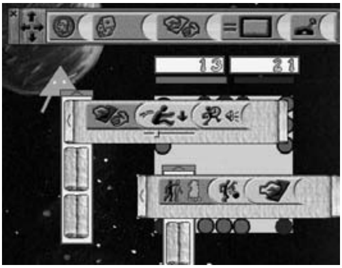

Figure 3: An example of linking rules in Pathways

In Pathways, the library of stones enables the user to make rules that will connect one object to another. So when, for example, the bouncing ball collides with a coloured, *blue* or *red* ball, this ball can instruct another object (by posting a message), to react. For example, the rule in Figure 3 can be read as ‘when I touch anything, I bounce off it and I show a *blue* message’ and the reaction of the other object is ‘when I see a *blue* message, I move a step downwards and I play a sound'. The small ball in the lottery machine moves continuously in 2 dimensions, a design decision made in order that children might visualise the global outcomes of the game. The final structure of the game as the child plays it, is illustrated in Figure 4.

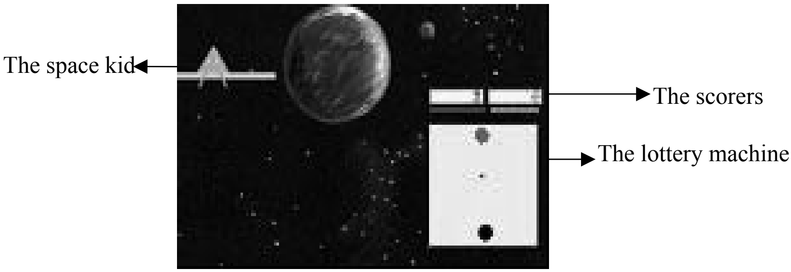

Figure 4: A screenshot of the game

Whilst individual collisions can be seen as single trials in a stochastic experiment, the totality of these movements gives an aggregate view of the long-term probability of the total events. In order to explore the connections children make between fairness and randomness, we began with a situation in which the children had to try to make the space kid move around a centre line in order to construct a ‘fair sample space’.

THE ‘CONCEPT’ OF THE GAME

The game gave children the opportunity, by manipulating the sample space and distribution, to identify intuitively whether an event is possible (whether it is impossible, certain or somewhere in between). The game consisted of two key pieces: a ‘lottery machine’ and a link between the local and global events. Diagram 1 gives the ‘concept’ of the game.

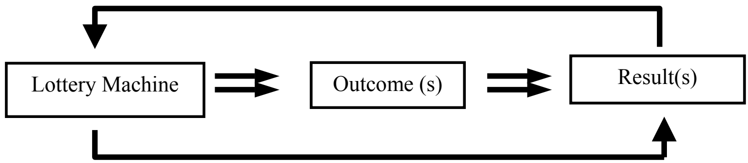

Diagram 1:

The concept of the game: the connection between the lottery machine, the outcomes and the results

Diagram 1 shows how the ingredients of the game are connected. The lottery machine generated an outcome and this affected the result of the game. The short arrows illustrate how children, by manipulating the lottery machine, were intended to experience the outcome of an individual event in the machine (i.e. a collision between two balls) and how this was connected to a single result in the game (i.e. effecting a movement of the space kid). The single outcome from the lottery machine provides an idea of a local event. The totality of the outcomes of the game gave a more aggregate view of the results and of the lottery machine's construction. The manipulations made via the lottery machine could have short-term and long-term outcomes within the game. For example, children could make decisions about their next change in the lottery machine based on the long-term results of their previous constructions.

THE CHOICE OF PATHWAYS SOFTWARE FOR DESIGNING THE GAME

Pathways was designed as a medium where children can build and modify games using the formalisation of rules as tools in a constructive process. This enabled the construction of the game, which afforded a simple means for programming the direct manipulation of objects, with which children could express meanings from actions and build new meanings of probabilistic ideas.

Hence, there were three reasons for the choice of Pathways: 1. evident, iconic rules that could be understood by children of this age 2. easily manipulated objects and 3. a clear message-passing mechanism providing a means of linking local and global events (see Figure 3). Thus, rules could be created specifying how each object of the game works. This afforded the children the opportunity to understand how the objects of the game were interrelated. It also allowed them to manipulate and link local and global events. Diagram 2 shows how the mathematical ideas and programming criteria are interrelated.

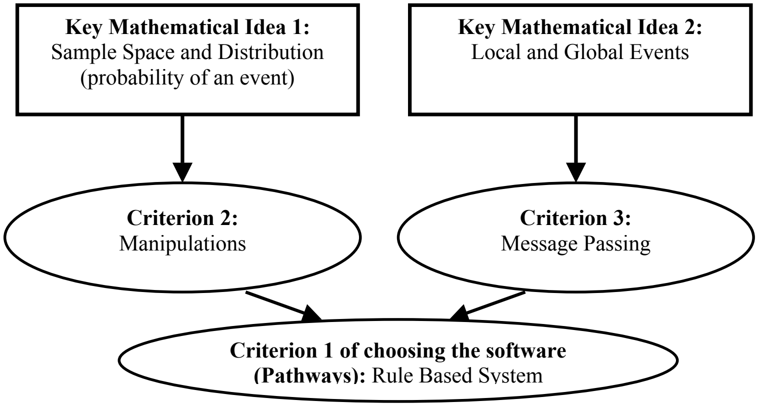

Diagram 2:

The interrelations between the mathematical ideas developed in the game and the criteria for Pathways choice

The key contribution of Pathways was that it allowed manipulations of the sample space and distribution, while its message-passing feature gave the opportunity to link local and global game events. Both criteria were related to the rule-based system on which Pathways (and therefore the game) was built.

