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AUTHOR	Boero, Paolo
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ABSTRACT

This paper discusses what the theory of abstraction is about, the need for a theory of abstraction in mathematics education, and the requirements that such a theory should meet. All three are reconsidered from a personal point of view. (KHR)

Leont'ev, A.N.: 1959, *Problemy razvitja psichiki*. Izd. Ak. Ped. Nauk RSFCR, Moskva.

Gravemeijer, K.: 1998, 'Developmental research as a research method'. In A. Sierpinska and J. Kilpatrick (eds.), *Mathematics Education as a Research Domain: A Search for Identity*. Dordrecht: Kluwer Academic Publishers.

ABSTRACTION: WHAT THEORY DO WE NEED IN MATHEMATICS EDUCATION?

Paolo Boero

I will divide my reaction into two parts. In the first part I will follow the grid *("what is the theory about"*, etc.). In the second part I will discuss the need for a theory of abstraction in mathematics education, and the requirements that, in my opinion, such a theory should meet, then I will reconsider the three theories from this personal point of view. In the sequel, G, GT, SHD will indicate the three theories (according to the initials of the family names of their authors).

According to the Grid...

Most of suggested reading criteria (*What is the theory about? What assumptions are being made? What does the theory claim? What terms are used and what do they mean? How has the theory been validated?*) are related to minimal, necessary requirements that each theory (as a "scientific theory") must meet in human sciences (psychology, anthropology, sociology, etc.). The second part of the last reading criterion (*What are the aims of the theory and what are its applications?*) refers to a specific challenge for theorists in mathematics education.

More or less explicitly each presented theory satisfies the first and the third requirement. Here it seems to me that theories G and SHD deal with subjects that are rather close to each other (a common title might be "abstraction in context", yet with a different meaning of the word "context" – see later), while in the case of GT the theory deals with mathematical content and related individual learning processes.

Different ways of satisfying the second requirement are followed within the above presentations of the three theories. In the case of GT, "assumptions" are intended as general assumptions derived from other theories in order to create an environment where the theory can develop and be better understood; in the case of G and SHD, general assumptions are internal to the theory and constitutive of the core of the theory itself. We can observe how G aims at self-sufficiency in the presentation of the theory, while SHD refers to existing general theories (especially "activity theory"). The second criterion poses some problems: in the case of G, what meanings may be attached to crucial terms in a self-sufficiency perspective? (See later). In the case of GT and SHD, what relationships to establish with related theories?

Concerning the latter problem, we can identify different attitudes in mathematics education research as well as in the GT and SHD presentations. In the GT case, an autonomous elaboration (the procept theory) is linked to existing general theories in the field of psychology (Piaget's *"Pseudo-empirical and reflective abstraction in arithmetic, algebra and calculus naturally focus on our notion of procept."*) and used to reinterpret some other theories in the field of mathematics education: *"We do not agree with Sfard or Dubinsky that the development invariably proceeds in a sequence we describe as procedure-process-procept"*. In my opinion the legitimacy of these links and re-interpretations should be carefully discussed. As concerns the specific section "*What assumptions are being made?*" in the GT presentation, in my opinion the need for this kind of discussion becomes ever stronger: for instance, beyond heuristic hints, what are the precise relationships between the "*semantic squeeze*" in Bruner's quotation, the *"reduction in brain area involved"*, considered in neurophysiology studies, and the construction and functioning of procepts? In the SHD case, activity theory is taken as a fundamental reference. In my opinion, the adoption of an activity theory reference paradigm needs to consider the teachers' role as constitutive of the "learning" process (in our case, of the "abstraction" process). Indeed in Vygotsky's seminal work it is well known that a crucial, recurring term is "*obucenie*", that means *"teaching and learning"*. Yet I see (in the article as well as in the SHD forum presentation) that this "joint activity" aspect is not sufficiently developed.

