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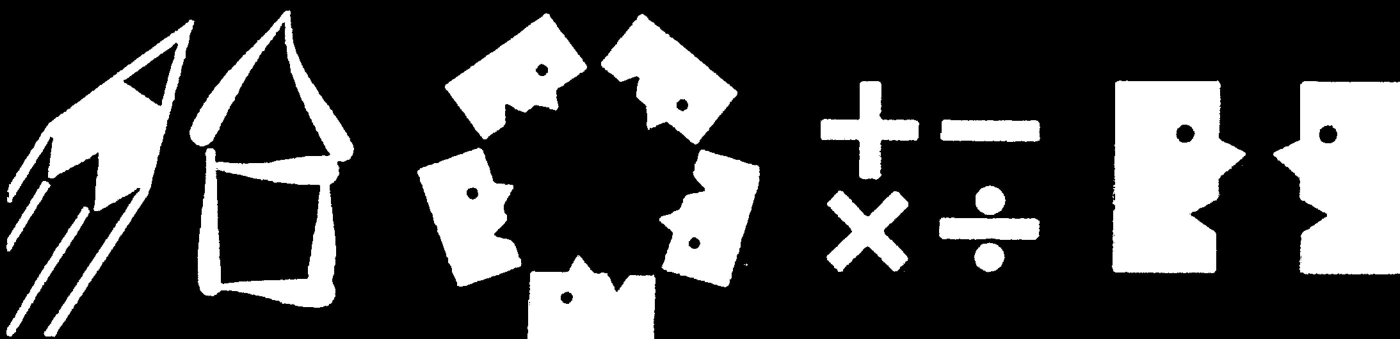
ABSTRACT

This toolkit contains a set of professional development materials whose goal is to help mathematics teachers in grades 6-10 learn to identify, describe, and foster algebraic thinking in their students. A core belief underlying the Toolkit is that good mathematics teaching begins with understanding how mathematics is learned, so these materials focus on how students think about mathematics and helping teachers to understand students' thinking through the analysis of different kinds of data such as student work and classroom observation. Instructional implications are also considered from the perspective of an understanding of how algebraic thinking develops. This material is a self-study guide and offers the following: (1) hands-on investigation, discussion, and reflection aimed at a deeper understanding of algebraic thinking; (2) language for talking and thinking about algebraic thinking; (3) structured approaches to gathering and analyzing data about students' mathematical thinking; (4) structured approaches to discussion among teachers about mathematics, student thinking, and other issues related to teachers' practice; and (5) mathematics problems that both elicit and develop algebraic thinking. (KHR)

ED 476 802

The Fostering Algebraic Thinking Toolkit
A Guide for Staff Development



- Introduction and Analyzing Written Student Work
- Asking Questions of Students
- Documenting Patterns of Student Thinking
- Listening to Students

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Fostering Algebraic THINKING TOOLKIT

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OVERVIEW

USING THE TOOLKIT

What You Need

- **A regular meeting time**
  The Toolkit materials are intended to be used by a small group of teachers who meet regularly over the course of a semester, a school year, or a longer period of time. (See the *Introduction* section for thoughts about different configurations.) About three hours should be allowed for each Toolkit session, and two to five weeks is a good period to have between sessions. Participants need some time to complete their homework, which usually involves doing an activity with students; however, for the sake of continuity, there should not be too long a gap between one session and the next.

- **6–12 participants**
  The ideal size for a Toolkit seminar group is between 6 and 12 teachers. We say this because of an interest in finding the right balance between richness of discussion and adequate participation by everyone. In a group smaller than 6 members, it can be difficult to sustain richness in discussion; however, in a group larger than 12, it can be difficult to spread participation across the entire group.

- **A facilitator**
  The Toolkit materials are written with the assumption that the group will have a designated facilitator and Facilitator Notes are included at the beginning of each module to support facilitation. Toolkit facilitators are not expected to be experts in algebraic thinking, or to have previous experience with the Toolkit. The facilitator may be a teacher, teacher leader, or administrator; the important thing is that the facilitator is willing to take on the responsibility of guiding the group through each session.

- **A meeting place**
  Toolkit seminars should take place in a room large enough to comfortably hold all of the participants. It is best if the seating can be arranged so that everyone can see and talk to one another (e.g., seats in a semi-circle rather than in rows), and so that it is easy for participants to work in groups of different sizes, and to switch seats to work with different people over the course of a session. An overhead projector should be available to facilitate discussion. Two of the modules (*Listening to Students* and *Asking Questions of Students*) require a VCR for some of their sessions.

- **A plan for using the Toolkit**
  We assume that you and your colleagues can read the *Introduction* section and use it to develop a plan for using the Toolkit. As you will see there, the Toolkit's modular structure means that you have different options for how to use it. In any case, however, your group should begin with the

Introductory Session and the *Analyzing Written Student Work* module. After that, you can move on to any of the other three modules, but participants should know how many sessions they are committing to when they sign up for the Toolkit seminars.

- **Toolkit materials**
  Blackline masters for all session materials are found behind the Facilitator Notes for each module. Before each session the facilitator will need to make copies of the session materials for all participants.

- **Support of a district mathematics coordinator, principal, or other administrator**
  Since the activities in the Toolkit are quite a bit different from the norm in staff development, teacher groups would be well advised to inform appropriate people in the administration and to seek their support, if not their involvement.

In order for you to describe the Toolkit succinctly for administrators, here is a brief summary of the content of each module and discussion about how they connect to one another.

The Modules

Analyzing Written Student Work is intended to follow directly after the *Introductory Session.* This module gives participants an opportunity to practice analyzing classroom data for evidence of algebraic thinking, and to become comfortable with the language of the algebraic habits of mind, before moving on to one of the other modules. To help with the analysis of written student work, we provide some student work examples and then provide guidelines for you and your colleagues to collect examples for analysis. Each of the three remaining modules builds on the work done in *Analyzing Written Student Work*, taking a new perspective and introducing new tools for gathering and analyzing data.