Our approach followed that of iterative design (diSessa, 1989), which facilitates, through the gradual refinement of computational tools, an increasingly fine focus on the primary issues involved in the restructuring process. In fact, the iteration involved three components: while each version of the game was informed by children's activities with the previous version, Pathways too was being refined and debugged, so that it could serve as a suitable medium in which to construct the game. This hardly made the iterative design process less complex, but it ensured a better match between the platform, the game and children's emergent activities.

METHODOLOGY FOR CHILDREN’S USE OF THE GAME

In keeping with the iterative design approach, the game was developed alongside the gathering of evidence for children’s use of the game. All interviews were videotaped and transcribed. Twenty-three children were involved in the final iteration of the main study. The children worked with the software individually for a period lasting between 2 and 3 hours. The role of the researcher was that of participant observer, interacting with the children in order to probe the reasons behind their answers and actions.

A BRIEF GLIMPSE OF CHILDREN'S USE OF THE GAME

The early iterations of the design process gave the researcher an opportunity to observe the characteristics of children’s ideas about randomness, and how they expressed these ideas. The dominant tendency was the use of patterns, as other related research has shown (Konold, 1989; Pratt, 2000). Piaget and Inhelder (1975) also claim that children under the age of 8 years old are most interested in the point of view that considers the pattern of the total number of balls and the 'effect' of each experiment on the next.

An example below demonstrates how the design of the game helped a child make the link between short-term movement of the white ball and long-term movement of other elements. In the following episode, Victoria (girl, aged 6 years) tries to construct a fair lottery machine that will keep the space kid near the yellow line.

Researcher: How did you arrange the balls?

Victoria: If it (the white ball) goes like this it might get the red (ball) and then the blue (ball) and then to move down like this or it may move like this and move up and get this one and then that one or to move in a different place and get this and this.

She starts the game.

V: Come on! Look! It (the white ball) gets red again! Now it (the space kid) moves down and then up and…Whatever I say it happens! Come on…It (the space kid) moves up and then down. Oh…no, we have more points for the red colour. I wanted to get one red and one blue. Go up, now move down…

R: Does it listen to you?

V: No! I will place somewhere else the white ball. Here! Let me start the game again.

V: Come on… (*she knocks on the table*), come on…Oh, not again. I can't control this white ball….

She stops the game.

Here, Victoria based her ideas on the previous movement of the white ball, thinking that it would follow the same path. As the game progressed, she began to realise that even if she was able to predict where the white ball ended up, she could not predict how it might get there (i.e. its path of movement), and she could not control or predict exactly its next move. Finally, Victoria focused on the movement of the space kid instead of the white ball.

At the beginning of the game, all the children participating in the main study tried to find patterns to 'explain' the movement of the white ball. These patterns were based on the path traced by the white ball in order to 'achieve' a hit. However, because of the continuous movement of the ball it quickly became impossible for them to predict a particular path. The characteristic of the game of connecting the short-term and long-term behaviours of the system, provided children with corresponding local and global events. The research findings (see also Paparistodemou, Noss & Pratt, 2002) show that the continuous movement of the ball links short-term and long-term behaviours, thus discouraging children from simply looking for patterns in randomness. It is conjectured that the components of the game, and the connections between them, helped children to connect "pieces" of the system mentally, and to introduce corresponding structure into their thinking about randomness.

It might have been expected that children this age would find it extremely difficult to construct a rich set of meanings for the challenging idea of randomness. One reason for this might be the noted tendency of young children to focus on controlling the 'thing' that delivers randomness. However, the episodes of the main study indicate that in the special dynamic medium of expression provided by the game, children did construct meanings for randomness, in the sense that they realised the need to control the outcome without controlling the random movement.

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INTEGRATING THE HISTORY OF MATHEMATICS IN EDUCATIONAL PRAXIS.

An Euclidean geometry approach to the solution of motion problems

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The integration of History in the educational practice can lead to the development of a series of activities exploiting genetic "moments" of the history of Mathematics. Utilizing genetic ideas that developed during the 14th century (Merton College, N. Oresme), activities are developed and mathematical models for solving problems related to uniform motions are proposed, using the graph of velocity vs. time. The view of the covered distance as the area of the figure between the time axis and the velocity curve allows for the use of concepts and propositions of the Euclidean geometry. The use of simple geometric transformations leads to equivalent motion problems of a real context. This approach, applied to a wider range of problems, can form the basis for the introduction of basic concepts of calculus (such as integral, derivative, and their interrelation), in the context of a program of instruction in Senior High School.