Let us come now to the most critical criterion: *What terms are used and what do they mean?* It is clear that the danger for a person like me, who was educated as a mathematician, did research in mathematics for some years and still teaches mathematics at the University level, is to apply such criterion in the strict way he generally uses when dealing with his students' mathematical performance. It is true that in the human sciences domain it is very difficult to give "definitions" in the same, strict sense. In most cases, definitions are reduced to some evocative words that suggest a meaning, and then the context provides the full meaning. But I think that within the same theoretical construction (a theory), or the presentation of a theory, a crucial term must have a rather precise meaning (in order to establish whether or not an object or a situation falls within its semantic domain) and keep it. From this point of view, I find that in G and SHD the meaning of some crucial terms is not sufficiently clear, while the meaning of other terms seems to change during the presentation of the theory. In particular, I refer to the following terms:

"Formal mathematics" (in G): "formal" according to a high level of formalisation? And/or according to a social (or academic) consensus about ways of presenting relevant concepts, validating statements, etc.?

"Mathematical reality" (in G): what is its psychological and epistemological status? A subjective construction (or re-construction)? A historically shared and inheritable production, rooted in mankind's needs and experiences? A set of shared conventions?

"Structure" (in SHD): one part of the axiomatic organisation of mathematical knowledge (e.g. "the structure of group")? And/or the overall organisation of mathematical knowledge? And/or the organisation of mathematical thought?

"Context" (in SHD): in mathematics education, like in psycholinguistics, the word "context" takes different meanings:

- that of "situation context": those factors affecting the mathematical performance that are related to the situatedness of the students' activity (including social relationships in the classroom, environmental factors, etc.);
- that of "task context": the task evokes specific "realities" and constraints; as a consequence, behaviours, schemes, etc. related to those "realities" are activated;
- that of "inner context": in this case the attention is focused on the (internal) representation of the subject's past and present experience.

These different meanings of the word "context" suggest different perspectives under which teaching and learning mathematics in the classroom can be considered. For instance, in the case of abstraction the second perspective suggests to choose peculiar tasks suitable for it, while the first perspective suggests to take into account the social interactions that the teacher must "orchestrate" in the classroom.

Concerning the *aims* and the *applications* of the three theories, they are very different. Here again there are strong analogies between G and SHD (the theories are intended to provide useful tools to plan and/or improve teaching projects, and better interpret what happens in the classrooms where planned teaching is implemented). In the case of GT, the focus is on interpretative aims and in particular on explaining *"why some students are so highly successful with symbols, whilst others are procedural at best"* (etc.). In my opinion, in mathematics education we need both types of theories, bearing in mind that a theory of the second type can develop in (or support) a theory of the first type, and that a theory of the first type can provide interesting research questions for theories of the second type.

Concerning *validation of theories*, it seems to me that (in relationship with their specific aims) each theory meets this requirement. However I must say that it is met not so much in the above papers as in the articles included in the references: this is an unavoidable, necessary consequence of the space limitations of presentations.

Do we Need a General Theory of Abstraction in Mathematics Education? What Kind of Theory?

Let us consider the following examples:

- a right-angled triangle is drawn on the blackboard; students draw right-angled triangles on their copybooks; the teacher illustrates Euclid's theorem;
- the teacher writes on the blackboard: $(uv)'=u'v+uv'$, then $\int x\sin x dx=$, then illustrates and justifies the well known method of integration of the $x\sin x$ function based on the law of derivation of products of functions;

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- the teacher draws a square on the blackboard, then one diagonal, then he proves (by the usual "reductio ad absurdum" proof) that the diagonal and the side of a square are incommensurable: .... $d^2=2s^2$, $(d/s)^2=2$, etc.
- the teacher establishes a 1-1 correspondence between the set of even numbers and the set of all natural numbers, then defines "infinite sets" as those sets which are equivalent to a proper subset.

In each of these cases mathematicians recognise some specific aspects of "abstraction". Mathematics educators have tried to deal with these aspects in different ways. For instance, C. Laborde and B. Capponi define a (geometric) figure as the set of couples (O,d_i), where O is the geometric object (e.g. the right-angled triangle) and d_i is one of the drawings that constitute the 'material representation' of the geometric object. Therefore the figure *"is the product of the abstraction process performed by the subject when, starting from a drawing (signifier) he or she thinks about the represented geometric object"*. This definition of figure is useful to deal with some difficulties that students meet when they approach geometrical reasoning (for instance, the reference to the peculiarities of a drawn right-angled triangle, or the stereotyped representation of the height of a triangle). These considerations are specific to the kind of abstraction inherent in the first situation. For the second situation an entirely different theoretical approach to abstraction is needed. Indeed, the 'material representation' is related to the represented mathematical object in a completely different manner: on one side the 'material representation' is much more distant from the mathematical object, on the other it becomes the starting point for a chain of transformations performed on the written expressions according to general syntactic rules. The third example shows some partial similarities both to the first example and to the second. Let us consider the fourth example: the idea of "epistemological obstacle" was elaborated in order to cope with students' difficulties inherent in "accepting" some cultural, "abstract" constructions like the "equivalence" between a set and a proper subset, which contradict our usual experiences about the sets of objects that we can describe extensively (i.e. by listing all their elements).