Listening to Students helps teachers deepen their understanding of student thinking by examining other aspects of their thought processes: the evolution of students' ideas and the thoughts that often do not get communicated on paper. This module complements *Analyzing Written Student Work*, and is a good choice for groups with a special interest in building their understanding of their students' thought processes, or groups that are concerned that their students' thinking does not come across well in written work.

Documenting Patterns of Student Thinking continues to use written student work as the data to analyze, but focuses on patterns of thinking across a class of students, rather than on individual student thinking. It also takes a step beyond just analyzing data for evidence of student thinking, to systematically recording the information gained through analysis. Finally, this module begins to address issues of alignment between goals for student learning and classroom assessment. Of the four modules, this one examines in the most depth the connections between algebraic habits of mind and the algebra learning expectations enunciated by the National Council of Teachers of Mathematics. This module is a good choice for groups who are interested in exploring how to keep track of the growth of algebraic thinking in their classes over time, and in investigating the alignment between learning goals and the activities they give their students.

Asking Questions of Students asks teachers to make the greatest leap from their work in *Analyzing Written Student Work*, with a change both in the type of student data considered (from written to real-time) and in the emphasis of the module (from understanding to fostering student thinking). Of the four, this module puts the least emphasis on looking at student thinking, and focuses most strongly on the relationship between teacher intentions and teacher questions. A group using this module must be able to arrange for pairs of teachers to observe one another in their classrooms, which may present some logistical difficulties. This is a good module for use by groups who are very comfortable with the kind of math activities used in the Toolkit and are looking for ways to adapt their teaching practice based on the understanding they developed in *Analyzing Written Student Work.*

What the Materials Provide

The participant materials for a module consist of a Module Overview and materials for each session. Each set of session materials contains:

- **A Session Agenda:** This page gives an overview of the activities in the session, with the amount of time allotted to each activity and an icon indicating the type of activity it is.
- **A Session Goals list:** This is a summary of what participants should expect to accomplish during the session.
- **Session Instructions:** This includes brief descriptions of the activities and logistical details, and can be a helpful reference during the agenda review.
- **Discussion sheets:** These sheets list questions to be discussed either in small groups or with the whole group. Each discussion activity is labeled with an icon indicating whether it is a full-group or small-group discussion. Discussion sheets may also include some relevant expository text or additional instructions.
- **Math Activity sheets:** These are math problems for participants to work on in small groups, generally followed by a discussion about the problem. When participants are expected to use the math problem with their students, a separate, clean copy is also provided.
- **A Homework sheet:** This gives instructions for the homework participants will do between this session and the next, typically a reading or an assignment to gather data such as student work.
- **A Mathematical Thinking Record** for any math activities used in the session: This is a journal-like set of sheets where participants can record information about the math activity, their work on it, and ideas from the discussions.

Each activity or discussion sheet is marked with an icon that shows the type of activity it is; these icons also appear next to the activity names in the agenda. A list of Toolkit icons follows.

Individual Work

Homework

Small Group Discussion

Full Group Discussion

Math Activity

In addition to the regular materials, the facilitator guide give information about the content and purpose of the sessions, and facilitation tips. The facilitator guide also include a section called Math Notes, which contains extra information about the math problems that appear in each session.

Session Structure

The four modules of the Toolkit have the same basic session structure. The amount of time allotted to each schedule component varies among modules, as well as among sessions in the same module. Agendas are provided indicating the amount of time to be spent on each component. Instructions describe any schedule elements that are unique to that session.

Check-in and Agenda Review

Each session begins with some time for group members to check in with one another and to review the Session Agenda. The facilitator may want to use this time to deal with logistical issues, hand out materials for the session, and so on.

As a way of focusing people's attention and reminding them of where the last session left off, the Check-in Activity includes a brief time for group discussion, prompted by a relevant question. If the group has other Toolkit-related topics it would like to discuss (for example, reporting back on a question that was brought up in a previous session), it is appropriate to discuss such topics instead of discussing the Check-in questions given in the materials.

Activities and Discussions

The bulk of the time in each session is devoted to activities and discussions pertaining to that session's focus. The kind of activities and the amount of time spent on discussion varies among sessions and modules. Activities and discussions are conducted in a variety of formats: whole group, small group, and individuals working on their own.

Break
Each session includes time for a brief break. Although the Session Agenda lists the break as occurring at a particular point in the session, the group may find it more convenient to move it to some other time.

Group Process Discussion
The Group Process discussion is a time for the group to talk about how the sessions are going, not in terms of the subjects being explored, but in terms of structure, group dynamics, or other "external" issues. It is a time to reflect on and provide feedback about the process of using the Toolkit together, and to voice any concerns that participants have about the way the group interacts. Questions for the Group Process discussions are provided.

Mathematical Thinking Records
The Mathematical Thinking Record is a way for participants to summarize and organize their thoughts about the mathematics and mathematical thinking they encounter in each session. The goal is for each participant to create a personal record of the math problems, as an aid toward remembering important information about the mathematics and the algebraic thinking involved in the problem, and to use for future reference.

The first two questions of the Mathematical Thinking Record have to do with the teachers' experience with the problem in the context of the Toolkit session. The third question refers to their students' experience. Time is allotted for teachers to make notes on this question at the following meeting, after they have given the problem to their students and discussed their students' thinking with their colleagues.

Next Steps
During this time, the facilitator should present the homework assignment for the next session and distribute any necessary materials. This is also a time to confirm scheduling for the next session and to take care of any logistical details that have come up.

Homework
Teachers will be asked to complete tasks between meetings, usually collecting data to be used in the following session. After some sessions, there will be a reading assignment. The reading may relate to topics in the completed session, or prepare teachers for the following session.