INTRODUCTION

Problems in context may have an important role in the development of cognitive functions, via the mathematization of real life situations. We focus on the designing of a series of activities for solving problems related to uniform motion. For the designing of these activities we adapt the idea of the integration of the history of Mathematics into the educational praxis, exploiting genetic "moments" (Furinghetti, 1997, 2002, Furingetti & Somaglia, 1998, Tzanakis & Arcavi, 2000, Tzanakis & Thomaidis, 2000, D' Ambrosio, 2001, Katz, 1997). This perspective is consistent with a more general view that the way in which mankind developed mathematical knowledge, is also the way in which individuals should acquire mathematical knowledge (Polya, 1963, Freudental, 1973, 1991).

The designing of the activities is inspired from the study methods of motion during the later Middle Ages (14th century, Merton College, N. Oresme). This methodology, as it is historically documented, was the genetic "moment " for mathematical concepts like function, graphics representation and the integral (Clagett, 1959, 1968, Gravemeijer & Doorman, 1999).

Oresme's geometrical representation model of motion, "reconstructed" in its modern form, corresponds to the (U, t) graph of velocity vs. time. In this "holistic" graph, velocity and time exist concurrently and so does distance covered, as the area of the figure between the curve and the time axis. The view of the distance covered as an area, allows the use of concepts and propositions of Euclidean geometry. We present a mathematical model for solving problems related to motion based solely on Euclidean geometry. This point of view is based on the equivalence of area of geometric figures and mainly on the invariance of these areas under translation. Simple geometrical transformations, which may be performed on the velocity graph, lead to equivalent mathematical situations and therefore to equivalent real problems.

The use of Euclidean geometry in the solution of such problems leads to the interaction of the embodied and the proceptual mathematical worlds (Tall, 2003) and allows for the study of motion based solely on the functional approach. It is a course that may lead students, through their intuition and through mathematization, to the creation of mental models in order to solve the given problems, and through generalization and formulation to the understanding of mathematical concepts.

As an example we apply our Euclidean geometry approach to the solution of a motion problem.

THE INTEGRATION OF HISTORY IN EDUCATIONAL PRACTICE

The integration of history in didactic practice helps the students to understand that mathematics is not a fixed finalized knowledge system, but a live development process, closely connected with other branches of science (Furinghetti, 2002, Furingetti & Somaglia, 1998, Tzanakis & Arcavi, 2000, Tzanakis & Thomaidis, 2000). It helps the student to understand that mistakes, doubts, intuitive arguments, controversies, and alternative approaches to problems are not only legitimate but also an integral part of mathematics in the making (Tzanakis & Arcavi, 2000, p. 205).

We used the integration of historical data in designing the activities in such a way that history is not visualized as the main element in the classroom. This type of integration is described as,

> … a reconstruction in which history enters implicitly, a teaching sequence is suggested in which use may be made of concepts, methods and notations that appeared later than the subject under consideration, keeping always in mind that the overall didactic aim is to understand mathematics in its modern form (Tzanakis & Arcavi, 2000, p. 210).

In our research firstly, we located the genetic historical "moments" for basic mathematical concepts, like function, graph and the integral. These genetic ideas were developed during the 14^{th} century by the Calculators (Merton College) and N. Oresme and were related to the study of motions. Secondly, we integrated these historical ideas in the development of a course consisting of activities that aim to the understanding of the basic concepts of calculus. Here we present the first part of this designing that aims to prepare the students for an intuitive approach to these concepts.

GENETIC HISTORICAL MOMENTS IN THE STUDY OF MOTION DURING THE 14^{TH} CENTURY

The mathematicians-logicians of Merton College at Oxford (1330 – 1340), are known as Calculators. The Calculators (William Heytesbury, Richard Swineshead, John Dumbleton) studied the motion of bodies. They introduced the idea of functional relations, in an attempt to describe qualitative magnitudes (velocity, distance, time), with quantitative measurable features (Gravemeijer & Doorman, 1999). They defined several kinds of motion, proposed theorems concerning motion and proved these mathematically (Clagett, 1959). Their proofs were based on Euclidean geometry. Swineshead in " De motu" defined uniform motion as follows:

> Uniform local motion is one in which in every equal part of the time an equal distance is described. (Clagett, 1959, p. 243)

Heytesbury in "Rules for Solving Sophisms" (Regule solvendi sophismata), defined uniform accelerated motion, as follows:

> For any motion whatever is uniformly accelerated (*uniformiter intenditur)* if, in each of any equal pars of the time whatsoever, it acquires an equal increment (*latitudo)* of velocity (Clagett, 1959, p. 237).