I add, as explicitly quoted in GT and in SHD, that general theories of abstraction already do exist in psychology.

A recurrent question for me, when reading the contributions for this panel, was: do we need a **general** theory of **mathematical** abstraction in **mathematics education**, i.e. a general theory suitable for describing and interpreting many typical phenomena of "abstraction" that intervene in teaching and learning mathematics and, possibly, controlling them (i.e. planning teaching in order to get the best results) by selecting pertinent variables and coming to predict the effects of actions on them?

In my opinion, a general theory of mathematical abstraction that would be of interest for mathematics education purposes should:

- cover most forms of abstraction currently met in teaching and learning mathematics at different school levels;

- interpret difficulties met by students in tackling abstraction in their approach to mathematical knowledge;
- point out relevant variables accessible to educational intervention;
- take into account relevant research in the field of epistemology of mathematics (as concerns reflection on abstraction) and cognitive sciences (as concerns general theories about abstraction, or specific theories about mathematical abstraction).

According to the first three requirements that I propose, each of the three theories offers some relevant contributions but also shows important weaknesses. G shows (through the example sketched in the PME contribution and other evoked examples) how some processes of abstraction can work, helping both the designers of teaching projects and teachers to plan and manage suitable classroom situations. But it seems to me that G does not cover processes of abstraction that are needed when a "break" is unavoidable in the transition from mathematical experience in real contexts to "formal mathematics". Moreover, the role of mediation played by the teacher, which is particularly crucial in this case, is not explicitly dealt with. GT is suitable to cover many "abstract" mathematical objects, but it seems to me that in my first example it does not provide much help in dealing with students' difficulties in managing the "abstract" notion of right-angled triangle (e.g. in the case of stereotyped images). And (being mainly a theory about mathematical objects in their relationships with generating processes) I see it might have some difficulties in dealing with the "abstraction" inherent in the activities (e.g. students' mathematical argumentation). SHD is a very general theory and it surely covers abstraction in a wide sense, but its current generality implies that some peculiarities are lost in specific cases. Perhaps in the future this theory will become suitable to deal in a productive way with all the examples provided at the beginning of this Section, but then the subject's individual processes should be better investigated in relation with the "situation context", the "task context" and the teacher's mediational strategies.

As concerns the fourth requirement that I propose, it seems to me that the three theories do not take sufficiently into account important, new streams of research in epistemology of mathematics, psychology and neurophysiology that are developing in different research communities. Recent joint studies in the fields of epistemology and neurophysiology (for an example the project "Geometry and cognition" at the ENS, in Paris) show the possibility of a convergence on the idea that even at the highest level of "abstraction" productive reasoning relies upon very "concrete", body related intuitions. This approach puts again into question, but within a new perspective (neurophysiology investigation), the idea of a purely conventional character of axioms and axiomatic theories; the anti-logicist positions have opposed this idea during the whole XX century under different perspectives (mainly philosophical or ideological or based on introspection). This approach also draws an almost entirely new picture of how high level professional mathematical activities are performed. It seems very interesting and promising that this stream of research goes in the same directions of some other streams of research in different disciplines (in particular, "embodied cognition", in psycholinguistics and psychology).

If it is true that "thinking abstract objects as if they were concrete" is possible, but "thinking in an abstract way" is impossible (as far as productive thinking is considered), then a theory of abstraction suitable for educational purposes should take charge of the whole complexity of the relationships between mathematical objects, the thinking processes concerning these objects, their "situatedness" in the classroom environment (the mediational role of the teacher being a crucial issue), and the most suitable "task contexts" for meaningful mathematical abstraction (both as concerns the mathematical content involved and the body-rooted processes). From this wide and very demanding point of view I think that the presented theories offer some important contributions, but we are still far from a comprehensive theoretical answer to the challenge of mathematical abstraction in mathematics education.

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