We have tried to keep the amount of homework minimal, knowing that teachers have busy schedules. However, if your group feels that it would like more continuity between the sessions and is willing to put in the extra time, the group could agree to some additional between-sessions work. An activity that can form a bridge between one session and the next, and between the sessions and the classroom experience, is to have the participants keep reflective journals.

REFERENCES

A few resources on algebra and algebraic thinking:

Burns, M. (1990). *A Collection of Math Lessons: From Grades 6 through 8.* White Plains, NY: Math Solution Publications (Distributed by Cuisinaire Company of America).

Education Development Center, Inc. (2000). *Connected Geometry.* Chicago, IL: Everyday Learning Corporation.

Friel, S. (Editor) (2001). *Navigating through Algebra in Grades 6–8.* Reston, VA: National Council of Teachers of Mathematics.

Gelfand, I. M. and Shen, A. (1993). *Algebra.* Boston, MA: Birkhäuser.

Lott, J. (Editor) (2001). *Navigating through Algebra in Grades 9–12.* Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Shell Centre for Mathematical Education. (1985). *The Language of Functions and Graphs.* Nottingham, England: Shell Centre for Mathematical Education. (Now available from QED of York (England) at +44-1904-424242 and <qed@enterprise.net>.)

Further reading on some of the math activities used in *The Fostering Algebraic Thinking Toolkit* can be found in:

Cai, J., Moyer, J., and Laughlin, C. (1998). "Algorithms for Solving Nonroutine Mathematical Problems." In *The Teaching and Learning of Algorithms in School Mathematics.* Reston, VA: National Council of Teachers of Mathematics. [The Staircase Problem, Crossing the River]

Driscoll, M. and Moyer, J. (2001). "Using Students' Work as a Lens on Algebraic Thinking." *Mathematics Teaching in the Middle School,* 3 (5), 282–287. [Crossing the River]

Kenney, P. A., and Zawojewski, J. S. (1995). "The Thinking of Students: Marcy's Dot Pattern." *Mathematics Teaching in the Middle School,* 3 (7), 474–477. [Marcy's Dots]

Schultz, H. S. (1999). "The Postage Stamp Problem, Number Theory, and the Programmable Calculator." *Mathematics Teacher,* 92 (1), 20–22. [Postage Stamps]

Slavit, D. (2001). "Revisiting a Difference of Squares." *Mathematics Teaching in the Middle School,* 6 (6), 378–381. [Differences of Squares]

Stylianou, D. A., Kenney, P. A., Silver, E. A., and Alacaci, C. (2000). "Gaining Insight into Students' Thinking through Assessment Tasks." *Mathematics Teaching in the Middle School,* 6 (2), 136–143. [Marcy's Dots]

For other math activities, you may want to search in:

Driscoll, M. (1999). *Fostering Algebraic Thinking.* Portsmouth, NH: Heinemann.

The Math Forum: http://forum.swarthmore.edu

Problems with a Point: http://www2.edc.org/MathProblems

INTRODUCTION

WHAT IS *THE FOSTERING ALGEBRAIC THINKING TOOLKIT*?

What It's About

Welcome to *The Fostering Algebraic Thinking Toolkit.* This *Introduction* describes the content of the materials and the principles and values that underlie its design. It is intended to help you and your colleagues fit the Toolkit to your own particular context. For a description of practical considerations, such as preparations for getting started, typical agendas, and so on, refer back to the *Overview.*

The Toolkit[1] is a set of professional development materials whose goal is to help mathematics teachers in grades 6–10 learn to identify, describe, and foster algebraic thinking in their students. A core belief underlying the Toolkit is that good mathematics teaching begins with understanding how mathematics is learned, and so these materials focus on how students think about mathematics and on helping you and your colleagues to understand students' thinking through the analysis of different kinds of data, such as student work and classroom observation. Instructional implications are also considered, from the perspective of an understanding of how algebraic thinking develops.

As you work with the Toolkit, you will use the concept of habits of mind that constitute algebraic thinking, and will increase your capacity to recognize evidence of these habits of mind in student work. Because the Toolkit also engages teachers in exploration and discussion of open-ended algebraic activities, you will also learn to recognize the workings of the habits of mind in your own and your colleagues' mathematical thinking.

The Toolkit is a self-study guide, intended for independent use by a professional development group of teachers, with the help of a teacher or administrator designated as the facilitator. It offers the following:

- Hands-on investigation, discussion, and reflection, aimed at a deeper understanding of algebraic thinking
- Language for talking and thinking about algebraic thinking (the algebraic habits of mind framework, also used in the book *Fostering Algebraic Thinking* [1999])
- Structured approaches to gathering and analyzing data about students' mathematical thinking
- Structured approaches to discussion among teachers about mathematics, student thinking, and other issues related to teachers' practice
- Mathematics problems that both elicit and develop algebraic thinking

The Toolkit's ideas and many materials are derived in good measure from three teacher-enhancement projects in which we at Education Development Center, Inc. (EDC) were involved. All three were National Science Foundation teacher-enhancement projects: 1. Linked Learning in Mathematics: Marquette University (1997-), ESI-9619366, involved Milwaukee teachers; 2. Leadership for Urban Mathematics Reform (LUMR): Education Development Center (1994-1997), ESI-9353449, involved teachers in Durham, Los Angeles, Milwaukee, St. Louis, San Diego, and Worcester; 3. Assessment Communities of Teachers (ACT): Pittsburgh Public Schools (1994-1997), ESI-9353622, involved teachers in Dayton, Memphis, Milwaukee, Pittsburgh, San Diego, and San Francisco.