Nicole Oresme, (1362), in "De configurationibus qualitatum" represented the variations of qualities, such as velocity, by geometrical figures. The basic idea of this representation is fairly simple: Geometric figures may be used in order to represent the quantity of a quality. As examples of Oresme's technique, let us consider the rectangle and right triangle in figure 1. Each figure measures the quantity of some quality (velocity). Line AB in either case represents the "extension" (time) of the quality. But in addition to extension, the "intensity" of the quality from point to point in the base line AB has to be represented. This, Oresme did by erecting lines perpendicular to the base line, the length of the lines varying as the intensity varies. Thus at every point along AB there is some intensity of the quality, and the sum of all these lines is the figure representing the quality.

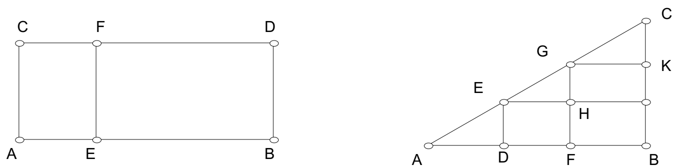


figure 1

Now the rectangle ABCD represents a uniform quality, since the lines AC, EF, BD represent the intensities of the quality at points A, E, and B (E being any point at all on AB), are equal, and thus the intensity of the quality is uniform throughout. In the case of the right triangle ABC it will be equally apparent that the lengths of the perpendicular lines representing intensities uniformly increase in length from zero at point A to BC at B, according to Merton College's definition of uniformly accelerated motion (AB is the time line). It is worth pointing out that Oresme designated the limiting line CD (or AC in the case of the triangle) as the "line of summit" or the "line of intensity". This corresponds to the "curve of motion" in modern analytic geometry. Oresme understood that the areas of the figures in the case of local motion represent the distances traveled in the corresponding times represented in each case by AB (M. Clagett, 1959, 1968).

THE DESIGNING OF THE ACTIVITIES

Tall (1996) argues that we should give greater emphasis to the visual representation of mathematical concepts. All over the world, the curricula emphasise the necessity of interconnecting the algebraic approaches with numerical and graphical ones, in the first contacts with functions and analysis. The triple: "numeric-graphic-algebraic" has become emblematic of different countries's projects. (Robert & Speer, 2001). Didactic research tends to offer arguments to support such a strategy by showing the role played in conceptualisation processes by flexibility between different semiotic registers of representations (Duval, 1995 as cited in Robert & Speer, 2001).

Taking this strategy into account and utilizing the described genetic historical ideas, we integrate the History of Mathematics in designing a range of problems related to uniform motion. For the solution of these problems we encourage the students to use the familiar Cartesian axes system and to focus on the graph of velocity vs. time. We encourage them to investigate the relation between "the area below the graph" and the distance covered in a period of time. We argue that the students, through guided reinvention (Freudental, 1991), can come to grips with the basic idea that the covered distance function can be expressed as the area of the graph function. From this geometric-graphical context, which represent the motion scenario, the students are asked to shift to the algebraic context and the algebraic formulas of the velocity and position functions. According to Tall (2003), at this point there is a shift from the embodied to the proceptual mathematical world (figure 2).

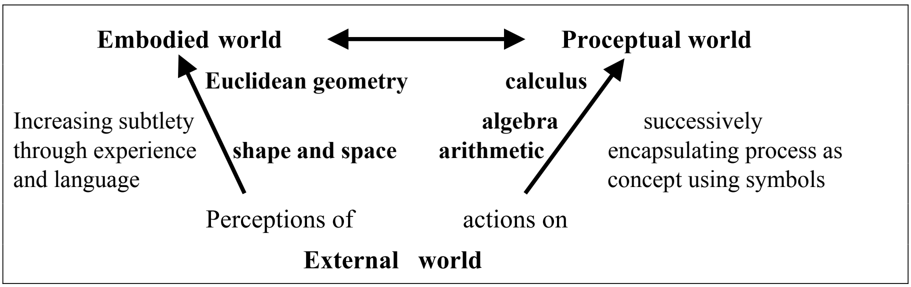


Figure 2

The covered distance of the moving body, viewed now as area, allows the use of concepts and propositions from Euclidean geometry in problem solving. Using the velocity – time graph, the students are able to solve uniform motion problems, based solely on Euclidean geometry.

We will present this solution model by an example. The solution to this problem is based on the equivalence of geometric figure areas and mainly on the invariance of these areas under translation. The geometric transformations that may be applied to the velocity graph lead to equivalent mathematical situations, and therefore to equivalent real problems. The understanding by the students that the same geometrical model can solve problems with different situational structures, can lead them to a classification of these problems.

AN EXAMPLE

Consider the following motion problem (cited in Yerushalmy and Gilead, 1999).

Problem 1.

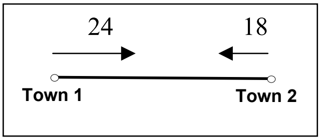
A biker traveled from town 1 to town 2 at an average speed of 24 km/hour. Arriving at town 2, she immediately turned back and traveled to town 1 at an average speed of 18 km/hour. The whole trip took 7 hours. How long was the trip in each direction?