Teachers are more accustomed to curriculum materials for students than they are accustomed to curriculum materials for teachers, so we think it important to reiterate that the Toolkit is *for teachers.* While it is the case that the materials will ask you to do activities with students, our primary focus is on your learning—in the belief that student learning will also be served. The Toolkit has been thoroughly teacher tested, and comments from the field have been inserted in this *Introduction* to illustrate the kinds of impact the Toolkit can have–for example:

> ***"....we do a problem, talk about it, how would our kids do this, we do it with the kids, we see how they do it, we talk about how they solved the problem, what their thinking was, so I think the Toolkit is exactly what it says. It's a toolkit, the tools for us to then take the next step and learn with our kids. And it's the discussion that I think is most valuable, from the problems to the kids and then back to the teachers."***

Toolkit Structure

The Fostering Algebraic Thinking Toolkit consists of an *Introductory Session,* a *Closing Session* and four thematic modules, each focusing on the collection and analysis of a different type of classroom data. Each module consists of four three-hour sessions. Three hours is our best estimate for the length of a session. During our field test, some study groups found that, for particular sessions, three hours were not enough for the extended and constructive discussions that they were engaged in, so they split those sessions in two. On occasion, your group may find it desirable to do the same.

All of the modules focus on helping teachers gain a better understanding of their students' algebraic thinking. The modules offer rich mathematical problems that encourage algebraic thinking. These problems become the means for collecting classroom data, which the teachers then analyze during the sessions. Teachers also work on problems together, examining their own thinking in the process.

Groups should start by working through the *Introductory Session* and the *Analyzing Written Student Work* module, before proceeding to the other modules. The three remaining modules can follow *Analyzing Written Student Work,* in any combination or order. The *Introductory Session,* the four modules, and the *Closing Session* are described below. Throughout the Toolkit, teachers are doing and discussing mathematics together, so that activity is assumed for all the module descriptions.

Introductory Session

This session introduces participants to the Toolkit, setting the stage for future sessions. You and colleagues could use this session as an "open house" for teachers who might be interested in participating and would like to find out more about the Toolkit. In this session, participants do the following:

- Familiarize themselves with how the study group members will work together
- Familiarize themselves with the format of Toolkit sessions and the Mathematical Thinking Record
- Explore the concept of algebraic habits of mind, in the context of several mathematics problems

Analyzing Written Student Work

This module picks up where the *Introductory Session* leaves off. It focuses on the development of skills in understanding algebraic thinking as evidenced in students' written work. In this module, participants do the following:

- Establish how they will discuss and analyze student work
- Analyze the algebraic thinking that can be seen in student written work on open-response math problems
- Become more familiar with using the language of the algebraic habits of mind to talk about algebraic thinking

Listening to Students

Building on participants' experience with analyzing algebraic thinking by examining written student work, this module focuses on analyzing the evidence of student thinking that often never makes it onto paper. In this module, participants do the following:

- Practice listening to students, with the goals of better understanding their thinking and determining differences between how students think and talk about mathematics and what they write down when they work on math problems
- Record and transcribe the dialogue among students working on math problems, and examine the evidence of algebraic thinking reflected in this data
- Consider steps that teachers can take to help students capitalize on the thinking potential shown in their problem solving

Asking Questions of Students

This module, building on the understanding of algebraic thinking that participants developed in *Analyzing Written Student Work*, helps them become more intentional about the questions they ask to elicit and foster algebraic thinking in their students. In this module, participants do the following:

- Observe one another in the classroom and discuss the observation afterwards, with the goal of helping one another reflect on the questions they ask their students
- Practice creating class plans that focus on the teacher's goals for a lesson and the questions the teacher can ask to help foster students' algebraic thinking
- Explore the different factors that influence the decision about what questions to ask

Documenting Patterns of Student Thinking

This module builds on the work done in *Analyzing Written Student Work* by helping teachers consider two related questions: "How do I assess the kind of algebraic thinking I have been examining in the Toolkit seminars?" and "What goals might I have for my students' growth in algebraic thinking, and how can I keep track of their collective progress toward those goals?" In this module, participants do the following:

- Understand the key role learning goals play in documenting student learning, in particular, the algebra expectations described in the National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics*
- Move from examining algebraic thinking on the individual level to a more systematic documentation of algebraic thinking patterns across a whole class
- Begin to think about adapting assessment tasks to be more in line with their goals for algebraic thinking.

Closing Session

This session, which is meant to be used by a group ready to move on from the Toolkit, gives participants the opportunity to reflect on what they have learned while using the Toolkit as well as look ahead to what their next steps will be, given their Toolkit experiences. In this session, participants do the following:

- Discuss what has been learned by using the Toolkit after reviewing the Toolkit Introduction
- Examine the mathematical thinking that has been evident across the Toolkit sessions, using the Mathematical Thinking Records as data
- Explore possibilities for how the group could continue or what next steps group members could take

Before going on to discuss the principles and values underlying *The Fostering Algebraic Thinking Toolkit*, we address briefly how its structure can be used and adapted in different settings. The Toolkit will typically be used in an ongoing, facilitated process employing two of the modules over the course of an academic year. Including the *Introductory Session*, this makes for a total of nine sessions during the academic year. With relatively even spacing, one session would take place every three weeks or so. This requires the following:

- Support of study group members to meet every three or four weeks for a three-hour session
- Arrangement of schedules so that teachers can observe one another and meet afterwards in pairs, when the *Asking Questions of Students* module is being used
- Someone to facilitate the seminars. This can be one person or a team of two. In several field-test sites, study groups were co-facilitated by two volunteers, an arrangement that appeared to work quite well

Other options may fit different contexts and teams of teachers. For example, a team may decide to use all four of the modules, spending more than one year doing so. After having used one module, a group may split up according to interest, with subgroups pursuing different modules. A team may focus on one module for more than four sessions. A team may even opt to use only a single four-week module. However, while this approach will still be useful, it will probably not be as fulfilling as the use of multiple modules, because of the modules' complementary nature. Finally, it is possible to repeat some of the sessions using math problems from *Fostering Algebraic Thinking* that do not appear in the Toolkit. A team may decide to take this option.

WHAT IS THE UNDERLYING PERSPECTIVE ON PROFESSIONAL DEVELOPMENT?

A Focus on Mathematical Thinking

A particular view of learning and teaching informs these materials, a view that we hope teachers will explore and use to inform their classroom practice. In this view of instruction, students are actively engaged in the learning process. The teacher relies less on explaining facts and procedures, and more on helping the students make meaning for themselves; less on waiting for the single answer the teacher wants or expects to hear, and more on listening to the students and trying to understand how they are thinking. However, the Toolkit does not attempt to directly teach a teaching methodology. Rather, it offers you and your colleagues the chance to reflect on your own practice, and to choose to adapt it—if you wish to—based on fresh insights into algebraic thinking and how your students' algebraic thinking is developing.

We think you will find that this process of studying, exchanging ideas with colleagues, reflecting, and, finally, adapting classroom practice will fit your goal of improving students' learning. A focus on mathematical thinking will benefit your students—in the long run, if not the short run.

> *"I think it's been wonderful because we've really sat down and talked about what types of thinking algebra involves. You don't really think about that when you're teaching it, you just kind of go through your routines and do it. This has made me stop and really think of how the kids might look at it and the different ways to approach a problem. Because it's so group involved, and we share all the different student work, and our work, and everything's collaborative, you just see so many different ideas and viewpoints. . . . "*

Analyzing Classroom Data

The Toolkit's approach to professional development emphasizes the systematic examination of data drawn from teachers' own classrooms. The Toolkit grounds reflections about learning in classroom data, specifically: student written work on assessment items and challenging math problems, transcripts of student mathematical conversations and explorations, and notes from teachers' observations of each other. You and colleagues will collect this kind of data and learn to examine it in depth in order to gain insights into your students' thinking.

You also will be able to draw on your experiences to develop everyday methods of probing more deeply into students' thinking. In addition, the Toolkit sessions will help participants build an understanding of algebraic thinking; this increased understanding is something you can use directly in your classroom, as well as apply to your own continued exploration of mathematics.

In their daily practice, teachers are asked to evaluate their students' work almost constantly. *The Fostering Algebraic Thinking Toolkit* offers participating teachers an opportunity, within its seminar meetings, to put the need for evaluation aside and to look at classroom data from a different perspective, one which they can integrate into their practice alongside evaluation.

> *"We're always trying to analyze why somebody didn't do something. What this has allowed me to do, helped me to do, made me more aware of the need to do, is analyze, not only the kids who are doing things wrong, but also analyzing the kids who are doing things right. To look at kids' work a little differently . . . it's sparked an interest, maybe, in actually looking more closely at . . . what do they think . . . That's been an interesting change in the way I work."*

Mathematics in the Toolkit

The Toolkit offers examples of the kind of mathematics problems that can elicit and develop algebraic thinking in middle school students (drawn, in many cases, from *Fostering Algebraic Thinking.*) It is not a teaching curriculum for the classroom; rather, the Toolkit's math problems serve as opportunities for teachers to gather data about algebraic thinking in their students, their colleagues, and themselves. The Toolkit's aim is to help teachers increase their understanding of algebraic thinking through examining real classroom data and reflecting on their own thinking.

The Toolkit is intended primarily for use by mathematics teachers in grades 6 through 10. As a group, students in these grades represent a wide range of ages and previous mathematical experience. Some students will find the Toolkit's math problems more difficult than others. However, most students will be able to benefit from working on these problems. Since the Toolkit is neither a mathematics curriculum nor a tool for evaluating students' performance, you needn't worry about whether the problems presented in the Toolkit are at the "right level" for your students. Occasionally, Toolkit field-test teachers found it necessary to pave the way for students' access to problems—e.g., clarifying vocabulary or assumptions in the problem statements—then generally were impressed by the quality and diversity of thinking once the students were attempting their own solutions

The point of having students do these problems, in conjunction with the Toolkit seminars, is for you and colleagues to gather information about how your students think algebraically. Any work that students do can offer this information, even if they don't finish a problem or don't manage to get the right answer. In fact, sometimes a teacher can get more insight into a student's thinking by looking at how the student arrived at an incorrect answer, or by identifying what the student struggled with.

A Collaborative Approach to Professional Development

> *"Most of the [inservices] that I've been to have mostly been like you sit and somebody presents you with some information. Not where you actually get involved, and you go and you do an activity with your class and then you bring it back and talk about it."*

The Fostering Algebraic Thinking Toolkit supports a collaborative approach to the examination of teaching practice. This experience is intended to be more collaborative and less predetermined than the typical situation, in which an expert delivers new content knowledge, demonstrates new methods, and so on. The Toolkit also aims to increase teachers' confidence in their ability to discuss and interpret student work with other teachers. Within a school or district, this common experience and approach to teaching algebra will promote further professional growth by enabling teachers to work and plan instruction together more productively.

Although the sessions are facilitated, there is no "instructor" involved; rather, the group uses the Toolkit as a guide for its own learning experience. The Toolkit does convey some new concepts through its written material but, for the most part, the learning that occurs in the sessions happens through the group's own work, discussions, and reflection. Participants must actively engage in the process by working on mathematics together, bringing in examples of student work and other data, observing one other, thinking hard about students' thinking, and sharing their ideas with the group. The facilitator's role as a meeting chair, rather than an expert, means that she or he will learn about and examine algebraic thinking along with the rest of the participants. All participants, including the facilitator, will need to take an experimental, exploratory attitude and be willing to struggle and figure things out on their own. Some may need to learn to hold back their impulse to rush to judgment; others may need to develop or learn to trust their own judgment instead of relying on an outside expert. The extent to which the facilitator and the participants understand and appreciate these differences will have a lot to do with how much they benefit from the sessions.