Traditional approach

Table for the trip

	time	speed	distance
Out	t	24	24t
Back	7- t	18	18(7 – t)

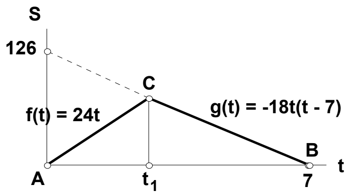
Traveling path



$$24t = 18(7 - t)$$

Figure 3. A traditional solution (cited in Yerushalmy and Gilead, 1999)

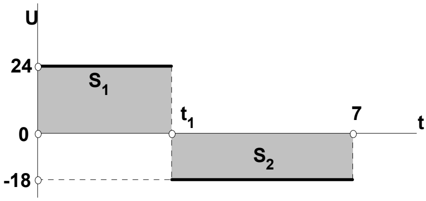
Functional approach (S, t)



$$f(t) \overset{?}{=} g(t)$$
$$24t = -18(t - 7)$$

Figure 4. A functional approach (cited in Yerushalmy and Gilead, 1999)

A "holistic" functional approach (U, t)



$$S(t) = \begin{cases} 24t, & t \in [0, t_1] \\ 24t_1 - 18(t - t_1), & t \in [t_1, 7]. \end{cases}$$

t_1 is the going out time.

$S_1 = S_2 \Rightarrow 24t_1 = \left|-18(7 - t_1)\right| \Rightarrow$

$\Rightarrow \ldots \Rightarrow t_1 = 3.$

Figure 5. A functional approach based on the "holistic" graph (U, t)

In this approach we have a shift from the embodied to the proceptual mathematical world. The formula of the position function results from the area of the rectangles and the equation for the solution of the problem is determined from the equality of the areas.

A Solution using Euclidean geometry

We give a solution to this problem, based solely on Euclidean geometry.

In figure 6, we have the representation of the problem, given that the two areas S_1, S_2 are equal ($S_1 = S_2$). Sketching two equal rectangles with area d, as shown in figure 7, we have, according to our hypothesis, that $S_1 + d = S_2 + d$. We know that the area $S_2 + d$ is the area of the rectangle with sides 7 and 18, hence is equal to 126 ($S_2 + d = 18 \cdot 7 = 126$). On the other hand, $S_1 + d$ is the area of the rectangle with sides 42 (24 +18) and t_1 (the going out time). Thus, from $S_1 + d = S_2 + d$, we get $126 = 42 \cdot t_1$ and consequently $t_1 = 3$.

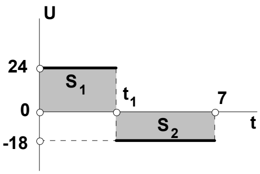


Figure 6.

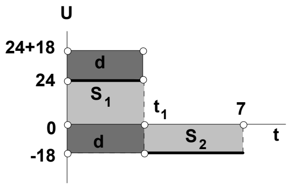


Figure 7.

Hence we gave a solution to the problem using the representative model of Oresme and simple geometric concepts.

Taking into account that the students have a good understanding of the invariance of area under translations since an early age, it is possible to lead them to equivalent real life problems, through graph transformations. For example (Figure 8):

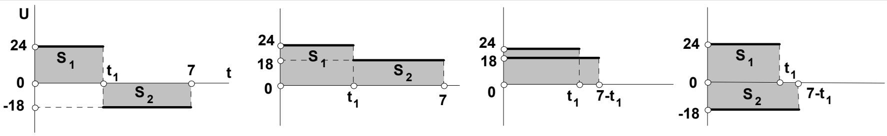


figure (a) (initial situation)

figure (b) (S_2 reflected in the t-axis)

figure (c) (S_2 reflected in the t-axis and translated by t_1 to the left)

figure (d) (S_2 translated by t_1 to the left)

Figure 8

We encourage the students to interpret the above graphs and to describe real problems of motion, corresponding to each graph. As an example, we give a real problem corresponding to figure (c).

Figure (c): Two bikers started their journey at the same time from town 1 to town 2 at an average speed 24 km/hour and 18 km / hour respectively. The time of the first biker was t_1 and the second t_2. If $t_1 + t_2$ is totally 7 hours, how long was the trip for the bikers?

The students are allowed to try several other geometric transformations and to express new equivalent real problems.

This approach applied to a wider range of uniform motion problems can lead the students to the statement that problems in which the constant velocities U_1 and U_2 are given and there is a known linear relation between the covered distances and the corresponding times (e.g. $S_2 = \kappa S_1 + \lambda$ and $t_2 = \mu t_1 + \nu$), can be solved with the same mathematical model. In fact, by employing geometric transformations, the basic unknown can be viewed as a side of a rectangle, with the area and the other side known.

Discussion

"Past never dies" said Fauvel (1991), arguing towards the integration of the history of mathematics and didactical praxis. A "past" such as Euclidean geometry, should be used appropriately in educational practice. By using simple geometric concepts, it is possible to create mathematical models for solving real problems and reach equivalent real situations via simple geometric transformations. New procedures and mathematical objects independent of the specific situation should emerge. The interconnection of the embodied and proceptual mathematical worlds (Tall, 2003), allows the deep understanding of concepts and leads to bridge the gap between informal and formal knowledge.