The Toolkit's vision of an ideal professional development experience mirrors its vision of an ideal classroom experience. Just as it is important for classroom learning to be based on student thinking and student participation, the learning in this professional development setting is based on the experience and ideas of the participating teachers. The Toolkit's sessions are designed to let teachers be as active and as interactive as possible. Through discussion and hands-on activities, teachers interpret and make connections between ideas from the Toolkit, their own experiences, and the perspectives of their colleagues. As the group explores mathematical thinking together, it builds on differences in how group members see and understand these ideas to come to a broader and richer picture of the possibilities for classroom mathematics teaching.

> *"To do mathematics as a group also helps us realize where our differences are and may raise an issue for another meeting. Sort of, 'Oh, I didn't realize you did it this way, I've been doing it this way, are we at odds with each other in how we're explaining things to kids?' Whenever you have an opportunity to really work hard at something together, it helps you as a school house and department."*

What Do We Mean by "Algebraic Thinking"?

> *"This isn't an algebra course. This is algebraic thinking, and you would be surprised how much you already teach it and don't recognize it."*

In 2000, the National Council of Teachers of Mathematics (NCTM) released its *Principles and Standards for School Mathematics* (NCTM, 2000). The document lists algebra learning expectations for middle grades students:

- Represent, analyze, and generalize a variety of patterns with tables, graphs, words, and, when possible, symbolic rules
- Relate and compare different forms of representation for a relationship
- Identify functions as linear or nonlinear and contrast their properties for tables, graphs, or equations
- Develop an initial conceptual understanding of different uses of variables
- Explore relationships between symbolic expressions and graphs of lines, paying particular attention to the meaning of intercept and slope
- Use symbolic algebra to represent situations and to solve problems, especially those that involve linear relationships
- Recognize and generate equivalent forms for simple algebraic expressions and solve linear equations
- Model and solve contextualized problems using various representations, such as graphs, tables, and equations
- Use graphs to analyze the nature of changes in quantities in linear relationships

Where does the Toolkit and its emphasis on algebraic thinking fit into the efforts to make these expectations a reality for students? For the middle grades, at least, the term "algebraic thinking" can be taken to mean the thinking that leads to these learning outcomes and keeps them alive for further learning. Before they are formally introduced to algebra, students have already begun to think algebraically, developing powerful lines of thinking that go into productive algebraic performance. This Toolkit is based on the belief that teachers can profit from paying attention to how algebraic thinking—however rough and unfinished—develops in their students and how it informs their mathematical work. The Toolkit is intended to provide several different ways for teachers to shed light on the development of algebraic thinking in their students.

The Algebraic Habits of Mind

Certain lines of thinking, when used habitually, lead students toward the learning outcomes listed above. The Toolkit focuses on three categories that seem to be critical to developing power in algebraic thinking. We call these categories "algebraic habits of mind," to emphasize the importance of students' developing the *habit* of applying these lines of thinking whenever they approach math problems.

The three habits of mind described in the Toolkit materials are as follows:

- **Doing/Undoing:** Effective algebraic thinking sometimes involves reversibility, that is, being able to undo mathematical processes as well as do them. In effect, it involves the capacity not only to use a process to get to a goal, but also to understand the process well enough to work backward from the answer to the starting point. So, for example, in a traditional algebraic setting, algebraic thinkers can not only solve an equation like $9x^2 - 16 = 0$, but also answer the question, "What is an equation with solutions 4/3 and -4/3?" Doing/Undoing also involves the capacity to take apart numbers and expressions as well as put them together. For example, students invoke this habit of mind when they realize they can regroup numbers into pairs that equal 101 to make the following computation simpler: "Compute: $1 + 2 + 3 + \ldots + 100$." They recognize that 101 can be decomposed into 100 + 1; 99 + 2; 98 + 3; and so on.

- **Building Rules to Represent Functions:** Critical to algebraic thinking is the capacity to recognize patterns and organize data to represent situations in which input is related to output by well-defined functional rules. For example, here is a functional rule that is computation-based: "Take an input number, multiply it by 4, and subtract 3." One way to represent this rule is by constructing a table of how the rule works for the natural numbers:

1	1
2	5
3	9
4	13
5	17
.	
.	
?	41

This habit of mind is a natural complement to Doing/Undoing, in that the capacity to understand how a functional rule works in reverse generally makes it a more accessible and useful process. For example, thinking about how to go from an output like 41 back to its input requires deeper thought about the rule connecting input to output than merely thinking about how to produce the output.

- **Abstracting from Computation:** This is the capacity to think about computations independently of particular numbers used. "Abstraction" has always been one of the most evident characteristics of algebra. But, just what is being abstracted? A good case can be made that thinking algebraically involves being able to *think about computations freed from the particular numbers they are tied to in arithmetic*—that is, abstracting system regularities from computation. For example, think of the question, "Which is the larger number, 5% of 7,000 or 7% of 5,000?" One way to answer this question is to use the numbers given and work through the two computations. Some people, however, know enough about the computational process they use in figuring such percentages to realize that it doesn't matter in what order the 5 and 7 appear—or any other pair of numbers, for that matter. They can reason from this knowledge that the answer to the question is that 5% of 7,000 and 7% of 5,000 are the same number. This is an example of Abstracting from Computation.

Awareness of the algebraic habits of mind can inform classroom practice in several ways:

- Examining student work from the perspective of algebraic habits of mind gives the teacher a deeper understanding of how the students are thinking and what fundamental struggles they may be going through.
- Considering the algebraic habits of mind prompts a style of assessment that acknowledges differences in student thinking and recognizes multiple kinds of productive algebraic thinking.
- Teachers can foster algebraic thinking in their students by making sure to ask the kind of questions that will help students develop and practice the habits, by providing opportunities for mathematical exploration, and by prompting classroom investigations and discussions that encourage different ways of finding answers.
- Some teachers may even find that it is useful to discuss the algebraic habits of mind directly with their students, to help students start thinking about their own thinking processes.