Finally through progressive mathematization, the need of formal mathematical thought must become clear to the students. We believe that the solution of motion problems through the velocity - time graph, could lead to a future didactic method for the introduction of the definite integral and fundamental theorem of calculus. This is exactly what the History of mathematics teaches us. The evolution of concepts during time can inspire a chain of teaching activities with specific goals, without abolishing the essence of mathematical knowledge.

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THE MAGIC CIRCLE OF THE TEXTBOOK – AN OPTION OR AN OBSTACLE FOR TEACHER CHANGE

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***Abstract**: There have been many efforts to change mathematics education and teachers. However this has proved to be very difficult. The starting point for this study was the question: Should we change people? There might be good reasons why things remain stable for longer periods. My aim is to approach the difficult question of teacher change from the perspective of stability and to figure out what constitutes the good and stable elements in school mathematics teaching and learning in the minds of teachers. The remarkable and complex role of textbooks is discussed in this research report.*

Introduction

There is an extensive amount of literature concerning teacher change which presents many aspects of the change process (e.g. Richardson & Placier 2001). During the last decades, new theoretical perspectives of teacher change in the context of mathematics education have been under development (see e.g. Fennema & Nelson 1997) and several empirical studies have been conducted on the change process. Recent findings confirm the previous findings about the complex nature of teacher change. It has become evident that teacher change is a long process that needs time and effort (e.g. Barnett & Friedman 1997; Borko et al. 2000).

The role of textbooks and other curriculum materials in these processes is not well known and there are somewhat contradictory views about the teacher - textbook - relationship. However, studies on the contribution of the textbook to teacher change processes give us reason to investigate this area much more closely (e.g. Sosniak & Stodolsky 1993; Remillard 2000; Collopy 2003). The influence of textbooks on classroom practices and teachers' work is not yet a solved problem.

Many efforts have been made to explain why teachers are not so easily changed (e.g. Guskey 2002). But only a few scholars have been interested in "unchange". Should we not ask why things remain stable and whether there is something valuable in stability? Cuban (1992) suggests that change and stability should not be examined as separate from each other. He points out that schools are deeply embedded in society. Many, but not all, of the impulses for changing schools arise in the larger society. Although schools are dealing with imposed changes, school organisations must preserve their stability – at least to some extent – in order to ensure that what happens today will probably occur tomorrow. Schools are responsible for maintaining order, instructing the young and producing students who have learned.

My aim has been to understand the difficult question of change from the perspective of stability. What constitutes the good and stable elements in school mathematics teaching and learning in the minds of teachers?

Participants and data collection

The data consists of semistructured interviews of nine elementary school teachers - seven female and two male teachers who worked at the same middle-sized urban school in Southern Finland. All of them were competent and qualified elementary school teachers. Other teachers who worked at this school and who did not participate in this study were not qualified teachers. The interviewed teachers had teaching experience ranging from one to seventeen years. Most of them had been only working in this school.

The interviews were conducted at the school in a quiet room. Each interview lasted about sixty minutes. The interviews were tape recorded and later transcribed by the author. I had three broad themes for the interview; otherwise, the discussion was quite open. The themes in the interviews were 1) The teacher's personal conception about the state of mathematics teaching; 2) The good and proper elements of mathematics teaching; 3) The elements in mathematics teaching which should not be changed in the case of any change.

Data analysis

The data was analysed qualitatively to find the stable key elements most important to teachers and to interpret the prerequisites of change. (Pehkonen, L. 2001). I began my analysis with a careful reading and rereading of the transcriptions of every respondent and writing initial remarks on the papers. Then I continued to organise the speech of every respondent under the interview themes.

During the initial analysis it became evident that there are three key elements which were dominating the teachers' talk about mathematics teaching – textbooks, teacher-centred teaching methods, and basic skills and facts. These formed a stable trinity in the minds of teachers. These were seen at the same time as the basis of good mathematics education and as a threat to it. And they were seen as obstacles and threats to development and change. They were also the main elements that teachers did not want to give up in case of any change.

I had not asked anything about textbooks. Anyway, every one of my informants spoke a good deal about them. It seems that textbooks have a remarkable and complex role in the change process. The relationship between teachers and textbooks is interesting. Remillard (1999) has shown that there are many differences inn how teachers interact with mathematics textbooks. She argues that teaching is multi-dimensional activity. Each dimension, or arena, as she calls the elements in her model, requires different types of decisions and support, and teachers are likely to use the textbooks differently in each arena. Even teachers who follow textbooks very closely make curriculum-development decisions. Textbooks - not even reform-oriented textbooks - do not operate on their own and naturally produce reforms. This paper concerns the portion of the data that is about textbooks. I decided to separate all the teachers' talk concerning textbooks for separate analysis. The method used to analyse this data was qualitative content analysis..