Bear in mind that the algebraic habits of mind are a language for describing algebraic thinking. You are not expected to "teach" the habits of mind to your students. If you decide to deal with them explicitly with your students, it is probably not useful to treat the algebraic habits of mind as a list of skills to be memorized by students or checked off on a list by the teacher. Instead, students may find it useful to reflect on their own thinking processes and the strategies they use to solve problems, as you will reflect on your own thinking during the Toolkit sessions.

Algebraic thinking develops slowly over time. You should not expect evidence of the algebraic habits of mind to look the same in ninth grade as it does in fifth grade. No one—teacher or student—is expected to develop productive thinking overnight. However, if you start looking more closely at classroom data for evidence of algebraic thinking, using the algebraic habits of mind as a tool to describe and understand your students' thought processes, you will be able to see how their thinking grows and deepens as time goes by.

> *"Doing arithmetic shortcuts, that's something I see a lot in 6th grade work, and point that out, that it's algebraic thinking. Again, when I talked about thinking about the rules independent of the numbers, and when they justify a rule. Today we were doing roots and square roots and talking about doing and undoing, and the inverse operation machine, I made reference to that as being algebraic thinking."*

Each habit of mind incorporates various productive lines of thinking that are part of developing a facility with algebra. The Algebraic Habits of Mind tables and diagram that follow show some of these features, grouped under three headings for the habits of mind they represent.

BUILDING RULES TO REPRESENT FUNCTIONS

Feature Name	Description	Example
Organizing information	Organizing information in ways useful for uncovering patterns and the rules that define the patterns	With only 7-cent and 10-cent stamps, can you make all postage amounts above $1.00? ("I'll make a table: one column is only combinations of 7-cent stamps and no 10-cent stamps, the next one combinations of one 10-cent stamp with different numbers of 7-cent stamps, and so on until I see whether any amounts of postage will be left out.")
Predicting patterns	Noticing a rule at work and trying to predict how it works	"The 1^{st} figure has 4 dots; the 2^{nd} has 10 dots; the 3^{rd} has 18 dots, and the 4^{th} has 28 dots. It looks like each stage is the number of that stage times that number plus 3. So, I predict the n^{th} figure will have $n(n + 3)$ dots."
Chunking the information	Looking for repeating chunks in information that reveal how a pattern works	How many even numbers are there in the 20^{th} row of Pascal's Triangle? ("Let's see. You can build the 20^{th} row from the 19^{th} row—each entry is the sum of the two entries right above it. I should be able to use that kind of repetition from each row to the next . . .")
Describing a rule	Describing the steps of a rule without using specific inputs	"To see if the Winter Olympics will be held in a particular year, divide that year by 4. If the remainder is 2, then the answer is yes. Otherwise, no."
Different representations	Wondering what different information about a situation or problem may be given by different representations, then trying the different representations	"The straight line $y = (\frac{7}{15})x + \frac{1}{3}$ passes through two points with integer coordinates (10, 5) and (-20, -9). Are there other points with integer coordinates on that line? One way to see is to build an x–y table, plugging in integers for x and figuring y, to see if any integers appear for y. I can also graph the straight line to help me see what promising values of x might be for the table."
Describing change	Describing change in a process or relationship	"What does this graph tell me about the speed of the car? In the early seconds it picks up speed pretty steadily, then it levels off for about 30 seconds, slows down for the next 30 seconds, then speeds up steadily again for the next 2 minutes."
Justifying a rule	Justifying why a rule works for "any number"	How many sections will a piece of paper have if you fold it in half *n* times? ("Let's see. Each time I fold the paper, each existing section becomes two sections. So, I'm really doubling the number of sections with each fold. So, if I fold *n* times, there will be 2^n sections.")

ABSTRACTING FROM COMPUTATION

Feature Name	Description	Example
Computational shortcuts	Looking for shortcuts in computation, based on an understanding of how operations work	What is 1 + 2 + 3 + . . . + 30 equal to? ("I can pair the 30 and 1, 29 and 2, 28 and 3, and so on. I end up with fifteen 31's, so the answer is 15 x 31= 465.")
Calculating without computing	Thinking about calculations independently of the particular numbers used	"If the average age on this team is 26.3 years old today, the average age will be 27.3 years a year from today, assuming the same players are on the team. That's because if I went to figure out the average, I'd be adding as many 1's as there are players."
Generalizing beyond examples	Going beyond a few examples to create generalized expressions, describe sets of numbers, or either state or conjecture the conditions under which particular mathematical statements are valid	"The whole numbers that leave remainder 1 when divided by 3 and remainder 2 when divided by 5 are 7, 22, 37, 52, ... Those numbers can be written as 7 + 15n, where n is any non-negative integer."
Equivalent expressions	Recognizing equivalence between expressions	"Here are two ways to divide 159 by 13. First I wrote 159 = 130 + 29 = 169 - 10. Since I knew that 130 is ten 13's and 169 is 13 squared, then I only needed to deal with the 29 and the 10 . . . "
Symbolic expressions	Expressing generalizations about operations symbolically	"So, for any numbers a and b, a × b × 0.01 will equal b × a × 0.01."
Justifying shortcuts	Using generalizations about operations to justify computational shortcuts	Which is larger, 5% of 7 billion dollars, or 7% of 5 billion dollars? ("I know that 5% of 7 billion dollars is the same as 7% of 5 billion dollars, because taking a percentage is the same as multiplying by 0.01, so both numbers are 5 × 7 × 0.01 × 1 billion, which will come out the same no matter what order you do the multiplication.")

DOING/UNDOING

Feature Name	Description	Example
Input from output	Finding input from output, or initial conditions from a solution	"The curve of $y = x^2 + 8$ will reach a height of 100 on the graph when x is somewhere between 9 and 10."
Working backward	Working the steps of a rule or procedure backward	When a number is divided by 3, the remainder is 1. When the number is divided by 5, the remainder is 3. When the number is divided by 7, the remainder is 1. What is the number? ("First of all, what numbers have a remainder of 1 when divided by 3 . . . ?")

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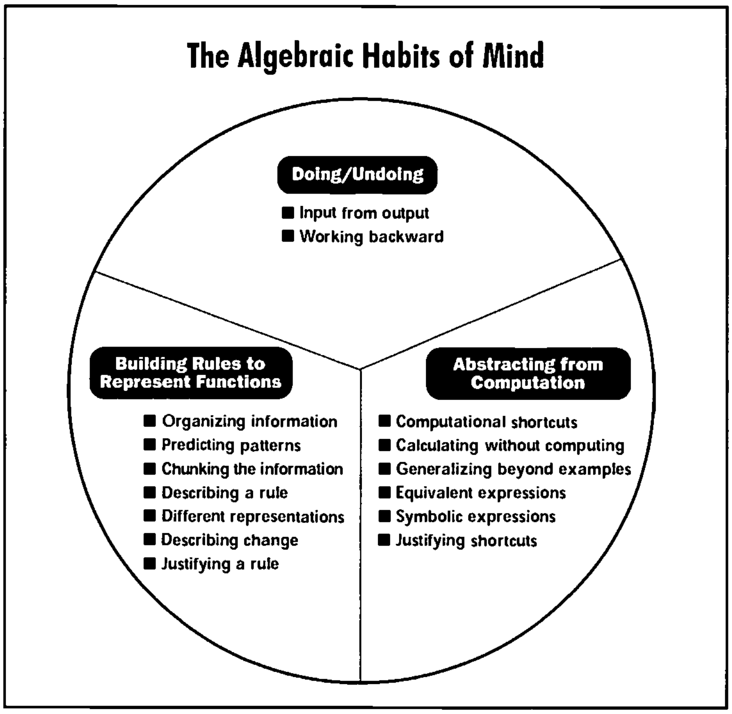

The Algebraic Habits of Mind in the Context of the NCTM Principles and Standards

Let's revisit the algebra expectations from the *NCTM Principles and Standards* listed earlier, this time using a habits-of-mind lens. If one looks at each expectation as a learning outcome and asks what habits of thinking could lead to this outcome, one can tease it apart to see several of the lines of thinking we listed for the three habits of mind. Take, for example:

- Represent, analyze, and generalize a variety of patterns with tables, graphs, words, and, when possible, symbolic rules

The following lines of thinking, which we associated with the habit of mind, Building Rules to Represent Functions, are relevant to this expectation:

- ➤ Organizing information in ways useful for uncovering patterns and the rules that define the patterns
- ➤ Noticing a rule at work, and trying to predict how it works
- ➤ Looking for repeating chunks in information that reveal how a pattern works
- ➤ Describing the steps of a rule without using specific inputs
- ➤ Justifying why a rule works for "any number"

Similarly, the following expectation/outcome relates to features of the habit of mind Abstracting from Computation:

- Recognize and generate equivalent forms for simple algebraic expressions and solve linear equations

Those features are:

- ➤ Thinking about calculations independently of the particular numbers used
- ➤ Looking for shortcuts in computation based on an understanding of how operations work
- ➤ Expressing generalizations about operations symbolically
- ➤ Using generalizations about operations to justify computational shortcuts
- ➤ Recognizing equivalence between expressions

Each of the remaining NCTM expectations can be similarly viewed with the habits-of-mind lens. The point behind this exercise is important, namely, that to help students meet these expectations for learning algebra, it is critical to understand and foster the lines of thinking that can propel them there.

Working with the Habits of Mind

Mathematics problems like those in the Toolkit invite a rich diversity in student responses. You will come across many pieces of student work that approach a problem differently, yet use the same feature of a habit of mind. One student may organize information by drawing a diagram, while another approaches the same problem by making a table. Both are using the "organizing information usefully" aspect of Building Rules to Represent Functions. From there, the students may take very different pathways to solving the problem, and the habits-of-mind lens is meant to help teachers gain a deeper understanding of the differences.

The habits-of-mind lens also is useful for thinking about the areas in which students may be having difficulty and in what ways their algebraic thinking is developing. Teachers are encouraged to compare the characteristics of productive algebraic thinking and the patterns that typically characterize students' development of algebraic thinking, as reflected in student work. Some characteristics of student work that may indicate difficulty with particular habits of mind are listed on the next page:

Habit of Mind	Indicator of Learner Difficulty
Building Rules to Represent Functions	• Developing rules that are incomplete or otherwise have little predictive power • Confusing the links among the verbal, symbolic, graphical, and tabular representations of rules • Failing to back up predictions about the rule with any convincing argument
Doing/Undoing	This habit of mind appears to be lacking in many students' thinking because of lack of practice. However, one impediment for many students is: • Failing to see that a rule based on computations is a rule that can be reversed, and instead, seeing these computational rules as strings of actions going in one direction
Abstracting from Computation	• Appearing to see calculations as only about the particular numbers used; shows no signs of thinking about calculations independently of the numbers used • Focusing only on the "right ways" to calculate, with no inclination to use shortcuts

Thinking about habits of mind can help teachers develop useful habits of their own: habits of listening, understanding, and teaching. To incorporate the idea of algebraic habits of mind into your teaching means seeing instruction as more than the transmission of facts, skills, and procedures. Instead, teachers thinking in terms of habits of mind treat instruction as a process of helping students form and practice productive thinking habits.

The algebraic habits of mind are a useful language for describing algebraic thinking. However, they are not the only language available. The goal of the Toolkit is to help teachers become comfortable with identifying, examining, and discussing algebraic thinking. Learning to look at algebraic thinking in terms of the three habits of mind is useful for this process, but it is not a goal in and of itself. (For further reading about the algebraic habits of mind, see *Fostering Algebraic Thinking*.)