Findings

Textbooks are important tools for teachers and how they speak about textbooks is plentiful and varied. The talk is considering and reflecting, but contains contradictory elements. The data reveals three qualitatively different ways to speak about the mathematics textbook: justification, criticism and guilt. The talk of justification is composed of ways to speak about the textbook in appreciation and its use in mathematics education and of positive accounts of textbook use. Teachers see the textbooks as a guarantee of a stable quality of teaching. Most of the teachers' talk consists of connected accounts of their textbook use from this perspective. This category includes also the talk about textbooks as a supporter for reforms. Teachers found that textbooks provided help and ideas for new ways to teach. The talk of criticism consists of the teachers' concern that textbooks are burdens and obstacles to their change process; textbooks tie down teachers' freedom to make their own decisions. Teachers felt that textbooks make them passive and mechanise the teaching in classrooms. The last category includes the teachers' talk of their feelings of guilt connected to excessive textbook use and concern about teachers' professional competency. In the following, these aspects are discussed in more detail.

Justification: 1) Textbooks as a guarantee of stable quality

Teachers have a basic belief that Finnish mathematics textbooks are very good. In the teachers' speech, textbooks represent the positive aspect of stability: the maintenance of a uniform quality of mathematics education. The textbooks help teachers to keep their teaching at an appropriate level; i.e. teaching is freer of inadequate and occasional changes. I could find out three different kinds of accounts of why textbooks have such a remarkable role in assuring uniform quality. First, teachers shared many accounts connected to the properties of the textbooks themselves. Teachers found mathematics textbooks to be logical and explicit. They contain the important basic facts and the tasks are connected to everyday life. So it is easy for both teachers and pupils to see the meaningfulness of mathematics. Moreover, textbooks were seen to be motivating; they are colourful and the exercises are varied. The books are easy enough, although there are challenges for most pupils. And one point that is very important to Finnish teachers: Textbooks educate pupils to work hard.

Second, there are accounts that are connected to teachers and to their work. Textbooks were seen as a method for teachers to keep their teaching logical and coherent. Textbooks help teachers in their plans and choices and they assist teachers in keeping the planned schedule. Finnish teachers are overloaded with work demands; they feel tired and they have the feeling that they cannot manage all the tasks that school organisations set for them (Santavirta et al. 2001). Mathematics textbooks help teachers with their work loads, because the books provide ready and sensible structures for lessons and enough exercises for pupils. Even when teachers are in a hurry and unable to prepare their lessons, or when they are busy with other work demands during the lesson, textbooks provide welcome help to teachers.

Textbooks can completely substitute for the teacher, at least now and then. One of the teachers said*:" The math books are so well-made that pupils do not need any help."*

The third category consists of those statements that are connected to pupils. The pupils' keen interest was seen as an evidence of high quality. Teachers of the youngest children told that children love their mathematics books and almost every teacher stated that pupils like the exercises in the books. As much of schoolwork is nowadays organised in small groups, teachers find that pupils love those peaceful moments when they are allowed to work alone and proceed at their own pace. And last but not least point is that with the help of books, children learn the facts they are expected to learn. That is enough for teachers.

Justification: 2) Textbooks can support change

One way to speak about textbooks was that of reform. Teachers believe that textbooks could help them to change the teaching of mathematics – at least somewhat. The main elements in teaching have remained stable throughout the years because they have proved to be satisfying. The fact is that pupils really seem to achieve what they are expected to achieve quite well, at least measured by official standards and tests. As Sharon says:

> *"Pupils know at least the facts measured by the tests very well. It is a different matter if you want to test for something else."*

But with the help of books and teachers' guides, teachers have adopted some new ideas and methods. Teachers appreciate some activities and problem-solving sessions – now and then, if they do not detract from learning of the basic facts and skills. Books are easy to use and are almost always at hand. That is why teachers believe they have a chance to influence what happens in classrooms. There is research-based evidence that some teachers can change their teaching practices with the help of a textbook and without additional support or feedback, but some others cannot (Colloby 2003). The influence is possible, but not self-evident.

Criticism: Textbooks as a burden and an obstacle for change

> *"The book is like a burden on your shoulders. It is not my affair that is in the book. I should always think how I will teach the lessons somebody else has planned."*

The textbook represents a supply of exercises to most teachers. School mathematics means that these supplies are used, and "teaching mathematics" means that somebody, usually the teacher, explains how these supplies are put into use. The interviewed teachers told that they usually follow the book very obediently. Teachers criticised textbooks, i.e. the supply of exercises, as being far too easy and simple and too congested. They feel uneasy about choosing the tasks or changing the order, even when they think it would be reasonable. Teachers very often make their pupils work out all the exercises in the book, even if the tasks are quite unsuitable – usually completely too easy - for their pupils. Everything in the book is carefully used and completed. Some teachers are really afraid to make any decisions or omit anything even if it seems reasonable:

> "*I don't know how to choose. The book is so good. But last year - I had the same book – we were really in a hurry. I was almost desperate: "How can we manage – we don't have time." I did not dare not to omit any tasks."* (Steven)
>
> *L: Why it is so difficult?*
>
> *S: I don't know. Everyone speaks how important it is to do well in mathematics. I don't know if one task is more important than another. What if I omit important tasks? I don't want to take the risk."*

Teachers say that on one hand textbooks tie them down, and on the other hand they make them passive. It is so easy to "obey" the books and not think independently. As Sue, a young teacher, said:

> *"The book helps so much. You don't have to think up the exercises yourself. But on the other hand, it ties you down."*

And Sylvia, a very reflective and experienced teacher who is interested in mathematics teaching stated:

> *"I have noticed that the book makes me passive so easily. The teacher's guide gives so much support for my work and the textbook, too. They really "tease" me not to think and not to consider how to construct this teaching unit for my pupils."*

Textbooks offer an easy solution to busy days, but this obedience to books can easily mean that mathematics teaching in classrooms becomes very mechanical. Teachers tell that they follow the books without thinking and considering, and the pupils complete almost every exercise. This process is repeated from lesson to lesson. If some pupils are very quick, teachers can easily give them more, similar exercises. The interaction between the teacher and pupils may remain very minimal.

Feelings of guilt and concern for professional competency

Teachers have recognised this state of affairs. Teachers are concerned about the power of textbooks. They feel guilty and they try to explain their course of action.

> *"It is really a silly thing to be bound to the book, but I am guilty myself. I should get free of it, but that requires so much hard work." (Sharon)*
>
> *"When you are tired – you feel guilty, as you know that you should be able to plan for yourself. But you are in a hurry, and you don't have time and you can't be bothered."* (Sylvia)

Teachers see the textbook authors as the most competent experts and they are a bit unsure about whether they are as competent as the writers to plan the lessons themselves. They are concerned that they have given up a part of their professionalism to the textbook authors. But the children are learning what they are expected to learn. Is that not enough?

> *"As if you have given a part of your teacherhood to the book:"* (Sylvia)

> *"I am a little concerned about my own professional competence - it needs some improvement, I think. Anyway, the children learn the multiplication tables - but I am not sure if it is sufficient."* (Steven)

One fact that all teachers agree on is that over the course of the years the pupils' attitudes turn against mathematics. Teachers tend to believe that it is due to a textbook-bound teaching of mathematics:

> *"Their attitudes gradually turn negative. We teachers should motivate pupils more. We could do something else than always make the kids fill in the textbook."* (Simon).

Discussion

Finnish teachers are very well educated. It is hard to get a place in a teacher education program and the education students are usually highly motivated. All teachers – even kindergarten teachers - are educated at universities and all primary school teachers achieve a master's degree in education. It takes about five years to complete the teacher education program. Finnish teacher education emphasises the academic aspects of education and research-based academic studies dominate the content of teacher education. Finnish teachers are academically well prepared for their task. However, even for these well-educated teachers, it seems hard to face changes in the curriculum and education (see also Pehkonen, E. 1993).

All teachers who participated in this study were good, competent teachers and innovative in many ways. They were enthusiastic about their work and wanted continuously develop it. But mathematics as a school subject seems to have a special character. Teachers do have ideas and views about mathematics education, but they are uncertain of how to cope with different conflicting pressures. Would it be better to meet the goals of academic mathematics educators and researchers or to satisfy the practical goals of achievement tests? They are not sure how much to respect the authority of textbook authors and to what extent to believe in their own professionalism. However, all the teachers reported that they do not have these contradictions in any other school subjects. What is matter with school mathematics?

Teachers are quite satisfied with their current practices, but it seems that teachers are somewhat confused. They do not know how much to rely on their own professional competence. They are happy to teach mathematics with the help of books, but feel inconvenienced if they must plan themselves. There are also some tasks in math books, but often marked with an asterisk (or by some other sign) to indicate that they are not the most important task. School administrators speak about "modern mathematics teaching" as an important matter, but what they then assess is mostly the "basics". Teachers are well informed and they do know about new trends.

Teachers have some kind of feeling that teacher educators and researchers do not respect their job. That makes them feel a bit guilty and inconvenient. But it is not a sufficient reason for change. They feel uneasy because they know the trends of modern mathematics teaching, but they are not sure whether these ideas are important or not. This finding is in line with Borko's and her colleagues' (2000) findings. Teachers have a kind of contradiction between what they speak and know and how they act.

The math book is like a magic circle – it could be a means for a change, but it seems more like an obstacle. Teachers want the mathematics textbooks to concentrate on the basics, since they believe the basics constitute good and proper mathematics teaching. Teachers want to do their job properly. One of their goals is that their pupils get good marks. And good marks are achieved by hard work and much practise. That is what teachers try to provide, and textbooks help them in this mission. They try to satisfy what parents, administrators, principals and teachers themselves believe is proper math teaching (see Cuban 1992). They have experience-based evidence that their teaching meets the set standards well enough.

Change is always a risk. How can we help teachers to take risks? Well-planned in-service education could help teachers cope with books and provide support and feedback.

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